General Relativity

Olaf Hohm

Exercise Sheet 2

April 28, 2019

Exercise 1: Charged particles in electrodynamics

Consider a particle of electric charge e whose world-line is parametrized by $x^\mu(\tau)$ with arbitrary parameter τ. Its current density is defined by

$$j^\mu(x) \; = \; e\int d\tau\, \dot{x}^\mu(\tau)\, \delta^{(4)}(x - x(\tau)) \; . \qquad (1)$$

[Note: the argument of $j^\mu(x)$ on the left-hand side denotes an arbitrary point $x \in \mathbb{M}$ in Minkowski space, which should not be confused with the function $x(\tau)$.]

1.1) Prove that j^μ is conserved, $\partial_\mu j^\mu = 0$, for instance by showing that for an arbitrary test function $\varphi(x)$ vanishing rapidly at infinity one has

$$\int d^4x\, \varphi(x)\, \partial_\mu j^\mu(x) \; = \; 0 \; . \qquad (2)$$

1.2) Is (1) reparametrization invariant?

1.3) Compute the equations of motion for the particle $x(\tau)$ and the electromagnetic field $A_\mu(x)$ as the Euler-Lagrange equations from the action

$$S[A(x), x(\tau)] \; = \; -m\int ds \; + \; \int d^4x\Big(-\frac{1}{4}\, F^{\mu\nu}F_{\mu\nu} \; - \; A_\mu\, j^\mu\Big) \; . \qquad (3)$$

Exercise 2: Energy-momentum tensor

Consider electrodynamics in vacuum described by the action (3) with $m = 0$ and $j^\mu = 0$.

2.1) Prove translation invariance by showing invariance of the Maxwell action under

$$\delta_a A_\mu \;=\; a^\nu \partial_\nu A_\mu \;, \tag{4}$$

where a^μ is a constant but otherwise arbitrary 4-vector.

2.2) Determine the energy-momentum tensor of the electromagnetic field from Noether's theorem and translation invariance as follows: add to (4) a gauge transformation with parameter $\Lambda = -a^\nu A_\nu$ and show that the resulting variations take the form

$$\delta_a A_\mu \;=\; a^\nu F_{\nu\mu} \;. \tag{5}$$

Then promote a^μ to a function of x, i.e., $a^\mu = a^\mu(x)$, and read off the energy momentum tensor from the variation

$$\delta_a S[A] \;=\; -\int d^4x\, \partial_\mu a_\nu\, T^{\mu\nu} \;. \tag{6}$$

2.3) Prove that the energy-momentum tensor is conserved *on-shell*, $\partial_\mu T^{\mu\nu} = 0$, upon using the vacuum Maxwell equations.

Exercise 3: Fierz-Pauli theory

In the lecture we determined the quadratic action for the gravitational field $h_{\mu\nu}$, given by the Fierz-Pauli action

$$S_{FP} \;=\; \int d^4x \Big(-\frac{1}{4}\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} + \frac{1}{2}\partial_\mu h^{\mu\nu}\partial^\rho h_{\nu\rho} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{4}\partial^\mu h \partial_\mu h \Big) \;, \tag{7}$$

where $h := \eta^{\mu\nu} h_{\mu\nu}$.

3.1) Show that this action and its general variation can be written as

$$S_{FP} \;=\; -\frac{1}{2}\int d^4x\, h^{\mu\nu} G_{\mu\nu}(h) \;, \qquad \delta S_{FP} \;=\; -\int d^4x\, \delta h^{\mu\nu} G_{\mu\nu}(h) \;, \tag{8}$$

where $G_{\mu\nu}$ is the (linearized) Einstein tensor defined as $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R\, \eta_{\mu\nu}$, where

$$R_{\mu\nu} \;:=\; -\frac{1}{2}\big(\Box h_{\mu\nu} - 2\,\partial^\rho \partial_{(\mu} h_{\nu)\rho} + \partial_\mu \partial_\nu h\big) \;, \qquad R \;:=\; \eta^{\mu\nu} R_{\mu\nu} \;. \tag{9}$$

3.2) What is the Bianchi identity implied by the gauge invariance $\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$? Verify it directly.