General Relativity

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Exercise Sheet 4

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Exercise 1: Relations for the metric tensor

In the lecture we postulated the gauge transformation of the metric tensor $g_{\mu\nu}$,

$$\delta_\xi g_{\mu\nu} \;=\; \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho} \; . \tag{1}$$

1.1) Compute the gauge transformation of the inverse metric $g^{\mu\nu}$.

1.2) Prove that the determinant $|g| \equiv \det g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$ can be written as

$$|g| \;=\; \frac{1}{4!}\,\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon^{\alpha\beta\gamma\delta} g_{\mu\alpha}\, g_{\nu\beta}\, g_{\rho\gamma}\, g_{\sigma\delta} \; ,$$

$$g^{\mu\nu} \;=\; \frac{1}{3!}\,\frac{1}{|g|}\,\varepsilon^{\mu\rho\sigma\kappa}\,\varepsilon^{\nu\alpha\beta\gamma}\, g_{\rho\alpha}\, g_{\sigma\beta}\, g_{\kappa\gamma} \; ,$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita symbol with $\varepsilon^{0123} = +1$.

1.3) Determine $\delta\sqrt{-g}$, where $g \equiv |g|$, under arbitrary variations $\delta g_{\mu\nu}$.

Exercise 2: Lie derivatives

Lie derivatives are defined as

$$\mathcal{L}_\xi V^\mu \;=\; \xi^\nu \partial_\nu V^\mu - \partial_\nu \xi^\mu V^\nu \; , \qquad \mathcal{L}_\xi W_\mu \;=\; \xi^\nu \partial_\nu W_\mu + \partial_\mu \xi^\nu W_\nu \; ,$$

and analogously for objects with an arbitrary number of upper and lower indices.

2.1) Prove that the Lie derivative operators $\mathcal{L}_\xi$ satisfy the algebra

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] \;=\; \mathcal{L}_{[\xi_1,\xi_2]} \; ,$$

where [,] on the left-hand side denotes the commutator. In the process, determine the Lie bracket $[\xi_1, \xi_2]$ defined by the right-hand side.

2.2) We defined the *non-covariant variation* $\Delta_\xi \equiv \delta_\xi - \mathcal{L}_\xi$, the difference of the actual gauge transformation and the covariant one given by the Lie derivative. Compute

$$\Delta_\xi \Gamma^\rho_{\mu\nu} \qquad \text{and} \qquad \Delta_\xi(\partial_\mu \Gamma^\rho_{\nu\sigma}) \; ,$$

where $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols for $g_{\mu\nu}$, and express the result, to the extent possible, with matrix notation for $\boldsymbol{\Gamma}_\mu \equiv (\boldsymbol{\Gamma}_\mu)^\rho{}_\nu \equiv \Gamma^\rho_{\mu\nu}$ and $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}^\mu{}_\nu \equiv \partial_\nu \xi^\mu$.

Exercise 3: Expansion of non-linear gravity around flat space

In the lecture it was claimed that the following action describes non-linear gravity:

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\Big(\frac{1}{4}g^{\mu\nu}\partial_\mu g^{\rho\sigma}\partial_\nu g_{\rho\sigma} - \frac{1}{2}g^{\mu\nu}\partial_\mu g^{\rho\sigma}\partial_\rho g_{\sigma\nu} + \partial_\mu g^{\mu\nu}\frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g} + g^{\mu\nu}\Big(\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\Big)\Big(\frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g}\Big)\Big)\,.$$

3.1) Expand the above action around Minkowski space to second order in h, writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,,$$

and show that it reproduces the Fierz-Pauli action.
Hint: Use the result of 1.3) in order to expand $\sqrt{-g}$.

3.2) Take the above expansion of $g_{\mu\nu}$ to be exact (i.e. no higher order terms of h are included on the right-hand side). Determine the complete gauge transformation $\delta_\xi h_{\mu\nu}$ for the fluctuation field from eq. (1) above.