General Relativity

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Exercise Sheet 3

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Exercise 1: Motion of massless particles

The equations for a massless particle parametrized by $x^\mu(\tau)$ are given by

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho \;=\; 0\,, \tag{1}$$

$$g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu \;=\; 0\,, \tag{2}$$

where $\Gamma^\rho_{\mu\nu}$ are the full Christoffel symbols for $g_{\mu\nu}$ and $\dot{} = \frac{d}{d\tau}$.

1.1) Prove that the equation of motion (1) is invariant under infinitesimal reparametrizations of the affine parameter λ,

$$x^\mu \to x^\mu + \delta x^\mu\,, \qquad \delta x^\mu \;=\; \lambda\dot{x}^\mu\,, \qquad \ddot{\lambda} \;=\; 0\,. \tag{3}$$

i.e., for redefinitions of the form $\lambda = a\tau + b$, with $a, b \in \mathbb{R}$ constant.

1.2) Prove that the constraint (2) is preserved under time evolution.

Exercise 2: Bending of light in gravitational field

In the lecture we showed that light is deflected in a gravitational field with potential $\phi = \phi(\vec{r})$. The deflection vector is given by the integral

$$\Delta\vec{\ell} \;=\; -2k\int \vec{\nabla}_\perp\phi\, dx \tag{4}$$

along the x-axis, where the gradient transverse $\vec{\nabla}_\perp$ is defined by

$$\vec{\nabla}_\perp\phi \;:=\; \vec{\nabla}\phi - k^{-2}(\vec{k}\cdot\vec{\nabla}\phi)\vec{k} \tag{5}$$

in terms of the null vector $k^\mu := \frac{dx^{(0)\mu}}{d\tau} = (k, \vec{k})$, $|\vec{k}|^2 = k^2$, with $x^{(0)\mu}$ the trajectory of the unperturbed light ray.

2.1) Take the unperturbed light ray to be along the x-axis and the gravitational potential to be sourced by a mass M,

$$\phi(\vec{r}) \;=\; -\frac{GM}{r}\,, \tag{6}$$

which we assume to be located at $(0, -b, 0)$. Determine the deflection angle $\alpha = \frac{|\Delta\vec{\ell}|}{k}$, using the integral (4) along the x-axis from $-\infty$ to $+\infty$.

2.2) Estimate the deflection of light passing by the sun, as becomes measurable during a total eclipse.