General Relativity

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Exercise Sheet 5

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Exercise 1: Identities of Riemannian Geometry

1.1) Take $\Gamma^\rho_{\mu\nu}$ to be an a priori undetermined connection (but still assumed to be symmetric in its two lower indices, $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$). Show that the *metricity* constraint

$$\nabla_\mu g_{\nu\rho} \;\equiv\; \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma} \;=\; 0$$

uniquely determines $\Gamma^\rho_{\mu\nu}$ (then called the Levi-Civita connection).

1.2) Let V^μ be a vector field. Verify that its covariant divergence can be written as

$$\nabla_\mu V^\mu \;=\; \frac{1}{\sqrt{-g}} \partial_\mu \big(\sqrt{-g}\, V^\mu \big) \; .$$

1.3) Prove that the variation of the Riemann tensor under arbitrary variations $\delta\Gamma^\rho_{\mu\nu}$ of the Christoffel symbols takes the covariant form

$$\delta R_{\mu\nu}{}^\rho{}_\sigma \;=\; \nabla_\mu \delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu \delta\Gamma^\rho_{\mu\sigma} \; .$$

1.4) Show that the gauge transformation of the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ with respect to ξ^μ can be written in terms of covariant objects as

$$\delta_\xi \Gamma^\rho_{\mu\nu} \;=\; \nabla_\mu \nabla_\nu \xi^\rho + \xi^\sigma R_{\sigma\mu}{}^\rho{}_\nu \; .$$

1.5)* Prove the algebraic Bianchi identity

$$R_{\mu\nu\rho\sigma} + R_{\nu\rho\mu\sigma} + R_{\rho\mu\nu\sigma} \;=\; 0 \; .$$

[*Hint: This identity can be understood as a consequence of* $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$*, using one of the relations above.*]

Exercise 2: Symmetries and Killing vectors

Symmetries in general relativity are generated by *Killing vector fields* (sometimes denoted Killing symmetries), whose Lie derivatives leave the metric invariant,

$$\mathcal{L}_K g_{\mu\nu} \;=\; 0\;. \tag{1}$$

2.1) Prove that the condition (1) is equivalent to the 'Killing equation'

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu \;=\; 0\;,$$

where as usual $K_\mu = g_{\mu\nu}K^\nu$.

2.2) Show that Killing symmetries form an algebra in the sense that if K_1 and K_2 are Killing vectors then the Lie bracket of vector fields $[K_1, K_2]$ is also a Killing vector.

2.3) Consider a sphere of radius R in three-dimensional Euclidean space $\mathbb{E}^3$ and determine its induced (two-dimensional) metric in some convenient (spherical) coordinates. How many Killing symmetries does this metric have?

2.4)* Prove that the second covariant derivative of a Killing vector K can be expressed in terms of the Riemann tensor as

$$\nabla_\mu \nabla_\nu K^\rho \;=\; -R_{\sigma\mu}{}^\rho{}_\nu K^\sigma\;.$$

Comment: the exercises marked by a * can be solved by a trick or else by a brute-force computation.