# TWO-POINT FUNCTIONS IN $\mathcal{N}=4$ SUPER YANG-MILLS WITH WILSON-LINE DEFECT 

Master's Thesis


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# TWO-POINT FUNCTIONS IN $\mathcal{N}=4$ SUPER YANG-MILLS WITH WILSON-LINE DEFECT 

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Since its first appearance in the 1970s [1, 2] following the development of conformal field theory (CFT), the conformal bootstrap program has lived several reincarnations. The modern idea of the bootstrap is to rely on the operator content of the theory and on numerical parameters called CFT data, which encode the dynamical information, in order to constrain CFTs in a non-perturbative fashion. CFTs are used for describing the behavior of quantum field theories and statistical systems near criticality, and in particular any quantum field theory can be seen as the perturbation of a UV conformal field theory. This suggests that some information about the Renormalization Group (RG) flow of quantum field theories is contained in the corresponding CFT. The conformal bootstrap has been very successful in 2 d theories [3], where the minimal models could be solved exactly. This is due to the presence of an extraordinary infinite-dimensional algebra called the Virasoro algebra, but an extension to theories in $d \geq 3$ appeared to be significantly more complicated. Until recently, it was not known how to approach the problem and little progress was made. But strong constraints on the CFT data of a theory can be obtained by assuming associativity of the operator algebra. This is most commonly referred to as crossing symmetry, and the renaissance of the conformal bootstrap came with the idea brought in [4] and based on [5] to derive bounds on the relevant physical quantities, instead of trying to solve the crossing equations for all the CFT data at once. Spectacular results were obtained since then, and in particular the conformal bootstrap has been used for setting bounds on the landscape of unitary CFTs [6, 7]. This has led to the discovery of new models, which for some of them do not have a known Lagrangian description yet [7]. Accessorily, the new techniques arising from this approach were used for deriving the most precise estimation of the critical exponents of the 3d Ising model to this day [8]. The conformal bootstrap can also be used in association to perturbation theory, where the computations have become increasingly tedious as higher-loop orders are being faced.

Maximally supersymmetric $\mathcal{N}=4$ Super Yang-Mills (SYM) theory in $4 d$ is a natural candidate for the conformal bootstrap program, since it is expected to be most effective for theories which are defined by a few properties only, such as its global symmetries [9]. $\mathcal{N}=4 \mathrm{SYM}$ has attracted a lot of attention since the emergence of the AdS/CFT correspondence, where it is believed to be dual to a type IIB string theory on $\mathrm{AdS}_{5} \times \mathrm{S}_{5}$ space, at least in the large $N$ limit [10]. This is particularly interesting, since gravity is inherent to string theory, while there is no trace of a spin-2 particle in the gauge theory. Another conjecture that makes $\mathcal{N}=4$ SYM all the more intriguing is the idea that it may be integrable [11], i.e. that it is possible to write down a system of equations encoding the quantities of interest in an exact fashion. Composite operators of special interest are scalar operators consisting of products of scalar fields, which are called single-trace operators, since taking the trace over the gauge-group indices is required in order to preserve gauge invariance. These operators are $1 / 2-B P S$, i.e. they preserve half of the supercharges. The two- and three-point functions of such operators are protected against radiative corrections [12, 13], which means that their conformal dimensions do not receive anomalous contributions.

As we have already discussed above, conformal symmetry imposes powerful constraints on the dynamics of a conformal field theory. However, this high degree of symmetry also leads to a loss of richness in structure. A certain class of extended operators called conformal defects presents the particularity of


Figure 1: Flowchart illustrating the concept behind this work schematically. The superconformal bootstrap associated to unitarity can be used for obtaining the functional shape of the two-point function in terms of superblocks. On the other hand, the two-point function can also be computed using perturbation theory, and comparing the result to the structure in superblocks yields the CFT data. Crossing symmetry relates the CFT data of the so-called bulk and defect channel expansions, and hence a large number of correlators can be obtained. The reader is invited to look at this chart once again after having read the full thesis.
breaking only mildly the conformal symmetry [14]. To be more precise, they preserve a large subgroup of the conformal symmetry of the vacuum, and hence the constraints of the defect-free theory are only partially relaxed. The correlators are constrained to a $S O(p+1,1) \times S O(q)$ symmetry, where $q$ is called the codimension of the defect and where $p+q=d$, and the bootstrap program can be extended to the study of such defect CFTs (see e.g. [15, 16]). A defect of particular interest in $\mathcal{N}=4$ SYM at large $N$ is the Maldacena-Wilson loop, which differs from the traditional Wilson loop in that scalar fields also couple to the loop [17]. This non-local operator has important applications in AdS/CFT, where it is believed to be dual to the area of minimal surface in $\mathrm{AdS}_{5}$ [18]. It turns out that the Maldacena-Wilson loop with an infinite-line geometry is also a $1 / 2$-BPS operator, and that its expectation value is just 1 at all orders in perturbation theory for the line geometry [19].

In presence of a defect, less constraints are present on the correlators and new crossing equations arise, which involve two-point functions. The two-point function of scalar operators was studied in the context of the defect conformal bootstrap in [14], while the superconformal case was done in [20] for defects of codimension one and three. 1/2-BPS line defects have been studied in e.g. [21, 22]. In this thesis, we repeat the analysis of $[14,20]$ specialized to the two-point function of single-trace operators in presence of the Maldacena-Wilson-line defect. This results in an expansion in superblocks, which fixes the functional form of the correlator non-perturbatively. Our goal is then to compute the twopoint function perturbatively, and to extract the (perturbative) CFT data by comparing the result to the superblock expansion. Because of crossing symmetry, it is not only the CFT data for the two-point functions that we obtain, but also the products of the CFT data for one-point and three-point functions. The flowchart in fig. 1 summarizes the concepts behind this work.

## Outline

This thesis is structured in four chapters. In the first one, we present the theoretical background necessary for understanding the content of the next chapters, including reviews of conformal field theory
and of supersymmetry. We also introduce conformal defects with a focus on the Maldacena-Wilson line.
The second chapter consists of the derivation of the superblocks for both the bulk and the defect channels, as well as a discussion of the crossing symmetry that relates the two CFT data sets. It also includes some considerations about the structure of the correlator, and we analyze its behavior in some limiting cases.

Chapter three contains the necessary ingredients of perturbation theory needed in order to compute the correlator with Feynman diagrams. It starts with a review of $\mathcal{N}=4 \mathrm{SYM}$ from the point of view of the action, then we present the relevant Feynman rules. Finally, we derive the scalar self-energy as well as the insertion rules related to the Wilson line and to the suitable vertices.

The last chapter is then dedicated to the computation of the correlator in perturbation theory up to next-to-leading order, and to the extraction of the CFT data for the defect channel. We conclude with the direct check of one coefficient.

After the conclusion, this thesis also contains three appendices, which respectively cover different aspects of group theory, the lists of the superblock coefficients and the analytical and numerical computations of the integrals encountered throughout this work.

## Conventions

All the computations are performed in 4 d Euclidean space, the only exception being the dimensional reduction performed from 10d Euclidean space in chapter 3.

In 4 dimensions, vectors are defined in the following way:

$$
x_{\mu}=(x, y, z, \tau),
$$

with $\tau$ the Euclidean time.
We will occasionally need the Fourier transform of propagators from momentum to position space in 4 dimensions, which is given by:

$$
\begin{equation*}
\int \frac{d^{4} p}{(2 \pi)^{4}} \frac{e^{i p \cdot x}}{p^{2}}=\frac{1}{(2 \pi)^{2} x^{2}} . \tag{0.1}
\end{equation*}
$$

See e.g. the appendix of [19] for an expression in arbitrary dimensions. This equation defines the convention that we will use throughout this work. Bosonic propagators are of the form $1 / x^{2}$ and are therefore Green's functions of the operator $\square$. In other words, they satisfy the following relation:

$$
\begin{equation*}
\square \frac{1}{x^{2}}=-(2 \pi)^{2} \delta^{(4)}(x) \tag{0.2}
\end{equation*}
$$

Note that we will often omit the superscript (4) of the $\delta$-function when the context leaves no ambiguity.

## CHAPTER 1

## Foundations

The content of this thesis strongly relies on the concepts presented in this chapter. Conformal field theory in 4 dimensions is first reviewed, including a short introduction to the conformal bootstrap. Supersymmetry is then covered, with an emphasis on maximally extended supersymmetry and the related superconformal algebra. Finally, we present conformal defects and define the extended operator called the Maldacena-Wilson line, which will be a central object of this work. Most of the material presented in this chapter is standard and can be found in greater detail in e.g. [14, 23, 24, 25].

### 1.1 Conformal Field Theory in 4d

This section is devoted to reviewing 4-dimensional conformal field theory (CFT), and is mostly based on [23] and [24]. We will work in Euclidean space, and thus the metric is defined as:

$$
g^{\mu \nu}(x)=\delta^{\mu \nu}
$$

with $\mu, v=1,2,3,4$.

## Poincaré Symmetry

An important subgroup of conformal symmetry is the Poincaré group. This is standard material in quantum field theory and will not be covered in depth here. The reader is invited to consult e.g. [26] for a more thorough analysis.

The Poincaré group consists of translations and Lorentz transformations, which are respectively defined by the following generators acting on functions:

$$
\begin{gather*}
P_{\mu} \equiv-i \partial_{\mu},  \tag{1.1a}\\
\left(J^{\rho \sigma}\right)^{\mu}{ }_{v} \equiv i\left(\delta^{\rho}{ }_{\nu} \delta^{\mu \sigma}-\delta^{\sigma}{ }_{\nu} \delta^{\mu \rho}\right) . \tag{1.1b}
\end{gather*}
$$

The infinitesimal form of the Lorentz transformations is given by:

$$
\Lambda^{\mu}{ }_{v}=\delta^{\mu}{ }_{v}+\frac{i}{2} \omega_{\rho \sigma}\left(J^{\rho \sigma}\right)^{\mu}{ }_{v}
$$

where $\omega_{\mu \nu}$ has to be antisymmetric with respect to $\mu \leftrightarrow v$. The Poincaré Lie algebra is then defined by the following commutation relations between the generators:

$$
\begin{gather*}
{\left[J^{\mu v}, P^{\rho}\right]=i\left(\delta^{\mu \rho} P^{v}-\delta^{v \rho} P^{\mu}\right),}  \tag{1.2a}\\
{\left[J^{\mu v}, J^{\rho \sigma}\right]=i\left(\delta^{\mu \rho} J^{v \sigma}+\delta^{v \sigma} J^{\mu \rho}-\delta^{v \rho} J^{\mu \sigma}-\delta^{\mu \sigma} J^{v \rho}\right)} \tag{1.2b}
\end{gather*}
$$

while all other possible commutators vanish. The commutator given in (1.2b) corresponds to $\mathfrak{s o}(3,1)^{1}$, which is referred to as the Lorentz algebra. Under an infinitesimal Lorentz transformation, the spin part of a field $\phi$ with components $\phi^{a}(a=1, \ldots, n)$ transforms as:

$$
\delta \phi^{a}=\frac{i}{2} \omega_{\mu v}\left(\mathcal{J}^{\mu \nu}\right)^{a}{ }_{b} \phi^{b},
$$

where $\mathcal{J}_{\mu \nu}$ has to satisfy (1.2b).
We wish now to review the finite-dimensional irreducible representations of the Lorentz algebra $\mathfrak{s o}(3,1)$. The simplest representation is the scalar representation $\phi$, which has an associated one-dimensional vector space and corresponds to $\mathcal{J}_{\mu \nu}=0$. Another important representation is the vector representation, for which the dimension is simply the number of spacetime dimensions $d$. In that case, the matrix $\mathcal{J}_{\mu \nu}$ is a $d \times d$-matrix, and is given by:

$$
\left(\mathcal{J}^{\rho \sigma}\right)_{v}^{\mu}=i\left(\delta_{v}^{\rho} \delta^{\mu \sigma}-\delta_{v}^{\sigma} \delta^{\mu \rho}\right)
$$

Higher-rank tensor representations can be constructed by considering tensor products of the vector representation. Such representations are in general reducible and can be decomposed into symmetric and antisymmetric tensors.

The Lorentz group admits another class of irreducible representations, which are called the spinor representations. Those can be constructed by use of the Clifford algebra, which is defined by:

$$
\left\{\gamma^{\mu}, \gamma^{v}\right\}=2 \delta^{\mu v} 0
$$

Here the $\gamma_{\mu}$ are called Dirac $\gamma$-matrices, and the simplest spinor representation can be obtained with:

$$
\mathcal{J}^{\mu v}=\frac{i}{4}\left[\gamma^{\mu}, \gamma^{v}\right] .
$$

This representation is called the Dirac spinor representation, and is reducible under the Lorentz algebra. Indeed the Dirac spinor $\psi$ can be projected onto two-component Weyl spinors $\psi_{L} \equiv\left(\psi_{L}, 0\right)$ (lefthanded) and $\psi_{R} \equiv\left(0, \psi_{R}\right)$ (right-handed). In chapter 3, we will encounter projections of Dirac spinors in 10 -dimensional space satisfying the following reality condition:

$$
\begin{equation*}
\psi^{*} \stackrel{!}{=} B C \psi \tag{1.3}
\end{equation*}
$$

where $B$ is a similarity transformation, i.e.:

$$
B \gamma_{\mu} B^{-1}=\gamma_{\mu}^{\dagger},
$$

and where $C$ is the charge conjugation matrix, defined by the following relation:

$$
C \gamma_{\mu} C^{-1}=-\gamma_{\mu}^{T}
$$

In this case, the associated particle, called Majorana fermion, is neutral under $\mathfrak{u}(1)$ transformations and is its own antiparticle.

[^0]The scalar, spinor and vector representations that we mentioned obey the following finite transformation laws:

$$
\begin{gathered}
\phi(x) \rightarrow \phi\left(\Lambda^{-1} x\right), \\
\psi^{\alpha}(x) \rightarrow S(\Lambda)^{\alpha}{ }_{\beta} \psi^{\beta}\left(\Lambda^{-1} x\right), \\
A^{\mu}(x) \rightarrow \Lambda^{\mu}{ }_{v} A^{v}\left(\Lambda^{-1} x\right) .
\end{gathered}
$$

To conclude, we note that the Poincaré group is not compact and hence there exists no unitary finitedimensional representation. This means that the representations have to be labeled by a continuous parameter, namely $p_{\mu}$.

## Conformal Symmetry

Let us now extend the Poincaré group to the full conformal group. The material presented here and in the next subsection is mostly based on [23]. The conformal group is defined as the group of anglepreserving transformations, i.e. the metric must transform in the following way:

$$
\begin{equation*}
g^{\prime \mu v}\left(x^{\prime}\right)=\kappa(x) g^{\mu v}(x) \tag{1.5}
\end{equation*}
$$

Note that, as mentioned in the previous subsection, the Poincaré group is a subgroup and corresponds to $\kappa=1$.

Conformal symmetry consists of 15 generators in 4 dimensions, which are the 10 generators of the Poincaré group complemented by the scalings $D$ and the special conformal transformations (SCT) $K_{\mu}$. These new generators are defined on functions as follows:

$$
\begin{gather*}
D \equiv-i x_{\mu} \partial^{\mu}  \tag{1.6a}\\
K^{\mu} \equiv i\left(x^{2} \partial^{\mu}-2 x^{\mu} x^{v} \partial_{v}\right) . \tag{1.6b}
\end{gather*}
$$

The conformal algebra is defined by the following commutation relations:

$$
\begin{gather*}
{\left[D, K^{\mu}\right]=-i K^{\mu},}  \tag{1.7a}\\
{\left[D, P^{\mu}\right]=i P^{\mu},}  \tag{1.7b}\\
{\left[K^{\mu}, P^{v}\right]=2 i\left(\delta^{\mu v} D-J^{\mu v}\right),}  \tag{1.7c}\\
{\left[K^{\mu}, J^{v \rho}\right]=i\left(\delta^{\mu v} K^{\rho}-\delta^{\mu \rho} K^{v}\right),} \tag{1.7d}
\end{gather*}
$$

to which we should also add the commutators of the Poincaré algebra. All other possible commutators vanish.

Scalar fields obey the following transformation laws:

$$
\begin{gathered}
D \phi(x)=i\left(x_{\mu} \partial^{\mu}+\Delta\right) \phi(x) \\
K_{\mu} \phi(x)=i\left(x^{2} \partial_{\mu}-2 x_{\mu} x_{v} \partial^{v}-2 x_{\mu} \Delta\right) \phi(x)
\end{gathered}
$$

where $\Delta$ is called the conformal or scaling dimension. It is defined by the action of a scale transformation on the field:

$$
\begin{equation*}
\phi(\lambda x)=\lambda^{-\Delta} \phi(x) \tag{1.9}
\end{equation*}
$$

The conformal group is obviously not compact, since the Poincaré group is not compact.
There exists a more concise notation for the conformal algebra if we allow for a ( $d+2$ )-embedding space. We start by defining a 6 -dimensional (flat) metric $\delta_{A B}(A, B=-1,0,1, \ldots, 4)$ and $J_{A B}$ such that:

$$
\begin{gathered}
J_{-1,0} \equiv D \\
J_{-1 \mu}+J_{0 \mu} \equiv P_{\mu} \\
J_{-1 \mu}-J_{0 \mu} \equiv K_{\mu} .
\end{gathered}
$$

It is straightforward to check that $J_{A B}$ satisfies the following commutation relation:

$$
\begin{equation*}
\left[J^{A B}, J^{C D}\right]=i\left(\delta^{A C} J^{B D}+\delta^{B D} J^{A C}-\delta^{B C} J^{A D}-\delta^{A D} J^{B C}\right) \tag{1.11}
\end{equation*}
$$

which corresponds to a $\mathfrak{s o}(5,1)$ algebra, i.e. a Lorentz algebra in six spacetime dimensions. This formulation of the conformal algebra is called the embedding formalism.

We will now discuss the representations of the conformal algebra. To that effect, we note that in unitary CFTs there exists a lower bound for $\Delta$. This is commonly referred to as the unitarity bound. Moreover, the generators $P_{\mu}$ increase $\Delta$, while $K_{\mu}$ decrease $\Delta$. Hence it is convenient to define fields which satisfy:

$$
\begin{equation*}
\left[K^{\mu}, \phi(0)\right]=0 \tag{1.12}
\end{equation*}
$$

i.e. the fields $\phi$ have the lowest dimension $\Delta$. Such fields are called conformal primaries. A conformal multiplet consists of a conformal primary and all its descendants, which are the fields with higher dimension $\Delta$ which can be constructed by applying $P_{\mu}$ an arbitrary number of times to the primary.

We conclude this discussion by noting that scalar conformal primaries have the following behavior under conformal transformations:

$$
\begin{equation*}
\phi(x) \rightarrow\left|\frac{\partial x^{\prime}}{\partial x}\right|^{-\Delta / d} \phi(x) \tag{1.13}
\end{equation*}
$$

with $d$ the number of spacetime dimensions as usual.

## Correlation Functions of Scalar Primaries

We wish now to investigate the consequences of conformal symmetry on the correlation functions of CFTs. In particular, we will focus on correlators of scalar primaries. The resulting constraints are very restrictive, and we find that the position dependence of two- and three-point functions is completely fixed up to a multiplicative constant.

We start by looking at two-point functions of scalar primaries $\mathcal{O}_{1}\left(x_{1}\right)$ and $\mathcal{O}_{2}\left(x_{2}\right)$. Because of Poincaré invariance, we know that the correlator can only be a function of the distance between $x_{1}$ and $x_{2}$ :

$$
\begin{equation*}
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right)\right\rangle=f\left(x_{12}^{2}\right) \tag{1.14}
\end{equation*}
$$

with $x_{12} \equiv x_{1}-x_{2}$. Moreover, scale invariance implies:

$$
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right)\right\rangle \stackrel{!}{=} \lambda^{\Delta_{1}+\Delta_{2}}\left\langle\mathcal{O}_{1}\left(\lambda x_{1}\right) \mathcal{O}_{2}\left(\lambda x_{2}\right)\right\rangle=\lambda^{\Delta_{1}+\Delta_{2}} f\left(\lambda^{2} x_{12}^{2}\right)
$$

and hence the correlator is now restricted to:

$$
f\left(x_{12}^{2}\right)=\frac{b_{12}}{\left(x_{12}^{2}\right)^{\left(\Delta_{1}+\Delta_{2}\right) / 2}}
$$

where $b_{12}=$ constant. We can also consider SCT invariance, which translates into:

$$
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right)\right\rangle \neq 0 \text { only for } \Delta_{1}=\Delta_{2},
$$

and thus the two-point function takes the following form:

$$
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right)\right\rangle= \begin{cases}\frac{b_{12}}{\left(x_{12}^{2}\right)^{\Delta / 2}}, & \text { if } \Delta_{1}=\Delta_{2} \equiv \Delta  \tag{1.15}\\ 0, & \text { otherwise }\end{cases}
$$

In a theory without defect, it is standard to absorb the constant $b_{12}$ in the fields such that the two-point function is completely fixed.

In very much the same way, conformal symmetry forces the three-point function to be:

$$
\begin{equation*}
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{3}\left(x_{3}\right)\right\rangle=\frac{c_{123}}{\left(x_{12}^{2}\right)^{\Delta_{123}}\left(x_{13}^{2}\right)^{\Delta_{132}}\left(x_{23}^{2}\right)^{\Delta_{231}}}, \tag{1.16}
\end{equation*}
$$

with $\Delta_{i j k} \equiv \Delta_{i}+\Delta_{j}-\Delta_{k}$. The three-point function is then also completely determined up to the constant $c_{123}$. This is a demonstration of the remarkable power of conformal symmetry.

Unfortunately, the shape of the four-point function is not fixed anymore by conformal symmetry, and the only thing that we can say about it is that it should have the following form:

$$
\begin{equation*}
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{3}\left(x_{3}\right) \mathcal{O}_{4}\left(x_{4}\right)\right\rangle=\frac{1}{\left(x_{12}^{2}\right)^{\left(\Delta_{1}+\Delta_{2}\right) / 2}\left(x_{34}^{2}\right)^{\left(\Delta_{3}+\Delta_{4}\right) / 2}} F(u, v), \tag{1.17}
\end{equation*}
$$

where $u$ and $v$ are called conformal or anharmonic ratios, and are defined as:

$$
\begin{equation*}
u \equiv \frac{x_{12}^{2} x_{34}^{2}}{x_{13}^{2} x_{24}^{2}}, \quad v \equiv \frac{x_{14}^{2} x_{23}^{2}}{x_{13}^{2} x_{24}^{2}} \tag{1.18}
\end{equation*}
$$

We make a few remarks to conclude this subsection. It is possible for the correlators to acquire an anomalous dimension at loop level, i.e.:

$$
\Delta \rightarrow \Delta^{(0)}+g \Delta^{(1)}+g^{2} \Delta^{(2)}+\ldots
$$

We will encounter anomalous dimensions when computing the two-point function in chapter 4 . We also note that one-point functions, which vanish in CFTs, will crucially not always be zero anymore once we add the defect. Two-point functions will also not be as constrained as in the case of a defect-free CFT.

## The Conformal Bootstrap

In this subsection, we present the most basic principles of the conformal bootstrap in absence of defects. We will build upon this knowledge in the next chapter when we introduce the defect conformal bootstrap. The conformal bootstrap is reviewed in e.g. [27].

We start by introducing the operator product expansion (OPE) [23], which consists of expanding a product of two local operators in terms of all the possible local operators of the theory. In principle, this expansion is not restricted to CFT, but conformal symmetry insures that the radius of convergence is greater than zero, which makes it a particularly interesting tool in this context.

In its most general form, the OPE of two local operators $\mathcal{O}_{1}\left(x_{1}\right)$ and $\mathcal{O}_{2}\left(x_{2}\right)$ reads:

$$
\begin{equation*}
\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right) \sim \sum_{k} f_{12 k}\left(x_{12}\right) \mathcal{O}_{k}\left(x_{2}\right) \tag{1.19}
\end{equation*}
$$



Figure 1.1: A representation of the OPE given in eq. (1.20). In the correlators of a conformal field theory, a product of operators can be expanded into a sum of conformal primaries $\mathcal{O}_{k}$ with an operator $C_{12 k}$ acting on them.
where the sum runs over all possible local operators, as mentioned above. Here $f_{12 k}$ is a function depending on $x_{12}^{2}$, and it is implied that the expansion holds within correlation functions only. Since all local operators can be constructed from conformal primaries, as we discussed previously, the OPE can be rewritten as follows:

$$
\begin{equation*}
\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right) \sim \sum_{k \text { prim. }} c_{12 k} C_{12 k}\left(x_{12}, \partial_{2}\right) \mathcal{O}_{k}\left(x_{2}\right), \tag{1.20}
\end{equation*}
$$

where the operator $C_{12 k}\left(x_{12}, \partial_{2}\right)$ encodes the construction of descendants, and where $c_{12 k}$ is a numerical factor. This operator can in principle be constructed with the help of two- and three-point functions, and most importantly it does depend only on $\Delta_{k}$ and $d$, but not on $c_{12 k}$.

We will now introduce the concept of conformal blocks. We first consider the canonical example of the four-point function, but later on we will focus on two-point functions in presence of a defect. The fourpoint function can be expanded with the fusions $\mathcal{O}_{1} \mathcal{O}_{2}$ and $\mathcal{O}_{3} \mathcal{O}_{4}$ by using the OPE given in (1.20). This results in:

$$
\begin{align*}
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{3}\left(x_{3}\right) \mathcal{O}_{4}\left(x_{4}\right)\right\rangle & =\sum_{k} c_{12 k} c_{34 k} C_{12 k}\left(x_{12}, \partial_{2}\right) C_{34 k}\left(x_{34}, \partial_{4}\right)\left\langle\mathcal{O}_{k}\left(x_{2}\right) \mathcal{O}_{k}\left(x_{4}\right)\right\rangle \\
& \stackrel{!}{=} \frac{1}{\left(x_{12}^{2}\right)^{\left(\Delta_{1}+\Delta_{2}\right) / 2}\left(x_{34}^{2}\right)^{\left(\Delta_{3}+\Delta_{4}\right) / 2}} F(u, v), \tag{1.21}
\end{align*}
$$

where the second line corresponds to the most general form that the four-point function can take in CFTs, and which was already given in eq. (1.17) together with (1.18). Note that from now on, $k$ will always mean that the sum runs over conformal primaries. This means that the function $F(u, v)$ takes the following form:

$$
\begin{equation*}
F(u, v)=\sum_{k} c_{12 k} c_{34 k} f_{k}^{12,34}(u, v), \tag{1.22}
\end{equation*}
$$

where the functions $f_{k}^{12,34}(u, v)$ are called conformal blocks and are group-theoretical functions, i.e. they satisfy a Casimir equation corresponding to the symmetry group. In the present case, the conformal blocks are eigenfunctions of the Casimir operator of the $\mathfrak{s o}(5,1)$ symmetry, which is the full conformal group in 4 dimensions expressed in the embedding formalism. Casimir equations and their solutions will be treated in more detail in the next chapter, for the case of a defect CFT.

The OPE is then completely determined if we know the quantum numbers $\left\{\Delta_{k}, s, c_{i j k}\right\}$, with $s$ the spin of the conformal primaries. This set is commonly referred to as CFT data. Moreover, we saw that $n$-point
correlators can be expanded in OPEs, and hence the knowledge of the spectrum of conformal primaries and of the CFT data suffices to completely determine the theory! Inversely, this means that a conformal field theory can be characterized in that way, and this set of rules is standardly called the axioms of conformal field theory (see e.g [28]).

However a serious impeachment to solving theories exactly is the fact that the number of unknowns is actually infinite. Fortunately, we have not used yet all the consistency conditions that we can extract from a CFT. Indeed, we have expanded the four-point function in only one way to fuse the operators together, but it is also possible to do the OPE with the fusions $\mathcal{O}_{1} \mathcal{O}_{3}$ and $\mathcal{O}_{2} \mathcal{O}_{4}$, and we expect the result to be the same. This is known as crossing symmetry, and leads to the following equality:

$$
\begin{equation*}
\sum_{k} c_{12 k} c_{34 k} f_{k}^{12,34}(u, v)=\sum_{k} c_{13 k} c_{24 k} f_{k}^{13,24}(u, v), \tag{1.23}
\end{equation*}
$$

which implies that the OPE coefficients are not independent of each other. This condition happened to be sufficient to solve the minimal models in 2 dimensions [29], but the problem appears to be considerably more difficult for $d \geq 3$. In the recent years, it was realized that these constraints greatly restrict the space of unitary CFTs [27], and this realization led to a string of new results, such as the most precise estimation of the critical exponent of the 3d Ising model so far [8].

### 1.2 Supersymmetry

Another class of symmetries playing an important role in modern quantum field theory is supersymmetry (SUSY). This section presents a short review of supersymmetry, focused towards massless realizations of the symmetry in the maximally extended theory. The superconformal algebra is also briefly discussed, as it is an essential element of the $\mathcal{N}=4$ Super Yang-Mills theory. Most of the content of this section is based on [24, 25].

Let us start by introducing some notation. In the SUSY formalism, a Dirac spinor is commonly defined as:

$$
\begin{equation*}
Q^{A} \equiv\binom{Q_{\alpha}^{A}}{\bar{Q}^{A \dot{\alpha}}}, \tag{1.24}
\end{equation*}
$$

where $\alpha=1,2, \dot{\alpha}=\dot{1}, \dot{2}$ are spinor indices and $A=1, \ldots, \mathcal{N}$, with $\mathcal{N}$ referring to the number of supercharges. $Q_{\alpha}^{A}$ is a left-handed Weyl spinor, while $\bar{Q}^{A \dot{\alpha}}$ is a right-handed one.

The Dirac $\gamma$-matrices are then defined as:

$$
\gamma^{\mu} \equiv\left(\begin{array}{cc}
0 & \left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} \\
\left(\bar{\sigma}^{\mu}\right)_{\dot{\alpha} \beta} & 0
\end{array}\right),
$$

with $\sigma^{\mu} \equiv\left(\mathbb{1}, \sigma^{i}\right)$ and $\bar{\sigma}^{\mu} \equiv\left(\mathbb{1},-\sigma^{i}\right)$, where $\sigma^{i}$ refers to the usual 3 -dimensional Pauli matrices.
Supersymmetry was originally introduced to bypass the famous Coleman-Mandula no-go theorem [30], which states that spacetime and internal symmetries can only be combined in a trivial way under a list of reasonable assumptions. One of these assumptions stipulates that the generators should be bosonic, and SUSY escapes the realm of validity of the theorem by allowing fermionic generators, as we will soon see in more detail.

## $\mathcal{N}=1$ SUSY

The simplest example of a supersymmetry is the case of one single supercharge, i.e. $\mathcal{N}=1$. The algebra of supersymmetry is a graded Lie algebra (also called superalgebra), which consists of the usual bosonic generators of the Lie algebra and of new fermionic generators. The generators are thus graded, i.e. bosonic generators have grade 0 while fermionic ones have grade 1. The product of two generators $\mathcal{O}_{1}, \mathcal{O}_{2}$ has grade $\left(g_{1}+g_{2}\right) \bmod 2$. As an example, the product of two fermionic generators results in a bosonic generator.

The grading implies that the algebra now contains both commutators and anticommutators. They are given by:

$$
\begin{equation*}
\left[\mathcal{O}_{1}, \mathcal{O}_{2}\right]_{ \pm}=\mathcal{O}_{1} \mathcal{O}_{2}-(-1)^{g_{1} g_{2}} \mathcal{O}_{2} \mathcal{O}_{1} \tag{1.25}
\end{equation*}
$$

where - refers to a commutator and + to an anticommutator. Clearly, this relation involves anticommutators only when both generators are fermionic, i.e. $g_{1}=g_{2}=+1$.

We note that, in spite of the name, a graded Lie algebra is not a Lie algebra, since the antisymmetry property is broken by the fermionic generators.

For $\mathcal{N}=1$, we drop the index $A$ and denote the only supercharge by $Q$. The structure of the superalgebra is very restricted, since we demand that it should be compatible with the Poincaré algebra. In the most general case, we have the following relations:

$$
\begin{gather*}
{\left[Q_{\alpha}, J^{\mu v}\right]=\left(\sigma^{\mu v}\right)_{\alpha}{ }^{\beta} Q_{\beta},}  \tag{1.26a}\\
{\left[\bar{Q}_{\dot{\alpha}}, J^{\mu v}\right]=\varepsilon_{\dot{\alpha} \dot{\beta}}\left(\bar{\sigma}^{\mu v}\right)^{\dot{\beta}} \overline{\mathcal{Q}}^{\dot{\gamma}},}  \tag{1.26b}\\
\left\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} P_{\mu}, \tag{1.26c}
\end{gather*}
$$

while all other (anti)commutation relations vanish. Of course these have to be supplemented with the usual Poincaré algebra. Note that the spinor indices are raised/lowered with the tensors $\varepsilon_{\alpha \beta}, \varepsilon_{\dot{\alpha} \dot{\beta}}$.

There is an additional global symmetry known as $R$-symmetry, which is a $\mathfrak{u}(1)$ automorphism and for which the transformation law for the supercharge reads:

$$
Q_{\alpha} \rightarrow e^{i p} Q_{\alpha}, \quad \bar{Q}_{\dot{\alpha}} \rightarrow e^{-i p} \bar{Q}_{\dot{\alpha}}
$$

where $p$ is a global parameter. The corresponding generator is denoted by $R$ and satisfies:

$$
\begin{equation*}
\left[Q_{\alpha}, R\right]=Q_{\alpha}, \quad\left[\bar{Q}_{\dot{\alpha}}, R\right]=-\bar{Q}_{\dot{\alpha}} . \tag{1.27}
\end{equation*}
$$

## Extended SUSY

We now consider extended supersymmetry, i.e. superalgebras with more than one supercharge. To that effect, we reinstate the index $A$ that we dropped in the previous subsection and denote the Dirac spinor as in eq. (1.24).

In that case, the superalgebra is given by:

$$
\begin{gather*}
{\left[Q_{\alpha}^{A}, J^{\mu v}\right]=\left(\sigma^{\mu v}\right)_{\alpha}{ }^{\beta} Q_{\beta}^{A}, \quad\left[\bar{Q}_{\dot{\alpha}}^{A}, J^{\mu v}\right]=\varepsilon_{\dot{\alpha} \dot{\beta}}\left(\bar{\sigma}^{\mu v}\right)_{\dot{\gamma}}^{\dot{\gamma}} \bar{Q}^{A \dot{\gamma}},}  \tag{1.28a}\\
\left\{Q_{\alpha}^{A}, \bar{Q}_{B \dot{\beta}}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} P_{\mu} \delta^{A}{ }_{B},  \tag{1.28b}\\
\left\{Q_{\alpha}^{A}, Q_{\beta}^{B}\right\}=\varepsilon_{\alpha \beta} Z^{A B}, \quad\left\{\bar{Q}_{A \dot{\alpha}}, \bar{Q}_{B \dot{\beta}}\right\}=\varepsilon_{\dot{\alpha} \dot{\beta}} \bar{Z}_{A B}, \tag{1.28c}
\end{gather*}
$$

while all other (anti)commutators vanish, except for the ones corresponding to the Poincaré algebra. We note that the two first lines are the same as eq. (1.26a), (1.26b) with indices $A, B$ attached where needed. But the third line is now different and involves the central charges $Z^{A B}, \bar{Z}^{A B}$ of the SUSY algebra, i.e. charges which commute with all the other generators. Note that the central charges must be antisymmetric with respect to $A \leftrightarrow B$.

As in the $\mathcal{N}=1$ case, there is also a R-symmetry present, which now takes the following form:

$$
Q_{\alpha}^{A} \rightarrow R_{B}^{A} Q_{\alpha}^{B}, \quad \bar{Q}_{A \dot{\alpha}} \rightarrow \bar{Q}_{B \dot{\alpha}}\left(R^{\dagger}\right)_{A}^{B},
$$

where $R$ is a $\mathcal{N} \times \mathcal{N}$-matrix rotating the supersymmetries. For the case $\mathcal{N}>1$, the R -symmetry is a global non-Abelian symmetry. $Q_{\alpha}^{A}$ transforms in the fundamental representation of $\mathfrak{u}(\mathcal{N})$, while $\bar{Q}_{A \dot{\alpha}}$ transforms in the corresponding complex conjugate representation.

The R-symmetry can be expressed via generators $T^{i}(i=1, \ldots, \mathcal{N})$ such that they fulfill:

$$
\begin{gather*}
{\left[T^{i}, T^{j}\right]=i f^{i j}{ }_{k} T^{k},}  \tag{1.29a}\\
{\left[Q_{\alpha}^{A}, T^{j}\right]=\left(B^{j}\right)^{A}{ }_{B} Q_{\alpha}^{B},}  \tag{1.29b}\\
{\left[\bar{Q}_{A \dot{\alpha}}, T^{j}\right]=-\left(B^{j}\right)_{A}{ }^{B} \bar{Q}_{B \dot{\alpha}},} \tag{1.29c}
\end{gather*}
$$

with the matrix $B$ satisfying:

$$
\left(B^{j \dagger}\right)_{B}^{A}=\left(B^{j}\right)_{A}{ }^{B} .
$$

## Massless Representations of SUSY Algebra

We now turn our attention to massless realizations of the SUSY algebra, since we will not consider massive SUSY in this work. It is useful to first determine the Casimir operators of the theory. In Poincaré algebra, they are given by $P^{2}$ and $W^{2}$, with $P_{\mu}$ the momentum and $W_{\mu} \equiv \frac{1}{2} \varepsilon_{\mu v \rho \sigma} J^{v \rho} P^{\sigma}$ the Pauli-Lubanski vector. While $P^{2}$ remains a Casimir operator of the superalgebra, it is not the case anymore for $W^{2}$. Instead, we can define a modified Pauli-Lubanski vector of the following form:

$$
\tilde{W}_{\mu} \equiv W_{\mu}-\frac{1}{4} \bar{Q}_{A \dot{\alpha}}\left(\bar{\sigma}_{\mu}\right)^{\dot{\alpha} \alpha} Q_{\alpha}^{A},
$$

and the new Casimir operator is given by:

$$
\begin{equation*}
\tilde{W}^{2} \equiv \tilde{C}_{\mu \nu} \tilde{C}^{\mu \nu} \tag{1.30}
\end{equation*}
$$

with the $\tilde{C}_{\mu \nu}$ defined as:

$$
\tilde{C}_{\mu \nu} \equiv \tilde{W}_{\mu} P_{v}-\tilde{W}_{v} P_{\mu}
$$

Since $P^{2}$ is Casimir operator of the SUSY algebra, we can already conclude that all the fields belonging to a same SUSY multiplet (or supermultiplet) must have the same mass, and in the case of interest we have $P^{2}=0$. We also note that, in a gauge theory, the generators of the gauge group commute with the supercharges, and hence all the fields in a given supermultiplet must be in the same representation of the gauge group. Finally, the number of bosonic degrees of freedom must be equal to the number of fermionic degrees of freedom in any supermultiplet.

Massless states are labeled as usual by their momentum $p_{\mu}$ and helicity $\lambda$. To construct the states explicitly, we go to the lightlike frame where $p_{\mu}=(0,0, E, i E)$. Recall that in our notation, the last


Figure 1.2: In $\mathcal{N}=4$ SUSY, successive applications of the operators $a^{A}$ and $a_{A}^{\dagger}$ result in states of helicity $\lambda=0, \pm 1 / 2, \pm 1$. The top state corresponds to the gauge field of spin 1 . Applying $a^{A}(A=1,2,3,4)$ results in 4 states of spin $1 / 2$. The next level $(\lambda=0)$ contains only 6 states because of doublons and of vanishing states.
component corresponds to Euclidean time, which is imaginary. Hence we have $P^{2}=0=W^{2}$. Computing the anticommutator $\left\{Q_{\alpha}^{A}, \bar{Q}_{B \dot{\beta}}\right\}$ results in:

$$
\begin{gather*}
\left\{Q_{1}^{A}, \bar{Q}_{B 1}\right\}=4 E \delta_{B}^{A},  \tag{1.31a}\\
\left\{Q_{2}^{A}, \bar{Q}_{B 2}\right\}=0, \tag{1.31b}
\end{gather*}
$$

which implies that $Q_{2}^{A}$ is realized trivially. Meanwhile, creation and annihilation operators can be defined with the help of $Q_{1}^{A}, Q_{A 1}$ as follows:

$$
\begin{equation*}
a^{A} \equiv \frac{Q_{1}^{A}}{2 \sqrt{E}}, \quad a_{A}^{\dagger} \equiv \frac{Q_{A 1}}{2 \sqrt{E}}, \quad \text { with }\left\{a^{A}, a_{B}^{\dagger}\right\}=\delta_{B}^{A} . \tag{1.32}
\end{equation*}
$$

$Q_{1}^{A}$ lowers the helicity by $1 / 2$, i.e. $Q_{1}^{A}|p, \lambda\rangle$ has helicity $-1 / 2$, while $Q_{A 1}$ raises the helicity by $1 / 2$. To construct a multiplet, we can therefore start with a vacuum state of lowest helicity $|\Omega\rangle$, i.e. a state which satisfies $Q_{1}^{A}|\Omega\rangle=0$, and create new states by acting with the creation/annihilation operators in all $2^{\mathcal{N}}$ possible ways.

## Maximally Extended SUSY

Now that we have reviewed extended supersymmetry and massless representations, we are ready to construct the maximally extended SUSY, i.e. the realization of supersymmetry with the largest number of supercharges with a multiplet representation of $\operatorname{spin} \leq 1$, that is without a graviton. This is clearly realized by $\mathcal{N}=4$, as explicitly represented in fig. 1.2. Applying the creation/annihilation operators in all possible ways, we find that the multiplet consists of 1 vector field, 4 Weyl fermions and 6 scalar fields, and it is easy to see that this multiplet contains 8 bosonic and 8 fermionic degrees of freedom. The R-symmetry group is $\mathfrak{s u}(4)_{R} \sim \mathfrak{s o}(6)_{R}$.

In this thesis, we will focus our attention on $\mathcal{N}=4$ Super Yang-Mills (SYM) theory. In the frame of the superconformal bootstrap, we do not need to write the action explicitly, but this will be required for extracting perturbative CFT data later on. The action as well as the insertion rules will be derived in detail in Chapter 3, and for now we only mention a few interesting aspects of the theory. As we already mentioned, all the particles are massless, and conformal symmetry is preserved at the quantum level. This has for immediate consequence that the $\beta$-function of the coupling vanishes. Moreover, $\mathcal{N}=4 \mathrm{SYM}$
is believed to be UV-finite in perturbation theory [31]. It is also invariant under the S-duality group $\mathfrak{s l}(2, \mathbb{Z})$, which implies a strong/weak coupling duality (also called Montonen-Olive duality) [32, 33]. These interesting properties have made the theory the subject of much attention in the recent years, as it came to be considered the hydrogen atom of the 21st century. $\mathcal{N}=4$ SYM lies at the crossroad between AdS/CFT, integrability and the superconformal bootstrap.

## Superconformal Algebra

We have seen that $\mathcal{N}=4$ SYM exhibits both conformal symmetry and supersymmetry, which are preserved at the quantum level. We will now study the resulting algebra, known as the superconformal algebra.

Naively, one might think that the algebra consists of the generators of the conformal group supplemented by the supercharges $Q_{\alpha}^{A}, \bar{Q}_{A \dot{\alpha}}$, but this is not quite correct. Indeed, in order to ensure closure of the algebra we need to add fermionic supercharges $S_{\alpha}^{A}, \bar{S}_{A \dot{\alpha}}$. The (anti)commutators of the corresponding $\mathfrak{s u}(2,2 \mid \mathcal{N})$ algebra are listed in appendix A.3. We only give here the relations involving $S_{\alpha}^{A}$ and $\bar{S}_{A \dot{\alpha}}$ :

$$
\begin{gather*}
\left\{S_{\alpha}^{A}, \bar{S}_{\dot{\beta} B}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} K_{\mu} \delta^{A}{ }_{B},  \tag{1.33a}\\
\left\{Q_{\alpha}^{A}, S_{\beta B}\right\}=\varepsilon_{\alpha \beta}\left(\delta_{B}^{A} D+R_{B}^{A}\right)+\frac{1}{2} \delta^{A}{ }_{B} J_{\mu v}\left(\sigma^{\mu v}\right)_{\alpha \beta} . \tag{1.33b}
\end{gather*}
$$

We consider now the representations of the superconformal algebra, and in particular we focus on local and gauge-invariant operators $\mathcal{O}(x)$. The operators are characterized by their conformal dimension $\Delta$ and their $\operatorname{spin} \mathcal{J}^{\mu \nu}$ :

$$
\begin{aligned}
{[D, \mathcal{O}(0)] } & =-i \Delta \mathcal{O}(0), \\
{\left[J^{\mu \nu}, \mathcal{O}(0)\right] } & =-\mathcal{J}^{\mu v} \mathcal{O}(0) .
\end{aligned}
$$

The most important class of such operators are called the superconformal primaries, which are defined to be the operators with the lowest $\Delta$ in a superconformal multiplet, i.e. they fulfill:

$$
\begin{equation*}
\left[S_{\alpha}^{A}, \mathcal{O}\right]_{ \pm}=0, \quad\left[\bar{S}_{A \dot{\alpha}}, \mathcal{O}\right]_{ \pm}=0 \quad \forall A, \alpha, \tag{1.35}
\end{equation*}
$$

since the $S$ 's lower the conformal dimension $\Delta$ (see the discussion on the unitarity bound in section 1.1). It is interesting to note that all superconformal primaries are conformal primaries, but the converse is not true. Descendants can be constructed as usual by applying any product of generators of the superconformal algebra to the primaries. We call superdescendants the descendants of superconformal primaries that are defined by:

$$
\begin{equation*}
\mathcal{O}^{\prime} \equiv[Q, \mathcal{O}] \tag{1.36}
\end{equation*}
$$

which leads to:

$$
\Delta_{\mathcal{O}^{\prime}}=\Delta_{\mathcal{O}}+\frac{1}{2}
$$

These operators are important since they are also conformal primaries.
An essential subset of superconformal primaries is the one consisting of chiral primaries, which on top of eq. (1.35) also fulfill the following condition:

$$
\begin{equation*}
\left[Q_{\alpha}^{A}, \mathcal{O}\right]_{ \pm}=0 \tag{1.37}
\end{equation*}
$$

i.e. the operators are annihilated by at least one of the supercharges. This is called the BPS condition ${ }^{2}$ [34, 35]. As a consequence, the operators are protected, which means that their conformal dimensions do not receive quantum corrections $[12,13]$.

We will focus in this work on single-trace operators of scalar primaries, which are defined as:

$$
\begin{equation*}
\mathcal{O}_{k}(x) \equiv u_{i_{1}} \ldots u_{i_{k}} \operatorname{Tr} \phi^{i_{1}} \ldots \phi^{i_{k}}, \tag{1.38}
\end{equation*}
$$

where the indices $i$ belong to the $\mathfrak{s o}(6) \mathrm{R}$-symmetry and where the trace acts on the gauge-group indices. The vectors $u_{i}$ are defined to keep track of the $\mathfrak{s o}(6)$-indices and are null-vectors, i.e. they satisfy $u^{2}=0$. These operators are $1 / 2-B P S$ operators, which means that they fullfill the BPS condition given in eq. (1.37) for half of the supercharges. The correlators of single-trace operators are thus protected, and hence the conformal dimension $\Delta=k$ does not receive anomalous contributions. 1/2-BPS operators saturate the unitarity bound, which implies that they belong to a short multiplet. In other words, the representation has null-states (i.e. states with zero norm), which can be safely removed from the multiplet.

Single-trace operators are the leading operators in the large $N$ limit (see chapter 3 for an introduction to the large $N$ expansion), and are characterized by their quantum numbers $\Delta, s$ (spin) and R-symmetry Dynkin labels [ $0, k, 0$ ] (see appendix A. 1 for a review of Dynkin labels). The dimension of the representation is given by the following formula:

$$
\operatorname{dim}[0, k, 0]=\frac{1}{12}(k+1)(k+2)^{2}(k+3) .
$$

In this thesis we focus on the case $k=2$, which has dimension 20.

It is also possible to construct multi-trace operators, i.e. products of single-trace operators. $1 / 4-$ and $1 / 8-$ BPS are realized by such operators. 1/4-BPS operators have conformal dimension $\Delta=k+2 l$ and Dynkin labels $[l, k, l](l \geq 1)$, while $1 / 8$-BPS operators are characterized by $\Delta=k+2 l+3 m$ and $[l, k, l+2 m](l \geq 2)$.

We conclude this review of supersymmetry by mentioning that there exists another larger class of operators which are not protected, i.e. for which the correlators do receive anomalous corrections. These operators are called long operators; they bear this name because it is always possible to produce new states with the $Q$ 's. We will discuss longs in greater detail in the next chapter in the context of the defect superconformal bootstrap.

### 1.3 Maldacena-Wilson-Line Defect

We introduce now the concept of conformal defect and discuss the consequences on a CFT when such extended objects are present in the vacuum. See [14] for an introduction to defects in conformal field theory. We focus on line defects, since we will exclusively deal with such objects in this thesis. We conclude by presenting the Maldacena-Wilson line, which is a $1 / 2$-BPS extended operator, and by investigating some of its most interesting properties.

[^1]
## Conformal Defects

Conformal defects are extended operators which preserve a large subgroup of the conformal symmetry. In this work we focus on flat defects ${ }^{3}$, for which the preserved symmetry is manifestly $\mathfrak{s o}(p+1,1) \times \mathfrak{s o}(q)$, with $p+q=d$. The quantum number associated to the $\mathfrak{s o}(p+1,1)$ symmetry is commonly referred to as the transuerse spin $s$, while the one corresponding to $\mathfrak{s o l}(q)$ is called the parallel spin. $q$ is called the codimension of the defect. Examples of flat defects include the boundary, which has codimension one, and the line, which has codimension three, and on which we will focus from now on.

The analysis of the conformal bootstrap in section 1.1 can be extended to the case of defect CFT (dCFT), and corresponding defect conformal blocks will play the same role. This will be investigated in the next chapter. In the case of a flat defect, the fact that the preserved symmetry factorizes leads to having defect conformal blocks satisfy two Casimir equations.

The line defect preserves the symmetry $\mathfrak{s o}(2,1) \times \mathfrak{s o}(3)$, where the first group corresponds to the 1 d conformal group on the line and the second one to the rotations orthogonal to the defect. When adding supersymmetry, we must also consider the $\mathfrak{s p}(4)_{R}$ R-symmetry, and together they form the $\mathfrak{o s p}(4 \mid 4)$ defect superalgebra. Representations of $\mathfrak{o s p}(4 \mid 4)$ are labeled by their conformal dimension $\hat{\Delta}$, transverse $\operatorname{spin} s$ as well as by the $\mathfrak{s p}(4)_{\mathrm{R}}$ Dynkin labels [a,b]. Following [20], we denote $1 / 2$-BPS multiplets of $\mathfrak{o s p}(4 \mid 4)$ by $(B, \pm)_{k}$, with $k$ labeling the $[0, k]$ irreducible representation of the superconformal primary. The notation is reviewed in more detail in section 2.2.

## Maldacena-Wilson Line

The defect that we will consider in this work is defined by the so-called Maldacena-Wilson loop, which is an extended operator defined by:

$$
\begin{equation*}
\mathcal{W}(C) \equiv \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \oint_{C} d \tau\left(i \dot{x}_{\mu} A^{\mu}+|\dot{x}| \theta_{i} \phi^{i}\right) \tag{1.39}
\end{equation*}
$$

with $A_{\mu} \equiv T_{a} A_{\mu}^{a}, \phi^{i} \equiv T^{a} \phi_{a}^{i}$, where $T^{a}$ is a generator of the gauge group of the SYM theory and $i$ a $\mathfrak{s o}(6)_{\mathrm{R}}$ index. We will show in chapter 3 how this operator can be constructed with dimensional reduction from the Wilson loop in 10 dimensions.

The Maldacena-Wilson loop has been studied for a great variety of geometries for the path $C$, mostly as a circular loop or as a line [19], but also with cusps (see e.g. [36, 37]). In this thesis we will always consider $C$ to be a straight infinite line.

The Maldacena-Wilson line preserves half of the supercharges, and is hence a $1 / 2$-BPS operator as well. As discussed in section 1.2, this implies that this is a protected operator which does not receive anomalous contributions to its conformal dimension. In particular, it was shown perturbatively in [19] that the expectation value of $\mathcal{W}(C)$ is simply:

$$
\begin{equation*}
\langle\mathcal{W}(C)\rangle=1 . \tag{1.40}
\end{equation*}
$$

This concludes the first chapter, which has been devoted to introducing conformal field theory, supersymmetry and defects separately. In chapter 2 we will bring these concepts together into a framework called the defect superconformal bootstrap.

[^2]
## CHAPTER 2

## The Superconformal Bootstrap with Line Defect

In this chapter we will use the building blocks presented in chapter 1 in order to bootstrap $\mathcal{N}=4$ SYM in presence of the Maldacena-Wilson-line defect. To that effect, we consider the simplest system possible, i.e. a two-point function of single-trace operators in presence of the defect. We start by reviewing the bootstrap for scalar operators without supersymmetry, from which we derive the spacetime conformal blocks. Considering the supersymmetric theory leads us to derive R-symmetry blocks, and we fix numerical coefficients between the components of the superblocks by using the superconformal Ward identities. These are obtained by requiring analyticity of the correlator. This chapter is then concluded by discussing a method for isolating the CFT data such that it can easily be compared to perturbative computations. For the most part, the two first sections present the works of [14] and [20].

### 2.1 The Conformal Bootstrap with Line Defect

It is easier to start by considering the two-point function without involving supersymmetry. We begin by describing the setup of the two-point function, and from there we describe how to obtain the conformal blocks and the new crossing symmetry equation that arises because of the defect. The preserved symmetry leads to Casimir equations, for which we present the solutions for both the defect and the bulk channels.

## Setup

The system we consider in this section consists of two scalar operators $\mathcal{O}_{1}\left(x_{1}\right)$ and $\mathcal{O}_{2}\left(x_{2}\right)$ in presence of the line defect defined in eq. (1.39), with the path $C$ being defined to be an infinite line. It is standard in defect CFT to not write explicitly the defect in the two-point function, i.e.:

$$
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right)\right\rangle \equiv\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{W}(C)\right\rangle
$$

A notation with double brackets is occasionally used (see e.g. [38]), in which the correlator is divided by the expectation value of the defect. We will not use this notation here, since $\langle\mathcal{W}(C)\rangle=1$ as we have seen at the end of chapter 1.

We define our coordinate system such that the Wilson line extends in the $\tau$-direction only. Moreover, we set the $x$-axis such that the operator $\mathcal{O}_{1}$ sits at $x_{1}=(1,0,0,0)$. By conformal symmetry, the second operator must lay in some $x y$-plane, and hence its coordinates are $x_{2}=(x, y, 0,0)$. This configuration is depicted in fig. 2.1.

As mentioned in section 1.3 , the symmetry preserved by the defect is $\mathfrak{s o}(2,1) \times \mathfrak{s o}(3)$. It is convenient to define the following two conformal cross-ratios:


Figure 2.1: The left figure represents the setup, with the bold line being the Maldacena-Wilson-line defect. The operators $\mathcal{O}\left(x_{1}\right)$ and $\mathcal{O}\left(x_{2}\right)$ lie in the xy-plane, and fixing $x_{1}=(1,0,0,0)$ leaves as only degrees of freedom the coordinates $x_{2}=(x, y, 0,0)$. The right figure shows the definition of the variable $\phi$, which is the angle formed by $x_{1}$ and $x_{2}$ in the $x y$-plane when taking the line defect as the origin.

$$
\begin{equation*}
\xi=\frac{x_{12}^{2}}{\left|x_{1}\right|\left|x_{2}\right|}, \quad \cos \phi=\frac{x_{1} \cdot x_{2}}{\left|x_{1}\right|\left|x_{2}\right|}, \tag{2.1}
\end{equation*}
$$

where $x_{12} \equiv x_{1}-x_{2}$, and where $\phi$ is the angle formed by the vectors $x_{1}$ and $x_{2}$ in the plane orthogonal to the defect, which in this case is simply the $x y$-plane (see fig. 2.1). Note that in the case of a defect of codimension one, the variable $\phi$ is not defined since no such angle can be formed. But only one cross-ratio would be needed, and $\phi$ can simply be abandoned there.

Since the system contains two degrees of freedom, it is also convenient to define complex coordinates of the following form:

$$
\begin{equation*}
z \equiv x+i y, \quad \bar{z} \equiv x-i y . \tag{2.2}
\end{equation*}
$$

The conformal cross-ratios of eq. (2.1) become:

$$
\begin{equation*}
\xi=\frac{(1-z)(1-\bar{z})}{\sqrt{z \bar{z}}}, \quad \cos \phi=\frac{1}{2} \frac{z+\bar{z}}{\sqrt{z \bar{z}}} \tag{2.3}
\end{equation*}
$$

We have seen in chapter 1 that conformal symmetry fixes the form of the two-point function in a CFT without defect. This is not the case anymore in presence of the Wilson line, and we express the corresponding correlator as:

$$
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right)\right\rangle=\frac{1}{\left|x_{1}^{\perp}\right|^{\Delta_{1}}\left|x_{2}^{\perp}\right|^{\Delta_{2}}} F(z, \bar{z}),
$$

where $x_{i}^{\perp}$ refers to the coordinates orthogonal to the defect. Since the operators $\mathcal{O}\left(x_{1}\right)$ and $\mathcal{O}\left(x_{2}\right)$ do not extend in the $\tau$-direction, we simply have $x_{i}^{\perp}=x_{i}$ for $i=1,2$.

## Defect Conformal Blocks and Crossing Symmetry

In section 1.1, the conformal bootstrap was built upon the concepts of OPE and crossing symmetry. Both ideas are also present in the defect CFT, although their form has been considerably modified by the presence of the defect.

A famous specificity of defect CFTs is that certain operators acquire a non-vanishing expectation value in presence of the defect, i.e.:

$$
\langle\mathcal{O}(x)\rangle=a_{\mathcal{O}}\left|x^{\perp}\right|^{-\Delta},
$$

with $a_{\mathcal{O}}$ being the corresponding CFT data. Note that only operators with even spin can acquire an expectation value [14].

The simplest crossing equation that can be built consists of two-point functions, which now involves a new type of OPE called defect OPE, in addition to the bulk OPE that was already introduced at the end of section 1.1. The bulk OPE gives the following expansion of the two-point function:

$$
\begin{equation*}
F(z, \bar{z})=\xi^{-\frac{\Delta_{1}+\Delta_{2}}{2}} \sum_{\mathcal{O}} c_{12 \mathcal{O}} a_{\mathcal{O}} f_{\Delta, J}(z, \bar{z}), \tag{2.4}
\end{equation*}
$$

where we call the functions $f_{\Delta, J}(z, \bar{z})$ bulk conformal blocks. The sum runs over conformal primaries as in eq. (1.20). In contrast to the four-point function of eq. (1.21), the CFT data consists now of the products of the three-point function coefficient $c_{12 \mathcal{O}}$ and of the one-point function coefficient $a_{\mathcal{O}}$ for a given bulk primary $\mathcal{O}$, for which $\Delta$ and $J$ are the quantum numbers. The bulk conformal blocks are eigenfunctions of the quadratic Casimir operator of the full conformal algebra $\mathfrak{s o}(5,1)$. Crucially, the CFT data $\left\{\Delta_{k}, c_{i j k}\right\}$ is not sufficient anymore for fixing the correlator in presence of the defect.

The defect possesses local excitations called defect operators, which we label $\hat{\mathcal{O}}_{i}$. Hats will always refer to the defect channel. Defect operators have conformal weights $\hat{\Delta}_{i}$, which clearly are not related by symmetry to the $\Delta$ 's of the bulk operators. This leads us to yet another operator product expansion, called the defect OPE:

$$
\begin{equation*}
\mathcal{O}(x)=\sum_{\mathcal{O}} b_{\mathcal{O} \mathcal{O}}\left|x^{\perp}\right|^{\hat{\Delta}-\Delta} \hat{\mathcal{O}}\left(x^{\|}\right), \tag{2.5}
\end{equation*}
$$

where the $b_{\mathcal{O O}}$ are coefficients of bulk-to-defect two-point functions, and where $x^{\|}$refers to the coordinates parallel to the defect. Only defect scalars $(j=0)$ are allowed to appear in the OPE [14], and it results in the following expansion of the correlator for the defect channel:

$$
\begin{equation*}
F(z, \bar{z})=\sum_{\hat{\mathcal{O}}} b_{1 \hat{\mathcal{O}}} b_{2 \hat{\mathcal{O}}} \hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z}), \tag{2.6}
\end{equation*}
$$

where the $\hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z})$ are called defect conformal blocks. They are eigenfunctions of the quadratic Casimir operator of the symmetry group preserved by the defect, i.e. $\mathfrak{s o (}(2,1) \times \mathfrak{s o}(3)$ in the case of the line.

It is now easy to write the defect crossing-symmetry equation for the two-point function. It can be represented pictorially as:
where the dotted lines indicate where the OPE is being performed, i.e. the left-hand side corresponds to the bulk channel and the right-hand side to the defect channel. This translates explicitly into::

$$
\begin{equation*}
\xi^{-\frac{\Delta_{1}+\Delta_{2}}{2}} \sum_{\mathcal{O}} c_{12 \mathcal{O}} a_{\mathcal{O}} f_{\Delta, J}(z, \bar{z}) \stackrel{!}{=} \sum_{\hat{\mathcal{O}}} b_{1 \mathcal{O}} b_{2 \mathcal{O}} \hat{f}_{\widehat{\Delta}, 0, s}(z, \bar{z}) . \tag{2.7}
\end{equation*}
$$

In this work we focus on the computation of the CFT data in the defect channel, with the idea of paving the way for the computation of the CFT data in the bulk channel as well. The next step is therefore to determine the (defect) conformal blocks $\hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z})$. For completeness, we also present the solutions of the Casimir equations for the (bulk) conformal blocks $f_{\Delta, J}(z, \bar{z})$.

## Defect Casimir Equation

As mentioned above, the defect conformal blocks $\hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z})$ are eigenfunctions of the algebra $\mathfrak{s o}(2,1) \times$ $\mathfrak{s o}(3)$, and since it is a direct product the defect Casimir equation factorizes accordingly and separates into two differential equations:

$$
\begin{align*}
& \left(\mathcal{L}^{2}+\hat{C}_{\hat{\Delta}, 0}\right) \frac{1}{\left|x_{1}\right|^{\Delta_{1}\left|x_{2}\right|^{\Delta_{2}}}} \hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z})=0  \tag{2.8a}\\
& \left(\mathcal{S}^{2}+\hat{C}_{0, s} \frac{1}{\left|x_{1}\right|^{\Delta_{1}\left|x_{2}\right|^{\Delta_{2}}}} \hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z})=0\right. \tag{2.8b}
\end{align*}
$$

where the eigenvalues of the Casimir operators are given by:

$$
\begin{equation*}
\hat{C}_{\hat{\Delta}, s} \equiv \hat{\Delta}(\hat{\Delta}-p)+s(s+q-2)=\hat{\Delta}(\hat{\Delta}-1)+s(s+1) . \tag{2.9}
\end{equation*}
$$

Note that we have inserted $p=1$ and $q=3$ in the second equality, since we focus on the line defect in this work. The Casimir operators are defined as:

$$
\mathcal{L}^{2} \equiv \frac{1}{2}\left(\mathcal{J}_{a b}^{\perp}\right)^{2}, \quad \mathcal{S}^{2} \equiv \frac{1}{2}\left(\mathcal{J}_{\alpha \beta}^{\|}\right)^{2},
$$

with $a, b=-1,0,4, \alpha, \beta=1,2,3$ and:

$$
\mathcal{J}_{A B} \equiv x_{A} \partial_{B}-x_{B} \partial_{A} .
$$

$\mathcal{J}^{\perp}$ corresponds to $\mathfrak{s o}(2,1)$ and $\mathcal{J}^{\|}$to $\mathfrak{s o}(3)$. Note that the operators act on only one of the points, e.g. $x_{2}$. In order to solve these equations, it is convenient to switch variables and to define:

$$
\chi \equiv \xi+2 \cos \phi .
$$

The Casimir equations can now be reformulated in the following way:

$$
\begin{gather*}
\left\{4 \cos \phi(1-\cos \phi) \partial_{\cos \phi}^{2}+2(1-3 \cos \phi) \partial_{\cos \phi}+s(s+1)\right\} \hat{f}_{\hat{\Delta}, 0, s}(\chi, \phi)=0  \tag{2.10a}\\
\left\{\left(4-\chi^{2}\right) \partial_{\chi}^{2}-2 \chi \partial_{\chi}+\hat{\Delta}(\hat{\Delta}-1)\right\} \hat{f}_{\hat{\Delta}, 0, s}(\chi, \phi)=0 \tag{2.10b}
\end{gather*}
$$

This set of equations (and the solution) are presented in [14] for the general case of $d$ dimensions and with a defect of codimension $q$. The dependence on $\chi$ and $\phi$ factorizes, and the complete solution can be obtained with a power series ansatz:

$$
\begin{equation*}
\hat{f}_{\hat{\Delta}, 0, s}(\chi, \phi)=A(\chi) B(\phi)=\sum_{k=0}^{\infty} a_{k} \chi^{k} \sum_{l=0}^{\infty} b_{l} \cos ^{l} \phi . \tag{2.11}
\end{equation*}
$$

Using Frobenius' method it is easy to find the following relation between the coefficients of the power series expansion of $A(\chi)$ :

$$
\frac{a_{k}}{a_{k-2}}=\frac{(\hat{\Delta}+k-2)(\hat{\Delta}-k+1)}{4 k(1-k)} .
$$

This corresponds to a hypergeometric function:

$$
\begin{equation*}
A(\chi)=\chi^{-\hat{\Delta}}{ }_{2} F_{1}\left(\frac{\hat{\Delta}}{2}, \frac{\hat{\Delta}}{2}, \hat{\Delta}+\frac{1}{2} ; \frac{4}{\chi^{2}}\right) . \tag{2.12}
\end{equation*}
$$

The part of the equation depending on the other variable can also be solved in the same way, and the complete solution reads:

$$
\begin{equation*}
\hat{f}_{\hat{\Delta}, 0, s}(\chi, \phi)=C(\hat{\Delta}, s) \chi^{-\hat{\Delta}}{ }_{2} F_{1}\left(\frac{s+1}{2},-\frac{s}{2}, \frac{1}{2} ; \sin ^{2} \phi\right){ }_{2} F_{1}\left(\frac{\hat{\Delta}}{2}, \frac{\hat{\Delta}}{2}, \hat{\Delta}+\frac{1}{2} ; \frac{4}{\chi^{2}}\right), \tag{2.13}
\end{equation*}
$$

with $C(\hat{\Delta}, s)$ a normalization constant. We can reintroduce $z$ and $\bar{z}$ by using the "superblock dictionary" given in the appendix of [20], and we obtain:

$$
\begin{equation*}
\hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z})=z^{\frac{\hat{\Delta}-s}{2}} \bar{z}^{\frac{\hat{\lambda}+s}{2}}{ }_{2} F_{1}\left(-s, \frac{1}{2}, \frac{1}{2}-s ; \frac{z}{\bar{z}}\right){ }_{2} F_{1}\left(\hat{\Delta}, \frac{1}{2}, \frac{1}{2}+\hat{\Delta} ; z \bar{z}\right) . \tag{2.14}
\end{equation*}
$$

## Bulk Casimir Equation

As mentioned before, in the rest of this work we focus solely on the defect channel expansion. Nevertheless, we give a short review of the bulk channel expansion for completeness.

The $f_{\Delta, J}(z, \bar{z})$ are eigenfunctions of the Casimir operator corresponding to the full conformal group, i.e. $\mathfrak{s o}(5,1)$ in the embedding formalism. The bulk Casimir equation reads:

$$
\begin{equation*}
\left(\mathcal{J}^{2}+C_{\Delta, J}\right) \frac{1}{\left|x_{1}\right|^{\Delta_{1} / 2}\left|x_{2}\right|^{\Delta_{2} / 2}} \xi^{-\frac{\Delta_{1}+\Delta_{2}}{2}} f_{\Delta, J}(z, \bar{z}) \stackrel{!}{=} 0 \tag{2.15}
\end{equation*}
$$

with the Casimir operator being defined as:

$$
\mathcal{J}^{2} \equiv \frac{1}{2}\left(\mathcal{J}_{A B}{ }^{(1)}+\mathcal{J}_{A B}{ }^{(2)}\right)^{2},
$$

with $\mathcal{J}^{A B}$ defined as in the previous subsection and $A, B=-1,0,1, \ldots, 4$. Note that the superscripts in the right-hand side indicate on which point the operator is acting. The eigenvalues are given by:

$$
\begin{equation*}
C_{\Delta, J} \equiv \Delta(\Delta-4)+J(J+2) \tag{2.16}
\end{equation*}
$$

for the case of the line in $d=4$ (see [14] for the general case).
We will not be able to solve this differential equation analytically. However it is possible to check that the following series expansion satisfies eq. (2.15):

$$
\begin{align*}
& f_{\Delta, J}(z, \bar{z})=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{4^{m-n}}{m!n!} \frac{\left(-\frac{J}{2}\right)_{m}\left(\frac{J}{2}\right)_{m}\left(\frac{2-J-\Delta}{2}\right)_{m}}{(-J)_{m}\left(\frac{3-J-\Delta}{2}\right)_{m}} \frac{\left(\frac{\Delta-1}{2}\right)_{n}^{2}\left(\frac{\Delta+J}{2}\right)_{n}}{(\Delta-1)_{n}\left(\frac{\Delta+J+1}{2}\right)_{n}} \frac{\left(\frac{\Delta+J}{2}\right)_{n-m}}{\left(\frac{\Delta+J-1}{2}\right)_{n-m}} \\
& \times{ }_{4} F_{3}\left(-n,-m, \frac{1}{2}, \frac{\Delta-J-2}{2}\right.\left., \frac{2-\Delta+J-2 n}{2}, \frac{\Delta+J-2 m}{2}, \frac{\Delta-J-1}{2} ; 1\right) \\
& \times{ }_{2} F_{1}\left(\frac{\Delta+J}{2}\right.\left.-m+n, \frac{\Delta+J}{2}-m+n, \Delta+J-2(m-n) ; 1-z \bar{z}\right) \\
& \times {[(1-z)(1-\bar{z})]^{\frac{\Delta-J}{2}+m+n}(1-z \bar{z})^{J-2 m}, } \tag{2.17}
\end{align*}
$$

where $(\cdot)_{k}$ refers to Pochhammer symbols, defined as:

$$
(x)_{k} \equiv \frac{\Gamma(x+k)}{\Gamma(x)}=x(x+1) \ldots(x+k-1) .
$$

It is also interesting to note that the bulk Casimir equation can be used for obtaining the R -symmetry blocks in the supersymmetric case and vice versa by analytic continuation. It is indeed shown in [20] that the two-point function with $1 / 2$-BPS defects of codimension-one (i.e. the boundary) and


Figure 2.2: Comparison of the superspace setup for the configurations with codimension-one defect (boundary, at the top) and codimension-three defect (line, at the bottom). The left side of the picture shows the configuration in spacetime for the boundary and in $R$-symmetry space for the line, while the right side depicts the boundary in $R$-symmetry space and the line in spacetime. The analytic continuation is performed following the double arrows in one direction or the other by following the prescriptions given in [20].
of codimension-three (the line) are intimately related, and that one can obtain the superblocks by (schematically) inverting the roles of spacetime and R -symmetry space, as depicted in fig. 2.2. The bulk Casimir equation is exactly solvable for the case $q=1$ and the corresponding superblocks read:

$$
\begin{equation*}
f_{\Delta, 0}(z)=(4 \xi)^{\Delta / 2}{ }_{2} F_{1}\left(\frac{\Delta+\Delta_{12}}{2}, \frac{\Delta-\Delta_{12}}{2} ; \Delta-1 ;-\xi\right) \tag{2.18}
\end{equation*}
$$

with $\xi$ defined as before, but with $z=\bar{z}$. We also defined $\Delta_{12} \equiv \Delta_{1}-\Delta_{2}$. We will come back to this result later on.

### 2.2 The Superconformal Two-Point Function

We specialize now the analysis of the previous section for the case of $\mathcal{N}=4 \mathrm{SYM}$ with the Maldacena-Wilson-line defect. Moreover, we add that the scalar operators should be single-trace operators as defined in eq. (1.38) with $k=2$ :

$$
\begin{equation*}
\mathcal{O}(x) \equiv u_{i} u_{j} \operatorname{Tr} \phi^{i}(x) \phi^{j}(x) \tag{2.19}
\end{equation*}
$$

The Maldacena-Wilson line is defined in eq. (1.39). We start this section by presenting superconformal Ward identities, which impose strong constraints on the supersymmetric two-point function. We then discuss the concept of superblocks, and in particular we show which operators are allowed to appear in the defect and bulk channels respectively. Finally, we derive the superblocks for the defect channel.

## Superconformal Ward Identities

It is convenient to use the complex coordinates defined in eq. (2.2) for discussing the superconformal Ward identities. We therefore need differential operators acting on complex variables, and we wish that
their behavior imitates the one of partial derivatives on real variables. This is fulfilled by the Wirtinger derivatives, which are defined as follows:

$$
\begin{align*}
\partial_{z} & \equiv \frac{1}{2}\left(\partial_{x}-i \partial_{y}\right),  \tag{2.20a}\\
\partial_{\bar{z}} & \equiv \frac{1}{2}\left(\partial_{x}+i \partial_{y}\right), \tag{2.20b}
\end{align*}
$$

and which have the effect that $z$ and $\bar{z}$ can be considered to be independent variables, i.e. $\partial_{z} \bar{z}=0$ and $\partial_{\bar{z}} z=0$. The Wirtinger derivatives behave just like ordinary partial derivatives in the case of a single complex variable, i.e. they are linear operators and fulfill the celebrated product and chain rules.

Since we now find ourselves in a supersymmetric setup, it is also convenient to construct a R-symmetry variable $\omega$ which, by analogy to the conformal ratio $\xi$ (see eq. (2.3)), can be defined as:

$$
\begin{equation*}
\frac{4 \omega}{(1-\omega)^{2}} \equiv \frac{\left(u_{1} \cdot \theta\right)\left(u_{2} \cdot \theta\right)}{\left(u_{1} \cdot u_{2}\right)} . \tag{2.21}
\end{equation*}
$$

We recall that the $u$ 's and $\theta$ correspond respectively to the single-trace operators and to the line defect, and that $u^{2}=0$ and $\theta^{2}=1$.

Superconformal invariance implies:

$$
\begin{equation*}
\left\langle\mathcal{O}\left(x_{1}\right) \mathcal{O}\left(x_{2}\right)\right\rangle=\frac{\left(u_{1} \cdot \theta\right)^{2}\left(u_{2} \cdot \theta\right)^{2}}{x_{1}^{2} x_{2}^{2}} F(z, \bar{z}, \omega) . \tag{2.22}
\end{equation*}
$$

The correlator is singular at $z=\omega$ and $\bar{z}=\omega$, and in order to keep it analytic we need to impose for the residue to vanish at these poles. This is called analyticity condition, and it results in the following superconformal Ward identities [20]:

$$
\begin{align*}
& \left.\left(\partial_{z}+\frac{1}{2} \partial_{\omega}\right) F(z, \bar{z}, \omega)\right|_{z=\omega}=0  \tag{2.23a}\\
& \left.\left(\partial_{\bar{z}}+\frac{1}{2} \partial_{\omega}\right) F(z, \bar{z}, \omega)\right|_{\bar{z}=\omega}=0 \tag{2.23b}
\end{align*}
$$

It is useful to define the following invariant:

$$
\begin{equation*}
\Omega \equiv \frac{(1-\omega)^{2}}{4 \omega} \frac{\sqrt{z \bar{z}}}{(1-z)(1-\bar{z})} \equiv \Omega_{\mathrm{R}} \Omega_{\mathrm{ST}}, \tag{2.24}
\end{equation*}
$$

where in the last equality ST stands for "spacetime" and $R$ for "R-symmetry". Any power of $\Omega$ fulfills the Ward identities, i.e.:

$$
\begin{equation*}
\left.\left(\partial_{z}+\frac{1}{2} \partial_{\omega}\right) \Omega^{k}\right|_{z=\omega}=0 \tag{2.25}
\end{equation*}
$$

We will soon see that it is very convenient to express the correlator in terms of $\Omega, \Omega_{\mathrm{R}}$ and $\Omega_{\mathrm{ST}}$.

## Superblocks

Let us investigate how to go from the conformal spacetime blocks of the previous section to superconformal blocks, also called superblocks. We introduce the $R$-symmetry blocks $h_{k}(\omega)$ and $\hat{h}_{k}(\omega)$, and sum over the blocks that can appear in the corresponding multiplets. Concretely, it means that the defect superblocks take the following form:

$$
\begin{equation*}
\hat{\mathcal{G}}_{\hat{\chi}}(z, \bar{z}, \omega)=\sum_{\hat{\Delta}, k, s} c_{\hat{\Delta}, k}(\hat{\chi}) \hat{h}_{k}(\omega) \hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z}), \tag{2.26}
\end{equation*}
$$

where $\hat{\chi}$ refers to a representation of the symmetry group preserved by the defect, which is now $\mathfrak{o s p}(4 \mid 4)$. The $c_{\hat{\Delta}, k}$ are coefficients, which we will be able to determine once we know the relevant quantum numbers for each allowed representation.

In the exact same way, the bulk superblocks read:

$$
\begin{equation*}
\mathcal{G}_{\chi}(z, \bar{z}, \omega)=\sum_{\Delta, k} c_{\Delta, k}(\chi) h_{k}(\omega) f_{\Delta, J}(z, \bar{z}) . \tag{2.27}
\end{equation*}
$$

In this case, the $\chi$ 's are representations of the full superconformal algebra.
The R-symmetry blocks can be determined in the same way as we proceeded for the spacetime blocks in the previous section, i.e. by solving the corresponding Casimir equation, with $\mathfrak{s u}(4)_{R}$ being the actual symmetry. As explained in section 2.1, it is possible to obtain the bulk R-symmetry blocks by analytic continuation from a system with boundary defect, by applying the substitution prescribed in [20] (see also fig. 2.2). This gives:

$$
\begin{equation*}
h_{k}(\omega)=\left(\frac{\omega}{(1-\omega)^{2}}\right)_{2}^{-k / 2} F_{1}\left(-\frac{k}{2},-\frac{k}{2},-k-1 ;-\frac{(1-\omega)^{2}}{4 \omega}\right) . \tag{2.28}
\end{equation*}
$$

The defect R-symmetry blocks are also solutions of the R-symmetry Casimir equation, and we obtain:

$$
\begin{equation*}
\hat{h}_{k}(\omega)=\left(\frac{(1-\omega)^{2}}{\omega}\right)_{2}^{k} F_{1}\left(-k-1,-k,-2(k+1) ;-\frac{4 \omega}{(1-\omega)^{2}}\right) \tag{2.29}
\end{equation*}
$$

We now have an explicit expression for all the spacetime and R-symmetry blocks, and we are left with two tasks: (i) determine which operators are allowed to appear in the defect and bulk OPEs, and (ii) determine the numerical coefficients $c$ of eq. (2.26) and (2.27).

Let us start with the first task for the defect channel. According to (2.5), we must include all the operators which can have a non-vanishing two-point function in presence of the defect. This analysis is explained in [20] for the case of the boundary defect, and by analytic continuation it can be extended to the line. The preserved symmetry is a $1 \mathrm{~d} \mathfrak{o s p}(4 \mid 4)$ algebra, which contains $\mathfrak{s o}(5)_{\mathrm{R}}$ and the 16 supercharges. $\mathfrak{o s p}(4 \mid 4)$ has bosonic subalgebra $\mathfrak{s l}(2, \mathbb{R}) \oplus \mathfrak{s u}(2) \oplus \mathfrak{u s p}(4)$, where the first component corresponds to 1 d conformal algebra along the line and the two last ones to the R -symmetry. The representations of $\mathfrak{o s p}(4 \mid 4)$ are uniquely characterized by the quantum numbers $\{\hat{\Delta}, n,[a, b]\}$, where $n$ labels the $(n+1)$ dimensional representation of $\mathfrak{s u}(2)$, and where $[a, b]$ are Dynkin labels for $\mathfrak{u s p}(4 \mid 4)$ (see appendix A. 1 for a short review of Dynkin labels).

It turns out that the following operators are allowed in the defect OPE (in addition to the identity operator 1):

- 1/2-BPS operators: $(B,+)_{1},(B,+)_{2}$, with $(B,+)_{b / 2} \equiv\left\{\frac{b}{2}, 0,[0, b]\right\}$;
- 1/4-BPS operators: $(B, 1)_{[0, s]},(B, 1)_{[1, s]}$, with $(B, 1)_{[b / 2, n]} \equiv\left\{\frac{b}{2}, n,[0, b]\right\}, s \geq 0$;
- long operators: $L_{[0, s]}^{\hat{\Delta}}$, with $L_{[0, b]}^{\hat{\Delta}, n} \equiv\{\hat{\Delta}, n,[0, b]\}, \hat{\Delta} \geq 1$ and $\hat{\Delta}-1 \geq s \geq 0$.

We saw in section 1.2 that BPS operators are short multiplets, which means that the number of states is finite once the null-states have been removed. Still the value of the spin $s$ is not bounded for the $1 / 4-$ BPS operators. We also have an infinite number of long operators which have to be included in the OPE.

The notation $(B,+)$ corresponds to chiral operators, while ( $B,-$ ) would refer to antichiral operators, but such operators are not compatible with $\mathfrak{o s p}(4 \mid 4)$ [20].

To summarize, this means that the defect channel expansion of the two-point function reads:

$$
\begin{equation*}
F(z, \bar{z}, \omega)=A+B \hat{\mathcal{G}}_{(B,+)_{1}}+C \hat{\mathcal{G}}_{(B,+)_{2}}+\sum_{s=0}^{\infty} D_{s} \hat{\mathcal{G}}_{(B, 1)_{[0, s]}}+\sum_{s=0}^{\infty} E_{s} \hat{\mathcal{G}}_{\left(B, 1_{[1, s]}\right.}+\sum_{\hat{\Delta}=1}^{\infty} \sum_{s=0}^{\hat{\Delta}-1} F_{\hat{\Delta}, s} \hat{\mathcal{G}}_{L_{[0, s]}^{\Delta}}, \tag{2.30}
\end{equation*}
$$

where the coefficients have the following relation to the CFT data:

$$
A \equiv b_{\mathcal{O} \mathbb{1}}=a_{\mathcal{O}}^{2}, \quad B \equiv b_{\mathcal{O}_{(B,+)_{1}}^{2}}^{2}, \quad \cdots
$$

The CFT data is not fixed by the superconformal Ward identities, and the goal of this work is precisely to compute these coefficients perturbatively up to next-to-leading order (NLO). But before we do so, we still have to fix the coefficients of eq. (2.26) and (2.27), and this will be done in the next subsection.

We mentioned before that the defect OPE is really the focus of this work. However, for completeness and in prevision of future work, we quickly review the bulk channel expansion as well. In that case, the operators present in the OPE must have a non-vanishing one-point function in presence of the defect (see eq. (2.4)). Note that the one-point function of a bulk operator is non-zero if and only if its corresponding superconformal primary has a non-zero one-point function [20]. Moreover, only scalar operators are allowed. We denote the representations of the R-symmetry $\mathfrak{s u}(4)_{R}$ with the Dynkin labels [ $q, p, r$ ], with all $q, p, r$ even.

The allowed operators are the following (in addition to the identity contribution $\mathbb{1}$ ):

- $1 / 2$-BPS operators $\mathcal{B}_{[0,2 k, 0]}$ with $k=1,2$;
- semishort blocks $\mathcal{C}_{[0, p, 0],(J, J)}$ with $p=0,2, J \geq 0$;
- long blocks $\mathcal{A}_{[0,0,0],(J, J)}^{\Delta}$ with $\Delta \geq 2(J+1), J \geq 0$,
with $J$ the spin of the operator, and where we refer to [20] for a thorough derivation.


## Superblock Coefficients

We now fix the coefficients of the defect channel given in eq. (2.26). The most efficient method is to write an ansatz based on the content of the exchanged multiplet, then apply the superconformal Ward identities and solve the resulting equations for the coefficients.

The complete list of coefficients is given in appendix B. We illustrate the method here only for the simplest case, which is the block corresponding to $(B,+)_{k}$. The multiplet content is:

$$
\{\hat{\Delta}, s, k\}=\{\{k, 0, k\},\{k+1,1, k-1\},\{k+2,0, k-2\}\},
$$

where $\hat{\Delta}$ is the conformal dimension, $s$ the transverse spin and $k$ the R-symmetry label. This translates into the following superblock:

$$
\begin{equation*}
\hat{\mathcal{G}}_{(B,+)_{k}}(z, \bar{z}, \omega)=a_{0} \hat{f}_{k, 0} \hat{h}_{k}+a_{1} \hat{f}_{k+1,1} \hat{h}_{k-1}+a_{2} \hat{f}_{k+2,0} \hat{h}_{k-2}, \tag{2.31}
\end{equation*}
$$

where we suppressed the dependence on $z, \bar{z}$ and $\omega$ on the RHS for compactness. The Ward identities imply:

$$
\left.\left(\partial_{z}+\frac{1}{2} \partial_{\omega}\right) \hat{\mathcal{G}}_{(B,+)_{k}}(z, \bar{z}, \omega)\right|_{z=\omega}=0
$$

and setting $a_{0} \equiv 1$ this can be exploited for deriving the following coefficients:

$$
\begin{gather*}
a_{1}=-\frac{2 k}{1+2 k},  \tag{2.32a}\\
a_{2}=\frac{16 k(k-1)(k+1)^{2}}{(2 k-1)(2 k+3)(1+2 k)^{2}} . \tag{2.32b}
\end{gather*}
$$

The other coefficients are obtained in the very same way for the other multiplets. Note that the resulting superblocks are all linear combinations of hypergeometric functions, since they are solutions of the Casimir equations.

It is important to notice that the superblocks corresponding to the long operators $L_{[0, s]}^{\hat{\Delta}}$ become $(B, 1)_{[0, s]}$ superblocks (with a minus sign) at the unitarity bound, i.e. at $\hat{\Delta}=s+1$. This is called multiplet shortening, and it implies that the coefficients $D_{s}$ and $F_{s+1, s}$ cannot be disentangled in certain situations, as we shall see later on. This has already been discussed in [20] for the case of the boundary defect. The long multiplet is the only one to not be BPS, and hence it is not protected against the conformal dimension receiving an anomalous correction. This spoils the multiplet shortening at higher order and can be used for distinguishing the contributions of longs at the unitarity bound and of $(B, 1)_{[0, s]}$ superblocks. In perturbation theory, this manifests itself by the appearance of log terms at higher orders in the coupling constant $g$.

The computation of the coefficients of the bulk channel is reserved for future work, but in principle they can be obtained in the very same way as demonstrated here for the defect channel.

### 2.3 CFT Data

In the previous section, we derived the superblocks explicitly for the defect and bulk channels, the last missing piece of the puzzle being the unknown CFT data. Remarkably we were able to do that nonperturbatively, and the aim of this section is to isolate the CFT data such that we are in a position to extract it from the perturbative computations to be done in chapter 4 . We start by showing that the R-symmetry dependence is already completely fixed, and that we can conveniently reformulate the superconformal Ward identities in terms of spacetime coordinates only. Then we present some useful limiting cases of the setup, which can be expanded near the line defect such that the CFT data stands out uniquely. Finally, we give the expansions explicitly and discuss how log terms appear as a consequence of anomalous corrections at higher order.

## R-Symmetry Channels

In this subsection we take a step back to generality and consider single-trace operators with $k$ scalar fields (and not just two), as it was originally defined in eq. (1.38). Although we only deal with the case $k=2$ in chapter 4 , this analysis will be useful for discussing the general case in the conclusion, and it does not bring unnecessary complications. In order to separate the R -symmetry dependence, we first list the external objects which carry $\mathfrak{s o}(6)_{\mathrm{R}}$ indices $(i=1, \ldots, 6)$ :

- $k \times u_{1}^{i}$ and $k \times u_{2}^{i}$ (from the single-trace operators $\mathcal{O}$ );
- $\infty \times \theta^{i}$ (from the Wilson line, with $\infty$ possible points on the line).

The possible $R$-symmetry channels, i.e. all the ways in which these objects can be contracted together, can be represented diagrammatically as follows:

where the lines connect R -symmetry indices, and do not represent per se propagators. These diagrams are in the $R$-symmetry space and should not be confused with the diagrams that will be presented in chapter 4 , which are spacetime diagrams. We will call the channel on the far left 0 -channel, since there is zero line connecting the operators to the defect. The next channel will be called the 1 -channel, since each operator is connected by one line to the defect. The rest follows analogously, and the last channel is thus the $k$-channel. There are in total $(k+1)$-channels contributing to the correlator.

The two-point function is then the sum of the contributions given by each $c$-channel, defined as:

$$
\begin{aligned}
\left\langle\mathcal{O}_{k}\left(x_{1}\right) \mathcal{O}_{k}\left(x_{2}\right)\right\rangle_{c} & =\left(u_{1} \cdot u_{2}\right)^{k-c}\left(u_{1} \cdot \theta\right)^{c}\left(u_{2} \cdot \theta\right)^{c} f_{c}(z, \bar{z}) \\
& \stackrel{\left(u_{1} \cdot \theta\right)^{k}\left(u_{2} \cdot \theta\right)^{k}}{(z \bar{z})^{k / 2}} F_{c}(z, \bar{z}, \omega),
\end{aligned}
$$

where the second equality is just (2.22) channelwise for an arbitrary number $k$ of scalar fields. Using the invariant $\Omega$ defined in eq. (2.24), we obtain the following relation between the $F$ 's and the $f$ 's:

$$
\begin{align*}
F_{c}(z, \bar{z}, \omega) & =\Omega_{\mathrm{R}}^{k-c}(z \bar{z})^{k / 2} f_{c}(z, \bar{z}) \\
& =\Omega^{k-c}(1-z)^{k-c}(1-\bar{z})^{k-c}(z \bar{z})^{c / 2} f_{c}(z, \bar{z}) \\
& \equiv \Omega^{k-c} g_{c}(z, \bar{z}), \tag{2.33}
\end{align*}
$$

where in the last line we have defined the convenient spacetime function $g_{c}(z, \bar{z})$, such that the full correlator can be formulated in a compact way as:

$$
\begin{equation*}
F(z, \bar{z}, \omega)=\sum_{c=0}^{k} \Omega^{k-c} g_{c}(z, \bar{z}) \tag{2.34}
\end{equation*}
$$

All the $\omega$ dependence is now contained in $\Omega$, and the only unknown functions left are the $g_{c}$ 's. The superconformal Ward identities of section 2.2 can now be rewritten in terms of $z$ and $\bar{z}$ only, and those will be referred to as reduced Ward identities:

$$
\begin{equation*}
\sum_{c=0}^{k}\left(\frac{1}{4} \frac{1-z}{1-\bar{z}} \frac{\sqrt{z \bar{z}}}{z}\right)^{k-c} \partial_{z} g_{c}(z, \bar{z}) \stackrel{!}{=} 0 \tag{2.35}
\end{equation*}
$$

The second Ward identity follows similarly. For the case $k=2$, which we treat in this work, we thus have three R -symmetry channels, and the correlator reads:

$$
\begin{equation*}
F(z, \bar{z}, \omega)=\sum_{c=0}^{2} \Omega^{2-c} g_{c}(z, \bar{z}) \tag{2.36}
\end{equation*}
$$

The reduced Ward identities take the following form:

$$
\begin{align*}
& \partial_{z} g_{0}(z, \bar{z})+\frac{1}{4} \frac{1-z}{1-\bar{z}} \frac{\sqrt{z \bar{z}}}{z} \partial_{z} g_{1}(z, \bar{z})+\frac{1}{16} \frac{(1-z)^{2}}{(1-\bar{z})^{2}} \frac{\bar{z}}{z} \partial_{z} g_{2}(z, \bar{z}) \stackrel{!}{=} 0,  \tag{2.37a}\\
& \partial_{\bar{z}} g_{0}(z, \bar{z})+\frac{1}{4} \frac{1-\bar{z}}{1-z} \frac{\sqrt{z \bar{z}}}{\bar{z}} \partial_{\bar{z}} g_{1}(z, \bar{z})+\frac{1}{16} \frac{(1-\bar{z})^{2}}{(1-z)^{2}} \frac{z}{\bar{z}} \partial_{\bar{z}} g_{2}(z, \bar{z}) \stackrel{!}{=} 0 . \tag{2.37b}
\end{align*}
$$

These powerful identities strongly constraint the spacetime dependence of the correlator, and they will play an important role in our perturbative computation of the two-point function in chapter 4.


Figure 2.3: Some limiting cases for the setup introduced at the beginning of section 2.1. The left figure represents the setup for the collinear limit $z=\bar{z}$, i.e. $x_{2} \equiv(x, 0,0,0)$. The middle figure corresponds to the case in which $z=-\bar{z}$ or $x_{2} \equiv(0, x, 0,0)$. The last picture corresponds to the case $x_{2} \equiv(x, k x, 0,0)$ for $k=1$.

## Limiting Cases

It is not always possible to compute the correlator for the most general case, and hence we would like to discuss some interesting limiting cases of the setup which may simplify the integrals.

The first obvious choice would be to place the operator $\mathcal{O}\left(x_{2}\right)$ on the $x$-axis, i.e. $x_{2} \equiv(x, 0,0,0)$ or $z=\bar{z}$. This is called the collinear limit and is depicted in the leftmost picture of fig. 2.3. The reduced Ward identities presented in the previous subsection greatly simplify in this limit, since $\left.\Omega\right|_{x=z=\bar{z}=\omega}=\operatorname{sgn} x$. For $x>0$, we therefore have:

$$
\begin{aligned}
& \left.\partial_{z}\left\{g_{0}(z, \bar{z})+\frac{1}{4} g_{1}(z, \bar{z})+\frac{1}{16} g_{2}(z, \bar{z})\right\}\right|_{\bar{z}=z} \stackrel{!}{=} 0, \\
& \left.\partial_{\bar{z}}\left\{g_{0}(z, \bar{z})+\frac{1}{4} g_{1}(z, \bar{z})+\frac{1}{16} g_{2}(z, \bar{z})\right\}\right|_{\bar{z}=z} \stackrel{!}{=} 0 .
\end{aligned}
$$

It follows by substituting (2.20b) and adding the two equations that:

$$
\left.\partial_{x}\left\{g_{0}(z, \bar{z})+\frac{1}{4} g_{1}(z, \bar{z})+\frac{1}{16} g_{2}(z, \bar{z})\right\}\right|_{z=x, \bar{z}=x} \stackrel{!}{=} 0,
$$

which is equivalent to:

$$
\begin{equation*}
16 g_{0}(x, x)+4 g_{1}(x, x)+g_{2}(x, x)=c_{1}, \tag{2.39a}
\end{equation*}
$$

where $c_{1}$ is a constant. The same derivation can be performed for the case $x<0$, and it results in:

$$
\begin{equation*}
16 g_{0}(x, x)-4 g_{1}(x, x)+g_{2}(x, x)=c_{2} \tag{2.39b}
\end{equation*}
$$

where $c_{2}$ is a distinct constant, i.e. it does not have to be equal to $c_{1}$. Eq. (2.39a) and (2.39b) provide a useful way of checking our results numerically, since the functions involved depend only on one variable.

There are other limits which reduce the system to one variable, such as the line $z=-\bar{z}$ or $x_{2} \equiv(0, x, 0,0)$. This limit allows us to reach information that may be lost in the collinear limit. In general, every limit $x_{2} \equiv(x, k x, 0,0)$ is useful and permits us to reach information which may be obscured in other limits. We will make use of these limits in section 4.2, and they are represented in fig. 2.3.

## Expansion of Superblocks

It is advantageous to reduce the correlator to functions of one variable using the limiting cases presented above, since it is easy to expand them at $x \sim 0$. We will show in this subsection what can be learned from such expansions.

We have already noted that BPS operators are protected against corrections at any order in perturbation theory, but that long operators acquire an anomalous dimension due to quantum effects, i.e.:

$$
\hat{\Delta}=\sum_{k=0}^{\infty} g^{2 k} \hat{\Delta}^{(2 k)}
$$

where only even powers of $g$ are allowed since propagators carry a $g^{2}$ dependence as the Feynman rules reveal (see next chapter). Each long operator characterized by $\{\hat{\Delta}, s\}$ gets a different correction, and hence we must label the anomalous dimensions as $\hat{\Delta}_{\hat{\Delta}, s}^{(2 k)}$. The expansion of a superblock of the type $L_{[0, s]}^{\hat{\Delta}}$ then yields:

$$
\begin{aligned}
& \hat{\mathcal{G}}_{L_{[0, s]}^{\hat{\Delta}}}(z, \bar{z})=\hat{\mathcal{G}}_{L_{[0, s]}^{\Delta(0)}}(z, \bar{z})+g^{2} \hat{\Delta}_{\hat{\Delta}, s}^{(2)} \partial_{\hat{\Delta}} \hat{\mathcal{G}}_{L_{[0, s]}^{\Delta(0)}}(z, \bar{z}) \\
&+g^{4}\left\{\frac{1}{2}\left(\hat{\Delta}_{\hat{\Delta}, s}^{(2)}\right)^{2} \partial_{\hat{\Delta}}^{2} \hat{\mathcal{G}}_{L_{[0, s]}^{\Delta(0)}}(z, \bar{z})+\hat{\Delta}_{\hat{\Delta}, s}^{(4)} \partial_{\hat{\Delta}} \hat{\mathcal{G}}_{L_{[0, s]}^{\Delta(0)}}(z, \bar{z})\right\}+\ldots
\end{aligned}
$$

Note that expanding until $\mathcal{O}\left(g^{4}\right)$ is sufficient in this case for reaching next-to-leading order, which is the goal of this work.

It is convenient to separate spacetime and R-symmetry dependence in the correlator in the following manner:

$$
F(z, \bar{z}, \omega)=\Omega_{\mathrm{R}}^{2}(\omega) \tilde{F}_{0}(z, \bar{z})+\Omega_{\mathrm{R}}(\omega) \tilde{F}_{1}(z, \bar{z})+\tilde{F}_{2}(z, \bar{z})
$$

with the $\tilde{F}$ 's being related to the $g$ 's by:

$$
\begin{equation*}
\tilde{F}_{c}(z, \bar{z}) \equiv \Omega_{\mathrm{ST}}^{2-c}(z, \bar{z}) g_{c}(z, \bar{z}) \tag{2.40}
\end{equation*}
$$

$\Omega_{\mathrm{R}}$ and $\Omega_{\mathrm{ST}}$ are defined by eq. (2.24). Each of the $g$ 's can be expanded in superblocks channelwise. The BPS superblocks are pure hypergeometric functions, hence they can be expanded in power series when assuming one of the limits of the previous subsections:

$$
\begin{equation*}
g_{c}^{\mathrm{BPS}}(x)=\sum_{k=0}^{\infty} a_{k} x^{k} . \tag{2.41}
\end{equation*}
$$

The notation on the left-hand side is somewhat sloppy, but its meaning should be clear: the function $g$ now depends only on one variable, whose definition depends on the chosen limit.

Due to anomalous dimensions, the long superblocks do not have an expansion only in power series, as they also contain log terms arising from expanding the prefactor $(z \bar{z})^{-\Delta}$ :

$$
\begin{equation*}
g_{c}^{\mathrm{L}}(x)=\sum_{k=0}^{\infty} b_{k} x^{k}+\log x \sum_{k=1}^{\infty} c_{k} x^{k} . \tag{2.42}
\end{equation*}
$$

It is easy to work out the relations between the coefficients in (2.41), (2.42) and the CFT data of (2.30) in a given limit using e.g. Mathematica. We do not give these expressions explicitly here, since they
are extremely lengthy albeit elementary, but as an example the first few terms of each channel in the limit $z=\bar{z}$ read:

$$
\begin{gather*}
\tilde{F}_{0}(x, x)=16 C x^{2}+8 g^{4} F_{1,0} \hat{\Delta}_{1,0}^{(2)} x^{3} \log x+\ldots  \tag{2.43a}\\
\tilde{F}_{1}(x, x)=4 B x-2\left(2 g^{2} F_{1,0} \hat{0}_{1,0}^{(2)}+g^{4} F_{1,0}\left(\hat{\Delta}_{1,0}^{(2)}\right)^{2}+2 g^{4} F_{1,0} \hat{0}_{1,0}^{(4)}\right) x^{2} \log x+\ldots  \tag{2.43b}\\
\tilde{F}_{2}(x, x)=A+\left(2 B-D_{0}+F_{1,0}\right) x+\left(g^{2} F_{1,0} \hat{0}_{1,0}^{(2)}+g^{4} F_{1,0} \hat{\Delta}_{1,0}^{(4)}\right) x \log x+\ldots \tag{2.43c}
\end{gather*}
$$

In perturbation theory, the correlator can also be expanded in the very same way and it is then possible to match the two expansions and solve for the CFT data. This is precisely what we will do in section 4.3.

## CHAPTER 3

## Perturbation Theory

Until now we have been able to learn a lot about the defect $\mathcal{N}=4$ SYM theory non-perturbatively by using the symmetries of the theory as well as the spectrum of operators; this illustrates the potential of the defect superconformal bootstrap. We will now return to more traditional quantum field theory techniques, and in particular we intend to compute the correlator up to next-to-leading order using perturbation theory. Combined with the bootstrap and crossing symmetry, it allows us to obtain a lot of perturbative results in one blow. In this chapter, we introduce the tools needed for the perturbative computation of the two-point function in presence of the defect, which will be performed in chapter 4. We first introduce the action, and in particular we show how $4 \mathrm{~d} \mathcal{N}=4 \mathrm{SYM}$ can be derived from 10 d $\mathcal{N}=1$ SYM with dimensional reduction. We then present the relevant Feynman rules, and discuss the large $N$ expansion, where $N$ corresponds to the range of the color indices of the gauge algebra $\mathfrak{u}(N)$. Finally, we derive elementary insertion rules which will be useful in the next chapter for computing the correlator.

## 3.1 $\mathcal{N}=4$ Super Yang-Mills

This section will be dedicated to introducing the perturbative foundations of $\mathcal{N}=4$ SYM, namely its action and the related Feynman rules. We have been avoiding to write down the action until now in the name of the conformal bootstrap, but for the perturbative computation of the correlator we find ourselves forced to take this road. The action will be derived by performing dimensional reduction of $\mathcal{N}=1$ SYM in 10 dimensions. This is presented in e.g. [24, 39]. We conclude this section with a discussion of the limit $N \rightarrow \infty$, in which a large number of Feynman diagrams can be discarded.

## From $10 d \mathcal{N}=1$ to $4 d \mathcal{N}=4$

We start by considering (classical) Yang-Mills theory in Euclidean 10 -dimensional space, with $\mathcal{N}=1$ supersymmetry. The field content is one bosonic gauge field $A_{M}(M=1, \ldots, 10)$ and one Majorana spinor $\psi$, which can be expressed as:

$$
\begin{gather*}
A_{M}(x) \equiv T_{a} A_{M}^{a}(x),  \tag{3.1a}\\
\psi(x) \equiv T_{a} \psi^{a}(x) . \tag{3.1b}
\end{gather*}
$$

The non-Abelian field strength reads:

$$
\begin{equation*}
F_{M N} \equiv \partial_{M} A_{N}-\partial_{N} A_{M}-i\left[A_{M}, A_{N}\right], \tag{3.1c}
\end{equation*}
$$

and we define as usual the covariant derivative to be:

$$
\begin{equation*}
D_{M} \cdot \equiv \partial_{M} \cdot-i\left[A_{M}, \cdot\right] \tag{3.1d}
\end{equation*}
$$

There is only one supersymmetry, for which the infinitesimal transformations are given by:

$$
\begin{gather*}
\delta_{\bar{\varepsilon}} A_{M}=-i \bar{\varepsilon} \Gamma_{M} \psi,  \tag{3.2a}\\
\delta_{\varepsilon} \psi=\frac{i}{2} F_{M N} \Gamma^{M N} \varepsilon, \tag{3.2b}
\end{gather*}
$$

with $\varepsilon, \bar{\varepsilon}$ constant Majorana spinors, while the $\Gamma_{M}$ refer to the ten 16-dimensional Dirac $\Gamma$-matrices in the Majorana representation. We also defined:

$$
\Gamma_{M N} \equiv \frac{i}{2}\left[\Gamma_{M}, \Gamma_{N}\right] .
$$

The $\Gamma$-matrices fulfill:

$$
\operatorname{Tr} \Gamma_{M} \Gamma_{N}=16 \delta_{M N}
$$

The spinor $\psi$ is a 16 -component Majorana spinor, i.e. it obeys the reality condition given in eq. (1.3).
The Yang-Mills action reads:

$$
\begin{equation*}
S=\frac{1}{g^{2}} \int d^{10} x \operatorname{Tr}\left(\frac{1}{2} F_{M N} F^{M N}+i \bar{\psi} \Gamma_{M} D^{M} \psi\right) \tag{3.3}
\end{equation*}
$$

It is easy to read the mass dimension of the fields and of the coupling constant:

$$
[g]=3, \quad[A]=1, \quad[\psi]=3 / 2 .
$$

We now wish to reduce (3.3) to a theory in 4 spacetime dimensions and with 6 internal dimensions. We decompose the index $M=1, \ldots, 10$ into $\mu=1, \ldots, 4$ and $i=5, \ldots, 10$. The compactification of the extra 6 -dimensional space is fulfilled by imposing the following conditions:

$$
\begin{gather*}
\partial_{i} A_{M}(x) \stackrel{!}{=} 0  \tag{3.4a}\\
\partial_{i} \psi(x) \stackrel{!}{=} 0 \tag{3.4b}
\end{gather*}
$$

We define the extra degrees of freedom of the gauge field as 6 real scalar fields, i.e.:

$$
\begin{equation*}
A_{i}(x) \equiv \phi_{i}(x) \tag{3.5}
\end{equation*}
$$

As a consequence, the field strength tensor decomposes into:

$$
\begin{aligned}
F_{M N} F^{M N} & =F_{\mu \nu} F^{\mu v}+2 F_{\mu i} F^{\mu i}+F_{i j} F^{i j} \\
& =F_{\mu \nu} F^{\mu v}+2 D_{\mu} \phi_{i} D^{\mu} \phi^{i}-\left[\phi_{i}, \phi_{j}\right]\left[\phi^{i}, \phi^{j}\right] .
\end{aligned}
$$

The first term is a typical field strength tensor for 4-dimensional non-Abelian Yang-Mills theory, and it contains the kinetic term for $A_{\mu}$ as well as cubic and quartic interactions involving the gauge fields only. The second term contains the kinetic term for the scalar fields, and also cubic and quartic mixing terms involving the $A_{\mu}$ and the $\phi^{i}$. Finally, the product of commutators is a $\phi^{4}$-like interaction term.

We now turn our attention to the second term of (3.3). In the same way as above, the $\Gamma$-matrices decompose into:

$$
\Gamma_{M} \equiv\left(\gamma_{\mu}, \Gamma_{i}\right),
$$

and we have:

$$
\bar{\psi} \Gamma^{M} D_{M} \psi=\bar{\psi} \gamma^{\mu} D_{\mu} \psi-i \bar{\psi} \Gamma^{i}\left[\phi_{i}, \psi\right] .
$$

The first term contains the kinetic term for the fermion as well as an interaction term QCD-like between the fermions and the gauge field. The second term corresponds to a Yukawa vertex between scalar fields and fermions.

We have now obtained a reduction of the 10d Super Yang-Mills to 4 dimensions, by introducing 6 scalar fields. This has of course not affected the number of bosonic and fermionic degrees of freedom, which remains the same as shown in section 1.2 ( 8 each).

We can also obtain the Maldacena-Wilson loop defined in eq. (1.39) from dimensional reduction, starting with a regular 10-dimensional gauge-invariant Wilson loop:

$$
\begin{equation*}
\mathcal{W}(C) \equiv \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp i \oint_{C} d x_{M} A^{M}(x), \tag{3.6}
\end{equation*}
$$

where $C$ is a closed path of integration. Introducing a parameter $\tau$ such that $d x_{M}=d \tau \dot{x}_{M}$ (with $\dot{x}$ the derivative of $x$ with respect to $\tau$ ) and splitting the dimensions into spatial and internal coordinates by using (3.5), the Wilson-loop operator becomes:

$$
\mathcal{W}(C)=\frac{1}{N} \operatorname{Tr} \exp i \oint_{C} d \tau\left(\dot{x}_{\mu} A^{\mu}+\dot{x}^{i} \phi_{i}\right) .
$$

To relate $\dot{x}^{\mu}$ and $\dot{x}^{i}$, we impose now that the path $C$ is a null-curve (or lightlike), i.e.:

$$
\dot{x}^{M} \dot{x}_{M}=\dot{x}^{\mu} \dot{x}_{\mu}+\dot{x}^{i} \dot{x}_{i} \stackrel{!}{=} 0,
$$

which implies that:

$$
\begin{equation*}
\left|\dot{x}_{i}\right|= \pm i|\dot{x}|, \tag{3.7}
\end{equation*}
$$

where by a slight abuse of notation we defined:

$$
|\dot{x}| \equiv \sqrt{\dot{x}^{\mu} \dot{x}_{\mu}}
$$

Choosing the minus sign solution of eq. (3.7), we obtain:

$$
\begin{equation*}
\mathcal{W}(C)=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \oint_{C} d \tau\left(i \dot{x}_{\mu} A^{\mu}+|\dot{x}| \theta_{i} \phi^{i}\right) \tag{3.8}
\end{equation*}
$$

which is exactly the definition of the Maldacena-Wilson loop given in eq. (1.39).
We will now briefly look at the quantized theory and discuss some properties of $\mathcal{N}=4 \mathrm{SYM}$.

## The Action

The quantization of the action can be done using the conventional Faddeev-Popov method, which is standard material and is reviewed in most QFT textbooks (see e.g. [26]). This introduces unphysical ghost fields $c$, and at the end of the day we can rewrite the action as:

$$
\begin{align*}
S=\frac{1}{g^{2}} \int d^{4} x \operatorname{Tr}\left\{\frac{1}{2} F_{\mu v} F^{\mu v}\right. & +D_{\mu} \phi_{i} D^{\mu} \phi^{i}-\frac{1}{2}\left[\phi_{i}, \phi_{j}\right]\left[\phi^{i}, \phi^{j}\right] \\
& \left.+i \bar{\psi} \gamma^{\mu} D_{\mu} \psi+\bar{\psi} \Gamma^{i}\left[\phi_{i}, \psi\right]+\partial_{\mu} \bar{c} D^{\mu} c+\xi\left(\partial_{\mu} A^{\mu}\right)^{2}\right\}, \tag{3.9}
\end{align*}
$$

where we have redefined the coupling constant $g$ such that it absorbs the remaining volume integral on the 6 -dimensional internal space:

$$
\frac{1}{g^{2}} \int d^{6} x_{i} \rightarrow \frac{1}{g^{2}}
$$

This redefinition turns $g$ into a dimensionless quantity. Note that we use the covariant gauge fixing condition $\partial^{\mu} A_{\mu}=0$, and that we work in the Feynman gauge (i.e. $\xi \equiv 1$ ). The action that we obtained is the same as in e.g. [19, 40, 41], and it has $\mathcal{N}=4$ supersymmetry.

After reduction, the (physical) field content is now: one gauge field, one Majorana field (which can be further decomposed into four Weyl spinors) and six scalar fields. All of these fields live in the adjoint representation of $\mathfrak{u}(N)$. Of course, the fields still have the same mass dimensions as before.

The SUSY transformations given in (3.2) still keep the action invariant, and they can be reduced to 4 d as well. $\mathcal{N}=4$ is believed to be an integrable theory, and it seems that supersymmetry plays a central role in this property [11]. As mentioned in chapter $1, \mathcal{N}=4$ preserves conformal symmetry at the quantum level, and hence its $\beta$-function vanishes. The corresponding superconformal algebra has already been discussed in section 1.2 , and the (anti)commutation relations are given explicitly in appendix A.3. It is really conformal symmetry that gives $\mathcal{N}=4$ SYM its special place in the realm of quantum field theories.

## Feynman Rules

Let us now derive the Feynman rules (in position space) that are relevant to this work. Let us start with some important definitions. The bosonic propagators are Green's functions of the operator $\square$, and it is useful to define the following expression:

$$
\begin{equation*}
I_{12} \equiv \frac{1}{(2 \pi)^{2} x_{12}^{2}}, \tag{3.10a}
\end{equation*}
$$

with $x_{i j} \equiv x_{i}-x_{j}$ as usual.

We will also encounter three-, four- and five-point massless Feynman integrals, which we define as follows:

$$
\begin{gather*}
Y_{123} \equiv \int d^{4} x_{4} I_{14} I_{24} I_{34},  \tag{3.10b}\\
X_{1234} \equiv \int d^{4} x_{5} I_{15} I_{25} I_{35} I_{45},  \tag{3.10c}\\
H_{13,24} \equiv \int d^{4} x_{56} I_{15} I_{35} I_{26} I_{46} I_{56} . \tag{3.10d}
\end{gather*}
$$

In the last expression we have defined $d^{4} x_{56} \equiv d^{4} x_{5} d^{4} x_{6}$ for brevity. The letter assigned to each integral makes sense when drawing the propagators. We will also encounter the following expression:

$$
\begin{equation*}
F_{13,24} \equiv \frac{\left(\partial_{1}-\partial_{3}\right) \cdot\left(\partial_{2}-\partial_{4}\right) H_{13,24}}{I_{13} I_{24}} . \tag{3.10e}
\end{equation*}
$$

The notation presented above has already been used in e.g. [41, 42]. The Y- and X-integrals have been solved analytically and can be found in appendix C.1. The H-integral seems to have no known closed form so far, but (3.10e) can fortunately be reduced to a sum of Y - and X-integrals, as shown in appendix C. 1 (see eq. (C.6)).

We will represent free propagators of scalar fields by solid lines, while solid lines bearing an arrow will correspond to fermions. Wavy lines are gauge fields, and dotted lines are ghost fields. As mentioned before, the bosonic (free) propagators are Green's functions of the D'Alembert operator $\square \equiv \partial^{\mu} \partial_{\mu}$, and hence their position dependence is simply given by (3.10a), with the relevant indices attached accordingly. The propagator for the Majorana field can be obtained similarly (see e.g. [43]), and all in all the propagators read:

$$
\begin{align*}
& \underset{i, a}{1} \underset{j, b}{2}=g^{2} \delta_{i j} \delta^{a b} I_{12},  \tag{3.11a}\\
& \underset{\mu, a}{1} \underset{v, b}{2} \underset{0}{2}=g^{2} \delta_{\mu \nu} \delta^{a b} I_{12},  \tag{3.11b}\\
& \xrightarrow[a]{1} \xrightarrow[b]{\stackrel{2}{0}}=i g^{2} \delta^{a b} \partial_{\Delta} I_{12}, \tag{3.11c}
\end{align*}
$$

where we have defined for brevity:

$$
\partial_{\Delta} \equiv \gamma \cdot \frac{\partial}{\partial \Delta}, \quad \Delta \equiv x_{1}-x_{2}
$$

with $\gamma_{\mu}$ the Dirac matrices.
The $\mathcal{N}=4$ SYM theory also contains the following 7 vertices:

which can be directly read from the action (3.9). Not all vertices will be relevant for this work, and the important couplings will be listed as insertion rules in the next section.

## Large $N$ Expansion

We will now discuss the expansion in the large $N$ limit, which was introduced by 't Hooft in [44]. The idea is to take the limit $N \rightarrow \infty$ ( $N$ being the number of color indices of the gauge group $U(N)$ ) and to consider only the leading graphs in that regime, which we will call planar diagrams for reasons that will soon become clear.

To see why it is possible and senseful to take this limit, let us define the so-called 't Hooft coupling:

$$
\lambda \equiv g^{2} N
$$

where $g$ is the coupling constant of the Yang-Mills theory that we encountered in e.g. eq. (3.9). When $\lambda$ is kept fixed, taking $N \rightarrow \infty$ results in a divergent factor:

$$
\frac{1}{g^{2}}=\frac{N}{\lambda} \rightarrow \infty \text { for } N \rightarrow \infty
$$

But the number of components $N^{2}$ in the fields also diverges as $N$ goes to $\infty$. In fact, what we obtain is a subtle cancellation of these two infinities such that the action remains finite [44]. The radius of convergence of a large $N$ expansion is known to be non-zero [45], although this can be spoiled by renormalization effects.

In Feynman diagrams, the $N$ dependence is carried through the color factors and we will now describe how to count the factors of $N$. It is easy to see from the Feynman rules given previously that a propagator carries a factor $1 / N$ and that a vertex contributes $N$. The Wilson line contains a prefactor $1 / N$ (see eq. (1.39)). Finally, a color index contraction adds a factor $N$. This relates the large $N$ expansion to the topologies of the Feynman diagrams, which can be characterized by their corresponding Euler's characteristic:

$$
\begin{equation*}
\chi \equiv V-E+F=2-2 G, \tag{3.12}
\end{equation*}
$$

where $V$ is the number of vertices, $E$ the number of edges (i.e. propagators), $F$ the number of faces (i.e. index contractions) and $G$ the genus of the diagram. Planar diagrams are thus the diagrams which have the lowest genus, i.e. the diagrams that can be drawn in 2 dimensions without crossing.

We now show with an example how to apply this rule to the present work. First, we cut the two loose ends of the Wilson line and use the double-line system [45] in order to count the number of color index contractions. $F$ is then obtained by counting the number of independent contour lines. The following diagram has $V=2, E=3, F=3$, and hence it is planar:


The diagram above can be made non-planar by exchanging the middle points on the Wilson line. This produces a diagram with $F=1$, which is hence non-planar:


The non-planarity of the diagram is manifest since it cannot be drawn without having lines crossing.

### 3.2 Insertion Rules

In this section, we present and derive all the insertion rules that will be needed in chapter 4 for computing the two-point function with line defect. We first look at insertions on the Wilson line up to 4 points. The self-energy correction to the scalar propagator is then computed at one loop, and we conclude this chapter by giving the different 3 - and 4 -point insertion rules that are relevant for this work.

## Wilson Line

The expression given in (1.39) for the Maldacena-Wilson line can be expanded in the following way:

$$
\begin{aligned}
\mathcal{W}(C)= & \frac{1}{N} \operatorname{Tr}\left\{\mathcal{P} \exp \oint_{C} d \tau\left(i \dot{x}^{\mu} A_{\mu}(x)+|\dot{x}| \theta_{i} \phi^{i}\right)\right\} \\
= & 1+\frac{1}{N} \operatorname{Tr} \oint_{C} d \tau\left(i A_{\mu}(x) \dot{x}^{\mu}+\phi^{i}(x)|\dot{x}| \theta_{i}\right) \\
& \quad+\frac{1}{2!N} \operatorname{Tr} \mathcal{P} \oint_{C} d \tau_{1} d \tau_{2}\left(i A_{\mu}\left(x_{1}\right) \dot{x}_{1}^{\mu}+\phi^{i}\left(x_{1}\right)\left|\dot{x}_{1}\right| \theta_{i}\right)\left(i A_{v}\left(x_{2}\right) \dot{x}_{2}^{v}+\phi^{j}\left(x_{2}\right)\left|\dot{x}_{2}\right| \theta_{j}\right) \\
& \quad \ldots
\end{aligned}
$$

in which each term refers to a certain number of points on the line. The first term corresponds to the tree-level insertion (i.e. no coupling between the line and the rest of the system), which is simply:

$$
\begin{equation*}
\mid=1 \tag{3.13}
\end{equation*}
$$

At first order, the Wilson-line insertions are proportional to the trace of one generator:

$$
\begin{equation*}
\{,\} \sim \sim \operatorname{Tr} T^{a} . \tag{3.14}
\end{equation*}
$$

This is zero for the gauge algebra $\mathfrak{s u}(N)$, as well as for $\mathfrak{u}(N)$ when there is at least one vertex in the diagram. Since we use $\mathfrak{u}(N)$ as an approximation for $\mathfrak{s u}(N)$, we ignore artefacts arising from (3.14) ${ }^{1}$.

At second order, we obtain our first non-trivial contribution for two scalar points on the line. We can use the cyclicity of the trace in order to remove the path-ordering as follows:

$$
\begin{align*}
\text { _- } & =\frac{1}{2!N} \theta_{i} \theta_{j} \operatorname{Tr} \int d \tau_{1} \int d \tau_{2}\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|\left\langle\mathcal{P}\left(\phi_{1}^{i} \phi_{2}^{j}\right) \ldots\right\rangle \\
& =\frac{1}{2!N} \theta_{i} \theta_{j}\left(\operatorname{Tr} T^{a} T^{b} \int_{-\infty}^{\infty} \int_{-\infty}^{\tau_{1}} d \tau_{2}+\operatorname{Tr} T^{b} T^{a} \int_{-\infty}^{\infty} \int_{\tau_{1}}^{\infty} d \tau_{2}\right)\left\langle\phi_{1, a}^{i} \phi_{2, b}^{j} \ldots\right\rangle \\
& =\frac{1}{4 N} \theta_{i} \theta_{j} \delta^{a b} \int d \tau_{1} \int d \tau_{2}\left\langle\phi_{1, a}^{i} \phi_{2, b}^{j} \ldots\right\rangle \tag{3.15}
\end{align*}
$$

It is important to understand that the diagram on the left-hand side refers to all possible path-orderings, as explicitly written on the right-hand side. In the same way, we can obtain the rule for two gluon points on the line:

$$
\begin{equation*}
\rho \sim \sim=-\frac{1}{4 N} \delta^{a b} \int d \tau_{1} \int d \tau_{2} \dot{x}_{1}^{\mu} \dot{x}_{2}^{v}\left\langle A_{1, \mu, a} A_{2, v, b} \ldots\right\rangle . \tag{3.16}
\end{equation*}
$$

Note that, in our setup, $\dot{x}_{i}^{\mu}=(0,0,0,1)$ for $i=1,2$. The insertion rule $\sim_{\sim}^{\sim}$ is not needed at NLO, but it would most probably become relevant at next-to-next-to-leading order (NNLO).

At third order, there is only one relevant insertion rule:

[^3]\[

$$
\begin{align*}
\sim \sim & \frac{i}{2!1!N} \theta_{i} \theta_{j} \operatorname{Tr} \int d \tau_{1} \int d \tau_{2} \int d \tau_{3}\left|\dot{x}_{1}\right| \dot{x}_{2}^{\mu}\left|\dot{x}_{3}\right|\left\langle\mathcal{P}\left(\phi_{1}^{i} A_{2, \mu} \phi_{3}^{j}\right) \ldots\right\rangle \\
= & \frac{i}{2!1!N} \theta_{i} \theta_{j} \int d \tau_{1} \int d \tau_{2} \int d \tau_{3}\left(\Theta\left(\tau_{123}\right) \operatorname{Tr} T^{a} T^{b} T^{c}+\Theta\left(\tau_{132}\right) \operatorname{Tr} T^{a} T^{c} T^{b}+\ldots\right) \\
& \quad \times \dot{x}_{2}^{\mu}\left\langle\phi_{1, a}^{i} A_{2, \mu, b} \phi_{3, c}^{j} \cdots\right\rangle \\
= & -\frac{1}{8 N} \theta_{i} \theta_{j} f^{a b c} \int d \tau_{1} \int d \tau_{2} \int d \tau_{3} \varepsilon\left(\tau_{1} \tau_{2} \tau_{3}\right) \dot{x}_{2}^{\mu}\left\langle\phi_{1, a}^{i} A_{2, \mu, b} \phi_{3, c}^{j} \ldots\right\rangle \\
& \quad+\frac{i}{8 N} \theta_{i} \theta_{j} d^{a b c} \int d \tau_{1} \int d \tau_{2} \int d \tau_{3} \dot{x}_{2}^{\mu}\left\langle\phi_{1, a}^{i} A_{2, \mu, b} \phi_{3, c}^{j} \ldots\right\rangle, \tag{3.17}
\end{align*}
$$
\]

where we defined the path-ordering symbols $\Theta\left(\tau_{i j k}\right)$ and $\varepsilon\left(\tau_{i} \tau_{j} \tau_{k}\right)$ as follows:

$$
\begin{gather*}
\Theta\left(\tau_{i j k}\right) \equiv \Theta\left(\tau_{i j}\right) \Theta\left(\tau_{j k}\right),  \tag{3.18a}\\
\varepsilon\left(\tau_{i} \tau_{j} \tau_{k}\right) \equiv \operatorname{sgn}\left(\tau_{i j}\right) \operatorname{sgn}\left(\tau_{i k}\right) \operatorname{sgn}\left(\tau_{j k}\right) . \tag{3.18b}
\end{gather*}
$$

The second definition is needed in order to account for the antisymmetry of $f^{a b c}$. The second term in (3.17) will always vanish in this work. Note that once again the diagrammatic representation of (3.17) includes all possible path-orderings.

Finally, we will also need the fourth-order insertion involving only scalar fields, i.e.:

$$
\begin{gather*}
=\frac{1}{4!N} \theta_{i} \theta_{j} \theta_{k} \theta_{l} \int d \tau_{1} \int d \tau_{2} \int d \tau_{3} \int d \tau_{4}\left(\Theta\left(\tau_{1234}\right) \operatorname{Tr} T^{a} T^{b} T^{c} T^{d}+\Theta\left(\tau_{1243}\right) \operatorname{Tr} T^{a} T^{b} T^{d} T^{c}+\ldots\right) \\
\times\left\langle\phi_{1, a}^{i} \phi_{2, b}^{j} \phi_{3, c}^{k} \phi_{4, d}^{l} \cdots\right\rangle \tag{3.19}
\end{gather*}
$$

with:

$$
\begin{equation*}
\Theta\left(\tau_{i j k l}\right) \equiv \Theta\left(\tau_{i j}\right) \Theta\left(\tau_{j k}\right) \Theta\left(\tau_{k l}\right) \tag{3.20}
\end{equation*}
$$

In the absence of vertices, this expression simply reduces to:

$$
\frac{1}{N} \theta_{i} \theta_{j} \theta_{k} \theta_{l} \operatorname{Tr} T^{a} T^{b} T^{c} T^{d} \int d \tau_{1} \int d \tau_{2} \int d \tau_{3} \int d \tau_{4} \Theta\left(\tau_{1234}\right)\left\langle\phi_{1, a}^{i} \phi_{2, b}^{j} \phi_{3, c}^{k} \phi_{4, d}^{l} \ldots\right\rangle .
$$

An identity relating the trace of 4 generators and the structure constants of the gauge group is given by eq. (A.18) in appendix A.2.

## Scalar Self-Energy

We now turn our attention to the one-loop correction of the scalar propagator. It consists of the following diagrams:


All the diagrams are easy to compute. The first one gives:

$$
\begin{aligned}
& =(-i)^{2} \frac{2}{g^{4}} \int d^{4} x_{3} d^{4} x_{4}\left\langle\phi_{1, i}^{a} \phi_{2, j}^{b} \operatorname{Tr} \partial_{\mu} \phi_{3}^{k} A_{3}^{\mu} \phi_{3, k} \operatorname{Tr} \partial_{v} \phi_{4}^{l} A_{4}^{v} \phi_{4, l}\right\rangle \\
& =2 g^{4} N \delta^{a b} \delta_{i j} Y_{112}+g^{4} N \delta^{a b} \delta_{i j} \frac{1}{(2 \pi)^{2} \varepsilon^{2}} \int d^{4} x_{3} I_{13} I_{23} .
\end{aligned}
$$

Note that the Y-integral of the first term is given explicitly in eq. (C.9) and contains a logarithmic divergence, while the second term contains a quadratic divergence encoded by the $1 / \varepsilon^{2}$. This factor arises when defining:

$$
\begin{equation*}
I_{33} \equiv \frac{1}{(2 \pi)^{2} \varepsilon^{2}} . \tag{3.21}
\end{equation*}
$$

This is called point-splitting regularization, i.e. the zero is replaced by an infinitesimal distance.
The second diagram reads:

where we recognize the same types of divergences as in the first diagram.
The last two diagrams differ only by a multiplicative factor, and give:


Those diagrams only contain quadratic divergences, and they would have vanished had we used dimensional regularization. Nevertheless they cancel each other when we sum up all the contributions together, and we are left with an expression containing only one log divergence:

$$
\begin{equation*}
\underset{i, a}{1} \underset{\substack{1 \\ j, b}}{2}=-2 g^{4} N \delta^{a b} \delta_{i j} Y_{112} . \tag{3.22}
\end{equation*}
$$

This expression is well-known, and is the same as the one given in e.g. [19, 40]. The expression for the gluon self-energy is very similar at one-loop and can also be found in [19, 40] (it will not be needed in this work).

## $n$-Point Insertions

We now derive the 3- and 4-point insertions that will be needed in chapter 4. The only 3-point insertion that is relevant is the vertex connecting two scalar fields and one gauge field, which is easy to obtain from the action (3.9):

$$
\begin{align*}
\underset{\mu, c}{3}{\underset{2}{j, b}}_{\rho_{j}^{i, a}}^{i} & =\frac{2 i}{g^{2}} \operatorname{Tr} T^{d}\left[T^{e}, T^{f}\right] \int d^{4} x_{4}\left\langle\phi_{1}^{i, a} \phi_{2}^{j, b} A_{3, \mu}^{c} \partial^{v} \phi_{4, d}^{k} \phi_{4, e, k} A_{4, v, f}\right\rangle \\
& =-g^{4} f^{a b c} \delta^{i j}\left(\partial_{1}-\partial_{2}\right)_{\mu} Y_{123} . \tag{3.23}
\end{align*}
$$

Another vertex that we need is the 4 -scalars coupling. Similarly to the 3 -vertex, it is straightforward to read the corresponding Feynman rule from the action and perform the Wick contractions to get:


We will also use the 4 -coupling between 2 scalars and 2 gluons. This vertex reads:


There are two more sophisticated 4-point insertions that we require. The first one reads:

with $F_{13,24}$ as defined in (3.10e).
The second (and last) insertion rule needed is the following:


Equipped with the Feynman rules and these insertion formulae, we are now ready to write the two-point function with line defect up to the next-to-leading order.

## CHAPTER 4

## Two-Point Function at Next-To-Leading Order and CFT Data

This final chapter presents the most important results of this thesis. The first section shows the topologies that are relevant up to order $\mathcal{O}\left(g^{8}\right)(\mathrm{NLO})$ and the corresponding Feynman diagrams to compute. We then compute the correlator up to leading order (LO), and show some intermediate results at NLO. In particular, the integral of the 2-channel could be solved in a closed form. Finally, we extract the CFT data order by order using the perturbative results and the expansions presented in section 2.3.

### 4.1 Feynman Diagrams

We start by collecting the diagrams which are relevant for the computation of the two-point function at $\mathcal{O}\left(g^{8}\right)$. We first show classes of vanishing diagrams, and discuss the non-renormalization of the disconnected two-point function. Then we derive a set of two equations to help us find the surviving topologies, and give the diagrams that we will need to compute in the subsequent sections.

## Vanishing Graphs

We already mentioned that graphs with only one point on the Wilson line are not to be considered in section 3.2. When there are two points on the Wilson line, the diagram vanishes if the two propagators coming from the line directly meet at a 3-point vertex. This happens because of the color factor:
where it is understood that the trace acts on $\phi_{1}$ and $\phi_{2}$ only. It follows analogously that:

$$
\begin{equation*}
\text { Ro- } O=\operatorname{Ci}_{8}^{\tan } O=0 \tag{4.1b}
\end{equation*}
$$

and in the very same way we also find that such a 3 -vertex contracted on the two fields of a single-trace operator produces a vanishing diagrams - or with less words:

$$
\begin{equation*}
\sim=0, \tag{4.1c}
\end{equation*}
$$

where the rightmost circle with the two points represents a single-trace operator (the circle being the trace).

## Non-Renormalization of the Defect-Free Two-Point Function

It is a well-known result that the two-point function of $1 / 2$-BPS operators does not renormalize at any loop-order [12, 13], i.e. the two-point function of single-trace operators $\mathcal{O}$ without defect reads:

$$
\begin{align*}
\left\langle\mathcal{O}\left(x_{1}\right) \mathcal{O}\left(x_{2}\right)\right\rangle & =f(g, N) \cdot \overbrace{0}^{\Omega} \\
& =f(g, N) \frac{g^{4} N^{2}}{2}\left(u_{1} \cdot u_{2}\right)^{2} I_{12}^{2} \\
& =\frac{\left(u_{1} \cdot \theta\right)^{2}\left(u_{2} \cdot \theta\right)^{2}}{x_{1}^{2} x_{2}^{2}} f(g, N) \frac{g^{4} N^{2}}{2^{5} \pi^{4}} \Omega^{2}, \tag{4.2}
\end{align*}
$$

with $f(g, N)$ a function depending on the regularization scheme and $\Omega$ as defined in (2.24). Note that here the expectation value does not include the defect, and hence it corresponds to the identity contribution in the full defect two-point function at order $\mathcal{O}\left(g^{4}\right)$, where $f(g, N)=1$. The $k=2$ operators find themselves in the same multiplet as the energy-momentum tensor $T_{\mu \nu}$, which does not renormalize, hence by supersymmetry $\left\langle\mathcal{O}\left(x_{1}\right) \mathcal{O}\left(x_{2}\right)\right\rangle$ is protected as well and its conformal dimension is preserved at the quantum level [46].

In order to make this discussion plausible, let us compute explicitly the two-point function at one-loop. It consists of the following diagrams:


We can use our insertion rules from the previous chapter as well as the identity (C.6) in order to figure out what each diagram corresponds to. After doing the Wick contractions, we easily find:

where for the second and third diagrams we took the limit $N \rightarrow \infty$. Adding these results together shows that the two-point function vanishes at one loop, i.e.:

$$
\begin{equation*}
\left.f(g, N)\right|_{\mathcal{O}\left(g^{6}\right)}=0, \tag{4.4}
\end{equation*}
$$

when using point-splitting regularization. This shows that we do not have to consider diagrams where the two operators and the Wilson line are disconnected. This is also the case in the context of the Ward identities, as even a non-zero $f(g, N)$ would only correspond to a constant shift in e.g. eq. (2.39a) and (2.39b).

Table 4.1: Solutions of topology eq. (4.5a) and (4.5b), excluding the vanishing graphs.

|  | $\mathcal{O}\left(g^{4}\right)$ | $\mathcal{O}\left(g^{6}\right)$ | $\mathcal{O}\left(g^{8}\right)$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{O}(\Gamma)$ | 4 | 6 | 8 | 8 | 8 | 8 | 8 |
| $\omega_{C}$ | 0 | 2 | 2 | 2 | 2 | 3 | 4 |
| $v_{3}$ | 0 | 0 | 0 | 2 | 0 | 1 | 0 |
| $v_{4}$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $p_{0}$ | 2 | 3 | 2 | 6 | 5 | 5 | 4 |
| $p_{1}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

## Topologies and Diagrams

We will now derive equations that will allow us to easily determine the allowed topologies. For any Feynman diagram $\Gamma$, it is convenient to define the following variables: $p_{0}$ is the number of tree-level propagators, $p_{1}$ is the number of one-loop propagators, $v_{3}$ is the number of 3 -vertices and $v_{4}$ the number of 4 -vertices. Moreover, let $\omega_{C}$ be the number of points on the Wilson line.

The first equation that we are looking for should relate the order of a graph $\Gamma$ to the numbers of propagators and vertices in it. This is straightforward to obtain, since (i) a tree-level propagator contributes $\mathcal{O}\left(g^{2}\right)$, (ii) a one-loop propagator contributes $\mathcal{O}\left(g^{4}\right)$, and (iii) any vertex contributes $\mathcal{O}\left(g^{-2}\right)$. This translates mathematically to:

$$
\begin{equation*}
\mathcal{O}(\Gamma) \stackrel{!}{=} 2\left(p_{0}+2 p_{1}-v_{3}-v_{4}\right) \tag{4.5a}
\end{equation*}
$$

where $\mathcal{O}(\Gamma)$ is the order of $\Gamma$ in $g$.
We need one more equation, since this one does not involve $\omega_{C}$. The (total) number of propagators $p \equiv p_{0}+p_{1}$ in a graph $\Gamma$ can only depend on (i) the number of points on the Wilson line $\omega_{C}$, (ii) the number of 3 - and 4 - vertices, and (iii) the number of fields in each $1 / 2$-BPS operator, here fixed at 2 . The third dependence means that we have at least 2 propagators in the diagrams at all times. Each point on the Wilson line contributes a half-propagator, from which the other half must end somewhere. Similarly, the 3 -vertices contribute three halves of a propagator and the 4 -vertices four halves. Writing this as an equation gives:

$$
\begin{equation*}
p^{!}=2+\frac{1}{2}\left(\omega_{C}+\sum_{n=3,4} n v_{n}\right), \tag{4.5b}
\end{equation*}
$$

where of course $p$ should be an integer. Moreover, we have seen in the previous section that (i) all diagrams with $\omega_{C}=0$ can be ignored at order $\mathcal{O}(\Gamma) \geq 6$, and (ii) all diagrams vanish when $\omega_{C}=1$.

Using the two equations and the constraints, we easily obtain all the topologies relevant up to order $\mathcal{O}\left(g^{8}\right)$. The corresponding configurations are gathered in table 4.1.

It is now straightforward to read the topologies of table 4.1 and dress the diagrams accordingly. The resulting (planar) configurations are summarized in table 4.2 and categorized in function of their R symmetry channel (see section 2.3).

We already know that the self-energy diagrams are log-divergent. We will show in section 4.2 that these divergences are in fact canceled by the ones arising from other diagrams, thus making the full expectation value finite.

Table 4.2: Connected diagrams for the computation of the two-point function with line defect up to next-to-leading order (NLO). The configurations are classified in function of their $R$-symmetry channel.
0-Channel

2-Channel


### 4.2 Perturbative Computation

We now compute the correlator for the two-point function with line-defect up to leading order, and show how far the integrals can be taken analytically for the next-to-leading order. We also show that the Ward identities are satisfied in a non-trivial way at NLO on the line $z=\bar{z}$ by using numerical computations.

## Identity and Leading Orders

We call the identity order the order at which the two-point function without defect contributes. There is only one diagram consisting of two propagators, and hence it is of order $\mathcal{O}\left(g^{4}\right)$.

The diagram is the following, and we refer to it as disconnected since it does not couple to the defect:

$$
\begin{align*}
\Omega & =\frac{1}{2}\left(u_{1} \cdot u_{2}\right)^{2} g^{4} N^{2} I_{12}^{2} \\
& =\frac{\left(u_{1} \cdot \theta\right)^{2}\left(u_{2} \cdot \theta\right)^{2}}{x_{1}^{2} x_{2}^{2}} \frac{g^{4} N^{2}}{2^{5} \pi^{4}} \Omega^{2} . \tag{4.6}
\end{align*}
$$

There is no integral to solve, and it is easy to see that the $g$-functions defined in section 2.3 are constant, i.e.:

$$
\begin{equation*}
g_{0}(z, \bar{z})=\frac{g^{4} N^{2}}{2^{5} \pi^{4}} \tag{4.7a}
\end{equation*}
$$

and:

$$
\begin{equation*}
g_{1}(z, \bar{z})=g_{2}(z, \bar{z})=0 \tag{4.7b}
\end{equation*}
$$

We saw at the end of section 2.3 that the most convenient formulation of the correlator in order to extract the CFT data was in terms of the $\tilde{F}$ 's, which are defined in eq. (2.40). The translation of the results given above simply reads:

$$
\begin{equation*}
\tilde{F}_{0}(z, \bar{z})=\frac{g^{4} N^{2}}{2^{5} \pi^{4}} \Omega_{\mathrm{ST}}^{2}, \tag{4.8}
\end{equation*}
$$

while the contributions to the other channels still vanish of course.
The leading order is the order at which the first connected diagram appears. Since the two-point function does not renormalize, we are left with only one contribution:

$$
\overbrace{}^{\{ }=\frac{\delta_{a b}}{4 N} \theta_{i} \theta_{j} u_{1, k} u_{1, l} u_{2, m} u_{2, n} \frac{\delta_{c d} \delta_{e f}}{4} \int d \tau_{3} \int d \tau_{4}\left\langle\left\langle\overline{\phi_{3}^{i, a} \phi_{4}^{j, b} \phi_{1}^{k, c} \phi_{1}^{l, d} \phi_{2}^{m, e}} \phi_{2}^{n, f}\right\rangle .\right.
$$

Doing the contractions, using the integral given in (C.1b) and inserting the invariant defined in (2.24), it is easy to obtain the following expression:

$$
\begin{equation*}
=\frac{\left(u_{1} \cdot \theta\right)^{2}\left(u_{2} \cdot \theta\right)^{2}}{x_{1}^{2} x_{2}^{2}} \frac{g^{6} N}{2^{7} \pi^{4}} \Omega . \tag{4.9}
\end{equation*}
$$

This translates into the $\tilde{F}$-function:

$$
\begin{equation*}
\tilde{F}_{1}(z, \bar{z})=\frac{g^{6} N}{2^{7} \pi^{4}} \Omega_{\mathrm{ST}}, \tag{4.10a}
\end{equation*}
$$

while the other channels vanish, i.e.:

$$
\begin{equation*}
\tilde{F}_{0}(z, \bar{z})=\tilde{F}_{2}(z, \bar{z})=0 \tag{4.10b}
\end{equation*}
$$

The full correlator up to order $\mathcal{O}\left(g^{6}\right)$ is therefore:

$$
\begin{equation*}
\left.\left\langle\mathcal{O}\left(x_{1}\right) \mathcal{O}\left(x_{2}\right)\right\rangle\right|_{\mathcal{O}\left(g^{6}\right)}=\frac{\left(u_{1} \cdot \theta\right)^{2}\left(u_{2} \cdot \theta\right)^{2}}{x_{1}^{2} x_{2}^{2}} \frac{g^{4} N^{2}}{2^{5} \pi^{4}}\left\{\Omega^{2}+\frac{g^{2}}{8 N} \Omega\right\}, \tag{4.11}
\end{equation*}
$$

which easily converts back to $\tilde{F}_{c}$ 's by dropping the leftmost prefactor and by replacing $\Omega$ by $\Omega_{\mathrm{ST}}$.
The Ward identities are manifestly fulfilled at this order, since the $F$-function only consists of powers of $\Omega$ (see eq. (2.25)).

## 2-Channel at Next-to-Leading Order

We now turn our attention to the more challenging computation of the correlator at next-to-leading order. In order to have compact expressions, it is convenient to define the following $R$-symmetry constant:

$$
\begin{equation*}
\lambda_{c} \equiv g^{8} N^{2}\left(u_{1} \cdot u_{2}\right)^{2-c}\left(u_{1} \cdot \theta\right)^{c}\left(u_{2} \cdot \theta\right)^{c} \tag{4.12}
\end{equation*}
$$

where $c=0,1,2$ refers to the channel just as it was the case in section 2.3.

We start by considering the 2 -channel, which is the simplest one since it consists of only one diagram without any vertex. After discarding the non-planar diagrams (this diagram was explicitly given as an example for the large $N$ limit in section 3.1) and doing the Wick contractions, it reads:

where it should be remembered that we considered all permutations of the points on the Wilson line when defining the insertion rule (3.19). $\Theta\left(\tau_{3456}\right)$ is defined in (3.20). This translates into:

$$
\tilde{F}_{2}(z, \bar{z}, \omega)=\frac{1}{8} g^{8} N^{2} I\left(1, x_{2}^{2}\right),
$$

with $I\left(x_{1}^{2}, x_{2}^{2}\right)$ the integral given above, and which is also defined in eq. (C.12). The diagram depends manifestly only on $x_{2}^{2}$, i.e. the only variable is the distance between the operator and the line, and the distance between the two bulk operators is irrelevant for this channel. The integral also exhibits an interesting inversion symmetry, i.e. it is invariant under the transformation $\left|x_{i}\right| \leftrightarrow 1 /\left|x_{i}\right|$ for $i=1,2$. This means that knowing the behavior of the channel for the range $(0,1)$ in the limit $z=\bar{z}$ is enough for knowing it for the entire $\mathbb{R}^{2}$.

The computation of the integral can be performed analytically, except for the last one-dimensional integral. We were able to find an exact expansion of the integral using numerical data, and from there it happened to be possible to guess the exact closed form. This procedure is detailed in appendix C.2, and we obtain the following expression:

$$
\begin{gather*}
\tilde{F}_{2}(z, \bar{z}, \omega)=\frac{1}{2^{12} \pi^{6}} g^{8} N^{2}\left\{3 \pi^{2}-4 i \pi \log 2+4 \tanh ^{-1} \sqrt{z \bar{z}}\left(\log z \bar{z}+4 \log 2-2 \tanh ^{-1} \sqrt{z \bar{z}}\right)\right. \\
+4 \log ^{2}(1-\sqrt{z \bar{z}})+2 \log (\sqrt{z \bar{z}}-1)(-2 \log (1-\sqrt{z \bar{z}})+\log (\sqrt{z \bar{z}}-1)+2 \log 2) \\
-2 \log (1+\sqrt{z \bar{z}}) \log 4(1+\sqrt{z \bar{z}})+4 \operatorname{Li}_{2}(-\sqrt{z \bar{z}})-4 \operatorname{Li}_{2} \sqrt{z \bar{z}} \\
\left.-4 \operatorname{Li}_{2} \frac{1}{2}(1-\sqrt{z \bar{z}})+4 \operatorname{Li}_{2} \frac{1}{2}(1+\sqrt{z \bar{z}})\right\} \tag{4.13}
\end{gather*}
$$

The corresponding $g$-function, useful for checking that the Ward identities are fulfilled, is plotted in fig. 4.1.

## 1-Channel at Next-to-Leading Order

The 1-channel contains many more diagrams, which now include vertices and that we therefore expect to be more difficult. Indeed it will be shown that the integrals are too hard to be solved, but that they can be computed numerically in the limit $z=\bar{z}$. The insertion rules used for reading the diagrams can all be found in section 3.2.

The first category of diagrams that we will treat are the divergent diagrams, where an infinity arises when we "pinch" the vertices towards one operator. We will refer to such diagrams as corner diagrams, in which we also include the self-energy diagrams for reasons that will soon become clear.

The $X$-diagram pinched at $x_{1}$ results in the following expression (after performing the contractions and taking the symmetry factors into account):


Figure 4.1: The left plot presents the numerical data gathered for the g-functions of each channel on the line $z=\bar{z}$. The dots represent the actual measurements, which can be found in table C.6, while the dashed lines just connect the dots and are here to guide the eye. Note that $g_{2}(x, x)$ is known analytically (see eq. (4.13)). On the right side of the figure, the superconformal Ward identities are tested with the data of the left plot following eq. (2.39a) and (2.39b). The identities are seen to be fulfilled, in that the function is constant for $x<0$ and $x>0$. The dots are the measurements, while the dashed lines indicate the constants $c_{1}=1 / 3 \cdot 2^{6} \pi^{4}$ and $c_{2}=0$.

$$
\begin{equation*}
=-\lambda_{1} \int d \tau_{3} \int d \tau_{4} I_{24} X_{1123} \tag{4.14}
\end{equation*}
$$

where we note that the pinching limit $X_{1123}$ is known analytically and given by eq. (C.10) in appendix C.1. But since some cancellation will occur, let us first collect the expressions associated to each diagram before attempting any computation. For now we simply note that this diagram is logarithmically divergent.

The $H$-diagram pinched at $x_{1}$ turns out to be:

$$
\begin{align*}
& =-\lambda_{1} I_{12} \int d \tau_{3} \int d \tau_{4} I_{13} I_{24} F_{12,13} \\
& =-\lambda_{1} I_{12} \int d \tau_{3} \int d \tau_{4} I_{13} I_{24}\left\{-\frac{X_{1123}}{I_{12} I_{13}}+\frac{Y_{112}}{I_{12}}+\frac{Y_{113}}{I_{13}}+\left(\frac{1}{I_{12}}+\frac{1}{I_{13}}-\frac{2}{I_{23}}\right) Y_{123}\right\}, \tag{4.15}
\end{align*}
$$

where in the second line we have made use of the pinched integral identity given in eq. (C.11). The three first terms are also logarithmically divergent.

The self-energy diagrams are straightforward to read using (3.22) and give:

$$
\begin{equation*}
\text { Q }=2 \lambda_{1} \int d \tau_{3} \int d \tau_{4} I_{13} I_{24} Y_{112} \tag{4.16}
\end{equation*}
$$

as well as:

$$
\begin{equation*}
\text { ? }=2 \lambda_{1} I_{12} \int d \tau_{3} \int d \tau_{4} I_{24} Y_{113} \tag{4.17}
\end{equation*}
$$

As we already know, these diagrams are also logarithmically divergent.
Comparing expressions in eq. (4.14-4.17), we notice that many terms occur several times with different signs. Hence it makes sense to group the diagrams in an upper-right corner diagram as follows:


Due to cancellations, the expression has considerably simplified and the integral is now finite! Note that this expression is in phase with the treatment of corner interactions performed in [42].

In a fully analogous way, we can treat the diagrams of the opposite corner, using the remaining half of the symmetric self-energy diagram:


The corner IYI-diagram at $x_{1}$ is defined as follows:

where we recall that all permutations of the legs connected to the line are considered. The path-ordering symbol is defined by eq. (3.18b).

This expression can be further simplified in the following way. Using integration by parts, one can rewrite $\left(\partial_{\tau_{2}}-\partial_{\tau_{5}}\right) Y_{245} \widehat{=}-\left(\partial_{\tau_{4}}+2 \partial_{\tau_{5}}\right) Y_{245}$. Since the derivatives now only act on the Wilson-line points, we can integrate by parts with respect to $\tau_{4}$ and $\tau_{5}$, and use the fact that:

$$
\partial_{\tau_{4}} \epsilon\left(\tau_{3} \tau_{4} \tau_{5}\right)=2\left(\delta\left(\tau_{45}\right)-\delta\left(\tau_{43}\right)\right),
$$

with $\tau_{i j} \equiv \tau_{i}-\tau_{j}$ as usual. The $\delta$-functions kill one $\tau$-integral, and we are left with:


We can combine this diagram with the remaining half of the corresponding self-energy to obtain the following upper-left corner diagram:


Once again, we see that the pinched integrals canceled, thus making this expression finite.
Similarly, we define the following lower-left corner diagram:


Another class of diagrams are the so-called symmetric diagrams, where the integrals are invariant under a permutation $x_{1} \leftrightarrow x_{2}$. Using the 4-point insertion rules from section 3.2, it is straightforward to write down the expressions corresponding to each diagram and to perform the Wick contractions. There is one $X$-diagram, which reads:

where the subscript 1 means that we only consider the 1-channel of the diagram. Indeed, the diagram contributes in principle also to the 0 -channel, as we will see in the next subsection.

There is also one $H$-diagram, which gives:

$$
\begin{equation*}
\}=-\lambda_{1} I_{12} \int d \tau_{3} \int d \tau_{4} I_{13} I_{24} F_{13,24}, \tag{4.24}
\end{equation*}
$$

where the F- and X-integrals are defined in section 3.1. Note that we have made use of the insertion rules (3.26) and of the fact that:

$$
\begin{equation*}
\int d \tau_{3} \int d \tau_{4}\left\{I_{13} I_{24} F_{13,24}+I_{14} I_{23} F_{14,23}\right\}=2 \int d \tau_{3} \int d \tau_{4} I_{13} I_{24} F_{13,24} \tag{4.25}
\end{equation*}
$$

In the same way, the symmetric IYI-diagram gives:


The $\tau_{3}$ - and $\tau_{5}$-integrals can be performed as follows: insert the definition (3.18b) of the path-ordering symbol $\epsilon\left(\tau_{3} \tau_{4} \tau_{5}\right)$, split e.g. the $\tau_{5}$-integral into pieces such that the signum functions involving $\tau_{5}$ can be eliminated, and integrate termwise. This results in the following expression:

$$
\int d \tau_{5} \epsilon\left(\tau_{3} \tau_{4} \tau_{5}\right) I_{25}=\frac{1}{(2 \pi)^{2}\left|x_{2}\right|}\left\{2\left(\tan ^{-1} \frac{\tau_{4}}{\left|x_{2}\right|}-\tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}\right)+\operatorname{sgn} \tau_{34} \pi\right\} .
$$

The $\tau_{3}$-integral is easy to do, and the IYI-diagram turns out to be:

$$
\begin{equation*}
=-\frac{1}{2} \frac{\lambda_{1}}{(2 \pi)^{3}\left|x_{1}\right|\left|x_{2}\right|} \int d \tau_{4}\left(\tan ^{-1} \frac{\tau_{4}}{\left|x_{2}\right|}-\tan ^{-1} \frac{\tau_{4}}{\left|x_{1}\right|}\right)\left(\partial_{\tau_{1}}-\partial_{\tau_{2}}\right) Y_{124 .} . \tag{4.26}
\end{equation*}
$$

There remains only a one-dimensional integral to do for this diagram. All the symmetric diagrams are finite, and hence the 1-channel is also finite on its own as expected.

Since the expressions for the Y-, X- and F-integrals are known analytically, we are left with onedimensional and two-dimensional integrals. We can group them accordingly, and we define:

$$
\begin{equation*}
F_{1}^{1 \mathrm{~d}}(z, \bar{z}, \omega) \equiv F_{\mathrm{IYI}}(z, \bar{z}, \omega)+F_{Y_{123}}(z, \bar{z}, \omega)+F_{Y_{124}}(z, \bar{z}, \omega), \tag{4.27}
\end{equation*}
$$

with:

$$
\begin{aligned}
F_{Y_{123}}(z, \bar{z}, \omega) & \equiv \lambda_{1} I_{12} \int d \tau_{3} \int d \tau_{4} I_{13} I_{24}\left(\frac{1}{I_{23}}-\frac{1}{I_{13}}\right) Y_{123} \\
& =\frac{\lambda_{1}}{4 \pi\left|x_{2}\right|} I_{12} \int d \tau_{3}\left(\frac{I_{13}}{I_{23}}-1\right) Y_{123}
\end{aligned}
$$

where we have used the elementary integral given in eq. (C.1b), and:

$$
F_{Y_{124}}(z, \bar{z}, \omega) \equiv \frac{\lambda_{1}}{4 \pi\left|x_{1}\right|} I_{12} \int d \tau_{4}\left(\frac{I_{24}}{I_{14}}-1\right) Y_{124} .
$$

$F_{\text {IYI }}$ is simply defined by the symmetric IYI-diagram of eq. (4.26).
Similarly, grouping the 2d integrals together results in:

$$
\begin{equation*}
F_{1}^{2 \mathrm{~d}}(z, \bar{z}, \omega) \equiv F_{X}(z, \bar{z}, \omega)+F_{Y_{134}}(z, \bar{z}, \omega)+F_{Y_{234}}(z, \bar{z}, \omega), \tag{4.28}
\end{equation*}
$$

where we have defined:

$$
F_{X}(z, \bar{z}, \omega) \equiv \lambda_{1} I_{12} \int d \tau_{3} \int d \tau_{4} I_{13} I_{24}\left(\frac{1}{I_{14} I_{23}}-\frac{1}{I_{13} I_{24}}-\frac{1}{I_{12} I_{34}}\right) X_{1234}
$$

as well as:

$$
F_{Y_{134}}(z, \bar{z}, \omega) \equiv \lambda_{1} I_{12} \int d \tau_{3} \int d \tau_{4} I_{13} I_{24}\left(\frac{1}{I_{13}}-\frac{1}{I_{14}}+\frac{1}{I_{34}}\right) Y_{134},
$$

and:

$$
F_{Y_{234}}(z, \bar{z}, \omega) \equiv \lambda_{1} I_{12} \int d \tau_{3} \int d \tau_{4} I_{13} I_{24}\left(\frac{1}{I_{24}}-\frac{1}{I_{23}}+\frac{1}{I_{34}}\right) Y_{234} .
$$

We will not be able to solve these integrals analytically. However it is possible to numerically integrate both (4.27) and (4.28) on the line $z=\bar{z} \equiv x$. This procedure is shown in detail in appendix C.2. In this limit, the integrals exhibit the same inversion symmetry $x \leftrightarrow 1 / x$ that we already encountered in our treatment of the 2 -channel. It is not clear how this symmetry extends to the whole of $\mathbb{R}^{2}$, and that would be an interesting thing to investigate in future work. In addition, we found numerically the following properties for some parts of the correlator:

$$
\begin{equation*}
g_{X}(x, x)=0 \quad \forall x \geq 0, \tag{4.29a}
\end{equation*}
$$

as well as:

$$
\begin{equation*}
g_{1}(x, x)-g_{X}(x, x)=\text { const. }=\frac{g^{8} N^{2}}{3 \cdot 2^{9} \pi^{4}} \quad \forall x \leq 0 . \tag{4.29b}
\end{equation*}
$$

The first relation simply means that the X-integrals do not contribute to the correlator for $x \geq 0$, while the second one implies that only the X-integrals are relevant for $x \leq 0$. We will not need to exploit these interesting properties in this work, but it would be great to see whether it can be used in the future for checking the results of this thesis.

Both analytical results (see eq. (2.42) in section 2.3) and numerical results indicate that the correlator corresponding to the 1 -channel takes the following form:

$$
\begin{equation*}
\tilde{F}_{1}(x, x)=\frac{g^{8} N^{2}}{3 \cdot 2^{9} \pi^{4}} x-\frac{g^{8} N^{2}}{2^{7} \pi^{6}} x^{2} \log x+\mathcal{O}\left(x^{2}\right) \tag{4.30}
\end{equation*}
$$

We were not able to perform the numerical integration precisely enough to obtain the closed form of the coefficients at higher-order. To be more specific, the problem is that the 2d integrals are difficult to perform numerically at a sufficient precision (see appendix C. 2 for a discussion).

## 0-Channel at Next-to-Leading Order

Similarly to the previous section, we will now write down the integrals of the 0 -channel and show that it is finite on its own. In particular, we conclude that only one diagram contributes at this order and that its corresponding $g$-function is constant on the line $z=\bar{z}$ for $x \leq 0$.

But let us start by reviewing the vanishing diagrams. There are two $X$-diagrams contributing to the 0 -channel, the scalars-to-scalars one giving:

where we recall that the subscript 0 is here to avoid including the 1 -channel part of the diagram, as we have already discussed in the last subsection. The gluons-to-scalars diagram reads:


The two diagrams consequently cancel each other:

and thus the X -diagrams do not contribute to the 0 -channel.
The next diagram that we study will be referred to as the YY-diagram. The two Y-integrals that it consists of are independent and hence we can factorize them and write:


It is easy to see that the integral vanishes for any $x_{2}$, since the integrand is antisymmetric with respect to $\tau_{3} \leftrightarrow-\tau_{3}$. This diagram therefore does not contribute to the 0 -channel and can be discarded.

The last diagram that we must consider is a $H$-diagram, which is more intricate because of the complicated look of the 4-point insertions given in eq. (3.26) and (3.27). However, noticing that:

$$
\int d \tau_{i} \partial_{\tau_{i}} I_{i j}=0
$$

allows the diagram to be simplified to the following compact expression:

Table 4.3: Leading CFT data coefficients at order $\mathcal{O}\left(g^{4}\right)$, obtained from the exact expression given in (4.8). The corresponding closed forms are given in the text.

| $A$ | $B$ | $C$ | $D_{s}-F_{s+1, s}$ | $E_{0}$ | $E_{1}$ | $E_{2}$ | $E_{3}$ | $F_{2,0}$ | $F_{3,0}$ | $F_{3,1}$ | $F_{4,0}$ | $F_{4,1}$ | $F_{4,2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | $\frac{g^{4} N^{2}}{2^{9} \pi^{4}}$ | 0 | $\frac{g^{4} N^{2}}{2^{7} \pi^{4}}$ | $\frac{g^{4} N^{2}}{2^{7} \pi^{4}}$ | $\frac{3 g^{4} N^{2}}{5 \cdot 2^{6} \pi^{4}}$ | $\frac{5 g^{4} N^{2}}{7 \cdot 2^{6} \pi^{4}}$ | $\frac{3 g^{4} N^{2}}{5 \cdot 2^{6} \pi^{4}}$ | 0 | $\frac{3 g^{4} N^{2}}{7 \cdot 2^{5} \pi^{4}}$ | $\frac{5 g^{4} N^{2}}{7 \cdot 3^{2} 2^{2} \pi^{4}}$ | 0 | $\frac{5 g^{4} N^{2}}{3^{2} \cdot 2^{5} \pi^{4}}$ |



This integral is the most difficult that we have encountered so far, since (i) the H-integral is not known analytically, and (ii) the fact that the derivatives are not contracted as in the 1-channel prevents us from reducing it to one-loop integrals using an identity such as the one given in eq. (C.6) (see footnote in appendix C. 1 for an informal explanation about why such an identity cannot exist). Note that the integral is finite, and hence the 0 -channel is finite on its own just like its siblings.

We will not be able to solve the 10d integral given in (4.33) analytically. On the line $z=\bar{z} \equiv x$, it can be reduced to a 4d integral by applying the integrals given in appendix C. 1 and using spherical coordinates as detailed in appendix C.2. It can be seen numerically that this integral also exhibits the inversion symmetry $x \leftrightarrow 1 / x$ observed in the other channels, and that was to be expected for the Ward identities to be fulfilled everywhere. Again, it is not clear what happens to the symmetry outside of the line $z=\bar{z}$.

It is not easy to obtain an expansion of (4.33) for $x \geq 0$, however we found numerically (see fig. 4.1) that:

$$
\begin{equation*}
g_{0}(x, x)=\text { const. }=-\frac{g^{8} N^{2}}{3 \cdot 2^{8} \pi^{4}} \quad \forall x \leq 0 \tag{4.34}
\end{equation*}
$$

and thus we have:

$$
\begin{equation*}
\tilde{F}_{0}(x, x)=-\frac{g^{8} N^{2}}{3 \cdot 2^{8} \pi^{4}} \frac{x^{2}}{(1-x)^{4}} \quad \forall x \leq 0 \tag{4.35}
\end{equation*}
$$

This is a very important result, since it allows us to expand the correlator at $x \sim 0$ from below and to obtain infinitely many terms, which can be compared to the expansion of the superblocks given in eq. (2.41) and (2.42) in order to extract the CFT data. This expansion reads:

$$
\begin{equation*}
\left.\tilde{F}_{0}(x, x)\right|_{x<0}=-\frac{g^{8} N^{2}}{3 \cdot 2^{11} \pi^{4}} \sum_{k=1}^{\infty} k(k+1)(k+2) x^{k+1} . \tag{4.36}
\end{equation*}
$$

Note that this expression contains no log term.

## Ward Identities in the Collinear Limit

We wish now to check whether the Ward identities that we presented in section 2.3 for the limiting case $z=\bar{z}$ are fulfilled by the expressions that we obtained in this section. We recall that the Ward identities take the following form:

$$
\begin{array}{ll}
16 g_{0}(x, x)+4 g_{1}(x, x)+g_{2}(x, x)=c_{1} & \forall x>0, \\
16 g_{0}(x, x)-4 g_{1}(x, x)+g_{2}(x, x)=c_{2} & \forall x<0,
\end{array}
$$



Figure 4.2: The plots show the matching between the exact correlator (dots) and the expansion in superblocks (solid lines) for the line $z=\bar{z}$. The left plot corresponds to the 0 -channel at identity order, with the coefficients given in table 4.3, while the right one represents the 1-channel at leading order, using the coefficients listed in table 4.4. The perfect agreement validates the closed forms given in eq. (4.42) and (4.47).
where the constants $c_{1}$ and $c_{2}$ do not have to be equal since the relation suffers a discontinuity at $x=0$. Indeed, the numerical data reveal that:

$$
\begin{gather*}
c_{1}=\frac{g^{8} N^{2}}{3 \cdot 2^{6} \pi^{4}},  \tag{4.38a}\\
c_{2}=0, \tag{4.38b}
\end{gather*}
$$

as it can be seen in fig. 4.1. This is a very important check of our results, and in addition to that it implies that it is sufficient to know two channels analytically in order to know the full correlator!

### 4.3 CFT Data

We are now ready to use the perturbative computation of the previous section for extracting the CFT data and to present the most important results of this thesis. We consider the expansions given above order by order and proceed to extract the coefficients of (2.30). In particular, we show that the knowledge that we gathered about the integrals in the last section suffices to extract the CFT data at next-toleading order and to bring us close to being able to reconstruct the full correlator.

## Identity Order

It was easy at order $\mathcal{O}\left(g^{4}\right)$ to obtain an exact analytical expression for the correlator. This result is given in eq. (4.8). Nevertheless, it is convenient to perform an expansion on the line $z=\bar{z}$ for reading the CFT data. The expansion gives:

$$
\begin{equation*}
\tilde{F}_{0}(x, x)=\frac{g^{4} N^{2}}{3 \cdot 2^{6} \pi^{4}} \sum_{k=1}^{\infty} k(k+1)(k+2) x^{k+1} \tag{4.39}
\end{equation*}
$$

Similarly, we can expand (4.8) on the line $z=-\bar{z}$, and we obtain:

$$
\begin{equation*}
\tilde{F}_{0}(i x,-i x)=\frac{g^{4} N^{2}}{2^{5} \pi^{4}} \sum_{k=1}^{\infty}(-1)^{k+1} x^{2 k} \tag{4.40}
\end{equation*}
$$

Table 4.4: Leading CFT data coefficients at order $\mathcal{O}\left(g^{6}\right)$, obtained by comparing the expansion of the correlator given in eq. (4.10) and the expansion in superblocks. The closed form is given in the text.

| $A$ | $B$ | $C$ | $D_{0}-F_{1,0}$ | $D_{1}-F_{2,1}$ | $D_{2}-F_{3,2}$ | $D_{3}-F_{4,3}$ | $E_{s}$ | $F_{\hat{\Delta}, s}(\hat{\Delta} \neq s+1)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $\frac{g^{6} N}{2^{9} \pi^{4}}$ | 0 | $\frac{g^{6} N}{3 \cdot 2^{7} \pi^{4}}$ | $\frac{3 g^{6} N}{5 \cdot 2^{8} \pi^{4}}$ | $\frac{g^{6} N}{7 \cdot 2^{6} \pi^{4}}$ | $\frac{g^{6} N}{2^{9} \pi^{4}}$ | 0 | 0 |

These expansions are easy to match to the expansions of superblocks presented in section 2.3. Let us illustrate the method with several examples, which can be derived using eq. (2.43). The coefficient $A$ appears only as the leading term in the expansion of the 2 -channel on the line $z=\bar{z}$, and since the latter vanishes at $\mathcal{O}\left(g^{4}\right)$ we find:

$$
\begin{equation*}
A \stackrel{!}{=} 0 \tag{4.41a}
\end{equation*}
$$

The coefficient $B$ is the leading term of the 1-channel, which also vanishes, and thus:

$$
\begin{equation*}
B \stackrel{!}{=} 0 \tag{4.41b}
\end{equation*}
$$

Similarly, the leading term of the 0-channel involves the coefficient $C$, but this time the channel is non-zero and from eq. (4.39) we obtain the following relation:

$$
\begin{equation*}
16 C \stackrel{!}{=} \frac{g^{4} N^{2}}{2^{5} \pi^{4}} \tag{4.41c}
\end{equation*}
$$

We can play the same game with the next orders in $x$ in order to obtain infinitely many coefficients. The expressions get more involved as higher powers of $x$ are considered, and that is why the expansion given in (4.40) is also needed. If enough CFT data is gathered, the closed form can be guessed and the result can be checked against the exact correlator with the superblocks.

Due to multiplet shortening (see section 2.3), we have at this order:

$$
\hat{\mathcal{G}}_{L_{[0, s]}^{s+1}}(z, \bar{z}, \omega)=-\hat{\mathcal{G}}_{(B, 1)_{[0, s]}}(z, \bar{z}, \omega),
$$

and hence the coefficients $D_{s}$ and $F_{s+1, s}$ in (2.30) cannot be distinguished. But the expansion has to be unique, so we expect the log terms of the LO and NLO to disentangle $D_{s}$ and $F_{s+1, s}$ at this order.

The leading CFT coefficients obtained with that method are listed in table 4.3. It is not hard to guess the closed form for the non-vanishing coefficients:

$$
\begin{gather*}
E_{s}=\frac{g^{4} N^{2}}{2^{8} \pi^{4}} \frac{(1+s)(2+s)}{1+2 s} \quad \forall s \geq 0,  \tag{4.42a}\\
F_{\hat{\Delta}, s}=\frac{g^{4} N^{2}}{2^{9} \pi^{4}} \frac{\Gamma(\hat{\Delta}+2) \Gamma(s+3 / 2)}{\Gamma(\hat{\Delta}+3 / 2) \Gamma(s+1)} \frac{(\hat{\Delta}-s)(\hat{\Delta}+s+1)}{(\hat{\Delta}+s)(\hat{\Delta}-s-1)} \quad \forall \hat{\Delta}-s \text { even. } \tag{4.42b}
\end{gather*}
$$

Note that $F_{\hat{\Delta}, s}=0$ for $\hat{\Delta}-s$ odd and $\hat{\Delta}-s \neq 1$. The closed forms can easily be checked by comparing the exact correlator with the expansion in superblocks. Both are plotted in fig. 4.2, and the perfect agreement validates eq. (4.42a) and (4.42b).

Table 4.5: Products of the CFT data $F_{s+1, s} \hat{\Delta}_{s+1, s}^{(2)}$ at order $\mathcal{O}\left(g^{6}\right)$ obtained with the $\log$ terms of the perturbative computation at NLO. The corresponding closed form is given in eq. (4.48).

| $F_{1,0} \hat{\Delta}_{1,0}^{(2)}$ | $F_{2,1} \hat{\Delta}_{2,1}^{(2)}$ | $F_{3,2} \hat{\Delta}_{3,2}^{(2)}$ | $F_{4,3} \hat{\Delta}_{4,3}^{(2)}$ | $F_{5,4} \hat{\Delta}_{5,4}^{(2)}$ | $F_{6,5} \hat{\Delta}_{6,5}^{(2)}$ | $F_{7,6} \hat{\Delta}_{7,6}^{(2)}$ | $F_{8,7} \hat{\Delta}_{8,7}^{(2)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{g^{6} N^{2}}{2^{9} \pi^{6}}$ | $\frac{g^{6} N^{2}}{2^{9} \pi^{6}}$ | $\frac{11 g^{6} N^{2}}{5 \cdot 2^{10} \pi^{6}}$ | $\frac{25 g^{6} N^{2}}{7 \cdot 3 \cdot 2^{9} \pi^{6}}$ | $\frac{137 g^{6} N^{2}}{9 \cdot 3 \cdot 2^{11} \pi^{6}}$ | $\frac{147 g^{6} N^{2}}{11 \cdot 5 \cdot 2^{10} \pi^{6}}$ | $\frac{363 g^{6} N^{2}}{13 \cdot 5 \cdot 2^{11} \pi^{6}}$ | $\frac{761 g^{6} N^{2}}{15 \cdot 7 \cdot 5 \cdot 2^{9} \pi^{6}}$ |

## Leading Order

The exact correlator is also known exactly at leading order and given in eq. (4.10). As for the identity order, we can expand the result on the line $z=\bar{z}$ at $x \sim 0$ :

$$
\begin{equation*}
\tilde{F}_{1}(x, x)=\frac{g^{6} N}{2^{7} \pi^{4}} \sum_{k=1}^{\infty} k x^{k} \tag{4.43}
\end{equation*}
$$

as well as on the line $z=-\bar{z}$ :

$$
\begin{equation*}
\tilde{F}_{1}(i x,-i x)=\frac{g^{6} N}{2^{7} \pi^{4}} \sum_{k=1}^{\infty}(-1)^{k-1} x^{2 k-1} \tag{4.44}
\end{equation*}
$$

where we assume in both cases $0 \leq x<1$.
The log terms in the expansion of the superblocks take the schematic form:

$$
\left(g^{2} \sum F_{\hat{\Delta}, s} \hat{\Delta}_{\hat{\Delta}, s}^{(2)}+\mathcal{O}\left(g^{4}\right)\right) x^{k} \log x,
$$

and at $\mathcal{O}\left(g^{6}\right)$ (i.e. $F_{\hat{\Delta}, s}$ is truncated at order $\mathcal{O}\left(g^{4}\right)$ ) this is equal to zero since there are no log terms in eq. (4.43) and (4.44). As an example, the first relation that we obtain is the following:

$$
g^{2} F_{1,0} \hat{\Delta}_{1,0}^{(2)}+g^{4} F_{1,0} \hat{\Delta}_{1,0}^{(4)} \stackrel{!}{=} 0
$$

This follows from the leading log term of the 2 -channel (see eq. (2.43) and (4.13)). For the LHS to be of order $\mathcal{O}\left(g^{6}\right)$, the $F_{1,0}$ of the first term must be of order $\mathcal{O}\left(g^{4}\right)$, while it is of order $\mathcal{O}\left(g^{2}\right)$ in the second one. Clearly the second term has to vanish, and hence we are left with the relation:

$$
F_{1,0} \hat{0}_{1,0}^{(2)}=0 .
$$

In fact, looking at the higher-order terms shows that:

$$
\begin{equation*}
F_{s+1, s} \hat{s}_{s+1, s}^{(2)}=0 \quad \forall s \geq 0 \tag{4.45}
\end{equation*}
$$

for $F_{s+1, s}$ at order $\mathcal{O}\left(g^{4}\right)$. This result would allow us to disentangle the $D_{s}$ and the $F_{s+1, s}$ of the previous section, if only we were able to show that $\hat{\Delta}_{s+1, s}^{(2)} \neq 0$ (we would then have $F_{s+1, s}=0$ ). This is in fact the case, as it will be shown with the results at next-to-leading order.

The next log term in the 2-channel leads to the following relation:

$$
-2 g^{2} F_{1,0} \hat{\Delta}_{1,0}^{(2)}+g^{2} F_{2,0} \hat{\Delta}_{2,0}^{(2)}+2 g^{2} F_{2,1} \hat{\Delta}_{2,1}^{(2)} \stackrel{!}{=} 0
$$

We know that $F_{2,0} \neq 0$ from the previous subsection, and using eq. (4.45) it is clear that:

$$
\hat{\Delta}_{2,0}^{(2)}=0 .
$$

The higher-order terms lead to the following generalization:


Figure 4.3: The plots show from left to right the $\tilde{F}$-functions of the 0 -, 1- and 2 -channels computed numerically (dots) on the line $z=\bar{z}$, and compared to the expansion in superblocks with the CFT data given in table 4.6 (solid lines). The discrepancy away from zero is expected, since the closed form is not known and hence the expansion is truncated early on. The remarkable agreement near zero seems to validate the coefficients that we managed to extract so far.

$$
\begin{equation*}
\hat{\Delta}_{\hat{\Delta}, s}^{(2)}=0 \quad \forall \hat{\Delta}-s \text { even. } \tag{4.46}
\end{equation*}
$$

This result means that the conformal dimensions of the long operators with $\hat{\Delta}-s$ even do not receive an anomalous contribution at $\mathcal{O}\left(g^{2}\right)$. This is discussed in further detail in the next subsection.

We proceed now to the matching of the power terms, in the same way as it was done at identity order. As before, we are not able to disentangle the $D_{s}$ and the $F_{s+1, s}$. The CFT data is given in table 4.4.

Again, the closed form for the coefficients $D_{s}-F_{s+1, s}$ is easy to guess and we find:

$$
\begin{equation*}
D_{s}-F_{s+1, s}=\frac{g^{6} N}{2^{8} \pi^{4}} \frac{1+s}{1+2 s} . \tag{4.47}
\end{equation*}
$$

The comparison between the exact correlator and the expansion in superblocks using the CFT data is shown in fig. 4.2, where the agreement leaves no doubt on the validity of (4.47).

## Next-to-Leading Order

At NLO the situation is a little different since we were not able to compute the full correlator analytically. However we will see that the information that we managed to obtain is sufficient for deriving an infinite amount of CFT data, even if once again the $D_{s}$ and $F_{s+1, s}$ are not distinguishable. Here we must use all of the limiting cases that were introduced in section 2.3, i.e. $z=\bar{z}, z=-\bar{z}$ and $y=k x$ for different values of $k$.

We start with the log terms of the 2-channel. We proceed in the same way as we did at LO. This allows us to reach two statements. First, we observe that the products $F_{s+1, s} \hat{\Delta}_{s+1, s}^{(2)}$ are non-vanishing for $F_{s+1, s}$ at order $\mathcal{O}\left(g^{6}\right)$, which implies that both $F_{s+1, s}$ and $\hat{\Delta}_{s+1, s}^{(2)}$ are non-zero for all $s \geq 0$. The last condition was needed for us to assess that $F_{s+1, s}=0$ at order $\mathcal{O}\left(g^{4}\right)$, as already discussed in the previous section. The products $F_{s+1, s} \Delta_{s+1, s}^{(2)}$ are given in table 4.5 for some values of $s$, and it is easy to guess the closed form:

$$
\begin{equation*}
F_{s+1, s} \hat{\Delta}_{s+1, s}^{(2)}=\frac{g^{6} N^{2}}{2^{9} \pi^{6}} \frac{1+s}{1+2 s} H_{s} . \tag{4.48}
\end{equation*}
$$

Unfortunately this still does not allow us to disentangle $D_{s}$ and $F_{s+1, s}$ at $\mathcal{O}\left(g^{6}\right)$. But this result is strikingly similar to eq. (4.47), and a conjecture based on that observation is discussed in the conclusion of this thesis.

Table 4.6: Leading CFT data coefficients at order $\mathcal{O}\left(g^{8}\right)$. Notice that the coefficients $D_{s}$ and $F_{s+1, s}$ cannot be distinguished because of multiplet shortening, as explained in the text. We have not been able to guess all of the closed forms yet.

| $A$ | $B$ | $C$ |
| :---: | :---: | :---: |
| $\frac{g^{8} N^{2}}{2^{12} \pi^{4}}$ | $\frac{g^{8} N^{2}}{3 \cdot 2^{11} \pi^{4}}$ | $-\frac{g^{8} N^{2}}{3 \cdot 2^{12} \pi^{4}}$ |


| $E_{0}$ | $E_{1}$ | $E_{2}$ | $E_{3}$ |
| :---: | :---: | :---: | :---: |
| $-\frac{g^{8} N^{2}\left(-3+\pi^{2}\right)}{3 \cdot 2^{10} \pi^{6}}$ | $-\frac{g^{8} N^{2}\left(-9+2 \pi^{2}\right)}{3 \cdot 2^{11} \pi^{6}}$ | $-\frac{g^{8} N^{2}\left(-65+12 \pi^{2}\right)}{5 \cdot 3 \cdot 2^{11} \pi^{6}}$ | $-\frac{g^{8} N^{2}\left(-145+24 \pi^{2}\right)}{7 \cdot 3^{2} \cdot 2^{12} \pi^{6}}$ |
| $E_{4}$ | $E_{5}$ | $E_{6}$ | $E_{7}$ |
| $-\frac{g^{8} N^{2}\left(-3899+600 \pi^{2}\right)}{9 \cdot 5 \cdot 3 \cdot 2^{13} \pi^{6}} \quad-\frac{g}{6}$ | $\frac{g^{8} N^{2}\left(-4109+600 \pi^{2}\right)}{11 \cdot 5^{2} \cdot 3 \cdot 2^{13} \pi^{6}}$ | $-\frac{g^{8} N^{2}\left(-419017+58800 \pi^{2}\right)}{13 \cdot 7 \cdot 5^{2} \cdot 3^{2} \cdot 2^{12} \pi^{6}}$ | $\underline{2}) \quad-\frac{g^{8} N^{2}\left(-288223+39200 \pi^{2}\right)}{7^{2} \cdot 5^{3} \cdot 2^{13} \pi^{6}}$ |
| $F_{2,0}$ | $F_{3,0}$ | $F_{3,1}$ | $F_{4,0}$ |
| $-\frac{g^{8} N^{2}}{5 \cdot 3^{4} \pi^{4}}$ | $\frac{23 g^{8} N^{2}}{7 \cdot 3^{2} \cdot 2^{7} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(5-4 \pi^{2}\right)}{7 \cdot 2^{10} \pi^{6}}$ | $-\frac{5 g^{8} N^{2}}{7 \cdot 3^{4} \cdot 2^{5} \pi^{4}}$ |
| $F_{4,1}$ | $F_{4,2}$ | $F_{5,0}$ | $F_{5,1}$ |
| $\frac{g^{8} N^{2}}{2^{8} \pi^{6}}$ | $\frac{5 g^{8} N^{2}\left(9-4 \pi^{2}\right)}{3^{3} \cdot 2^{10} \pi^{6}}$ | $\frac{139 g^{8} N^{2}}{11 \cdot 5^{2} \cdot 3 \cdot 2^{6} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(7-10 \pi^{2}\right)}{11 \cdot 5 \cdot 3^{3} \cdot 2^{4} \pi^{6}}$ |
| $F_{5,2}$ | $F_{5,3}$ | $F_{6,0}$ | $F_{6,1}$ |
| $\frac{29 g^{8} N^{2}}{7^{2} \cdot 2^{7} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(91-30 \pi^{2}\right)}{11 \cdot 3 \cdot 2^{10} \pi^{6}}$ | $-\frac{7 g^{8} N^{2}}{13 \cdot 11 \cdot 5 \cdot 3^{2} \cdot 2^{2} \pi^{4}}$ | $\frac{263 g^{8} N^{2}}{13 \cdot 11 \cdot 5 \cdot 3 \cdot 2^{5} \pi^{6}}$ |
| $F_{6,2}$ | $F_{6,3}$ | $F_{6,4}$ | $F_{7,0}$ |
| $\frac{5 g^{8} N^{2}\left(83-60 \pi^{2}\right)}{13 \cdot 11 \cdot 3 \cdot 2^{8} \pi^{6}}$ | $\frac{931 g^{8} N^{2}}{13 \cdot 3^{3} \cdot 2^{9} \pi^{6}}$ | $\frac{7 g^{8} N^{2}\left(439-120 \pi^{2}\right)}{13 \cdot 5 \cdot 3 \cdot 2^{12} \pi^{6}}$ | $\frac{191833 g^{8} N^{2}}{13 \cdot 11 \cdot 7^{2} \cdot 5^{3} \cdot 3^{4} \pi^{6}}$ |
| $F_{7,1}$ | $F_{7,2}$ | $F_{7,3}$ | $F_{7,4}$ |
| $\frac{g^{8} N^{2}\left(761-1680 \pi^{2}\right)}{13 \cdot 11 \cdot 7 \cdot 5^{3} \cdot 2^{5} \pi^{6}}$ | $\frac{2201 g^{8} N^{2}}{13 \cdot 7 \cdot 3^{4} \cdot 2^{6} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(3341-1680 \pi^{2}\right)}{13 \cdot 5^{2} \cdot 3^{5} \cdot 2^{5} \pi^{6}}$ | $\frac{397 g^{8} N^{2}}{11 \cdot 5^{2} \cdot 2^{8} \pi^{6}}$ |
| $F_{7,5}$ | $F_{8,0}$ | $F_{8,1}$ | $F_{8,2}$ |
| $\frac{7 g^{8} N^{2}\left(35011-8400 \pi^{2}\right)}{5^{3} \cdot 3^{3} \cdot 2^{11} \pi^{6}}$ | $-\frac{24 g^{8} N^{2}}{17 \cdot 13 \cdot 11 \cdot 7 \cdot 5 \pi^{4}}$ | $\frac{50381 g^{8} N^{2}}{17 \cdot 13 \cdot 7 \cdot 5^{2} \cdot 3^{4} \cdot 2^{2} \pi^{6}}$ | $\frac{3 g^{8} N^{2}\left(531-560 \pi^{2}\right)}{17 \cdot 13 \cdot 7 \cdot 5^{3} \cdot 2^{4} \pi^{6}}$ |
| $F_{8,3}$ | $F_{8,4}$ | $F_{8,5}$ | $F_{8,6}$ |
| $\frac{5251 g^{8} N^{2}}{17 \cdot 11^{2} \cdot 5 \cdot 3 \cdot 2^{5} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(845-336 \pi^{2}\right)}{17 \cdot 11 \cdot 3^{2} \cdot 2^{8} \pi^{6}}$ | $\frac{8871 g^{8} N^{2}}{17 \cdot 13^{2} \cdot 2^{9} \pi^{6}}$ | $-\frac{g^{8} N^{2}\left(130213+25200 \pi^{2}\right)}{17 \cdot 7 \cdot 5^{2} \cdot 2^{11} \pi^{6}}$ |


| $D_{0}-F_{1,0}$ | $D_{1}-F_{2,1}$ | $D_{2}-F_{3,2}$ | $D_{3}-F_{4,3}$ |
| :---: | :---: | :---: | :---: |
| $\frac{g^{8} N^{2}\left(6+\pi^{2}-12 \log 2\right)}{3^{2} \cdot 1^{10} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(9+2 \pi^{2}-36 \log 2\right)}{3^{2} \cdot 2^{10} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(89+20 \pi^{2}-440 \log 2\right)}{5^{2} \cdot 2^{12} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(755+168 \pi^{2}-4200 \log 2\right)}{7^{2} \cdot 3^{2} \cdot 2^{11} \pi^{6}}$ |
| $D_{4}-F_{5,4}$ | $D_{5}-F_{6,5}$ | $D_{6}-F_{7,6}$ | $D_{7}-F_{8,7}$ |
| $\frac{g^{8} N^{2}\left(3271+720 \pi^{2}-1928 \log 2\right)}{9^{2} \cdot 3 \cdot 2^{14} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(60529+13200 \pi^{2}-388880 \log 2\right)}{11^{2} \cdot 5^{2} \cdot 3 \cdot 2^{13} \pi^{6}}$ | $\frac{g^{8} N^{2}\left(504599+109200 \pi^{2}-3397680 \log 2\right)}{13^{2} \cdot 5^{2} \cdot 3^{2} \cdot 2^{4} 4 \pi^{6}}$ | $\frac{g^{8} N^{2}\left(109373+23502 \pi^{2}-767088 \log 2\right)}{15^{2} \cdot 7^{2} \cdot 3 \cdot 2^{12} \pi^{6}}$ |

The second statement is the following:

$$
\begin{equation*}
\hat{\Delta}_{\hat{\Delta}, s}^{(4)}=0 \quad \forall \hat{\Delta}-s \text { even. } \tag{4.49}
\end{equation*}
$$

We have already seen in eq. (4.46) that long operators with $\hat{\Delta}-s$ even do not receive an anomalous contribution $\hat{\Delta}^{(2)}$ neither. This means that such operators are protected at least up to this order. This is a rather surprising result, which should be investigated in more detail in future work.

Let us now consider the power terms in order to extract the CFT data at order $\mathcal{O}\left(g^{8}\right)$. The coefficients $A, B$ and $C$ can be obtained as discussed for the identity order, and the results are given in table 4.6. For the other coefficients, we consider the expansion of the 2 -channel in all the limiting cases that we mentioned, and in particular we consider the line $y=k x$ for all positive integer $k \leq 8$. We also have the expansion of the 0 -channel for $x \leq 0$ in the collinear limit (see eq. (4.36)). We are now well-acquainted with the method used for obtaining the coefficients, and solving order by order we were able to extract the $E_{s}$ up to $s=7$, the $F_{\hat{\Delta}, s}(\hat{\Delta} \neq s+1)$ up to $\hat{\Delta}=8$ and the now also familiar $D_{s}-F_{s+1, s}$ up to $s=7$. These results are the most important ones of this thesis and are gathered in table 4.6.

Note that we have not been able to find a closed form for all the coefficients yet, and filling this gap will be the main focus of future work. So far we could only guess the closed form of the coefficients $E_{s}$, which reads:

$$
\begin{equation*}
E_{s}=\frac{g^{8} N^{2}}{2^{8} \pi^{6}} \frac{(1+s)(2+s)}{1+2 s}\left(H_{s+2}^{(2)}-\frac{H_{s+2}}{2+s}-\frac{\pi^{2}}{6}\right) . \tag{4.50}
\end{equation*}
$$

The last term has the same form as eq. (4.42a), which suggests the presence of a $\Omega^{2}$ term. It is easy to check that it is the case, since (4.42b) is fulfilled by the $1 / \pi^{4}$ part of the coefficients $F_{\Delta, s}$ with $\Delta-s$ even, and the coefficient $C$ behaves exactly as in (4.41c). In the same way, we can probe (and confirm) the presence of $\Omega$ by comparing eq. (4.47) with the $1 / \pi^{4}$ terms of $D_{s}-F_{s+1, s}$.

Comparisons of the numerical results with the expansions in superblocks are shown in fig. 4.3, and the plots seem to validate the results of tables 4.5 and 4.6. Accessorily, it also reveals that $f(g, N)$ (see eq. (4.4)) vanishes at $\mathcal{O}\left(g^{8}\right)$ as well in point-splitting regularization. In any case, we are now in position to obtain as many coefficients as we wish, the only limitation being the computing time necessary to expand the superblocks at a high-enough order in $x$. Therefore we expect to extract many more coefficients in the near future, with the hope that we will be able to understand their closed form.

## Direct Computation of Coefficient $A$

To conclude this work, we would like to perform a direct computation of the coefficient $A$. The goal is to see whether we obtain independently the same coefficient as given in table 4.6. This is easy to do since the physical meaning of $A$ is just:

$$
A \equiv a_{\mathcal{O}}^{2}
$$

i.e. it is the square of the coefficient for the one-point function of a single-trace operator $\mathcal{O}(x)$ in presence of the line defect. We have to consider the following correlator:

$$
\begin{equation*}
\langle\mathcal{O}(x)\rangle=\frac{(u \cdot \theta)^{2}}{x^{2}} F(x, \omega), \tag{4.51}
\end{equation*}
$$

where $F$ is defined similarly to eq. (2.22).
At leading order there is only one diagram, which is easy to compute:

$$
\begin{align*}
\square & =\frac{1}{N}(u \cdot \theta) g^{4} \operatorname{Tr} T^{a} T^{b} \operatorname{Tr} T_{a} T_{b} \int d \tau_{3} \int d \tau_{4} I_{13} I_{14} \\
& =\frac{(u \cdot \theta)^{2}}{x^{2}} \frac{g^{4} N}{2^{6} \pi^{2}} \tag{4.52}
\end{align*}
$$

This result matches perfectly the coefficient $A$ that we provided in table 4.6 and reinforces our trust in the CFT data that we derived. Note that this is the easiest check of the CFT data that can be performed independently, but we can do the same type of computation for other coefficients. It will be one of the focuses of future work to compare our results with such direct calculations.

## Conclusion and Outlook

We conclude this thesis with a brief summary of what has been achieved, and some thoughts about computations that should be done in the near future in order to complement the current results.

Following mostly the works of [14, 20], we have rederived the superblocks for the two-point function in defect $\mathcal{N}=4$ SYM, with the defect being specialized to the case of the Maldacena-Wilson line. The new results consist of the perturbative computation of the correlator at up to leading order while gathering important informations about the next-to-leading order. In particular, an analytical expression for the 2-channel could be obtained, and restricting ourselves to some limiting cases we were able to derive the CFT data at this order. So far we have not been able to determine the closed form of all the coefficients, but we are fairly confident that this can be achieved in the very near future after having extracted more coefficients. Still it was possible to confirm the presence of $\Omega$ and $\Omega^{2}$ terms. We performed various checks of the validity of the results, and in particular we saw that the integrals computed numerically satisfy the superconformal Ward identities in the collinear limit $z=\bar{z}$. Moreover, it was easy to obtain directly (and independently) the coefficient $A$, and we reached the same result as the one presented in table 4.6. The CFT data allows us to know a large number of two-point functions at order $\mathcal{O}\left(g^{4}\right)$ without having to compute them independently, and this illustrates how the conformal bootstrap can be used in association to perturbation theory in order to indirectly compute correlators in a defect CFT.

Clearly the most urgent task left to accomplish is to find the closed form for the coefficients $F_{\hat{\Delta}, s}$ and $D_{s}-F_{s+1, s}$. We are not limited in the number of coefficients that we can produce in addition to table 4.6 , hence it is reasonable to expect that guessing the closed form is an accessible target. Once this is done, we will be in possession of the full correlator at NLO, and therefore we can go back to the R-symmetry channels and give an analytical expression for each of them, which can easily be compared to the numerical data presented in this thesis.

There are several ways in which we can complete the checking of our results. The most obvious and straightforward one would be to extend the numerical computations to the whole $\mathbb{R}^{2}$ plane and confirm that the Ward identities are indeed fulfilled everywhere. At the same time, it would shed light on the nature of the inversion symmetry that we observed for each channel independently, as it is so far not clear how it would manifest itself outside of the collinear limit. It would also be interesting to directly compute other coefficients, in the same way as we did for $A$, and compare such independent results with table 4.6. In particular, we have seen in section 4.3 that the defect long operators with $\hat{\Delta}-s$ even satisfy:

$$
\Delta_{\hat{\Delta}, s}^{(2)}=\hat{\Delta}_{\hat{\Delta}, s}^{(4)}=0,
$$

possibly suggesting that such operators are protected for a reason yet unknown. This should be checked explicitly, and might be easy to understand diagrammatically.

It is unfortunate that the coefficients $D_{s}$ and $F_{s+1, s}$ could not be distinguished at orders $\mathcal{O}\left(g^{6}\right)$ and $\mathcal{O}\left(g^{8}\right)$. It seems likely that this can only be realized by computing at least the log terms at NNLO, and maybe looking at the 2 -channel would be enough to reach that goal. At order $\mathcal{O}\left(g^{6}\right)$, we notice an
extraordinary similarity between the expressions that we found for $D_{s}-F_{s+1, s}$ (see eq. (4.47)) and for $F_{s+1, s} \hat{\Delta}_{s+1, s}^{(2)}$ (eq. (4.48)). The following conjecture arises naturally:

$$
\hat{\Delta}_{s+1, s}^{(2)} \stackrel{?}{=} \frac{N}{2} H_{s}, \quad D_{s} \stackrel{?}{=} 0 .
$$

Of course this is only one possibility out of infinitely many solutions, and it should be checked explicitly.
In this work we have looked at the case of single-trace operators with two scalar fields. However it should not be hard to generalize the results to the case of $k$ scalar fields, since the number of Rsymmetry channels remains locked at three at $\mathrm{NLO}^{1}$ (see section 2.3). In particular, the table of topologies 4.1 remains basically unchanged and thus the integrals are very similar (in most cases they are perfectly identical) to the case $k=2$.

The most important piece of work remaining to be done is to consider the bulk channel expansion, which was left aside in the course of chapter 2. But even in that case, the hard part of the work is already done, and using a method similar to what we presented in section 4.3 should deliver the products $a_{\mathcal{O}} c_{12 \mathcal{O}}$ for the bulk operators $\mathcal{O}$ listed in section 2.2, i.e. the operators which can have a non-vanishing one-point function. Moreover, we should also be able to extract the anomalous dimensions of many operators, and in that case they can be compared to the literature since they are related to bulk operators (see e.g. [47]).

Finally, Wilson lines appear to play an important role in other frameworks, and in particular it is known to be an invariant of the Nicolai map [48], which consists in formulating $\mathcal{N}=4 \mathrm{SYM}$ without anticommuting variables, with the help of a non-local and non-linear mapping to a free Maxwell theory. It would be interesting to investigate how the axioms of the conformal bootstrap could be related to this approach, and to see whether a completely rigorous construction of $\mathcal{N}=4$ SYM is possible at the non-perturbative level.

[^4]
## APPENDIX A

## Lie Algebras and Superconformal Algebra

This appendix is dedicated to reviewing the concepts of Lie algebras and of the superconformal algebra. The first and third sections are mostly based on the appendix of [24]. We start with an introduction to Lie groups and algebras, including a discussion of the representation theory, in particular for the special cases of interest $\mathfrak{u}(N)$ and $\mathfrak{s u}(4)_{\mathrm{R}}$. We then prove trace and structure constant identities that are useful for the computation of the Feynman diagrams. Finally, we review the (anti)commutation relations of the superconformal algebra.

## A. 1 Lie Groups and Lie Algebras

In this section, we introduce Lie groups and Lie algebras, review some useful properties and investigate the representation theory of Lie algebras. We conclude with a focus on the gauge algebra $\mathfrak{u}(N)$ and on the R -symmetry algebra $\mathfrak{s u}(4)_{\mathrm{R}}$.

## Definitions and Properties

A Lie group is a smooth manifold $G$ with a group structure that can be either Abelian or non-Abelian. A Lie algebra is a vector space $\mathfrak{g}$ over some field $\mathbb{F}$ with an operation $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called Lie bracket, which satisfies $\forall \alpha, \beta \in \mathbb{F}, \forall x, y, z \in \mathfrak{g}$ :

> (i) bilinearity: $[\alpha x+\beta y, z]=\alpha[x, z]+\beta[y, z]$,
> (ii) antisymmetry: $[x, y]=-[y, x]$,
> (iii) Jacobi identities: $[x,[y, z]]+[y,[z, x]]+[z,[x, y]]=0$.

Lie algebras are related to Lie groups, in that they correspond to the tangent space $T_{0}(G)$ at the identity element $\mathbb{1}$ of $G$. Consequently, there can exist for a given Lie algebra $\mathfrak{g}$ more than one corresponding Lie group $G$, i.e. the mapping between Lie groups and Lie algebras is not one-to-one.

Any element $x$ of $\mathfrak{g}$ can be expressed as $x=x_{a} T^{a}$, with $x_{a} \in \mathbb{F}$ and $T^{a}$ a basis vector of $\mathfrak{g}(a=1, \ldots, \operatorname{dim} \mathfrak{g})$. These basis vectors are called generators of the Lie algebra $\mathfrak{g}$, and they satisfy the following commutation relation:

$$
\begin{equation*}
\left[T^{a}, T^{b}\right]=i f_{c}^{a b} T^{c}, \tag{A.2}
\end{equation*}
$$

where the $f^{a b}{ }_{c}$ are called structure constants and also satisfy Jacobi identities. In the case where the Lie group is Abelian, then all the structure constants vanish. The definition of the structure constant follows from (A.2):

$$
\begin{equation*}
f^{a b c}=-2 i \operatorname{Tr}\left(\left[T^{a}, T^{b}\right] T^{c}\right) \tag{A.3}
\end{equation*}
$$

It is useful to define the following constant as well:

$$
\begin{equation*}
d^{a b c}=2 \operatorname{Tr}\left(\left\{T^{a}, T^{b}\right\} T^{c}\right) \tag{A.4}
\end{equation*}
$$

We would like now to specialize our analysis to the case of simple and semi-simple Lie algebras. In order to do that, we first need to define ideals (also called invariant subalgebras), which are subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$ such that:

$$
[x, y] \in \mathfrak{h} \text { and }\left[y_{1}, y_{2}\right] \in \mathfrak{h} \quad \forall x \in \mathfrak{g} \forall y, y_{1}, y_{2} \in \mathfrak{h} .
$$

Then a simple Lie algebra is a non-Abelian Lie algebra $\mathfrak{g}$ for which $\{0\}$ and $\{\mathfrak{g}\}$ are the only ideals. A Lie algebra is called semi-simple if its only Abelian ideal is $\{0\}$.

A semi-simple Lie algebra can always be expressed as the direct sum of simple Lie algebras. To be more specific, the direct sum of algebras is defined as follows:

$$
\mathfrak{g}_{1} \oplus \mathfrak{g}_{2} \equiv\left\{x_{1}+x_{2} \mid x_{1} \in \mathfrak{g}_{1}, x_{2} \in \mathfrak{g}_{2}\right\} \text { with }\left[x_{1}+x_{2}, y_{1}+y_{2}\right]=\left[x_{1}, x_{2}\right]+\left[y_{1}, y_{2}\right] \quad \forall x_{1}, x_{2} \in \mathfrak{g}_{1} \forall y_{1}, y_{2} \in \mathfrak{g}_{2} .
$$

For a Lie algebra $\mathfrak{g}$, there always exists a maximal set of linearly independent generators $h_{i} \in \mathfrak{g}(i=$ $1, \ldots, l$ ) which commute with each other, i.e.:

$$
\left[h_{i}, h_{j}\right]=0 \quad \forall i, j=1, \ldots, l .
$$

Such a subalgebra is called the Cartan subalgebra, and $l$ is referred to as the rank of the Lie algebra $\mathfrak{g}$.
Finally, the Casimir operators of a Lie algebra are operators $\mathcal{C}$ such that:

$$
[\mathcal{C}, x]=0 \quad \forall x \in \mathfrak{g},
$$

i.e. they commute with all the elements of the Lie algebra. We find $l$ Casimir operators for a semisimple Lie algebra of rank $l$. Of particular interest is the quadratic Casimir operator, which is defined as:

$$
\begin{equation*}
\mathcal{C}_{2} \equiv \kappa_{a b} T^{a} T^{b} \tag{A.5}
\end{equation*}
$$

where $\kappa_{a b}$ is called the Killing form and is defined as:

$$
\kappa_{a b} \equiv-f_{a}^{c d} f_{b c d}
$$

## Representations of Lie Algebras

We review now the representation theory of Lie algebras. A representation of a Lie algebra $\mathfrak{g}$ is a map $D: \mathfrak{g} \rightarrow \operatorname{Mat}(N, \mathbb{F})$ assigning to each element $x \in \mathfrak{g}$ a $N \times N$-matrix, such that:

$$
\begin{equation*}
D([x, y])=[D(x), D(y)] \quad \forall x, y \in \mathfrak{g} . \tag{A.6}
\end{equation*}
$$

$N$ is called the dimension of the representation, and should not be confused with dim $\mathfrak{g}$. Such a representation is denoted by $\mathbf{N}$. We call a faithful representation a representation such that the map $D$ is injective.

We will now list some important representations. The simplest one is called the trivial representation (or singlet) and is defined by $D: \mathfrak{g} \rightarrow \operatorname{Mat}(1, \mathbb{R})$ with $D(x) \equiv 0 \forall x \in \mathfrak{g}$. It is obviously not faithful, and is
denoted by $\mathbf{1}$ since it is one-dimensional. Next we can also obtain a representation by defining $D \equiv \mathrm{id}$. In that case, the representation is faithful and is of dimension $N$. This is called the fundamental representation, denoted by $\mathbf{N}$. Finally, another very useful representation is the adjoint representation. The map $D$ is in that case defined as $D: \mathfrak{g} \rightarrow \operatorname{GL}(\mathfrak{g})$ with $D(x)(y) \equiv[x, y] \forall x \in \mathfrak{g}$ fixed and $\forall y \in \mathfrak{g}$. The generators of this representation are given by:

$$
\begin{equation*}
\left(T_{a}{ }^{\text {adj }}\right)_{c}^{b}=i f_{a c}{ }^{b} . \tag{A.7}
\end{equation*}
$$

The adjoint representation is $\operatorname{dim} \mathfrak{g}$-dimensional, and is faithful in the case of semi-simple algebras. We call irreducible representations (or irreps) representations which cannot be brought into a block-diagonal form by a transformation of the form $D(\mathfrak{g}) \mapsto P^{-1} D(\mathfrak{g}) P$.

We can classify the representations of a semi-simple Lie algebra by using Dynkin labels, building upon the concept of Cartan subalgebra that we introduced in the previous subsection. It is convenient to express the Lie algebra in the so-called Cartan-Weyl form, which consists of decomposing $\mathfrak{g}$ into the generators $h_{i}(i=1, \ldots, l)$ of the Cartan subalgebra and the remaining generators, which we label $e_{\rho}$. It is elementary to see that any $x \in \mathfrak{g}$ can be expressed as:

$$
x=x^{i} h_{i}+x^{\rho} e_{\rho} .
$$

The commutation relations thus become:

$$
\begin{gathered}
{\left[h_{i}, e_{\rho}\right]=\alpha_{i} e_{\rho},} \\
{\left[e_{\rho}, e_{\sigma}\right]= \begin{cases}n_{\rho \sigma} e_{\rho+\sigma}, & \text { if } \rho \neq \sigma \\
\alpha^{i} h_{i}, & \text { if } \rho=-\sigma\end{cases} }
\end{gathered}
$$

where the $n_{\rho \sigma}$ are normalization constants, and where the $\alpha$ are $l$-dimensional vectors called the roots of $\mathfrak{g}$. We also call the weight space the space of such roots.

Roots have some important properties. First, if $\alpha$ is a root, then $-\alpha$ is also a root, and thus we need only to consider positive roots, i.e. roots with the first component $\alpha_{1}$ being positive. We then call simple roots positive roots $\alpha_{i}$ which cannot be decomposed into a sum of positive roots. Hence we can express any positive root $\beta$ as a sum of simple roots:

$$
\beta=\sum n^{i} \alpha_{i},
$$

with $n^{i} \in \mathbb{N}$. We can define a reciprocal basis $\eta^{j}$ with:

$$
\frac{\left\langle\eta^{j}, \alpha_{i}\right\rangle}{\left\langle\eta^{i}, \alpha_{i}\right\rangle}=\delta_{i}^{j}
$$

An irreducible representation of a Lie algebra can be characterized by its highest-weight vector $\Lambda \equiv$ $\sum m_{j} \eta^{j}$, and in particular by the coefficients $m_{j}$. The Dynkin labels are defined as the collection of these coefficients, such that $\left[m_{1}, \ldots, m_{l}\right]$ characterizes the representation.

## $\mathfrak{u}(N)$ and $\mathfrak{s u}(4)_{\mathbf{R}}$

We specialize now the considerations of the previous subsections to the cases of $\mathfrak{u}(N)$ and $\mathfrak{s u}(4)_{\mathrm{R}}$. In this thesis, $\mathfrak{u}(N)$ is used as a gauge group of Yang-Mills theory, while $\mathfrak{s u}(4)_{\mathrm{R}}$ is the R -symmetry group.

The unitary group $U(N)$ is defined as the group of unitary complex $N \times N$ matrices, i.e. matrices $U$ which satisfy:

$$
\begin{equation*}
U^{\dagger} U=\mathbb{1} \tag{A.9}
\end{equation*}
$$

It is easy to see that the dimension of the unitary group is $\operatorname{dim} U(N)=N^{2}$. The special unitary group $S U(N)$ is defined by the matrices $U \in U(N)$ with the additional requirement that $\operatorname{det} U=+1$. This results in the loss of one degree of freedom in comparison to the $U(N)$ group, and thus the dimension of the special unitary group is $\operatorname{dim} S U(N)=N^{2}-1$.

Both $U(N)$ and $S U(N)$ are connected to the identity and hence are Lie groups. Their elements can be expressed infinitesimally as:

$$
U=\mathbb{1}+i \alpha_{a} T^{a}+\mathcal{O}\left(\alpha^{2}\right)
$$

with $T^{a}$ the generators of the groups and $\alpha$ an infinitesimal real parameter. As a consequence, the generators must satisfy:

$$
\begin{equation*}
T^{a} \stackrel{!}{=}\left(T^{a}\right)^{\dagger} \tag{A.10}
\end{equation*}
$$

$U(N)$ and $S U(N)$ can be formulated such that they share $N^{2}-1$ traceless generators, i.e. $U(N)$ has an extra generator, which we define to be $T^{0}$ and which is not traceless. In the large $N$ limit (see section 3.1), $S U(N)$ tends towards $U(N)$, and hence we directly work with $U(N)$ as gauge group in this thesis, while we ignore the artefacts arising from the fact that $\operatorname{Tr} T^{0} \neq 0$.

Let us now move to the corresponding Lie algebras $\mathfrak{u}(N)$ and $\mathfrak{s u}(N)$. We first normalize the generators such that:

$$
\begin{equation*}
\operatorname{Tr} T^{a} T^{b}=\frac{\delta^{a b}}{2} \tag{A.11}
\end{equation*}
$$

while the commutator is given by eq. (A.2). The quadratic Casimir is:

$$
\begin{equation*}
\mathcal{C}_{2} \equiv T^{a} T_{a}=\frac{N}{2} \mathbb{d} . \tag{A.12}
\end{equation*}
$$

All the fields of $\mathcal{N}=4 \mathrm{SYM}$ are in the adjoint representation of the gauge group.
We will now discuss the representations of the R -symmetry algebra $\mathfrak{s u}(4)_{\mathrm{R}} . \mathfrak{s u}(4)_{\mathrm{R}}$ has rank $l=3$, which corresponds to the number of diagonal generators in the Cartan-Weyl basis. Hence the irreducibles representations of the R-symmetry are characterized by the Dynkin labels [ $m_{1}, m_{2}, m_{3}$ ]. The dimension of the representation is given by the following formula:

$$
\operatorname{dim}\left[m_{1}, m_{2}, m_{3}\right]=\frac{1}{12}\left(m_{1}+1\right)\left(m_{2}+2\right)\left(m_{3}+1\right)\left(m_{1}+m_{2}+2\right)\left(m_{2}+m_{3}+2\right)\left(m_{1}+m_{2}+m_{3}+3\right) .
$$

Of particular interest for this work are the representations of BPS operators (see section 1.2 for the definition). It turns out that $1 / 2$-BPS operators correspond to the Dynkin labels [ $0, k, 0](k \geq 2, \Delta=k$ ), $1 / 4$-BPS operators to the labels $[l, k, l](l \geq 1, \Delta=k+2 l)$, and 1/8-BPS operators are characterized by $[l, k, l+2 m](m \geq 1, \Delta=k+2 l+3 m)$.

## A. $2 \mathfrak{u}(N)$ Identities

In this section, we will review and prove identities related to the $\mathfrak{u}(N)$ algebra. They are separated into two groups: trace identities, i.e. traces of generators, and structure constant identities, which involve the tensors $f^{a b c}$ and $d^{a b c}$.

## Trace Identities

We start with trace identities. First, it is useful to derive the following completeness relation:

$$
\begin{equation*}
T_{i j}^{a} T_{a, l k}=\frac{1}{2} \delta_{i k} \delta_{j l} \tag{A.13}
\end{equation*}
$$

To see that, note that any complex matrix $M$ can be expressed as:

$$
M=m_{a} T^{a},
$$

with $m_{a} \in \mathbb{C}$, since the generators of $\mathfrak{u}(N)$ form a basis of the $N \times N$ matrices. Thus we have:

$$
\operatorname{Tr} M T^{a}=m_{b} \operatorname{Tr} T^{a} T^{b}=\frac{m^{a}}{2}
$$

and from this follows that we can write $M$ as:

$$
M=2 \operatorname{Tr}\left(M T^{a}\right) T_{a}
$$

or in index notation:

$$
M_{i j}=M^{k l} \delta_{i k} \delta_{j l} \stackrel{!}{=} 2 M^{k l} T_{l k}^{a} T_{a, i j}
$$

From this expression we can extract eq. (A.13).
The completeness relation allows us to obtain the two following help identities:

$$
\begin{align*}
& \operatorname{Tr} T^{a} A \operatorname{Tr} T_{a} B=T_{i j}^{a} T_{a, l k} A^{j i} B^{k l}=\frac{1}{2} \operatorname{Tr} A B,  \tag{A.14a}\\
& \operatorname{Tr} T^{a} A T_{a} B=T_{i j}^{a} T_{a, l k} A^{j l} B^{i k}=\frac{1}{2} \operatorname{Tr} A \operatorname{Tr} B . \tag{A.14b}
\end{align*}
$$

It is straightforward to find the expression for the trace of one generator. The trace of $T^{a}$ is non-zero only for $a=0$, and we have:

$$
\left(\operatorname{Tr} T^{a}\right)^{2}=\operatorname{Tr} T^{a} \mathbb{1} \operatorname{Tr} T_{a} \mathbb{1}=\frac{N}{2} .
$$

Thus we find:

$$
\begin{equation*}
\operatorname{Tr} T^{a}=\sqrt{\frac{N}{2}} \delta^{a 0} \tag{A.15}
\end{equation*}
$$

Note that, as mentioned before, the generators of the $\mathfrak{s u}(N)$ algebra are traceless and hence are zero for all $a$.

The trace of two generators has been defined by the normalization condition (A.11). When the indices are contracted, we have the following equality:

$$
\begin{equation*}
\operatorname{Tr} T^{a} T_{a}=\frac{N^{2}}{2} \tag{A.16}
\end{equation*}
$$

which is the trace of the Casimir operator defined in (A.12).
For three generators, we will make use of the constants defined in (A.3) and (A.4). We find that:

$$
\begin{aligned}
d^{a b c}+f^{a b c} & =2\left\{\operatorname{Tr}\left(\left\{T^{a}, T^{b}\right\} T^{c}\right)+\operatorname{Tr}\left(\left[T^{a}, T^{b}\right] T^{c}\right)\right\} \\
& =4 \operatorname{Tr} T^{a} T^{b} T^{c}
\end{aligned}
$$

from which follows:

$$
\begin{equation*}
\operatorname{Tr} T^{a} T^{b} T^{c}=\frac{1}{4}\left(d^{a b c}+i f^{a b c}\right) \tag{A.17}
\end{equation*}
$$

For the trace of four generators, we start by writing:

$$
T^{a} T^{b} T^{c} T^{d}=\frac{1}{4}\left(\left[T^{a}, T^{b}\right]+\left\{T^{a} \cdot T^{b}\right\}\right)\left(\left[T^{c}, T^{d}\right]+\left\{T^{c} . T^{d}\right\}\right)
$$

and we note that:

$$
\left\{T^{a}, T^{b}\right\}=2 \operatorname{Tr}\left(\left\{T^{a}, T^{b}\right\} T_{c}\right) T^{c}=d^{a b}{ }_{c} T^{c} .
$$

Inserting this in the previous equation and taking the trace, it is straightforward to obtain:

$$
\begin{equation*}
\operatorname{Tr} T^{a} T^{b} T^{c} T^{d}=\frac{1}{8}\left(d^{a b e} d^{c d e}-f^{a b e} f^{c d e}+i f^{a b e} d^{c d e}+i d^{a b e} f^{c d e}\right) \tag{A.18}
\end{equation*}
$$

## Structure Constant Identities

We will now consider identities which involve the structure constants $f^{a b c}$ and $d^{a b c}$.
We start by noting that:

$$
\operatorname{Tr}\left[T^{a}, T^{b}\right]=\operatorname{Tr} T^{a} T^{b}-\operatorname{Tr} T^{b} T^{a}=0 \stackrel{!}{=} i f_{c}^{a b} \operatorname{Tr} T^{c} \propto f^{a b 0}
$$

and thus:

$$
\begin{equation*}
f^{a b 0}=0 . \tag{A.19}
\end{equation*}
$$

The product of two structure constants with two indices free out of the six can be obtained as follows:

$$
\begin{aligned}
f^{a c d} f_{c d}^{b} & =-4 \operatorname{Tr}\left(\left[T^{a}, T^{c}\right] T^{d}\right) \operatorname{Tr}\left(\left[T^{b}, T_{c}\right] T_{d}\right) \\
& =-2 \operatorname{Tr}\left(\left[T^{a}, T^{c}\right]\left[T^{b}, T_{c}\right]\right) \\
& =-2 \operatorname{Tr} T^{a} \operatorname{Tr} T^{b}+2 N \operatorname{Tr} T^{a} T^{b},
\end{aligned}
$$

where in the second line we have used (A.14a), and in the third line (A.14b). The remaining traces are given in the previous subsection, and we obtain:

$$
\begin{equation*}
f^{a c d} f_{c d}^{b}=N\left(\delta^{a b}-\delta^{a 0} \delta^{b 0}\right)=N \delta^{\tilde{a} \tilde{b}} \tag{A.20}
\end{equation*}
$$

where $\tilde{a}, \tilde{b} \equiv 1, \ldots, N$ are $\mathfrak{s u}(N)$ indices (one generator less). This is used in e.g. [40]. By abuse of notation we often drop the tilde in this thesis. This is harmless in the large $N$ limit.

From the previous result we immediately obtain the case in which all indices are contracted:

$$
\begin{equation*}
f^{a b c} f_{a b c}=N\left(N^{2}-1\right) \sim N^{3}, \tag{A.21}
\end{equation*}
$$

where the last equality holds in the large $N$ limit.

The product of two $d^{a b c}$ can be obtained in exactly the same way. For the case in which two indices are kept free, we find:

$$
\begin{equation*}
d^{a c d} d_{c d}^{b}=N\left(\delta^{a b}+\delta^{a 0} \delta^{b 0}\right) \tag{A.22}
\end{equation*}
$$

The full contraction becomes:

$$
\begin{equation*}
d^{a b c} d_{a b c}=N\left(N^{2}+1\right) \sim N^{3}, \tag{A.23}
\end{equation*}
$$

where again the last equality is relevant in the limit $N \rightarrow \infty$.
Finally, we would like to show that the product of one $f$ and one $d$ vanishes when two indices are kept free:

$$
f^{a c d} d_{c d}^{b}=f^{a c d} d_{d c}^{b}=-f^{a d c} d_{d c}^{b}=-f^{a c d} d_{c d}^{b}
$$

where we have used $d^{a b c}=d^{a c b}$ and $f^{a b c}=-f^{a c b}$, and in the last equality we have renamed the indices. It follows:

$$
\begin{equation*}
f^{a c d} d^{b}{ }_{c d}=0 . \tag{A.24}
\end{equation*}
$$

## A. 3 Superconformal Algebra

For completeness we list in this section all the (anti)commutation relations of the superconformal algebra $\mathfrak{s u}(2,2 \mid 4)$ introduced in section 1.2. We recall that two generators obey the following relation:

$$
\left[\mathcal{O}_{1}, \mathcal{O}_{2}\right]_{ \pm}=\mathcal{O}_{1} \mathcal{O}_{2}-(-1)^{g_{1} g_{2}} \mathcal{O}_{2} \mathcal{O}_{1}
$$

where $g_{i}$ refers to the grade of the corresponding generator ( 0 for bosonic, 1 for fermionic). It follows that anticommutators arise only in the case where $g_{1}=g_{2}=+1$, i.e. when both generators are fermionic.

We start by listing all the relations that involve the generators of the Lorentz group:

$$
\begin{gather*}
{\left[J^{\mu \nu}, J^{\rho \sigma}\right]=-i\left(\eta^{\mu \rho} J^{v \sigma}-\eta^{\mu \sigma} J^{v \rho}-\eta^{v \rho} J^{\mu \sigma}+\eta^{v \sigma} J^{\mu \rho}\right),}  \tag{A.25a}\\
{\left[J^{\mu v}, P^{\rho}\right]=i\left(\eta^{\mu \rho} P^{v}-\eta^{v \rho} P^{\mu}\right),}  \tag{A.25b}\\
{\left[J^{\mu v}, K^{\rho}\right]=i\left(\eta^{\mu \rho} K^{v}-\eta^{\nu \rho} K^{\mu}\right), \quad\left[J^{\mu v}, D\right]=0,}  \tag{A.25c}\\
{\left[J^{\mu \nu}, Q_{\alpha}^{A}\right]=-\left(\sigma^{\mu v}\right)_{\alpha}{ }^{\beta} Q_{\beta}^{A}, \quad\left[J^{\mu v}, \bar{Q}_{A \dot{\alpha}}\right]=-\varepsilon_{\dot{\alpha} \dot{\beta}}\left(\bar{\sigma}^{\mu v}\right)^{\dot{\beta}} \dot{\dot{\gamma}} \bar{Q}_{A}^{\dot{\gamma}},}  \tag{A.25d}\\
{\left[J^{\mu v}, S_{\alpha}^{A}\right]=-\left(\sigma^{\mu v}\right)_{\alpha}{ }_{\alpha}^{\beta} S_{\beta}^{A}, \quad\left[J^{\mu v}, \bar{S}_{\dot{\alpha}}^{A}\right]=-\varepsilon_{\dot{\alpha} \dot{\beta}}\left(\bar{\sigma}^{\mu v}\right)^{\dot{\beta}} \overline{\dot{\gamma}}^{A} \dot{\gamma} .} \tag{A.25e}
\end{gather*}
$$

The remaining commutators of the conformal algebra obey:

$$
\begin{gather*}
{\left[P^{\mu}, P^{v}\right]=0, \quad\left[K^{\mu}, K^{v}\right]=0,}  \tag{A.26a}\\
{\left[K^{\mu}, P^{v}\right]=2 i\left(\eta^{\mu v} D-J^{\mu v}\right),}  \tag{A.26b}\\
{\left[D, P^{\mu}\right]=i P^{\mu}, \quad\left[D, K^{\mu}\right]=-i K^{\mu} .} \tag{A.26c}
\end{gather*}
$$

Anticommutation relations concern only the Poincaré and fermionic supercharges $Q, \bar{Q}, S, \bar{S}$ :

$$
\begin{gather*}
\left\{Q_{\alpha}^{A}, \bar{Q}_{B \dot{\beta}}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} P_{\mu} \delta^{A}{ }_{B}, \quad\left\{Q_{\alpha}^{A}, Q_{\beta}^{B}\right\}=\varepsilon_{\alpha \beta} Z^{A B}, \quad\left\{\bar{Q}_{A \dot{\beta}}, \bar{Q}_{B \dot{\beta}}\right\}=\varepsilon_{\dot{\alpha} \dot{\beta}} \bar{Z}_{A B},  \tag{A.27a}\\
\left\{S_{A \alpha} \bar{S}_{\beta}^{B}\right\}=2\left(\sigma^{\mu}\right)_{\alpha \dot{\beta}} K_{\mu} \delta^{B}{ }_{A}, \quad\left\{S_{A \alpha}, S_{B \beta}\right\}=\left\{\bar{S}_{\dot{\alpha}}^{A}, \bar{S}_{\beta}^{B}\right\}=0,  \tag{A.27b}\\
\left\{Q_{\alpha}^{A}, S_{B \beta}\right\}=2 \varepsilon_{\alpha \beta} \delta^{A}{ }_{B} D-i\left(\sigma^{\mu v}\right)_{\alpha}{ }^{\gamma} \varepsilon_{\gamma \beta} J_{\mu \nu} \delta^{A}{ }_{B}-4 i \varepsilon_{\alpha \beta} B^{i A}{ }_{B} T_{i},  \tag{A.27c}\\
\left\{\bar{Q}_{A \dot{\alpha}}, \bar{S}_{\dot{\beta}}^{B}\right\}=2 \varepsilon_{\dot{\alpha} \dot{\beta} \dot{\beta}} \delta^{B}{ }_{A} D-i\left(\sigma^{\mu \nu}\right)^{\dot{\gamma}}{ }_{\dot{\beta}} \varepsilon_{\dot{\alpha} \dot{\gamma}} J_{\mu \nu} \delta^{B}{ }_{A}+4 i \varepsilon_{\dot{\alpha} \dot{\beta}} B^{i}{ }_{A}^{B} T_{i},  \tag{A.27d}\\
\left\{Q_{\alpha}^{A}, \bar{S}_{\dot{\beta}}^{B}\right\}=\left\{\bar{Q}_{A \dot{\alpha}}, S_{B \beta}\right\}=0 . \tag{A.27e}
\end{gather*}
$$

where the $B^{\prime} s$ are defined by the commutators between the supercharges and the R -symmetry generators $T^{i}$, as given below. We first list the commutation relations involving the supercharges and the remaining generators of the conformal group:

$$
\begin{gather*}
{\left[Q_{\alpha}^{A}, D\right]=-\frac{i}{2} Q_{\alpha}^{A}, \quad\left[\bar{Q}_{A \dot{\alpha}}, D\right]=-\frac{i}{2} \bar{Q}_{A \dot{\alpha}},}  \tag{A.28a}\\
{\left[Q_{\alpha}^{A}, P^{\mu}\right]=0, \quad\left[\bar{Q}_{A \dot{\alpha}}, P^{\mu}\right]=0,}  \tag{A.28b}\\
{\left[Q_{\alpha}^{A}, K^{\mu}\right]=i\left(\sigma^{\mu}\right)_{\alpha \dot{\alpha}} \bar{S}^{A \dot{\alpha}}, \quad\left[\bar{Q}_{A \dot{\alpha}}, K^{\mu}\right]=-i \varepsilon_{\dot{\alpha} \dot{\beta}}\left(\bar{\sigma}^{\mu}\right)^{\dot{\beta} \alpha} S_{A \alpha},}  \tag{A.28c}\\
{\left[D, S_{A \alpha}\right]=-\frac{i}{2} S_{A \alpha}, \quad\left[D, \bar{S}_{\dot{\alpha}}^{A}\right]=-\frac{i}{2} \bar{S}_{\dot{\alpha}}^{A},}  \tag{A.28d}\\
{\left[S_{A \alpha}, P^{\mu}\right]=-i\left(\sigma^{\mu}\right)_{\alpha}^{\alpha} \bar{Q}_{A \dot{\alpha}}, \quad\left[\bar{S}_{\dot{\alpha}}^{A}, P^{\mu}\right]=i \varepsilon_{\dot{\alpha} \dot{\beta}}\left(\bar{\sigma}^{\mu}\right)^{\dot{\beta} \gamma} Q_{\gamma}^{A},}  \tag{A.28e}\\
{\left[S_{A \alpha}, K^{\mu}\right]=0, \quad\left[\bar{S}_{\dot{\alpha}}^{A}, K^{\mu}\right]=0 .} \tag{A.28f}
\end{gather*}
$$

Finally, we cover the commutators involving the R-symmetry generators $T$ :

$$
\begin{gather*}
{\left[J^{\mu v}, T^{i}\right]=\left[P^{\mu}, T^{i}\right]=\left[K^{\mu}, T^{i}\right]=\left[D, T^{i}\right]=0,\left[T^{i}, T^{j}\right]=i f^{i j}{ }_{k} T^{k},}  \tag{A.29a}\\
{\left[Q_{\dot{\alpha}}^{A}, T^{i}\right]=B_{B}^{i A}{ }_{B} Q_{\dot{\alpha}}^{B}, \quad\left[\bar{S}_{\dot{\alpha}}^{A}, T^{i}\right]=B_{B}^{i A} \bar{S}_{\dot{\alpha}}^{B} .} \tag{A.29b}
\end{gather*}
$$

## APPENDIX B

## Superblock Coefficients

In this appendix, we give the coefficients that relate the blocks inside the superblocks, following eq. (2.26). We recall the definition of the $R$-symmetry variable:

$$
\Omega_{\mathrm{R}} \equiv \frac{(1-\omega)^{2}}{4 \omega} .
$$

We found in section 2.1 that defect spacetime blocks (which are the solutions of the defect Casimir equations) take the following form:

$$
\hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z})=z^{\frac{\hat{\Delta}-s}{2}} \bar{z}^{\frac{\hat{\Delta}+s}{2}}{ }_{2} F_{1}\left(-s, \frac{1}{2}, \frac{1}{2}-s ; \frac{z}{\bar{z}}\right){ }_{2} F_{1}\left(\hat{\Delta}, \frac{1}{2}, \hat{\Delta}+\frac{1}{2} ; z \bar{z}\right) .
$$

Similarly, we showed in section 2.2 that the defect $R$-symmetry blocks are:

$$
\hat{h}_{k}(\omega)=\left(4 \Omega_{R}\right)^{k}{ }_{2} F_{1}\left(-1-k,-k,-2(1+k) ;-\Omega_{R}^{-1}\right) .
$$

It is explained in section 2.2 that the superblock corresponding to a representation $\hat{\chi}$ of the $\mathfrak{o s p}(4 \mid 4)$ algebra read:

$$
\hat{\mathcal{G}}_{\hat{\chi}}(z, \bar{z}, \omega)=\sum_{\hat{\Delta}, k, s} c_{\hat{\Delta}, k}(\hat{\chi}) \hat{h}_{k}(\omega) \hat{f}_{\hat{\Delta}, 0, s}(z, \bar{z}) .
$$

The allowed operators are listed in section 2.2. The superblock corresponding to the identity operator $\mathbb{1}$ is just 1 .

## B. 1 Coefficients of the $(B,+)_{k}$ Superblocks

The method used to find the coefficients is explained at the end of section 2.2 and illustrated for the case $(B,+)_{k}$. The following coefficients were obtained:

$$
\begin{gather*}
a_{0}=1,  \tag{B.1a}\\
a_{1}=-\frac{2 k}{1+2 k},  \tag{B.1b}\\
a_{2}=\frac{16 k(k-1)(k+1)^{2}}{(2 k-1)(2 k+3)(1+2 k)^{2}} . \tag{B.1c}
\end{gather*}
$$

## B. 2 Coefficients of the $(B, 1)_{[k, s]}$ Superblocks

We move now our attention to $(B, 1)_{[k, s]}$, for which we make the following ansatz based on the content of the multiplet:

$$
\begin{align*}
& \hat{\mathcal{G}}_{(B, 1)_{[k, s]}}(z, \bar{z}, \omega)=b_{0} \hat{f}_{3+k+s, s} \hat{h}_{k-2}+\left(b_{1,1} \hat{f}_{2+k+s, s-1}+b_{1,2} \hat{f}_{2+k+s, s+1}+b_{1,3} \hat{f}_{4+k+s, s+1}\right) \hat{h}_{k-1} \\
&+\left(b_{2,1} \hat{f}_{1+k+s, s}+b_{2,2} \hat{f}_{3+k+s, s}+b_{2,3} \hat{f}_{3+k+s, s+2}\right) \hat{h}_{k}+b_{3} \hat{f}_{2+k+s, s+1} \hat{h}_{k+1} . \tag{B.2}
\end{align*}
$$

Applying the Ward identities, we find the following solution for the coefficients (setting $b_{2,1} \equiv-1$ ):

$$
\begin{gather*}
b_{0}=\frac{16 k(k+1)(k-1)(2+k+s)}{(1-2 k)(1+2 k)^{2}(5+2 k+2 s)},  \tag{B.3a}\\
b_{1,1}=-\frac{8 k s^{2}}{(1+2 k)(1+2 s)(1-2 s)},  \tag{B.3b}\\
b_{1,2}=\frac{8 k(2+k)(2+k+s)}{(1+2 k)(3+2 k)(5+2 k+2 s)},  \tag{B.3c}\\
b_{1,3}=\frac{8 k(2+k+s)(3+k+s)^{2}(1+2 s)}{(1+2 k)(1+s)(5+2 k+2 s)^{2}(7+2 k+2 s)},  \tag{B.3d}\\
b_{2,1}=-1,  \tag{B.3e}\\
b_{2,2}=-\frac{16 k(1+s)(2+k+s)^{2}}{(1+2 k)(3+2 s)(3+2 k+2 s)(5+2 k+2 s)},  \tag{B.3f}\\
b_{2,3}=-\frac{(2+k+s)(1+2 s)}{(1+s)(5+2 k+2 s)},  \tag{B.3g}\\
b_{3}=\frac{1+2 s}{2(1+s)} . \tag{B.3h}
\end{gather*}
$$

## B. 3 Coefficients of the $L_{[k, s]}^{\hat{\Delta}}$ Superblocks

For the long operators, we have the following ansatz based on the multiplet content of $L_{[k, s]}^{\hat{\Delta}}$ :

$$
\begin{align*}
\hat{\mathcal{G}}_{L_{[k, s]}^{\hat{\Delta}}}(z, \bar{z}, \omega)= & c_{0} \hat{f}_{\widehat{\Delta}+2, s} \hat{h}_{k+2}+\left(c_{1,1} \hat{f}_{\widehat{\Delta}+1, s+1}+c_{1,2} \hat{f}_{\widehat{\Delta}+3, s+1}+c_{1,3} \hat{f}_{\widehat{\Delta}+1, s-1}+c_{1,4} \hat{f}_{\widehat{\Delta}+3, s-1}\right) \hat{h}_{k+1} \\
& +\left(c_{2,1} \hat{f}_{\widehat{\Delta}, s}+c_{2,2} \hat{f}_{\widehat{\Delta}+2, s+2}+c_{2,3} \hat{f}_{\widehat{\Delta}+2, s}+c_{2,4} \hat{f}_{\widehat{\Delta}+2, s-2}+c_{2,5} \hat{f}_{\widehat{\Delta}+4, s}\right) \hat{h}_{k} \\
& +\left(c_{3,1} \hat{f}_{\hat{\Delta}+3, s-1}+c_{3,2} \hat{f}_{\hat{\Delta}+1, s-1}+c_{3,3} \hat{f}_{\hat{\Delta}+3, s+1}+c_{3,4} \hat{f}_{\hat{\Delta}+1, s+1}\right) \hat{h}_{k-1}+c_{4} \hat{f}_{\hat{\Delta}+2, s} \hat{h}_{k-2} . \tag{B.4}
\end{align*}
$$

Repeating the same method, we find the following solution:

$$
\begin{gather*}
c_{0}=-\frac{(1+k+s-\hat{\Delta})(k-s-\hat{\Delta})}{(1-k+s+\hat{\Delta})(k+s-\hat{\Delta})},  \tag{B.5a}\\
c_{1,1}=-\frac{k+s-\hat{\Delta}}{1+k+s-\hat{\Delta}},  \tag{B.5b}\\
c_{1,2}=-\frac{4(\hat{\Delta}+2)^{2}(3+k+s+\hat{\Delta})}{(4+k+s+\hat{\Delta})(3+2 \hat{\Delta})(5+2 \hat{\Delta})},  \tag{B.5c}\\
c_{1,3}=-\frac{16 s^{2}(\hat{\Delta}+2)^{2}(2+k-s+\hat{\Delta})}{\left(4 s^{2}-1\right)(3+k-s+\hat{\Delta})(3+2 \hat{\Delta})(5+2 \hat{\Delta})}, \tag{B.5d}
\end{gather*}
$$

$$
\begin{align*}
& c_{1,4}=-\frac{4 s^{2}(1-k+s+\hat{\Delta})}{\left(4 s^{2}-1\right)(\hat{\Delta}-k+s)},  \tag{B.5e}\\
& c_{2,1}=\frac{(k+s-\hat{\Delta})(k-s-\hat{\Delta}-1)}{(k-s-\hat{\Delta})(1+k+s-\hat{\Delta})},  \tag{B.5f}\\
& c_{2,2}=\frac{(k+s-\hat{\Delta})(3+k+s+\hat{\Delta})}{(1+k+s-\hat{\Delta})(4+k+s+\hat{\Delta})},  \tag{B.5g}\\
& c_{2,3}=-\frac{8}{(1+2 k)(5+2 k)(1+2 s)(3+2 \hat{\Delta})} \\
& \times\left\{-\frac{(1+s)(-3+2 s-2 \hat{\Delta})(1+s-\hat{\Delta})(k+s-\hat{\Delta})(1+\hat{\Delta})(1-k+s+\hat{\Delta})(3+k+s+\hat{\Delta})}{(3+2 s)(1+k+s-\hat{\Delta})(1+2 \hat{\Delta})}\right. \\
& +\frac{s(-1+2 s-2 \hat{\Delta})(-3+s-\hat{\Delta})(-2-k+s-\hat{\Delta})(2+\hat{\Delta})(1-k+s+\hat{\Delta})(3+k+s+\hat{\Delta})}{(2 s-1)(3+k-s+\hat{\Delta})(5+2 \hat{\Delta})} \\
& +\frac{(1+s)(-2-k+s-\hat{\Delta})(k+s-\hat{\Delta})(2+\hat{\Delta})(4+s+\hat{\Delta})(3+k+s+\hat{\Delta})(3+2 s+2 \hat{\Delta})}{(3+2 s)(4+k+s+\hat{\Delta})(5+2 \hat{\Delta})} \\
& \left.+\frac{s(-2-k+s-\hat{\Delta})(k+s-\hat{\Delta})(1+\hat{\Delta})(s+\hat{\Delta})(1-k+s+\hat{\Delta})(5+2 s+2 \hat{\Delta})}{(2 s-1)(-k+s+\hat{\Delta})(1+2 \hat{\Delta})}\right\},  \tag{B.5h}\\
& c_{2,4}=-\frac{16 s^{2}(s-1)^{2}(-2-k+s-\hat{\Delta})(1-k+s+\hat{\Delta})}{(1-2 s)^{2}(2 s-3)(1+2 s)(3+k-s+\hat{\Delta})(-k+s+\hat{\Delta})},  \tag{B.5i}\\
& c_{2,5}=\frac{16(2+\hat{\Delta})^{2}(3+\hat{\Delta})^{2}(2+k-s+\hat{\Delta})(3+k+s+\hat{\Delta})}{(3+k-s+\hat{\Delta})(4+k+s+\hat{\Delta})(3+2 \hat{\Delta})(5+2 \hat{\Delta})^{2}(7+2 \hat{\Delta})},  \tag{B.5j}\\
& c_{3,1}=-\frac{64 k(2+k) s^{2}(-1+k-s-\hat{\Delta})(2+\hat{\Delta})^{2}(2+k-s+\hat{\Delta})(3+k+s+\hat{\Delta})}{(1+2 k)(3+2 k)(2 s-1)(2 s+1)(k-s-\hat{\Delta})(3+k-s+\hat{\Delta})(4+k+s+\hat{\Delta})(3+2 \hat{\Delta})(5+2 \hat{\Delta})}  \tag{B.5k}\\
& c_{3,2}=-\frac{16 k(2+k) s^{2}(-1+k-s-\hat{\Delta})(k+s-\hat{\Delta})(2+k-s+\hat{\Delta})}{(1+2 k)(3+2 k)(2 s-1)(2 s+1)(k-s-\hat{\Delta})(1+k+s-\hat{\Delta})(3+k-s+\hat{\Delta})},  \tag{B.5l}\\
& c_{3,3}=-\frac{16 k(2+k)(k+s-\hat{\Delta})(2+\hat{\Delta})^{2}(2+k-s+\hat{\Delta})(3+k+s+\hat{\Delta})}{(1+2 k)(3+2 k)(1+k+s-\hat{\Delta})(3+k-s+\hat{\Delta})(4+k+s+\hat{\Delta})(3+2 \hat{\Delta})(5+2 \hat{\Delta})},  \tag{B.5m}\\
& c_{3,4}=-\frac{4 k(2+k)(-1+k-s-\hat{\Delta})(k+s-\hat{\Delta})(3+k+s+\hat{\Delta})}{(1+2 k)(3+2 k)(k-s-\hat{\Delta})(1+k+s-\hat{\Delta})(4+k+s+\hat{\Delta})},  \tag{B.5n}\\
& c_{4}=\frac{16 k(k+1)(k-1)(2+k)(-1+k-s-\hat{\Delta})(k+s-\hat{\Delta})(2+k-s+\hat{\Delta})(3+k+s+\hat{\Delta})}{(2 k+1)^{2}(2 k-1)(3+2 k)(k-s-\hat{\Delta})(1+k+s-\hat{\Delta})(3+k-s+\hat{\Delta})(4+k+s+\hat{\Delta})} . \tag{B.5o}
\end{align*}
$$

These coefficients can also be found in the appendix of [20] for the case of a codimension-one defect, and the superblocks are obtained by inserting the spacetime and R-symmetry blocks as well as the coefficients in eq. (2.31), (B.2) and (B.4). Note that, as mentioned in the main text, the conformal dimensions $\hat{\Delta}$ of (B.4) can receive anomalous corrections, which give rise to log terms.

The coefficients of the bulk channel can be derived in the very same way, although we reserve this analysis for future work.

## APPENDIX C

## Integrals

This appendix is dedicated to the various integrals that appear throughout this thesis. We start by presenting standard integrals, i.e. integrals which are either elementary or already have a known solution in the literature. Then we present the numerical computation of the Feynman integrals, and in particular we show how to retrieve the full analytical solution for the integral of the 2 -channel. We also show the progress made for the other channels. Finally, we gather the measurements done in this work, i.e. the numerical data that has been collected for obtaining the expansions of the integrals and therefore the CFT data.

## C. 1 Standard Integrals

This section consists of the integrals that are either elementary or which can be found in the literature. In particular, we present the solutions of the conformal integrals $\mathrm{Y}, \mathrm{X}$ and F , as well as a powerful integral identity that reduces certain two-loop integrals to a sum of one-loop ones. Finally, we give pinching limits of these integrals which are relevant for the computations performed in chapter 4.

## Elementary Integrals

We list here the elementary integrals encountered in this work. We often make use of the following relation:

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d \tau}{\left(x^{2}+\tau^{2}\right)^{v}}=\frac{\Gamma(v-1 / 2)}{\Gamma(v)} \frac{\sqrt{\pi}}{\left(x^{2}\right)^{v-1 / 2}} . \tag{C.1a}
\end{equation*}
$$

We often run into the special case $v=1$, where the expression reduces to:

$$
\begin{equation*}
\int_{-\infty}^{\infty} \frac{d \tau}{x^{2}+\tau^{2}}=\frac{\pi}{|x|} \tag{C.1b}
\end{equation*}
$$

Because of the path-ordering the limits of integration are often not infinite. We sometimes face the following situation:

$$
\begin{equation*}
\int_{\tau_{i}}^{\infty} \frac{d \tau_{j}}{x^{2}+\tau_{j}^{2}}=\frac{1}{|x|}\left(\frac{\pi}{2}-\tan ^{-1} \frac{\tau_{i}}{|x|}\right) \tag{C.1c}
\end{equation*}
$$

It also happens that both limits of integration are finite, and in this case the following expression holds:

$$
\begin{equation*}
\int_{\tau_{i}}^{\tau_{j}} \frac{d \tau_{k}}{x^{2}+\tau_{k}^{2}}=\frac{1}{|x|}\left(\tan ^{-1} \frac{\tau_{j}}{|x|}-\tan ^{-1} \frac{\tau_{i}}{|x|}\right) . \tag{C.1d}
\end{equation*}
$$

## Conformal Integrals

The 3- and 4-point massless integrals in Euclidean space are conformal and have been solved analytically (see e.g. [49, 50]). The so-called $X$-integral is given by:

$$
\begin{equation*}
X_{1234}=\frac{1}{16 \pi^{2}} I_{13} I_{24} \Phi(r, s) \tag{C.2}
\end{equation*}
$$

where we have defined:

$$
\begin{gather*}
\Phi(r, s) \equiv \frac{1}{A} \operatorname{Im}\left\{\operatorname{Li}_{2} e^{i \varphi} \sqrt{\frac{r}{s}}+\log \sqrt{\frac{r}{s}} \log \left(1-e^{i \varphi} \sqrt{\frac{r}{s}}\right)\right\},  \tag{C.3a}\\
e^{i \varphi} \equiv i \sqrt{-\frac{1-r-s-4 i A}{1-r-s+4 i A}}, \quad A \equiv \frac{1}{4} \sqrt{4 r s-(1-r-s)^{2}},  \tag{C.3b}\\
r \equiv \frac{I_{13} I_{24}}{I_{12} I_{34}}, \quad s \equiv \frac{I_{13} I_{24}}{I_{14} I_{23}} . \tag{C.3c}
\end{gather*}
$$

The Y-integral can easily be obtained from this expression by taking the following limit:

$$
\begin{equation*}
Y_{123}=\lim _{x_{4} \rightarrow \infty}(2 \pi)^{2} x_{4}^{2} X_{1234}=\frac{1}{16 \pi^{2}} I_{12} \Phi(r, s), \tag{C.4}
\end{equation*}
$$

where here the conformal ratios are defined as:

$$
\begin{equation*}
r \equiv \frac{I_{12}}{I_{13}}, \quad s \equiv \frac{I_{12}}{I_{23}} . \tag{C.5}
\end{equation*}
$$

We note that both integrals are finite when the points are distinct. Furthermore, eq. (C.3a) implies that the function $\Phi$ vanishes in the limit $r \rightarrow \infty$ and $s \rightarrow \infty$, and that $\Phi(r, s)=\Phi(1 / r, s / r) / r$ [41]. The latter simply means that the conformal ratios can be defined arbitrarily, as long as consistency is respected.

## A Powerful Integral Identity

We recall that the $F$-integral is defined as:

$$
F_{13,24} \equiv \frac{\left(\partial_{1}-\partial_{3}\right) \cdot\left(\partial_{2}-\partial_{4}\right) H_{13,24}}{I_{13} I_{24}} .
$$

It was shown in [41] that this integral can be reduced to a sum of conformal integrals in the following way:

$$
\begin{align*}
F_{13,24}=\frac{X_{1234}}{I_{12} I_{34}}-\frac{X_{1234}}{I_{14} I_{23}}+\left(\frac{1}{I_{14}}-\frac{1}{I_{12}}\right) Y_{124} & +\left(\frac{1}{I_{23}}-\frac{1}{I_{34}}\right) Y_{234} \\
& +\left(\frac{1}{I_{23}}-\frac{1}{I_{12}}\right) Y_{123}+\left(\frac{1}{I_{14}}-\frac{1}{I_{34}}\right) Y_{134} \tag{C.6}
\end{align*}
$$

or diagrammatically:


We wish to give here a detailed proof of this identity. The basic idea is to transform the left-hand side of eq. (C.6) to momentum space, and then transform back to position space the resulting expression term by term. Using (0.1), it is straightforward to show that the Fourier transform gives:

$$
\begin{equation*}
\tilde{F}_{13,24}=-(2 \pi)^{8}\left(\partial_{p_{1}}-\partial_{p_{3}}\right)^{2}\left(\partial_{p_{2}}-\partial_{p_{4}}\right)^{2} \frac{\left(p_{1}-p_{3}\right) \cdot\left(p_{2}-p_{4}\right)}{p_{1}^{2} p_{2}^{2} p_{3}^{2} p_{4}^{2}\left(p_{1}+p_{3}\right)^{2}} \delta(P), \tag{C.7}
\end{equation*}
$$

with $P \equiv p_{1}+p_{2}+p_{3}+p_{4}$. To obtain this expression, it is important to realize that:

$$
x_{13}^{2} x_{24}^{2} e^{i \Sigma p \cdot x}=\left(\partial_{p_{1}}-\partial_{p_{3}}\right)^{2}\left(\partial_{p_{2}}-\partial_{p_{4}}\right)^{2} e^{i \sum p \cdot x} .
$$

To perform the derivatives in (C.7), we note that:

$$
\begin{aligned}
\left(\partial_{p_{1}}-\partial_{p_{3}}\right)_{\mu} f\left(p_{1}, p_{3}\right) \delta(P) & =\delta(P)\left(\partial_{p_{1}}-\partial_{p_{3}}\right)_{\mu} f\left(p_{1}, p_{3}\right)+f\left(p_{1}, p_{3}\right)\left(\partial_{p_{1}}-\partial_{p_{3}}\right)_{\mu} \delta(P) \\
& =\delta(P)\left(\partial_{p_{1}}-\partial_{p_{3}}\right)_{\mu} f\left(p_{1}, p_{3}\right),
\end{aligned}
$$

since $\left(\partial_{p_{1}}-\partial_{p_{3}}\right)_{\mu} \delta(P)=0$. This means that $\delta(P)$ can be moved back and forth to our liking in (C.7), and this fact can be used to simplify the derivatives. Using the Green's equation (0.2), we obtain:

$$
\begin{gather*}
\tilde{F}_{13,24}=64(2 \pi)^{8} \frac{\left(p_{1} \cdot p_{2}\right) \cdot\left(p_{3} \cdot p_{4}\right)}{p_{1}^{4} p_{2}^{4} p_{3}^{4} p_{4}^{4}} \delta(P)-64(2 \pi)^{8} \frac{\left(p_{1} \cdot p_{4}\right) \cdot\left(p_{2} \cdot p_{3}\right)}{p_{1}^{4} p_{2}^{4} p_{3}^{4} p_{4}^{4}} \delta(P) \\
+(2 \pi)^{12} \frac{1}{p_{3}^{4}} \delta(P) \delta\left(p_{1}\right) \delta\left(p_{2}\right)+3 \text { similar terms } \tag{C.8}
\end{gather*}
$$

We note that the factor $\left(p_{1}+p_{3}\right)^{-2}$ disappeared in all of the terms. This factor was a mark of the twoloop nature of the F-integral, and its absence suggests that the expression can be reduced to one-loop integrals ${ }^{1}$. The first two terms correspond to the leading terms of the X-integrals, as it can be seen by Fourier transforming the latter:

$$
\begin{gathered}
\mathrm{FT}\left(\frac{X_{1234}}{I_{12} I_{34}}\right)=64(2 \pi)^{8} \frac{\left(p_{1} \cdot p_{2}\right)\left(p_{3} \cdot p_{4}\right)}{p_{1}^{4} p_{2}^{4} p_{3}^{4} p_{4}^{4}} \delta(P)+8(2 \pi)^{10} \frac{p_{1} \cdot p_{2}}{p_{1}^{4} p_{2}^{4} p_{4}^{2}} \delta(P) \delta\left(p_{3}\right)+3 \text { similar terms } \\
+(2 \pi)^{12} \frac{1}{p_{2}^{4}} \delta(P) \delta\left(p_{1}\right) \delta\left(p_{3}\right)+3 \text { similar terms. }
\end{gathered}
$$

The first term is exactly the same as in (C.8), but it comes with a bunch of other terms without which we would live just as good. The terms of the form $f\left(p_{i}, p_{j}, p_{k}\right) \delta(P) \delta\left(p_{l}\right)$ are the leading terms of Y-integrals. The corresponding Fourier transform gives:

$$
\mathrm{FT}\left(\frac{Y_{124}}{I_{12}}\right)=-8(2 \pi)^{10} \frac{p_{1} \cdot p_{2}}{p_{1}^{4} p_{2}^{4} p_{4}^{2}} \delta(P) \delta\left(p_{3}\right)-(2 \pi)^{12} \frac{1}{p_{2}^{4}} \delta(P) \delta\left(p_{1}\right) \delta\left(p_{3}\right)-(2 \pi)^{12} \frac{1}{p_{1}^{4}} \delta(P) \delta\left(p_{2}\right) \delta\left(p_{3}\right) .
$$

It is now only a matter of gathering the correct combination of X and Y . Choosing terms such as those with one $\delta$-function vanish results precisely in the right-hand side of (C.6).

[^5]
## Pinching Limits

We will give here the pinching limits of the Y-, X- and F-integrals described in the previous sections. The procedure is very simple, and we will show it explicitly for the Y-integral only, and just give the results for the other integrals.

We use point-splitting regularization, and define:

$$
Y_{122} \equiv \lim _{x_{3} \rightarrow x_{2}} Y_{123}, \quad \quad \lim _{x_{3} \rightarrow x_{2}} I_{23} \equiv \frac{1}{(2 \pi)^{2} \epsilon^{2}} .
$$

In this limit, the conformal ratios are now given by

$$
r=1, \quad s=(2 \pi)^{2} \epsilon^{2} I_{12} .
$$

Inserting this in (C.4) and expanding up to order $\mathcal{O}\left(\log \epsilon^{2}\right)$, we obtain:

$$
\begin{equation*}
\bigcirc \equiv Y_{112}=Y_{122}=-\frac{1}{16 \pi^{2}} I_{12}\left(\log \frac{\epsilon^{2}}{x_{12}^{2}}-2\right) . \tag{C.9}
\end{equation*}
$$

This result coincides with the expression given in [42].
Similarly, the pinching limit of the X-integral reads:

$$
\begin{equation*}
\bigcirc \equiv X_{1123}=-\frac{1}{16 \pi^{2}} I_{12} I_{13}\left(\log \frac{\epsilon^{2} x_{23}^{2}}{x_{12}^{2} x_{13}^{2}}-2\right), \tag{C.10}
\end{equation*}
$$

which is again the same as in [42].
Finally, the pinching limit $x_{2} \rightarrow x_{1}$ of the F-integral gives:

$$
\begin{align*}
-\beta & \equiv F_{13,14}=F_{14,13}=-F_{13,41} \\
& =-\frac{X_{1134}}{I_{13} I_{14}}+\frac{Y_{113}}{I_{13}}+\frac{Y_{114}}{I_{14}}+\left(\frac{1}{I_{13}}+\frac{1}{I_{14}}-\frac{2}{I_{34}}\right) Y_{134} . \tag{C.11}
\end{align*}
$$

## C. 2 Numerical Integration

This section contains details about the numerical integrations performed in this thesis. We first show how to obtain an analytical expression for the integral of the 2 -channel at NLO. We then describe in some detail the computations that can be done in the 1-channel, and we present some partial results for the 1 d integrals. Finally, we conclude with a description of how to handle the H-integral of the 0 -channel, in particular for the line $z=\bar{z}$ such that it can be integrated numerically.

## Integral of the 2-Channel

As discussed in section 4.2, the only integral that needs to be computed for the 2-channel is the following:

$$
\begin{equation*}
I\left(x_{1}^{2}, x_{2}^{2}\right) \equiv x_{1}^{2} x_{2}^{2} \int d \tau_{3} \int d \tau_{4} \int d \tau_{5} \int d \tau_{6} \Theta\left(\tau_{3456}\right)\left(I_{13} I_{25}+I_{15} I_{23}\right)\left(I_{14} I_{26}+I_{16} I_{24}\right) \tag{C.12}
\end{equation*}
$$

with $\Theta\left(\tau_{3456}\right)$ defined in (3.20). We first note that the integral does not depend on $x_{12}$, but only on the distances between the operators and the line defect. As a consequence, the integral is symmetric with



Figure C.1: The left plot compares the numerical integration of eq. (C.13) (dots) on the line $z=\bar{z}$ with the analytical expression given in (C.16) (solid line). The agreement is perfect and validates the analytical result. The right plot shows the numerical data (dots) for the $1 d$ integrals of the 1-channel on the line $z=\bar{z}$ (see eq. (C.17)), while the solid line corresponds to the expansion up to order $\mathcal{O}\left(x^{11}\right)$ with the coefficients of table C.1. We observe a near-perfect agreement in the neighborhood of $x \sim 0$, but we have not been able to guess a closed form yet for the whole curve, hence the discrepancy near $x \sim 1$.
respect to $x_{2} \leftrightarrow-x_{2}$. Moreover, the integral possesses a hidden symmetry with respect to inversion, i.e. $I\left(x_{1}^{2}, x_{2}^{2}\right)$ is invariant with respect to $x_{i} \leftrightarrow 1 / x_{i}$. This symmetry implies that knowing the channel in the range $(0,1)$ is enough for knowing it everywhere.

In order to do the integral, we first perform the $\tau_{6}$ - and $\tau_{4}$-integrals and we get:

$$
\begin{aligned}
I\left(x_{1}^{2}, x_{2}^{2}\right)= & \frac{\left|x_{1}\right|\left|x_{2}\right|}{128 \pi^{4}} \int d \tau_{3} \int d \tau_{5} \Theta\left(\tau_{35}\right)\left(I_{13} I_{25}+I_{15} I_{23}\right)\left\{\pi\left(\tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}-\tan ^{-1} \frac{\tau_{5}}{\left|x_{1}\right|}\right)\right. \\
& \left.\quad+\tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}\left(2 \tan ^{-1} \frac{\tau_{5}}{\left|x_{1}\right|}+\pi\right)+\tan ^{-1} \frac{\tau_{5}}{\left|x_{2}\right|}\left(2 \tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}-\pi\right)-4 \tan ^{-1} \frac{\tau_{5}}{\left|x_{1}\right|} \tan ^{-1} \frac{\tau_{5}}{\left|x_{2}\right|}\right\} .
\end{aligned}
$$

We can perform one more integration analytically by treating it term by term. The first integral gives:

$$
\begin{aligned}
I_{1}\left(x_{1}^{2}, x_{2}^{2}\right) & =\frac{\left|x_{1}\right|\left|x_{2}\right|}{128 \pi^{3}} \int d \tau_{3} \int d \tau_{5} \Theta\left(\tau_{35}\right)\left(I_{13} I_{25}+I_{15} I_{23}\right)\left(\tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}-\tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}\right) \\
& =\frac{1}{256 \pi^{5}} \int d \tau_{3} \tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}\left\{\left|x_{1}\right| I_{13} \tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}+\left|x_{2}\right| I_{23} \tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}\right\},
\end{aligned}
$$

where we have performed the $\tau_{3}$-integral in the second term and relabeled $\tau_{5}$ to $\tau_{3}$ in the second line.
The second integral reads:

$$
\begin{aligned}
& I_{2}\left(x_{1}^{2}, x_{2}^{2}\right)= \frac{\left|x_{1}\right|\left|x_{2}\right|}{128 \pi^{4}} \int d \tau_{3} \int d \tau_{5} \Theta\left(\tau_{35}\right)\left(I_{13} I_{25}+I_{15} I_{23}\right)\left\{\begin{array}{l}
\tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}\left(2 \tan ^{-1} \frac{\tau_{5}}{\left|x_{1}\right|}+\pi\right)
\end{array}\right. \\
&\left.+\tan ^{-1} \frac{\tau_{5}}{\left|x_{2}\right|}\left(2 \tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}-\pi\right)\right\}
\end{aligned} \quad \begin{aligned}
& =\frac{\left|x_{1}\right|\left|x_{2}\right|}{64 \pi^{4} \int d \tau_{5} \int d \tau_{3}\left(I_{13} I_{25}+I_{15} I_{23}\right) \tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|} \tan ^{-1} \frac{\tau_{5}}{\left|x_{1}\right|}} \\
& \quad+\frac{\left|x_{1}\right|\left|x_{2}\right|}{128 \pi^{4}}\left(\int d \tau_{5} \int d \tau_{3} \Theta\left(\tau_{35}\right)-\int d \tau_{5} \int d \tau_{3} \Theta\left(\tau_{53}\right)\right)\left(I_{13} I_{25}+I_{15} I_{23}\right) \tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|} .
\end{aligned}
$$

The first term vanishes and we are left with:

Table C.1: Coefficients for the expansion of $I\left(1, x_{2}^{2}\right)$ following the ansatz given in eq. (C.14) and obtained numerically by computing (C.15a) and (C.15b). The coefficient $a_{0}$ has to be multiplied by $g^{8} N^{2}$, the $a_{k}$ 's for $k \geq 1$ by $g^{8} N^{2} / 2^{8} \pi^{6}$ and the $b_{k}$ 's have a missing factor $g^{8} N^{2} / 2^{9} \pi^{6}$. Guessing the closed form leads to the expression given in (C.16).

| $a_{0}$ | $a_{1}$ | $a_{2}$ | $a_{3}$ | $a_{4}$ | $a_{5}$ | $a_{6}$ | $a_{7}$ | $a_{8}$ | $a_{9}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{2^{12} \pi^{4}}$ | $\log 2-\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{\log 2}{3}+\frac{1}{9}$ | $-\frac{1}{3}$ | $\frac{\log 2}{5}+\frac{13}{100}$ | $-\frac{23}{90}$ | $\frac{\log 2}{7}+\frac{71}{588}$ | $-\frac{22}{105}$ | $\frac{\log 2}{9}+\frac{71}{648}$ |
| $a_{10}$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $a_{16}$ | $a_{17}$ | $a_{18}$ | $a_{19}$ |
| $-\frac{563}{3150}$ | $\frac{\log 2}{11}+\frac{1447}{14520}$ | $-\frac{1627}{10395}$ | $\frac{\log 2}{13}+\frac{617}{6760}$ | $-\frac{88069}{630630}$ | $\frac{\log 2}{15}+\frac{1061}{12600}$ | $-\frac{5692}{45045}$ | $\frac{\log 2}{17}+\frac{12657}{161840}$ | $-\frac{1593269}{13783770}$ | $\frac{\log 2}{19}+\frac{132931}{1819440}$ |
| $b_{1}$ | $b_{2}$ | $b_{3}$ | $b_{4}$ | $b_{5}$ | $b_{6}$ | $b_{7}$ | $b_{8}$ | $b_{9}$ | $b_{10}$ |
| 1 | 0 | $\frac{1}{3}$ | 0 | $\frac{1}{5}$ | 0 | $\frac{1}{7}$ | 0 | $\frac{1}{9}$ | 0 |

$$
I_{2}\left(x_{1}^{2}, x_{2}^{2}\right)=\frac{1}{256 \pi^{5}} \int d \tau_{3} \tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}\left\{\left|x_{1}\right| I_{13} \tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}+\left|x_{2}\right| I_{23} \tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}\right\} .
$$

The remaining term can also be reduced to a one-dimensional integral:

$$
\begin{aligned}
I_{3}\left(x_{1}^{2}, x_{2}^{2}\right)= & -\frac{\left|x_{1}\right|\left|x_{2}\right|}{32 \pi^{4}} \int d \tau_{3} \int d \tau_{5}\left(I_{13} I_{25}+I_{15} I_{23}\right) \tan ^{-1} \frac{\tau_{5}}{\left|x_{1}\right|} \tan ^{-1} \frac{\tau_{5}}{\left|x_{2}\right|} \\
= & -\frac{1}{256 \pi^{5}} \int d \tau_{3} \tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|} \tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}\left\{\left|x_{1}\right| I_{13}\left(\pi-2 \tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}\right)\right. \\
& \left.+\left|x_{2}\right| I_{23}\left(\pi-2 \tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}\right)\right\} .
\end{aligned}
$$

Putting everything together, the integral becomes:

$$
\begin{aligned}
& I\left(x_{1}^{2}, x_{2}^{2}\right)=\frac{1}{256 \pi^{6}} \int d \tau_{3}\left\{\left|x_{1}\right| I_{13} \tan ^{-2} \frac{\tau_{3}}{\left|x_{2}\right|}\left(2 \tan ^{-1} \frac{\tau_{3}}{\left|x_{1}\right|}+\pi\right)\right. \\
&\left.+\left|x_{2}\right| I_{23} \tan ^{-2} \frac{\tau_{3}}{\left|x_{1}\right|}\left(2 \tan ^{-1} \frac{\tau_{3}}{\left|x_{2}\right|}+\pi\right)\right\} .
\end{aligned}
$$

The terms cubic in $\tan ^{-1}$ vanish because of antisymmetry. The integral reduces therefore to the following compact expression:

$$
\begin{equation*}
I\left(x_{1}^{2}, x_{2}^{2}\right)=\frac{1}{256 \pi^{5}} \int d \tau_{3}\left\{\left|x_{1}\right| I_{13} \tan ^{-2} \frac{\tau_{3}}{\left|x_{2}\right|}+\left|x_{2}\right| I_{23} \tan ^{-2} \frac{\tau_{3}}{\left|x_{1}\right|}\right\} . \tag{C.13}
\end{equation*}
$$

The inversion symmetry mentioned above can be made manifest by substituting $\tau_{3} \rightarrow \tau_{3} /\left|x_{2}\right|$.
We were not able to solve this integral analytically, but with the help of numericals it is still possible to obtain the closed form. We start with the following ansatz, which is based on the expansion of the superblocks given in (2.41) and (2.42):

$$
\begin{equation*}
I\left(1, x^{2}\right)=\sum_{k=0}^{\infty} a_{k} x^{k}+\log x \sum_{k=1}^{\infty} b_{k} x^{k}, \tag{C.14}
\end{equation*}
$$



Figure C.2: The numerical data corresponding to $g_{X}(x, x)$ is presented in the left plot, in which we notice that it vanishes for all $x \geq 0$. The right plot shows the function $g_{1}(x, x)-g_{X}(x, x)$, and in this case it is constant for $x \leq 0$.
where we have defined $\left|x_{1}\right|=1$ and $\left|x_{2}\right| \equiv x$ to lighten the notation. If this expression holds, the coefficients obey the following relations:

$$
\begin{gather*}
a_{k}=\frac{1}{k!} \lim _{x \rightarrow 0} \partial_{x}^{k}\left\{I\left(1, x^{2}\right)-\log x \sum_{l=1}^{k-1} b_{l} \frac{x^{2 l-1}}{2 l-1}\right\},  \tag{C.15a}\\
b_{k}=\frac{1}{k!} \lim _{x \rightarrow 0}\left\{x \partial_{x}^{k+1} I\left(1, x^{2}\right)+\sum_{l=1}^{k-1}(-1)^{k-l+1}(k-l)!l!x^{l-k}\right\} . \tag{C.15b}
\end{gather*}
$$

Hence the coefficients can be computed numerically for decreasing $x$ until convergence. The convergence also confirms the validity of the ansatz given in (C.14). The coefficients are given in table C.1, while the numerical data can be found in table C.3. We managed to obtain accurate enough data to be able to guess the closed form for all the coefficients. Moreover, the resulting series are all identifiable and we could guess the closed form of the full integral:

$$
\begin{gather*}
I\left(1, x_{2}^{2}\right)=\frac{1}{2^{12} \pi^{6}}\left\{3 \pi^{2}-4 i \pi \log 2+4 \tanh ^{-1} \sqrt{z \bar{z}}\left(\log z \bar{z}+4 \log 2-2 \tanh ^{-1} \sqrt{z \bar{z}}\right)\right. \\
+4 \log ^{2}(1-\sqrt{z \bar{z}})+2 \log (\sqrt{z \bar{z}}-1)(-2 \log (1-\sqrt{z \bar{z}})+\log (\sqrt{z \bar{z}}-1)+2 \log 2) \\
-2 \log (1+\sqrt{z \bar{z}}) \log 4(1+\sqrt{z \bar{z}})+4 \operatorname{Li}_{2}(-\sqrt{z \bar{z}})-4 \operatorname{Li}_{2} \sqrt{z \bar{z}} \\
\left.-4 \operatorname{Li}_{2} \frac{1}{2}(1-\sqrt{z \bar{z}})+4 \operatorname{Li}_{2} \frac{1}{2}(1+\sqrt{z \bar{z}})\right\} . \tag{C.16}
\end{gather*}
$$

It is worth checking that we got the closed form right. Fig. C. 1 shows a plot of the numerical data and of eq. (C.16), which match perfectly. We have therefore obtained an exact analytical expression for the 2-channel.

## Integrals of the 1-Channel

As mentioned in the main text, an analytical expression for the integrals of the 1-channel could not be obtained, neither analytically nor numerically. The channel consists of 1 -dimensional and of 2 dimensional integrals. Using the same ansatz as in eq. (C.14) for the limiting case $z=\bar{z}$, we were able to obtain several coefficients for the 1-dimensional integrals. Unfortunately, we have not been able to guess a closed form for the series yet. Even worse, for the 2 -dimensional integrals we have only been able to extract the leading coefficients so far.

The expression for the 1d integrals that we wish to compute is defined as:

$$
\begin{equation*}
g_{1}^{1 \mathrm{~d}}(x, x) \equiv g_{\mathrm{IYI}}(x, x)+g_{Y_{123}}(x, x)+g_{Y_{124}}(x, x), \tag{C.17}
\end{equation*}
$$

which is related to (4.27) by the usual relation given in (2.36). The numerical data for the coefficients is given in table C.4, while the closed forms can be found in table C.2. We were able to derive the closed form for all of the sequences except for the second term in the $\alpha_{k}$ 's with $k$ odd. The rest of the expression reads:

$$
\begin{equation*}
g_{1}^{1 \mathrm{~d}}(x, x)=\frac{1}{3 \cdot 2^{11} \pi^{4}}+\frac{1}{2^{8} \pi^{6}} \log x \tanh ^{-1} x+\frac{\log 2}{2^{7} \pi^{6}} \tanh ^{-1} x-\frac{3}{2^{8} \pi^{6}} \tanh ^{-2} x+\text { missing piece } . \tag{C.18}
\end{equation*}
$$

The expansion with the known coefficients is compared to the numerical data for the full integral in fig. C.1. Obtaining more coefficients should make it possible to guess the missing term, and that is one of the objectives of future work.

As warned before, the situation is even worse regarding the 2 d integrals. The $g$-function corresponding to eq. (4.28) in the collinear limit is defined as:

$$
\begin{equation*}
g_{1}^{2 \mathrm{~d}}(x, x) \equiv g_{X}(x, x)+g_{Y_{134}}(x, x)+g_{Y_{234}}(x, x) \tag{C.19}
\end{equation*}
$$

For $x \geq 0$, we find numerically that:

$$
g_{X}(x, x)=0,
$$

while for $x \leq 0$ we notice that:

$$
g_{1}(x, x)-g_{X}(x, x)=\text { const. }=\frac{g^{8} N^{2}}{3 \cdot 2^{9} \pi^{4}}
$$

These results are shown explicitly in fig. C.2. Using the ansatz of eq. (C.14) to which we are now used to, we could derive only the two first coefficients of the integrals for $x \geq 0$ so far, i.e.:

$$
\begin{equation*}
g_{1}^{2 \mathrm{~d}}(x, x)=\frac{g^{8} N^{2}}{3 \cdot 2^{9} \pi^{4}}-\frac{g^{8} N^{2}}{2^{7} \pi^{6}} x \log x+\mathcal{O}(x) \tag{C.20}
\end{equation*}
$$

We hope to be able to optimize the integration algorithms in order to obtain more coefficients in the future. Putting together this result with the previous one for the 1d integrals results in the expression given in (4.30). The full channel along the line $z=\bar{z}$ is plotted in fig. 4.1 with the numerical data of table C.6.

## Integral of the 0-Channel

We now discuss the case of the H -integral in the 0 -channel, which is by far the hardest that we have to consider in this work. It is a 10 -dimensional integral with two $\tau$-derivatives, acting on $x_{1}$ and $x_{2}$ :

$$
\begin{equation*}
\text { SO }=-4 \lambda_{0} I_{12} \partial_{\tau_{1}} \partial_{\tau_{2}} \int d \tau_{3} \int d \tau_{4} H_{13,24} \tag{C.21}
\end{equation*}
$$

with $H_{13,24}$ defined in (3.10d). Mathematica is unable to handle such a monster, and we must simplify it before having a chance to feed it to the computer.

Integration by parts can be used for removing the $\tau_{2}$-derivative:

Table C.2: Coefficients for the $1 d$ integrals of the 1-channels based on the same ansatz as the one used in eq. (C.14). $a_{0}$ should be multiplied by $g^{8} N^{2}$, while the other $a_{k}$ 's are missing a factor $g^{8} N^{2} / 2^{7} \pi^{6}$. The $b_{k}$ 's have to be multiplied by $g^{8} N^{2} / 2^{8} \pi^{6}$.

| $a_{0}$ | $a_{1}$ | $a_{2}$ | $a_{3}$ | $a_{4}$ | $a_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\frac{1}{3 \cdot 2^{11} \pi^{4}}$ | $-\log 2-\frac{1}{2}$ | 3 | $-\frac{\log 2}{3}-\frac{8}{9}$ | 2 | $-\frac{\log 2}{5}-\frac{211}{150}$ |
| $a_{6}$ | $a_{7}$ | $a_{8}$ | $a_{9}$ | $a_{10}$ | $a_{11}$ |
| $\frac{23}{15}$ | $-\frac{\log 2}{7}-\frac{1703}{1470}$ | $\frac{44}{35}$ | $-\frac{\log 2}{9}-\frac{11213}{11340}$ | $\frac{563}{525}$ | $-\frac{\log 2}{11}+\frac{131995}{152460}$ |
| $b_{1}$ | $b_{2}$ | $b_{3}$ | $b_{4}$ | $b_{5}$ | $b_{6}$ |
| -1 | 0 | $-\frac{1}{3}$ | 0 | $-\frac{1}{5}$ | 0 |

$$
\begin{aligned}
\partial_{\tau_{2}} H_{13,24} & =\int d^{4} x_{56} I_{15} I_{35} \partial_{\tau_{2}} I_{26} I_{46} I_{56} \\
& =-\int d^{4} x_{56} I_{15} I_{35} I_{26}\left(\partial_{\tau_{4}}+\partial_{\tau_{5}}\right) I_{46} I_{56}
\end{aligned}
$$

Since $\int d \tau_{4} \partial_{\tau_{4}} I_{46}=0$, we can drop the first term in the last line. Using integration by parts with respect to the $x_{5}$-integral, we obtain:

$$
\partial_{\tau_{2}} H_{13,24} \widehat{=}-\partial_{\tau_{1}} H_{13,24},
$$

where the $\widehat{=}$ means that this equality is to be understood as valid in the context of eq. (C.21) only. Here we have used again the fact that $\int d \tau_{3} \partial_{\tau_{3}} I_{35}=0$.

We have now:

where we do not keep trace of the numerical prefactors for compactness (they will be reinstated in the final result). The Y-integral is known analytically, so we now find ourselves facing a 6 -dimensional integral.

The derivatives give:

$$
\partial_{\tau_{1}}^{2} I_{15}=2(2 \pi)^{2} I_{15}^{2}\left(4(2 \pi)^{2} \tau_{5}^{2} I_{15}-1\right),
$$

and it is easy to do the $\tau_{3}$-integral using (C.1b):

where $\vec{x}$ means that the $\tau$-component is zero, i.e. $\vec{x} \equiv(x, y, z, 0)$.
This is as far as we can go for a general $x_{2}$. Going to the limiting case $x_{2}=(x, 0,0,0)$ encourages us to introduce 3 d spherical coordinates for $y_{5}, z_{5}, \tau_{5}$, because $y_{5}$ and $z_{5}$ now always appear in the form
$y_{5}^{2}+z_{5}^{2}$. The integration is independent of the azimuthal angle and we can kill one integral in exchange of a $2 \pi$ factor. What we have now is:

$$
\begin{equation*}
\underbrace{\infty}_{0}=\frac{4}{(2 \pi)^{6}} I_{12} \int_{0}^{\infty} d r \int_{0}^{\pi} d \theta \int_{-\infty}^{\infty} d x \int_{-\infty}^{\infty} d \tau_{4} \frac{r^{2} \sin \theta}{R\left(d^{2}\right)^{2}}\left(\frac{4 r^{2} \cos ^{2} \theta}{d^{2}}-1\right) Y_{245} \tag{C.22}
\end{equation*}
$$

where we have dropped the index 5 in order to keep our expression compact, and where all the prefactors have been reintroduced. The functions $R$ and $d$ are defined as follows:

$$
\begin{aligned}
& R(x, r, \theta):=\sqrt{x^{2}+r^{2} \sin ^{2} \theta} \\
& d^{2}(x, r):=(1-x)^{2}+r^{2}
\end{aligned}
$$

We are thus left with a hard 4-dimensional integral, and at that point we cannot go further analytically. MATHEMATICA is able to handle this expression, although it is hard to obtain high-enough accuracy. The behavior of the integral is shown in fig. 4.1, while the numerical data is gathered in table C.6. The integral has the same interesting inversion symmetry $x \leftrightarrow 1 / x$ as the other channels, and we also notice that it is constant for $x \leq 0$. This remarkable feature is exploited in section 4.3 for extracting the CFT data without having to know the full correlator analytically.

## C. 3 Measurements

In this section, we present the numerical measurements done throughout the work, i.e. the numerical data resulting from computing the integrals with MATHEMATICA.

Table C.3: Coefficients for the ansatz (C.14) of the integral of the 2-channel, obtained by computing eq. (C.15a) and (C.15b) until convergence. The numerical values should be multiplied by $10^{-6}$, and the bars on the last digit indicate the uncertainty. The corresponding closed forms can be found in table C.1.

| $a_{k}$ | Numerical value | $a_{k}$ | Numerical value | $b_{k}$ | Numerical value |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{0}$ | $2.506343323897542 \overline{8}$ | $a_{10}$ | $-0.7262040047936072 \overline{0}$ | $b_{1}$ | $2.03156537753096151 \overline{6}$ |
| $a_{1}$ | $0.78478224958661216 \overline{7}$ | $a_{11}$ | $0.6609455282623829 \overline{1}$ | $b_{2}$ | $<10^{-19}$ |
| $a_{2}$ | $-2.03156537753096151 \overline{6}$ | $a_{12}$ | $-0.63595129759362662 \overline{6}$ | $b_{3}$ | $0.6771884591769871 \overline{7}$ |
| $a_{3}$ | $1.39024151515718267 \overline{5}$ | $a_{13}$ | $0.58749296478910426 \overline{4}$ | $b_{4}$ | $<10^{-19}$ |
| $a_{4}$ | $-1.354376918353974343 \overline{9}$ | $a_{14}$ | $-0.56742600648169053 \overline{1}$ | $b_{5}$ | $0.4063130755061923 \overline{0}$ |
| $a_{5}$ | $1.091476523581564 \overline{7} \cdot$ | $a_{15}$ | $0.5298979157063065 \overline{3}$ | $b_{6}$ | $<10^{-19}$ |
| $a_{6}$ | $-1.038355637404713663 \overline{6}$ | $a_{16}$ | $-0.51342746715090389 \overline{4}$ | $b_{7}$ | $0.290223625361565 \overline{9}$ |
| $a_{7}$ | $0.89295150388992 \overline{0}$ | $a_{17}$ | $0.4834322502284853 \overline{0}$ | $b_{8}$ | $<10^{-19}$ |
| $a_{8}$ | $-0.85132263439392673 \overline{0}$ | $a_{18}$ | $-0.46965817588270517 \overline{0}$ | $b_{9}$ | $0.2257294863923290 \overline{6}$ |
| $a_{9}$ | $0.75811622339793493 \overline{9}$ | $a_{19}$ | $0.445087215392601 \overline{8}$ | $b_{10}$ | $<10^{-19}$ |

Table C.4: Coefficients for the ansatz (C.14) adapted to the sum of the $1 d$ integrals of the 1 -channel (see eq. (C.17)). Here also these numerical values are obtained by computing eq. (C.15a) and (C.15b) until convergence. The values should be multiplied by $10^{-6}$, and the corresponding closed forms are given in table C.2. The bar on the last digit indicates the uncertainty.

| $a_{k}$ | Numerical value | $a_{k}$ | Numerical value | $b_{k}$ | Numerical value |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $a_{0}$ | $1.67089 \overline{6}$ | $a_{6}$ | $6.230133 \overline{8}$ | $b_{1}$ | $-4.0631307 \overline{5}$ |
| $a_{1}$ | $9.69582600 \overline{9}$ | $a_{7}$ | $-5.51182155 \overline{1}$ | $b_{2}$ | $<10^{-9}$ |
| $a_{2}$ | $12.18939226 \overline{5}$ | $a_{8}$ | $5.1079358 \overline{1}$ | $b_{3}$ | $-1.35437 \overline{7}$ |
| $a_{3}$ | $-9.10090864 \overline{9}$ | $a_{9}$ | $-4.643481 \overline{6}$ | $b_{4}$ | $<10^{-9}$ |
| $a_{4}$ | $8.126261 \overline{5}$ | $a_{10}$ | $4.3572 \overline{2}$ | $b_{5}$ | $-0.812626 \overline{1}$ |
| $a_{5}$ | $-6.842009 \overline{6}$ | $a_{11}$ | $-4.0297 \overline{9}$ | $b_{6}$ | $<10^{-9}$ |

Table C.5: Coefficients analogous to the ones of table C.4, but for the sum of $2 d$ integrals of the 2 -channel (see eq. (C.19)). The numerical values should be multiplied by $10^{-6}$.

| $a_{k}$ | Numerical value | $b_{k}$ | Numerical value |
| :---: | :---: | :---: | :---: |
| $a_{0}$ | $5.0126866 \overline{7}$ | $b_{0}$ | $-4.063 \overline{1}$ |

Table C.6: Numerical integration of the $\tilde{F}$-functions for the 0 - and 1-channels on the line $z=\bar{z} \equiv x$. The corresponding $g$-functions are plotted in fig. 4.1. The values for the $\tilde{F}$ 's should be multiplied by $10^{-7}$. The bar on the last digit indicates the uncertainty of the measurement.

| $x$ | -0.9999 | $-1 / 2$ | $-1 / 3$ | $-1 / 4$ | $-1 / 5$ | $-1 / 10$ | $-1 / 100$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\tilde{F}_{0}$ | $-8.3 \overline{5}$ | $-6.6 \overline{0}$ | $-4.7 \overline{0}$ | $-3.4 \overline{2}$ | $-2.5 \overline{8}$ | $-0.9 \overline{1}$ | $-0.1 \overline{1}$ |
| $x$ | $1 / 100$ | $1 / 10$ | $1 / 5$ | $1 / 4$ | $1 / 3$ | $1 / 2$ | 0.9999 |
| $\tilde{F}_{0}$ | $-0.0 \overline{1}$ | $-1.5 \overline{9}$ | $-7.8 \overline{1}$ | $-13.7 \overline{5}$ | $-30.5 \overline{1}$ | $-121.6 \overline{0}$ | 0 |
|  |  |  |  |  |  |  |  |
| $x$ | -0.9999 | $-1 / 2$ | $-1 / 3$ | $-1 / 4$ | $-1 / 5$ | $-1 / 10$ | $-1 / 100$ |
| $\tilde{F}_{1}$ | $8.35 \overline{5}$ | $7.91 \overline{1}$ | $7.28 \overline{6}$ | $6.64 \overline{3}$ | $6.07 \overline{2}$ | $4.16 \overline{7}$ | $0.62 \overline{2}$ |
| $x$ | $1 / 100$ | $1 / 10$ | $1 / 5$ | $1 / 4$ | $1 / 3$ | $1 / 2$ | 0.9999 |
| $\tilde{F}_{1}$ | $0.70 \overline{9}$ | $9.36 \overline{5}$ | $23.90 \overline{6}$ | $33.81 \overline{3}$ | $56.19 \overline{0}$ | $144.16 \overline{0}$ | $>10^{2}$ |

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## Declaration of Authorship

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Julien Barrat



[^0]:    ${ }^{1}$ In Euclidean space this is just $\mathfrak{s o}(4)$ of course, but in this work we use the notation usually reserved to Minkowski space for another purpose: the 1 in $\mathfrak{s o}(3,1)$ actually refers to the dimension parallel to the line defect, which as we will see extends in the $\tau$-direction. It is useful to keep this notation in defect theory, for example to distinguish the terms in the product $\mathfrak{s o}(2,1) \times \mathfrak{s o}(3)$ (see section 1.3).

[^1]:    ${ }^{2}$ BPS $=$ Bogomol'nyi-Prasad-Sommerfield

[^2]:    ${ }^{3}$ Spherical defects are discussed in e.g. [14].

[^3]:    ${ }^{1}$ This makes no difference for the two-point function that we are considering, but could affect the computation for the two-point function of operators not having the same number of scalar fields in the trace, e.g. $k_{1}=2$ and $k_{2}=3$ in eq. (1.38). In this case it is easy to imagine one diagram with one point on the Wilson line and no vertex. Such a graph would vanish with $\mathfrak{s u}(N)$ as a gauge group, but not with $\mathfrak{u}(N)$.

[^4]:    ${ }^{1}$ Note that we expect the difference to be greater at NNLO, where a fourth R-symmetry channel would appear for $k \geq 3$ but not for $k=2$.

[^5]:    ${ }^{1}$ This feature allows us to assess that there is no similar integral identity for the H-diagram of the direct channel given in eq. (4.33). Because of the uncontracted indices, there always remains a term containing $\left(p_{1}+p_{3}\right)^{-2}$, which cannot be generated neither by X - or Y-integrals.

