General Relativity
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Disclaimer: the following has not yet been proofread.

Preliminaries

Calculus of Variations
A function maps a number x and to another number $f(x)$:

$$\mathbb{R} \ni x \mapsto f(x) \in \mathbb{R}\,. \tag{1}$$

The derivative of a function is defined as:

$$f'(x) = \frac{df}{dx} := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\,. \tag{2}$$

The Taylor expansion of a function is:

$$f(x+h) - f(x) = f'(x)h + \mathcal{O}(h^2)\,, \tag{3}$$

$$h = \Delta x\,, \quad \delta f(x) = f(x+h) - f(x) = f'(x)\Delta x\,. \tag{4}$$

Let us extend this concept to functionals. A functional S maps a function $\varphi(x)$ to a number $S[\varphi(x)] \in \mathbb{R}$. Let $\varphi(x) \to \varphi(x) + \epsilon\xi(x)$, where ϵ is small. The variational derivative of the functional $S[\varphi(x)]$ is

$$\delta_\varphi S := S[\varphi(x) + \epsilon\xi(x)] - S[\varphi(x)] \equiv \int dx \frac{\delta S}{\delta\varphi} \epsilon\xi(x) + \mathcal{O}(\epsilon^2)\,. \tag{5}$$

This is the implicit definition of functional derivative $\frac{\delta S}{\delta \varphi}$. Usually, we denote $\delta\varphi(x) \equiv \epsilon\xi(x)$. The variational principle states that $\delta S \stackrel{!}{=} 0$, $\forall \delta\varphi$. If we demand the variational principle,

$$\delta S = \int dx \frac{\delta S}{\delta\varphi}\delta\varphi = 0 \;\text{ only if }\; \frac{\delta S}{\delta\varphi} = 0\,, \tag{6}$$

one obtains the equations of motion.

Consider the example:

$$S[\varphi(x)] = \int dx\varphi^2(x)\,. \tag{7}$$

We compute:

$$\begin{aligned} S[\varphi(x) + \epsilon\xi(x)] &= \int dx(\varphi(x) + \epsilon\xi(x))^2 \\ &= \int dx(\varphi(x)^2 + 2\epsilon\varphi(x)\xi(x) + \mathcal{O}(\epsilon^2)) = S[\varphi(x)] + \int dx 2\varphi(x)\epsilon\xi(x)\,, \end{aligned} \tag{8}$$

$$\delta S = \int dx 2\varphi(x)\epsilon\xi(x) \Longrightarrow \frac{\delta S}{\delta\varphi(x)} = 2\varphi(x)\,. \tag{9}$$

You can see that taking functional derivatives is similar to taking ordinary derivatives. Let us consider the arbitrary function $f(x)$,

$$S[\varphi(x)] := \int dx f(\varphi(x)) \tag{10}$$

$$\begin{aligned} S[\varphi(x) + \epsilon\xi(x)] &= \int dx f(\varphi(x) + \epsilon\xi(x)) \\ &= \int dx\big(f(\varphi(x)) + f'(\varphi(x))\epsilon\xi(x) + \mathcal{O}(\epsilon^2)\big) \end{aligned} \tag{11}$$

$$\delta_\varphi S(x) = \int dx f'(\varphi(x))\delta\varphi(x) = \int dx f'(\varphi(x))\epsilon\xi(x) \tag{12}$$

Above we have applied the chain rule and the Leibniz rule. Consider another example:

$$S[\varphi] = \int dx e^{-\alpha\varphi(x)}(\varphi')^2\,, \tag{13}$$

$$\begin{aligned} \delta_\varphi S &= \int dx\Big\{ -\alpha\delta\varphi e^{-\alpha\varphi}(\varphi')^2 + e^{-\alpha\varphi}(2\varphi'\delta(\varphi')\Big\} \\ &= \int dx\Big\{ -\alpha\delta\varphi e^{-\alpha\varphi}(\varphi')^2 + e^{-\alpha\varphi}(2\varphi'\delta\Big(\frac{d\varphi}{dx}\Big)\Big\} \\ &= \int dx\Big\{ -\alpha\delta\varphi e^{-\alpha\varphi}(\varphi')^2 + e^{-\alpha\varphi}(2\varphi'\delta\frac{d}{dx}(\delta\varphi)\Big)\Big\} \\ &= \int dx\delta\varphi\Big\{ -\alpha e^{-\alpha\varphi}(\varphi')^2 - \frac{d}{dx}\Big(2e^{-\alpha\varphi}\varphi'\Big)\Big\}\,. \end{aligned} \tag{14}$$

Now let us review the Dirac delta distribution. Let $x \in \mathbb{R}$ be fixed, then $S[\varphi] = \varphi(x) \in \mathbb{R}$ is the Dirac delta functional, related to the more familiar physicist's notation by

$$S[\varphi] = \int dx'\delta(x - x')\varphi(x') \tag{15}$$

$$\delta S[\varphi] = \delta\varphi(x) = \int dx' \delta(x-x')\delta\varphi(x') = \int dx' \frac{\delta S}{\delta\varphi(x')}\delta\varphi(x') \tag{16}$$

$$\frac{\delta S[\varphi]}{\delta\varphi(x')} = \frac{\delta\varphi(x)}{\delta\varphi(x')} = \delta(x-x') \tag{17}$$

$\frac{\delta\varphi(x)}{\delta\varphi(x')} = \delta(x-x')$ is, so to say, the infinite-dimensional form of familiar relation $\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu{}_\nu$ from multi-variable calculus.

Lecture 1

Introduction

Lecture 2

Special Relativity

Let $\mathbb{M}$ be Minkowski space (which is $\mathbb{R}^4$ as a vector space). Take $x^\mu \in \mathbb{M}$, $\mu, \nu = 0, ..., 3$. Then let us define:

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,, \tag{18}$$

$$x^2 = \langle x, x\rangle = \eta_{\mu\nu}x^\mu x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2\,. \tag{19}$$

It is useful to think about invariant objects. There is an invariant notion of distance between two points in Minkowski spacetime:

$$I(p,q) = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu\,. \tag{20}$$

Instead of rulers and compasses, in special relativity, we use clocks and light rays.
Exercise:
Let A and B be two observers. A moves and is constantly sending light rays in the direction of B. Assume that the light ray that hits B is at some point reflected back towards A. Show that the invariant spacetime interval is $I(p,q) = -ab$.

Motion in Minkowski Space

Consider a general curve. How do we determine the invariant interval between p and q for a general curve? Let us introduce coordinates, $x^\mu(\tau)$, parameterized by $\tau \in I \subset R$. This is the parameterization of the curve c, for $I = [a,b]$, $x(a) = p$, $x(b) = q$. What is the invariant length along the curve? Assume this is a timelike curve, meaning that the derivative at each point is timelike, $\dot{x}^2 < 0$. The invariant length of a section along the curve is

$$\Delta s_i = \sqrt{-\eta_{\mu\nu}\Delta x^\mu x^\nu}\,. \tag{21}$$

The total length of the curve is thus

$$S(p,q) = \sum_i \Delta s_i = \sum_i \sqrt{-\eta_{\mu\nu}\Delta x_i^\mu \Delta x_i^\nu}\,. \tag{22}$$

In the continuum limit the sum becomes an integral:

$$S(p,q) = \int_c ds = \int_I \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau\,. \tag{23}$$

This is the invariant length between p and q. (23) is a good candidate to determine the equation of motion of a free particle. Let us introduce the notation,

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau}\,,\ \dot{x}^2 = \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,. \tag{24}$$

Let us then write the action,

$$S[x(\tau)] = \int \sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\tau\,. \tag{25}$$

Consider an arbitrary, small variation $x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)$. The equations of motion are

$$\dot{u}^\mu = \frac{du^\mu}{d\tau} = 0\,, \quad \text{where } u^\mu := \frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}}\,. \tag{26}$$

u^μ is known as the 4-velocity. You see immediately as a consequence of the definition of u^μ that

$$u^2 = -1\,. \tag{27}$$

Exercise: show that $\delta S = \int d\tau \delta x^\mu \frac{du_\mu}{d\tau}$.

Now we have determined the equations of motion for a free particle. How many solutions should we expect? How many initial data can we specify? (24) is a second order differential equation for four functions x^μ. Naively, we could specify $x^\mu(0) = x_0^\mu$ and $\dot{x}^\mu(0) = u_0^\mu$. This is not how it works, because there is one constraint,

$$u_\mu \dot{u}^\mu = 0\,, \tag{28}$$

which is identically satisfied (by using the definition of u^μ). One can see this by (27),

$$0 = \frac{d}{d\tau}u^2 = 2u_\mu\dot{u}^\mu\,. \tag{29}$$

Thus, there is a redundancy in the formulation. This comes from reparameterization invariance. Consider the reparameterization, $\tau \to \tau' = f(\tau)$. x^μ is a scalar and transforms as

$$x^{\mu\prime}(\tau') = x^\mu(\tau)\,. \tag{30}$$

Consequently,

$$\frac{dx^{\mu\prime}(\tau')}{d\tau'} = \frac{dx^\mu(\tau)}{d\tau'} = \frac{dx^\mu}{d\tau}\frac{d\tau}{d\tau'}\,. \tag{31}$$

The action (25) is invariant under reparameterization:

$$\begin{aligned} S[x'(\tau')] &= \int d\tau' \sqrt{-\eta_{\mu\nu}\frac{dx^{\mu\prime}}{d\tau'}\frac{dx^{\nu\prime}}{d\tau'}} \\ &= \int d\tau' \sqrt{-\eta_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}}\left|\frac{d\tau}{d\tau'}\right| \\ &= \int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \\ &= S[x(\tau)]\,. \end{aligned} \tag{32}$$

This explains why we have the constraint, because it can be interpreted as a Bianchi identity for reparameterization invariance as follows: Instead of performing a finite reparameterization, consider an infinitesimal reparameterization, $\tau \to \tau' = \tau - \lambda(\tau)$, where λ is an arbitrary and small function of τ.
Exercise: denote $\delta x^\mu = x^{\mu\prime}(\tau) - x^\mu(\tau)$; prove that $\delta x^\mu = \lambda \dot{x}^\mu$.
The variation that is induced by this change of parameterization reads

$$\delta_\lambda S = \int d\tau \lambda(\tau)\dot{x}^\mu \dot{u}_\mu \stackrel{!}{=} 0\,,\ \forall\lambda(\tau)\,. \tag{33}$$

This one-dimensional reparameterization invariance, also called diffeomorphism invariance, implies the Bianchi identity, $\dot{x}^\mu \dot{u}_\mu = 0$. It is often convenient to introduce a parameter for the curve in the formulation of an action principle, but the price to pay is a certain redundancy. We can make a choice, called "gauge fixing." We can choose $\tau \to s(\tau)$, and simply take the invariant length itself to be the parameter to parametrize the curve:

$$\begin{aligned} s(\tau) &= \int_0^\tau ds = \int_0^\tau d\tau' \sqrt{-\dot{x}^\mu \dot{x}_\mu}\,, \\ \frac{ds}{d\tau} &= \sqrt{-\dot{x}^2}\,. \end{aligned} \tag{34}$$

The 4-velocity then takes a simple form,

$$u^\mu = \frac{dx^\mu}{d\tau}\frac{1}{\sqrt{-\dot{x}^2}} = \frac{1}{\sqrt{-\dot{x}^2}}\frac{dx^\mu}{ds}\frac{ds}{d\tau} = \frac{dx^\mu}{ds}\,. \tag{35}$$

Then $\dot{x}^\mu(s)\dot{x}_\mu(s) = -1$. The initial conditions in terms of proper time can be specified:

$$x^\mu(0) = x_0^\mu\,, \quad \dot{x}^\mu(0) = u_0^\mu\,, \quad u_0^\mu u_{0\mu} = -1\,. \tag{36}$$

The equations of motion read

$$\frac{d^2 x^\mu}{ds^2} = 0\,, \tag{37}$$

whose solution is a straight line in Minkowski space,

$$x^\mu(s) = u_0^\mu s + x_0^\mu\,. \tag{38}$$

An alternative parameterization is to demand $\tau = x^0 = t$, t being the time coordinate, so $x^\mu(\tau) = \big(t(\tau), x^i(\tau)\big) = \big(t, x^i(t)\big)$. Every point on the line is labelled by its projection onto the $x^0 = t$ axis. Then

$$\dot{x}^\mu = \left(1, \frac{dx^i}{dt}\right) = (1, v^i)\,, \tag{39}$$

where v^i is the conventional 3-velocity. The invariant length,

$$\sqrt{-\dot{x}^2} = \sqrt{1-|v|^2} = \frac{1}{\gamma}\,, \tag{40}$$

contains the familiar gamma factor $\gamma = \frac{1}{\sqrt{1-v^2}}$. The 4-velocity is given by

$$u^\mu(t) = (\gamma, \gamma v^i)\,. \tag{41}$$

Then we can define the 4-momentum,

$$p^\mu := m u^\mu = (\gamma m, \gamma m v^i)\,, \quad \text{where } m \text{ is the rest mass.} \tag{42}$$

The 4-momentum satisfies

$$p^\mu p_\mu = -m^2\,. \tag{43}$$

Consider the action for a particle in terms of these parameters.

$$\begin{aligned} S_{particle} &= -mc\int ds \\ &= -mc\int d\tau\sqrt{1-\frac{v^2}{c^2}} \end{aligned} \tag{44}$$

The sign in front of the integral is conventional so that the extremum of $S_{particle}$ is a minimum. Note that this action is not manifestly Lorentz invariant, but of course it is Lorentz invariant by construction. Let us expand the action in $\frac{1}{c^2}$, to see the relativistic corrections:

$$S_{particle} \simeq \frac{1}{c}\left[-mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\frac{1}{c^2}\right)\right]. \tag{45}$$

Maxwell's Theory

Consider Maxwell's theory of electrodynamics in the Lorentz covariant formulation. The electromagnetic field is described by a 4-potential, $A_\mu(x)$ on Minkowski space $\mathbb{M}$, for which the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $\partial_\mu := \frac{\partial}{\partial x^\mu}$. Let us split the indices, $\mu = (0, i)$, $i = 1, 2, 3$. Think of a specific observer with respect to a rest frame. With respect to the observer:

$$\begin{aligned} F_{00} &= 0\,, \\ F_{0i} &= E_i\,, \\ F_{ij} &= -\epsilon_{ijk}B^k\,, \end{aligned} \tag{46}$$

where E_i is the electric field, B_i is the magnetic field, and ϵ_{ijk} is the Levi-Civita symbol. The above can be collected in a matrix:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} . \tag{47}$$

There is a gauge symmetry:

$$A_\mu \to A_\mu + \delta A_\mu \,, \quad \delta A_\mu = \partial_\mu \Lambda \tag{48}$$

where $\Lambda = \Lambda(x)$ is arbitrary function of x. Since $F_{\mu\nu}$ is gauge invariant (by the fact that partial derivatives commute), it is natural to write a quadratic term in $F_{\mu\nu}$.

$$S[A] = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\int d^4x(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{49}$$

$\partial^\mu := \eta^{\mu\nu}\partial_\nu$ and $A^\mu = \eta^{\mu\nu}A_\nu$. (49) is a good candidate for the action, whose variation reads

$$\begin{aligned} \delta S &= -\frac{1}{2}\int d^4x F^{\mu\nu}\delta F_{\mu\nu} = -\frac{1}{2}\int d^4x F^{\mu\nu}\left[\partial_\mu(\delta A_\nu) - \partial_\nu(\delta A_\mu)\right] \\ &= -\int d^4x F^{\mu\nu}\partial_\mu \delta A_\nu = \int d^4x \delta A_\nu \partial_\mu F^{\mu\nu} = 0 \end{aligned} \tag{50}$$

Thus, the field equations are

$$\partial_\mu F^{\mu\nu} = 0\,. \tag{51}$$

These are Maxwell's equations in a vacuum written in covariant form. How do we couple the electromagnetic field to charged matter? Consider the action:

$$S[x, A] = \int_c -mds - eA_\mu dx^\mu - \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}d^4x\,. \tag{52}$$

Let us write the part of the action describing the interaction as

$$S_{\text{int}} = -e\int A_\mu(x(\tau))\dot{x}^\mu d\tau\,. \tag{53}$$

We need to check that this is gauge invariant:

$$\delta S_{int} = -e\int \partial_\mu \Lambda \dot{x}^\mu d\tau = e\int -\frac{\partial\Lambda}{\partial x^\mu}\frac{dx^\mu}{d\tau}d\tau = e\int -\frac{d\Lambda}{d\tau}d\tau = 0\,, \tag{54}$$

assuming boundary conditions so that $\Lambda(a) = \Lambda(b) = 0$ as $a, b \to \infty$, $\Lambda \to 0$. Consider only the gauge transformations that have certain assumptions near infinity. The equations of motion for x^μ can be derived:

$$\begin{aligned}\delta_x S_{int} &= -e\int\Big(\partial_\nu A_\mu \delta x^\nu \dot{x}^\mu + A_\mu \frac{d}{d\tau}(\delta x^\mu)\Big)d\tau \\ &= -e\int\Big(\partial_\nu A_\mu \delta x^\nu \dot{x}^\mu - \partial_\nu A_\mu \dot{x}^\nu \delta x^\mu\Big)d\tau \\ &= -e\int \delta x^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu)\dot{x}^\nu d\tau \\ &= -e\int \delta x^\mu F_{\mu\nu}\dot{x}^\nu d\tau\,.\end{aligned} \tag{55}$$

The full equation of motion is

$$-m\dot{u}_\mu - eF_{\mu\nu}\dot{x}^\nu = 0\,. \tag{56}$$

Lecture 3

Electrodynamics
Consider a particle of mass m, charge e, described by $x^\mu(\tau)$, and a 4-vector field $A_\mu(x)$ on $\mathbb{M}$. The action describing the system is

$$S[x,A] = \int_c \big(-mds - eA_\mu dx^\mu\big) - \frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu}\,, \tag{57}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. The curve, along which the particle moves, responds according to the vector potential of the electromagnetic field. Let us define a current density,

$$j^\mu(x) := e\int d\tau \dot{x}^\mu(\tau)\delta^{(4)}(x - x(\tau))\,. \tag{58}$$

Here, remember that $x(\tau)$ is the parametrized curve and that x is the coordinate of a point in space. The interaction term in (57) can be written in terms of the current.

$$\begin{aligned}-\int d^4x A_\mu(x) j^\mu(x) &= -e\int d\tau \dot{x}^\mu(\tau)\int d^4x \delta^{(4)}(x - x(\tau))A_\mu(x) \\ &= -e\int d\tau \dot{x}^\mu(\tau)A_\mu(x(\tau)) \\ &= -e\int A_\mu dx^\mu\end{aligned} \tag{59}$$

We can then rewrite the A-dependent part of (57) as

$$S[A] = \int d^4x\Big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - A_\mu j^\mu\Big)\,, \tag{60}$$

where

$$\partial_\mu F^{\mu\nu} = j^\nu\,. \tag{61}$$

Exercise: show that for any test function $\phi(x)$, $\int d^4x\phi(x)\partial_\mu j^\mu = 0$.
The action is invariant under the transformations:

$$\begin{aligned} A_\mu &\to A_\mu + \delta A_\mu \\ \delta A_\mu &= \partial_\mu \Lambda\,. \end{aligned} \tag{62}$$

A Bianchi identity arises from the gauge variation of the action:

$$\delta S[A] = \int d^4x \delta A_\nu \partial_\mu F^{\mu\nu} = \int d^4x \partial_\nu \Lambda \partial_\mu F^{\mu\nu} = -\int d^4x \Lambda(x) \partial_\nu \partial_\mu F^{\mu\nu}\,. \tag{63}$$

The action is gauge invariant, that is, $\delta S[A] = 0$, and

$$\partial_\mu \partial_\nu F^{\mu\nu} = 0\,. \tag{64}$$

(64) is called a Bianchi identity. This is identically satisfied, since

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu F^{\mu\nu} = \partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\mu\nu} = 0\,. \tag{65}$$

Comment:
In the literature you will find the "Bianchi identity" for $F_{\mu\nu}$ as

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} \equiv 3\partial_{[\mu} F_{\nu\rho]} = 0\,, \tag{66}$$

which is identically satisfied. Here we have introduced the notation,

$$\begin{aligned} A_{[\mu\nu]} &:= \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) \\ A_{(\mu\nu)} &:= \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})\,. \end{aligned} \tag{67}$$

The continuity equation can now be derived. The zeroth component of the current density is the charge density, and the current vector encodes the current.

$$j^\mu = (j^0, j^i) = (\rho, \vec{j}) \tag{68}$$

Using the Bianchi identity and (61),

$$\partial_\mu j^\mu = \partial_t \rho + \mathrm{div}\vec{j} = 0\,. \tag{69}$$

Solutions of the Maxwell Equations
Consider the rest frame of some observer in which there are static charges, so the current density is $j^\mu = (\rho(\vec{x}), \vec{0})$. We use the ansatz, $A^\mu = (\phi(\vec{x}), \vec{0})$, where $\phi(\vec{x})$ is a scalar potential. Using (61),

$$j^\nu = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)\,. \tag{70}$$

For $\nu = 0$,

$$\rho = \Big(-\frac{\partial^2}{\partial t^2} + \Delta \Big)\phi - \partial_t(\partial_\mu A^\mu)\,, \tag{71}$$

which results in the Poisson equation,

$$\Delta\phi = \rho\,. \tag{72}$$

For a point-charge at $\vec{v} = 0$: $\rho(\vec{r}) = e\delta(r)$, $\Delta\left(\frac{1}{r}\right) = -4\pi\delta(r)$, the solution to (72) is

$$\phi(\vec{r}) = -\frac{e}{4\pi r}\,, \tag{73}$$

which is the familiar Coulomb potential.

Electromagnetic Waves

Let us assume that there is no matter, i.e. Maxwell equations in a vacuum. In Newtonian theory, if there is no matter nothing happens. But for electromagnetism we have nontrivial equations,

$$E^{\mu} := \partial_{\nu}F^{\nu\mu} = \Box A^{\mu} - \partial^{\mu}(\partial\cdot A) = 0 \tag{74}$$

where $\partial\cdot A = \partial_{\mu}A^{\mu}$.

Let us identify field configurations which are gauge equivalent. Not all 4 components of A^{μ} are physical.

From the Bianchi identity: $\partial_{\mu}E^{\mu} = 0$. As a rule of thumb, each local gauge symmetry removes 2 degrees of freedom (per point in space). For A_{μ} we have $4-2=2$ d.o.f.. The proof can be carried out in light-cone coordinates. Light-cone coordinates are given by:

$$\begin{aligned} x^{+} &= \frac{1}{\sqrt{2}}(x^0 + x^1)\,,\\ x^{-} &= \frac{1}{\sqrt{2}}(x^0 - x^1)\,. \end{aligned} \tag{75}$$

The invariant interval is then

$$-(x^0)^2 + \sum_{i=1}^{3}(x_i)^2 = -2x^{+}x^{-} + \sum_{\alpha=1}^{2}(x_{\alpha})^2 = \eta'_{\mu\nu}x^{\mu}x^{\nu}\,, \tag{76}$$

where

$$\eta'_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{77}$$

Here we have renamed x^0 and x^3 to x^1 and x^2. The vector potential can be written in components as $A^{\mu}(x) = (A^{+}, A^{-}, A^{\alpha})$. In Fourier space,

$$A^{\mu}(x) = \int d^4k a^{\mu}(k)e^{ik_{\nu}x^{\nu}} \tag{78}$$

$$\Lambda(x) = \int d^4k\lambda(k)e^{ik_{\nu}x^{\nu}}\,. \tag{79}$$

The gauge transformations are then given by:

$$\delta_\lambda a_\mu(k) = ik_\mu \lambda(k)\,. \tag{80}$$

$E_\mu \equiv 0$ gives the gauge invariant equation:

$$k^2 a^\mu - k^\mu (k \cdot a) = 0\,. \tag{81}$$

Assume we pick $k^+ \neq 0$,

$$\delta a^+(k) = ik^+ \lambda(k)\,. \tag{82}$$

We can then choose $\lambda = i\frac{a^+(k)}{k^+}$ to impose the gauge-fixing condition ("light-cone gauge")

$$a^+(k) = 0\,. \tag{83}$$

We still have three components left. There is one more to be eliminated. By analyzing the gauge invariant equation, with $\mu = +$,

$$0 = k^2 a^+ - k^+(k \cdot a) \;\to k \cdot a = 0 \tag{84}$$

Let us write out this condition with the η' metric,

$$0 = \eta'_{\mu\nu} k^\mu a^\nu = -k^+ a^- - k^- a^+ - k^\alpha a^\alpha \tag{85}$$

a^- is entirely dependent on k^+, k^α and a^α: $a^- = \frac{1}{k^+} k^\alpha a^\alpha$. Let now $\mu = \alpha$, then

$$k^2 a^\alpha(k) = 0 \tag{86}$$

which is solved by $k^2 = 0$. This now shows that we have two independent degrees of freedom, i.e. two independent solutions of the Maxwell equations, $A^\alpha(x) = a^\alpha(k) e^{ikx}$ for $k^2 = 0$.
This is the dynamical part of the Maxwell equations. Why don't we always choose light-cone coordinates? In the process, we give up manifest Lorentz invariance.

Energy-Momentum Tensor

The energy-momentum tensor can be obtained thourugh an application of Noether's Theorem. From time translation invariance (no explicit t-dependence), one can prove that energy is conserved. From spatial translation invariance (no explicit x-dependence), one can prove that momentum is conserved. Similarly, for a relativistic field theory with Lagrangian $\mathcal{L}[\phi, \partial\phi]$, with ϕ generically denoting the fields, we demand that there is no explicit x^μ dependence. We require four-dimensional translational invariance under $x^\mu \to x^\mu + a^\mu$, where $a^\mu = \text{const}$. Then the fields transform as

$$\phi(x) \to \phi'(x) := \phi(x + a) = \phi(x) + a^\mu \partial_\mu \phi(x) + \ldots\,, \tag{87}$$

and the infinitesimal variation is

$$\delta_a \phi(x) = \phi'(x) - \phi(x) = a^\mu \partial_\mu \phi\,. \tag{88}$$

The invariance condition is

$$\delta_a S = 0\,, \quad \text{for } a = \text{const.} \tag{89}$$

Noether's theorem can now be derived by a trick: We first promote a^μ to a spacetime dependent vector $a^\mu(x)$. Then, of course, the action is generally no longer invariant, but we can still conclude that its variation must be writable as

$$\delta_a S(x) = -\int d^4x \partial_\mu a_\nu T^{\mu\nu}\,. \tag{90}$$

Indeed, for constant a^μ we then have $\delta_a S = 0$, as assumed. Second, we can now prove that $T^{\mu\nu}$ is conserved in the sense of satisfying $\partial_\mu T^{\mu\nu} = 0$. To this end we integrate by parts,

$$\delta_a S(x) = \int d^4x a_\nu \partial_\mu T^{\mu\nu} \doteq 0\,, \tag{91}$$

and use that on-shell the action is invariant under arbitrary variations. Since this holds for arbitrary $a(x)$, we infer the conservation equation

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{92}$$

$T_{\mu\nu}$ is called the energy-momentum tensor.
As a remark, for a macroscopic body/fluid, $T^{\mu\nu} = (p+\rho)u^\mu u^\nu - p\eta^{\mu\nu}$.
Exercise: show that for

$$S[A] = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu}\,, \tag{93}$$

the energy-momentum tensor is $T^{\mu\nu} = F^{\mu\rho}F^\nu{}_\rho - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}$, *for the transformations given by* $\delta A_\mu = a^\nu \partial_\nu A_\mu + \partial_\mu \Lambda$, $\Lambda = -a^\nu A_\nu$. *Hint: rewrite* $\delta A_\mu = a^\nu F_{\nu\mu}$.

Comment: from $\partial_\mu T^{\mu\nu} \doteq 0$, *the conserved charges are energy density,* $M = \int d^3x T_{00}$ *and momentum density,* $p^i = \int d^3x T^{i0}$.

Lecture 4

Relativistic Field Theory of Gravity
In analogy to Maxwell's theory (see (60)),

$$\begin{aligned} S[A] &= -\frac{1}{4}\int d^4x\big(F^{\mu\nu}F_{\mu\nu} + A_\mu j^\mu\big) \\ &= \int d^4x\Big(-\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 - A_\mu j^\mu\Big)\,. \end{aligned} \tag{94}$$

Recall the invariance under the gauge transformation, $A_\mu \to A_\mu + \delta A_\mu$, $\delta A_\mu = \partial_\mu \Lambda$. The energy-momentum tensor is proper relativistic presentation of mass/energy, which should be conserved, i.e. $\partial^\mu T_{\mu\nu} = 0$. For gravity, the idea is to take a symmetric tensor, $h_{\mu\nu} = h_{\nu\mu}$, on Minkowski space $\mathbb{M}$. In the action, we want the structure to be of this sort,

$$S[h] = \int d^4x(\text{“}\partial h \partial h\text{”} + h_{\mu\nu}T^{\mu\nu}) \,, \tag{95}$$

in order to immediately couple to the stress-energy tensor and to obtain second-order equations of motion. In analogy to electrodynamics, we expect to have a gauge symmetry, which shifts the metric tensor by a symmetric gradient,

$$\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \,, \tag{96}$$

where S is gauge-invariant if the stress-energy tensor is conserved. The kinetic terms for h should be quadratic in h, with two derivatives ∂_μ, and be Lorentz invariant. Comparing with (94) and defining a Lorentz invariant object which is the trace of $h_{\mu\nu}$, $h := \eta^{\mu\nu}h_{\mu\nu}$, we try to write down all possible allowed terms:

$$S[h] = \int d^4x \bigg(a\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} + b\partial_\mu h^{\mu\nu}\partial^\rho h_{\rho\nu} + c\partial_\mu h^{\mu\nu}\partial_\nu h + d\partial^\mu h \partial_\mu h \bigg) \,. \tag{97}$$

We demand the invariance under the transformation (96). Let us also introduce the notation, $\Box := \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial^\mu\partial_\mu$. We have four coefficients and we want to fix them by demanding gauge invariance:

$$\begin{aligned} 0 \stackrel{!}{=} \delta_\xi S[h] &= \int d^4x \bigg(2a\partial^\mu h^{\nu\rho}\delta_\xi(\partial_\mu h_{\nu\rho}) + 2b\partial_\mu h^{\mu\nu}\delta_\xi(\partial^\rho h_{\rho\nu}) \\ &\quad + c\delta_\xi(\partial_\mu h^{\mu\nu})\partial_\nu h + c\partial_\mu h^{\mu\nu}\delta_\xi(\partial_\nu h) + 2d\partial^\mu h \delta_\xi(\partial_\mu h) \bigg) \\ &= \int d^4x \bigg(4a\partial^\mu h^{\nu\rho}\partial_\mu\partial_\nu\xi_\rho + 2b\partial_\mu h^{\mu\nu}\left[\Box\xi_\nu + \partial_\nu(\partial\cdot\xi)\right] \\ &\quad + c\big[\Box\xi^\nu + \partial^\nu(\partial\cdot\xi)\big]\partial_\nu h + 2c\partial_\mu h^{\mu\nu}\partial_\nu(\partial\cdot\xi) + 4d\partial^\mu h\partial_\mu\partial\cdot\xi \bigg) \,, \end{aligned} \tag{98}$$

where we have used

$$\delta_\xi(\partial_\mu h^{\mu\nu}) = \Box\xi^\nu + \partial_\mu\partial^\nu\xi^\mu \,. \tag{99}$$

We are free to rescale the action by an overall factor, so we can choose one of the coefficients. We fix $a = -\frac{1}{4}$. Can we now fix b, c, and d? We expect the following terms to cancel:

$$-\partial^\mu h^{\nu\rho}\partial_\mu\partial_\nu\xi_\rho + 2b\partial_\mu h^{\mu\nu}\Box\xi_\nu \,. \tag{100}$$

By integration by parts, we find that $b = \frac{1}{2}$. By isolating terms with $\partial\cdot\xi$ and $h^{\mu\nu}$,

$$2b\partial_\mu h^{\mu\nu}\partial_\nu(\partial\cdot\xi) + c\partial_\mu h^{\mu\nu}\partial_\nu(\partial\cdot\xi) = 0 \,, \tag{101}$$

we find that $c = -\frac{1}{2}$. By observing the other terms containing h and $\partial\cdot\xi$,

$$-\frac{1}{2}\Box\xi^\nu\partial_\nu - \frac{1}{2}\partial^\nu(\partial\cdot\xi)\partial_\nu h + 4d\partial^\mu h\partial_\mu(\partial\cdot\xi) \,, \tag{102}$$

and after integration by parts, we find that $d = \frac{1}{4}$. Inserting all the coefficients back into (97), we obtain

$$S_{FP} = \int d^4x \bigg(-\frac{1}{4}\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\nu}\partial^\rho h_{\rho\nu} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{4}\partial^\mu h\partial_\mu h\bigg) . \tag{103}$$

This is known as the Fierz-Pauli action. Let us couple the gravitational field to some matter.

$$S = S_{FP}[h] + 8\pi G \int d^4x h_{\mu\nu}T^{\mu\nu} + \text{“matter kinetic energy”} , \tag{104}$$

where G is Newton's constant. The Fierz-Pauli action can be written as follows,

$$S_{FP} = -\frac{1}{2}\int d^4x h^{\mu\nu}G_{\mu\nu}(h) , \tag{105}$$

where $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu} = R_{\rho\mu}{}^{\rho}{}_{\nu}$ is the Ricci tensor,

$$R_{\mu\nu\rho\sigma} := -\partial_\mu\partial_{[\rho}h_{\sigma]\nu} + \partial_\nu\partial_{[\rho}h_{\sigma]\mu} \tag{106}$$

is the Riemann tensor, $R := R^\mu{}_\mu$ is the Ricci scalar.
Exercise: prove that (103) can be written as (105).
The variation of the Fierz-Pauli action is

$$\delta S_{FP} = -\int d^4x \delta h^{\mu\nu}G_{\mu\nu}(h) . \tag{107}$$

The Einstein tensor satisfies the Bianchi identity $\partial^\mu G_{\mu\nu} \equiv 0$. The gravitational field equations from (104) are:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} = 8\pi G T_{\mu\nu} . \tag{108}$$

For completeness, let us take the trace of the field equations:

$$R - \frac{1}{2}R\eta^{\mu\nu}\eta_{\mu\nu} = -R = 8\pi G T , \tag{109}$$

(since $\eta^{\mu\nu}\eta_{\mu\nu} = 4$) where $T = \eta_{\mu\nu}T^{\mu\nu}$. We can insert this into (108) and write

$$R_{\mu\nu} = 8\pi G\bigg(T_{\mu\nu} - \frac{1}{2}T\eta_{\mu\nu}\bigg) . \tag{110}$$

Gravitational Waves (in Vacuum)
In vacuum, the field equations are

$$G_{\mu\nu} = 0 . \tag{111}$$

$h_{\mu\nu}$ is a symmetric tensor in four dimensions and thus has 10 components. ξ_μ has 4 components. From the gauge symmetry,

$$h_{\mu\nu} \simeq h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu . \tag{112}$$

The four constraints and four components of the gauge parameter reduce the physical degrees of freedom to $10 - 4 - 4 = 2$.

Exercise: prove that $h_{\mu\nu}$ only has 2 physical degrees of freedom using light-cone gauge.

Let us assume the Lorentz gauge fixing condition

$$\partial^\mu \bar{h}_{\mu\nu} = 0 \,, \text{where } \bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \,. \tag{113}$$

The gauge transformation of this condition is

$$\begin{aligned} \delta(\partial^\mu \bar{h}_{\mu\nu}) &= \partial^\mu \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2} [2(\partial \cdot \xi) \eta_{\mu\nu}] \right) \\ &= \Box \xi_\nu + \partial_\nu (\partial \cdot \xi) - \partial_\nu (\partial \cdot \xi) \\ &= \Box \xi_\nu \,. \end{aligned} \tag{114}$$

If $\Box \xi_\nu = 0$, we maintain the gauge fixing condition. We can express the linearized Einstein tensor in terms of $\bar{h}$. We can use the relation, $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \eta_{\mu\nu}$, which one can derive from the definition of $\bar{h}$. The linearized Einstein tensor is then rewritten as

$$G_{\mu\nu} = -\frac{1}{2} \Box \bar{h}_{\mu\nu} + \partial_{(\mu} \partial^\rho \bar{h}_{\nu)\rho} - (\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma}) \eta_{\mu\nu} \,. \tag{115}$$

We can set any divergence of $\bar{h}$ to zero by the gauge fixing condition. Inserting this into the vacuum field equation (111), we obtain the wave equation,

$$\Box \bar{h}_{\mu\nu} = 0 \,. \tag{116}$$

This can be solved by the plane-wave ansatz:

$$\bar{h}_{\mu\nu} = C_{\mu\nu} e^{ikx} \,, \tag{117}$$

where $C_{\mu\nu}$ is constant. Inserting the ansatz into (116), we find that k must be a null vector, i.e. $k^2 = 0$. By inserting the ansatz into the gauge condition (113),

$$0 = \partial^\mu \bar{h}_{\mu\nu} = i k^\mu C_{\mu\nu} e^{ikx} \quad \rightarrow \quad k^\mu C_{\mu\nu} = 0 \,. \tag{118}$$

Let us choose a basis in which $k^\mu = (\omega, 0, 0, \omega)$, such that $k^\mu k_\mu = 0$ and describes a wave traveling in the x^3 direction. Taking C to be traceless, the two degrees of freedom are given by C_{11}, C_{12}:

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{119}$$

Gravitostatic Case

Let the energy-momentum tensor encode some matter distribution of particles that do not move at high speeds:

$$T_{\mu\nu} = \rho u_\mu u_\nu \,, \tag{120}$$

where the 4-velocities are $u^\mu = (1,0,0,0)$ and $u_\mu = (-1,0,0,0)$ for $x^\mu(\tau) = (\tau,0,0,0)$. We want to investigate what the gravitational field of this point particle is. The field equations are then:

$$G_{\mu\nu} = -\frac{1}{2}\Box\bar{h}_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{121}$$

The static solution is given by

$$\bar{h}_{00} = -4\phi(\vec{r}) \tag{122}$$

and the wave equation gives the Poisson equation:

$$\Box\phi(\vec{r}) = \nabla\phi(\vec{r}) = 4\pi G\rho\,. \tag{123}$$

Inserting the density of a point particle of mass m,

$$\rho(r) = m\delta(r)\,, \tag{124}$$

and using $\nabla\left(\frac{1}{r}\right) = -4\pi\delta(r)$, we obtain the Newtonian potential,

$$\phi(r) = -G\frac{m}{r}\,. \tag{125}$$

We have recovered Newton's theory of gravity as a special case of the theory we have developed here. We can write down the matrix form of h by using

$$\bar{h} := \eta^{\mu\nu}\bar{h}_{\mu\nu} = \eta^{00}\bar{h}_{00} = -\bar{h}_{00} = 4\phi\,, \tag{126}$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\bar{h}\eta_{\mu\nu} = \bar{h}_{\mu\nu} - 2\phi\eta_{\mu\nu}\,, \tag{127}$$

so that

$$h_{\mu\nu} = \begin{pmatrix} -2\phi & 0 & 0 & 0 \\ 0 & -2\phi & 0 & 0 \\ 0 & 0 & -2\phi & 0 \\ 0 & 0 & 0 & -2\phi \end{pmatrix}. \tag{128}$$

Lecture 5

Field Theory of Gravity

We have so far introduced the field $h_{\mu\nu}$ and we have constructed an action that is manifestly Lorentz invariant and that has this symmetry given by $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. We have also been able to define an invariant object, the Riemann tensor, $R_{\mu\nu\rho\sigma}$, (see (106)) which is the analog of the gauge invariant electromagnetic field strength $F_{\mu\nu}$. We have also defined the Ricci tensor, $R_{\mu\nu}$, and the Ricci scalar, $R = \eta^{\mu\nu}R_{\mu\nu}$.

Now we will consider massive point particles in a gravitational field $h_{\mu\nu}(x)$. In order to do this, we will switch on an interaction term,

$$S_{\text{int}} = \frac{1}{2}\int d^4x h_{\mu\nu}T^{\mu\nu}\,. \tag{129}$$

Let the worldline of the point particle x be parameterized by τ. The energy-momentum tensor then reads

$$T^{\mu\nu} := m \int d\tau n^{-1} \dot{x}^{\mu}(\tau) \dot{x}^{\nu}(\tau) \delta^{(4)}(x - x(\tau)) \,, \tag{130}$$

where n, transforming as $n(\tau') = \frac{d\tau}{d\tau'} n(\tau)$, is known as the lapse function. This is analagous to the current density which was used to couple a charged particle to the electromagnetic field. Note that $n(\tau)$ was introduced to maintain the reparameterization invariance of $T^{\mu\nu}$. Inserting the ansatz (130) into (129).

$$\begin{aligned} S_{\text{int}} &= \frac{1}{2} m \int d\tau n^{-1} \dot{x}^{\mu}(\tau) \dot{x}^{\nu}(\tau) \int d^4 x h_{\mu\nu} \delta^{(4)}(x - x(\tau)) \\ &= \frac{1}{2} m \int d\tau n^{-1} h_{\mu\nu}(x(\tau)) \dot{x}^{\mu}(\tau) \dot{x}^{\nu}(\tau) \,, \end{aligned} \tag{131}$$

Recall the point particle action and insert a mass term m:

$$S_{particle} = \frac{1}{2} \int d\tau \big(n^{-1} \eta_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} - m^2 n \big) \,. \tag{132}$$

The equation of motion for n is

$$n = \frac{1}{m} (\sqrt{-\dot{x}^2}) \,, \tag{133}$$

which simplifies (132) to

$$S_{particle} = -m \int d\tau (\sqrt{-\dot{x}^2}) = -m \int ds \,. \tag{134}$$

We rewrite (131) as

$$S_{int} = \frac{1}{2} \int d\tau \big\{ n^{-1} (\eta_{\mu\nu} + m h_{\mu\nu}(x(\tau)) \big\} \dot{x}^{\mu}(\tau) \dot{x}^{\nu}(\tau) - m^2 n(\tau) \,. \tag{135}$$

So far we have taken the Minkowski metric to encode the geometry and define the light-cones. In general relativity, the geometry is encoded by the deviation of the Minkowksi metric. Let us introduce the notation,

$$g_{\mu\nu} := \eta_{\mu\nu} + m h_{\mu\nu} \,, \tag{136}$$

and write the action in terms of $g_{\mu\nu}$.

$$S = \frac{1}{2} \int d\tau \big\{ n^{-1} g_{\mu\nu}(x(\tau)) \dot{x}^{\mu}(\tau) \dot{x}^{\nu}(\tau) - m^2 n(\tau) \big\} \tag{137}$$

Now our task is to find the equation of motion of this particle and to find out how the particle responds to the gravitational field. Taking the variation of the action with respect to $x(\tau)$ and using the chain rule and integration by parts, we obtain

$$
\begin{aligned}
\delta_x S &= \frac{1}{2}\int d\tau n^{-1}\Big[\delta\big(g_{\mu\nu}(x)\big)\dot{x}^\mu\dot{x}^\nu + 2g_{\mu\nu}(x(\tau))\Big(\frac{d}{d\tau}\delta x^\mu\Big)\dot{x}^\nu\Big] \\
&= \frac{1}{2}\int d\tau n^{-1}\Big[\partial_\rho g_{\mu\nu}\delta x^\rho \dot{x}^\mu\dot{x}^\nu + \delta x^\mu\Big(-2\frac{d}{d\tau}\big(g_{\mu\nu}(x(\tau))n^{-1}\dot{x}^\nu\big)\Big)\Big] \\
&= \frac{1}{2}\int d\tau \delta x^\mu\Big(n^{-1}\partial_\mu g_{\nu\rho}\dot{x}^\nu\dot{x}^\rho - 2n^{-1}\partial_\rho g_{\mu\nu}\dot{x}^\nu\dot{x}^\rho - 2g_{\mu\nu}\frac{d}{d\tau}u^\nu\Big),
\end{aligned}
\tag{138}
$$

where $u^\mu := \frac{1}{n}\dot{x}^\mu$. The equation of motion given from $\delta_x S$ is:

$$
\begin{aligned}
g_{\mu\nu}\frac{du^\nu}{d\tau} &= \frac{1}{2}n^{-1}\big(\partial_\mu g_{\nu\rho} - 2\partial_\rho g_{\mu\nu}\big)\dot{x}^\nu\dot{x}^\rho \\
&= -n^{-1}\frac{1}{2}\big(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\nu\rho}\big)\dot{x}^\nu\dot{x}^\rho\,.
\end{aligned}
\tag{139}
$$

We define

$$
\Gamma_{\nu\rho|\mu} := \frac{1}{2}\big(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\nu\rho}\big) \tag{140}
$$

as the Christoffel symbol. Let us assume that $g_{\mu\nu}$ is invertible, and the inverse is written as $g^{\mu\nu}$ which satisfies $g^{\mu\nu}g_{\nu\rho} = \delta^\mu{}_\rho$. If $g_{\mu\nu}$ has an inverse, then the Christoffel symbol can be written (in the more common notation) as:

$$
\Gamma^\rho_{\mu\nu} := \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})\,. \tag{141}
$$

Writing the equation of motion in terms of the Christoffel symbol,

$$
\frac{1}{n}\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = 0\,. \tag{142}
$$

This is the geodesic equation. Let us consider the particle moving in a static gravitational field and assume that it is moving slowly and its worldline is described by $x^\mu(t) = (t, x^i(t))$, where t is the coordinate time.

$$
u^\mu = \frac{1}{n}\dot{x}^\mu = m\gamma(t, v^i) \tag{143}
$$

Let us assume that $v \ll 1$, i.e. v is much smaller than the speed of light, so $\gamma \simeq 1$ and $u^0 \gg u^i$. The equation of motion for the $\mu = i$ component is then

$$
\begin{aligned}
\frac{d}{d\tau}\Big(m\gamma\frac{dx^i}{dt}\Big) &= -\frac{1}{m\gamma}\Gamma^i_{00}u^0u^0 = -m\gamma\Gamma^i_{00} \\
\Rightarrow \frac{d^2x^i}{dt^2} &= -\Gamma^i_{00}\,.
\end{aligned}
\tag{144}
$$

Computing Γ^i_{00} from (125) and (128),

$$
\Gamma^i_{00} = \frac{1}{2}\delta^{ij}(-\partial_j h_{00}) = \partial^i\phi\,. \tag{145}
$$

Inserting this into the equation of motion,

$$\frac{d^2x^i}{dt^2} = -(\mathrm{grad}\,\phi)^i \tag{146}$$

Notice that this equation is independent of the rest mass of the particle, m. This is known as the Equivalence Principle.

Lecture 6

In the last lecture, we introduced the energy-momentum tensor (130) which we used to couple the massive point particle to the gravitational field via the interaction given by (129). Let us take a moment now to derive the energy-momentum tensor from Noether's theorem and translation invariance of the action of a free massive particle in Minkowski space:

$$S = \frac{1}{2}\int d\tau \big(n^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 n\big)\,. \tag{147}$$

The action is invariant under translations:

$$x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau)\,, \quad \delta x^\mu = a^\mu\,, \quad (a = const.)\,. \tag{148}$$

Promote the translation parameter to have spacetime dependence, $a^\mu = a^\mu(x)$. The variation of x^μ becomes:

$$\delta\dot{x}^\mu = \frac{d}{d\tau}(\delta x^\mu) = \frac{d}{d\tau}(a^\mu(x(\tau)) = \partial_\nu a^\mu \dot{x}^\nu \tag{149}$$

$$\begin{aligned}
\delta_a S &= \int d\tau n^{-1}\eta_{\mu\nu}\partial_\rho a^\mu \dot{x}^\rho \dot{x}^\nu \\
&= \int d\tau (\partial_\mu a_\nu)(x(\tau))\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) \\
&= \int d^4x \partial_\mu a_\nu(x) \int d\tau n^{-1}\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)\delta^{(4)}(x - x(\tau)) \\
&\sim \int d^4x \partial_\mu a_\nu T^{\mu\nu}\,,
\end{aligned} \tag{150}$$

where then $T^{\mu\nu} := m\int d\tau n^{-1}(\tau)\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)\delta^{(4)}(x - x(\tau))$.

Gravitational Deflection of Light

We will now look at post-Newtonian effects of general relativity, in particular the bending of light in a gravitational field. In order to do that, we will consider massless particles. Let us use the equations of motion. Recall the geodesic equation (equation of motion for x^μ),

$$\frac{1}{n}\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = 0\,, \quad u^\mu := \frac{1}{n}\dot{x}^\mu\,, \tag{151}$$

and from $\delta_n S$,

$$\begin{aligned}\delta_n S &= \frac{1}{2}\int d\tau(-n^{-2}\delta n g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2\delta n) \\ &= \frac{1}{2}\int d\tau n^{-2}\delta n(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 n^2)\,,\end{aligned} \tag{152}$$

the equation of motion is

$$n^2 m^2 = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,. \tag{153}$$

When $m^2 \neq 0$, (153) can be written as

$$n(\tau) = \frac{1}{m}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,. \tag{154}$$

When we take the limit $m \to 0$, (153) becomes

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0\,. \tag{155}$$

Since we are dealing with light rays, we indeed expect their trajectories to be null with respect to the full metric. Because of reparameterization invariance, we can fix $n = const.$ and we can write (151) as

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,. \tag{156}$$

Exercise: show that these equations are invariant under reparameterizations, $\delta x^\mu = \lambda\dot{x}^\mu$, provided $\ddot{\lambda} = 0$ (λ is an affine parameter, $\lambda(\tau) = a\tau + b$, where a, $b = const.$).
Now consider the situation in the figure, in which a source emits light near a massive body, say, the sun, with mass M. Let the impact parameter, i.e. the distance between unperturbed light ray and massive object, to be b. Let us denote the unperturbed light ray by $x^{(0)}(\tau)$. An observer sees the bent light with angle α. Let us assume that the gravitational field is weak and we will only consider the small deviation of the light ray to first-order. Let the light ray be described as the sum of the unperturbed light ray and its deviation, $x^{(1)}(\tau)$: $x^\mu(\tau) = x^{(0)}(\tau) + x^{(1)\mu}(\tau)$. Let us define:

$$k^\mu := \frac{dx^{(0)\mu}}{d\tau}\,, \quad l^\mu = \frac{dx^{(1)\mu}}{d\tau}\,; \tag{157}$$

explicitly, $\dot{x}^\mu = k^\mu + l^\mu$. Let us expand

$$0 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \eta_{\mu\nu}k^\mu k^\nu + 2\eta_{\mu\nu}k^\mu l^\nu + h_{\mu\nu}k^\mu k^\nu\,, \tag{158}$$

and keep in mind that the first term of the expansion on the right-hand side is zeroth order in perturbations and the second and third terms are first order. Thus we have two equations:

$$k^\mu k_\mu = 0\,, \tag{159}$$

$$2\eta_{\mu\nu}k^\mu l^\nu + h_{\mu\nu}k^\mu k^\nu = 0\,. \tag{160}$$

In the weak-field regime, the Christoffel symbol is simply given by

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\nu\rho})\,. \tag{161}$$

Let us decompose $k^\mu = (k^0, \vec{k})$, where $(k^0)^2 = |\vec{k}|^2 \equiv k^2$ since k^μ is a null vector. We require the components of the Christoffel symbol.

$$\begin{aligned}
\Gamma^0_{0i} &= -\frac{1}{2}(\partial_0 h_{i0} + \partial_i h_{00} - \partial_0 h_{0i}) = \partial_i \phi\,, \\
\Gamma^i_{00} &= \frac{1}{2}\delta^{ij}(-\partial_j h_{00}) = \partial^i \phi\,, \\
\Gamma^i_{jk} &= \frac{1}{2}\delta^{il}(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{ij}) = -2\delta^i_{(j}\partial_{k)}\phi + \partial^i\phi\delta_{jk}\,.
\end{aligned} \tag{162}$$

Here we have recalled the matrix form of $h_{\mu\nu}$ in (128), in which the off-diagonal elements are zero, e.g. $h_{0i} = h_{i0} = 0$. Now we are ready to address the first-order equation by inserting (157) and (162) into (156):

$$\frac{dl^\mu}{d\tau} = -\Gamma^\mu_{\nu\rho}k^\nu k^\rho\,; \tag{163}$$

$$\mu = 0\,, \quad \frac{dl^0}{d\tau} = -2\Gamma^0_{0i}k^0 k^i = -2\partial_i\phi k^0 k^i$$
$$= -2k\vec{k}\cdot\vec{\nabla}\phi\,, \tag{164}$$
$$\mu = i\,, \quad \frac{dl^i}{d\tau} = -\Gamma^i_{jk}k^j k^k - \Gamma^i_{00}k^0 k^0$$
$$= (2\delta^i_j\partial_k\phi - \partial^i\phi\delta_{jk})k^j k^k - \partial^i\phi k^2\,. \tag{165}$$

Rewriting in vector notation,

$$\frac{d\vec{l}}{d\tau} = -2k^2\vec{\nabla}_\perp\phi\,, \tag{166}$$

where we used a split of the gradient into parallel and transverse components, $\vec{\nabla}\phi = \vec{\nabla}_\parallel\phi + \vec{\nabla}_\perp\phi$, where

$$\begin{aligned}
\vec{\nabla}_\parallel &= k^{-2}(\vec{k}\cdot\vec{\nabla}\phi)\vec{k}\,, \\
\vec{\nabla}_\perp &= \vec{\nabla}\phi - k^{-2}(\vec{k}\cdot\vec{\nabla}\phi)\vec{k}\,, \quad \text{such that} \quad \vec{k}\cdot\vec{\nabla}_\perp\phi = 0\,.
\end{aligned} \tag{167}$$

We claim that (164) is solved by

$$l^0 = -2k\phi\,, \quad (l^0 = 0 \quad \text{for} \quad \phi = 0)\,. \tag{168}$$

Indeed,

$$\begin{aligned}
\frac{dl^0}{d\tau} &= -2k\frac{\partial\phi}{\partial x^i}\frac{dx^i}{d\tau} \simeq -2k\frac{\partial\phi}{\partial x^i}\frac{dx^{i(0)}}{d\tau} \\
&= -2kk^i\partial_i\phi\,.
\end{aligned} \tag{169}$$

From (160),

$$
\begin{aligned}
-kl^0 + \vec{k}\cdot\vec{l} &= 2k^2\phi\,, \\
2k^2\phi + \vec{k}\cdot\vec{l} &= 2k^2\phi\,, \quad \Rightarrow \vec{k}\cdot\vec{l} = 0\,,
\end{aligned}
\tag{170}
$$

and indeed we do expect the deviation of the light ray to be perpendicular to the unperturbed light ray. The total deviation of the light ray can be computed as the following integral, where we used (166) and we have assumed the unperturbed light ray to be $x^{(0)\mu}(\tau) = k(\tau,\tau,0,0)$, so that $x(\tau) = k\tau$:

$$
\Delta\vec{l} = \int \frac{d\vec{l}}{d\tau}d\tau = -2k^2\int \vec{\nabla}_\perp\phi d\tau = -2k\int \vec{\nabla}_\perp \phi dx\,. \tag{171}
$$

We can then determine the angle of the deflection, α, since $|\Delta\vec{l}| = \alpha k$ as demonstrated in the figure.
Exercise: compute α for $\phi = -\frac{GM}{r}$.
The gravitational deflection of light was one of the first experimental confirmations of general relativity in the weak field approximation.

Lecture 7

Towards a Nonlinear Theory
Let us revisit the analogy between electromagnetism and general relativity. The photon is a massless particle of spin-1. This means that the particle has two physical polarizations. Without a gauge symmetry, it would carry an extra polarization. Gauge invariance is mandatory in order to describe a spin-1 particle with two physical polarizations. The relative coefficients of the Fierz-Pauli action are fixed by demanding the gauge invariance, $\delta_\xi h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. Gravitational waves have two physical polarizations.

The interaction of the point particle and the electromagnetic field gives a conserved current, j^μ (as shown in an exercise following equation (58)). We will now show that the energy-momentum tensor for gravity is not conserved. Let us remind ourselves that the energy-momentum tensor is:

$$
T^{\mu\nu}(x) = \int d\tau \frac{1}{n}\dot{x}^\mu\dot{x}^\nu\delta^4(x - x(\tau))\,. \tag{172}
$$

Let us consider a test-vector v_ν and the following integral:

$$
\begin{aligned}
\int d^4x v_\nu \partial_\mu T^{\mu\nu} &= \int d^4x v_\nu \partial_\mu \int d\tau \frac{1}{n}\dot{x}^\mu \dot{x}^\nu \delta^4(x - x(\tau)) \\
&= -\int d^4x \partial_\mu v_\nu \int d\tau \frac{1}{n}\dot{x}^\mu \dot{x}^\nu \delta^4(x - x(\tau)) \\
&= -\int d\tau \partial_\mu v_\nu \Big|_{x(\tau)} \frac{1}{n}\dot{x}^\mu \dot{x}^\nu \\
&= -\int d\tau \left(\frac{d}{d\tau}v_\nu\right)\frac{1}{n}\dot{x}^\nu \\
&= -\int d\tau \left(\frac{d}{d\tau}v_\nu\right) u^\nu \\
&= \int d\tau v_\nu \frac{d}{d\tau}u^\nu \\
&= \int d^4x\, v_\nu(x) \int d\tau\, \delta^{(4)}(x - x(\tau))\frac{du^\nu}{d\tau}\,.
\end{aligned}
\tag{173}
$$

We have thus obtained an equation for the divergence of $T^{\mu\nu}$:

$$
\partial_\mu T^{\mu\nu} = \int d\tau\, \delta^{(4)}(x - x(\tau))\frac{du^\nu}{d\tau}\,. \tag{174}
$$

If the particle were free, the right-hand side would be zero. But now the particle interacts with the gravitational field, with its interaction described in the action as

$$
S = \int d\tau \left(\frac{1}{2n}\dot{x}^\mu \dot{x}_\mu - \frac{n}{2}m^2\right) + \int d^4x h_{\mu\nu}T^{\mu\nu}\,. \tag{175}
$$

In contrast, for electromagnetism one finds that the current is conserved, i.e. $\partial_\mu j^\mu = 0$, regardless of whether the particle is free or not. To make sense of this, we must recall that the charged particles generate an electric current, but the electromagnetic field itself is not charged, i.e. does not source another electromagnetic field. The photon is a neutral particle. On the contrary, the gravitational field $h_{\mu\nu}$ must generate energy and momentum, but it also couples to the energy-momentum tensor. The particle in a gravitational field exchanges energy with the gravitational field, whereas a charged particle in an electromagnetic field does not transfer its charge into the electromagnetic field because the electromagnetic field is not charged. Gravity is a non-linear, self-coupled theory.

We can rewrite (175) as

$$
S[x;h] = S_{FP} + \int d\tau \left\{\frac{1}{2n}\dot{x}^\mu \dot{x}_\mu - \frac{n}{2}m^2 + \frac{1}{2n}h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right\}. \tag{176}
$$

Varying with respect to x and gauge-fixing $n(\tau) = 1$, the equation of motion is the geodesic equation,

$$
E_\mu \equiv \ddot{x}_\mu + \Gamma_{\nu\rho|\mu}\dot{x}^\nu \dot{x}^\rho = 0\,. \tag{177}
$$

Taking the gauge variation of $\Gamma_{\nu\rho,\mu}$,

$$\begin{aligned}\delta_\xi \Gamma_{\nu\rho|\mu} &= \frac{1}{2}(\partial_\nu\partial_\mu\xi_\rho + \partial_\nu\partial_\rho\xi_\mu + \partial_\rho\partial_\mu\xi_\nu + \partial_\rho\partial_\nu\xi_\mu - \partial_\mu\partial_\nu\xi_\rho - \partial_\mu\partial_\rho\xi_\nu) \\ &= \partial_\nu\partial_\rho\xi_\mu\,,\end{aligned} \tag{178}$$

one sees that the Christoffel symbol is not a gauge invariant object. The gauge variation of the geodesic equation is

$$\begin{aligned}\delta_\xi(\ddot{x}_\mu + \Gamma_{\nu\rho,\mu}\dot{x}^\nu\dot{x}^\rho) &= \dot{x}^\nu\dot{x}^\rho\partial_\nu\partial_\rho\xi_\mu \\ &= \dot{x}^\nu\frac{d}{d\tau}\partial_\nu\xi_\mu \\ &= \frac{d^2}{d\tau^2}\xi_\mu - \ddot{x}^\nu\partial_\nu\xi_\mu \\ &= \ddot{\xi}_\mu + \mathcal{O}(\xi h)\,.\end{aligned} \tag{179}$$

Remember that $h_{\mu\nu}$ and ξ_μ are both first order in perturbations. The equation of motion of the particle is not gauge invariant.

Let us try to amend this. Suppose the gauge transformation of $h_{\mu\nu}$ includes $\delta x^\mu = -\xi^\mu(x(\tau))$, i.e. simultaneously change the trajectory of the particle and the field $h_{\mu\nu}$.

$$\delta_\xi^{new}(E_\mu) = \delta\ddot{x}_\mu + \ddot{\xi}_\mu + 2\Gamma_{\nu\rho,\mu}\dot{x}^\nu\delta\dot{x}^\rho \tag{180}$$

Since $\delta\ddot{x}^\mu = -\ddot{\xi}^\mu$ and the last term is second order in perturbations, $\delta_\xi^{new}(E_\mu) = 0$. What can we learn from this? For $\delta_\xi h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, we must change the trajectory $x^\mu(\tau) \to x^\mu(\tau) - \xi^\mu(x(\tau))$. This means that the interval $ds^2 := \eta_{\mu\nu}dx^\mu dx^\nu$ is not invariant. We are forced to realize, for the theory to make sense, that the true invariant interval is taken from the curved geometry, that is, $ds^2 := g_{\mu\nu}dx^\mu dx^\nu$, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Let us see if the quantity $g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$ is invariant:

$$\begin{aligned}\delta_\xi(\eta_{\mu\nu} + h_{\mu\nu}(x))\dot{x}^\mu\dot{x}^\nu &= 2\dot{x}^\mu\delta\dot{x}^\nu\eta_{\mu\nu} + (\delta_\xi h_{\mu\nu})\dot{x}^\mu\dot{x}^\nu + \mathcal{O}(h\xi) \\ &= -2\eta_{\mu\nu}\dot{x}^\mu\dot{\xi}^\nu + (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu)\dot{x}^\mu\dot{x}^\nu \\ &= -2\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\lambda\partial_\lambda\xi^\nu + 2\partial_\mu\xi_\nu\dot{x}^\mu\dot{x}^\nu \\ &= 0\,.\end{aligned} \tag{181}$$

The newly defined invariant interval is indeed gauge invariant. This is due to the presence of gravity. (175) can now be rewritten as

$$S = \int d\tau\left\{\frac{1}{2n}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{n}{2}m^2\right\}. \tag{182}$$

Now we will try to find the full gauge transformation of the full metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

(without assuming that $h_{\mu\nu}$ is small) with $\delta_\xi x^\mu = -\xi^\mu(\tau)$, where ξ^μ is infinitesimal:

$$\begin{aligned} 0 = \delta_\xi S &= \frac{1}{2}\int d\tau n^{-1}\big(\partial_\rho g_{\mu\nu}\delta_\xi x^\rho \dot{x}^\mu \dot{x}^\nu + (\delta_\xi g_{\mu\nu})\dot{x}^\mu\dot{x}^\nu + 2g_{\mu\nu}\dot{x}^\mu\delta\dot{x}^\nu\big) \\ &= \frac{1}{2}\int d\tau n^{-1}\big(-\xi^\rho\partial_\rho g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - 2g_{\mu\nu}\dot{x}^\mu\dot{\xi}^\nu + \delta_\xi g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\big) \\ &= \frac{1}{2}\int d\tau n^{-1}\big(-\xi^\rho\partial_\rho g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - 2g_{\mu\nu}\dot{x}^\mu\dot{x}^\lambda\partial_\lambda\xi^\nu + \delta_\xi g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\big)\,. \end{aligned} \tag{183}$$

We see that the infinitesimal transformation of $g_{\mu\nu}$ must be

$$\delta_\xi g_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + \partial_\nu\xi^\rho g_{\mu\rho} + \partial_\mu\xi^\rho g_{\nu\rho}\,. \tag{184}$$

One can check that in the weak field limit, i.e. for $h_{\mu\nu}$ small, the gauge transformation of $g_{\mu\nu}$ to zeroth order in h (neglecting $\mathcal{O}(\xi h)$), reproduces $\delta_\xi h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. How does one find the finite transformation of $g_{\mu\nu}$? An infinitesimal transformation , $\delta_\xi x^\mu = -\xi^\mu(x)$, gives $x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$, i.e. a small local translation. A finite transformation is $x^\mu \to x'^\mu(x)$. To find the finite transformation for $g_{\mu\nu}$, we can inspect the finite transformation of the action:

$$\begin{aligned} S' &= \int d\tau \frac{1}{2n} g'_{\mu\nu}(x')\dot{x}'^\mu\dot{x}'^\nu \\ &= \int d\tau \frac{1}{2n} g'_{\mu\nu}(x')\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\rho}\dot{x}^\lambda\dot{x}^\rho \\ &= S = \int d\tau\frac{1}{2n}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\,, \end{aligned} \tag{185}$$

where we have used $\frac{x'^\mu(x)}{d\tau} = \frac{dx^\mu}{d\tau}\frac{\partial x'^\mu}{\partial x^\nu}$ in the second line, and in the last line we have equated the transformed action to the original action. For this equality to be true, the finite transformation of $g_{\mu\nu}$ must be

$$g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}g_{\lambda\rho}(x)\,. \tag{186}$$

Lecture 8

Our task today is to find a nonlinear action which is invariant under the gauge transformations in (184) for $g_{\mu\nu}$. For this we need the following definitions:

$$\begin{aligned} &g^{\mu\nu}\ \text{ is the inverse of }\ g_{\mu\nu}:\ \ g^{\mu\rho}g_{\rho\nu} = \delta^\mu{}_\nu\,, \\ &g \equiv |g| \equiv \det g_{\mu\nu} < 0\,,\ \ \text{for } (-,+,+,+) \text{ signature}\,. \end{aligned} \tag{187}$$

A useful formula is

$$|g| = \frac{1}{4!}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\,, \tag{188}$$

where ϵ is the Levi-Civita symbol and $\epsilon^{0123} = +1$. A useful way of writing $g^{\mu\nu}$ is

$$g^{\mu\nu} = \frac{1}{3!}\frac{1}{|g|}\varepsilon^{\mu\rho\sigma\kappa}\varepsilon^{\nu\alpha\beta\gamma}g_{\rho\alpha}g_{\alpha\beta}g_{\kappa\gamma}\,. \tag{189}$$

Exercise: prove (188) and (189).
We can then write the gauge transformation of the determinant $|g|$:

$$\begin{aligned}\delta_\xi|g| &= \frac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}(\delta_\xi g_{\mu\alpha})g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta} \\ &= |g|g^{\mu\nu}\delta_\xi g_{\mu\nu} \\ &= |g|g^{\mu\nu}(\xi^\rho\partial_\rho g_{\mu\nu} + 2\partial_\mu\xi^\rho g_{\rho\nu}) \\ &= \xi^\rho\partial_\rho|g| + 2\partial_\mu\xi^\mu|g|\,.\end{aligned} \tag{190}$$

We see that the square-root of minus the determinant $\sqrt{-g}$ transforms as:

$$\delta_\xi\sqrt{-g} = \xi^\rho\partial_\rho\sqrt{-g} + \partial_\rho\xi^\rho\sqrt{-g} = \partial_\rho(\xi^\rho\sqrt{-g})\,. \tag{191}$$

Suppose we had a function, i.e. a scalar $F(g)$, transforming as $\delta_\xi F = \xi^\rho\partial_\rho F$. Then the action

$$S = \int d^4x\sqrt{-g}F(g) \tag{192}$$

is gauge invariant:

$$\begin{aligned}\delta_\xi S &= \int d^4x\big(\partial_\rho(\xi^\rho\sqrt{-g})F + \sqrt{-g}\xi^\rho\partial_\rho F\big) \\ &= \int d^4x\partial_\rho\big(\xi^\rho\sqrt{-g}F\big) \\ &= 0\,.\end{aligned} \tag{193}$$

The next step is to construct a function out of the metric tensor and its derivatives that has this property. Can we write a two-derivative action for $g_{\mu\nu}$? We claim that, since we can always integrate by parts, we can at most have first-order derivatives of $g_{\mu\nu}$: $S \sim (\partial g)(\partial g)$. We have assumed this before when deriving the Fierz-Pauli action. In addition, we claim that there are only four independent structures, since

$$|g|^{-1}\partial_\mu|g| = g^{\rho\sigma}\partial_\mu g_{\rho\sigma}\,. \tag{194}$$

Here we give the action that we want:

$$\begin{aligned}S = \int d^4x\Bigg\{\sqrt{-g}\bigg[&\frac{1}{4}g^{\mu\nu}\partial_\mu g^{\rho\sigma}\partial_\nu g_{\rho\sigma} - \frac{1}{2}g^{\mu\nu}\partial_\mu g^{\rho\sigma}\partial_\rho g_{\sigma\nu} + \partial_\mu g^{\mu\nu}\left(\frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g}\right) \\ &+ g^{\mu\nu}\left(\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\right)\left(\frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g}\right)\bigg]\Bigg\} + S_{\text{matter}}\,.\end{aligned} \tag{195}$$

S_{matter} describes the coupling to matter, for instance,

$$S_{\text{Maxwell}} = -\frac{1}{4}\int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{196}$$

where we have promoted $\int d^4x \to \int d^4x\sqrt{-g}$ and $\eta^{\mu\nu} \to g^{\mu\nu}$ from its special relativity form. We will prove that (195) is gauge invariant once we develop some geometrical methods.

Differential Geometry I: Quick and Dirty

Let us start with Lie derivatives. A Lie derivative $\mathcal{L}_\xi$ with respect to ξ^μ acting on a scalar (a function) is

$$\mathcal{L}_\xi f := \xi^\mu \partial_\mu f\,. \tag{197}$$

A Lie derivative acting on a vector field is

$$\mathcal{L}_\xi V^\mu := \xi^\nu \partial_\nu V^\mu - \partial_\nu \xi^\mu V^\nu \tag{198}$$

and on a co-vector it is

$$\mathcal{L}_\xi W_\mu := \xi^\nu \partial_\nu W_\mu + \partial_\mu \xi^\nu W_\nu\,. \tag{199}$$

We can generalize these rules to a general tensor $T^{\mu_1\cdots\mu_s}{}_{\nu_1\cdots\nu_r}$,

$$\begin{aligned}\mathcal{L}_\xi T^{\mu_1\cdots\mu_s}{}_{\nu_1\cdots\nu_r} := {}& \xi^\rho \partial_\rho T^{\mu_1\cdots\mu_s}{}_{\nu_1\cdots\nu_r} - \partial_\rho \xi^{\mu_1} T^{\rho\mu_2\cdots\mu_s}{}_{\nu_1\cdots\nu_r} - \cdots - \partial_\rho \xi^{\mu_s} T^{\mu_1\cdots\mu_{s-1}\rho}{}_{\nu_1\cdots\nu_r} \\ & + \partial_{\nu_1}\xi^\rho T^{\mu_1\cdots\mu_s}{}_{\rho\nu_2\cdots\nu_r} + \cdots + \partial_{\nu_r}\xi^\rho T^{\mu_1\cdots\mu_s}{}_{\nu_1\cdots\nu_{r-1}\rho}\,.\end{aligned} \tag{200}$$

The Lie derivative has the following properties:

$$1)\ \ [\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1,\xi_2]}\,,\ \text{where}\ \ [\xi_1,\xi_2]^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu = \mathcal{L}_{\xi_1}\xi_2^\mu = -\mathcal{L}_{\xi_2}\xi_1^\mu\,, \tag{201}$$

$$2)\ \ \mathcal{L}_\xi (V^\mu W_\mu) = (\mathcal{L}_\xi V^\mu) W_\mu + V^\mu \mathcal{L}_\xi W_\mu = \xi^\nu \partial_\nu (V^\mu W \mu)\,. \tag{202}$$

In words, the Lie derivative 1) obeys the Leibniz rule and 2) is consistent with index contractions. How is this related to the gauge-transformation of $g_{\mu\nu}$? A collection of functions $T^{\mu_1\cdots\mu_s}{}_{\nu_1\cdots\nu_r}$ is a tensor field of rank (r,s), provided it transforms as

$$\delta_\xi T^{\mu_1\cdots\mu_s}{}_{\nu_1\cdots\nu_r} = \mathcal{L}_\xi T^{\mu_1\cdots\mu_s}{}_{\nu_1\cdots\nu_r}\,. \tag{203}$$

Thus, $g_{\mu\nu}$ is a $(2,0)$ tensor field and $g^{\mu\nu}$ is a $(0,2)$ tensor field. Is there a way to take one or two derviatives of $g_{\mu\nu}$ such that the result is also a tensor field? Can we build tensors from derivatives of $g_{\mu\nu}$? Let us introduce the non-covariant variation,

$$\Delta_\xi := \delta_\xi - \mathcal{L}_\xi\,. \tag{204}$$

This notation allows us to keep track of the non-covariant parts of the variations of objects. Note that Δ_ξ acts trivially on tensors, e.g. $\Delta_\xi g^{\mu\nu} \equiv 0$, and Δ_ξ behaves as a variation: $\Delta_\xi(R\cdot S) = \Delta_\xi R \cdot S + R\Delta_\xi S$. Let us see how derivatives of $g_{\mu\nu}$ transform.

$$\begin{aligned}\delta_\xi(\partial_\mu g_{\nu\rho}) &= \partial_\mu(\delta_\xi g_{\nu\rho}) \\ &= \partial_\mu(\xi^\sigma \partial_\sigma g_{\nu\rho} + \partial_\nu \xi^\sigma g_{\sigma\rho} + \partial_\rho \xi^\sigma g_{\nu\sigma}) \\ &= \xi^\sigma \partial_\sigma \partial_\mu g_{\nu\rho} + \partial_\mu \xi^\sigma \partial_\sigma g_{\nu\rho} + \partial_\nu \xi^\sigma \partial_\mu g_{\sigma\rho} + \partial_\rho \xi^\sigma \partial_\mu g_{\nu\sigma} \\ &\quad + \partial_\mu \partial_\nu \xi^\sigma g_{\sigma\rho} + \partial_\mu \partial_\rho \xi^\sigma g_{\nu\sigma}\end{aligned} \tag{205}$$

We can write the non-covariant variation of $\partial_\mu g_{\nu\rho}$,

$$\Delta_\xi(\partial_\mu g_{\nu\rho}) = \partial_\mu\partial_\nu\xi^\sigma g_{\sigma\rho} + \partial_\mu\partial_\rho\xi^\sigma g_{\nu\sigma} \,. \tag{206}$$

If we antisymmetrize μ and ν, we could eliminate the first term of $\Delta_\xi(\partial_\mu g_{\nu\rho})$, but the second term would not vanish. We conclude that no combination of $\partial_\mu g_{\nu\rho}$ is a tensor. We can work with the Christoffel symbol $\Gamma^\rho_{\mu\nu}$ and we find that $\Delta_\xi\Gamma^\rho_{\mu\nu} = \partial_\mu\partial_\nu\xi^\rho$.
Exercise: explicitly compute the non-covariant variation of the Christoffel symbol,

$$\delta_\xi\Gamma^\rho_{\mu\nu} = \partial_\mu\partial_\nu\xi^\rho + \xi^\sigma\partial_\sigma\Gamma^\rho_{\mu\nu} + \partial_\mu\xi^\sigma\Gamma^\rho_{\sigma\nu} + \partial_\nu\xi^\sigma\Gamma^\rho_{\sigma\mu} - \partial_\sigma\xi^\rho\Gamma^\sigma_{\mu\nu} \,. \tag{207}$$

Exercise: prove

$$\Delta_\xi(\partial_\mu\Gamma^\rho_{\nu\sigma}) = \partial_\mu\partial_\nu\partial_\sigma\xi^\rho + \partial_\mu\partial_\nu\xi^\lambda\Gamma^\rho_{\lambda\sigma} + \partial_\mu\partial_\sigma\xi^\lambda\Gamma^\rho_{\nu\lambda} - \partial_\mu\partial_\lambda\xi^\rho\Gamma^\lambda_{\nu\sigma} \,. \tag{208}$$

It is useful to use matrix notation:

$$\boldsymbol{\Gamma}_\mu \equiv (\boldsymbol{\Gamma}_\mu)^\rho{}_\nu \equiv \Gamma^\rho_{\mu\nu} \,, \tag{209}$$

$$\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}^\mu{}_\nu \equiv \partial_\nu\xi^\mu \,. \tag{210}$$

In matrix notation,

$$\begin{aligned} \Delta_\xi(\partial_\mu\boldsymbol{\Gamma}_\nu) &= \partial_\mu\partial_\nu\boldsymbol{\sigma} + \partial_\mu\partial_\nu\xi^\rho\boldsymbol{\Gamma}_\rho - [\partial_\mu\boldsymbol{\sigma}, \boldsymbol{\Gamma}_\nu] \,, \\ \Delta_\xi(\partial_{[\mu}\boldsymbol{\Gamma}_{\nu]}) &= -[\partial_{[\mu}\boldsymbol{\sigma}, \boldsymbol{\Gamma}_{\nu]}] \,. \end{aligned} \tag{211}$$

We can then write the Riemann tensor:

$$\boldsymbol{\mathcal{R}}_{\mu\nu} := \partial_\mu\boldsymbol{\Gamma}_\nu - \partial_\nu\boldsymbol{\Gamma}_\mu + [\boldsymbol{\Gamma}_\mu, \boldsymbol{\Gamma}_\nu] \,. \tag{212}$$

Unpacking to index form,

$$R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \,. \tag{213}$$

We compute the non-covariant variation,

$$\begin{aligned} \Delta_\xi\boldsymbol{\mathcal{R}}_{\mu\nu} &= \Delta_\xi(2\partial_{[\mu}\boldsymbol{\Gamma}_{\nu]}) + 2[\Delta_\xi\boldsymbol{\Gamma}_{[\mu}, \boldsymbol{\Gamma}_{\nu]}] \\ &= -2[\partial_{[\mu}\boldsymbol{\sigma}, \boldsymbol{\Gamma}_{\nu]}] + 2[\partial_{[\mu}\boldsymbol{\sigma}, \boldsymbol{\Gamma}_{\nu]}] \\ &= 0 \,. \end{aligned} \tag{214}$$

Therefore,

$$\delta_\xi R_{\mu\nu}{}^\rho{}_\sigma = \mathcal{L}_\xi R_{\mu\nu}{}^\rho{}_\sigma \,. \tag{215}$$

We have proved that the non-covariant variation of the Riemann tensor is zero and that the Riemann tensor is a $(3,1)$ tensor. We can construct the Ricci tensor, $R_{\mu\nu} := R_{\rho\mu}{}^\rho{}_\nu$, which is a symmetric $(2,0)$ tensor. We can also construct the Ricci scalar, $R := g^{\mu\nu}R_{\mu\nu}$, which transforms as $\delta_\xi R = \xi^\mu\partial_\mu R$. The Riemann tensor also has the properties ($R_{\mu\nu\rho\sigma} := g_{\rho\lambda}R_{\mu\nu}{}^\lambda{}_\sigma$):

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} \tag{216}$$

and also satisfies the Bianchi identity:

$$R_{\mu\nu\rho\sigma} + R_{\nu\rho\mu\sigma} + R_{\rho\mu\nu\sigma} \equiv 0\,. \tag{217}$$

We can then write the gauge-invariant action by using the Ricci scalar,

$$S = \int d^4x \sqrt{-g} R\,. \tag{218}$$

(218) is called the Einstein-Hilbert action. The Einstein-Hilbert action is equivalent to (195) up to total derivatives.

Lecture 9

Today we will continue our discussion of differential geometry. The gauge transformation for $g_{\mu\nu}$ is

$$\delta_\xi g_{\mu\nu} \equiv \mathcal{L}_\xi g_{\mu\nu} \tag{219}$$

and $g_{\mu\nu}$ is a $(2,0)$ tensor. In the previous lecture we showed that the Christoffel symbol is not a tensor:

$$\delta_\xi \Gamma^\rho_{\mu\nu} = \partial_\mu \partial_\nu \xi^\rho + \mathcal{L}_\xi \Gamma^\rho_{\mu\nu}\,. \tag{220}$$

We showed that the Riemann tensor is a $(3,1)$ tensor ,

$$R_{\mu\nu}{}^\rho{}_\sigma = 2\big(\partial_{[\mu} \Gamma^\rho_{\nu]\sigma} + \Gamma^\rho_{\lambda[\mu} \Gamma^\lambda_{\nu]\sigma}\big)\,. \tag{221}$$

From this we have then constructed the Einstein-Hilbert action in (218). In order to couple matter fields to gravity, we need to define covariant derivatives. Let us consider the vector field A^μ of Maxwell's theory and its gauge transformation:

$$\begin{aligned}
\delta_\xi A_\mu &= \mathcal{L}_\xi A_\mu = \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu\,,\\
\delta_\xi(\partial_\mu A_\nu) &= \partial_\mu(\xi^\rho \partial_\rho A_\nu + \partial_\nu \xi^\rho A_\rho)\\
&= \xi^\rho \partial_\rho(\partial_\mu A_\nu) + \partial_\mu \xi^\rho \partial_\rho A_\nu + \partial_\nu \xi^\rho \partial_\mu A_\rho + \partial_\mu \partial_\nu \xi^\rho A_\rho\,,\\
\Delta_\xi(\partial_\mu A_\nu) &= \partial_\mu \partial_\nu \xi^\rho A_\rho\,.
\end{aligned} \tag{222}$$

We would like to define a covariant derivative ∇_μ such that $\nabla_\mu A_\nu$ transforms as a tensor,

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\rho_{\mu\nu} A_\rho\,. \tag{223}$$

The noncovariant variation of $\nabla_\mu A_\nu$ vanishes:

$$\Delta_\xi(\nabla_\mu A_\nu) = \partial_\mu \partial_\nu \xi^\rho A_\rho - \Delta_\xi \Gamma^\rho_{\mu\nu} A_\rho = 0. \tag{224}$$

We have thus successfully defined a covariant derivative. The covariant derivative of A^μ is

$$\nabla_\mu A^\nu \equiv \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho\,. \tag{225}$$

For a scalar f, the partial derivative $\partial_\mu f$ transforms as a covariant vector. Whenever a special relativistic theory is coupled to gravity, as a general rule, the Minkowski metric $\eta_{\mu\nu}$ should be replaced by $g_{\mu\nu}$ and the partial derivative ∂_μ should be replaced by ∇_μ. The electromagnetic field tensor for example becomes

$$\begin{aligned} F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - 2\Gamma^\rho_{[\mu\nu]} A_\rho \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \,. \end{aligned} \tag{226}$$

Let us now discuss some properties of covariant derivatives. First, the metric $g_{\mu\nu}$ is covariantly constant, meaning

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \,. \tag{227}$$

Second, the commutator of ∇_μ gives the Riemann tensor $\boldsymbol{\mathcal{R}}_{\mu\nu}$,

$$[\nabla_\mu, \nabla_\nu] = \boldsymbol{\mathcal{R}}_{\mu\nu} \,. \tag{228}$$

Thus, $[\nabla_\mu, \nabla_\nu] \neq 0$, unlike the commutator of the usual partial derivatives $[\partial_\mu, \partial_\nu] = 0$. For instance, $[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma$, since

$$\begin{aligned} [\nabla_\mu, \nabla_\nu] V^\rho &= \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu\lambda} \nabla_\nu V^\lambda - (\mu \leftrightarrow \nu) \\ &= \partial_\mu \big(\partial_\nu V^\rho + \Gamma^\rho_{\nu\lambda} V^\lambda \big) + \Gamma^\rho_{\mu\lambda} \big(\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\sigma} V^\sigma \big) - (\mu \leftrightarrow \nu) \\ &= \partial_\mu \Gamma^\rho_{\nu\lambda} V^\lambda + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} V^\sigma - (\mu \leftrightarrow \nu) \\ &= \big(\partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} \big) V^\lambda \\ &= R_{\mu\nu}{}^\rho{}_\lambda V^\lambda \,. \end{aligned} \tag{229}$$

Third, the variations of the Christoffel symbol and the Riemann tensor under $\delta g_{\mu\nu}$ are tensors:

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \big(\nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \delta g_{\mu\nu} \big) \,, \tag{230}$$

$$\begin{aligned} \delta R_{\mu\nu}{}^\rho{}_\sigma &= \nabla_\mu \delta\Gamma^\rho_{\nu\sigma} - \nabla_\nu \delta\Gamma^\rho_{\mu\sigma} \\ \delta R_{\mu\nu} &= \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\mu \delta\Gamma^\rho_{\rho\nu} \,. \end{aligned} \tag{231}$$

The Christoffel symbol itself is not a tensor, but its variation is a tensor. There are two kinds of Bianchi identities for the Riemann tensor. One is algebraic and one is differential. The algebraic Bianchi identity for $R_{\mu\nu\rho\sigma} \equiv g_{\rho\lambda} R_{\mu\nu}{}^\lambda{}_\sigma$ is

$$R_{\mu\nu\rho\sigma} + R_{\nu\rho\mu\sigma} + R_{\rho\mu\nu\sigma} \equiv 0 \,. \tag{232}$$

This can also be written in a more compact form,

$$R_{[\mu\nu\rho]\sigma} \equiv 0 \,. \tag{233}$$

The differential Bianchi identity is

$$\nabla_\mu R_{\nu\rho}{}^\sigma{}_\lambda + \nabla_\nu R_{\rho\mu}{}^\sigma{}_\lambda + \nabla_\rho R_{\mu\nu}{}^\sigma{}_\lambda \equiv 0 \,. \tag{234}$$

The proof of this is the Jacobi identity,

$$\big[[\nabla_\mu, \nabla_\nu], \nabla_\rho\big] + \big[[\nabla_\nu, \nabla_\rho], \nabla_\mu\big] + \big[[\nabla_\rho, \nabla_\mu], \nabla_\nu\big] \equiv 0 \,. \tag{235}$$

(234) can be written as $\nabla_{[\mu} \boldsymbol{\mathcal{R}}_{\nu\rho]} \equiv 0$. Our goal now is to derive the Einstein equations from the Einstein-Hilbert action (218). In order to do this, we have to remember to take all the contributions of the metric.

$$\begin{aligned} \delta S_{EH} &= \delta \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} \\ &= \int d^4x \left(\frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} R + \sqrt{-g} \delta g^{\mu\nu} R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} (\nabla_\rho \delta \Gamma^\rho_{\mu\nu} - \nabla_\mu \delta \Gamma^\rho_{\rho\nu}) \right) \\ &= \int d^4x \sqrt{-g} \delta g_{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \end{aligned} \tag{236}$$

Exercise: show that $\int dx \sqrt{-g} \nabla_\mu V^\mu = \int d^4x \partial_\mu (\sqrt{-g} V^\mu) = 0$.
Thus we have obtained the full non-linear extension of the vacuum Einstein equation:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \,, \tag{237}$$

$G_{\mu\nu}$ is the Einstein tensor. The matter couplings, i.e. in $S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R + S_{matter}$, would result in a source term to the Einstein equation. Recall that the G in front of the integral is Newton's constant. This gives us a new definition of the energy-momentum tensor $T_{\mu\nu}$,

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} \,. \tag{238}$$

The variation of the action can be written in terms of $T_{\mu\nu}$,

$$\begin{aligned} \delta S &= \frac{1}{16\pi G} \int d^4x \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} + \int d^4x \frac{\delta S_{matter}}{\delta g^{\mu\nu}} \delta g^{\mu\nu} \\ &= \frac{1}{16\pi G} \int d^4x \sqrt{-g} \delta g^{\mu\nu} \big(G_{\mu\nu} - 8\pi G T_{\mu\nu} \big) \,, \end{aligned} \tag{239}$$

which then gives

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{240}$$

$G_{\mu\nu}$ satisfies the differential Bianchi identity,

$$\nabla^\mu G_{\mu\nu} \equiv 0 \,, \tag{241}$$

which is consistent with $\nabla^\mu T_{\mu\nu} = 0$ on-shell (as a consequence of matter field equations).

Differential Geometry II: Classical Approach

We want to develop a kind of calculus on curved surfaces. Consider a curved smooth space ("manifold") M, embedded in D-dimensional flat space $\mathbb{R}^D$ with coordinates Z^M, $M = 1, ..., D$ and basis e_M, and a constant metric G_{MN}. We will describe an arbitrary surface by means of a parametrization. We have already briefly discussed the one-dimensional curve and its parameterization. As a one-dimensional curve is described by one parameter, an n-dimensional surface is described by n parameters. Let us try to draw such a surface called M in $\mathbb{R}^D$. We parameterize the surface by functions

$$Z^M(x^\mu)\,, \quad \mu = 0, ..., d-1\,. \tag{242}$$

The dimension of the manifold is $\dim(M) = d$ if $\frac{\partial Z^M}{\partial x^\mu}$ are all linearly independent. We can draw a surface tangent to the point p. The tangent space T_pM at p is spanned by:

$$\left.\frac{\partial Z^M}{\partial x^\mu}\right|_p e_M\,, \quad \mu = 0, ..., d-1\,. \tag{243}$$

A general vector of M is written as

$$V = V^\mu \left.\frac{\partial Z^M}{\partial x^\mu}\right|_p e_M\,. \tag{244}$$

A vector field V,

$$V(x) = V^\mu(x)\frac{\partial Z^M(x)}{\partial x^\mu} e_M\,, \tag{245}$$

assigns a vector $V_p \in T_pM$ to each point $p \in M$. Is there a natural metric to measure the length of such vectors? The flat metric of the full space induces a metric on the manifold M. If V and W are vector fields, we can define

$$\langle V, W\rangle \equiv V^\mu \frac{\partial Z^M}{\partial x^\mu} W^\nu \frac{\partial Z^N}{\partial x^\nu} G_{MN} = V^\mu W^\nu g_{\mu\nu}\,, \tag{246}$$

where $g_{\mu\nu} \equiv \frac{\partial Z^M}{\partial x^\mu}\frac{\partial Z^N}{\partial x^\nu} G_{MN}$ is called the induced metric. We do not care about the way we parameterize surfaces, we only care about the shape of the surface (its curvature). Let us connect to our previous lectures by considering a reparameterization $x^\mu \to x'^\mu(x)$,

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial Z^M}{\partial x'^\mu}\frac{\partial Z^N}{\partial x'^\nu} G_{MN} \\ &= \frac{\partial Z^M}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial Z^N}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x'^\nu} G_{MN} \\ &= \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)\,, \end{aligned} \tag{247}$$

and on the vector field,

$$V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)\,. \tag{248}$$

The infinitesimal transformation $x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$ acting on a vector V^{μ} is

$$\delta_{\xi} V^{\mu} \equiv V'^{\mu}(x) - V^{\mu}(x) \equiv \mathcal{L}_{\xi} V^{\mu} \,. \tag{249}$$

We see now that the gauge transformations are general coordinate transformations on a curved space M.

Lecture 10

Parallel Transport

We will discuss the geometrical interpretation of the Christoffel symbol Γ as a connection and the Riemann tensor R as a curvature. Consider a vector space $\mathbb{R}^D$. Let there be a vector V^M at point p and a vector at point q in the same direction. With respect to a basis, these vectors have the same components. You can think of a vector as an equivalence class of parallel arrows. On flat space, we do not care whether the vector is at point p or point q. On a curved surface, we may have two different tangent surfaces at two different points. Mathematically, T_pM and T_qM have a priori nothing to do with each other and are different spaces. It does not make sense to ask: given a vector in T_pM, what is the corresponding vector at q? Let us draw a curve from point p to point q. Suppose we want to move a vector in T_pM to point q. What is the canonically associated vector at point q in T_qM?

In flat space, there is a natural way to move the vector around. If we choose to parallel transport the vector according to the ambient space, it is not necessarily in T_qM at point q. Let $p = Z(x)$ and $q = Z(x + \delta x)$, where δx is small. We take the projection of the parallel transported vector at q onto the tangent space T_qM. In coordinates, let

$$V^M(Z(x)) = V^{\mu} \frac{\partial Z^M}{\partial x^{\mu}} \,. \tag{250}$$

At point $q = Z(x + \delta x)$,

$$V^M = V_{\parallel}^M + V_{\perp}^M \,, \quad V_{\parallel}^M \in T_pM \,. \tag{251}$$

Let us determine $V_{\parallel}^M$ and $V_{\perp}^M$. First, there must be a K^{μ} so that

$$V_{\parallel}^M = K^{\mu} \frac{\partial Z^M}{\partial x^{\mu}}(x + \delta x) \,. \tag{252}$$

Second, $V_{\perp}^M$ is orthogonal to any vector at $T_qM \in W^{\mu} \frac{\partial Z^M}{\partial x^{\mu}}$, so

$$0 = G_{MN} V_{\perp}^M W^{\mu} \frac{\partial Z^N}{\partial x^{\mu}} \quad \forall W^{\mu} \,, \tag{253}$$

$$V_{\perp}^M \frac{\partial Z_M}{\partial x^{\mu}}(x + \delta x) = 0 \,. \tag{254}$$

We contract (251) with $\frac{\partial Z_M}{\partial x^\nu}(x+\delta x)$,

$$\begin{aligned} V^M \frac{\partial Z_M}{\partial x^\nu} &= K^\mu \frac{\partial Z^M}{\partial x^\mu}\frac{\partial Z_M}{\partial x^\nu} = K^\mu g_{\mu\nu} = K_\nu \\ &= V^\mu \frac{\partial Z^M}{\partial x^\mu}\frac{\partial Z_M}{\partial x^\nu}(x+\delta x) \\ &= V^\mu \frac{\partial Z^M}{\partial x^\mu}\left(\frac{\partial Z_M}{\partial x^\nu}(x) + \frac{\partial^2 Z_M}{\partial x^\nu \partial x^\rho}\delta x^\rho\right) \\ &= V^\mu \left(g_{\mu\nu} + \frac{\partial Z^M}{\partial x^\mu}\frac{\partial^2 Z_M}{\partial x^\nu \partial x^\rho}\delta x^\rho\right). \end{aligned} \tag{255}$$

The parallel transport of V_μ at x to $x+\delta x$ yields K_μ and the difference can be written as:

$$\delta V_\mu = K_\mu - V_\mu = V^\nu \partial_\nu Z^M \partial_\mu \partial_\rho Z_M \delta x^\rho . \tag{256}$$

Is there an intrinsic formula for δV_μ that is independent of $Z^M(x)$? Can we rewrite (256) in terms of $g_{\mu\nu}$? To see that there is such a formula, recall

$$\begin{aligned} g_{\mu\nu} &= \partial_\mu Z^M \partial_\nu Z_M , \\ \partial_\rho g_{\mu\nu} &= \partial_\mu \partial_\rho Z^M \partial_\nu Z_M + \partial_\mu Z^M \partial_\nu \partial_\rho Z_M . \end{aligned} \tag{257}$$

so that we can compute

$$\Gamma_{\mu\nu|\rho} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) = \partial_\mu \partial_\nu Z^M \partial_\rho Z_M . \tag{258}$$

We can then rewrite (256) as

$$\delta V_\mu = V^\nu \Gamma_{\mu\rho|\nu} \delta x^\rho = \Gamma^\rho_{\mu\nu} V_\rho \delta x^\nu . \tag{259}$$

An important property is that the norm $g^{\mu\nu}V_\mu V_\nu$ is preserved under parallel transport:

$$\begin{aligned} \delta(g^{\mu\nu}V_\mu V_\nu) &= \partial_\rho g^{\mu\nu}\delta x^\rho V_\mu V_\nu + 2g^{\mu\nu}\delta V_\mu V_\nu \\ &= -g^{\mu\alpha}g^{\nu\beta}\partial_\rho g_{\alpha\beta}\delta x^\rho V_\mu V_\nu + 2g^{\mu\nu}\Gamma^\sigma_{\mu\rho}V_\sigma \delta x^\rho V_\nu \\ &= -g^{\mu\alpha}g^{\nu\beta}\partial_\rho g_{\alpha\beta}\delta x^\rho V_\mu V_\nu + (\partial_\mu g_{\rho\sigma} + \partial_\rho g_{\mu\sigma} - \partial_\sigma g_{\mu\rho})V^\sigma V^\mu \\ &= -V^\alpha V^\beta \partial_\rho g_{\alpha\beta}\delta x^\rho + \delta x^\rho \partial_\rho g_{\mu\sigma} V^\sigma V^\mu \\ &= 0 . \end{aligned} \tag{260}$$

Demanding

$$\delta(W^\mu V_\mu) = \delta W^\mu V_\mu + W^\mu \Gamma^\rho_{\mu\nu} V_\rho \partial x^\nu \overset{!}{=} 0 , \tag{261}$$

one finds

$$\delta W^\mu = -\Gamma^\mu_{\nu\rho} W^\nu \delta x^\rho . \tag{262}$$

In matrix notation, $(\Gamma_\mu)^\nu_\rho = \Gamma^\nu_{\mu\rho}$ and

$$\delta W = -\delta x^\mu \Gamma_\mu(W)\,. \tag{263}$$

If we parallel transport a vector from the north pole of a sphere to the equator, then parallel transport to some other point on the equator, and finally parallel transport back to the initial point on the north pole, we can see that the final transported vector is not the same as the initial vector. This is one expression of curvature.

If we start with a vector at point p and parallel transport to another point, and to another point, and so on, in flat space, we obtain a straight line. In curved space, the curve we obtain is what we call a geodesic. Let u^μ be the vector $u^\mu \equiv \frac{dx^\mu}{d\tau}$. We use the equation (262),

$$\delta u^\mu + \Gamma^\mu_{\nu\rho} u^\nu \delta x^\rho = 0\,. \tag{264}$$

Dividing by $\delta\tau$ one obtains the geodesic equation:

$$\begin{aligned} \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu \frac{dx^\rho}{d\tau} &= 0\,, \\ \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} &= 0\,. \end{aligned} \tag{265}$$

Let us take a curve γ, parameterized as $x^\mu(\tau)$, and the vector field V^μ.

$$\begin{aligned} &V^\mu(\tau) := V^\mu(x(\tau)) \\ &\dot V^\mu(\tau) = \partial_\tau V^\mu(x(\tau))\,. \end{aligned} \tag{266}$$

Note that $\dot V^\mu(\tau)$ is not a vector field:

$$V^\mu(\tau + \delta\tau) - V^\mu(\tau) \simeq \partial_\tau V^\mu \delta\tau\,. \tag{267}$$

This failure can be understood as a consequence of illegally comparing vectors at two different points. In order to compare $V^\mu(\tau+\delta\tau)$ to $V^\mu(\tau)$ we parallel transport the latter:

$$\begin{aligned} V^\mu(\tau + \delta\tau) - (V^\mu - \Gamma^\mu_{\nu\rho} V^\nu \delta x^\rho) &\equiv \nabla_\tau V^\mu \delta\tau \\ &= \delta\tau(\partial_\tau V^\mu + \Gamma^\mu_{\nu\rho} V^\nu \dot x^\rho) \\ &= \delta\tau(\nabla_\tau V^\mu)\,, \end{aligned} \tag{268}$$

where

$$\nabla_\tau \equiv \dot x^\nu \nabla_\nu V^\mu = \dot x^\mu(\partial_\nu V^\mu + \Gamma^\mu_{\nu\rho} V^\rho)\,. \tag{269}$$

In the following we also use the following notation for the parallel transport of a vector from p to q:

$$V(q) = V(p) - \delta x^\mu \Gamma_\mu(V(p))\,. \tag{270}$$

We want to consider now what happens when we parallel transport around a closed curve. Consider the parallel transport of a vector V from point $p \to q \to r$ and $p \to s \to r$ in the figure.

$$\begin{aligned}
V(q) &= V - \delta_1 x^\mu \Gamma_\mu(V)\,, \quad V \equiv V(p)\,, \\
V(r) &= V(q) - \delta_2 x^\mu \Gamma_\mu(x+\delta_1 x)\big(V(q)\big) \\
&= V - \delta_1 x^\mu \Gamma_\mu(V) - \delta_2 x^\mu (\Gamma_\mu(x) + \delta_1 x^\nu \partial_\nu \Gamma_\mu)(V - \delta_1 x^\rho \Gamma_\rho(V))\,, \\
&= V - (\delta_1 x^\mu + \delta_2 x^\mu)\Gamma_\mu(V) - \delta_1 x^\mu \delta_2 x^\nu (\partial_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu)(V)
\end{aligned} \tag{271}$$

From $p \to s \to r$, we have to exchange $\delta_1 x^\mu \leftrightarrow \delta_2 x^\mu$ in the formula (271) we just computed.

$$V'(r) = V - (\delta_2 x^\mu + \delta_1 x^\mu)\Gamma_\mu(V) - \delta_1 x^\mu \delta_2 x^\nu (\partial_\nu \Gamma_\mu - \Gamma_\mu \Gamma_\nu)(V)\,. \tag{272}$$

We can take the difference

$$\Delta V := V(r) - V'(r) = \delta_1 x^\mu \delta_2 x^\nu (\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu)(V)\,. \tag{273}$$

The terms in the parantheses are precisely the matrix notation of the Riemann tensor,

$$\Delta V = -\delta_1 x^\mu \delta_2 x^\nu \boldsymbol{\mathcal{R}}_{\mu\nu}(V)\,. \tag{274}$$

Lecture 11

The Schwarzschild Solution

What is the gravitational field around a star or a planet? We set the energy momentum tensor to zero since we are in vacuum. The Einstein equations become

$$R_{\mu\nu} = 0 \tag{275}$$

since

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{276}$$

and $R = 0$. Since we have spherical symmetry, it is intuitive to work in spherical coordinates. This means that we work in coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$. The general ansatz is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}(r,t)dt^2 + 2g_{01}(r,t)drdt + g_{11}(r,t)dr^2 + W(r,t)r^2 d\Omega^2\,, \tag{277}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. This is the most general ansatz that is compatible with spherical symmetry. Now we will look for simplifications. One simplification is Birkhoff's theorem, which states that upon using the vacuum Einstein equations, the solution is necessarily stationary. This implies that one can choose the metric $g_{\mu\nu}$ to be independent of time. This simplifies the ansatz to

$$ds^2 = g_{00}(r)dt^2 + 2g_{01}(r)drdt + g_{11}(r)dr^2 + W(r)r^2 d\Omega^2\,. \tag{278}$$

We can next reparameterize time, $t \to t' = t - f(r)$, and $dt' = dt - f'(r)dr$, where $' \equiv \frac{\partial}{\partial r}$.

$$g_{00}(dt)^2 = g_{00}\big((dt')^2 + 2f'(r)drdt + (f'(r))^2 dr^2\big)\,. \tag{279}$$

The off-diagonal term $2(g_{01} + g_{00}f')$ can be set to zero by choosing

$$f = -\int^r \frac{g_{01}(r')}{g_{00}(r')} dr' \tag{280}$$

We can choose radial coordinates

$$\tilde{r} = \sqrt{W(r)} \tag{281}$$

which sets $W = 1$. We can finally write (278) as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega^2 \tag{282}$$

or, in matrix notation,

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\nu(r)} & 0 & 0 & 0 \\ 0 & e^{2\lambda(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{283}$$

and

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2\nu(r)} & 0 & 0 & 0 \\ 0 & e^{-2\lambda(r)} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}. \tag{284}$$

The Christoffel symbols for this metric are:

$$\begin{aligned} &\Gamma^1_{00} = \nu' e^{2\nu-2\lambda}\,, && \Gamma^0_{10} = \nu'\,, && \Gamma^1_{11} = \lambda'\,, \\ &\Gamma^2_{12} = \Gamma^3_{13} = r^{-1}\,, && \Gamma^1_{22} = -re^{-2\lambda}\,, && \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta} = \cot\theta\,, \\ &\Gamma^1_{33} = -r\sin^2\theta e^{-2\lambda}\,, && \Gamma^2_{33} = -\sin\theta\cos\theta\,. \end{aligned} \tag{285}$$

The Ricci tensor is

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\mu \Gamma_\nu + \Gamma_\lambda \Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu}\,, \tag{286}$$

where $\Gamma_\nu = \Gamma^\rho_{\rho\nu} = (0, \nu' + \lambda' + \frac{2}{r}, \cot\theta, 0)$.

$$\begin{aligned} R_{00} &= \partial_1\Gamma^1_{00} + \Gamma_1\Gamma^1_{00} - \Gamma^\rho_{0\lambda}\Gamma^\lambda_{0\rho} \\ &= \left\{(\nu'' + \nu'(2\nu' - 2\lambda'))e^{2\nu-2\lambda} + \left(\nu' + \lambda' + \frac{2}{r}\right)\nu' e^{2\nu-2\lambda} - 2(\nu')^2 e^{2\nu-2\lambda}\right\} \\ &= \left(\nu'' + (\nu')^2 - \lambda'\nu' + \frac{2\nu'}{r}\right)e^{2\nu-2\lambda} \end{aligned} \tag{287}$$

$$
\begin{aligned}
R_{00} &= \left(\nu'' + (\nu')^2 - \lambda'\nu' + \frac{2\nu'}{r}\right)e^{2\nu-2\lambda} \\
R_{11} &= -\nu'' + \nu'\lambda' - (\nu')^2 + 2\frac{\lambda'}{r} \\
R_{22} &= -(1 + r\nu' - r\lambda')e^{2\lambda} + 1 \\
R_{33} &= \sin^2\theta R_{22}\,.
\end{aligned} \tag{288}
$$

Now to solve $R_{\mu\nu} = 0$,

$$
e^{-2\nu+2\lambda}R_{00} + R_{11} = \frac{2}{r}(\nu' + \lambda') = 0 \tag{289}
$$

Therefore,

$$
\nu'(r) + \lambda'(r) = 0 \tag{290}
$$

and

$$
\nu(r) = -\lambda(r) + const. \tag{291}
$$

In the limit $r \to \infty$, $\nu, \lambda \to 0$ therefore $const. = 0$ and hence

$$
\nu = -\lambda\,. \tag{292}
$$

Next,

$$
R_{22} = 0 \tag{293}
$$

implies

$$
\begin{aligned}
1 + 2r\nu'(r)e^{2\nu} &= 1\,, \\
\frac{\partial}{\partial r}(re^{2\nu(r)}) &= 1\,,
\end{aligned} \tag{294}
$$

$$
re^{2\nu(r)} = r + C\,, \tag{295}
$$

$$
g_{00} = -e^{2\nu} = -\left(1 + \frac{C}{r}\right). \tag{296}
$$

The integration constant can be fixed

$$
g_{\mu\nu}dx^\mu dx^\nu = -\left(1 + \frac{C}{r}\right)dt^2 + \cdots \simeq (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu\,, \tag{297}
$$

by comparing this with (128),

$$
h_{00} = -2\phi = \frac{2GM}{r}\,, \tag{298}
$$

therefore

$$
C = -2GM\,. \tag{299}
$$

Finally, we have found the Schwarzschild solution:

$$
ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\,. \tag{300}
$$

This is valid for $r > r_s = 2GM$, where r_s is the Schwarzschild radius.

Time Dilatation and Redshift

Let there be two stationary observers, Alice and Bob, at distances, r_A and r_B. Let $x^A(t) = (t, r_A, 0, 0)$ and $x^\mu_B(t) = (t, R_B, 0, 0)$. Let $r_A > r_B \gg r_s$. Then $\dot{x}^\mu_A = (1, 0, 0, 0)$ and $\dot{x}^\mu_B = (1, 0, 0, 0)$. Let us compute

$$\begin{aligned} \tau_A &= \int_0^{\Delta\tau} \sqrt{-g_{\mu\nu}\dot{x}^\mu_A \dot{x}^\nu_A} dt = \int_0^{\Delta\tau} \sqrt{-g_{00}} dt \\ &= \int_0^{\Delta\tau} \left(1 - \frac{2GM}{r_A}\right)^{1/2} dt = \left(1 - \frac{2GM}{r_A}\right)^{1/2} \Delta t \end{aligned} \tag{301}$$

There is a similar formula for τ_B. The elapsed time for B is smaller than that for A:

$$\frac{\tau_B}{\tau_A} = \frac{\left(1 - \frac{2GM}{r_B}\right)^{1/2}}{\left(1 - \frac{2GM}{r_A}\right)^{1/2}} < 1\,. \tag{302}$$

This implies redshift because the frequency $\omega \sim \frac{1}{time}$, and hence the ratio

$$\begin{aligned} \frac{\omega_A}{\omega_B} = \frac{\tau_B}{\tau_A} &\simeq \left(1 - \frac{GM}{r_B}\right)\left(1 + \frac{GM}{r_A}\right) \\ &\simeq 1 - \frac{GM}{r_B} + \frac{GM}{r_A} < 1\,. \end{aligned} \tag{303}$$

Black Holes

Now let us talk about black holes. On light cones, $ds^2 = 0$ and therefore

$$dt = \pm\left(1 - \frac{2GM}{r}\right)^{-1} dr\,. \tag{304}$$

If $r \to \infty$, $dt = \pm dr$. Very far from the source, we have Minkowski space. When $r \to 2GM$, we can see from this formula, $dt \gg dr$. The light cone structure changes when we get close to the black hole. What happens to a light ray emitted close to the event horizon? Let us assume that there is a second observer who is falling into the black hole. At the event horizon, the light cone actually aligns with the event horizon. We want to see what happens to a freely falling observer in such a spacetime. A freely falling observer falls on a geodesic. We have to fix some parameterization for the curve, $x^\mu(\tau) = (t(\tau), r(\tau), 0, 0)$. Let τ be proper time and therefore $\dot{x}^2 = -1$. Let

$$u^\mu \equiv \frac{dx^\mu}{d\tau} \equiv (u^0, u^1, 0, 0)\,. \tag{305}$$

The geodesic equation for u^μ is

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu_{\nu\rho} u^\nu u^\rho\,. \tag{306}$$

Then

$$\frac{du^0}{d\tau} = -2\Gamma^0_{10}u^1u^0 = -2\frac{d\nu}{dr}\frac{dr}{d\tau}u^0 = -2\frac{d\nu}{d\tau}u^0\,, \tag{307}$$

$$e^{2\nu}\frac{du^0}{d\tau} + 2\frac{d\nu}{d\tau}e^{2\nu}u^0 = 0\,, \tag{308}$$

i.e.

$$\frac{d}{d\tau}(e^{2\nu}u^0) = 0\,, \tag{309}$$

therefore

$$e^{2\nu}u^0 \equiv K = constant. \tag{310}$$

$$g_{\mu\nu}u^\mu u^\nu = -e^{2\nu}(u^0)^2 + e^{-2\nu}(u')^2 \equiv -1 \tag{311}$$

$$(e^{2\nu}u^0)^2 - (u')^2 = e^{2\nu} = 1 - \frac{r_S}{r} \tag{312}$$

Now we can solve this expression for the components of u.

$$\begin{aligned} u^0 &= K\left(1 - \frac{r_s}{r}\right)^{-1}, \\ u^1 &= -\left(K^2 - 1 + \frac{r_s}{r}\right)^{1/2}. \end{aligned} \tag{313}$$

When $r \gg r_s$, $u^0 = K$ and $u^1 = -\sqrt{K^2-1}$. Thus for $K \neq 1$, there is an initial radial velocity. Let us take the initial radial velocity to be zero so that $K = 1$. Let us assume that the observer is close to the horizon so that $r = r_s + \epsilon$, $\frac{\epsilon}{r_s} \ll 1$.

$$\frac{dt}{dr} = \frac{u^0}{u^1} = -\frac{1}{1-\frac{r_s}{r}}\left(\frac{r}{r_s}\right)^{1/2} = -\frac{1}{1-\frac{r_s}{r_s+\epsilon}}\left(1+\frac{\epsilon}{r_s}\right)^{1/2} \tag{314}$$

$$\frac{dt}{dr} \simeq -\frac{r_s}{\epsilon} = -\frac{r_s}{r-r_s} \tag{315}$$

Upon integration of (315),

$$t = -r_s\ln(r - r_s) + const. \tag{316}$$

And therefore when $r \to r_s = 2GM$, we have $t \to \infty$. Therefore it will take an infinite time for an outside observer to see the in-falling observer to reach the horizon.

Lecture 12

We will continue our discussion of black holes. Recall the Schwarzschild solution in (300). Let us call $x^\mu = (t, r, \theta, \phi)$ the Schwarzschild coordinates, which is valid for $r > r_s$. Let us imagine two observers, A and B. A is stationary at a fixed radius from the black hole. B starts closer to the center of the black hole and falls freely toward the black hole. If B wants to communicate with A, B has to send light signals. The closer one is to the event horizon, the more upwards the light cones are tilted. Closer to the black hole, the light cones become narrower.

What is the experienced time of the observer that falls into the black hole? From the last lecture (313),

$$\begin{aligned}\frac{1}{u^1} = \frac{d\tau}{dr} &= -\left(K^2 - 1 + \frac{r_S}{r}\right)^{1/2} \overset{K=1}{=} -\left(\frac{r_S}{r}\right)^{1/2}, \\ \tau = \int d\tau &= \int \frac{d\tau}{dr} dr = -\sqrt{r_S}\int_{r_i}^{r_S} r^{-1/2} dr \\ &= -2\sqrt{r_S}[r^{1/2}]_{r_i}^{r_S} = 2\sqrt{r_S}(\sqrt{r_i} - \sqrt{r_S}) > 0\,.\end{aligned} \tag{317}$$

Observer B experiences a finite proper time to get to the event horizon, whereas A had measured an infinite time. This seems to be paradoxical.

Is there an extension of the Schwarzschild solution that goes beyond the event horizon? To investigate this, we look at a different coordinate system. For light cones,

$$ds^2 = 0\,, \tag{318}$$

$$dt = \pm\left(1 - \frac{2GM}{r}\right)^{-1} dr\,. \tag{319}$$

The *tortoise coordinates* are defined as:

$$r^* \equiv r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \tag{320}$$

This gives us

$$\begin{aligned} dr^* &= dr + 2GM \frac{1}{\frac{r}{2GM} - 1}\frac{dr}{2GM} \\ &= \frac{1}{1 - \frac{2GM}{r}} dr\,. \end{aligned} \tag{321}$$

Then (319) can be written as

$$dt = \pm dr^*\,. \tag{322}$$

This suggests us to introduce the *null coordinates*:

$$\begin{aligned} v &= t + r^* \quad \text{(advanced)} \\ u &= t - r^* \quad \text{(retarded)}\,. \end{aligned} \tag{323}$$

The trick is now to look for a coordinate system in which $r = r_s = 2GM$ is not so special. Let us make the coordinate transformation,

$$t \to v = t + r^*\,, \tag{324}$$

and leave the spatial coordinates (r, θ, ϕ) unchanged. We then compute

$$dv = dt + dr^* = dt + \left(1 - \frac{2GM}{r}\right)^{-1} dr\,, \tag{325}$$

$$-\left(1 - \frac{2GM}{r}\right)dv^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 - 2dtdr - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2\,, \tag{326}$$

$$2dvdr = 2dtdr + 2\left(1 - \frac{2GM}{r}\right)^{-1} dr^2\,. \tag{327}$$

ds^2 becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2\,. \tag{328}$$

We have now rewritten the Schwarzschild solution using one of the null coordinates. The coordinates used to write this metric (328) are called Eddington-Finkelstein coordinates. (328) can be written in terms of a metric $g_{\alpha\beta}$, where $\det g_{\alpha\beta} = -1$:

$$ds^2 = g_{\alpha\beta}(r)dx^\alpha dx^\beta + r^2 d\Omega^2\,; \quad x^\alpha = (v, r)\,, \quad \alpha, \beta = 0, 1\,, \tag{329}$$

$$g_{\alpha\beta} = \begin{pmatrix} -\left(1 - \frac{2GM}{r}\right) & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{330}$$

If we evaluate on the event horizon,

$$g_{\alpha\beta}(r = 2GM) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{331}$$

the metric is perfectly regular. The singularity in the original coordinates is then known as a coordinate singularity. Another example of a coordinate singularity arises when we try to describe the Euclidean 2-plane in polar coordinates, in which the metric is

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}\,, \tag{332}$$

which has a coordinate singularity at $r = 0$. In Cartesian coordinates, we know that such a singularity does not exist.

A test of whether a singularity is "real" or due to the coordinate system is to compute a curvature invariant such as

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}\,. \tag{333}$$

The curvature goes to infinity as r approaches zero. $r = 0$ is indeed a real physical singularity.

Gravitational Waves

We return to gravitational waves in order to work out how they interact with test particles. In previous lectures we discussed the linearized Einstein equations (recall (111 - 116)) and its solution:

$$\bar{h}_{\mu\nu} = C_{\mu\nu} e^{ikx} \,, \tag{334}$$

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,, \tag{335}$$

where $k^\mu = (\omega, 0, 0, \omega)$ is null and describes a wave traveling in the x^3 direction. Recall that we had implemented a gauge fixing $k^\mu C_{\mu\nu} = 0$, which gave us $\partial^\mu \bar{h}_{\mu\nu} = 0$, $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$. The component of the Riemann tensor (106) that we require is

$$R_{\mu 00\sigma} = \frac{1}{2} \partial_0^2 h_{\mu\sigma} \,. \tag{336}$$

Let there be a particle x^μ whose trajectory is given by the geodesic equation,

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} \,. \tag{337}$$

We need to have a bunch of test particles and investigate their relative motion. Let us assume there is a second particle, $x^\mu + \delta x^\mu$, whose corresponding geodesic equation is

$$0 = \frac{d^2 (x^\mu + \delta x^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x + \delta x) \frac{d(x^\nu + \delta x^\nu)}{d\tau} \frac{d(x^\rho + \delta x^\rho)}{d\tau} \,. \tag{338}$$

Let us take the difference of the two equations to first order in δx^μ:

$$0 = \frac{d^2(\delta x^\mu)}{d\tau^2} + \delta x^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} + 2\Gamma^\mu_{\nu\rho} \frac{d(\delta x^\nu)}{d\tau} \frac{dx^\rho}{d\tau} \,. \tag{339}$$

We want to rewrite this in a covariant form. Suppose V^μ is a vector along the curve x^μ. We defined the covariant derivative ∇ on V^μ as

$$\nabla V^\mu \equiv \frac{dV^\mu}{d\tau} + \dot{x}^\nu \Gamma^\mu_{\nu\rho} V^\rho \,. \tag{340}$$

Now we can specialize to $V^\mu \equiv \delta x^\mu$. One can prove that (339) is then equivalent to

$$\nabla^2 \delta x^\mu + R_{\sigma\nu}{}^\mu{}_\rho \delta x^\sigma \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \,. \tag{341}$$

This is also known as the geodesic deviation equation. For slowly moving test particles, the four-velocity is approximately given by $u^\mu = \dot{x}^\mu \simeq (1, 0, 0, 0)$, and $\tau \simeq t$. To first order in δx^μ and h, (341) is

$$\begin{aligned} &\frac{d^2}{dt^2} \delta x^\mu + R_{\sigma 0}{}^\mu{}_0 \delta x^\sigma = 0 \\ \Rightarrow\ &\frac{d^2(\delta x^\mu)}{dt^2} - \frac{1}{2} \left(\frac{d^2}{dt^2} h^\mu{}_\nu \right) \delta x^\nu = 0 \,. \end{aligned} \tag{342}$$

Let us assume first that $h_\times = 0$. Then (342) becomes

$$\frac{d^2}{dt^2}(\delta x^1) - \frac{1}{2}\frac{d^2}{dt^2}(h_+ e^{ikx})\delta x^1 = 0 \tag{343}$$

Moving the δx^1 inside the derivative of the second term, using that the difference is of higher order in h, one finds

$$\frac{d^2}{dt^2}\left[\left(1 - \frac{1}{2}h_+ e^{ikx}\right)\delta x^1\right] = 0\,. \tag{344}$$

The solution of this is

$$\left(1 - \frac{1}{2}h_+ e^{ikx}\right)\delta x^1 = a + bt\,. \tag{345}$$

Making the physical assumption that the solution does not grow linearly, we find the perturbative solution,

$$\delta x^1 \simeq \left(1 + \frac{1}{2}h_+ e^{ikx}\right)a^1\,, \tag{346}$$

where a^1 is the initial separation. Similarly,

$$\delta x^2 \simeq \left(1 - \frac{1}{2}h_+ e^{ikx}\right)a^2\,. \tag{347}$$

We assume the initial configuration of the particles is in a circle. When the gravitational wave comes through, one can see that this corresponds to the particles oscillating in a $+$ formation. Let us next assume that $h_+ = 0$ and $h_\times \neq 0$. One finds that

$$\begin{aligned} \delta x^1 &\simeq a^1 + \frac{1}{2}h_\times e^{ikx}a^2\,, \\ \delta x^2 &\simeq a^2 + \frac{1}{2}h_\times e^{ikx}a^1\,. \end{aligned} \tag{348}$$

These correspond to the particles oscillating in a $\times$ formation. One can define

$$\begin{aligned} h_R &= \frac{1}{\sqrt{2}}(h_+ + ih_\times)\,, \\ h_L &= \frac{1}{\sqrt{2}}(h_+ - ih_\times)\,. \end{aligned} \tag{349}$$

A pure h_R (h_L) wave rotates the particles in the right-handed (left-handed) sense.

Lecture 13

Production of Gravitational Waves

Recall from an earlier lecture that we solved the linearized Einstein equation and used a convenient redefinition of $h_{\mu\nu}$,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\,, \tag{350}$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$. With the Lorenz gauge,

$$\partial^\mu \bar{h}_{\mu\nu} = 0\,, \tag{351}$$

$$G_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu}\,. \tag{352}$$

The Einstein equation becomes

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \tag{353}$$

One can solve this equation by the method of Green's functions,

$$\Box G(x-y) = \delta^{(4)}(x-y)\,, \quad \Box \equiv \Box_x = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\,. \tag{354}$$

Thus the solution is

$$\bar{h}_{\mu\nu} = -16\pi G \int d^4y\, G(x-y) T_{\mu\nu}(y) \tag{355}$$

This should be familiar from electrodynamics. The retarded Green's function can be written explicitly as

$$G(x) = -\frac{1}{4\pi|\vec{x}|}\delta(|\vec{x}| - x^0)\theta(x^0)\,, \quad x \equiv x^\mu = (x^0, \vec{x})\,. \tag{356}$$

$\theta(x)$ is the Heaviside function,

$$\theta(x) = \begin{cases} 1, & x > 0\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{357}$$

This is to make sure that the retarded Green's function vanishes for $x^0 \leq 0$.

$$\bar{h}_{\mu\nu} = 4G \int d^4y \frac{1}{|\vec{x}-\vec{y}|}\delta\big(|\vec{x}-\vec{y}| - (x^0 - y^0)\big)\theta(x^0 - y^0) T_{\mu\nu}(y) \tag{358}$$

Perform $\int dy^0$, $y^0 \to x^0 - |\vec{x}-\vec{y}|$, $\theta(x^0 - y^0) = 1$.

$$\bar{h}_{\mu\nu}(x) = 4G \int d^3y \frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}(t - |\vec{x}-\vec{y}|, \vec{y}) \tag{359}$$

At a linearized level we can compute the gravitational field. The gravitational field at some given time t only depends on earlier times. If you imagine a matter distribution far away, it takes the corresponding travel time for the light:

$$t_r := t - |\vec{x}-\vec{y}|\,, \tag{360}$$

which is called retarded time. We want to derive the quadrupole formula which expresses the gravitational field in terms of T^{00}. It is often convenient to work in Fourier space:

$$\begin{aligned}\tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x}) &\equiv \frac{1}{\sqrt{2\pi}}\int dt e^{i\omega t}\bar{h}_{\mu\nu}(t) \\ &= \frac{4G}{\sqrt{2\pi}}\int dt d^3y e^{i\omega t}\frac{1}{|\vec{x}-\vec{y}|}T_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y}) \\ &= \frac{4G}{\sqrt{2\pi}}\int dt_r d^3y e^{i\omega t_r}e^{i\omega|\vec{x}-\vec{y}|}\frac{T_{\mu\nu}(t_r,\vec{y})}{|\vec{x}-\vec{y}|} \\ &\equiv 4G\int d^3y e^{i\omega|\vec{x}-\vec{y}|}\frac{\tilde{T}_{\mu\nu}(\omega,\vec{y})}{|\vec{x}-\vec{y}|}\end{aligned} \tag{361}$$

In the third line, we have changed integration variables to t_r. Let us imagine an observer at a distance r from matter, whose dimensions are described by δr, $\delta r \ll r$. The exponential,

$$\frac{e^{i\omega|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \simeq \frac{e^{i\omega r}}{r}\,, \quad r \text{ fixed}\,. \tag{362}$$

(361) becomes

$$\tilde{\bar{h}}_{\mu\nu}(\omega,\vec{x}) = 4G\frac{e^{i\omega r}}{r}\int d^3y\tilde{T}_{\mu\nu}(\omega,\vec{y})\,. \tag{363}$$

The goal is to compute $\tilde{\bar{h}}_{ij}$. We use the gauge condition,

$$\begin{aligned}\partial^\mu\bar{h}_{\mu\nu} &= 0\,, \\ \partial^0\bar{h}_{0\nu} &= -\partial^i\bar{h}_{i\nu} \\ \tilde{\bar{h}}_{0\nu} &= -\frac{1}{\omega}\partial^i\tilde{\bar{h}}_{i\nu}\,.\end{aligned} \tag{364}$$

The trick one can use to simplify this is to consider the conservation of the energy-momentum tensor, $\partial^\mu T_{\mu\nu} = 0$, which implies that the spatial divergence of $\tilde{T}$ is

$$-\partial_k\tilde{T}^{k\mu} = i\omega\tilde{T}^{0\mu}\,. \tag{365}$$

One can rewrite the integral,

$$\begin{aligned}\int d^3y\tilde{T}^{ij}(\omega,\vec{y}) &= \int d^3y\big(\partial_k(y^i\tilde{T}^{kj}) - y^i\partial_k\tilde{T}^{kj}\big) \\ &= i\omega\int d^3y y^i\tilde{T}^{0j} \\ &= \frac{i\omega}{2}\int d^3y(y^i\tilde{T}^{0j} + y^j\tilde{T}^{0i}) \\ &= \frac{i\omega}{2}\int d^3y\big(\partial_l(y^iy^j\tilde{T}^{0l}) - y^iy^j\partial_l\tilde{T}^{0l}\big) \\ &= \frac{i\omega}{2}\int d^3y\big(\partial_l(y^iy^j\tilde{T}^{0l}) + i\omega y^iy^j\tilde{T}^{00}\big) \\ &= -\frac{\omega^2}{2}\int d^3y y^iy^j\tilde{T}^{00}\end{aligned} \tag{366}$$

The trick here was to rewrite such that we get the divergence of the energy-momentum tensor. The boundary terms are zero. In the last step we have used the symmetry of the energy-momentum tensor. Let us define the quadrupole moment tensor

$$I^{ij} \equiv \int d^3y y^i y^j T^{00}(t,\vec{y}) \tag{367}$$

In summary, we have shown

$$\tilde{\bar{h}}_{ij}(\omega,\vec{x}) = -2G\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}(\omega)\,, \tag{368}$$

where $\tilde{I}_{ij}$ is the Fourier transform of the quadrupole moment tensor. Let us compute the inverse Fourier transform.

$$\begin{aligned}
\bar{h}_{ij}(t,\vec{x}) &= \frac{1}{\sqrt{2\pi}} \int d\omega e^{-i\omega t} \tilde{\bar{h}}_{ij}(\omega,\vec{x}) \\
&= -\frac{2G}{r}\frac{1}{\sqrt{2\pi}} \int d\omega \omega^2 e^{-i\omega(t-r)} \tilde{I}_{ij}(\omega) \\
&= -\frac{2G}{r}\frac{1}{\sqrt{2\pi}} \int d\omega \Big(-\frac{d^2}{dt^2} e^{-i\omega(t-r)} \Big) \tilde{I}_{ij}(\omega)
\end{aligned} \tag{369}$$

By taking the second time-derivative outside of the integral, we are left with the $\bar{h}$ in terms of the quadrupole moment tensor:

$$\bar{h}_{ij} = \frac{2G}{r}\frac{d^2}{dt^2} I_{ij}(t-r)\,. \tag{370}$$

(370) is called the quadrupole formula. This formula tells you how a given matter distribution encoded in T^{00} gives rise to a gravitational field.

Binary Star System

Let us consider two masses of equal mass M in the $x^1 - x^2$ plane. Let one of the masses, a, be at $(R,0)$ and the other, b, at $(-R,0)$. From

$$\frac{GM^2}{(2R)^2} = \frac{Mv^2}{R}\,, \tag{371}$$

the magnitude of the velocity of each mass is

$$v = \sqrt{\frac{GM}{4R}} \tag{372}$$

and the corresponding angular frequency is

$$\Omega \equiv \frac{2\pi}{T} = \left(\frac{GM}{4R^3}\right)^{1/2}. \tag{373}$$

The motion of each mass is described:

$$\begin{aligned} x_a^1 &= R\cos(\Omega t)\,, \quad x_a^2 = R\sin(\Omega t)\,. \\ x_b^1 &= -R\cos(\Omega t)\,, \quad x_b^2 = -R\sin(\Omega t)\,. \end{aligned} \tag{374}$$

The component of the energy momentum tensor, T^{00} is then

$$T^{00}(t,\vec{x}) = M\delta(x^3)\big[\delta(x^1 - R\cos(\Omega t))\delta(x^2 - R\sin(\Omega t)) + \delta(x^1 + R\cos(\Omega t))\delta(x^2 + R\sin(\Omega t))\big]\,. \tag{375}$$

There are two delta functions that localize towards the orbit on which these objects move.

$$\begin{aligned} I^{11} &= \int d^3y y^1 y^1 M\delta(y^2)\big[\delta(y^1 - R\cos(\Omega t))\delta(y^2 - R\sin\Omega t) + \dots\big] \\ &= M(R\cos\Omega t)^2 \end{aligned} \tag{376}$$

Working out all the components,

$$\bar{h}_{ij} = \frac{8GM}{r}\Omega^2 R^2 \begin{pmatrix} -\cos(2\Omega t_r) & -\sin(2\Omega t_r) & 0 \\ -\sin(2\Omega t_r) & \cos(2\Omega t_r) & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{377}$$

Energy of Gravitational Waves
In order to compute the energy of gravitational waves, we must determine an energy-momentum tensor for $h_{\mu\nu}$. Starting with the Einstein tensor, we expand in powers of h, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

$$G_{\mu\nu} = G^{(1)}_{\mu\nu}(h) + G^{(2)}_{\mu\nu}(h) + \cdots = 8\pi G T_{\mu\nu} \tag{378}$$

The $T_{\mu\nu}$ on the right-hand side is the energy-momentum tensor for all other matter. $G^{(2)}(h)$ is quadratic in h. We can rewrite the equation as

$$G^{(1)}_{\mu\nu} = 8\pi G(T_{\mu\nu} + t_{\mu\nu}) + \cdots\,, \tag{379}$$

where

$$t_{\mu\nu} \equiv -\frac{1}{8\pi G}G^{(2)}_{\mu\nu}(h) = -\frac{1}{8\pi G}\left(R^{(2)}_{\mu\nu} - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}\eta_{\mu\nu}\right)\,. \tag{380}$$

This motivates us to identify $t_{\mu\nu}$ as the energy-momentum tensor of the linearized gravitational field. In fact, when evaluated for solutions $h_{\mu\nu}$ of the linearized vacuum Einstein equations, $G^{(1)}_{\mu\nu}(h) = 0$, it is conserved as we prove now. The full Bianchi identity reads

$$\nabla_\mu G^{\mu\nu} = \partial_\mu G^{\mu\nu} + \Gamma^\mu_{\mu\rho}G^{\rho\nu} + \Gamma^\nu_{\mu\rho}G^{\mu\rho} \equiv 0\,. \tag{381}$$

Expanding this to second order in h using (378), we obtain two equations:

$$\partial_\mu G^{(1)\mu\nu} = 0\,, \tag{382}$$

$$\partial_\mu G^{(2)\mu\nu} = -\Gamma^{(1)\mu}_{\mu\rho} G^{(1)\rho\nu} - \Gamma^{(1)\nu}_{\mu\rho} G^{(1)\mu\rho} \, . \tag{383}$$

Thus, on shell, for which $G^{(1)}_{\mu\nu}(h) = 0$, we have

$$\partial_\mu t^{\mu\nu} = -\frac{1}{8\pi G} \partial_\mu G^{(2)\mu\nu}(h) = 0 \, . \tag{384}$$

Unfortunately, $t_{\mu\nu}$ is not gauge invariant under the lowest-order gauge transformations appropriate for the linearized fluctuations $h_{\mu\nu}$. To see this we recall that the full gauge transformations of $G_{\mu\nu}$ are given by:

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu} \, . \tag{385}$$

Expanding to second order in h with (378), and recalling that ξ is of the same order as h,

$$\delta^{(0)}_\xi G^{(1)}_{\mu\nu} = 0 \, , \tag{386}$$

$$\delta^{(0)}_\xi G^{(2)}_{\mu\nu} + \delta^{(1)}_\xi G^{(1)}_{\mu\nu} = \mathcal{L}_\xi G^{(1)}_{\mu\nu} \, , \tag{387}$$

where we expanded the gauge variations as $\delta_\xi = \delta^{(0)}_\xi + \delta^{(1)}_\xi + \cdots$ in powers of h, with

$$\begin{aligned} \delta^{(0)}_\xi h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \, , \\ \delta^{(1)}_\xi h_{\mu\nu} &= \mathcal{L}_\xi h_{\mu\nu} \, . \end{aligned} \tag{388}$$

On-shell we have $G^{(1)}_{\mu\nu}(h) = 0$ and thus (387) reduces to

$$\delta^{(0)}_\xi G^{(2)}_{\mu\nu} + \delta^{(1)}_\xi G^{(1)}_{\mu\nu} = 0 \, . \tag{389}$$

Next, we use that $G^{(1)}_{\mu\nu}$ by definition depends linearly on its argument so that we can write, using (388), $\delta^{(1)}_\xi G^{(1)}_{\mu\nu}(h) = G^{(1)}_{\mu\nu}(\mathcal{L}_\xi h)$, and thus

$$\delta^{(0)}_\xi G^{(2)}_{\mu\nu} = -G^{(1)}_{\mu\nu}(\mathcal{L}_\xi h) \, . \tag{390}$$

All in all, $t_{\mu\nu}$ transforms on-shell as

$$\delta_\xi t_{\mu\nu} = \frac{1}{8\pi G} G^{(1)}_{\mu\nu}(\mathcal{L}_\xi h) \, , \tag{391}$$

where we suppressed the superscript $^{(0)}$ as it is understood that we consider the lowest-order gauge transformations. Thus, $t_{\mu\nu}$ is not gauge invariant and hence does not provide meaningful physical information. This problem is usually circumvented by introducing an averaging procedure over several wavelength for which gauge invariance is recovered. This state of affairs is sometimes expressed with the slogan that "gravitational energy cannot be localized in space". Denoting the averaging schematically by brackets $\langle \cdots \rangle$ one finally defines the energy-momentum tensor of gravitational waves as

$$t_{\mu\nu} \equiv -\frac{1}{8\pi G} \langle G^{(2)}_{\mu\nu}(h) \rangle \, . \tag{392}$$

We do not have to specify the averaging $\langle \cdots \rangle$ beyond it being defined by an integral so that any gradients average to zero: $\langle \partial_\alpha (\cdots) \rangle = 0$. Gauge invariance then follows immediately from (391) and recalling that the first-order "Einstein operator" reads

$$G^{(1)}_{\mu\nu}(a) = -\frac{1}{2}\left[\Box a_{\mu\nu} - 2\partial_{(\mu}\partial^\rho a_{\nu)\rho} + \partial_\mu \partial_\nu a + (\partial^\rho \partial^\sigma a_{\rho\sigma} - \Box a)\eta_{\mu\nu}\right] , \tag{393}$$

so that $\langle G^{(1)}_{\mu\nu}(a) \rangle = 0$ for arbitrary symmetric tensor $a_{\mu\nu}$.

The formula (392) can be used to compute the energy of gravitational waves and hence the energy loss of a binary star system due to emission of gravitational waves. The period of the binary system changes accordingly and has been measured to be in perfect agreement with the predictions of general relativity.

Addendum: Precession of Perihelia

We want to analyze the motion of freely falling objects in a spherically symmetric gravitational field $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ of the form

$$h_{\mu\nu} \;=\; \begin{pmatrix} -2\phi(r) & 0 & 0 & 0 \\ 0 & -2\psi(r) & 0 & 0 \\ 0 & 0 & -2\psi(r) & 0 \\ 0 & 0 & 0 & -2\psi(r) \end{pmatrix} , \tag{394}$$

where ϕ and ψ are two a priori independent functions of the radial coordinate r, in order to exhibit the precession of perihelia, whose effect for Mercury was one of the first spectacular confirmations of general relativity.

We use the equations of motion for a point mass (the geodesic equation) in the form

$$\frac{d}{d\tau}(g_{\mu\nu}u^\nu) = \frac{1}{2}(\partial_\mu g_{\nu\rho})u^\nu u^\rho \ , \tag{395}$$

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the 4-velocity with proper time τ. This form of the equation can be quickly seen to be equivalent to the more familiar form involving Christoffel symbols by working out the derivative of $g_{\mu\nu}$ on the left-hand side with the chain rule and bringing it to the right-hand side. Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ this equation reduces to

$$\frac{d}{d\tau}(\eta_{\mu\nu}u^\nu + h_{\mu\nu}u^\nu) = \frac{1}{2}\partial_\mu h_{\nu\rho}\, u^\nu u^\rho \ . \tag{396}$$

Let the curve be parameterized by $x^\mu(\tau) = (t(\tau), x^i(\tau))$, so that $u^\mu \equiv (\dot{t}, \dot{x}^i)$, $\dot{} \equiv \frac{\partial}{\partial\tau}$. We now evaluate the four equations (396) for (394). For $\mu = 0$,

$$\frac{d}{d\tau}\big((-1-2\phi)\dot{t}\big) = \frac{1}{2}\partial_0 h_{\nu\rho}u^\nu u^\rho = 0 \ , \tag{397}$$

using that $h_{\mu\nu}$ does not depend explicitly on time. Thus, we infer the conservation law

$$(1+2\phi)\dot{t} = E = \text{const.} \tag{398}$$

where E can be identified with energy. Next, for $\mu = i$,

$$\frac{d}{d\tau}\big((1-2\psi)\delta_{ij}\dot{x}^j\big) = \frac{1}{2}\partial_i h_{\nu\rho}u^\nu u^\rho = -\partial_i\phi\, \dot{t}^2 - \partial_i\psi\, |\dot{\mathbf{x}}|^2 \ , \tag{399}$$

using obvious 3-vector notation with $|\mathbf{x}|^2 \equiv \delta_{ij}x^i x^j$. Employing the chain rule for differentiation with respect to $r = \sqrt{\delta_{ij}x^i x^j}$ one obtains

$$\partial_i\phi = \phi'(r)\frac{\partial r}{\partial x^i} = \frac{\phi'(r)}{r}\delta_{ij}x^j \tag{400}$$

and similarly for ψ. Thus, (399) can be written as

$$\frac{d}{d\tau}\big((1-2\psi)\dot{x}^i\big) = -\frac{1}{r}(\phi'(r)\dot{t}^2 + \psi'(r)|\dot{\mathbf{x}}|^2)x^i \ , \tag{401}$$

cancelling the Kronecker delta on both sides. With this formula we can now prove that the three quantities

$$L_i \equiv -(1-2\psi)\epsilon_{ijk}\,\dot{x}^j x^k \tag{402}$$

are conserved. (Here ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol.) Indeed,

$$\frac{dL_i}{d\tau} = -\frac{d}{d\tau}\big((1-2\psi)\dot{x}^j\big)\epsilon_{ijk}x^k - (1-2\psi)\epsilon_{ijk}\,\dot{x}^j\dot{x}^k = 0 \ , \tag{403}$$

since both terms vanish separately by total antisymmetry of ϵ_{ijk} (in the first term one has to use that the derivative is proportional to x^j by (401) and hence this term is proportional to $\epsilon_{ijk}x^jx^k = 0$). The conserved quantities can be identified with angular momentum. We can thus assume that the motion is confined to a plane, which we may as well identify with the $z=0$ plane. The conservation of $L \equiv L_3$ then reads

$$\frac{d}{d\tau}\big((1-2\psi)(\dot{x}y - \dot{y}x)\big) = 0 \ . \tag{404}$$

We next replace (x,y) by polar coordinates (r,θ), so that the 3-vector reads

$$\mathbf{x} = (r\cos\theta, r\sin\theta, 0) \ , \tag{405}$$

from which one infers by a quick computation

$$\begin{aligned} |\dot{\mathbf{x}}|^2 &= \dot{r}^2 + r^2\dot{\theta}^2 \\ \dot{x}y - \dot{y}x &= -r^2\dot{\theta} \ . \end{aligned} \tag{406}$$

Thus, the conserved angular momentum reads

$$L \equiv (1-2\psi)r^2\dot{\theta} = \text{const.} \tag{407}$$

We have one more conserved quantity we can use, $g_{\mu\nu}u^\mu u^\nu = -1$,

$$-(1+2\phi)\dot{t}^2 + (1-2\psi)|\dot{\mathbf{x}}|^2 = -1 \ . \tag{408}$$

We now use (398) and (406) to write this as

$$-\frac{E^2}{(1+2\phi)} + (1-2\psi)(\dot{r}^2 + r^2\dot{\theta}^2) = -1 \, . \tag{409}$$

Next, we multiply by $(1-2\psi)$ and rewrite

$$\Big(\frac{E^2}{1+2\phi} - 1\Big)(1-2\psi) = (1-2\psi)^2(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{L^2}{r^4\dot{\theta}^2}(\dot{r}^2 + r^2\dot{\theta}^2) \ , \tag{410}$$

using (407) in the last step. We can finally eliminate the dependence on proper time:

$$\left(\frac{E^2}{1+2\phi}-1\right)\frac{1-2\psi}{L^2}=\frac{1}{r^4}\left(\frac{\dot{r}^2}{\dot{\theta}^2}+r^2\right)=\frac{1}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2+r^2\right)\ . \tag{411}$$

This is the differential equation we want to solve. It can be simplified by introducing the new variable $\rho\equiv\frac{1}{r}$,

$$\Big(\frac{E^2}{1+2\phi}-1\Big)\frac{1-2\psi}{L^2}=\left(\frac{d\rho}{d\theta}\right)^2+\rho^2\ . \tag{412}$$

So far we have not made any assumption on the functions $\phi(r)$ and $\psi(r)$, but now we want to solve (412) for the actual functions arising in general relativity through the Schwarzschild solution. We first have to rewrite the Schwarzschild solution in coordinates that are appropriate for comparison with (394), where all three spatial coordinates are on equal footing. The corresponding coordinates for the Schwarzschild solution are referred to as isotropic coordinates, for which one defines a new radial function r' by

$$r=r'\left(1+\frac{r_s}{4r'}\right)^2\ , \tag{413}$$

where $r_s=2GM$. From this one computes

$$\begin{aligned}dr&=\left(1-\frac{r_s}{4r'}\right)\left(1+\frac{r_s}{4r'}\right)dr'\ ,\\ 1-\frac{r_s}{r}&=\frac{\big(1-\frac{r_s}{4r'}\big)^2}{\big(1+\frac{r_s}{4r'}\big)^2}\ .\end{aligned} \tag{414}$$

The Schwarzschild solution in isotropic coordinates then reads

$$ds^2=-\frac{\big(1-\frac{r_s}{4r}\big)^2}{\big(1+\frac{r_s}{4r}\big)^2}dt^2+\left(1+\frac{r_s}{4r}\right)^4\left(dx^2+dy^2+dz^2\right)\ , \tag{415}$$

where we finally dropped the prime on r. Introducing again $\rho\equiv\frac{1}{r}$ and expanding in powers of $r_s\rho$, this reads

$$ds^2=-\left(1-r_s\rho+\frac{1}{2}r_s^2\rho^2+\cdots\right)dt^2+\left(1+r_s\rho+\frac{3}{8}r_s^2\rho^2+\cdots\right)\left(dx^2+dy^2+dz^2\right)\ . \tag{416}$$

We can finally read off ϕ and ψ to second order in $r_s\rho$:

$$\begin{aligned}\phi&=-\frac{1}{2}r_s\rho+\frac{1}{4}r_s^2\rho^2+\cdots\ ,\\ \psi&=-\frac{1}{2}r_s\rho-\frac{3}{16}r_s^2\rho^2+\cdots\ .\end{aligned} \tag{417}$$

Returning now to the equation (412) we want to solve, we expand the left-hand side to the same order:

$$\frac{E^2-1}{L^2}+\frac{2E^2-1}{L^2}r_s\rho+\frac{15E^2-3}{8L^2}r_s^2\rho^2+\cdots=\left(\frac{d\rho}{d\theta}\right)^2+\rho^2 \tag{418}$$

This equation can be solved by bringing it to the form of an harmonic oscillator with an external force by differentiating with respect to θ on both sides and then dividing by $2\frac{d\rho}{d\theta}$:

$$\frac{2E^2-1}{2L^2}r_s + \frac{15E^2-3}{8L^2}r_s^2\rho = \frac{d^2\rho}{d\theta^2} + \rho \tag{419}$$

or in explicit harmonic oscillator form:

$$\frac{d^2\rho}{d\theta^2} + \omega^2\rho = \frac{2E^2-1}{2L^2}r_s \ , \tag{420}$$

where

$$\omega \equiv \sqrt{1 - \frac{15E^2-3}{8L^2}r_s^2} = 1 - \frac{15E^2-3}{16L^2}r_s^2 + \cdots \tag{421}$$

To first order in r_s we thus have $\omega = 1$, in which case (420) can be quickly verified to be solved by

$$\rho = \frac{2E^2-1}{2L^2}r_s\left(1 + e\cos\theta\right) \ , \tag{422}$$

where e is an integration constant. This is the well-known form of a parameterized ellipse with radial and angular coordinates (r,θ) (but measured from one of the focal points!):

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta} \ , \tag{423}$$

where a is the semi-major axis and $e = \sqrt{1-\frac{b^2}{a^2}}$ the eccentricity. We have thus recovered, to lowest order, Kepler's law that closed orbits follow an ellipse. In particular, $\rho(\theta)$ is 2π-periodic, confirming that the orbits are closed. This changes once we go to second-order, in which case we have $\omega \neq 1$ and so the general solution of (420),

$$\rho = \frac{2E^2-1}{2L^2\omega^2}r_s\left(1 + e\cos(\omega\theta)\right) \ , \tag{424}$$

is no longer 2π periodic. In order to estimate the relative angle measuring the failure of the orbit to be closed we assume that the object is slowly moving so that its energy reduces to that corresponding to its rest mass. We then have $E \simeq mc^2 = 1$ in natural units, and

$$\omega = 1 - \frac{3}{4L^2}r_s^2 + \cdots \ . \tag{425}$$

We now ask: what is the total angle that the object has to cover in order for the radial distance r (or equivalently ρ) to return to their initial value (say at the perihelion). Denoting this angle by $2\pi + \Delta\theta$ one infers from (424)

$$\omega(2\pi + \Delta\theta) = 2\pi \, . \tag{426}$$

Thus,

$$\Delta\theta = 2\pi\left(\frac{1}{\omega} - 1\right) = 2\pi\left(1 + \frac{3}{4L^2}r_s^2 + \cdots - 1\right) \simeq \frac{6\pi}{4L^2}(2GM)^2 = \frac{6\pi G^2M^2}{L^2} \ , \tag{427}$$

which for mercury precisely accounts for the observed discrepancy.