# General Relativity

Olaf Hohm

Exercise Sheet 6

July 2, 2019

**Exercise 1: Motion in spherically symmetric gravitational field**

We want to analyze the motion of freely falling objects in a spherically symmetric gravitational field $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ of the form

$$h_{\mu\nu} \;=\; \begin{pmatrix} -2\phi(r) & 0 & 0 & 0 \\ 0 & -2\psi(r) & 0 & 0 \\ 0 & 0 & -2\psi(r) & 0 \\ 0 & 0 & 0 & -2\psi(r) \end{pmatrix} , \tag{1}$$

where $\phi$ and $\psi$ are two a priori independent functions of the radial coordinate $r$.

1.1) Show that the geodesic equation is generally equivalent to

$$\frac{d}{d\tau}(\eta_{\mu\nu}u^\nu + h_{\mu\nu}u^\nu) = \frac{1}{2}\partial_\mu h_{\nu\rho}\, u^\nu u^\rho \;, \tag{2}$$

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity with proper time $\tau$.
*Remark: In this equation $h_{\mu\nu}$ need not be small.*

1.2) Let the curve be parameterized by $x^\mu = (t,x,y,z)$, so that $u^\mu = (\dot t, \dot x, \dot y, \dot z)$, $\dot{} \equiv \frac{\partial}{\partial\tau}$. Evaluate the four equations (2) for (1) and prove the conservation law

$$\frac{d}{d\tau}\big((1-2\psi)(\dot x y - \dot y x)\big) \;=\; 0 \;.$$

Assume next that the motion takes place in the $z=0$ plane and replace $(x,y)$ by polar coordinates $(r,\theta)$. Show conservation of angular momentum:

$$L \equiv (1-2\psi)r^2\dot\theta = \text{const.}$$

1.3) Upon introducing $\rho \equiv \frac{1}{r}$ prove that the motion is governed by the equation

$$\Big(\frac{K^2}{1+2\phi} - 1\Big)\frac{1-2\psi}{L^2} \;=\; \Big(\frac{d\rho}{d\theta}\Big)^2 + \rho^2 \;, \tag{3}$$

where $K$ is an integration constant.
*Hint: Recall the normalization $g_{\mu\nu}u^\mu u^\nu = -1$.*

1.4) Perform an expansion of (3) in powers of $\phi$. Show that the Newtonian limit is obtained to first order in $\phi$ for $\phi = \psi = -\frac{GM}{r}$, i.e., that this reproduces Kepler's elliptical orbits. Show that to next order, and taking the corrections implied by the Schwarzschild solution into account, the equations predict a precession of perihelia.