General Relativity

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Exercise Sheet 1

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Exercise 1: Motion of free particle and reparametrization invariance

Consider a particle moving along a curve in Minkowski space parametrized by $x^\mu(\tau)$, $\tau \in \mathbb{R}$. The action principle reads

$$S[x(\tau)] \; = \; \int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \; , \tag{1}$$

where the dot denotes differentiation with respect to τ.

1.1) Show that the Euler-Lagrange equations for (1) are

$$\dot{u}^\mu \; = \; 0 \; , \quad \text{where} \quad u^\mu \; := \; \frac{\dot{x}^\mu}{\sqrt{-\dot{x}^2}} \; . \tag{2}$$

1.2) Consider an infinitesimal reparametrization $\tau \rightarrow \tau' = \tau - \lambda(\tau)$, where λ is a small but otherwise arbitrary function of τ, and take $x(\tau)$ to be a scalar under these transformations: $x'(\tau') = x(\tau)$. Prove that the infinitesimal variations are given by

$$\delta_\lambda x^\mu \; := \; x'^\mu(\tau) - x^\mu(\tau) \; = \; \lambda(\tau)\dot{x}^\mu(\tau) \; . \tag{3}$$

[Hint: For infinitesimal variations one can ignore higher-order terms in λ.]

1.3) Prove that the action (1) is reparametrization invariant under (3).

1.4) Prove that under (3)

$$\delta_\lambda u^\mu \; = \; \lambda \dot{u}^\mu \; . \tag{4}$$

Consider now an alternative action for $x^\mu(\tau)$ and a new dynamical variable $n(t)$ ('lapse function' or 'einbein'),

$$S[x(\tau), n(\tau)] \; = \; -\frac{1}{2}\int d\tau \big(n^{-1}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - n\big) \tag{5}$$

1.5)* Determine the equations of motion for x and n and prove that they are equivalent to the equations of motion for (1).

1.6)* How should n transform under reparametrizations so that (5) is invariant?

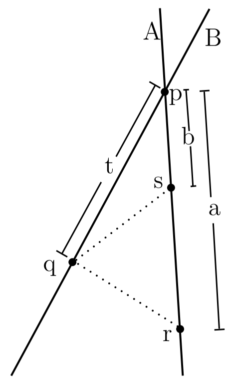

Figure 1

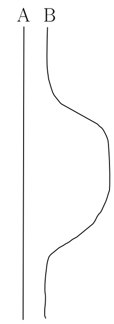

Figure 2

Exercise 2: Twin Paradox

Assume that two observers A and B moving in Minkowski space relative to each other meet at a point ("event") p, as shown in figure 1. Moreover, there are light rays passing back and forth, which the observers want to use in order to determine whether the clock of the other observer ticks at the same or a different rate.

2.1) The spacetime interval between events q and p indicated in the figure can be directly determined by observer B from his clock reading between these events: $I(p,q) = -t^2$. Show that the invariant interval can also be determined by observer A as the product of the two times indicated in the figure, so that:

$$I(p,q) \;=\; -ab\,. \tag{6}$$

2.2) A measures the clock rate of B as follows: A observes at event s the light emitted from q and thus B's clock reading at event q. Next, at event p, A observes B's clock reading directly. Finally, A compares B's elapsed time between q and p with his own elapsed time (between s and p). Show that from A's point of view B's clocks run *faster*.

2.3) B measures the clock rate of A as follows: B observes at event q the light emitted from r and thus A's clock reading at event r. Next, at event p, B observes A's clock reading directly. Finally, B compares A's elapsed time between r and p with his own elapsed time (between q and p). Show that from B's point of view A's clocks run *faster*.

2.4) A starts having doubts whether his procedure was the "correct" one to compare the clock rates. A decides to determine the event q' that is simultaneous with q (in his rest frame!) and compare his elapsed time between q' and p with B's elapsed time between q and p. Show that then from A's point of view B's clocks run *slower*. By symmetry, the same follows if B decides to compare the clock rates using the new procedure.

2.5) Assume now that A and B are identical twins being initially at rest with respect to each other. B starts moving and eventually turns around, as sketched in figure 2. After the twins reunite, who is younger, A or B?