From Yangian Symmetry to Factorized Scattering

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Chapter 1

Motivation and Outline

Integrable models such as the harmonic oscillator, the Kepler problem and the Heisenberg spin chain are characterized by – an often unexpected – beauty and simplicity of the structures emerging in their solution. These features allow for a detailed understanding of their fundamental properties and make them an ideal starting point for the study of more general, i.e. non-integrable, models. The origin and form of appearance of the simplifications in integrable models is diverse and there exist several definitions of integrability. For finite-dimensional classical mechanical systems the various approaches are closely related and can be united to a universal definition of classical integrability called Liouville integrability. By contrast, their connection for infinite-dimensional models, but also finite-dimensional quantum systems, is not always evident. In this thesis we study several perspectives on quantum integrability. We focus on its definition via a tower of commuting local charges that are conserved and an infinite-dimensional symmetry algebra called Yangian. We discuss these versions of integrability mainly in the context of factorized scattering. It makes the scattering of an arbitrary number of particles reducible to the successive scattering of two particles. This simplification of a model's scattering properties is a common feature of integrable quantum field theories (QFTs) and quantum spin chains. In fact, this motivates the definition of quantum integrability via factorization of scattering itself – a definition which is considered to be the least problematic. Nevertheless, the formulation of quantum integrability via hidden symmetries is more desirable. Therefore, we investigate in this thesis how factorized scattering originates from defining symmetry properties of quantum integrable models. For (1+1)-dimensional QFTs with massive particles it was already shown that the existence of a tower of conserved local charges directly implies factorization of scattering. We review this proof briefly on the basis of [1–4] in order to motivate the subsequent discussion.

Factorized Scattering in Integrable Quantum Field Theories

Let us consider a (1+1)-dimensional relativistic QFT with massive particles. For such a continuous model with infinitely many degrees of freedom, integrability is often defined via the existence of infinitely many commuting independent conserved local charges $\mathcal{Q}_s$ of different Lorentz spin s satisfying[1]

$$\mathcal{Q}_s \, |a, u\rangle \sim p^s \, |a, u\rangle \, . \tag{1.1}$$

[1]Note that this definition of quantum integrability is problematic since there exists no consistent notion of independent operators in quantum systems.

Here the vector $|a, u\rangle$ is associated to the wave-packet of a one-particle state. a denotes the set of quantum numbers characterizing the particle's type. u is its rapidity which is associated to the light-cone momentum $p = m_a e^u$ via the mass m_a. Due to locality the charges $\mathcal{Q}_s$ act on m-particle states as

$$\mathcal{Q}_s |a_1, u_1; a_2, u_2; ...; a_m, u_m\rangle \sim \sum_{n=1}^{m} p_n^s |a_1, u_1; a_2, u_2; ...; a_m, u_m\rangle . \tag{1.2}$$

Note that these multi-particle states are asymptotic in the sense that the particles are largely separated compared to the typical interaction range of the model. Similar to the notation of the one-particle state, the a_i characterize the m particles which move with rapidities u_i, $i = 1, ..., m$.

Demanding the conservation of $\mathcal{Q}_s$ constrains a scattering process in such a way that its incoming state $|in\rangle$ and outgoing state $|out\rangle$ obey

$$\langle in|\mathcal{Q}_s|in\rangle = \langle out|\mathcal{Q}_s|out\rangle . \tag{1.3}$$

Using (1.2) this gives the constraint

$$\sum_{i=1}^{m_{in}} p_i^s = \sum_{f=1}^{m_{out}} p_f^s, \tag{1.4}$$

where m_{in} and m_{out} are the numbers of incoming and outgoing particles. They have momenta $\{p_i\}$, $i = 1, ..., m_{in}$, and $\{p_f\}$, $f = 1, ..., m_{out}$, respectively. For integrable QFTs with infinitely many degrees of freedom this relation must hold for infinitely many s. Consequently, the set of incoming and outgoing momenta is identical and the particle number is conserved during a scattering process, i.e. $m_{in} = m_{out}$. In fact, (1.3) also implies factorization of scattering. This statement can be verified by the introduction of a quantity called S-matrix. It characterizes scattering processes and will be of great importance throughout this thesis. It maps the *out*-state of a scattering process to the *in*-state via[2]

$$|in\rangle = \mathrm{S}\,|out\rangle . \tag{1.5}$$

Together with the conservation equation (1.3) this results in $[\mathrm{S}, \mathcal{Q}_s] = 0$ or

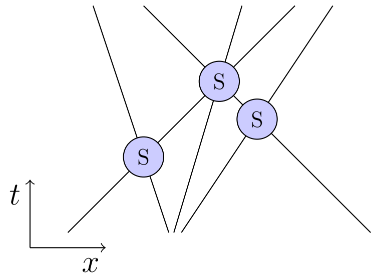
$$\langle in|\mathrm{S}|out\rangle = \langle in|e^{-i\alpha\mathcal{Q}_s}\mathrm{S}e^{i\alpha\mathcal{Q}_s}|out\rangle \qquad \forall\alpha \in \mathbb{R}. \tag{1.6}$$

The term on the left hand side is the transition amplitude of the process $|in\rangle \to |out\rangle$. Interpreting $\mathcal{Q}_s$ as generator of finite symmetry transformations $e^{i\alpha\mathcal{Q}_s}$ implies that the term on the right hand side of (1.6) corresponds to the transition amplitude between the symmetry transformed *in*- and *out*-states. The equality of both amplitudes for arbitrary *in*- and *out*-states implies the equality of the scattering amplitudes in the original and symmetry transformed system.

This constraint can be analysed geometrically[3]. For this purpose, we examine the Fourier-transformed multi-particle states $|a_1, x_1; ...; a_m, x_m\rangle$ in position space. The x_i denote the approximate positions of the wave-packets. The evolution of this vector in time, i.e. the propagation of the particles through space, can be illustrated in a diagram such as

[2]This definition is in accordance with the standard convention where S maps the outgoing state into the incoming state, rather than vice versa.

[3]A more algebraic discussion can be found in [3].



with space- and time-direction as indicated. The lines correspond to the positions of the asymptotically free wave-packets. Their slope is proportional to the velocities of the particles which, for simplicity, shall all have equal masses in the diagram. The shaded circles are associated to scattering processes via the action of the S-matrix. We already used the above conclusion that there is no particle creation or annihilation during a scattering process and that the set of momenta for the incoming and outgoing states is the same.

Using this picture we can easily interpret (1.6) for $s = 1$. Due to (1.1) the charge $\mathcal{Q}_1$ corresponds to the momentum operator. As a generator of symmetry transformations, it generates translations in space, i.e.

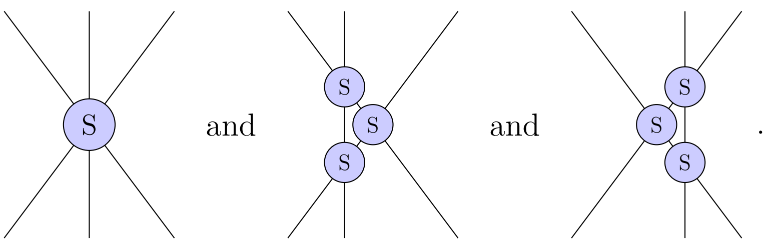
$$e^{i\alpha\mathcal{Q}_1} |a_1, x_1; a_2, x_2; ...; a_m, x_m\rangle \sim |a_1, x_1 + \alpha; a_2, x_2 + \alpha; ...; a_m, x_m + \alpha\rangle . \tag{1.7}$$

The parameter $\alpha \in \mathbb{R}$ is a measure for the magnitude of the displacement. Geometrically the action of the operator $e^{i\alpha Q_1}$ on a multi-particle state corresponds to a parallel shift of the particles' lines in space. As a consequence, the conservation equation (1.6) for $s = 1$ in position space implies the translation invariance of scattering processes as the one shown above.

What kind of symmetry transformations do the remaining charges $\mathcal{Q}_s$ for $s > 1$ generate? In order to understand this, one assumes that the wave-function of a single particle with approximate position x_1 and momentum p_1 is a Gaussian wave-packet in position space [3, 5]. One finds[4] that the operators $e^{i\alpha\mathcal{Q}_s}$ for $s > 1$ displace wave-packets of momentum p_1 by an amount $-\alpha s p_1^{s-1}$ rather than by a constant. Since the particles in a scattering process have different momenta, they will be shifted by different amounts

$$e^{i\alpha\mathcal{Q}_s} |a_1, x_1; ...; a_m, x_m\rangle \sim |a_1, x_1 - \alpha s p_1^{s-1}; ...; a_m, x_m - \alpha s p_m^{s-1}\rangle . \tag{1.8}$$

As a result, we may displace each m-particle scattering process in such a way that it only contains two-particle scattering processes. In particular, (1.6) for $s > 1$ implies the equality of the transition amplitudes of



This phenomenon is called factorization of scattering since the m-particle S-matrix factorizes into a product of several two-particle S-matrices. The equality of the last

[4]See [3] or the analogue discussion for spin chains in section 3.4.

two diagrams illustrates the famous quantum Yang-Baxter equation and ensures the consistency of the factorization.

Note that this result strongly relies on having a single space-direction. The Coleman-Mandula theorem [6] states that for $d > 2$ the S-matrix is trivial, i.e. $|in\rangle \sim |out\rangle$, if there exists a single conserved higher-order charge. Similar to the above charges $\mathcal{Q}_s$ for $s > 1$, a higher-order charge generates momentum-dependent shifts of a particle's trajectory. The conservation of such a charge results in the equality of the transition amplitude of some scattering process and the symmetry-transformed one. Since in more than one space-direction we may shift all lines apart such that they do not cross, the transition amplitude is only non-zero for $|in\rangle \sim |out\rangle$.

Factorized Scattering for Integrable Spin Chains

The above proof for the factorization of scattering in (1+1) dimensions as a result of the hidden symmetries associated to integrability is remarkable. Similar to QFT-models, one may define integrability in the context of spin chains via the existence of a tower of conserved local charges. Interestingly, factorized scattering is occurring in these models as well. This gives the motivation to translate the above proof into the language of spin chains which we will do in this thesis. This discussion will reveal the close connection between conserved local charges and factorization of scattering for integrable spin chains.

From Yangian Symmetry to Factorized Scattering

In the main part of this thesis we investigate another formulation of quantum integrability that is based on the existence of an infinite-dimensional symmetry algebra called Yangian. For many integrable models the existence of infinitely many commuting conserved local charges and the invariance of the model's S-matrix under a Yangian are closely related. In these models factorization of the S-matrix can be understood as a consequence of a conserved Yangian via the existence of local charges. Nevertheless, the close connection between local charges and the Yangian is not always manifest which we will review on the basis of [7]. In these cases it is not clear how the Yangian algebra and factorization of scattering are connected and one might ask whether the factorization can also be understood as a direct consequence of imposing its invariance under Yangian symmetry. This will be the issue of the main part of this thesis. In the analysis we focus on specific symmetry algebras: We look at the Yangians $Y[\mathfrak{su}(\mathrm{n})]$, $Y[\mathfrak{su}(1|1)]$ and $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$ in their fundamental representations. The latter are particularly relevant in the context of the AdS/CFT correspondence which we want to briefly review now.

AdS/CFT Correspondence and Integrable Spin Chains

The AdS/CFT correspondence is a useful tool in the investigation of strongly-interacting conformal field theories (CFTs). The duality can be used to relate strong coupling QFT-observables to weak coupling string theory calculations and vice versa. The most prominent example in this context is $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) which is supposed to be dual to type-II B string theory living on the Anti-de-Sitter (AdS) space $AdS_5 \times S^5$. An interesting region for the study of this conjectured duality is the planar limit where integrable structures occur on both sides of the duality. On the CFT-side integrable spin chains arise in the eigenvalue problem of the dilatation operator. In particular, in $\mathcal{N} = 4$ SYM an integrable spin chain with $\mathfrak{psu}(2,2|4)$ symmetry algebra appears [8, 9]. A subsector of this model is an integrable long-range $\mathfrak{su}(1|2)$ spin chain. The scattering of the so-called magnons, which are the quasi-particles in spin chain models, is described by $\mathfrak{su}(1|1)$-invariant S-matrices. In this thesis we lift the invariance of the S-matrix under the Lie superalgebra $\mathfrak{su}(1|1)$ to an invariance under the corresponding Yangian $Y[\mathfrak{su}(1|1)]$ and check whether it implies factorization of scattering.

The S-matrix of the $\mathfrak{psu}(2,2|4)$ spin chain of $\mathcal{N} = 4$ SYM is the tensor-product of two S-matrices which are invariant under $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ with two bosonic and two fermionic particles in the fundamental representation [10]. This motivates the investigation of the latter algebra. In [11] it was already shown that the corresponding $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$-invariant S-matrix is also invariant under the associated Yangian. In this thesis we explore the connection between the invariance of the S-matrix under this Yangian and factorization of scattering.

Relevance of Yangian-Invariants in Other Contexts

In the following, we mainly discuss the Yangian in the context of integrable spin chains having a Yangian-invariant S-matrix. Nevertheless, Yangian-invariant objects occur also in other contexts. In [12] it is shown that the tree-level scattering amplitudes in four-dimensional $\mathcal{N} = 4$ SYM exhibit a Yangian structure corresponding to the Lie symmetry algebra $\mathfrak{psu}(2, 2|4)$. In fact, the corresponding generators act similarly to Yangian generators in spin chain models on sites defined by the fields in the color ordered amplitude. Another domain where Yangian-invariants are relevant is the hexagon construction of correlation functions in planar $\mathcal{N} = 4$ SYM. In [13] it is shown that the $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$-invariant S-matrix that we will discuss in this thesis is a building block in the hexagon approach to correlation functions. As proposed by the authors of this paper, it could be interesting to examine whether their approach – that includes demanding factorization of the form factors – can be reformulated in terms of the Yangian. Thus, the study of the implications of imposing Yangian invariance of an object is relevant in a much broader field than that of integrable spin chain models.

Outline

Having motivated the main parts of this thesis, let us outline its contents. We start with an introduction to quantum integrability in chapter 2. This is done with special focus on spin chains and their most important toy model – the Heisenberg spin chain. During its discussion, we will encounter factorization of scattering. In order to reveal the origin of this simplicity of the model's dynamics, we then introduce the Lax formalism. Doing so, we show that the Heisenberg spin chain has infinitely many conserved local charges. This discussion is the motivation for several definitions of integrability in the context of general quantum systems.

Afterwards, we move on to more general spin chains and their S-matrices in chapter 3. We show the connection between the existence of local conserved charges and factorized scattering by translating the argumentation for massive relativistic QFTs given in section 1.1 into the language of spin chains.

In chapter 4 we introduce the formulation of integrability that is based on the infinite-dimensional symmetry algebra called Yangian. We motivate it by showing its close connection to the existence of local charges for the Heisenberg spin chain. We end this chapter by developing the Yangian constraints on the S-matrix.

In the following chapters 5-7 we explicitly evaluate these constraints and investigate the results with particular focus on consistent factorization. We begin with a fairly simple example – the fundamental representation of the Yangian $Y[\mathfrak{su}(\mathrm{n})]$ that is associated to the Lie algebra $\mathfrak{su}(\mathrm{n})$. Its discussion will help us to set the ground for more challenging algebras.

We proceed with the Yangian $Y[\mathfrak{su}(1|1)]$ whose Lie superalgebra $\mathfrak{su}(1|1)$ allows for a wider range of structure. In its fundamental representation it not only contains a bosonic but also a fermionic particle and thus it is a good starting point for the discussion of Yangian-symmetric S-matrices of supersymmetric models. First we investigate this symmetry in the context of conventional models where the Hamiltonian is invariant under $\mathfrak{su}(1|1)$. We then move on to more exotic models which are mainly studied in the context of integrable spin chains that arise in the planar limit of CFTs relevant for the AdS/CFT correspondence. Here the Hamiltonian is not invariant under an external symmetry algebra but is rather a part of it, see e.g. [14]. Since its eigenvalues depend on the momenta of the magnons, the representation of this algebra becomes momentum-dependent. This feature is often called dynamical, in contrast to the conventional undynamical case.

The last algebra that we discuss is $\mathfrak{su}(2|2)$. It contains two bosonic and two fermionic particles in its fundamental representation. Furthermore, it allows for length-changing operators. This feature can be captured in the representation of the algebra by making it dynamical. In order to obtain the correct eigenvalues of the Hamiltonian, it is necessary to extend the algebra by two central charges to $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$. We will comment on this in more detail in chapter 7 and calculate the Yangian-invariant S-matrices in this setup.

In chapter 8 we summarize the results of this thesis and conclude with a brief outlook on possible future directions of research.

Chapter 2

Review of Integrability for Spin Chains

The field of integrability emerged together with the search for exactly solvable classical mechanical systems since integrability and solvability often go hand in hand. Nevertheless, both terms emphasize different facets when trying to understand the dynamics of a system. As discussed above, integrability is an intrinsic feature of a system resulting in unexpected simplicity of its equations and is thus closely related to symmetries. In contrast to this, solvability describes one's ability to solve a model's dynamics to the very end. Unfortunately, there exists no universally applicable definition of integrability for all physical systems. Integrability of classical mechanical systems with a finite number of degrees of freedom is well-defined (cf. appendix A) whereas its notion for quantum systems and systems with infinitely many degrees of freedom is less satisfactory.

In this chapter we review some aspects of integrability for spin chains with particular focus on the Heisenberg spin chain. We start by discussing this model and its dynamics in sections 2.1 and 2.2. Afterwards, we move on to the origin of the arising simplicity. Doing so, we introduce the Lax formalism and show that the Heisenberg spin chain allows for a tower of conserved local charges in section 2.3 which motivates one definition of quantum integrability. In section 2.4 we derive an important equation in the context of integrable quantum models – the quantum Yang-Baxter equation – which will also be relevant in our discussion of factorized scattering. The sections 2.3 and 2.4 focus on integrability for a special class of integrable spin chains, the so-called fundamental spin chains. In the last section 2.5 of this chapter we discuss integrability in the context of a general, i.e. non-fundamental, spin chain.

2.1 Definition of the Heisenberg Spin Chain

An $\mathfrak{su}(\mathrm{n})$ spin chain consists of N sites, each equipped with a vector space transforming under a representation of $\mathfrak{su}(\mathrm{n})$. It has either periodic, cyclic, open or infinite boundary conditions. Such a model corresponds to a theory whose Hamiltonian is $\mathfrak{su}(\mathrm{n})$ invariant. Let us focus on the Heisenberg spin-1/2-chain in this section. It is a special type of $\mathfrak{su}(\mathrm{n})$ spin chains for n= 2 and lattice sites transforming in the fundamental representation of $\mathfrak{su}(2)$ associated to spin 1/2. Its discussion will help us to set up and illustrate the framework for more general spin chains that are relevant in the following. This section is developed in close analogy to [15–17].

The State Space

The natural coordinates for a system consisting of localized spins are spin variables. Since we focus on the fundamental representation of $\mathfrak{su}(2)$, the spin state at each site can be written as a linear combination of spin-up $|{\uparrow}\rangle$ and -down $|{\downarrow}\rangle$ vectors. They span a 2-dimensional Hilbert space denoted by $\mathscr{H}$. The total state of the spin chain is described by the tensor product of all of the individual vectors. It lives in the Hilbert space $\mathscr{H}^{\otimes N}$ having 2^N basis vectors, e.g.

$$|m_1\rangle \otimes ... \otimes |m_n\rangle \otimes ... \otimes |m_N\rangle =: |m_1 \; ... \; m_n \; ... \; m_N\rangle \tag{2.1}$$

where m_n is either $\uparrow$ or $\downarrow$.

Spin Operators

Now let us turn to operators acting on these states. An operator acting non-trivially on one site only is denoted by

$$\mathcal{X}_n := \mathbb{I} \otimes \mathbb{I} \otimes ... \otimes \underset{n\text{th site}}{\mathcal{X}} \otimes ... \otimes \mathbb{I} \tag{2.2}$$

with $n = 1, ..., N$. $\mathbb{I} \in \mathscr{H}$ is the identity matrix.

Take as an example the spin operators[1] $\mathcal{S}_n^a$ which measure the spin of the nth site of the chain in the direction a with $a = x, y, z$. These operators satisfy the usual angular momentum commutation relations

$$[\mathcal{S}_{n_1}^a, \mathcal{S}_{n_2}^b] = i\delta_{n_1 n_2}\varepsilon^{abc}\mathcal{S}_{n_1}^c \tag{2.3}$$

locally. ε^{abc} denotes the completely antisymmetric tensor with $\varepsilon^{xyz} = 1$. The smallest non-trivial representation of the corresponding Lie algebra $\mathfrak{su}(2)$ associated to spins $s = 1/2$ maps the three generators at each site to the Pauli matrices σ^a

$$\mathcal{S}_n^a = \tfrac{\sigma^a}{2} \tag{2.4}$$

($\hbar = 1$) with

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

Thus, the one-site Hilbert space $\mathscr{H}$ is the space $\mathbb{C}^2$ of all 2×2 hermitian matrices with identity matrix

$$\mathbb{I} = (\sigma^a)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.6}$$

The basis of the states at each site can be chosen to consist of the eigenvectors of σ_z that are

$$|{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.7}$$

The total spin operators of a spin chain are given by

$$\mathcal{S}^a = \sum_{n=1}^{N} \mathcal{S}_n^a. \tag{2.8}$$

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[1]Note the difference between the spin operators denoted by $\mathcal{S}$ and the S-matrix denoted by S.

The Heisenberg Hamiltonian

Having a notation for states of the spin chain and spin operators we may proceed with the spin chain's dynamics. A Heisenberg spin chain is characterized by the Hamiltonian

$$\mathcal{H} = \sum_n \left(\tfrac{1}{4} - \mathcal{S}_n^a \mathcal{S}_{n+1}^a \right). \tag{2.9}$$

For spin chains with periodic boundaries the sum over n runs from 1 to N and we identify the spins $\mathcal{S}_{N+n}^a$ and $\mathcal{S}_n^a$. For open spin chains n varies between 1 and $N-1$. The Hamiltonian commutes with the total spin operators $\mathcal{S}^a$, $[\mathcal{H}, \mathcal{S}^a] = 0$, and thus the Heisenberg spin chain is indeed $\mathfrak{su}(2)$ symmetric.

The Hamiltonian only includes nearest-neighbor spin-spin interactions. One gets a deeper insight into these interactions by introducing raising and lowering spin operators $\mathcal{S}_n^\pm = \mathcal{S}_n^x \pm i\mathcal{S}_n^y$ which satisfy the usual raising and lowering conditions

$$\begin{aligned} &\mathcal{S}^+ \left|\uparrow\right\rangle = 0, && \mathcal{S}^- \left|\uparrow\right\rangle = \left|\downarrow\right\rangle, && \mathcal{S}^z \left|\uparrow\right\rangle = +\tfrac{1}{2} \left|\uparrow\right\rangle, \\ &\mathcal{S}^+ \left|\downarrow\right\rangle = \left|\uparrow\right\rangle, && \mathcal{S}^- \left|\downarrow\right\rangle = 0, && \mathcal{S}^z \left|\downarrow\right\rangle = -\tfrac{1}{2} \left|\downarrow\right\rangle. \end{aligned} \tag{2.10}$$

Using these operators we can rewrite (2.9) as

$$\mathcal{H} = \sum_n \left(\tfrac{1}{4} - \tfrac{1}{2}(\mathcal{S}_n^+ \mathcal{S}_{n+1}^- + \mathcal{S}_n^- \mathcal{S}_{n+1}^+) - \mathcal{S}_n^z \mathcal{S}_{n+1}^z \right). \tag{2.11}$$

Thus, the Hamiltonian allows for scattering of neighboring $\left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle$ leaving the total spin eigenvalue s^z of $\mathcal{S}^z$ constant in time $[\mathcal{H}, \mathcal{S}^z] = 0$. Furthermore, the spin chain's energy eigenvalue gets smaller the more spins are aligned, i.e. the model describes a ferromagnetic spin chain.

2.2 The Coordinate Bethe Ansatz

In order to understand the physics of a spin chain, we need to find the energy spectrum and eigenstates of the Hamiltonian. Since $\dim(\mathscr{H}^{\otimes N}) = 2^N$, the Hamiltonian $\mathcal{H}$ is a $2^N \times 2^N$ matrix. It can become rather involved to diagonalize such a matrix by brute force to obtain the energy spectrum. In fact, there exists a more convenient approach called *Coordinate Bethe Ansatz* [18] for the Heisenberg spin chain. In this section we outline this approach for periodic spin chains on the basis of [4], [15] and [19, 20].

Translational and $\mathfrak{su}(2)$ Invariance

By confining this discussion to homogeneous spin chains with periodic boundary conditions, we gain translational symmetry of the system. This symmetry is manifest for the Heisenberg spin chain since the Hamiltonian $\mathcal{H}$ in (2.9) for periodic boundaries commutes with the shift operator $\mathcal{U}$ that shifts the chain by one site, i.e. $[\mathcal{H}, \mathcal{U}] = 0$. Shifting a chain of length N by N sites must re-establish its original form, i.e.

$$\mathcal{U}^N = \mathbb{I}^{\otimes N} \tag{2.12}$$

where $\mathbb{I}^{\otimes N}$ denotes the identity in $\mathscr{H}^{\otimes N}$. Therefore the eigenvalues U_k of $\mathcal{U}$ are

$$U_k = e^{ik} \quad \text{with} \quad k = \tfrac{2\pi}{N}\, n,\ n = 0, ..., N-1 \tag{2.13}$$

which is invariant under $k \to k + 2\pi$. We identify the total momentum operator[2] $\mathcal{K}$ as the generator of translations, i.e.

$$\mathcal{U} = e^{i\mathcal{K}}. \tag{2.14}$$

Thus, the variables k in (2.13) coincide with the momentum eigenvalues and the periodicity condition results in the quantization of the total momentum of the chain.

The Vacuum State

The Coordinate Bethe Ansatz exploits the fact that the model has even more symmetry. The invariance under $\mathfrak{su}(2)$ symmetry transformations can be used to decompose the space $\mathscr{H}^{\otimes N}$ of eigenstates into $N+1$ subspaces, called spin-multiplets, consisting of states with the same eigenvalue s^z of $\mathcal{S}^z$. The subspace with highest $s^z = N/2$ consists of a single vector

$$|0\rangle = |\uparrow\uparrow \dots \uparrow\rangle \,. \tag{2.15}$$

It is called Bethe reference state and – since this eigenvector of $\mathcal{S}^z$ is non-degenerate – it is also an eigenvector of $\mathcal{H}$ with vanishing energy $\mathcal{H}\,|0\rangle = 0$. This state corresponds to the ferromagnetic vacuum of a magnetic spin chain. Applying the shift operator to this state leaves it unchanged, $\mathcal{U}\,|0\rangle = |0\rangle$, i.e. the total momentum k is 0.

The First Excited State

The spin-multiplet with $s^z = N/2 - 1$ consists of states with a single spin down. Let us denote its position in the chain by n and the vector associated to a pure state in this multiplet by

$$|n\rangle = |\uparrow\uparrow \dots \uparrow\underset{n\text{th site}}{\downarrow}\uparrow \dots \uparrow\rangle \,. \tag{2.16}$$

This subspace is N-dimensional and the Hamiltonian $\mathcal{H}$ acts inside the multiplet. We need to diagonalize it, i.e. find a basis of eigenvectors, in order to obtain the energy spectrum. The Bethe ansatz proposes that the energy eigenstates are plane waves of the form

$$|k\rangle = \sum_{n=1}^{N} e^{ikn}\,|n\rangle \,. \tag{2.17}$$

Note that $|k\rangle$ denotes a state in momentum space, whereas $|n\rangle$ is a vector in position space. The state (2.17) is indeed an eigenstate with energy eigenvalue

$$\mathcal{H}\,|k\rangle = 2\sin^2\tfrac{k}{2}\,|k\rangle \,. \tag{2.18}$$

Acting with the shift operator defined by $\mathcal{U}\,|n\rangle = |n-1\rangle$ on such a state yields

$$\mathcal{U}\,|k\rangle = \sum_{n=1}^{N} e^{ikn}\,|n-1\rangle = \sum_{n=1}^{N} e^{ik(n+1)}\,|n\rangle = e^{ik}\,|k\rangle \,. \tag{2.19}$$

[2]Strictly speaking, $\mathcal{K}$ is a quasi-momentum operator since the translational symmetry is not continuous.

Accordingly, $|k\rangle$ is also an eigenvector of $\mathcal{U}$ and we identify k with the momentum discussed above which is why we already denoted it with the same letter. Demanding periodicity of $|k\rangle$, i.e. its invariance under $n \to n + N$, we obtain the momentum quantization condition discussed above

$$e^{ikN} = 1. \tag{2.20}$$

Consequently, k can be identified as the quasi-momentum of the quasi-particle with spin down that can hop between neighboring sites in a vacuum of spin up states. This state is often called *one-magnon state*.

The Second Excited State

Let us move on to the spin multiplet $s^z = N/2 - 2$ consisting of states with two spins down. Similar to (2.16) we denote such a spin chain by

$$|n_1, n_2\rangle = |\uparrow\uparrow \dots \uparrow\underset{n_1}{\downarrow}\uparrow \dots \uparrow\underset{n_2}{\downarrow}\uparrow \dots \uparrow\rangle \tag{2.21}$$

with $1 \leq n_1 < n_2 \leq N$. The Bethe ansatz for the energy eigenstates corresponding to two excitations is

$$|k_1, k_2\rangle = \sum_{1 \leq n_1 < n_2 \leq N} \left(e^{i(k_1 n_1 + k_2 n_2)} + \mathrm{S}(k_1, k_2) e^{i(k_2 n_1 + k_1 n_2)} \right) |n_1, n_2\rangle \,. \tag{2.22}$$

This is not automatically an energy eigenvector. One finds that the factor $\mathrm{S}(k_1, k_2)$ must satisfy

$$\mathrm{S}(k_1, k_2) = \frac{u_1 - u_2 - i}{u_1 - u_2 + i} \tag{2.23}$$

where we introduced the rapidity variables

$$u_j = \tfrac{1}{2} \cot \tfrac{k_j}{2} \qquad \Leftrightarrow \qquad e^{ik_j} = \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}. \tag{2.24}$$

Calculating the associated energy eigenvalues gives

$$\mathcal{H}\, |k_1, k_2\rangle = \left(2 \sum_{j=1}^{2} \sin^2 \tfrac{k_j}{2} \right) |k_1, k_2\rangle \,, \tag{2.25}$$

which is the sum of two single-magnon energies corresponding to momenta k_1 and k_2. Thus, this state is called *two-magnon state* and the two summands in the ansatz (2.22) can be interpreted as spin chain states containing two magnons of momenta k_1 and k_2. In the first term the magnon of momentum k_2 is to the right of the magnon with momentum k_1 and vice versa for the second term. The factor connecting both terms is S and is called *scattering matrix* (or *S-matrix*) which is – in this model – just a number. Demanding translational invariance of $|k_1, k_2\rangle$ under the shifts $n_{1,2} \to n_{1,2} + N$ yields the quantization of the total momentum $k_1 + k_2$

$$e^{i(k_1 + k_2)N} = 1. \tag{2.26}$$

By imposing invariance under $n_1 \to n_1 + N$ we obtain constraints on $\mathrm{S}(k_1, k_2)$

$$\mathrm{S}(k_1, k_2) = e^{ik_2 N} \qquad\qquad \mathrm{S}(k_2, k_1) = e^{ik_1 N}, \tag{2.27}$$

i.e. a magnon picks up a pure phase factor when shifted around the chain due to scattering with the other magnon. If we let both magnons travel around the chain we end up with the original situation $\mathrm{S}(k_1,k_2)\mathrm{S}(k_2,k_1)=1$. Combining (2.23) and (2.27) yields the so-called *Bethe equations* for two magnons

$$\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^N=\prod_{\substack{k=1\\k\neq j}}^{2}\frac{u_j-u_k+i}{u_j-u_k-i},\qquad j=1,2 \tag{2.28}$$

that enable us to determine the allowed momenta k_1 and k_2.

Higher Excited States

One can carry out this procedure for every number of magnons $m=0,...,N$ corresponding to the multiplet of spin $s^z=N/2-m$. The Bethe ansatz for the corresponding eigenstates $|k_1,k_2,...,k_m\rangle$ is a generalization of the eigenvectors (2.15), (2.17) and (2.22) discussed above. Take as an example $m=3$ and define the vector

$$|k_1,k_2,k_3\rangle_o:=\sum_{1\leq n_1<n_2<n_3\leq N}e^{i(k_1n_1+k_2n_2+k_3n_3)}\,|n_1,n_2,n_3\rangle\,. \tag{2.29}$$

Then, the Bethe ansatz is given by

$$\begin{aligned}|k_1,k_2,k_3\rangle=&|k_1,k_2,k_3\rangle_o+\mathrm{S}_{213}\,|k_2,k_1,k_3\rangle_o+\mathrm{S}_{132}\,|k_1,k_3,k_2\rangle_o\\&+\mathrm{S}_{231}\,|k_2,k_3,k_1\rangle_o+\mathrm{S}_{312}\,|k_3,k_1,k_2\rangle_o+\mathrm{S}_{321}\,|k_3,k_2,k_1\rangle_o\end{aligned} \tag{2.30}$$

where each factor S_{ijk} in general depends on all momenta $\mathrm{S}_{ijk}=\mathrm{S}_{ijk}(k_1,k_2,k_3)$. Demanding that this state is an eigenstate of the Hamiltonian fixes these coefficients and we obtain [20]

$$\begin{aligned}|k_1,k_2,k_3\rangle=&|k_1,k_2,k_3\rangle_o+\mathrm{S}_{12}\,|k_2,k_1,k_3\rangle_o+\mathrm{S}_{23}\,|k_1,k_3,k_2\rangle_o\\&+\mathrm{S}_{12}\mathrm{S}_{13}\,|k_2,k_3,k_1\rangle_o+\mathrm{S}_{23}\mathrm{S}_{13}\,|k_3,k_1,k_2\rangle_o+\mathrm{S}_{12}\mathrm{S}_{13}\mathrm{S}_{23}\,|k_3,k_2,k_1\rangle_o\end{aligned} \tag{2.31}$$

with $\mathrm{S}_{ij}:=\mathrm{S}(k_i,k_j)$ which is the two-particle S-matrix satisfying

$$\mathrm{S}(k_j,k_k)=\frac{u_j-u_k+i}{u_j-u_k-i}. \tag{2.32}$$

This implies that the overall scattering factor for magnons going around the chain consists of two-magnon S-matrices for each scattering between magnons along the way. This factorization of multi-particle S-matrices into two-particle S-matrices works for all m and is a characteristic feature of integrable models that we focus on in this thesis. The associated energy eigenvalues for arbitrary m are given by

$$\mathcal{H}\,|k_1,k_2,...,k_m\rangle=\frac{1}{2}\sum_{j=1}^{m}\frac{1}{u_j^2+\frac{1}{4}}\,|k_1,k_2,...,k_m\rangle \tag{2.33}$$

in agreement with the above results. Demanding translational invariance of the eigenstates yields the total momentum conservation and fixes products of S-matrices

$$e^{ik_jN}=\prod_{\substack{k=1\\k\neq j}}^{m}\mathrm{S}(k_j,k_k). \tag{2.34}$$

Combining (2.32) and (2.34) yields the generalization of (2.28) as

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^N = \prod_{\substack{k=1\\k\neq j}}^{m} \frac{u_j - u_k + i}{u_j - u_k - i}, \qquad j = 1, ..., m. \tag{2.35}$$

These so-called *Bethe-equations* constrain the allowed momenta.

Thus, we encounter a remarkable simplicity of the equations describing the dynamics of the Heisenberg spin chain. Its whole spectrum and scattering information is encoded in a set of algebraic relations: The calculation of energies is reduced to the calculation of allowed momenta and the corresponding one-magnon energies. The scattering of m magnons reduces to the successive scattering of two magnons. This simplicity of the model's solution hints on its integrability whose origin we want to discuss in the following section.

2.3 Lax Operator and Monodromy Matrix

In this section we introduce the quantum version of the classical Lax formalism[3] along the lines of [15–17]. It is crucial in the discussion of spin chain models whose dynamics are unexpectedly simple because it provides the natural framework in which these simplifications can be traced back to symmetries. Furthermore, it will help us to motivate various definitions of quantum integrability. Since we want to discuss general spin chain models in the following, we develop this section with no special focus on the Heisenberg spin chain. Nevertheless, we refer to it whenever it is useful as an example.

The Lax Operator

Let us begin by denoting the Hilbert space at a site n of a general spin chain model by $\mathscr{H}_n$. Then the complete state space of a spin chain of length N is $\mathscr{H}^{\otimes N}$ and is given by the tensor product of the individual Hilbert spaces

$$\mathscr{H}^{\otimes N} = \bigotimes_{n=1}^{N} \mathscr{H}_n. \tag{2.36}$$

In this thesis we only study spin chains whose Hilbert spaces $\mathscr{H}_n$ are isomorphic. Whenever it is useful, we denote the single-site Hilbert space by $\mathscr{H}_0$ with

$$\mathscr{H}_n \equiv \mathscr{H}_0 \qquad \forall\, n \in \{1, ..., N\}. \tag{2.37}$$

In the Lax formalism the discussion of the dynamics of the model is based on the *Lax operator* $\mathrm{L}(\lambda)$ rather than the Hamiltonian $\mathcal{H}$. It acts on the tensor product of two spaces $\mathrm{L}_{n,a}$: $\mathscr{H}_n \otimes \mathscr{V}_a \to \mathscr{H}_n \otimes \mathscr{V}_a$, i.e. on the physical Hilbert space $\mathscr{H}_n$ at site n of the chain and on an auxiliary space $\mathscr{V}_a$. Moreover, it depends on the *spectral parameter* λ which is a complex continuous variable whose role becomes clear in the following. The Lax operator for the Heisenberg spin chain is given by

$$\mathrm{L}_{n,a}(\lambda) = -i\lambda(\mathbb{I} \otimes \mathbb{I})_{n,a} + (\mathcal{S}^b \otimes \sigma^b)_{n,a} \tag{2.38}$$

and acts on $\mathscr{H}_n = \mathbb{C}^2$ and $\mathscr{V}_a = \mathbb{C}^2$. It has two features which make the Heisenberg spin chain a so-called *fundamental model* [7]:

[3] A brief review of the classical Lax formalism is given in the appendix A.2.

- The auxiliary space $\mathscr{V}_a$ and physical space $\mathscr{H}_0$ are identical, i.e. $\mathscr{V}_a \equiv \mathscr{H}_0$.
- The Lax operator corresponds to the permutation operator $\mathbb{P}$ on $\mathscr{H}_0 \otimes \mathscr{H}_0$ at a point $\lambda = \lambda_0$, i.e. $\mathrm{L}(\lambda_0) = \mathbb{P}$.

For the Heisenberg spin chain we identify the point λ_0 by introducing the permutation operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ at sites n_1 and n_2

$$\mathbb{P}_{n_1,n_2} = \tfrac{1}{2}(\mathbb{I} \otimes \mathbb{I} + \sigma^a \otimes \sigma^a)_{n_1,n_2}. \tag{2.39}$$

In order to confirm that this is indeed the permutation operator, one can explicitly act on a two-spin state represented by the vectors in (2.7). This operator satisfies

$$\begin{aligned}
&\mathbb{P}_{n,n_1}\mathbb{P}_{n,n_2} = \mathbb{P}_{n_1,n_2}\mathbb{P}_{n,n_1} = \mathbb{P}_{n,n_2}\mathbb{P}_{n_2,n_1}, \\
&\mathbb{P}_{n_1,n_2} = \mathbb{P}_{n_2,n_1}, \\
&\mathbb{P}_{n_1,n_2}\mathbb{P}_{n_1,n_2} = \mathbb{I}_{n_1,n_2}.
\end{aligned} \tag{2.40}$$

Using $\mathbb{P}$ we may rewrite the expression in (2.38) as

$$\mathrm{L}_{n,a}(\lambda) = -i\left(\lambda - \tfrac{i}{2}\right)\mathbb{I}_{n,a} + \mathbb{P}_{n,a} \tag{2.41}$$

and identify the Lax operator (2.38) at $\lambda_0 = \frac{i}{2}$ with the permutation operator.

Monodromy and Transfer Matrix

In the following we want to answer two questions: How can we approach the concept of integrability for a model on the basis of the Lax operator and how is such an operator related to a spin chain at all? In particular, what is the connection between the object in (2.38) and the Heisenberg spin chain discussed in section 2.1 and 2.2? In order to answer these questions, let us begin by promoting $\mathrm{L}_{n,a}$ to an operator acting non-trivially on the whole spin chain

$$\mathrm{T}_a(\lambda) = \mathrm{L}_{1,a}(\lambda)\mathrm{L}_{2,a}(\lambda)...\mathrm{L}_{N,a}(\lambda). \tag{2.42}$$

This object is called *monodromy matrix* and acts on $\mathscr{H}^{\otimes N} \otimes \mathscr{V}_a$. Tracing over the auxiliary space gives the *transfer matrix*

$$t(\lambda) = \mathrm{tr}_a(\mathrm{T}_a(\lambda)) \tag{2.43}$$

which only acts on the physical spaces.

Local Operators from the Monodromy

Interestingly, we can use the point $\lambda = \lambda_0$ of fundamental models to find a quantity that we already discussed in the previous sections – the shift operator $\mathcal{U}$. For this purpose, let us calculate the monodromy matrix (2.42) at the point λ_0 for general fundamental spin chain models. Rearranging the permutation operators with the help of (2.40) yields

$$\begin{aligned}
\mathrm{T}_a(\lambda_0) &= \mathbb{P}_{1,a}\mathbb{P}_{2,a}\mathbb{P}_{3,a}...\mathbb{P}_{N,a} = \mathbb{P}_{2,a}\mathbb{P}_{1,2}\mathbb{P}_{3,a}...\mathbb{P}_{N,a} \\
&= ... = \mathbb{P}_{N,a}\mathbb{P}_{N-1,N}\mathbb{P}_{N-2,N-1}...\mathbb{P}_{2,3}\mathbb{P}_{1,2}.
\end{aligned} \tag{2.44}$$

Now we can take the trace over the auxiliary space using $\mathrm{tr}_a \mathbb{P}_{N,a} = \mathbb{I}_N$. For the Heisenberg spin chain this can be verified easily by writing $\mathbb{P}_{N,a}$ as a matrix in the auxiliary space

$$\mathbb{P}_{N,a} = \frac{1}{2}\begin{pmatrix} \mathbb{I}+\sigma^z & \sigma^x - i\sigma^y \\ \sigma^x + i\sigma^y & \mathbb{I}-\sigma^z \end{pmatrix}. \tag{2.45}$$

This yields

$$t\left(\lambda_0\right) = \mathbb{P}_{N-1,N} \ldots \mathbb{P}_{2,3}\mathbb{P}_{1,2} \equiv \mathcal{U}, \tag{2.46}$$

i.e. the transfer matrix t evaluated at $\lambda = \lambda_0$ gives the left-shift operator $\mathcal{U}$. Using $\mathcal{U} = e^{i\mathcal{K}}$ with momentum operator $\mathcal{K}$ we find

$$\mathcal{K} = -i \ln\left(t(\lambda_0)\right), \tag{2.47}$$

where ln denotes the matrix logarithm for t in matrix representation. Thus, the Lax operator of fundamental spin chain models contains the shift operator $\mathcal{U}$ and its generator $\mathcal{K}$.

Let us examine the next order of the expansion of t around λ_0. We concentrate on the Heisenberg spin chain for which we know the explicit λ-dependence of L. Differentiating T_a with respect to λ and using (2.40) gives

$$\left.\frac{\mathrm{d}\mathrm{T}_a}{\mathrm{d}\lambda}\right|_{\lambda=\frac{i}{2}} = -i\sum_n \mathbb{P}_{1,a}\ldots\mathbb{P}_{n-1,a}\mathbb{P}_{n+1,a}\ldots\mathbb{P}_{N,a} = -i\sum_n \mathbb{P}_{N,a}\mathbb{P}_{N-1,N}\ldots\mathbb{P}_{n-1,n+1}\ldots\mathbb{P}_{2,3}\mathbb{P}_{1,2} \tag{2.48}$$

and thus we find

$$\begin{aligned} t(\tfrac{i}{2})^{-1}\left.\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right|_{\lambda=\frac{i}{2}} &= -i\left(\mathbb{P}_{1,2}\ldots\mathbb{P}_{m-2,m-1}\mathbb{P}_{m-1,m}\mathbb{P}_{m,m+1}\mathbb{P}_{m+1,m+2}\ldots\mathbb{P}_{N-2,N-1}\mathbb{P}_{N-1,N}\right)\cdot \\ &\qquad \cdot\sum_n\left(\mathbb{P}_{N-1,N}\mathbb{P}_{N-2,N-1}\ldots\mathbb{P}_{n+1,n+2}\mathbb{P}_{n-1,n+1}\mathbb{P}_{n-2,n-1}\ldots\mathbb{P}_{1,2}\right) \\ &= -i\sum_n \mathbb{P}_{n,n+1}. \end{aligned} \tag{2.49}$$

This corresponds to the Heisenberg Hamiltonian (2.9) since we may rewrite it in terms of the permutation operator as

$$\mathcal{H} = \tfrac{1}{2}\sum_n\left(\mathbb{I}_{n,n+1} - \mathbb{P}_{n,n+1}\right). \tag{2.50}$$

Hence we can identify the second term in the expansion of t around λ_0 with the Heisenberg Hamiltonian up to a trivial constant, e.g. for the closed spin chain

$$\mathcal{H} = \tfrac{N}{2} - \tfrac{i}{2}\left.\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left(t(\lambda)\right)\right|_{\lambda=\frac{i}{2}}. \tag{2.51}$$

This justifies the statement that the Lax operator (2.38) is associated to the Heisenberg spin chain. For general spin chain models the first order of the expansion of t in the form $\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\left(t(\lambda)\right)\right|_{\lambda=\lambda_0}$ is typically connected to the Hamiltonian of the system. In [21] it was shown that the remaining charges in the expansion of the form

$$\mathcal{Q}_r = -\tfrac{i}{r!}\left.\tfrac{\mathrm{d}^{r-1}}{\mathrm{d}\lambda^{r-1}}\ln(t(\lambda))\right|_{\lambda=\lambda_0} \tag{2.52}$$

are local of degree r for the XYZ spin-1/2 chain, i.e. they only act on r neighboring sites. For fundamental models this construction is supposed to be analogous [7], in particular the charge $\mathcal{Q}_2$ associated to the Hamiltonian only contains nearest-neighbor interactions.

Conserved Charges from the Monodromy

Having shown the connection of Lax operators and Hamiltonians of spin chain models, we still need to answer the following question: Why should we prefer basing the discussion of a spin chain model on a Lax operator rather than a Hamiltonian? In fact, the great advantage of defining a system via the Lax operator is that there exists a procedure that enables us to obtain a tower of conserved operators from this object for certain models. This large amount of symmetry leads to the simplicity of the model's solution and can be used as a definition of quantum integrability. We want to demonstrate this in the following.

Let us begin by introducing the so-called *RLL-relation*

$$\mathrm{R}_{a,b}(\lambda_1 - \lambda_2)\mathrm{L}_{n,a}(\lambda_1)\mathrm{L}_{n,b}(\lambda_2) = \mathrm{L}_{n,b}(\lambda_2)\mathrm{L}_{n,a}(\lambda_1)\mathrm{R}_{a,b}(\lambda_1 - \lambda_2) \tag{2.53}$$

with the *R-operator* $\mathrm{R}_{a,b}(\lambda)$: $\mathscr{V}_a \otimes \mathscr{V}_b \to \mathscr{V}_a \otimes \mathscr{V}_b$. This equation is defined on $\mathscr{H}_n \otimes \mathscr{V}_a \otimes \mathscr{V}_b$ with auxiliary spaces $\mathscr{V}_{a,b}$. The existence of an R-operator satisfying (2.53) for a specific Lax operator has tremendous implications on the model's dynamics. This can be shown by rewriting this equation into a formula for the transfer matrix. In order to do so, let us repeatedly use it on all sites of the chain such that we obtain the so-called *RTT-relation*

$$\mathrm{R}_{a,b}(\lambda_1 - \lambda_2)\mathrm{T}_a(\lambda_1)\mathrm{T}_b(\lambda_2) = \mathrm{T}_b(\lambda_2)\mathrm{T}_a(\lambda_1)\mathrm{R}_{a,b}(\lambda_1 - \lambda_2). \tag{2.54}$$

For an invertible R-matrix this implies

$$\mathrm{T}_a(\lambda_1)\mathrm{T}_b(\lambda_2) = \mathrm{R}_{a,b}(\lambda_1 - \lambda_2)^{-1}\mathrm{T}_b(\lambda_2)\mathrm{T}_a(\lambda_1)\mathrm{R}_{a,b}(\lambda_1 - \lambda_2) \tag{2.55}$$

which can be traced over in both auxiliary spaces yielding

$$[t(\lambda_1), t(\lambda_2)] = 0. \tag{2.56}$$

Expanding this relation in λ_1 and λ_2 generates all possible commutators of the operators produced in an expansion of t. Equating coefficients reveals that these commutators have to vanish. In particular, since the Hamiltonian is one of the operators in the expansion around λ_0, all operators in any expansion are conserved, i.e. commute with the Hamiltonian. Therefore, the transfer matrix generates a tower of commuting conserved quantities. Using (2.52) one may construct N local charges, i.e. the system allows for exactly the same number of conserved charges as there are degrees of freedom. This leads to our first formulation of *quantum integrability*: We call a spin chain of length N integrable if there exists a tower of N local independent conserved operators whose commutators vanish[4]. This is in close analogy to the definition of integrability for continuous quantum field theories from above (see chapter 1) and classical theories (see appendix A).

For the Heisenberg spin chain one finds that

$$\mathrm{R}_{a,b}(\lambda) = \lambda\mathbb{I}_{a,b} + i\mathbb{P}_{a,b} \tag{2.57}$$

satisfies (2.53). The existence of this R-operator implies the existence of a tower of conserved commuting operators and thus it is integrable in the above sense. As we already discussed above, the momentum $\mathcal{Q}_1 = \mathcal{K}$ and energy operator $\mathcal{Q}_2 \sim$

[4]As already mentioned above, this formulation is not suitable as a general definition of quantum integrability since the notion of independence of operators in quantum models is problematic.

$\mathcal{H}$ are among these charges. It is not clear whether the higher charges contain physical information but they generate hidden symmetries which are the basis of the model's integrability. The Lax formalism can even be used to efficiently calculate the spectrum of the Heisenberg spin chain by a variation of the Coordinate Bethe Ansatz called *Algebraic Bethe Ansatz*. We do not discuss it in detail but sketch some results in the appendix B.

2.4 The Quantum Yang-Baxter Equation

The crucial ingredient in the proof that a spin chain is integrable in the above sense is the existence of an R-matrix satisfying the RLL-relation. Therefore, one often reformulates integrability in the following way: A spin chain model with Lax matrix $\mathrm{L}(\lambda)$ is integrable if there exists an R-matrix R: $\mathscr{V}_a \otimes \mathscr{V}_b \to \mathscr{V}_a \otimes \mathscr{V}_b$ which satisfies the RLL-relation (2.53)[5]. This definition can be used to find new integrable quantum models. In order to demonstrate this, let us first develop a consistency condition for the RLL-relation. We start with the combination $\mathrm{L}_{n,a}(\lambda_1)\mathrm{L}_{n,b}(\lambda_2)\mathrm{L}_{n,c}(\lambda_3)$ of three Lax matrices acting on the space $\mathscr{H}_n \otimes \mathscr{V}_a \otimes \mathscr{V}_b \otimes \mathscr{V}_c$ and rewrite it using (2.53) as

$$\begin{aligned} \mathrm{L}_a\mathrm{L}_b\mathrm{L}_c &= \mathrm{R}_{a,b}^{-1}\mathrm{R}_{a,b}\mathrm{L}_a\mathrm{L}_b\mathrm{L}_c = \mathrm{R}_{a,b}^{-1}\mathrm{L}_b\mathrm{L}_a\mathrm{L}_c\mathrm{R}_{a,b} \\ &= \dots = \left(\mathrm{R}_{b,c}\mathrm{R}_{a,c}\mathrm{R}_{a,b}\right)^{-1}\mathrm{L}_c\mathrm{L}_b\mathrm{L}_a\left(\mathrm{R}_{b,c}\mathrm{R}_{a,c}\mathrm{R}_{a,b}\right). \end{aligned} \tag{2.58}$$

Here we drop the index n on the Lax operators and the dependence on the spectral parameters. On the other hand, we may also rearrange this combination of Lax operators as

$$\mathrm{L}_a\mathrm{L}_b\mathrm{L}_c = \mathrm{L}_a\mathrm{R}_{b,c}^{-1}\mathrm{R}_{b,c}\mathrm{L}_b\mathrm{L}_c = \dots = \left(\mathrm{R}_{a,b}\mathrm{R}_{a,c}\mathrm{R}_{b,c}\right)^{-1}\mathrm{L}_c\mathrm{L}_b\mathrm{L}_a\left(\mathrm{R}_{a,b}\mathrm{R}_{a,c}\mathrm{R}_{b,c}\right) \tag{2.59}$$

and thus we find the *quantum Yang-Baxter equation* (qYBE)

$$\mathrm{R}_{a,b}(\lambda_1-\lambda_2)\mathrm{R}_{a,c}(\lambda_1-\lambda_3)\mathrm{R}_{b,c}(\lambda_2-\lambda_3) = \mathrm{R}_{b,c}(\lambda_2-\lambda_3)\mathrm{R}_{a,c}(\lambda_1-\lambda_3)\mathrm{R}_{a,b}(\lambda_1-\lambda_2). \tag{2.60}$$

Here we reintroduced the dependence on the spectral parameters. Any solution $\mathrm{R}(\lambda)$ to the qYBE with $\mathrm{R}(0) = \mathbb{P}$ defines a fundamental integrable quantum model with Lax operator

$$\mathrm{L} : \mathscr{H}_n \otimes \mathscr{V}_a \to \mathscr{H}_n \otimes \mathscr{V}_a \qquad \text{and} \qquad \mathrm{L}_{n,a}(\lambda) = \mathrm{R}_{a,n}(\lambda - \lambda_0) \tag{2.61}$$

for $\mathscr{H}_n \equiv \mathscr{V}_a$ in the version of integrability discussed at the beginning of this section. This can easily be verified by checking the RLL-relation starting from the qYBE (2.60) and using (2.61). Due to the existence of a point λ_0 with $\mathrm{L}(\lambda_0) = \mathbb{P}$ and the equivalence of the one-site Hilbert space $\mathscr{H}_0$ and the auxiliary space $\mathscr{V}_a$, the model is fundamental.

It will turn out in the following that the qYBE and the R-matrix are crucial in the study of factorization of scattering for integrable quantum models. Indeed, this equation is associated to the consistency of the factorization which we already commented on in chapter 1.

[5]This is the formulation used by Faddeev et al in [7]. Note that for non-fundamental models this formulation is problematic, see the discussion in section 2.5.

2.5 Fundamental and Non-Fundamental Models

In the previous sections we often restricted ourselves to fundamental models. In the following we want to show the close connection between the existence of a tower of commuting conserved local charges and the factorization of scattering for both fundamental and non-fundamental models. For this purpose, let us briefly review from [7] how it is possible to construct local charges for integrable models that do not satisfy the above properties.

We start with a model with Lax operator $\mathrm{L}(\lambda)$ acting on $\mathscr{H}_n \otimes \mathscr{V}_a$. It shall be integrable in the sense that there is an R-matrix acting on $\mathscr{V}_a \otimes \mathscr{V}_b$ and satisfying the RLL-relation (2.53). For fundamental models the transfer matrix built from the Lax operator generates a tower of commuting conserved charges. Via the proof in [21] it should be possible to find a formulation in which these act locally on neighbouring sites. For non-fundamental models it is less clear whether the charges (2.52) built from the transfer matrix act locally since the equality of the permutation operator and the Lax operator at the expansion point λ_0 is crucial in the proof of locality of (2.52) given in [7, 21]. But there exists a way to construct local charges for non-fundamental models that we want to review on the basis of the original paper [7]. For this purpose, one needs to find an R-matrix $\mathbf{R}(\lambda)$ on the physical space

$$\mathbf{R} : \mathscr{H}_n \otimes \mathscr{H}_0 \to \mathbf{R} : \mathscr{H}_n \otimes \mathscr{H}_0 \tag{2.62}$$

that fulfills $\mathbf{R}(\lambda_0) = \mathbb{P}$ at a point $\lambda = \lambda_0$ and which satisfies a version of the RLL-relation

$$\mathbf{R}_{n,0}(\lambda_1 - \lambda_2)\mathrm{L}_{n,a}(\lambda_1)\mathrm{L}_{0,a}(\lambda_2) = \mathrm{L}_{0,a}(\lambda_2)\mathrm{L}_{n,a}(\lambda_1)\mathbf{R}_{n,0}(\lambda_1 - \lambda_2) \tag{2.63}$$

acting on $\mathscr{H}_n \otimes \mathscr{H}_0 \otimes \mathscr{V}_a$. Then we may define a version of the monodromy matrix

$$\mathbf{T}_0(\lambda) = \mathbf{R}_{N,0}(\lambda)\mathbf{R}_{N-1,0}(\lambda)...\mathbf{R}_{1,0}(\lambda) \tag{2.64}$$

that is a matrix in $\mathscr{H}_0$. Tracing over this space gives

$$\boldsymbol{t}(\lambda) = \mathrm{tr}_0\left(\mathbf{T}_0(\lambda)\right). \tag{2.65}$$

This object is similar to the transfer matrix in (2.43) for fundamental models and thus the operator $\boldsymbol{t}(\lambda)$ generates operators $\boldsymbol{Q}_r$

$$\boldsymbol{Q}_r = -\tfrac{i}{r!}\,\tfrac{\mathrm{d}^{r-1}}{\mathrm{d}\lambda^{r-1}}\ln(\boldsymbol{t}(\lambda))\Big|_{\lambda=\lambda_0}. \tag{2.66}$$

Their construction is equivalent to the charges in (2.52) and thus, by the proof in [21], they are supposed to be local.

One might ask whether these operators are conserved with respect to the original model's Hamiltonian $\mathcal{Q}_r$ for some r. In order to check this, let us rewrite (2.63)

$$\mathbf{R}_{n,0}(\lambda_3)\mathrm{L}_{n,a}(\lambda_1)\mathrm{L}_{0,a}(\lambda_1 - \lambda_3) = \mathrm{L}_{0,a}(\lambda_1 - \lambda_3)\mathrm{L}_{n,a}(\lambda_1)\mathbf{R}_{n,0}(\lambda_3) \tag{2.67}$$

with a change of variables $\lambda_3 = \lambda_1 - \lambda_2$. Using this equation on all sites n we find

$$\mathbf{T}_0(\lambda_3)\mathrm{T}_a(\lambda_1)\mathrm{L}_{0,a}(\lambda_1 - \lambda_3) = \mathrm{L}_{0,a}(\lambda_1 - \lambda_3)\mathrm{T}_a(\lambda_1)\mathbf{T}_0(\lambda_3) \tag{2.68}$$

that includes both the monodromy matrix $\mathrm{T}_a(\lambda)$ of the original model and the version for non-fundamental models $\mathbf{T}_0(\lambda)$. Multiplying this equation by the inverse of $\mathrm{L}_{0,a}(\lambda_1-\lambda_3)$ and taking the trace $\mathrm{tr}_{0,a}$ over the space $\mathscr{V}_a \otimes \mathscr{H}_0$ yields

$$[t(\lambda_1), \boldsymbol{t}(\lambda_3)] = 0. \tag{2.69}$$

Expanding this relation in λ_1 and λ_3 and equating coefficients implies the vanishing of the commutators of the original $\mathcal{Q}_r$ and new charges $\mathbf{Q}_r$. In particular, the $\mathbf{Q}_r$ commute with the Hamiltonian of the original system and thus they are indeed conserved.

In order to show that the charges $\mathbf{Q}_r$ are not only conserved, but also build a tower of commuting charges, we derive the qYBE for the alternative R-matrices $\mathbf{R}(\lambda)$. This is done analogously to the procedure in the previous section 2.4 by rewriting a combination of Lax operators acting on $\mathscr{H}_0 \otimes \mathscr{H}_{\bar{0}} \otimes \mathscr{H}_1 \otimes \mathscr{V}_a$ as

$$\mathrm{L}_0\mathrm{L}_{\bar{0}}\mathrm{L}_1 = \mathbf{R}_{0,\bar{0}}^{-1}\mathbf{R}_{0,\bar{0}}\mathrm{L}_0\mathrm{L}_{\bar{0}}\mathrm{L}_1 = ... = (\mathbf{R}_{\bar{0},1}\mathbf{R}_{0,1}\mathbf{R}_{0,\bar{0}})^{-1}\mathrm{L}_1\mathrm{L}_{\bar{0}}\mathrm{L}_0(\mathbf{R}_{0,\bar{0}}\mathbf{R}_{0,1}\mathbf{R}_{\bar{0},1}) \tag{2.70}$$

where we drop the index a for the auxiliary space and the dependence on the spectral parameter. On the other hand, we have

$$\mathrm{L}_0\mathrm{L}_{\bar{0}}\mathrm{L}_1 = ... = (\mathbf{R}_{0,\bar{0}}\mathbf{R}_{0,1}\mathbf{R}_{\bar{0},1})^{-1}\mathrm{L}_1\mathrm{L}_{\bar{0}}\mathrm{L}_0(\mathbf{R}_{\bar{0},1}\mathbf{R}_{0,1}\mathbf{R}_{0,\bar{0}}) \tag{2.71}$$

which yields together with (2.70)

$$\mathbf{R}_{0,\bar{0}}(\lambda_1-\lambda_2)\mathbf{R}_{0,1}(\lambda_1-\lambda_3)\mathbf{R}_{\bar{0},1}(\lambda_2-\lambda_3) = \mathbf{R}_{\bar{0},1}(\lambda_2-\lambda_3)\mathbf{R}_{0,1}(\lambda_1-\lambda_3)\mathbf{R}_{0,\bar{0}}(\lambda_1-\lambda_2) \tag{2.72}$$

where we reintroduced the dependence on spectral parameters. This alternative qYBE can be repeatedly used on all sites which gives

$$\mathbf{R}_{0,\bar{0}}(\lambda_1-\lambda_2)\mathbf{T}_0(\lambda_1-\lambda_3)\mathbf{T}_{\bar{0}}(\lambda_2-\lambda_3) = \mathbf{T}_{\bar{0}}(\lambda_2-\lambda_3)\mathbf{T}_0(\lambda_1-\lambda_3)\mathbf{R}_{0,\bar{0}}(\lambda_1-\lambda_2). \tag{2.73}$$

Assuming that $\mathbf{R}_{0,\bar{0}}$ is invertible we may multiply by the inverse and afterwards trace over the spaces $\mathscr{H}_0$ and $\mathscr{H}_{\bar{0}}$ which yields

$$[\boldsymbol{t}(\lambda_1-\lambda_3), \boldsymbol{t}(\lambda_2-\lambda_3)] = 0. \tag{2.74}$$

By similar arguments as for fundamental models, this equation implies that the charges $\mathbf{Q}_r$ commute. Thus they build a tower of conserved and commuting local charges.

This discussion shows that the existence of an R-matrix that satisfies the RLL-relation (2.53) for a non-fundamental model with Lax operator $\mathrm{L}(\lambda)$ does not necessarily imply the existence of a tower of local charges. We showed in this section that one may construct them if one assumes the existence of an alternative R-matrix $\mathbf{R}(\lambda)$ that satisfies (2.63). It is not clear whether each integrable non-fundamental model allows for such an object and thus allows for a tower of local charges.

Chapter 3

Spin Chain S-Matrices

In the previous chapter we introduced some basic concepts of integrability in the context of spin chains and have encountered the factorization of scattering for the Heisenberg spin chain. This is a common feature of many integrable spin chain models that we want to investigate in more detail in the following. In order to do so, we introduce the quantity that characterizes scattering processes in these models – the spin chain's S-matrix. For this purpose, we develop the notion of the vacuum and particle states for general spin chains in section 3.1. Afterwards in 3.2, we discuss the symmetry algebra of asymptotic states and proceed in section 3.3 by introducing the S-matrix. This discussion is partly based on [22, 23]. Then we move on to the connection of the existence of a tower of conserved local charges in a spin chain model and factorization of scattering in section 3.4.

3.1 Vacuum and Asymptotic States

The S-matrix is a quantity that relates asymptotic states. In the context of continuous QFTs, asymptotic states consist of asymptotically free particles, i.e. their separation is large compared to their interaction range. They propagate on a vacuum without interacting with it – it is just the stage on which the scattering process takes place. In order to define the S-matrix for spin chains, we first need to translate this picture of vacua, particles and asymptotic states into the language of spin chains. We discuss these concepts in the context of $\mathfrak{g}$-invariant spin chains. These spin chains consist of particles transforming under some representation of the algebra $\mathfrak{g}$ and are associated to Hamiltonians invariant under this algebra. We will call $\mathfrak{g}$ the *full symmetry algebra* in the following in order to distinguish it from the residual symmetry algebra which we introduce in the next section. The simplest example of such a spin chain is the previously discussed Heisenberg spin chain which is an $\mathfrak{su}(2)$ spin chain whose particles transform in the fundamental representation. In contrast to the discussion of this model, we will not specify the spin chain models by introducing Hamiltonians, but base the discussion on their symmetry properties.

Spin Chain Vacua

What is the vacuum for a spin chain? Clearly the notion of space void of particles is not suitable for a lattice model. Let us take inspiration from the discussion of the Heisenberg spin chain. There we already encountered the notion of "vacuum" and "(quasi-)particle". We defined one spin configuration ($|\uparrow\rangle$) as vacuum orientation and a linear combination of states with a single reverse orientation ($|\downarrow\rangle$) as one-particle state called magnon. We were able to assign a (quasi-)momentum to the motion of this particle on the vacuum. Furthermore, we were able to build multi-particle states. We may generalize this notion of the vacuum and excitations by picking a specific spin orientation from the representation as vacuum configuration. This orientation shall be denoted by a_0 in the following such that the vacuum state of the spin chain is given by

$$|0\rangle = |a_0 \ldots a_0\ a_0 \ldots a_0\rangle \tag{3.1}$$

similar to (2.15) of the Heisenberg spin chain. The remaining possible states of a site are excitations which we want to denote by a.

One-Particle and Asymptotic States

Just as in the discussion of the Heisenberg spin chain in (3.2), one may introduce linear combinations of the form

$$|a, u\rangle = \sum_{n=1}^{N} e^{ikn}\,|a, n\rangle \tag{3.2}$$

with

$$|a, n\rangle = |a_0 \ldots a_0 \underset{n\text{th site}}{a} a_0 \ldots a_0\rangle\,. \tag{3.3}$$

We interpret these as quasi-particles – the so-called magnons – moving along the chain with momentum k and corresponding rapidity $u = u(k)$.

These one-particle states can be used to build asymptotic m-particle states. They are multi-particle states consisting of m magnons that are largely separated on the chain, i.e.

$$|a_1, u_1; a_2, u_2; \ldots; a_m, u_m\rangle := \sum_{n_1 \ll \ldots \ll n_m} e^{i(k_1 n_1 + k_2 n_2 + \ldots + k_m n_m)}\,|\ldots \underset{\substack{\uparrow\\ n_1}}{a_1} \ldots \underset{\substack{\uparrow\\ n_2}}{a_2} \ldots \underset{\substack{\uparrow\\ n_m}}{a_m} \ldots\rangle\,. \tag{3.4}$$

The symbols $a_i,\ i = 1, \ldots, m$ denote the excitations of the m particles moving with momentum k_i and corresponding rapidities $u_i = u_i(k_i)$.

3.2 Residual Symmetry Algebra

Picking a specific spin orientation as vacuum configuration breaks the symmetry of the model. Only some generators of the full symmetry algebra $\mathfrak{g}$ will preserve the vacuum and number of excitations. They correspond to a subalgebra $\mathfrak{g}_r$ of $\mathfrak{g}$ which is called *residual symmetry algebra*. Let us illustrate this for an $\mathfrak{su}(3)$ spin chain.

Example: $\mathfrak{su}(3)$ Spin Chain

In the fundamental representation the $\mathfrak{su}(3)$ generators at each site are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$
$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.6}$$

The matrices λ_3 and λ_8 span the Cartan subalgebra and can be diagonalized simultaneously. The state at each site of the spin chain is a linear combination of the three eigenstates of the Cartan operators

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |s\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.7}$$

By choosing $|s\rangle$ as vacuum spin configuration, there are two types of excitations $|u\rangle$ and $|d\rangle$. Only the generators

$$\{\lambda_1, \lambda_2, \lambda_3\} \tag{3.8}$$

transform between them and preserve the vacuum state. The corresponding residual symmetry algebra is $\mathfrak{su}(2)$ which is obvious since the generators λ_1, λ_2 and λ_3 embed the Pauli matrices into 3×3-matrices. Thus the excitations transform as spin-1/2-particles.

The full symmetry algebra $\mathfrak{su}(3)$ generates two more residual $\mathfrak{su}(2)$ subalgebras. One consists of the generators

$$\{\lambda_4, \lambda_5, \frac{1}{\sqrt{2}}(\sqrt{3}\lambda_8 + \lambda_3)\} \tag{3.9}$$

corresponding to a vacuum $|d\rangle$ and the last subalgebra consists of

$$\{\lambda_6, \lambda_7, \frac{1}{\sqrt{2}}(\sqrt{3}\lambda_8 - \lambda_3)\} \tag{3.10}$$

corresponding to the vacuum $|u\rangle$.

In general, one can show that each Lie algebra $\mathfrak{su}$(n) contains Lie subalgebras $\mathfrak{su}$(n-1), see e.g. [24]. Therefore, for each $\mathfrak{su}$(n) spin chain it is possible to define a vacuum state and the n-1 excitations transform under a residual symmetry algebra $\mathfrak{su}$(n-1). In a concrete model with known Hamiltonian, the state of lowest energy is chosen to be the vacuum state.

Example: The $\mathfrak{su}(1|2)$ Spin Chain

Let us illustrate this concept for a super spin chain, i.e. a spin chain corresponding to a Lie superalgebra. One of the simplest examples is the $\mathfrak{su}(1|2)$ spin chain. In its fundamental representation the particles' states are given by a linear combination of three basis states denoted by $\mathcal{Z}$, ϕ and ψ. The first two are bosonic and the latter is fermionic. If we choose $\mathcal{Z}$ as vacuum orientation, the residual algebra of the excitations ϕ and ψ is $\mathfrak{su}(1|1)$, cf. [14]. We discuss this algebra in section 6.1.

Example: The $\mathfrak{su}(2|3)$ Spin Chain

Let us move on to a more complex spin chain model, the $\mathfrak{su}(2|3)$ spin chain. This spin chain is interesting in the context of $\mathcal{N} = 4$ SYM with superconformal algebra $\mathfrak{psu}(2,2|4)$ which we already discussed in chapter 1. In the fundamental representation the Hilbert space at each site consists of five possible spin orientations $\{\phi_1, \phi_2, \phi_3|\psi_1, \psi_2\}$, where the first three represent bosonic states and the last two fermionic states. Choosing the bosonic complex combination $\mathcal{Z} = \phi_2 + i\phi_3$ as vacuum state reduces the amount of symmetry for the excitations. This state minimizes the energy eigenvalue of the corresponding Hamiltonian that is given in [25]. The residual symmetry subalgebra of the excitations $\{\phi_1, \phi_2|\psi_1, \psi_2\}$ is $\mathfrak{su}(2|2)$ [10]. We will discuss this Lie superalgebra in section 7.1.

The Hilbert Space of Asymptotic States

Under a residual symmetry algebra $\mathfrak{g}_r$, which preserves the number of excitations, vacuum and spin chain length, the Hilbert space $\mathscr{H}_{asym}$ of asymptotic states decomposes into subspaces[1]

$$\mathscr{H}_{asym}^{\otimes N} = (V_0) \;\oplus\; (V_1) \;\oplus\; (V_1 \otimes V_2) \;\oplus ... \tag{3.11}$$

The space V_0 is the vacuum space and only contains the state (3.1). V_1 contains all one-magnon states (3.2) and $V_1 \otimes V_2 \otimes ... \otimes V_m$ contains all m-magnon asymptotic states (3.4). Note that the sum in (3.11) does not go all the way to $m = N$ but stops as soon as the number of magnons is not compatible with an asymptotic state anymore. In the next section we introduce the $m \to m$ S-matrix which we define as an operator that maps inside the subspaces $\bigotimes_{n=1}^{m} V_n$ for $m \geq 2$.

3.3 Spin Chain S-Matrices

Incoming and Outgoing States

As we already discussed in chapter 1, it is a special feature of theories in (1+1) dimensions that we may order particles with respect to their velocities. We want to use this feature to define incoming and outgoing states in a scattering process. An incoming state in an m-particle scattering process is an asymptotic state

$$|a_1, u_1; a_2, u_2; ...; a_m, u_m\rangle_{in} \in \mathscr{H}_{in} \tag{3.12}$$

where the magnon of type a_1 moves behind the magnon of type a_2 etc. The corresponding rapidities are such that $u_1 > u_2 > ... > u_m$ and all particles participate in the scattering process. The Hilbert space $\mathscr{H}_{in}$ contains all possible incoming states. Similarly, for an outgoing state of an m-particle scattering process

$$|a_1, u_1; a_2, u_2; ...; a_m, u_m\rangle_{out} \in \mathscr{H}_{out} \tag{3.13}$$

the magnon a_1 moves behind a_2 etc., but the rapidities are in the reverse order $u_1 < u_2 < ... < u_m$. Thus no scattering is possible. All outgoing states form the Hilbert space $\mathscr{H}_{out}$.

[1] Note that there occur subtleties for spin chain models with length-changing symmetry generators. This length-changing effect can be captured by making the algebra momentum-dependent, see chapter 7.

Definition of the S-Matrix

The m-particle S-matrix is an operator that maps an m-particle outgoing state into an m-particle incoming state

$$|a_1, u_1; a_2, u_2; ...; a_m, u_m\rangle_{in} = \mathrm{S}^{a_1\,a_2\,...\,a_m}_{b_1\,b_2\,...\,b_m}(u_i, v_i)\,|b_1, v_1; b_2, v_2; ...; b_m, v_m\rangle_{out} \tag{3.14}$$

and thus characterizes the scattering of m magnons. In general, an S-matrix may depend on all the rapidities u_i and v_i, $i = 1, ..., m$, of both states. The object defined in (3.14) is the generalization of the coefficients in the Coordinate Bethe Ansatz discussed in section 2.2. For general periodic spin chains, the S-matrices are the prefactors in the asymptotic Bethe Ansatz.

The S-matrix maps from the asymptotic Hilbert space $\mathscr{H}_{out}$ of outgoing states to $\mathscr{H}_{in}$ of incoming states

$$\mathrm{S} : \mathscr{H}_{out} \to \mathscr{H}_{in}. \tag{3.15}$$

These Hilbert spaces correspond to the previously discussed m-particle asymptotic Hilbert spaces (3.11). Let us label the single-particle Hilbert spaces by the rapidity u of the particle as V_u.[2] Thus the S-matrix is an operator that maps as

$$\mathrm{S} : V_{v_1} \otimes V_{v_2} \otimes ... \otimes V_{v_m} \to V_{u_1} \otimes V_{u_2} \otimes ... \otimes V_{u_m} \tag{3.16}$$

with incoming rapidities $u_1 > u_2 > ... > u_m$ and outgoing rapidities $v_1 < v_2 < ... < v_m$. Take as an example a free theory, i.e. the particles do not interact at all and they may only overtake each other. In this case, the S-matrix simply permutes the one-particle Hilbert spaces

$$\mathrm{S} : V_{u_m} \otimes ... \otimes V_{u_2} \otimes V_{u_1} \to V_{u_1} \otimes V_{u_2} \otimes ... \otimes V_{u_m} \tag{3.17}$$

and acts on a two-particle state as

$$\mathrm{S}\,|a_2, u_2; a_1, u_1\rangle_{out} \sim |a_1, u_1; a_2, u_2\rangle_{in}\,. \tag{3.18}$$

Here the particle with quantum numbers a_1 and rapidity u_1 overtakes the particle a_2 with smaller rapidity u_2 without an interaction.

The symmetries of the spin chain model under consideration restrict the form of the S-matrix. In particular, since it is the map inside the subspaces of the decomposed asymptotic Hilbert space (3.11), it must be invariant under the residual algebra $\mathfrak{g}_r$ of the model. We will use this feature in the following chapters to constrain the S-matrix.

3.4 From Conserved Local Charges to Factorized Scattering

Having defined spin chain S-matrices, we can now investigate how the existence of a tower of commuting conserved local charges $\mathcal{Q}_r$ constrains a scattering process. As discussed in chapter 1, in a massive relativistic QFT the existence of infinitely many conserved charges of different Lorentz spin s constrains the theory's S-matrices in such a way that

[2] In chapter 6 and 7 there will be further quantum numbers labeling these spaces.

- they allow for no particle creation and annihilation,
- there are equal sets of rapidities for the initial and final state,
- the m-particle S-matrix factorizes into two-particle S-matrices.

In the discussion of scattering for the Heisenberg spin chain in section 2.2 we observed the same features. Indeed, one can translate the proof for integrable massive relativistic QFTs into the language of integrable spin chains which we will do in the following by transferring the argumentation given in [3].

The crucial ingredient in the proof of factorization is the definition of integrability via a tower of conserved local charges. For massive relativistic QFTs with infinitely many degrees of freedom we demanded the existence of infinitely many charges $\mathcal{Q}_s$ that all have different Lorentz spin s. They act on one-particle states as (1.1). Let us formulate integrability in the context of spin chains of length N as the existence of N local conserved commuting charges $\mathcal{Q}_r$ that act as

$$\mathcal{Q}_r\,|a,u\rangle \sim k^r\,(1+\mathcal{O}(k))\,|a,u\rangle \tag{3.19}$$

with $u=u(k)$ and $r\in\mathbb{N},\ 1\leq r\leq N$.

Before proceeding with the proof of factorization of S-matrices, let us study how this definition of integrability fits into the framework of integrability that we introduced in chapter 2. For fundamental models with Lax matrix L=L(λ) the existence of a tower of local charges $\mathcal{Q}_r$ of the form (2.52) can be used as a definition of integrability. For non-fundamental models we may construct local charges $\mathbf{Q}_r$ as in (2.66). Indeed, the local charges obtained in these ways are supposed to act on one-particle states as in (3.19)[3]. Let us motivate this statement by studying once more the Heisenberg spin chain. Calculating the normalized eigenvalues q_r of $\mathcal{Q}_r$ from the eigenvalues of the transfer matrix (B.4) and identifying the Bethe roots and rapidities[4] gives single-particle eigenvalues q_r of the form

$$q_r(u)=\frac{i}{r}\left(\frac{1}{(u+\lambda_0)^{r-1}}-\frac{1}{(u-\lambda_0)^{r-1}}\right) \tag{3.20}$$

with $\lambda_0=\frac{i}{2}$. Note that for $r=2$ we rediscover the one-magnon energy $E=E(u)$ given in (B.7) in the appendix B. Using the relation (2.24) and Taylor-expanding around $k=0$, we find that q_r indeed scales as

$$q_r(u(k))\sim k^r\left(1+\mathcal{O}(k^2)\right). \tag{3.21}$$

This result is in accordance with the assumption in (3.19).

Having motivated this definition of integrability for spin chains, let us use it to analyse the implications from demanding the conservation of $\mathcal{Q}_r$ during scattering processes

$$\langle\mathcal{Q}_r\rangle_{in}=\langle\mathcal{Q}_r\rangle_{out}\,. \tag{3.22}$$

[3] In fact, that might be only true up to a trivial additive constant. We discard this constant in the following by normalizing $\mathcal{Q}_r$ in such a way that its eigenvalue $q_r=q_r(u)$ satisfies $q_r(u(k=0))=0$ for a concrete model with known $u=u(k)$.

[4] For a comment on this identification see the end of appendix B.

The incoming and outgoing states are denoted by $|...\rangle_{in}$ and $|...\rangle_{out}$ and correspond to the asymptotic states in (3.12) and (3.13), respectively. Due to locality the charges act on asymptotic states as

$$\mathcal{Q}_r \left|a_1, u_1; ...; a_m, u_m\right\rangle \sim \sum_{n=1}^{m} k_n^r \left(1 + \mathcal{O}(k_n^{r+1})\right) \left|a_1, u_1; ...; a_m, u_m\right\rangle . \tag{3.23}$$

with rapidities $u_i = u(k_i)$. The conservation equation (3.22) implies the equality of the sum of the individual eigenvalues of the charges in the incoming and outgoing state. Since these eigenvalues q_r scale like k^r only up to higher-order terms, the following argumentation for spin chain models is more subtle than the proof for massive relativistic QFTs given in chapter 1. There the one-particle eigenvalues q_s of $\mathcal{Q}_s$ are $q_s \sim p^s$. In order to circumvent this difficulty, we assume that there are linear combinations Q_r of the original $\mathcal{Q}_r$ that act on one-particle states as eigenoperators with eigenvalues $\sim k^r$. Their existence is not obvious for infinitely long spin chains which, however, are of purely theoretical nature anyway. By contrast, for finite, i.e. physical, spin chains their construction is less straightforward due to the finiteness of these linear combinations. For incoming momenta $\{k_i\}$ and outgoing momenta $\{k_f\}$ in an $m_{in} \to m_{out}$ scattering process the existence of these linear combinations Q_r implies

$$\sum_{i=1}^{m_{in}} k_i^r = \sum_{f=1}^{m_{out}} k_f^r \quad \forall\ r > 0 \tag{3.24}$$

where the sum on the left hand side is over all initial momenta and the sum on the right hand side over all final momenta. Note that we included the momentum conservation equation given by (3.24) for $r = 1$. This is the analogue of the relation (1.4) of the massive relativistic QFT. Similarly to its discussion, we can use (3.24) for $r > 0$ to conclude the conservation of the magnon number and the equivalence of the initial and final sets of momenta.

We now focus on the question whether (3.22) also implies factorization of scattering. Introducing the S-matrix via (3.14) and demanding the conservation of the local charges gives

$$\langle in|\mathrm{S}|out\rangle = \langle in|e^{-i\alpha\mathcal{Q}_r}\mathrm{S}e^{i\alpha\mathcal{Q}_r}|out\rangle \quad \forall\ \alpha \in \mathbb{Z},\ 1 \le r \le N. \tag{3.25}$$

For $r = 1$ and $\mathcal{Q}_1 = \mathcal{K}$, the transformation $e^{i\alpha\mathcal{K}}$ corresponds to the shift operator $\mathcal{U}^\alpha$ that moves the excitations on the chain by α sites. This results in a constant shift of all magnons. Thus the conservation equation (3.25) for $r = 1$ implies shift invariance of the S-matrix. In order to investigate the implications of (3.25) for higher-order charges $r > 1$, we describe the magnons moving along the spin chain via a lattice version of the Gaussian wave packet. This is in close connection to the proof for massive relativistic QFTs. By straightforward discretization, the wave-function of a one-magnon state moving with momentum k_1 is given by

$$\psi(n) \propto \int \mathrm{d}k e^{ik(n-n_1)} e^{-c^2(k-k_1)^2} \tag{3.26}$$

in lattice space with $c \in \mathbb{R}$. Note that $n \in \mathbb{N},\ 1 \le n \le N$ denotes the sites of the chain. For $n_1 \in \mathbb{N},\ 1 \le n_1 \le N$ the Gaussian wave-packet is centered around site n_1. Otherwise, i.e. for general $n_1 \in \mathbb{R},\ 1 \le n_1 \le N$, the maximal probability of finding the excitation is at the site corresponding to the nearest integer of n_1. Note that for finite spin chains the momentum integral becomes a summation over the allowed momenta on the chain.

Acting with the symmetry operator $e^{i\alpha Q_r}$ on the wave-function (3.26) changes it to

$$\bar{\psi}(n) \propto \int \mathrm{d}k e^{ik(n-n_1)} e^{c^2(k-k_1)^2} e^{i\alpha k^r}. \tag{3.27}$$

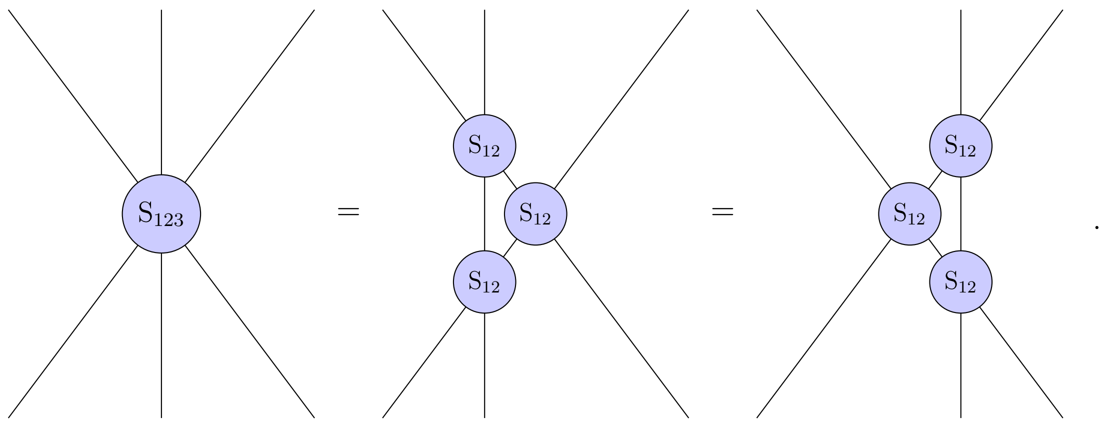
The integrand can be Taylor-expanded around k_1 since this region gives the biggest contribution to the integral

$$k^r = k_1^r + r k_1^{r-1}(k-k_1) + \mathcal{O}((k-k_1)^2). \tag{3.28}$$

The same also works for finite spin chains – here only the terms with momentum k close to k_1 contribute in the sum. (3.28) gives

$$\bar{\psi}(n) \propto \int \mathrm{d}k e^{ik(n-n_1+\alpha r k_1^{r-1})} e^{-c^2(k-k_1)^2} e^{i\alpha(1-r)k_1^r} \tag{3.29}$$

and thus the wave-packet is shifted in space by an amount $-\alpha r k_1^{r-1}$. Note that this shift is momentum-dependent. Since the magnons in a general scattering process have different momenta, they get shifted by different amounts under a transformation $e^{i\alpha Q_r}$ for $r > 1$. As a result, we can shift scattering events of more than two magnons apart such that they only contain two-particle scattering events. By the conservation equation (3.25), the probability amplitude for the original and shifted process must be the same, i.e. m-magnon S-matrices must factorize into two-magnon S-matrices. In particular, a three-magnon S-matrix factorizes into three two-magnon S-matrices that have to satisfy a consistency equation



(3.30)

Thus the definition of integrability via the existence of local charges of the form (3.19) implies factorization of scattering for the physical finite spin chains by arguments similar to the ones for massive relativistic QFTs given in chapter 1.

Chapter 4

The Yangian

As we already realized in the previous chapters, integrability is closely related to symmetries: The presence of a tower of commuting conserved local charges is one possible defining characteristics of quantum integrable models. Another version is the existence of hidden symmetries that form a quantum algebra called *Yangian*. We define it in section 4.1. That definition once more contains a tower of conserved charges. However, in this case these charges act non-locally and satisfy non-trivial commutation relations. In section 4.2 we study the Yangian for a concrete example – the Heisenberg spin chain. This discussion will reveal the close connection between the existence of conserved Yangian generators and local charges $\mathcal{Q}_r$. Nevertheless, we will argue that this connection is not always manifest. At the end of this chapter we investigate the constraints resulting from the presence of Yangian symmetries on the model's S-matrix. This will help us to analyse the scattering behavior of particles in various integrable models in the following chapters 5-7.

4.1 Definition of the Yangian

We begin this chapter with a review of the Yangian algebra on the basis of [17] and [26–28].

Generators of the Yangian Algebra

The Yangian is an algebra containing an infinite number of generators that are arranged in infinitely many levels. The first set contains the *level-0 generators* J^a. They span a finite-dimensional simple Lie algebra $\mathfrak{g}$ with structure constants $f^{ab}{}_c$

$$[\mathrm{J}^a, \mathrm{J}^b] = f^{ab}{}_c \mathrm{J}^c. \tag{4.1}$$

Hence the dimensionality of this level is given by the dimension $\dim(\mathfrak{g})$ of the Lie algebra, i.e. $a = 1, ..., \dim(\mathfrak{g})$. The Killing form κ^{ab} and its inverse raises and lowers indices. The second set of generators contains the same number of elements. These are called *level-1 generators* and are denoted by $\hat{\mathrm{J}}^a$. They satisfy a commutation relation with the level-0 generators of the form

$$[\mathrm{J}^a, \hat{\mathrm{J}}^b] = f^{ab}{}_c \hat{\mathrm{J}}^c. \tag{4.2}$$

All further levels are successively generated by commuting the generators of the previous level. Take as an example the level-2 generators which are given by $\hat{\hat{\mathrm{J}}}^a \sim$

$f^a{}_{bc}[\hat{\mathrm{J}}^b, \hat{\mathrm{J}}^c]$. This procedure is only constrained by the *Serre-relations*

$$\begin{aligned}
&[\hat{\mathrm{J}}^a, [\hat{\mathrm{J}}^b, \mathrm{J}^c]] = g^{abc}{}_{def}\{\mathrm{J}^d, \mathrm{J}^e, \mathrm{J}^f\}, \\
&[[\hat{\mathrm{J}}^a, \hat{\mathrm{J}}^b], [\mathrm{J}^r, \hat{\mathrm{J}}^s]] + [[\hat{\mathrm{J}}^r, \hat{\mathrm{J}}^s], [\mathrm{J}^a, \hat{\mathrm{J}}^b]] = (g^{abc}{}_{def} f^{rs}_c + g^{rsc}{}_{def} f^{ab}_c)\{\mathrm{J}^d, \mathrm{J}^e, \mathrm{J}^f\},
\end{aligned} \tag{4.3}$$

where

$$g^{abc}{}_{def} = \tfrac{1}{24} f^{ai}{}_d f^{bj}{}_e f^{ck}{}_f f_{ijk}, \qquad \{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k. \tag{4.4}$$

Thus, although the Yangian is an infinite-dimensional algebra, it is spanned by a finite set of operators – the level-0 and level-1 generators – that must satisfy the Jacobi-like relations in (4.3). Since a Yangian algebra is always based on the underlying Lie algebra $\mathfrak{g}$, it is denoted by $Y[\mathfrak{g}]$.

The Coproduct Structure

An important property of the Yangian is its coproduct structure. If we think of J^a and $\hat{\mathrm{J}}^a$ as symmetry generators acting on a one-particle state, the action on two-particle states is specified by the coproduct Δ with

$$\begin{aligned}
\Delta(1) &= 1 \otimes 1, \\
\Delta(\mathrm{J}^a) &= \mathrm{J}^a \otimes 1 + 1 \otimes \mathrm{J}^a, \\
\Delta(\hat{\mathrm{J}}^a) &= \hat{\mathrm{J}}^a \otimes 1 + 1 \otimes \hat{\mathrm{J}}^a + f^a{}_{bc} \mathrm{J}^b \otimes \mathrm{J}^c.
\end{aligned} \tag{4.5}$$

Just like in the usual Lie algebra, the action of a level-0 generator on a two-particle state is the sum of the level-0 generators acting on both particles separately. The action of the level-1 generator is slightly more involved. It contains the separate action of level-1 generators on the first and second particle but also includes the simultaneous action of level-0 generators on both particles. Note that the last line in (4.5) is sensitive to a rescaling of the generators due to an imbalance of the number of generators and structure constants. For the action on an m-particle state these relations generalize to

$$\begin{aligned}
\Delta^{m-1}(\mathrm{J}^a) &= \sum_{n=1}^{m} \mathrm{J}^a_n, \\
\Delta^{m-1}(\hat{\mathrm{J}}^a) &= \sum_{n=1}^{m} \hat{\mathrm{J}}^a_n + f^a{}_{bc} \sum_{n_1=1}^{m} \sum_{n_2=1}^{n_1-1} \mathrm{J}^b_{n_2} \mathrm{J}^c_{n_1}.
\end{aligned} \tag{4.6}$$

Here the generators J^a_n and $\hat{\mathrm{J}}^a_n$ act non-trivially on the nth particle and trivially on the remaining $m-1$ particles, e.g.

$$\mathrm{J}^a_n = \underset{1}{1} \otimes \underset{2}{1} \otimes \underset{\dots}{\dots} \otimes \underset{n}{\mathrm{J}^a} \otimes \underset{\dots}{\dots} \otimes \underset{m}{1}. \tag{4.7}$$

The coproduct structure reveals that the Yangian is an algebra of non-local charges: While the coproduct of the level-0 generators acts locally on single spaces, the coproduct of the level-1 generator acts non-locally on two spaces simultaneously.

The Evaluation Representation

One possible representation of the Yangian generators is the *evaluation representation* $\rho_{\bar{u}}$. It lifts the level-0 representation ρ of the underlying Lie algebra (4.1) to a representation of both the level-0 and level-1 Yangian generators via

$$\rho_{\bar{u}}(\mathrm{J}^a) = \rho(\mathrm{J}^a) \qquad \rho_{\bar{u}}(\hat{\mathrm{J}}^a) = \bar{u}\rho(\mathrm{J}^a). \tag{4.8}$$

For theories with moving particles the role of the parameter $\bar{u}$ is taken by the rapidity u of the particle that is characterized by the set of quantum numbers a, i.e.

$$\rho_{\bar{u}}(\mathrm{J}^a)\,|a,u\rangle = \rho(\mathrm{J}^a)\,|a,u\rangle \qquad \rho_{\bar{u}}(\hat{\mathrm{J}}^a)\,|a,u\rangle = \bar{u}\rho(\mathrm{J}^a)\,|a,u\rangle \tag{4.9}$$

with $\bar{u} \propto u$ up to a model-dependent constant[1]. See e.g. [17] for a discussion of this connection between both parameters in a massive, relativistic, two-dimensional QFT. In order to be a representation for the whole Yangian algebra, the Serre-relations (4.3) have to be satisfied for $\rho_{\bar{u}}$. For $\bar{u} = 0$ the left hand sides of these relations vanish trivially. Thus the Lie algebra representation ρ of the right hand side has to vanish. This constraint reduces the number of Lie algebra representations ρ that are suitable for the construction of evaluation representations of the form (4.8). Acting with the Yangian generators in the evaluation representation on multi-particle states is defined via the coproduct (4.6) as

$$\begin{aligned}
&\rho_{\bar{u}}^{\otimes m}(\Delta^{m-1}\mathrm{J}^a)\,|a_1,u_1;...;a_m,u_m\rangle = \sum_{n=1}^{m} \rho^{\otimes m}(\mathrm{J}^a_n)\,|a_1,u_1;...;a_m,u_m\rangle \\
&\rho_{\bar{u}}^{\otimes m}(\Delta^{m-1}\hat{\mathrm{J}}^a)\,|a_1,u_1;...;a_m,u_m\rangle \\
&\qquad = \left(\sum_{n=1}^{m} \bar{u}_n \rho^{\otimes m}(\mathrm{J}^a_n) + \tfrac{1}{2} f^a{}_{bc} \sum_{1\le n_1<n_2\le m} \rho^{\otimes m}(\mathrm{J}^b_{n_1})\rho^{\otimes m}(\mathrm{J}^c_{n_2})\right) |a_1,u_1;...;a_m,u_m\rangle\,.
\end{aligned} \tag{4.10}$$

Here $\rho_{\bar{u}}^{\otimes m}$ and $\rho^{\otimes m}$ denote the representations on a tensor product of length m.

Note that there exist several realizations of the Yangian, c.f. [17]. In particular, the generators of the Yangian algebra were first discovered in the context of the RTT-relation (2.54). In the corresponding RTT-realization one defines the monodromy matrix T as the solution of the RTT-relation for a given R-matrix, which is in the case of the fundamental representation of $\mathfrak{su}(\mathrm{n})$ Yang's R-matrix

$$\mathrm{R} = \mathbb{I} + \tfrac{\mathbb{P}}{\lambda}. \tag{4.11}$$

This solution T=T(λ) can be expanded for large λ which yields the generators of the Yangian algebra satisfying the same commutation relations as the generators discussed in this section. We show this explicitly for the Heisenberg spin chain in the next section. In this version of the Yangian, the close connection between the Yangian and integrability via its definition by the R-matrix satisfying (2.54) is visible.

[1] In the following chapters we will not distinguish between the parameter $\bar{u}$ and u. Thus, when making use of the results of this thesis, one has to bear in mind a potential model-dependent factor. Importantly, since u is shifted under a Lorentz boost, the parameter $\bar{u}$ is also additive.

4.2 The Yangian of the Heisenberg Spin Chain

In this section we want to motivate an enlargement of the set of definitions of quantum integrability by a formulation including the Yangian which is: The theory's S-matrices are Yangian invariants. In order to do so, we discuss this statement for the Heisenberg spin chain[2].

Yangian Invariance of the Heisenberg Spin Chain

We start by investigating its monodromy along the lines of [17]. The monodromy defined by (2.42) for the Lax operator (2.38) is a matrix in auxiliary space with matrix elements

$$\begin{aligned}\mathrm{T}^{\alpha}_{\beta}(\lambda) = &\left(\lambda\mathbb{I}_1 \otimes \mathbb{I}^{\alpha}_{\gamma_1} + i\mathcal{S}^{a_1}_1 \otimes (\sigma^{a_1})^{\alpha}_{\gamma_1}\right)\cdot\left(\lambda\mathbb{I}_2 \otimes \mathbb{I}^{\gamma_1}_{\gamma_2} + i\mathcal{S}^{a_2}_2 \otimes (\sigma^{a_2})^{\gamma_1}_{\gamma_2}\right)\cdot \\ &\dots\cdot\left(\lambda\mathbb{I}_N \otimes \mathbb{I}^{\gamma_{N-1}}_{\beta} + i\mathcal{S}^{a_N}_N \otimes (\sigma^{a_N})^{\gamma_{N-1}}_{\beta}\right). \end{aligned} \tag{4.12}$$

Expanding it around $\lambda_0 = \frac{i}{2}$ and tracing over the auxiliary space gives local charges. Let us now do the expansion for large λ. This yields

$$\begin{aligned}\mathrm{T}^{\alpha}_{\beta}(\lambda) = \lambda^N &\left(\frac{i}{\lambda}\sum_{n=1}^{N}\mathcal{S}^a_n - \frac{1}{\lambda^2}f^{abc}\sum_{1\le n_1<n_2\le N}\mathcal{S}^b_{n_1}\mathcal{S}^c_{n_2}\right)\otimes(\sigma^a)^{\alpha}_{\beta} \\ &+ \lambda^N\left(\mathbb{I} - \frac{1}{\lambda^2}\sum_{1\le n_1<n_2\le N}\mathcal{S}^a_{n_1}\mathcal{S}^a_{n_2}\right)\otimes\mathbb{I}^{\alpha}_{\beta} + \lambda^N\mathcal{O}\left(\frac{1}{\lambda^3}\right), \end{aligned} \tag{4.13}$$

where we used $\sigma^a\sigma^b = f^{abc}\sigma^c + \delta^{ab}$. The leading contribution in the first term corresponds to the total spin operator $\sum_{n=1}^{N}\mathcal{S}^a_n$ which is an invariant of the Hamiltonian. The emergence of the generators of the $\mathfrak{su}(2)$ symmetry algebra in the expansion of the monodromy for large λ points towards the exceptionality of this point which is why we proceed with the analysis of the subleading contribution of this expansion.

In fact, we show that the terms in the first summand build the Yangian algebra $Y[\mathfrak{su}(2)]$. Its level-0 generators in the evaluation representation ρ_u is given by the fundamental representation discussed in section 2.1

$$\rho_u(\mathrm{J}^a) = \mathcal{S}^a. \tag{4.14}$$

The structure constants are $f^{abc} = i\epsilon^{abc}$. The action on N sites is given by the total spin operators

$$\rho_u^{\otimes N}(\Delta^{N-1}\mathrm{J}^a) = \sum_{n=1}^{N}\mathcal{S}^a_n \tag{4.15}$$

that we discovered in the first term of the expansion. Let us now look for the level-1 generators $\hat{\mathrm{J}}^a$ of $Y[\mathfrak{su}(2)]$. On N sites in the evaluation representation ρ_u for $u = 0$ they are given by

$$\rho_0^{\otimes N}(\Delta^{N-1}\hat{\mathrm{J}}^a) = \frac{1}{2}f^{abc}\sum_{1\le n_1<n_2\le N}\mathcal{S}^b_{n_1}\mathcal{S}^c_{n_2}. \tag{4.16}$$

[2]Note that in [29] the relation of local charges and non-local Yangian charges in principal chiral $(1+1)$-dimensional models are investigated.

Indeed, this operator corresponds to the term in the second order of the expansion in the first summand in (4.13). Further powers in the expansion of the monodromy will give the higher-level generators. As we already discussed in the last section, the level-0 and level-1 generators in the evaluation representation span the whole set of Yangian generators. The second summand in (4.13) consists of the identity and powers of the level-0 generator which can be seen by calculating

$$\sum_{1\le n_1<n_2\le N} \mathcal{S}^a_{n_1}\mathcal{S}^a_{n_2} = \tfrac{1}{2}\sum_{n_1,n_2=1}^{N} \mathcal{S}^a_{n_1}\mathcal{S}^a_{n_2} - \tfrac{1}{2}\sum_{n=1}^{N} \mathcal{S}^a_{n}\mathcal{S}^a_{n} = \tfrac{1}{2}\mathcal{S}^a\mathcal{S}^a - \tfrac{1}{8}\mathbb{I}. \tag{4.17}$$

Thus we proved that the Yangian $Y[\mathfrak{su}(2)]$ naturally arises in the monodromy of the Heisenberg spin chain. Its generators commute with the Hamiltonian (2.9) for infinitely long spin chains and thus the Yangian is a symmetry algebra of the model. For finite chains this is only true up to boundary terms, i.e. the Yangian is a symmetry algebra only in the bulk. Therefore, magnons moving along Heisenberg spin chains must transform under the Yangian and the S-matrix – which maps outgoing magnons onto incoming magnons – must be a Yangian-invariant.

Fundamental and Non-Fundamental Spin Chain Models

We have shown that for the Heisenberg spin chain the conserved local charges and Yangian generators follow from the same monodromy. Thus the definition of integrability via a conserved Yangian automatically yields a tower of conserved local charges. We conclude by the arguments in section 3.4 that scattering processes in this model factorize. This argumentation holds for all spin chains whose monodromy generates both Yangian and local operators, i.e. for all fundamental models. This implicit proof of factorized scattering from a conserved Yangian is also possible for integrable QFTs with Yangian and local charges following from the same monodromy via the argumentation outlined in chapter 1.

Nevertheless, it is not clear whether this implicit proof is possible for all quantum models which are integrable via a conserved Yangian. Take as an example the non-fundamental models that we discussed in section 2.5. Although there exists a technique to construct local charges from an alternative monodromy, it is not clear whether one always finds the alternative R-matrix $\mathbf{R}$ that is essential in this construction. Thus it makes sense to investigate whether Yangian symmetry directly implies factorization of scattering. This study is also particularly relevant in the context of Yangian-invariant scattering amplitudes of $\mathcal{N}=4$ SYM and the hexagon approach to form factors as discussed in the introductory chapter.

4.3 Yangian Invariant S-Matrices

For integrable spin chain models with Yangian symmetry the magnons transform under the Yangian $Y[\mathfrak{g}_r]$ corresponding to the residual Lie algebra $\mathfrak{g}_r$. This symmetry has important implications on the S-matrix which relates multi-magnon states. The conservation of the Yangian charges during a scattering process constrains the m-particle S-matrix such that

$$\text{level 0:}\ \ [\Delta^{m-1}\mathrm{J}^a, \mathrm{S}] = 0 \tag{4.18}$$

$$\text{level 1:}\ \ [\Delta^{m-1}\hat{\mathrm{J}}^a, \mathrm{S}] = 0. \tag{4.19}$$

We investigate these constraints and their implications on S-matrices for the three different Yangian algebras $Y[\mathfrak{su}(\mathrm{n})]$, $Y[\mathfrak{su}(1|1)]$ and $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$ in detail in the following chapters. Doing so we focus on $2 \to 2$ and $3 \to 3$ S-matrices and check their consistent factorization. In fact, the results of this discussion are not only valid for the corresponding spin chain models. The results can be used for any Yangian-invariant model with an S-matrix whose particles transform under the evaluation representation (4.9) of the above algebras in the fundamental representation.

Chapter 5

$Y[\mathfrak{su}(\mathrm{n})]$-Invariant S-Matrices

In this chapter we carry out the explicit analysis of the constraints (4.18) and (4.19) for the Yangian generators of $Y[\mathfrak{su}(\mathrm{n})]$. For this purpose, we begin with a discussion of the algebra, its generators and its coproduct structure in section 5.1. This will help us to set up a framework for the explicit realization of the Yangian constraints. We then move on to the solution of these constraints for the two- and three-particle S-matrix in 5.2 and 5.3 and discuss the results with focus on consistent factorization.

5.1 The Algebra $Y[\mathfrak{su}(\mathrm{n})]$ and Its Fundamental Representation

Generators of $\mathfrak{su}(\mathrm{n})$

Let us begin by reviewing the algebra $\mathfrak{su}(\mathrm{n})$ and introducing a suitable basis for its generators. A basis of the algebra $\mathfrak{u}(\mathrm{n})$ is spanned by the n^2 operators $\bar{\mathfrak{R}}^a_b$, $a, b = 1, 2, ..., \mathrm{n}$, satisfying

$$[\bar{\mathfrak{R}}^a_b, \bar{\mathfrak{R}}^c_d] = \delta^c_b \bar{\mathfrak{R}}^a_d - \delta^a_d \bar{\mathfrak{R}}^c_b. \tag{5.1}$$

The fundamental representation of these operators in bra-ket notation is

$$\rho\left(\bar{\mathfrak{R}}^a_b\right) = |a\rangle\langle b| \tag{5.2}$$

with $\langle a|b\rangle = \delta^b_a$. The generators $\mathfrak{R}^a_b$ of $\mathfrak{su}(\mathrm{n})$ satisfy the same commutation relation

$$[\mathfrak{R}^a_b, \mathfrak{R}^c_d] = \delta^c_b \mathfrak{R}^a_d - \delta^a_d \mathfrak{R}^c_b. \tag{5.3}$$

and are moreover traceless. To form traceless operators, we may subtract the identity operator $\rho(1) = \sum_c |c\rangle\langle c|$ from (5.2) such that

$$\rho\left(\mathfrak{R}^a_b\right) = |a\rangle\langle b| - \delta^a_b \frac{1}{\mathrm{n}} \sum_c |c\rangle\langle c|\,. \tag{5.4}$$

As expected, we obtain $\mathrm{n}^2 - 1$ independent generators which may be arranged in the following sets:

$$\begin{aligned}
\mathfrak{R}_{\mathrm{o}} &= \{\mathfrak{R}^a_b, 1 \le a = b \le \mathrm{n}-1\} \\
\mathfrak{R}_{-} &= \{\mathfrak{R}^a_b, 1 \le a < b \le \mathrm{n}\} \\
\mathfrak{R}_{+} &= \{\mathfrak{R}^a_b, 1 \le b < a \le \mathrm{n}\}.
\end{aligned} \tag{5.5}$$

The set $\mathfrak{R}_{\mathrm{o}}$ contains the Cartan operators that generate the maximal abelian subalgebra of $\mathfrak{su}(\mathrm{n})$. The operators in $\mathfrak{R}_+$ can be interpreted as raising operators and similarly $\mathfrak{R}_-$ contains the lowering operators, cf. the example below.

$Y[\mathfrak{su}(\mathrm{n})]$ in the Evaluation Representation

Now let us turn to the Yangian that is associated to $\mathfrak{su}(\mathrm{n})$. The level-0 generators are the $\mathfrak{su}(\mathrm{n})$-generators $\mathfrak{R}^a_b$. For the evaluation representation ρ_u (4.8) we use (5.4) as the Lie algebra representation ρ, i.e.

$$\rho_u(\mathfrak{R}^a_b) = \rho(\mathfrak{R}^a_b). \tag{5.6}$$

In this representation a single particle may take n "spin" orientations labeled by a

$$|a\rangle\,,\;\; a = 1, ..., \mathrm{n}. \tag{5.7}$$

These states are the eigenstates of the Cartan operators of $\mathfrak{R}_\mathrm{o}$. The action of the level-1 generators $\hat{\mathfrak{R}}^a_b$ in the evaluation representation makes it necessary to introduce an evaluation parameter. We identify it with the rapidity u and label each one-particle state as

$$|a, u\rangle\,. \tag{5.8}$$

These states shall form an orthonormal set, i.e. $\langle a, u|b, v\rangle = \delta^a_b \delta_{u,v}$. The operators in (5.4) do not change the rapidity of a particle. The corresponding level-1 generators $\hat{\mathfrak{R}}^a_b$ of $Y[\mathfrak{su}(\mathrm{n})]$ in the evaluation representation (4.8) are given by

$$\rho_u(\hat{\mathfrak{R}}^a_b)\,|a, u\rangle = u\rho(\mathfrak{R}^a_b)\,|a, u\rangle\,. \tag{5.9}$$

Example: $Y[\mathfrak{su}(2)]$

Take as an example the evaluation representation for the Yangian $Y[\mathfrak{su}(2)]$. The Lie algebra $\mathfrak{su}(2)$ is spanned by a single Cartan operator $\mathfrak{R}^1_1$, as well as the ladder operators $\mathfrak{R}^1_2$ and $\mathfrak{R}^2_1$. They satisfy the commutation relations (5.3), i.e.

$$[\mathfrak{R}^1_1, \mathfrak{R}^2_1] = -\mathfrak{R}^2_1, \qquad [\mathfrak{R}^1_1, \mathfrak{R}^1_2] = +\mathfrak{R}^1_2, \qquad [\mathfrak{R}^2_1, \mathfrak{R}^1_2] = -2\mathfrak{R}^1_1. \tag{5.10}$$

By comparing these with the commutation relations of the spin operators $\mathcal{S}^z$, $\mathcal{S}^+$ and $\mathcal{S}^-$ from 2.1

$$[\mathcal{S}^z, \mathcal{S}^+] = \mathcal{S}^+, \qquad [\mathcal{S}^z, \mathcal{S}^-] = -\mathcal{S}^-, \qquad [\mathcal{S}^+, \mathcal{S}^-] = 2\mathcal{S}^z \tag{5.11}$$

we can identify

$$\mathcal{S}^z \sim -\mathfrak{R}^1_1,\; \mathcal{S}^+ \sim \mathfrak{R}^2_1,\; \mathcal{S}^- \sim \mathfrak{R}^1_2. \tag{5.12}$$

The two single-particle states are $|1, u\rangle$ and $|2, u\rangle$ which correspond to the previously introduced eigenstates $|{\downarrow}\rangle$ and $|{\uparrow}\rangle$ of the Pauli matrices (2.7). We obtain the analogous relations to (2.10) for $\mathfrak{su}(2)$

$$\begin{aligned} &-\rho(\mathfrak{R}^1_1)\,|1, u\rangle = -\tfrac{1}{2}\,|1, u\rangle\,, \qquad &&\rho(\mathfrak{R}^2_1)\,|1, u\rangle = |2, u\rangle\,, \qquad &&\rho(\mathfrak{R}^1_2)\,|1, u\rangle = 0, \\ &-\rho(\mathfrak{R}^1_1)\,|2, u\rangle = +\tfrac{1}{2}\,|2, u\rangle\,, \qquad &&\rho(\mathfrak{R}^2_1)\,|2, u\rangle = 0, \qquad &&\rho(\mathfrak{R}^1_2)\,|2, u\rangle = |1, u\rangle\,. \end{aligned} \tag{5.13}$$

The action of the level-1 generator is given by the above relations multiplied by the rapidity u of the particle as (5.9).

Coproduct Structure of $Y[\mathfrak{su}(\mathbf{n})]$

In order to calculate the S-matrix corresponding to the scattering of $2 \to 2$ and $3 \to 3$ particles, we need the action of the Yangian generators on two- and three-particle states. As discussed in the previous chapter, this is defined via the coproduct (4.6). For the level-0 generators we find

$$\begin{aligned} \Delta \mathfrak{R}^a_b &= \mathfrak{R}^a_b \otimes 1 + 1 \otimes \mathfrak{R}^a_b, \\ \Delta^2 \mathfrak{R}^a_b &= \mathfrak{R}^a_b \otimes 1 \otimes 1 + 1 \otimes \mathfrak{R}^a_b \otimes 1 + 1 \otimes 1 \otimes \mathfrak{R}^a_b \end{aligned} \tag{5.14}$$

and for the level-1 generators

$$\begin{aligned} \Delta \hat{\mathfrak{R}}^a_b &= \hat{\mathfrak{R}}^a_b \otimes 1 + 1 \otimes \hat{\mathfrak{R}}^a_b + \tfrac{1}{4} f^{ace}_{bdf}\, \mathfrak{R}^d_c \otimes \mathfrak{R}^f_e, \\ \Delta^2 \hat{\mathfrak{R}}^a_b &= \hat{\mathfrak{R}}^a_b \otimes 1 \otimes 1 + 1 \otimes \hat{\mathfrak{R}}^a_b \otimes 1 + 1 \otimes 1 \otimes \hat{\mathfrak{R}}^a_b + \\ &\quad \tfrac{1}{4} f^{ace}_{bdf} \left(\mathfrak{R}^d_c \otimes \mathfrak{R}^f_e \otimes 1 + \mathfrak{R}^d_c \otimes 1 \otimes \mathfrak{R}^f_e + 1 \otimes \mathfrak{R}^d_c \otimes \mathfrak{R}^f_e \right) \end{aligned} \tag{5.15}$$

with constants f^{ace}_{bdf} defined by

$$f^{ace}_{bdf} = \delta^c_b \delta^e_d \delta^a_f - \delta^a_d \delta^e_b \delta^c_f. \tag{5.16}$$

The coproduct structure in (5.15) can be derived by evaluating the term[1] $f^a{}_{bc} \mathrm{J}^b \otimes \mathrm{J}^c$ from (4.6) for $\mathfrak{su}(2)$ in the representation via spin operators $\mathcal{S}^a$ given in (2.4). Take as an example the x-component $f^x{}_{ab} \mathcal{S}^a \otimes \mathcal{S}^b$. Using the identifications (5.12) gives

$$\begin{aligned} f^x{}_{ab} \mathcal{S}^a \otimes \mathcal{S}^b &= i \mathcal{S}^y \otimes \mathcal{S}^z - i \mathcal{S}^y \otimes \mathcal{S}^z \\ &= \tfrac{1}{2} \left(\mathcal{S}^+ \otimes \mathcal{S}^z - \mathcal{S}^- \otimes \mathcal{S}^z - \mathcal{S}^z \otimes \mathcal{S}^+ + \mathcal{S}^z \otimes \mathcal{S}^- \right) \\ &\sim \tfrac{1}{2} \left(-\mathfrak{R}^2_1 \otimes \mathfrak{R}^1_1 + \mathfrak{R}^1_2 \otimes \mathfrak{R}^1_1 + \mathfrak{R}^1_1 \otimes \mathfrak{R}^2_1 - \mathfrak{R}^1_1 \otimes \mathfrak{R}^1_2 \right) \\ &= \tfrac{1}{4} \left(f^{2ac}_{1bd} \mathfrak{R}^b_a \otimes \mathfrak{R}^d_c + f^{1ac}_{2bd} \mathfrak{R}^b_a \otimes \mathfrak{R}^d_c \right). \end{aligned} \tag{5.17}$$

Identifying $\mathcal{S}^x \sim \frac{1}{2}(\mathfrak{R}^2_1 + \mathfrak{R}^1_2)$ we verify the coproduct structure in (5.15). Similar calculations can be done for the y- and z-component. For an arbitrary $\mathfrak{su}(\mathrm{n})$ the same identification is true as long as we normalize the operators $\mathcal{S}^a$ via $\mathrm{tr}(\mathcal{S}^a \mathcal{S}^b) = \frac{1}{2}\delta^{ab}$. These identifications ensure that the result is in the same convention as the results of [17].

$Y[\mathfrak{su}(\mathbf{n})]$-Constraints on the S-Matrix

Having a representation of the Yangian generators as well as the formulas for their action on multi-particle states, we can make the constraints on S given by (4.18) and (4.19) explicit. For the $Y[\mathfrak{su}(\mathrm{n})]$-invariant m-particle S-matrix $\mathrm{S}_{12\ldots m}$ they are given by

$$\text{level 0:} \qquad \left[\sum_{n=1}^{m} (\mathfrak{R}^a_b)_n, \mathrm{S}_{12\ldots m}(u_i, v_i) \right] = 0 \tag{5.18}$$

$$\begin{aligned} \text{level 1:} \qquad & \left(\sum_{n=1}^{m} u_n (\mathfrak{R}^a_b)_n + \frac{1}{4} f^{ace}_{bdf} \sum_{n_1=1}^{m} \sum_{n_2=1}^{n_1-1} (\mathfrak{R}^d_c)_{n_2} (\mathfrak{R}^f_e)_{n_1} \right) \mathrm{S}_{12\ldots m}(u_i, v_i) = \\ & \mathrm{S}_{12\ldots m}(u_i, v_i) \left(\sum_{n=1}^{m} v_n (\mathfrak{R}^a_b)_n + \frac{1}{4} f^{ace}_{bdf} \sum_{n_1=2}^{m} \sum_{n_2=1}^{n_1-1} (\mathfrak{R}^d_c)_{n_2} (\mathfrak{R}^f_e)_{n_1} \right). \end{aligned} \tag{5.19}$$

[1] Note that the indices a, b, c in $f^a{}_{bc} \mathrm{J}^b \otimes \mathrm{J}^c$ in the fundamental representation take the values x, y, z whereas the indices in $f^{ace}_{bdf}\, \mathfrak{R}^d_c \otimes \mathfrak{R}^f_e$ run from $1, \ldots, \mathrm{n}$ with $\mathrm{n} = 3$.

Here the u_i are the rapidities of the incoming and v_i of the outgoing particles. As we discussed in the previous chapters, it is a feature of integrable theories in (1+1) dimensions that the sets of incoming rapidities v_i and outgoing rapidities u_i of a scattering process are equal. Nevertheless, we do not postulate this feature here but check whether Yangian symmetry yields it automatically. Thus, the level-0 condition (5.18) implies that the S-matrix is invariant under $\mathfrak{su}$(n) transformations. The level-1 condition (5.19) is a bilocal constraint. Higher orders of Yangian generators do not impose further constraints on S as long as the Serre-relations are satisfied.

5.2 Solution of the Level-0 Constraint

Let us turn to the explicit analysis of the Yangian constraints (5.18) and (5.19) on S. We calculate the S-matrix for two- and three-particle scattering processes but the discussion can be easily extended to the scattering of more particles. (5.18) restricts the S-matrix to be a linear combination of $\mathfrak{su}$(n)-invariant operators. There are several possibilities to obtain these. We will discuss two approaches in the following.

5.2.1 $\mathfrak{su}$(n)-Invariant Operators from the Symmetric Group $\mathbb{S}_m$

First we use the symmetric group $\mathbb{S}_m$. It contains all permutations of a set of m objects. One can show that the permutation operators projecting onto these permutations span the space of all linear invariants of $\mathfrak{su}$(n) over the tensor product of m Hilbert spaces (see e.g. [30, 31]). Thus we realize that the level-0 condition constrains the S-matrix to be a linear combination of $\mathbb{S}_m$ permutation operators that permute m particles.

Invariant Operators of Length 1

Let us examine the $\mathbb{S}_m$ permutation operators for $m = 1, 2, 3$. The invariant operator on a single site corresponding to $m = 1$ is given by

$$\mathcal{P}_1 = \sum_a |a\rangle \langle a| \tag{5.20}$$

where we sum over all excitation types. It acts as an identity map concerning the particle's quantum number a, i.e. $\mathcal{P}_1 |a\rangle = |a\rangle$. An $\mathfrak{su}$(n)-invariant tensor acting on a single particle moving with rapidity u and conserving its momentum is thus given by[2]

$$\mathcal{I}_1(u) = A_1(u) \sum_a |a, u\rangle \langle a, u| \tag{5.21}$$

with arbitrary factor $A_1(u)$. For $A_1 \equiv 1$ this operator can be used to define the particle number operator via

$$\mathcal{L} := \sum_{n=1}^{m} \left(\sum_u \mathcal{I}_1(u)|_{A\equiv 1} \right)_n = \sum_{n=1}^{m} 1 \otimes 1 \otimes ... \otimes \underset{\substack{\uparrow \\ n\text{th site}}}{\sum_u \mathcal{I}_1(u)|_{A\equiv 1}} \otimes ... \otimes 1. \tag{5.22}$$

[2] In fact, the level-0 constraint that arises from the invariance of S under the Lie-algebra symmetry does not require the labeling of one- and multi-particle states by the rapidity u. Nevertheless, since this will become relevant at level-1 in the evaluation representation of the Yangian, we already introduce the notation including u here.

Here we sum over all allowed rapidities[3] u. This operator acts as

$$\mathcal{L}\,|a_1,u_1;a_2,u_2;...;a_m,u_m\rangle = m\,|a_1,u_1;a_2,u_2;...;a_m,u_m\rangle\,. \tag{5.23}$$

Invariant Operators of Length 2

For $m=2$ there exist two permutation operators which we denote by $\mathcal{P}_{12}$ and $\mathcal{P}_{21}$ and represent their action on a two-particle state by

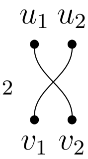
$$\mathcal{P}_{12} = \sum_{a_1,a_2} |a_1;a_2\rangle\langle a_1;a_2|\,, \qquad \mathcal{P}_{21} = \sum_{a_1,a_2} |a_2;a_1\rangle\langle a_1;a_2|\,. \tag{5.24}$$

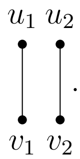
$\mathcal{P}_{12}$ is the identity map for two particles and thus describes a trivial permutation, whereas $\mathcal{P}_{21}$ really permutes the two particles' types. An $\mathfrak{su}(\mathrm{n})$-invariant operator is the superposition of both permutation operators. Introducing rapidities and allowing the invariant operator to change these yields the general form

$$\mathcal{I}_{12}(u_{1,2};v_{1,2}) = A_{12}\,|a_2,u_1;a_1,u_2\rangle\langle a_1,v_1;a_2,v_2| + B_{12}\,|a_1,u_1;a_2,u_2\rangle\langle a_1,v_1;a_2,v_2|\,. \tag{5.25}$$

Here the summation over excitation types is implicit. The coefficients A_{12} and B_{12} may depend on $u_{1,2}$ and $v_{1,2}$. Denoting the momenta of the particles by $k_1 = k(u_1)$, $k_2 = k(u_2)$, $p_1 = k(v_1)$ and $p_2 = k(v_2)$, momentum conservation restricts the rapidities via $k_1 + k_2 = p_1 + p_2$. For the S-matrix we will later on demand $u_1 > u_2$ and $v_1 < v_2$ due to the convention regarding the ordering of rapidities in incoming and outgoing states as discussed in section (3.3).

Let us introduce a diagrammatic way to represent the action of $\mathcal{I}_{12}$ on a two-particle state using permutation diagrams of the form

$\mathcal{I}_{12}(u_{1,2};v_{1,2}) = A_{12}$ 

$+B_{12}$ 

. (5.26)

Here the two dots at the bottom of each diagram represent the particles before the action of the operator, i.e. in case of the S-matrix corresponding to the outgoing state with rapidities $v_1 < v_2$. The state after the action of the operator is represented by the top dots which are associated to the incoming state with rapidities $u_1 > u_2$ for the S-matrix. The lines represent the action of the corresponding permutation operators and connect particles of the same type. Let us analyse these diagrams in the context of S-matrices, i.e. for the ordering of rapidities as $u_1 > u_2$ and $v_1 < v_2$. Then, the first diagram illustrates a process where the faster particle of rapidity u_1 overtakes the second particle of rapidity u_2 without a change of type but redistribution of momenta to v_1 and v_2 with $v_1 < v_2$. In contrast to this, the second diagram represents a scattering process where the two particles interchange their type, i.e. the ordering of the quantum numbers a_i is the same in the initial and final state.

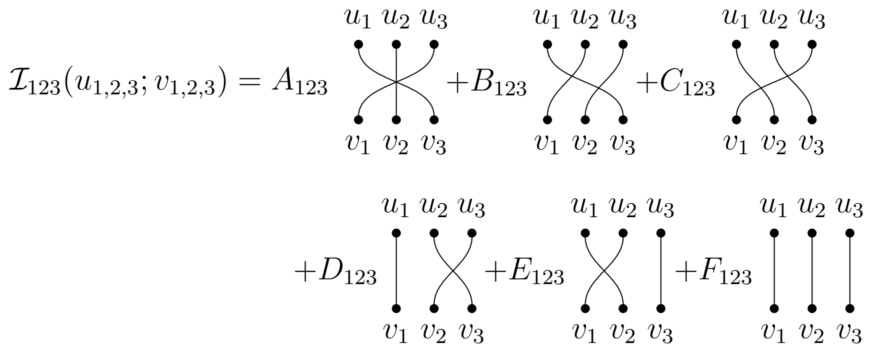
[3] A discrete set of allowed rapidities is a special feature of periodic spin chains and field theories, see e.g. the discussion in section 2.2. For infinite spin chains or unbounded field theories there might be no quantization of momentum and thus we would need to integrate over the allowed range of rapidities.

Invariant Operators of Length 3

Now let us move on to $m = 3$. Here we find $3! = 6$ permutation operators that permute the excitation types of three particles

$$\begin{aligned} \mathcal{P}_{123} &= |a_1; a_2; a_3\rangle \langle a_1; a_2; a_3| \,, & \mathcal{P}_{213} &= |a_2; a_1; a_3\rangle \langle a_1; a_2; a_3| \,, \\ \mathcal{P}_{132} &= |a_1; a_3; a_2\rangle \langle a_1; a_2; a_3| \,, & \mathcal{P}_{231} &= |a_2; a_3; a_1\rangle \langle a_1; a_2; a_3| \,, \\ \mathcal{P}_{312} &= |a_3; a_1; a_2\rangle \langle a_1; a_2; a_3| \,, & \mathcal{P}_{321} &= |a_3; a_2; a_1\rangle \langle a_1; a_2; a_3| \,. \end{aligned} \tag{5.27}$$

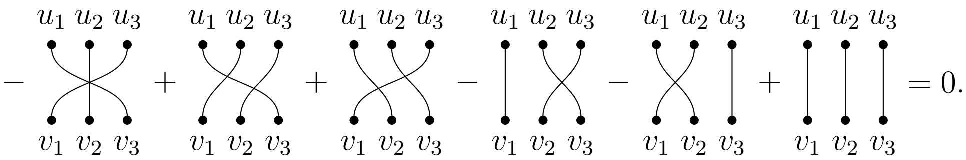
Thus an $\mathfrak{su}$(n)-invariant operator of length 3 can be written as a linear combination



(5.28)

with coefficients $A_{123}, ..., F_{123}$ which may be rapidity-dependent.

Note that the number of orthogonal $\mathfrak{su}$(n)-invariant operators does not necessarily coincide with the number of permutations in $\mathbb{S}_m$ given by $m!$. In fact, this is only true as long as $m \leq n$. Take as an example the $\mathfrak{su}(2)$-invariant operator $\mathcal{I}_{123}$. Here the permutation operators obey a relation which may be represented by the permutation diagrams as

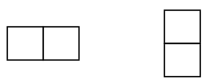


(5.29)

The left hand side of this equation is an operator that antisymmetrizes three excitation types. Since for $\mathfrak{su}(2)$ there are only two allowed values for the a_i, this operator vanishes. Thus for $\mathfrak{su}(2)$ the invariant operator $\mathcal{I}_{123}$ can be written as a linear combination of an arbitrary set of five permutation operators.

5.2.2 $\mathfrak{su}$(n)-Invariant Operators from Young tableaux

We may also obtain invariant operators by constructing them as hermitian Young projection operators and transition operators. In [31] it is shown that these span the whole space of $\mathfrak{su}$(n)-invariants. Let us illustrate this construction for invariant operators of length 2 and 3. Any two-particle state transforming under $\mathfrak{su}$(n) can be depicted by the Young diagrams as



corresponding to the two Young tableaux

$$\ytableaushort{12} \qquad \ytableaushort{1,2}\,.$$

Thus there exist two Young projection operators that project onto this completely symmetric and completely antisymmetric representation. They span the whole space of $\mathfrak{su}$(n) invariant operators of length 2. Now let us move on to $m = 3$ where the states correspond to three Young diagrams

$$\ydiagram{3} \qquad \ydiagram{2,1} \qquad \ydiagram{1,1,1}$$

and four Young tableaux

$$\ytableaushort{123} \qquad \ytableaushort{12,3} \qquad \ytableaushort{13,2} \qquad \ytableaushort{1,2,3}\,.$$

We can construct four Young projectors projecting onto the three-particle states corresponding to these tableaux. In [32] it was shown that tableaux of the same shape correspond to equivalent representations. Thus the authors argue that there must exist transition operators that project between these representations. In the case of $m = 3$, there are two transition operators corresponding to two tableaux of the same shape. Together with the four Young projection operators they span a fully orthogonal basis of invariants. Note that for n=2 the completely antisymmetric representation for $m = 3$ is empty. The corresponding Young projection operator is the null operator and there are five orthogonal operators in agreement with the above discussion. Thus, in this approach we immediately get the correct number of orthogonal operators without looking for relations between the permutation operators.

This procedure can be applied for all particle numbers m. There exists a standard method that allows us to build the Young projection and transition operators explicitly, see e.g. [32]. In the following, we use the approach via the permutation diagrams since their implementation in `Mathematica` is easier. We use the invariants in (5.26) and (5.28) and further constrain them by the level-1 constraint (5.19).

5.3 Solution of the Level-1 Constraint

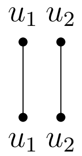
Having found the most general form of the $\mathfrak{su}$(n)-invariant operators for length 2 and 3, we proceed with the level-1 constraint (5.19). In the appendix D we briefly show how to set up a `Mathematica` notebook that contains the level-0 ansatz from section 5.3 and how to implement the level-1 constraints. In this chapter we only discuss the results.

Two-Particle S-Matrix

Evaluating explicitly the constraint for the case of two scattering particles using the symbolic computation program `Mathematica` gives two solutions for the coefficients and rapidities of the ansatz (5.26). The first one is

$$(1) \quad v_1 = u_1,\ v_2 = u_2,\ A^{(1)}_{12} = 0,\ B^{(1)}_{12} = c(u_1, u_2)\ \delta_{u_1,v_1}\delta_{u_2,v_2}, \tag{5.30}$$

$$\mathcal{I}^{(1)}_{12}(u_{1,2};v_{1,2}) = c(u_1,u_2)\ \delta_{u_1,v_1}\delta_{u_2,v_2}$$



$$\mathcal{I}^{(2)}_{12}(u_{1,2};v_{1,2}) = c(u_1,u_2)\ \delta_{u_1,v_2}\delta_{u_2,v_1}$$

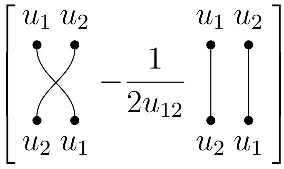


Table 5.1: $\mathfrak{su}$(n)-invariant operators on length 2

where c is an unknown overall factor that may depend on the rapidities. In terms of permutation diagrams, this result can be represented by the operator $\mathcal{I}^{(1)}_{12}$ given in Table 5.1. This operator projects a state to itself up to a factor, i.e. initial and final state coincide, and it does not describe the scattering of particles. This would-be scattering occurs if the particle with smaller momentum moves behind the particle with larger momentum and they never meet, i.e. in our conventions $u_1 < u_2$. Alternatively, the scattering did not yet take place. Thus, this operator cannot be identified with the S-matrix we are looking for.

The second solution of the constraint (5.19) is

$$(2) \quad v_1 = u_2,\ v_2 = u_1,\ A^{(2)}_{12} = c(u_1,u_2)\ \delta_{u_1,v_2}\delta_{u_2,v_1},\ B^{(2)}_{12} = -\frac{c(u_1,u_2)}{2(u_1-u_2)}\delta_{u_1,v_2}\delta_{u_2,v_1}, \tag{5.31}$$

where c is again an undetermined overall factor. In terms of the permutation diagrams this result can be illustrated as the operator $\mathcal{I}^{(2)}_{12}$ in Table 5.1. Here we introduced the variable u_{ij} for the difference of rapidities

$$u_{ij} = u_i - u_j. \tag{5.32}$$

This operator describes the real scattering process between two particles: The rapidities in the final state are exchanged, i.e. one particle has overtaken the second due to $u_1 > u_2$. After the action of the operator the state is a linear combination of two states. The first one is the state where the particles simply overtake each other without changing their type. In the second case they overtake and exchange their type. Both operators in $\mathcal{I}^{(2)}_{12}$ are related by a coefficient depending on the difference of both rapidities u_{ij}. We denote this operator by

$$\mathrm{S}_{12}(u_i;v_i) = \mathcal{I}^{(2)}_{12}(u_i;v_i). \tag{5.33}$$

Since S_{12} only depends on the rapidity difference u_{ij} of the incoming particles, we will write $\mathrm{S}_{12} = \mathrm{S}_{12}(u_{ij})$ in the following which corresponds to $\mathcal{I}^{(2)}_{12}(u_i;v_i)$ from Table 5.2 with dropped Kronecker-deltas.

Thus the Yangian constraint relates the coefficients A_{12} and B_{12} in (5.26) but does not fix the overall factor. The set of momenta is conserved and we do not need to impose momentum conservation. Furthermore, the S-matrix only depends on the rapidity differences of the incoming particles. Note that the result is valid for all n.

Three-Particle S-Matrix

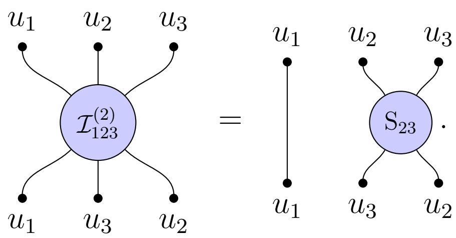
Now let us turn to the 3-particle S-matrix. Explicitly evaluating (5.19) yields a set of constraints on the coefficients and rapidities in (5.28) that is solved by six solutions. Once more, this constraint restricts the set of incoming and outgoing momenta to be the same. We do not have to impose momentum conservation. Let us discuss the solutions obtained via `Mathematica` by directly representing them as permutation diagrams which can be found in Table 5.2. In fact, we will find that only one of these solutions can be interpreted as the real 3→3 S-matrix of the model.

The simplest solution is given by $\mathcal{I}^{(1)}_{123}$. Similar to $\mathcal{I}^{(1)}_{12}$ this matrix does not scatter particles since the initial and final state are equivalent up to a factor. This represents the situation where the particle of smallest rapidity moves behind the other two and the fastest excitation in front of the others, i.e. $u_1 < u_2 < u_3$.

The next solution is represented by $\mathcal{I}^{(2)}_{123}$. Here only the particles of momenta u_2 and u_3 scatter and the particle of momentum u_1 remains unchanged. Thus, this operator describes a scattering event with rapidities $u_1 < u_3 < u_2$, where the particle of rapidity u_1 never reaches the others. It corresponds to the two-particle S-matrix S_{12} illustrated in Table 5.1 with one particle unchanged, i.e.

$$\mathcal{I}^{(2)}_{123} = \mathbb{I}_1 \mathrm{S}_{23}(u_{23}). \tag{5.34}$$

The operator $\mathbb{I}_1$ acts on the first position of a three-particle state as an identity map. $\mathrm{S}_{23}(u_{23})$ is the two-particle S-matrix from (5.33) acting on the second and third position of a two-particle state. In this notation we omit the dependence of $\mathcal{I}^{(2)}_{123}$ on the rapidities v_i. This equation can be illustrated in a diagrammatic way as[4]



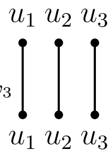
(5.35)

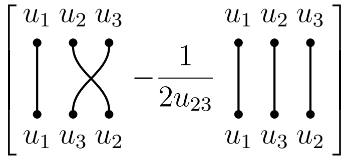
The diagram on the left hand side represents a scattering process with incoming rapidities u_1, u_2, u_3. The dots at the top of the diagram represent the incoming particles and the dots at the bottom the outgoing particles. The operator $\mathcal{I}^{(2)}_{123}$ maps the outgoing state onto the incoming state. This action of $\mathcal{I}^{(2)}_{123}$ can be depicted as the diagram on the right hand side of (5.35). The particle of rapidity u_1 is unchanged while the remaining particles scatter. A similar situation is represented by $\mathcal{I}^{(3)}_{123}$. Here the rapidities are ordered as $u_2 < u_1 < u_3$ such that only the two particles with rapidities u_1 and u_2 scatter. Once more, this operator corresponds to the two-particle S-matrix S_{12} in the form

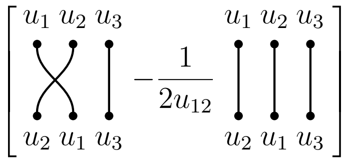
$$\mathcal{I}^{(3)}_{123} = \mathrm{S}_{12}(u_{12})\mathbb{I}_3 \tag{5.36}$$

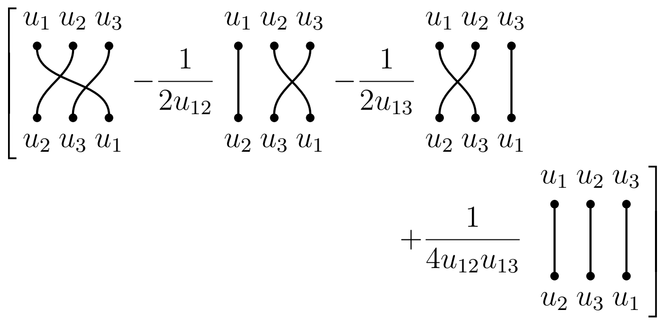
with S-matrix S_{12} scattering the first two particles of a three-particle state and $\mathbb{I}_3$ being the identity map for the third particle. In a diagrammatic way this operator

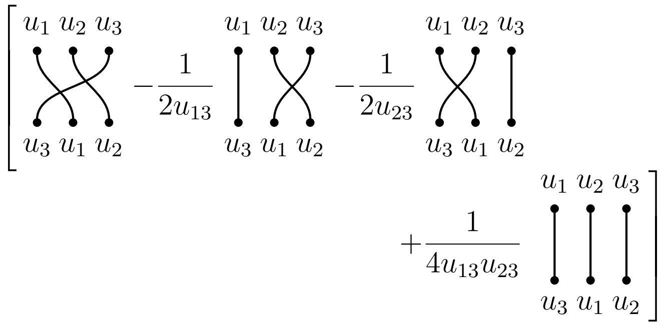
[4]Note that these relations are only true up to overall constants that we neglect for convenience.

$\mathcal{I}^{(1)}_{123}(u_i; v_i) = c(u_i)\delta_{u_1,v_1}\delta_{u_2,v_2}\delta_{u_3,v_3}$ 

$\mathcal{I}^{(2)}_{123}(u_i; v_i) = c(u_i)\delta_{u_1,v_1}\delta_{u_2,v_3}\delta_{u_3,v_2}$ 

$\mathcal{I}^{(3)}_{123}(u_i; v_i) = c(u_i)\delta_{u_1,v_2}\delta_{u_2,v_1}\delta_{u_3,v_3}$ 

$\mathcal{I}^{(4)}_{123}(u_i; v_i) = c(u_i)\delta_{u_1,v_3}\delta_{u_2,v_1}\delta_{u_3,v_2}$ 

$\mathcal{I}^{(5)}_{123}(u_i; v_i) = c(u_i)\delta_{u_1,v_2}\delta_{u_2,v_3}\delta_{u_3,v_1}$ 

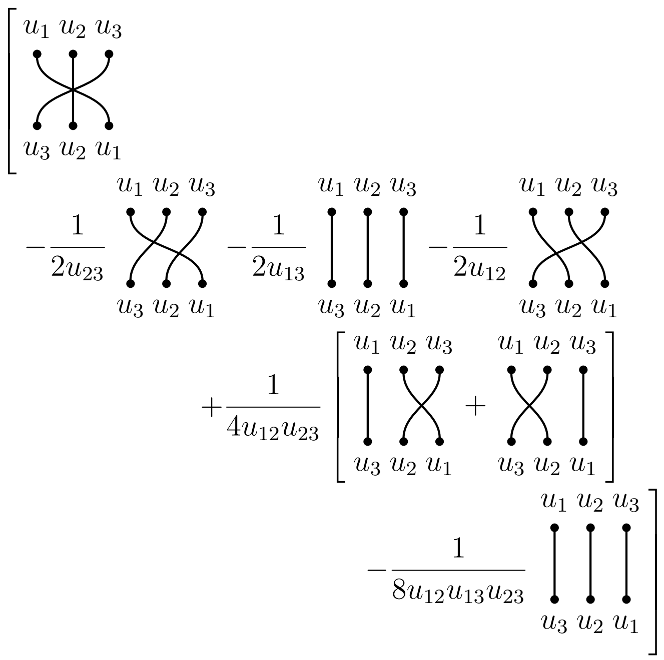
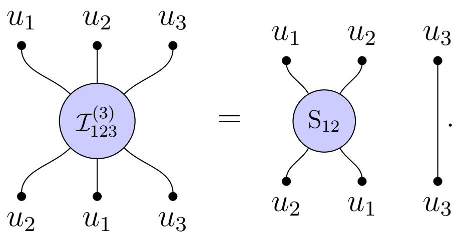
$\mathcal{I}^{(6)}_{123}(u_i; v_i) = c(u_i)\delta_{u_1,v_3}\delta_{u_2,v_2}\delta_{u_3,v_1}$ 

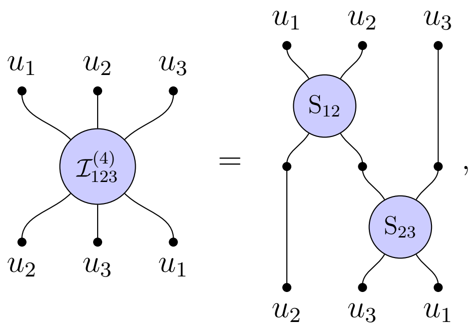
Table 5.2: $\mathfrak{su}$(n)-invariant operators on length 3

acts as



(5.37)

Another solution of (5.19) can be illustrated as $\mathcal{I}^{(4)}_{123}$. This operator represents scattering processes where the particle with rapidity u_1 overtakes the other two but the two remaining particles do not scatter, i.e. $u_1 > u_3 > u_2$. This can be illustrated as

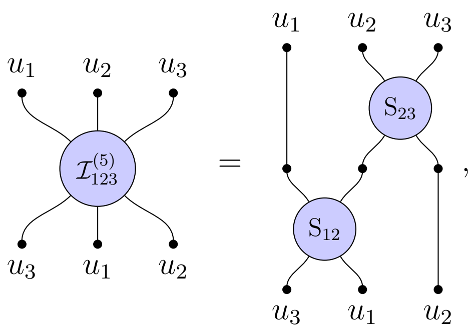


(5.38)

i.e. this operator factorizes as

$$\mathcal{I}^{(4)}_{123}(u_i; v_i) = \mathrm{S}_{12}(u_{12})\mathrm{S}_{23}(u_{13}) \tag{5.39}$$

which we verified explicitly by using the explicit form of $\mathcal{I}^{(4)}_{123}$ and S_{12} given in (5.33). A similar operator is $\mathcal{I}^{(5)}_{123}$. Here the rapidities are ordered as $u_2 > u_1 > u_3$ and the two faster particles overtake the slowest but do not scatter themselves. This can be illustrated by



(5.40)

i.e. the operator $\mathcal{I}^{(5)}_{123}$ factorizes into two-particle S-matrices as

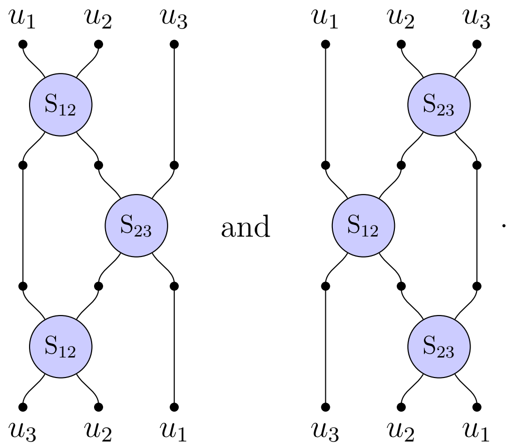
$$\mathcal{I}^{(5)}_{123}(u_i; v_i) = \mathrm{S}_{23}(u_{23})\mathrm{S}_{12}(u_{13}). \tag{5.41}$$

Note that one might also interpret the first five solutions as incomplete scattering events.

The last solution to (5.19) is $\mathcal{I}^{(6)}_{123}$. Here the rapidities of the state after applying the operator are in the reverse order due to $u_1 > u_2 > u_3$, i.e. this operator describes the real 3→3 scattering we are looking for and thus we call it

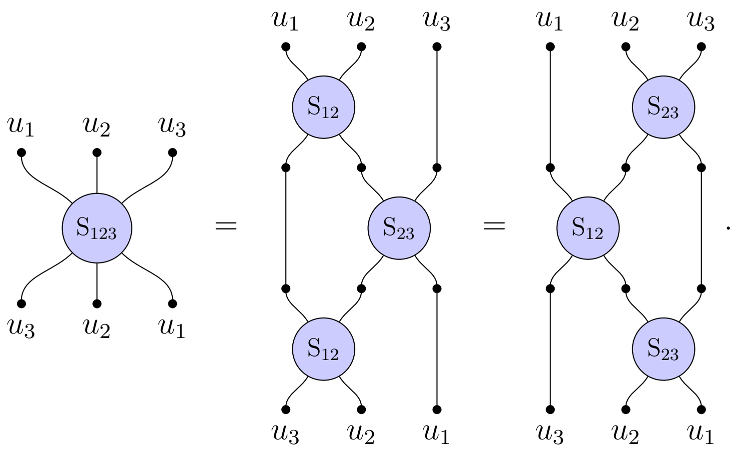
$$\mathrm{S}_{123}(u_i; v_i) = \mathcal{I}^{(6)}_{123}(u_i; v_i). \tag{5.42}$$

Similar to the operators above one may check whether this operator factorizes. In fact, there are two possibilities for this factorization which can be illustrated as



(5.43)

In the left diagram the two particles with rapidities u_1 and u_2 scatter first whereas in the second diagram the particles with rapidities u_2 and u_3 scatter first. Consistent factorization of the 3-particle S-matrix thus demands



(5.44)

In terms of S-matrices we may formulate these relations as

$$\mathrm{S}_{123}(u_1, u_2, u_3) = \mathrm{S}_{12}(u_{12})\mathrm{S}_{23}(u_{13})\mathrm{S}_{12}(u_{23}) = \mathrm{S}_{23}(u_{23})\mathrm{S}_{12}(u_{13})\mathrm{S}_{23}(u_{12}) \tag{5.45}$$

where we once more drop the dependence on the outgoing rapidities v_i since they are determined completely by the incoming rapidities u_i. For $\mathrm{R} = \mathbb{P}\mathrm{S}$ the second equality corresponds to the qYBE that was introduced in (2.60). In order to verify the factorization of S_{123} one has to calculate the right-hand sides of (5.45) for the two-particle S-matrix S_{12} from Table 5.1. Doing so, we find that the three-particle S-matrix indeed factorizes. Furthermore, the two-particle S-matrix satisfies the qYBE for S. This implies consistent factorization.

Discussion of the Results

Let us summarize some important features of the resulting S-matrices. In all discussed cases the Yangian fixes the linear combination of permutation operators up to an overall factor which cannot be obtained by demanding its invariance under a symmetry algebra with generators $\mathfrak{J}$ in the form $[\mathrm{S}, \mathfrak{J}] = 0$. This so-called *dressing* *factor* can be determined by imposing unitarity and crossing relations, see e.g. [33] for a discussion of this factor in integrable field theories and, in particular, the dressing factor of integrable spin chains arising from $\mathcal{N} = 4$ SYM.

Furthermore, Yangian symmetry restricts the set of outgoing momenta to be the same as the set of incoming momenta, i.e.

$$\{u_i\} = \{v_i\}. \tag{5.46}$$

This is a substantial feature of scattering processes in (1+1)-dimensional integrable models that we already discussed above. Moreover, note that the S-matrices only depend on the differences of rapidities. In a Lorentz-invariant continuum theory with one spatial direction, this statement reflects the boost invariance of the S-matrix provided that the rapidities are additive under Lorentz boosts. Interestingly, this feature occurs in general $Y[\mathfrak{su}(\mathrm{n})]$-invariant S-matrices. In the context of spin chains this hints on the existence of a lattice generalization of the continuous Lorentz symmetry, see [34] and [35] for the construction of discrete Lorentz boost operators. In fact, this feature of the two-particle S-matrix can be understood directly on the basis of the Yangian by an argument given in [11]. The level-1 constraint (5.19) for a general level-0 generator J and level-1 generator $\hat{\mathrm{J}}$ and equal sets of rapidities is given by

$$\left[\Delta\hat{\mathrm{J}}, \mathrm{S}_{12}\right] = \left(u_1 \mathrm{J} \otimes 1 + u_2 1 \otimes \mathrm{J}\right) \mathrm{S}_{12} - \mathrm{S}_{12}\left(u_2 \mathrm{J} \otimes 1 + u_1 1 \otimes \mathrm{J}\right) + \left[\mathrm{J} \otimes \mathrm{J}, \mathrm{S}_{12}\right] = 0. \tag{5.47}$$

Here $\mathrm{J}\otimes\mathrm{J}$ shall represent the bilocal term in (4.5). This constraint can be rearranged using the level-0 constraint (5.18)

$$\left[\mathrm{J} \otimes 1 + 1 \otimes \mathrm{J}, \mathrm{S}_{12}\right] = 0 \tag{5.48}$$

as

$$-\left[u_{12} 1 \otimes \mathrm{J}, \mathrm{S}_{12}\right] + \left[\mathrm{J} \otimes \mathrm{J}, \mathrm{S}_{12}\right] = 0. \tag{5.49}$$

Thus the rapidity-dependence of the two-particle S-matrix is governed by an equation that contains only the relative rapidity. This implies $\mathrm{S}_{12}(u_1, u_2) = \mathrm{S}_{12}(u_{12})$. For factorized m-particle $Y[\mathfrak{su}(\mathrm{n})]$-invariant S-matrices this implies their invariance under uniform rapidity shifts.

Furthermore, we showed that the resulting $Y[\mathfrak{su}(\mathrm{n})]$-invariant three-particle S-matrix S_{123} factorizes into three two-particle S-matrices consistently. There are similar relations for the factorization of S-matrices describing the scattering of more than three particles. For example a four-particle scattering process factorizes into one three-particle and three two-particle scattering events. Note that the operators $\mathcal{I}^{(1)}_{123}$,...,$\mathcal{I}^{(6)}_{123}$ correspond to the coefficients in the asymptotic Bethe ansatz of $\mathfrak{su}(\mathrm{n})$ spin chain models of the form (2.30) and factorize similar to the coefficients of the Heisenberg spin chain, see section 2.2.

In the following two chapters we perform the analysis of the constraints (4.18) and (4.19) for the Yangians corresponding to the Lie superalgebras $\mathfrak{su}(1|1)$ and $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$.

Chapter 6

$Y[\mathfrak{su}(1|1)]$-Invariant S-Matrices

We now turn to the Yangian corresponding to $\mathfrak{su}(1|1)$. Its discussion is a good intermediate step before the investigation of $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ since it only contains a single boson and a single fermion in its fundamental representation. Furthermore, it allows for a dynamic representation. We begin this chapter with a review on this algebra and the corresponding Yangian in section 6.1. We develop the explicit constraints on S-matrices in a theory with $Y[\mathfrak{su}(1|1)]$ as the particles' symmetry algebra. In section 6.2 we analyse the solution of the Yangian constraints for the undynamic representation of this algebra. In 6.3 we do so for the dynamic case. We focus on the two-particle and three-particle S-matrix and analyse the results with respect to consistent factorization.

6.1 The Algebra $Y[\mathfrak{su}(1|1)]$ and its Fundamental Representation

Generators of $\mathfrak{su}(1|1)$ and the Fundamental Representation

Let us briefly review the Lie superalgebra $\mathfrak{su}(1|1)$ on the basis of [14]. It consists of the supersymmetry generators $\mathfrak{Q}$ and $\mathfrak{S}$, the outer automorphism $\mathfrak{B}$ and the central charge $\mathfrak{C}$. These generators satisfy the commutation relations

$$\{\mathfrak{Q}, \mathfrak{S}\} = \mathfrak{C}, \quad [\mathfrak{B}, \mathfrak{Q}] = -2\mathfrak{Q}, \quad [\mathfrak{B}, \mathfrak{S}] = +2\mathfrak{S}. \tag{6.1}$$

The remaining Lie brackets vanish. The fundamental representation ρ of this algebra consists of one bosonic state $|\phi\rangle$ and one fermionic state $|\psi\rangle$. The generators of $\mathfrak{su}(1|1)$ act on these single-particle states as[1]

$$\begin{aligned}
\mathfrak{Q}\,|\phi\rangle &= q\,|\psi\rangle\,, & \mathfrak{Q}\,|\psi\rangle &= 0,\\
\mathfrak{S}\,|\phi\rangle &= 0, & \mathfrak{S}\,|\psi\rangle &= \tfrac{c}{q}\,|\phi\rangle\,,\\
\mathfrak{B}\,|\phi\rangle &= (b+1)\,|\phi\rangle\,, & \mathfrak{B}\,|\psi\rangle &= (b-1)\,|\psi\rangle\,,\\
\mathfrak{C}\,|\phi\rangle &= c\,|\phi\rangle\,, & \mathfrak{C}\,|\psi\rangle &= c\,|\psi\rangle
\end{aligned} \tag{6.2}$$

with parameters b, c and q. In the notation we used in the previous chapter for the $Y[\mathfrak{su}(\mathrm{n})]$ generators we may write the generators in the fundamental representation

[1]Note that we drop ρ from now on. It indicated whenever we used the operators in the fundamental representation. In the following it will be clear whether we use the generators as general objects spanning the Yangian algebra or as operators acting on a state.

as

$$\mathfrak{Q} = q\,|\psi\rangle\,\langle\phi|\,, \qquad \mathfrak{S} = \tfrac{c}{q}\,|\phi\rangle\,\langle\psi|\,,$$
$$\mathfrak{B} = (b+1)\,|\phi\rangle\,\langle\phi| + (b-1)\,|\psi\rangle\,\langle\psi|\,, \quad \mathfrak{C} = c\,|\phi\rangle\,\langle\phi| + c\,|\psi\rangle\,\langle\psi|\,. \tag{6.3}$$

The central charge $\mathfrak{C}$ is proportional to the identity operator $1 = |\phi\rangle\,\langle\phi| + |\psi\rangle\,\langle\psi|$.

Coproduct Structure of $Y[\mathfrak{su}(1|1)]$

The generators $\mathfrak{Q}$, $\mathfrak{S}$, $\mathfrak{B}$, $\mathfrak{C}$ are the level-0 generators of the Yangian algebra $Y[\mathfrak{su}(1|1)]$. Their action in the evaluation representation is given by (6.2). The level-1 generators $\hat{\mathrm{J}}$ are denoted by $\hat{\mathfrak{Q}}$, $\hat{\mathfrak{S}}$, $\hat{\mathfrak{B}}$, $\hat{\mathfrak{C}}$ and act on single-particle states as [14]

$$\hat{\mathrm{J}}\,|\phi,u\rangle = iu\mathrm{J}\,|\phi,u\rangle \qquad \hat{\mathrm{J}}\,|\psi,u\rangle = iu\mathrm{J}\,|\psi,u\rangle \tag{6.4}$$

with rapidity u and level-0 generator J corresponding to $\hat{\mathrm{J}}$.

In order to evaluate the level-0 constraint of the two- and three-particle S-matrix, we need the action of the level-0 generators on two and three particles which is given by the coproduct

$$\Delta\mathfrak{Q} = \mathfrak{Q}\otimes 1 + (-1)^F\otimes\mathfrak{Q}, \qquad \Delta\mathfrak{S} = \mathfrak{S}\otimes 1 + (-1)^F\otimes\mathfrak{S},$$
$$\Delta\mathfrak{B} = \mathfrak{B}\otimes 1 + 1\otimes\mathfrak{B}, \qquad \Delta\mathfrak{C} = \mathfrak{C}\otimes 1 + 1\otimes\mathfrak{C}. \tag{6.5}$$

Their generalization on length 3 is listed in the appendix in equation (C.1). Here, one has to take into account the fermionic nature of the generators $\mathfrak{Q}$ and $\mathfrak{S}$. It results in factors $(-1)^F$ whenever they pass through a particle. For bosonic particles we have $F = 0$ and for fermionic particles $F = 1$, i.e.

$$(-1)^F\,|\phi\rangle = +\,|\phi\rangle \qquad (-1)^F\,|\psi\rangle = -\,|\psi\rangle\,. \tag{6.6}$$

This operator ensures the correct statistics of the fermionic and bosonic particles.

The coproduct structure of the level-1 generators can be obtained from Table 2 from [11] which shows the coproduct of the level-1 generators of $\mathfrak{su}(2|2)$. $\mathfrak{su}(1|1)$ is a subalgebra of $\mathfrak{su}(2|2)$ that only contains one boson and one fermion rather than two. By setting all indices a, b, α, β to 1 instead of 1, 2 and including the supersymmetric grading, we conclude

$$\begin{aligned}
\Delta\hat{\mathfrak{Q}} &= \hat{\mathfrak{Q}}\otimes 1 + (-1)^F\otimes\hat{\mathfrak{Q}} + \tfrac{1}{2}\mathfrak{Q}\otimes\mathfrak{C} - \tfrac{1}{2}(-1)^F\mathfrak{C}\otimes\mathfrak{Q},\\
\Delta\hat{\mathfrak{S}} &= \hat{\mathfrak{S}}\otimes 1 + (-1)^F\otimes\hat{\mathfrak{S}} - \tfrac{1}{2}\mathfrak{S}\otimes\mathfrak{C} + \tfrac{1}{2}(-1)^F\mathfrak{C}\otimes\mathfrak{S},\\
\Delta\hat{\mathfrak{B}} &= \hat{\mathfrak{B}}\otimes 1 + 1\otimes\hat{\mathfrak{B}} - (-1)^F\mathfrak{S}\otimes\mathfrak{Q} - (-1)^F\mathfrak{Q}\otimes\mathfrak{S},\\
\Delta\hat{\mathfrak{C}} &= \hat{\mathfrak{C}}\otimes 1 + 1\otimes\hat{\mathfrak{C}}.
\end{aligned} \tag{6.7}$$

For the action on states including more particles this can be easily generalized. We present the results on length 3 in equation (C.2). With these relations we can explicitly use (4.18) and (4.19) to constrain the S-matrix. We will do this analysis in the following sections.

As discussed above, this algebra is particularly relevant in the $\mathfrak{g} = \mathfrak{su}(1|2)$ subsector of the $\mathfrak{psu}(2,2|4)$ spin chain of $\mathcal{N} = 4$ SYM. The sites of this supersymmetric spin chain are either in one of the two bosonic states $\mathcal{Z}$ and ϕ or in the fermionic state ψ. By choosing $\mathcal{Z}$ as vacuum there are two excited states corresponding to the states $|\phi\rangle\,, |\psi\rangle$ of the fundamental representation of $\mathfrak{su}(1|1)$ discussed above. Thus the algebra $\mathfrak{su}(1|1)$ assumes the role of the residual algebra $\mathfrak{g}_r$ discussed in section 3.2.

6.2 Undynamic $Y[\mathfrak{su}(1|1)]$-Invariant S-Matrices

As already indicated in chapter 1, the algebra $\mathfrak{su}(1|1)$ can be discussed both in its dynamical and undynamical representation. Let us first study the implications of demanding Yangian invariance for an undynamic representation of $Y[\mathfrak{su}(1|1)]$, i.e. it does not depend on the rapidities of the particles and the parameters c, q, b in (6.2) are constants. This is relevant for conventional spin chain models. For convenience, we normalize the algebra such that

$$c = 1 \tag{6.8}$$

which also corresponds to the conventions in [14]. Since q is only an unphysical scaling factor between the bosonic and fermionic particle we set it to $q = 1$ for convenience. This set-up corresponds to models whose particles' dynamics is governed by a Hamiltonian that is invariant under the external symmetry algebra $\mathfrak{su}(1|1)$.

6.2.1 Two-Particle S-Matrix

Now let us turn to the analysis of the level-0 constraint for a two-particle S-matrix. It is the map

$$\mathrm{S}_{12}(u_{1,2}; v_{1,2}) : (1|1)_{v_1} \otimes (1|1)_{v_2} \to (1|1)_{u_1} \otimes (1|1)_{u_2} \tag{6.9}$$

with different sets of incoming and outgoing rapidities satisfying $u_1 > u_2$ and $v_1 < v_2$. They are constrained by momentum conservation which is

$$p(u_1) + p(u_2) = p(v_1) + p(v_2). \tag{6.10}$$

In order to evaluate this constraint explicitly one has to assume a concrete dependency between the momentum and rapidity, i.e. $p = p(u)$. This relation characterizes the specific model under consideration.

Solution of the Level-0 Constraints

We start analysing explicitly the constraint (4.18) using the coproduct structure (6.5) of the $Y[\mathfrak{su}(1|1)]$ Yangian generators. We performed the explicit evaluation of these constraints using the program `Mathematica`. We show some important aspects of the implementation in the appendix D and discuss the results in this chapter.

Since the central charge $\mathfrak{C}$ corresponds to the identity operator of the representation, the constraint $[\Delta\mathfrak{C}, \mathrm{S}_{12}] = 0$ puts no restriction on S_{12}. The level-0 constraint for the generator $\mathfrak{B}$ restricts the S-matrix to keep the number of bosons and fermions constant, i.e. the S-matrix simply permutes the particles[2]

$$\begin{aligned}
\mathrm{S}_{12}(u_i, v_i)\,|\phi, v_1; \phi, v_2\rangle &= A_{12}\,|\phi, u_1; \phi, u_2\rangle\,, \\
\mathrm{S}_{12}(u_i, v_i)\,|\phi, v_1; \psi, v_2\rangle &= B_{12}\,|\psi, u_1; \phi, u_2\rangle + C_{12}\,|\phi, u_1; \psi, u_2\rangle\,, \\
\mathrm{S}_{12}(u_i, v_i)\,|\psi, v_1; \phi, v_2\rangle &= D_{12}\,|\phi, u_1; \psi, u_2\rangle + E_{12}\,|\psi, u_1; \phi, u_2\rangle\,, \\
\mathrm{S}_{12}(u_i, v_i)\,|\psi, v_1; \psi, v_2\rangle &= F_{12}\,|\psi, u_1; \psi, u_2\rangle
\end{aligned} \tag{6.11}$$

[2]In the following we will neglect the indication *in* and *out* on the multi-particle states that were introduced in (3.12) and (3.13). Nevertheless, we will keep in mind that the S-matrix maps between states of ordered rapidities.

with unknown coefficients $A_{12}, ..., F_{12}$. Demanding the invariance of S_{12} under $\Delta\mathfrak{Q}$ and $\Delta\mathfrak{S}$ sets four of the $A_{12}, ..., F_{12}$ in relation to the remaining two coefficients. Choosing A_{12} and F_{12} as free coefficients yields

$$B_{12} = D_{12} = \frac{1}{2}\left(A_{12} - F_{12}\right), \qquad C_{12} = E_{12} = \frac{1}{2}\left(A_{12} + F_{12}\right). \tag{6.12}$$

Thus, after imposing the level-0 constraint there remain two degrees of freedom which are manifested in the free coefficients A_{12} and F_{12}. This corresponds to the existence of two Casimirs of length 2 for the algebra $\mathfrak{su}(1|1)$, see [14] for their construction.

Solution of the Level-1 Constraint

Let us move on to the level-1 constraint (4.19) corresponding to $\hat{\mathfrak{C}}$, $\hat{\mathfrak{B}}$, $\hat{\mathfrak{Q}}$ and $\hat{\mathfrak{S}}$ with coproduct structure given in (6.7). Imposing $[\Delta\hat{\mathfrak{Q}}, \mathrm{S}_{12}] = 0$ reduces the degrees of freedom by relating the coefficients A_{12} and F_{12} via

$$F_{12} = A_{12}\frac{i - u_{12}}{i - v_{12}}. \tag{6.13}$$

Thus there only remains an unknown overall factor in the S-matrix. The level-1 constraint imposed by $\Delta\hat{\mathfrak{C}}$ demands the conservation of the sum of rapidities for the incoming and outgoing state, i.e.

$$v_1 + v_2 = u_1 + u_2. \tag{6.14}$$

These rapidities are further constrained by the Yangian generator $\hat{\mathfrak{B}}$ whose associated level-1 constraint only allows for the two solutions

$$\begin{aligned} &(1) \quad v_1 = u_1, \; v_2 = u_2 \\ &(2) \quad v_1 = u_2, \; v_2 = u_1, \end{aligned} \tag{6.15}$$

i.e. an undynamic-$Y[\mathfrak{su}(1|1)]$-invariant operator restricts two-particle processes to not change the set of rapidities. Note that in the discussion of $Y[\mathfrak{su}(\mathrm{n})]$-invariants of length 2 we also obtained two solutions and their interpretation is analogous as we will see in the following.

The first solution in (6.15) corresponds to the invariant operator

$$\mathcal{I}_{12} \propto |\phi_1;\phi_2\rangle\langle\phi_1;\phi_2| + |\phi_1;\psi_2\rangle\langle\phi_1;\psi_2| + |\psi_1;\phi_2\rangle\langle\psi_1;\phi_2| + |\psi_1;\psi_2\rangle\langle\psi_1;\psi_2|, \tag{6.16}$$

where $|\phi_1\rangle := |\phi, u_1\rangle$ etc. It is the identity map on two sites, i.e.

$$\mathcal{I}_{12}(u_1, u_2) : (1|1)_{u_1} \otimes (1|1)_{u_2} \to (1|1)_{u_1} \otimes (1|1)_{u_2}. \tag{6.17}$$

Thus this operator does not correspond to the S-matrix we are looking for since the incoming state and outgoing state coincide.

In the second solution, the rapidities are in the reverse order after the application of the corresponding operator and thus it yields the S-matrix. It permutes the subspaces of the two particles[3]

$$\mathrm{S}_{12}(u_{1,2}) : (1|1)_{u_2} \otimes (1|1)_{u_1} \to (1|1)_{u_1} \otimes (1|1)_{u_2}. \tag{6.18}$$

[3]In fact, the S-matrix is often defined as an operator permuting the one-particle Hilbert spaces. Since we want to check whether imposing its invariance under a specific Yangian restricts the S-matrix in such a way that it preserves the sets of momenta, we do not demand this here. It naturally follows from the Yangian invariance of the S-matrix.

Its coefficients in (6.11) are given by

$$\begin{aligned}
A_{12} &= -\mathrm{S}^0_{12}\delta_{v_1,u_2}\delta_{v_2,u_1}\frac{i+u_{12}}{i-u_{12}}, & B_{12} &= -\mathrm{S}^0_{12}\delta_{v_1,u_2}\delta_{v_2,u_1}\frac{u_{12}}{i-u_{12}},\\
C_{12} &= -\mathrm{S}^0_{12}\delta_{v_1,u_2}\delta_{v_2,u_1}\frac{i}{i-u_{12}}, & D_{12} &= -\mathrm{S}^0_{12}\delta_{v_1,u_2}\delta_{v_2,u_1}\frac{u_{12}}{i-u_{12}},\\
E_{12} &= -\mathrm{S}^0_{12}\delta_{v_1,u_2}\delta_{v_2,u_1}\frac{i}{i-u_{12}}, & F_{12} &= -\mathrm{S}^0_{12}\delta_{v_1,u_2}\delta_{v_2,u_1}.
\end{aligned} \tag{6.19}$$

They only depend on the differences of the incoming rapidities. The invariance under $\mathfrak{S}$ does not impose further constraints. This S-matrix satisfies the qYBE given by the second equality in (5.45). This was checked by explicit calculation.

6.2.2 Three-Particle S-Matrix

Let us move on to the $Y[\mathfrak{su}(1|1)]$-invariant three-particle S-matrix for undynamic representations. It is the map

$$\mathrm{S}_{123}(u_{1,2,3}; v_{1,2,3}) : (1|1)_{v_1} \otimes (1|1)_{v_2} \otimes (1|1)_{v_3} \to (1|1)_{u_1} \otimes (1|1)_{u_2} \otimes (1|1)_{u_3} \tag{6.20}$$

with $u_1 > u_2 > u_3$ and $v_1 < v_2 < v_3$.

Solution of the Level-0 Constraint

Similar to the $2 \to 2$ scattering case, the level-0 constraint for $\mathfrak{C}$ is automatically satisfied for all possible S_{123} since $\Delta^2\mathfrak{C}$ is proportional to the identity map on length 3. Moreover, demanding $[\Delta^2\mathfrak{B}, \mathrm{S}_{123}] = 0$ restricts the S-matrix to keep the number of bosonic and fermionic particles constant, i.e.

$$\begin{aligned}
\mathrm{S}_{123}\,|\phi_1\phi_2\phi_3\rangle &= A_{123}\,|\phi_3\phi_2\phi_1\rangle\\
\mathrm{S}_{123}\,|\phi_1\phi_2\psi_3\rangle &= B_{123}\,|\psi_3\phi_2\phi_1\rangle + C_{123}\,|\phi_3\psi_2\phi_1\rangle + D_{123}\,|\phi_3\phi_2\psi_1\rangle\\
\mathrm{S}_{123}\,|\phi_1\psi_2\phi_3\rangle &= E_{123}\,|\phi_3\psi_2\phi_1\rangle + F_{123}\,|\phi_3\phi_2\psi_1\rangle + G_{123}\,|\psi_3\phi_2\phi_1\rangle\\
\mathrm{S}_{123}\,|\psi_1\phi_2\phi_3\rangle &= H_{123}\,|\phi_3\phi_2\psi_1\rangle + K_{123}\,|\phi_3\psi_2\phi_1\rangle + L_{123}\,|\psi_3\phi_2\phi_1\rangle\\
\mathrm{S}_{123}\,|\psi_1\psi_2\phi_3\rangle &= M_{123}\,|\phi_3\psi_2\psi_1\rangle + N_{123}\,|\psi_3\phi_2\psi_1\rangle + O_{123}\,|\psi_3\psi_2\phi_1\rangle\\
\mathrm{S}_{123}\,|\psi_1\phi_2\psi_3\rangle &= P_{123}\,|\psi_3\phi_2\psi_1\rangle + Q_{123}\,|\psi_3\psi_2\phi_1\rangle + R_{123}\,|\phi_3\psi_2\psi_1\rangle\\
\mathrm{S}_{123}\,|\phi_1\psi_2\psi_3\rangle &= T_{123}\,|\psi_3\psi_2\phi_1\rangle + U_{123}\,|\psi_3\phi_2\psi_1\rangle + V_{123}\,|\phi_3\psi_2\psi_1\rangle\\
\mathrm{S}_{123}\,|\psi_1\psi_2\psi_3\rangle &= W_{123}\,|\psi_3\psi_2\psi_1\rangle
\end{aligned} \tag{6.21}$$

with 20 unknown coefficients $A_{123}, ..., W_{123}$. These are constrained by imposing the invariance of S_{123} under $\mathfrak{Q}$ and $\mathfrak{S}$. We obtain

$$\begin{aligned}
D_{123} &= A_{123} - B_{123} - C_{123}, & G_{123} &= A_{123} - E_{123} - F_{123},\\
H_{123} &= B_{123} + C_{123} - F_{123}, & K_{123} &= A_{123} - C_{123} - E_{123},\\
L_{123} &= -B_{123} + E_{123} + F_{123}, & N_{123} &= -B_{123} - C_{123} + 2F_{123} - M_{123},\\
O_{123} &= -A_{123} + C_{123} + 2E_{123} + M_{123}, & P_{123} &= -B_{123} + E_{123} + M_{123},\\
Q_{123} &= -B_{123} + F_{123} - M_{123}, & &\\
R_{123} &= A_{123} - B_{123} - 2C_{123} - E_{123} + F_{123} - M_{123}, & &\\
T_{123} &= C_{123} - F_{123} + M_{123}, & &\\
U_{123} &= A_{123} - B_{123} - C_{123} - E_{123} - M_{123}, & V_{123} &= E_{123} - F_{123} + M_{123},\\
W_{123} &= -A_{123} + B_{123} + 2C_{123} + 2E_{123} - 2F_{123} + 3M_{123}.
\end{aligned} \tag{6.22}$$

Thus, there remain six free coefficients $A_{123}, B_{123}, C_{123}, E_{123}, F_{123}, M_{123}$ and we conclude that there exist six Casimirs for length 3.

Solution of the Level-1 Constraint

The general form of the $\mathfrak{su}(1|1)$-invariant S_{123} given in (6.21) and (6.22) gets further constrained by demanding its invariance under the level-1 generators of $Y[\mathfrak{su}(1|1)]$. A vanishing commutator of S_{123} with $\Delta^2\hat{\mathfrak{C}}$ implies the conservation of the sum of rapidities

$$v_1 + v_2 + v_3 = u_1 + u_2 + u_3. \tag{6.23}$$

The constraint imposed by $\hat{\mathfrak{Q}}$ results in the relations

$$\begin{aligned}
E_{123} &= A_{123}\frac{i - v_1 + u_2}{i - v_{12}} - C_{123}\frac{2i - v_{13}}{i - v_{12}},\\
F_{123} &= A_{123}\frac{u_3 - v_3}{i - v_{12}} + (B_{123} + C_{123})\frac{2i - v_{13}}{i - v_{12}},\\
M_{123} &= A_{123}\frac{u_3 - v_3}{i - v_{12}} + (B_{123} + C_{123})\frac{i - u_2 + v_3}{i - v_{12}} - C_{123}\frac{u_3 - v_3}{i - v_{12}}
\end{aligned} \tag{6.24}$$

such that there remain three free coefficients $A_{123}, B_{123}, C_{123}$. From the combined constraints of vanishing commutators of S_{123} with $\Delta^2\hat{\mathfrak{B}}$ and $\Delta^2\hat{\mathfrak{S}}$ we obtain a set of 35 equations that constrain the free coefficients and rapidities. Unfortunately, we were not able to solve these for general b using `Mathematica`. The kernel was quit during the evaluation, most likely due to an insufficient internal memory capacity (16GB). Nevertheless, calculating the result for different values of b such as $b = 0, 1, 2$ suggests that the solutions are independent of b, similar to the two-particle case. We obtain a set of six solutions with equal sets of incoming and outgoing rapidities

$$\begin{aligned}
&(1) \quad v_1 = u_1,\ v_2 = u_2,\ v_3 = u_3, \quad && B_{123} = C_{123} = 0\\
&(2) \quad v_1 = u_1,\ v_2 = u_3,\ v_3 = u_2, \quad && B_{123} = 0,\ C_{123} = A_{123}\frac{u_{23}}{i + u_{23}}\\
&(3) \quad v_1 = u_2,\ v_2 = u_1,\ v_3 = u_3, \quad && B_{123} = C_{123} = 0\\
&(4) \quad v_1 = u_2,\ v_2 = u_3,\ v_3 = u_1, \quad && B_{123} = A_{123}\frac{u_{12}u_{13}}{(i + u_{12})(i + u_{13})},\\
& && C_{123} = A_{123}\frac{iu_{13}}{(i + u_{12})(i + u_{13})}\\
&(5) \quad v_1 = u_3,\ v_2 = u_1,\ v_3 = u_2, \quad && B_{123} = 0,\ C_{123} = A_{123}\frac{u_{23}}{(i + u_{23})}\\
&(6) \quad v_1 = u_3,\ v_2 = u_2,\ v_3 = u_1, \quad && B_{123} = A_{123}\frac{u_{12}u_{13}}{(i + u_{12})(i + u_{13})},\\
& && C_{123} = A_{123}\frac{iu_{13}}{(i + u_{12})(i + u_{13})}.
\end{aligned} \tag{6.25}$$

Once more, the number of solutions is the same as in the discussion of $Y[\mathfrak{su}(\mathrm{n})]$-invariant S-matrices. Similarly, the solutions (1),...,(5) are scattering processes which do not respect the ordering of rapidities of the real 3→3 S-matrix. Let us briefly discuss the solutions to see the similarity between the results. The first solution in (6.25) corresponds to the identity map on three sites, i.e. the particles do not scatter. The second and third solution correspond to operators that only scatter the latter two and first two particles of a three-particle state leaving the other particle unchanged. In the fourth and fifth solution one particle overtakes the other two particles. The sixth solution has the correct ordering of rapidities and represents the real 3→3 S-matrix. Let us present the complete result for the coefficients in (6.21)

$$
\begin{aligned}
A_{123} &= -\mathrm{S}^0_{123}\frac{(i+u_{12})(i+u_{13})(i+u_{23})}{(i-u_{12})(i-u_{13})(i-u_{23})}, & B_{123} &= -\mathrm{S}^0_{123}\frac{u_{12}u_{13}(i+u_{23})}{(i-u_{12})(i-u_{13})(i-u_{23})},\\
C_{123} &= -\mathrm{S}^0_{123}\frac{iu_{13}(i+u_{23})}{(i-u_{12})(i-u_{13})(i-u_{23})}, & D_{123} &= -\mathrm{S}^0_{123}\frac{i(i+u_{12})(i+u_{23})}{(i-u_{12})(i-u_{13})(i-u_{23})},\\
E_{123} &= -\mathrm{S}^0_{123}\frac{-i+u_{12}(i+u_{13})u_{23}}{(i-u_{12})(i-u_{13})(i-u_{23})}, & F_{123} &= -\mathrm{S}^0_{123}\frac{i(i+u_{12})u_{13}}{(i-u_{12})(i-u_{13})(i-u_{23})},\\
G_{123} &= -\mathrm{S}^0_{123}\frac{iu_{13}(i+u_{23})}{(i-u_{12})(i-u_{13})(i-u_{23})}, & H_{123} &= -\mathrm{S}^0_{123}\frac{(i+u_{12})u_{13}u_{23}}{(i-u_{12})(i-u_{13})(i-u_{23})},\\
K_{123} &= -\mathrm{S}^0_{123}\frac{(i+u_{12})u_{13}}{(i-u_{12})(i-u_{13})(i-u_{23})}, & L_{123} &= -\mathrm{S}^0_{123}\frac{i(i+u_{12})(i+u_{23})}{(i-u_{12})(i-u_{13})(i-u_{23})},\\
M_{123} &= -\mathrm{S}^0_{123}\frac{u_{12}u_{13}}{(i-u_{12})(i-u_{13})}, & N_{123} &= -\mathrm{S}^0_{123}\frac{iu_{13}}{(i-u_{12})(i-u_{13})},\\
O_{123} &= -\mathrm{S}^0_{123}\frac{i}{(i-u_{13})}, & P_{123} &= -\mathrm{S}^0_{123}\frac{-i+u_{12}(i-u_{13})u_{23}}{(i-u_{12})(i-u_{13})(i-u_{23})},\\
Q_{123} &= -\mathrm{S}^0_{123}\frac{iu_{13}}{(i-u_{13})(i-u_{23})}, & R_{123} &= -\mathrm{S}^0_{123}\frac{iu_{13}}{(i-u_{12})(i-u_{13})},\\
T_{123} &= -\mathrm{S}^0_{123}\frac{u_{13}u_{23}}{(i-u_{13})(i-u_{23})}, & U_{123} &= -\mathrm{S}^0_{123}\frac{iu_{13}}{(i-u_{13})(i-u_{23})},\\
V_{123} &= -\mathrm{S}^0_{123}\frac{i}{(i-u_{13})}, & W_{123} &= -\mathrm{S}^0_{123},
\end{aligned}
\tag{6.26}
$$

where we drop the Kronecker deltas $\delta_{v_1,u_3}\delta_{v_2,u_2}\delta_{v_3,u_1}$. This S-matrix is determined up to an overall factor and factorizes into three two-particle S-matrices of the form (6.19). This was checked by explicit calculation.

Discussion of the Results

The resulting $Y[\mathfrak{su}(1|1)]$-invariant two- and three-particle S-matrices share many features with the $Y[\mathfrak{su}(\mathrm{n})]$-invariant S-matrices that we discussed in the previous chapter. Let us list important properties of the S-matrices in (6.19) and (6.26):

- They are completely determined up to the dressing factor. In order to obtain this overall factor, one has to impose unitarity and crossing relations.
- They conserve the set of incoming momenta in a scattering process and thus automatically ensure momentum conservation.
- They only depend on relative rapidities which can be understood by the discussion at the end of section 5.3. This feature reflects the boost invariance of Lorentz-invariant theories and hints on the existence of a quasi-boost for spin chains.
- The three-particle S-matrix factorizes into three two-particle S-matrices. These satisfy the qYBE (5.45). Thus, the Yangian $Y[\mathfrak{su}(1|1)]$ in its undynamic rep-

resentation implies that multi-particle scattering processes factorize into two-particle scattering processes consistently.

6.3 Dynamic $Y[\mathfrak{su}(1|1)]$-Invariant S-Matrices

In a specific sector of the planar limit of $\mathcal{N} = 4$ SYM, a long-range spin chain with full symmetry algebra $\mathfrak{su}(1|2)$ and residual symmetry algebra $\mathfrak{su}(1|1)$ occurs. It does not only contain nearest-neighbor interactions, but also interactions of a larger number of neighboring sites. In this model the central charge $\mathfrak{C}$ corresponds to the Hamiltonian $\mathcal{H}$ via [14]

$$\mathfrak{C} = \mathfrak{C}_0 + \lambda \mathcal{H}(\lambda) \tag{6.27}$$

with coupling constant[4] λ. Therefore, the Hamiltonian is a part of the symmetry algebra in this model, whereas in conventional models the Hamiltonian is invariant under an external symmetry algebra. Using (6.27) we can interpret the coefficient c as the energy of the spin chain. Since the eigenvalues of the Hamiltonian depend on the momentum of the magnons moving along the spin chain, c becomes rapidity-dependent. This makes the representation (6.2) of $\mathfrak{su}(1|1)$ dynamic. Together with the hypercharge b this parameter labels the representations of asymptotic states. The coefficient q only contains unphysical degrees of freedom which correspond to similarity transformations.

This model motivates the investigation of the implications of imposing dynamic Yangian constraints on S-matrices. Doing so, we assume that all coefficients of the $\mathfrak{su}(1|1)$ generators become rapidity dependent, i.e. $b = b(u)$, $c = c(u)$, $q = q(u)$. Then the representation of an asymptotic state is given by

$$(1|1)_{u_1,c(u_1),b(u_1),q(u_1)} \otimes (1|1)_{u_2,c(u_2),b(u_2),q(u_2)} \otimes ... \otimes (1|1)_{u_m,c(u_m),b(u_m),q(u_m)} \tag{6.28}$$

and the S-matrix is a map between them. The conventional spin chain discussed in the previous section can be obtained by putting $c = q = 1$ and $b = const.$

6.3.1 Two-Particle S-Matrix

We begin this discussion with an analysis of the dynamic Yangian constraints on a two-particle S-matrix which is the map

$$\begin{aligned} \mathrm{S}_{12}(u_{1,2}; v_{1,2}) : (1|1)_{v_1,c(v_1),b(v_1),q(v_1)} \otimes (1|1)_{v_2,c(v_2),b(v_2),q(v_2)} \\ \to (1|1)_{u_1,c(u_1),b(u_1),q(u_1)} \otimes (1|1)_{u_2,c(u_2),b(u_2),q(u_2)}. \end{aligned} \tag{6.29}$$

Solution of the Level-0 Constraint

We analyse the constraint (4.18) using the coproduct structure of the $Y[\mathfrak{su}(1|1)]$ Yangian generators in (6.5). The level-0 constraint for the generator $\mathfrak{C}$ restricts the two-particle S-matrix S_{12} to preserve the sum of the individual eigenvalues c of the incoming and outgoing particles

$$c(v_1) + c(v_2) = c(u_1) + c(u_2). \tag{6.30}$$

[4]Note that we used to denote the spectral parameter by the same symbol. Nevertheless, since we will not refer to the spectral parameter in the following, we use the standard convention here, i.e. denote the coupling constant by λ.

In the interpretation of c as energy, this equation corresponds to the conservation of energy during a scattering process.

Similar to the undynamic case, the level-0 constraint for the generator $\mathfrak{B}$ implies that bosons and fermions in a scattering process get permuted. Thus we can make the ansatz (6.11) with unknown coefficients $A_{12}, ..., F_{12}$ once more. Moreover, we find that the sum of the b's of the individual particles has to be conserved, i.e.

$$b(v_1) + b(v_2) = b(u_1) + b(u_2). \tag{6.31}$$

Demanding the invariance of S_{12} under $\Delta\mathfrak{Q}$ and $\Delta\mathfrak{S}$ relates the coefficients in the ansatz such that there remain two degrees of freedom. Choosing A_{12} and F_{12} as free coefficients yields

$$\begin{aligned}
B_{12} &= \frac{A_{12}c(u_2)q(v_1)q(v_2) - F_{12}c(v_2)q(u_1)q(u_2)}{(c(u_1)+c(u_2))q(v_2)q(u_2)}, \\
C_{12} &= \frac{A_{12}c(u_2)q(v_1)q(v_2) + F_{12}c(v_1)q(u_1)q(u_2)}{(c(u_1)+c(u_2))q(v_1)q(u_2)}, \\
D_{12} &= \frac{A_{12}c(u_1)q(v_1)q(v_2) - F_{12}c(v_1)q(u_1)q(u_2)}{(c(u_1)+c(u_2))q(v_1)q(u_1)}, \\
E_{12} &= \frac{A_{12}c(u_1)q(v_1)q(v_2) + F_{12}c(v_2)q(u_1)q(u_2)}{(c(u_1)+c(u_2))q(v_2)q(u_1)}.
\end{aligned} \tag{6.32}$$

Setting $c = q = 1$ we encounter the result of the undynamic case (6.12). The remaining two degrees of freedom correspond to the existence of two length-2 Casimirs of the algebra $\mathfrak{su}(1|1)$ as before, cf. [14].

Solution of the Level-1 Constraint

We proceed with the level-1 constraint (4.19) for the Yangian generators $\hat{\mathfrak{C}}$, $\hat{\mathfrak{B}}$, $\hat{\mathfrak{Q}}$ and $\hat{\mathfrak{S}}$ whose coproduct is given in (6.7). The level-1 constraint imposed by $\hat{\mathfrak{C}}$ implies

$$v_1 c(v_1) + v_2 c(v_2) = u_1 c(u_1) + u_2 c(u_2). \tag{6.33}$$

For a known (energy) dependence $c = c(u)$ the equations (6.30) and (6.33) completely determine the rapidities v_1 and v_2 of the outgoing state as functions of the incoming rapidities u_1 and u_2.

The invariance under $\hat{\mathfrak{Q}}$ relates the coefficients A_{12} and F_{12} as

$$F_{12} = A_{12}\frac{q(v_1)q(v_2)}{q(u_1)q(u_2)}\frac{c(u_1)+c(u_2)+2i(v_1-v_2)}{c(u_1)+c(u_2)+2i(u_1-u_2)}. \tag{6.34}$$

It is convenient to reparameterize the rapidities and central charges as [14]

$$\begin{aligned}
u_i &= \tfrac{1}{2}(x_i^+ + x_i^-) \qquad & c(u_i) &= -i(x_i^+ - x_i^-) \\
v_i &= \tfrac{1}{2}(y_i^+ + y_i^-) \qquad & c(v_i) &= -i(y_i^+ - y_i^-),
\end{aligned} \tag{6.35}$$

where $x_i^\pm = x^\pm(u_i)$ and $y_i^\pm = y^\pm(v_i)$. For the $\mathfrak{su}(1|2)$ sector of $\mathcal{N} = 4$ SYM these parameters are connected to the incoming momenta p_i and outgoing momenta k_i via

$$\frac{x_i^+}{x_i^-} = e^{ip_i}, \qquad \frac{y_i^+}{y_i^-} = e^{ik_i}. \tag{6.36}$$

In a concrete model with known central charge $c = c(u)$, one may explicitly calculate these functions. For the $\mathfrak{su}(1|2)$ long-range spin chain of $\mathcal{N} = 4$ SYM the new parameters collectively denoted by $x^\pm$ are related to the model's coupling constant λ by

$$x^+ + \frac{\lambda}{x^+} - x^- - \frac{\lambda}{x^-} = i. \tag{6.37}$$

For the undynamic case with $c = 1$ we have

$$x_i^\pm = u_i \pm \tfrac{i}{2} \qquad y_i^\pm = v_i \pm \tfrac{i}{2}. \tag{6.38}$$

Rearranging the result (6.34) using the parametrization (6.35) yields

$$F_{12} = -A_{12} \frac{q(u_1)q(u_2)}{q(v_1)q(v_2)} \frac{x_1^- - x_2^+}{y_2^+ - y_1^-} \tag{6.39}$$

and thus we obtain for the coefficients of S_{12} in (6.11)

$$\begin{aligned}
A_{12} &= \mathrm{S}^0_{12} \frac{y_2^+ - y_1^-}{y_2^- - y_1^+}, \\
B_{12} &= \mathrm{S}^0_{12} \frac{q(u_1)}{q(v_2)} \frac{-(x_2^+ - x_1^-)(x_2^+ - x_2^-) + (y_2^+ - y_1^1)(y_2^+ - y_2^-)}{(y_2^- - y_1^+)(x_1^+ + x_2^+ - x_1^- - x_2^-)}, \\
C_{12} &= \mathrm{S}^0_{12} \frac{q(u_2)}{q(v_2)} \frac{+(x_1^+ - x_1^-)(x_2^+ - x_1^-) + (y_2^+ - y_1^-)(y_2^+ - y_2^-)}{(y_2^- - y_1^+)(x_1^+ + x_2^+ - x_1^- - x_2^-)}, \\
D_{12} &= \mathrm{S}^0_{12} \frac{q(u_2)}{q(v_1)} \frac{-(x_1^+ - x_1^-)(x_2^+ - x_1^-) + (y_1^+ - y_1^-)(y_2^+ - y_1^-)}{(y_2^- - y_1^+)(x_1^+ + x_2^+ - x_1^- - x_2^-)}, \\
E_{12} &= \mathrm{S}^0_{12} \frac{q(u_1)}{q(v_1)} \frac{+(x_2^+ - x_1^-)(x_2^+ - x_2^-) + (y_1^+ - y_1^-)(y_2^+ - y_1^-)}{(y_2^- - y_1^+)(x_1^+ + x_2^+ - x_1^- - x_2^-)}, \\
F_{12} &= -\mathrm{S}^0_{12} \frac{q(u_1)q(u_2)}{q(v_1)q(v_2)} \frac{x_1^- - x_2^+}{y_2^- - y_1^+}.
\end{aligned} \tag{6.40}$$

Here we chose a specific parametrization of A_{12} with respect to the unknown overall factor S^0_{12}.

The constraints from vanishing commutators of S_{12} with $\Delta\hat{\mathfrak{B}}$ and $\Delta\hat{\mathfrak{S}}$ yield a set of four solutions. The trivial solution is

$$u_1 = v_1 \qquad u_2 = v_2 \tag{6.41}$$

which does not correspond to an S-matrix since the associated operator maps like the identity matrix (6.16) on two sites with

$$\mathcal{I}_{12} : (1|1)_{u_1,c_1,b_1,q_1} \otimes (1|1)_{u_2,c_2,b_2,q_2} \to (1|1)_{u_1,c_1,b_1,q_1} \otimes (1|1)_{u_2,c_2,b_2,q_2} \tag{6.42}$$

and $c_i := c(u_i),\ b_i := b(u_i),\ q_i := q(u_i)$.

The second solution is given by

$$u_1 = v_2 \qquad u_2 = v_1. \tag{6.43}$$

Thus, the corresponding operator maps as

$$\mathrm{S}_{12} : (1|1)_{u_2,c_2,b_2,q_2} \otimes (1|1)_{u_1,c_1,b_1,q_1} \to (1|1)_{u_1,c_1,b_1,q_1} \otimes (1|1)_{u_2,c_2,b_2,q_2}, \tag{6.44}$$

i.e. it permutes the Hilbert spaces of the two particles. For $u_1 > u_2$ this solution is the 2→2 S-matrix which preserves the set of rapidities. Substituting this result into (6.40) gives the two-particle S-matrix (6.11) with coefficients

$$\begin{aligned} A_{12} &= \mathrm{S}^0_{12}\delta_{u_1,v_2}\delta_{u_2,v_1}\frac{x_1^+ - x_2^-}{x_1^- - x_2^+}, & B_{12} &= \mathrm{S}^0_{12}\delta_{u_1,v_2}\delta_{u_2,v_1}\frac{x_1^+ - x_2^+}{x_1^- - x_2^+}, \\ C_{12} &= \mathrm{S}^0_{12}\delta_{u_1,v_2}\delta_{u_2,v_1}\frac{q_2}{q_1}\frac{x_1^+ - x_1^-}{x_1^- - x_2^+}, & D_{12} &= \mathrm{S}^0_{12}\delta_{u_1,v_2}\delta_{u_2,v_1}\frac{x_1^- - x_2^-}{x_1^- - x_2^+}, \\ E_{12} &= \mathrm{S}^0_{12}\delta_{u_1,v_2}\delta_{u_2,v_1}\frac{q_1}{q_2}\frac{x_2^+ - x_2^-}{x_1^- - x_2^+}, & F_{12} &= -\mathrm{S}^0_{12}\delta_{u_1,v_2}\delta_{u_2,v_1}. \end{aligned} \tag{6.45}$$

This is the well-known S-matrix of the $\mathfrak{su}(1|2)$ sector, cf. [14]. It satisfies the qYBE given by the second equality in (5.45) which we checked by direct computation. For $c = q = 1$ this result simplifies to (6.19).

There are further solutions for which the set of rapidities in the incoming and outgoing state do not coincide. We obtain the two sets

$$\begin{aligned} &v_1 + v_2 = u_1 + u_2 \\ &v_2 - v_1 = \pm\tfrac{i}{2}(c(u_1) - c(u_2)) \\ &v_1 b(v_1) + v_2 b(v_2) = u_1 b(u_1) + u_2 b(u_2) \end{aligned} \tag{6.46}$$

which only differ by a sign in the second equality. The last equation ensures the constraint $[\Delta\hat{\mathfrak{B}}, \mathrm{S}_{12}] = 0$. For the long-range $\mathfrak{su}(1|2)$ spin chain of $\mathcal{N} = 4$ SYM the first two constraints ensure momentum conservation which can be phrased as

$$e^{i(p_1+p_2)} = e^{i(k_1+k_2)} \Leftrightarrow \frac{c(u_1) - 2iu_1}{c(u_1) + 2iu_1}\frac{c(u_2) - 2iu_2}{c(u_2) + 2iu_2} = \frac{c(v_1) - 2iv_1}{c(v_1) + 2iv_1}\frac{c(v_2) - 2iv_2}{c(v_2) + 2iv_2} \tag{6.47}$$

using (6.36).

Note that the two solutions in (6.46) correspond to the two physical situations $v_1 > v_2$ and $v_1 < v_2$. The latter represents a scattering process. For a known dependence $c = c(u)$ one can identify the solution corresponding to the S-matrix. For $c = 1$ the second equation reduces to

$$v_1 = v_2 = u_1 = u_2 \tag{6.48}$$

which is an unphysical constraint for a scattering process. Therefore, we did not discuss this solution of the Yangian constraints in the previous section. Nevertheless, one can understand the solutions in (6.46) as deformations of this trivial case.

The first equation together with (6.30) and (6.33) puts strong constraints on the outgoing rapidities v_1, v_2 of a scattering process. In fact, if $c = c(u)$ can be expanded in a power series in u these three equations are only solved by $v_1, v_2 \in \{u_1, u_2\}$. Then (6.46) can be discarded since it does not enlarge the set of solutions but corresponds to a further constrained version of the original solutions (6.41) and (6.43).

6.3.2 Three-Particle S-Matrix

We now turn to the discussion of the 3→3 scattering process described by the S-matrix S_{123}. It acts on the Hilbert space (6.28) for $m = 3$ as

$$\begin{aligned} \mathrm{S}_{123}(u_i; v_i) : (1|1)_{v_1,c(v_1),b(v_1),q(v_1)} \otimes (1|1)_{v_2,c(v_2),b(v_2),q(v_2)} \otimes (1|1)_{v_3,c(v_3),b(v_3),q(v_3)} \\ \to (1|1)_{u_1,c(u_1),b(u_1),q(u_1)} \otimes (1|1)_{u_2,c(u_2),b(u_2),q(u_2)} \otimes (1|1)_{u_3,c(u_3),b(u_3),q(u_3)}. \end{aligned} \tag{6.49}$$

Solution of the Level-0 Constraint

Similar to the level-0 constraint for the two-particle S-matrix, a vanishing commutator of S_{123} with $\Delta^2\mathfrak{C}$ demands that

$$c(v_1) + c(v_2) + c(v_3) = c(u_1) + c(u_2) + c(u_3). \tag{6.50}$$

This implies energy conservation for the $\mathfrak{su}(1|2)$ sector of $\mathcal{N} = 4$ SYM.

Demanding the level-0 constraint for $\mathfrak{B}$ restricts the S-matrix to only permute the bosons and fermions, i.e. we can make the same ansatz (6.21) as for the undynamic case. It has 20 unknown coefficients $A_{123}, ..., W_{123}$. Furthermore, the sum of the hypercharges has to be conserved

$$b(u_1) + b(u_2) + b(u_3) = b(v_1) + b(v_2) + b(v_3). \tag{6.51}$$

The two remaining level-0 constraints with $\mathfrak{Q}$ and $\mathfrak{S}$ set 14 of the coefficients in relation to the remaining 6. Putting $c = q = 1$ reduces the result to the undynamic solution (6.22). Thus we find that the algebra $\mathfrak{su}(1|1)$ allows for six Casimirs of length 3.

Solution of the Level-1 Constraint

Let us move on to the solution of the level-1 constraints of $Y[\mathfrak{su}(1|1)]$. The constraint $[\Delta^2\hat{\mathfrak{C}}, \mathrm{S}_{123}] = 0$ implies

$$v_1 c(v_1) + v_2 c(v_2) + v_3 c(v_3) = u_1 c(u_1) + u_2 c(u_2) + u_3 c(u_3). \tag{6.52}$$

Unfortunately, obtaining the solution to the remaining constraints $[\Delta^2\hat{\mathfrak{Q}}, \mathrm{S}_{123}] = [\Delta^2\hat{\mathfrak{B}}, \mathrm{S}_{123}] = [\Delta^2\hat{\mathfrak{S}}, \mathrm{S}_{123}] = 0$ was beyond our computational power. These constraints are given by 35 equations restricting the six free coefficients and rapidities $v_{1,2,3}$ directly and indirectly via the functions $c = c(u), b = b(u), q = q(u)$. Solving this set of equations for the undynamic case with $c = q = 1$ and $b = 0, 1, 2$ using `Mathematica` took several minutes. For the general case we could find no solutions because the internal capacity of the computer was insufficient such that the evaluation aborted after two weeks. Therefore, we restrict the following discussion to the question whether the remaining constraints allow for a 3→3 S-matrix that preserves the set of rapidities with

$$v_1 = u_3, \qquad v_2 = u_2, \qquad v_3 = u_1. \tag{6.53}$$

The constraint $[\Delta^2\hat{\mathfrak{B}}, \mathrm{S}_{123}] = 0$ imposes a relation on five of the remaining six coefficients such that the whole S-matrix is determined up to an overall factor. This result does not get further constrained by imposing $[\Delta^2\hat{\mathfrak{Q}}, \mathrm{S}_{123}] = [\Delta^2\hat{\mathfrak{S}}, \mathrm{S}_{123}] = 0$. Using `Mathematica` we checked that the resulting matrix factorizes into three two-particle S-matrices.

Discussion of Results

Let us summarize the results of this section:

- We completely determined the two-particle $Y[\mathfrak{su}(1|1)]$-invariant and rapidity-conserving S-matrix (6.40) up to the dressing factor. This overall factor cannot be obtained by imposing its invariance under a Yangian.

- We also found additional solutions which only preserve the sum of rapidities. Together with the constraints from an invariance under $\mathfrak{C}$ and $\hat{\mathfrak{C}}$, this imposes strong constraints on the outgoing rapidities of a scattering process. For physical models that allow for a power series of $c = c(u)$ in u these are only solved for conserved rapidities, i.e. there are no additional solutions. Thus we rediscovered the conservation of rapidities which is a common feature of integrable models.

- The two-particle S-matrix satisfies the qYBE.

- The Yangian constraints on length 3 together with the assumption of conserved rapidities completely determine the three-particle S-matrix. It factorizes into the two-particle S-matrices that preserve the rapidities and satisfy the qYBE, i.e. it factorizes consistently.

- We could not answer the question whether the Yangian constraints on length 3 are only solved by an S-matrix which preserves the set of rapidities or whether there exist further solutions.

- The resulting S-matrices do not depend on relative rapidities only. We can understand this feature from the discussion at the end of chapter 5. In the dynamic representation of the Yangian, the generators J in (5.49) become rapidity-dependent and thus the rapidity-dependence of S_{12} is governed by an equation that depends on the absolute value of both incoming rapidities u_1 and u_2. This implies that such a model does not correspond to a Lorentz-invariant theory.

- By putting $c = q = 1$ and $b = const.$, we were able to obtain the S-matrices that we calculated in the last section and which are invariant under the undynamical Yangian $Y[\mathfrak{su}(1|1)]$.

Note that the dynamic level-0 and level-1 Yangian constraints of the generators $\mathfrak{C}$ and $\hat{\mathfrak{C}}$ form local rapidity-dependent constraints on the S-matrix. It could be interesting to check whether the higher levels of this central charge also have local coproducts. Then the Yangian would allow for an infinite tower of local conserved charges and would thus imply particle number and rapidity conservation, as well as factorization of scattering directly via the arguments given in section 3.4. This would unite the approach to integrability via conserved local charges and via the conservation of a Yangian for the corresponding models.

Chapter 7

$Y\,[\mathfrak{su}(2|2)]$-Invariant S-Matrices

In this chapter, we analyse the Yangian constraints on S-matrices for $Y[\mathfrak{su}(2|2)]$ and $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$, respectively. We already gave a brief review of its importance in the context of the AdS/CFT correspondence in the introductory chapter 1. Due to the existence of two bosonic and two fermionic particles in its fundamental representation and the possibility of length-changing generators, this algebra is more advanced than $\mathfrak{su}(1|1)$. In section 7.1 we begin with a review on the algebra $\mathfrak{su}(2|2)$ and its central extension. Doing so, we discuss two realizations in physical models and thus motivate the subsequent analysis of both the undynamic and dynamic representation of this algebra. In sections 7.2 and 7.3 we perform the explicit analysis for two- and three-particle S-matrices and investigate whether the results allow for consistent factorization. As above, we concentrate here on the discussion of the results and comment on a possible implementation in `Mathematica` in the appendix D.

7.1 The Algebra $\mathfrak{su}(2|2)\ltimes\mathbb{R}^2$ and its Representations

Generators of $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ and the Fundamental Representation

We begin this chapter with a discussion of the Lie superalgebra $\mathfrak{su}(2|2)$ on the basis of [11,36]. It consists of the $\mathfrak{su}(2)\times\mathfrak{su}(2)$ generators $\mathfrak{R}^a_b$ and $\mathfrak{L}^\alpha_\beta$, the supersymmetry generators $\mathfrak{Q}^\alpha_b$, $\mathfrak{S}^a_\beta$ and the central charge $\mathfrak{C}$. Note that we use Latin indices $a, b, c, ...$ to denote bosonic degrees of freedom and Greek indices $\alpha, \beta, \gamma, ...$ for fermionic degrees of freedom. These generators satisfy the commutation relations

$$\begin{aligned}
&[\mathfrak{R}^a_b, \mathfrak{R}^c_d] = \delta^c_b \mathfrak{R}^a_d - \delta^a_d \mathfrak{R}^c_b, && [\mathfrak{L}^\alpha_\beta, \mathfrak{L}^\gamma_\delta] = \delta^\gamma_\beta \mathfrak{L}^\alpha_\delta - \delta^\alpha_\delta \mathfrak{L}^\gamma_\beta,\\
&[\mathfrak{R}^a_b, \mathfrak{Q}^\gamma_d] = -\delta^a_d \mathfrak{Q}^\gamma_b + \tfrac{1}{2}\delta^a_b \mathfrak{Q}^\gamma_d, && [\mathfrak{L}^\alpha_\beta, \mathfrak{Q}^\gamma_d] = +\delta^\gamma_\beta \mathfrak{Q}^\alpha_d - \tfrac{1}{2}\delta^\alpha_\beta \mathfrak{Q}^\gamma_d,\\
&[\mathfrak{R}^a_b, \mathfrak{S}^c_\delta] = +\delta^c_b \mathfrak{S}^a_\delta - \tfrac{1}{2}\delta^a_b \mathfrak{S}^c_\delta, && [\mathfrak{L}^\alpha_\beta, \mathfrak{S}^c_\delta] = -\delta^\alpha_\delta \mathfrak{S}^c_\beta + \tfrac{1}{2}\delta^\alpha_\beta \mathfrak{S}^c_\delta,\\
&\{\mathfrak{Q}^\alpha_b, \mathfrak{S}^c_\delta\} = \delta^c_b \mathfrak{L}^\alpha_\delta + \delta^\alpha_\delta \mathfrak{R}^c_b + \delta^c_b \delta^\alpha_\delta \mathfrak{C}. &&
\end{aligned} \tag{7.1}$$

The remaining Lie brackets vanish. It will become obvious in the following that one has to extend this algebra by two central charges $\mathfrak{P}$ and $\mathfrak{K}$ in order to obtain the correct S-matrix for the planar limit of $\mathcal{N} = 4$ SYM. These modify the anticommutators as

$$\{\mathfrak{Q}^\alpha_b, \mathfrak{Q}^\gamma_d\} = \varepsilon^{\alpha\gamma}\varepsilon_{bd}\mathfrak{P}$$
$$\{\mathfrak{S}^a_\beta, \mathfrak{S}^c_\delta\} = \varepsilon^{ac}\varepsilon_{\beta\delta}\mathfrak{K}. \tag{7.2}$$

Demanding that $\mathfrak{P}$ and $\mathfrak{K}$ annihilate physical states constrains the eigenvalues of these operators and guarantees that the physical result is compatible with an $\mathfrak{su}(2|2)$ invariance of the model.

The fundamental representation of this algebra contains two bosonic states $|\phi^a\rangle$ and two fermionic states $|\psi^\alpha\rangle$ with $a, \alpha = 1, 2$. The $\mathfrak{su}(2)$ generators $\mathfrak{R}^a_b$ and $\mathfrak{L}^\alpha_\beta$ act analogously to (5.4) as

$$\mathfrak{R}^a_b\,|\phi^c\rangle = \delta^c_b\,|\phi^a\rangle - \tfrac{1}{2}\,|\phi^c\rangle\,,$$
$$\mathfrak{L}^\alpha_\beta\,|\psi^\gamma\rangle = \delta^\gamma_\beta\,|\phi^\alpha\rangle - \tfrac{1}{2}\,|\phi^\gamma\rangle\,. \tag{7.3}$$

Note that we continue not to indicate that we use the operators in the fundamental representation ρ. In this representation the supersymmetry generators transform bosons into fermions and vice versa with coefficients a, b, c, d as

$$\mathfrak{Q}^\alpha_a\,|\phi^b\rangle = a\;\delta^b_a\,|\psi^\alpha\rangle\,, \qquad \mathfrak{Q}^\alpha_a\,|\psi^\beta\rangle = b\;\varepsilon^{\alpha\beta}\varepsilon_{ab}\,|\phi^b\rangle\,,$$
$$\mathfrak{S}^a_\alpha\,|\phi^b\rangle = c\;\varepsilon^{ab}\varepsilon_{\alpha\beta}\,|\psi^\beta\rangle\,, \qquad \mathfrak{S}^a_\alpha\,|\psi^\beta\rangle = d\;\delta^\beta_\alpha\,|\phi^a\rangle\,. \tag{7.4}$$

Via the anticommutation relations (7.2), the eigenvalues P and K of the central charges $\mathfrak{P}$ and $\mathfrak{K}$ are given by

$$P = ab, \qquad K = cd. \tag{7.5}$$

The anticommutator $\{\mathfrak{Q}^\alpha_b, \mathfrak{S}^c_\delta\}$ in (7.1) demands that the eigenvalue C of $\mathfrak{C}$ is

$$C = \tfrac{1}{2}(ad + bc) \tag{7.6}$$

and that the coefficients a, b, c, d satisfy

$$ad - bc = 1. \tag{7.7}$$

Realization of $\mathfrak{su}(2|2)$ in the $\mathfrak{su}(2|3)$ sector of $\mathcal{N} = 4$ SYM

In the $\mathfrak{su}(2|3)$ sector of $\mathcal{N} = 4$ SYM the algebra $\mathfrak{su}(2|2)$ appears as residual symmetry algebra $\mathfrak{g}_r$ of the excitations of the spin chain. Interestingly, there are multi-particle states such as $|\phi^{[1}\phi^2\phi^{3]}\rangle$ and $|\psi^{[1}\psi^{2]}\rangle$ which have the same quantum numbers and energies but different lengths [25]. This allows for fluctuations which make the length of the spin chain a dynamical variable. This feature is obvious in another notation of the fundamental representation discussed in [10] given by

$$\mathfrak{Q}^\alpha_a\,|\phi^b\rangle = a\;\delta^b_a\,|\psi^\alpha\rangle\,, \qquad \mathfrak{Q}^\alpha_a\,|\psi^\beta\rangle = \tilde{b}\;\varepsilon^{\alpha\beta}\varepsilon_{ab}\,|\phi^b\mathcal{Z}^+\rangle\,,$$
$$\mathfrak{S}^a_\alpha\,|\phi^b\rangle = \tilde{c}\;\varepsilon^{ab}\varepsilon_{\alpha\beta}\,|\psi^\beta\mathcal{Z}^-\rangle\,, \qquad \mathfrak{S}^a_\alpha\,|\psi^\beta\rangle = d\;\delta^\beta_\alpha\,|\phi^a\rangle\,. \tag{7.8}$$

The generators $\mathfrak{Q}^\alpha_a$ and $\mathfrak{S}^a_\alpha$ act on fields by insertion and annihilation of vacuum states. $\mathcal{Z}^+$ inserts an additional vacuum site and $\mathcal{Z}^-$ deletes one. But this length-changing can also be captured by using the so-called braiding factor[1] $\mathcal{U}$, cf. [37, 38], with

$$|\mathcal{Z}^\pm\Psi\rangle = e^{\mp ip}\,|\Psi\mathcal{Z}^\pm\rangle =: \mathcal{U}^{\mp 2}\,|\Psi\rangle\,. \tag{7.9}$$

[1] Note that we denoted the shift operator on spin chains by the same symbol. Since we do not refer to this quantity any more in the following, there will be no risk of confusion.

Here Ψ is either ϕ or ψ and the coefficients $\tilde{b}$ and $\tilde{c}$ are related to b and c via the eigenvalue $U = U(p)$ of $\mathcal{U}$ as

$$\tilde{b} = b\, U^{-2} \qquad \tilde{c} = c\, U^{+2}. \tag{7.10}$$

Since the eigenvalue U of $\mathcal{U}$ is momentum-dependent, the representation in (7.4) is also momentum-dependent. Thus this discussion shows the close connection of length-changing effects and dynamic representations for this model.

If we had not introduced the two extra central charges $\mathfrak{P}$ and $\mathfrak{K}$, i.e. for $P = K = 0$ on each site, we would have obtained

$$C = \pm\tfrac{1}{2} \tag{7.11}$$

using (7.5) and (7.6). In the $\mathfrak{su}(2|3)$ sector of $\mathcal{N} = 4$ SYM the central charge $\mathfrak{C}$ is associated to the energy of the chain [10] and is not quantized as in (7.11). In fact, due to the additional generators $\mathfrak{P}$ and $\mathfrak{K}$ this contradiction is resolved. One can show that the one-magnon energy for non-vanishing eigenvalues P and K at single sites and vanishing $P = K = 0$ on the whole cyclic spin chain is given by [10]

$$C = \pm\tfrac{1}{2}\sqrt{1 + 16\alpha\beta \sin^2\left(\tfrac{p}{2}\right)} \tag{7.12}$$

where α and β are constants satisfying

$$ab = g\alpha\left(1 - e^{ip}\right) \qquad\qquad cd = \tfrac{\beta}{g}\left(1 - e^{-ip}\right). \tag{7.13}$$

The product $\alpha\beta = g^2$ corresponds to the square of the coupling constant g of the model, i.e. (7.11) is true at leading order. This model motivates the discussion of S-matrices that are invariant under a dynamic representation of the Yangian $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$. We discuss these in section 7.3.

Realization of $\mathfrak{su}(2|2)$ in a Condensed Matter System

In fact, the algebra $\mathfrak{su}(2|2)$ is also relevant in the context of strongly correlated electron systems on a one-dimensional lattice, see [39]. In this model each site is a superposition of four possible electronic states and two of these are fermionic. There are no length-changing effects such that the undynamic representation with $\mathcal{U} = 1$ becomes relevant and one may directly put $P = K = 0$ at each site, i.e. $C = \pm\frac{1}{2}$. Thus, there is no need to make the representation (7.4) rapidity-dependent. The R-matrix of this model corresponds to the $Y[\mathfrak{su}(2|2)]$-invariant S-matrix which motivates the investigation of the undynamic Yangian $Y[\mathfrak{su}(2|2)]$, see [40]. Since the symmetry algebra of this model corresponds to the leading order contribution of the dynamic representation, we will check in the following sections whether the dynamic S-matrices contain the undynamic results.

Coproduct Structure of $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$

Now let us move on to the coproduct structure of the Yangian corresponding to $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$. Its level-0 generators are the Lie superalgebra generators $\mathfrak{R}^a_b$, $\mathfrak{L}^\alpha_\beta$, $\mathfrak{Q}^\alpha_b$, $\mathfrak{S}^a_\beta$ and $\mathfrak{C}$, as well as the central charges $\mathfrak{P}$ and $\mathfrak{K}$ of the central extension. Their action on single-particle states is given by (7.3) and (7.4) in the evaluation representation. The corresponding level-1 generators $\hat{\mathrm{J}}$ are $\hat{\mathfrak{R}}^a_b$, $\hat{\mathfrak{L}}^\alpha_\beta$, $\hat{\mathfrak{Q}}^\alpha_b$, $\hat{\mathfrak{S}}^a_\beta$, $\hat{\mathfrak{C}}$, $\hat{\mathfrak{P}}$ and $\hat{\mathfrak{K}}$. Their action on single particles in this representation is given by

$$\hat{\mathrm{J}}\,|\phi,u\rangle = igu\mathrm{J}\,|\phi,u\rangle \qquad \hat{\mathrm{J}}\,|\psi,u\rangle = igu\mathrm{J}\,|\psi,u\rangle \tag{7.14}$$

with rapidity u and J being the level-0 generator that is associated to $\hat{\mathrm{J}}$.

In order to evaluate explicitly the constraints (4.18) and (4.19), we need the coproduct structure of the Yangian generators. For the level-0 generators acting on two sites it is [11]

$$\begin{aligned}
&\Delta\mathfrak{C} = \mathfrak{C}\otimes 1 + 1\otimes\mathfrak{C}, && \Delta\mathfrak{R}^a_b = \mathfrak{R}^a_b\otimes 1 + 1\otimes\mathfrak{R}^a_b,\\
&\Delta\mathfrak{P} = \mathfrak{P}\otimes 1 + \mathcal{U}^{+2}\otimes\mathfrak{P}, && \Delta\mathfrak{L}^\alpha_\beta = \mathfrak{L}^\alpha_\beta\otimes 1 + 1\otimes\mathfrak{L}^\alpha_\beta,\\
&\Delta\mathfrak{K} = \mathfrak{K}\otimes 1 + \mathcal{U}^{-2}\otimes\mathfrak{K}, && \Delta\mathfrak{Q}^\alpha_b = \mathfrak{Q}^\alpha_b\otimes 1 + \mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\alpha_b,\\
& && \Delta\mathfrak{S}^a_\beta = \mathfrak{S}^a_\beta\otimes 1 + \mathcal{U}_F^{-1}\otimes\mathfrak{S}^a_\beta
\end{aligned}\tag{7.15}$$

with $\mathcal{U}_F := (-1)^F\,\mathcal{U}$. 1 is the identity operator of $\mathfrak{su}(2|2)$ and $(-1)^F$ is the fermionic grading operator, i.e.

$$1\,|\phi^a\rangle = |\phi^a\rangle\,,\quad 1\,|\psi^\alpha\rangle = |\psi^\alpha\rangle\,,\quad (-1)^F\,|\phi^a\rangle = +\,|\phi^a\rangle\,,\quad (-1)^F\,|\psi^\alpha\rangle = -\,|\psi^\alpha\rangle\,. \tag{7.16}$$

$(-1)^F$ takes care of the correct statistics when anticommuting fermionic particles and the fermionic operators $\mathfrak{Q}^\alpha_b$ and $\mathfrak{S}^a_\beta$. The abelian braiding operator $\mathcal{U}$ includes length-changing effects and can be set to 1 for conventional spin chains. The action on a three-particle state can be obtained from (7.15) by making use of (4.10) and incorporating all fermionic grading operators $(-1)^F$ and the braiding factors $\mathcal{U}$. The result can be found in the appendix in equation (C.3). The coproduct structure of the level-1 generators is given by

$$\begin{aligned}
\Delta\hat{\mathfrak{C}} =&\hat{\mathfrak{C}}\otimes 1 + 1\otimes\hat{\mathfrak{C}} + \tfrac{1}{2}\mathfrak{P}\,\mathcal{U}^{-2}\otimes\mathfrak{K} - \tfrac{1}{2}\mathfrak{K}\,\mathcal{U}^{+2}\otimes\mathfrak{P},\\
\Delta\hat{\mathfrak{P}} =&\hat{\mathfrak{P}}\otimes 1 + \mathcal{U}^{+2}\otimes\hat{\mathfrak{P}} - \mathfrak{C}\,\mathcal{U}^{+2}\otimes\mathfrak{P} + \mathfrak{P}\otimes\mathfrak{C},\\
\Delta\hat{\mathfrak{K}} =&\hat{\mathfrak{K}}\otimes 1 + \mathcal{U}^{-2}\otimes\hat{\mathfrak{K}} + \mathfrak{C}\,\mathcal{U}^{-2}\otimes\mathfrak{K} - \mathfrak{K}\otimes\mathfrak{C},\\
\Delta\hat{\mathfrak{R}}^a_b =&\hat{\mathfrak{R}}^a_b\otimes 1 + 1\otimes\hat{\mathfrak{R}}^a_b + \tfrac{1}{2}\mathfrak{R}^a_c\otimes\mathfrak{R}^c_b - \tfrac{1}{2}\mathfrak{R}^c_b\otimes\mathfrak{R}^a_c\\
&-\tfrac{1}{2}\mathfrak{S}^a_\gamma\,\mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\gamma_b - \tfrac{1}{2}\mathfrak{Q}^\gamma_b\,\mathcal{U}_F^{-1}\otimes\mathfrak{S}^a_\gamma\\
&+\tfrac{1}{4}\delta^a_b\;\mathfrak{S}^d_\gamma\,\mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\gamma_d + \tfrac{1}{4}\delta^a_b\;\mathfrak{Q}^\gamma_d\,\mathcal{U}_F^{-1}\otimes\mathfrak{S}^d_\gamma,\\
\Delta\hat{\mathfrak{L}}^\alpha_\beta =&\hat{\mathfrak{L}}^\alpha_\beta\otimes 1 + 1\otimes\hat{\mathfrak{L}}^\alpha_\beta - \tfrac{1}{2}\mathfrak{L}^\alpha_\gamma\otimes\mathfrak{L}^\gamma_\beta + \tfrac{1}{2}\mathfrak{L}^\gamma_\beta\otimes\mathfrak{L}^\alpha_\gamma\\
&+\tfrac{1}{2}\mathfrak{Q}^\alpha_c\,\mathcal{U}_F^{-1}\otimes\mathfrak{S}^c_\beta + \tfrac{1}{2}\mathfrak{S}^c_\beta\,\mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\alpha_c\\
&-\tfrac{1}{4}\delta^\alpha_\beta\;\mathfrak{Q}^\delta_c\,\mathcal{U}_F^{-1}\otimes\mathfrak{S}^c_\delta - \tfrac{1}{4}\delta^\alpha_\beta\;\mathfrak{S}^c_\delta\,\mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\delta_c,\\
\Delta\hat{\mathfrak{Q}}^\alpha_b =&\hat{\mathfrak{Q}}^\alpha_b\otimes 1 + \mathcal{U}_F^{+1}\otimes\hat{\mathfrak{Q}}^\alpha_b - \tfrac{1}{2}\mathfrak{L}^\alpha_\gamma\,\mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\gamma_b + \tfrac{1}{2}\mathfrak{Q}^\gamma_b\otimes\mathfrak{L}^\alpha_\gamma\\
&-\tfrac{1}{2}\mathfrak{R}^c_b\,\mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\alpha_c + \tfrac{1}{2}\mathfrak{Q}^\alpha_c\otimes\mathfrak{R}^c_b - \tfrac{1}{2}\mathfrak{C}\,\mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\alpha_b + \tfrac{1}{2}\mathfrak{Q}^\alpha_b\otimes\mathfrak{C}\\
&+\tfrac{1}{2}\varepsilon^{\alpha\gamma}\varepsilon_{bd}\;\mathfrak{P}\,\mathcal{U}_F^{-1}\otimes\mathfrak{S}^d_\gamma - \tfrac{1}{2}\varepsilon^{\alpha\gamma}\varepsilon_{bd}\;\mathfrak{S}^d_\gamma\,\mathcal{U}^{+2}\otimes\mathfrak{P},\\
\Delta\hat{\mathfrak{S}}^a_\beta =&\hat{\mathfrak{S}}^a_\beta\otimes 1 + \mathcal{U}_F^{-1}\otimes\hat{\mathfrak{S}}^a_\beta + \tfrac{1}{2}\mathfrak{R}^a_c\,\mathcal{U}_F^{-1}\otimes\mathfrak{S}^c_\beta - \tfrac{1}{2}\mathfrak{S}^c_\beta\otimes\mathfrak{R}^a_c\\
&+\tfrac{1}{2}\mathfrak{L}^\gamma_\beta\,\mathcal{U}_F^{-1}\otimes\mathfrak{S}^a_\gamma - \tfrac{1}{2}\mathfrak{S}^a_\gamma\otimes\mathfrak{L}^\gamma_\beta + \tfrac{1}{2}\mathfrak{C}\,\mathcal{U}_F^{-1}\otimes\mathfrak{S}^a_\beta - \tfrac{1}{2}\mathfrak{S}^a_\beta\otimes\mathfrak{C}\\
&-\tfrac{1}{2}\varepsilon^{ac}\varepsilon_{\beta\delta}\;\mathfrak{K}\,\mathcal{U}_F^{+1}\otimes\mathfrak{Q}^\delta_c + \tfrac{1}{2}\varepsilon^{ac}\varepsilon_{\beta\delta}\;\mathfrak{Q}^\delta_c\,\mathcal{U}^{-2}\otimes\mathfrak{K}.
\end{aligned}\tag{7.17}$$

On a three-particle state they act as shown in eq. (C.4) in the appendix C. The representation of the Yangian $Y[\mathfrak{su}(2|2)]$ without central extension can be obtained by setting the eigenvalues of $\mathfrak{P}$ and $\mathfrak{K}$ to 0 and of $\mathcal{U}$ to 1.

$\mathfrak{su}(1|1)$ as Subalgebra of $\mathfrak{su}(2|2)$

Note that the algebra $\mathfrak{su}(2|2)$ contains $\mathfrak{su}(1|1)$ as a subalgebra. One can obtain the $\mathfrak{su}(1|1)$ generators by setting the bosonic and fermionic indices to $a = b = \alpha = \beta = 1$ and $|\phi\rangle := |\phi^1\rangle$ and $|\psi\rangle := |\psi^1\rangle$. Then the $\mathfrak{su}(1|1)$ generators $\mathfrak{Q}$ and $\mathfrak{S}$ can be obtained from $\mathfrak{Q}^\alpha_a$ and $\mathfrak{S}^a_\alpha$ by putting the representation coefficients in (7.4) to $a = q$ and $d = \frac{c}{q}$. The outer automorphism $\mathfrak{B}$ can be obtained from the linear combination

$$\frac{b-1}{1-\frac{1}{\mathrm{n}}}\mathfrak{L}^1_1 + \frac{b+1}{1-\frac{1}{\mathrm{n}}}\mathfrak{R}^1_1 \tag{7.18}$$

and the central charge $\mathfrak{C}$ from $\mathfrak{su}(1|1)$ is given via the combination of $\mathfrak{su}(2|2)$ generators of the form

$$\mathfrak{C} + \mathfrak{R}^1_1 + \mathfrak{L}^1_1. \tag{7.19}$$

This close connection of both algebras will allow us to derive the $Y[\mathfrak{su}(1|1)]$-invariant S-matrices discussed in the previous chapter from the $\mathfrak{su}(2|2)$ results.

7.2 Undynamic $Y[\mathfrak{su}(2|2)]$-Invariant S-Matrices

In this section we calculate the $Y[\mathfrak{su}(2|2)]$-invariant two- and three-particle S-matrices for the Yangian in its undynamic representation.

7.2.1 Two-Particle S-Matrix

The two-particle S-matrix is the map between incoming and outgoing two-particle states that transform under $Y[\mathfrak{su}(2|2)]$, i.e.

$$\mathrm{S}_{12}(u_{1,2}; v_{1,2}) : (2|2)_{v_1} \otimes (2|2)_{v_2} \to (2|2)_{u_1} \otimes (2|2)_{u_2} \tag{7.20}$$

with different sets of incoming and outgoing rapidities satisfying $u_1 > u_2$ and $v_1 < v_2$.

Solution of the Level-0 Constraint

The constraint (4.18) for the central charge $\mathfrak{C}$ does not restrict the form of S_{12} since it is an abelian operator. Constraining the S-matrix by demanding a vanishing commutator with the generators $\Delta\mathfrak{R}^a_b$ and $\Delta\mathfrak{L}^\alpha_\beta$ restricts it to be of the form

$$\begin{aligned}
\mathrm{S}_{12}\,|\phi^a, v_1; \phi^b, v_2\rangle &= A_{12}\,|\phi^{\{a}, u_1; \phi^{b\}}, u_2\rangle + B_{12}\,|\phi^{[a}, u_1; \phi^{b]}, u_2\rangle \\
&\qquad + \tfrac{1}{2}C_{12}\varepsilon^{ab}\varepsilon_{\alpha\beta}\,|\psi^\alpha, u_1; \psi^\beta_1, u_2\rangle \\
\mathrm{S}_{12}\,|\phi^a, v_1; \psi^\beta, v_2\rangle &= D_{12}\,|\psi^\beta, u_1; \phi^a, u_2\rangle + E_{12}\,|\phi^a, u_1; \psi^\beta, u_2\rangle \\
\mathrm{S}_{12}\,|\psi^\alpha, v_1; \phi^b, v_2\rangle &= F_{12}\,|\psi^\alpha, u_1; \phi^b, u_2\rangle + G_{12}\,|\phi^b, u_1; \psi^\alpha, u_2\rangle \\
\mathrm{S}_{12}\,|\psi^\alpha, v_1; \psi^\beta, v_2\rangle &= H_{12}\,|\psi^{\{\alpha}, u_1; \psi^{\beta\}}, u_2\rangle + K_{12}\,|\psi^{[\alpha}, u_1; \psi^{\beta]}, u_2\rangle \\
&\qquad + \tfrac{1}{2}L_{12}\varepsilon^{\alpha\beta}\varepsilon_{ab}\,|\phi^a, u_1; \phi^b, u_2\rangle
\end{aligned} \tag{7.21}$$

with ten unknown coefficients $A_{12}, ..., L_{12}$. This ansatz corresponds to the ansatz in Table 1 of [10] where the rapidities are conserved $v_1 = u_2$ and $v_2 = u_1$. Demanding a vanishing commutator of S_{12} with $\Delta\mathfrak{Q}^\alpha_a$ implies

$$\begin{aligned}
&B_{12} = H_{12}, \quad C_{12} = 0, \quad D_{12} = \tfrac{1}{2}(A_{12} - H_{12}), \quad E_{12} = \tfrac{1}{2}(A_{12} + H_{12}), \\
&K_{12} = A_{12}, \quad L_{12} = 0, \quad G_{12} = \tfrac{1}{2}(A_{12} - H_{12}), \quad F_{12} = \tfrac{1}{2}(A_{12} + H_{12}).
\end{aligned} \tag{7.22}$$

The resulting S-matrix automatically commutes with $\Delta\mathfrak{S}^\alpha_a$. We conclude that the Yangian level-0 constraints fix the S-matrix up to two degrees of freedom corresponding to two Casimirs of $\mathfrak{su}(2|2)$. Note that due to $C_{12} = L_{12} = 0$ the S-matrix permutes the particles in a scattering event. This result is in accordance with the level-0 result for the $\mathfrak{su}(1|1)$-invariant S-matrix given in (6.11) and (6.12). Note that the construction of the Casimirs from a Killing form κ^{ab} of the algebra is non-trivial since it vanishes, see [41].

Solution of the Level-1 Constraint

Let us further constrain the two-particle S-matrix by imposing $[\Delta\hat{\mathrm{J}}, \mathrm{S}_{12}] = 0$. For $\hat{\mathrm{J}} = \hat{\mathfrak{C}}$ this constraint is satisfied as long as the sum of rapidities in the incoming and outgoing state is identical

$$v_1 + v_2 = u_1 + u_2. \tag{7.23}$$

From $[\Delta\hat{\mathfrak{R}}^a_b, \mathrm{S}_{12}] = 0$ we obtain a set of two solutions

$$\begin{aligned} &(1) \quad v_1 = u_1,\ v_2 = u_2,\ H_{12} = A_{12} \\ &(2) \quad v_1 = u_2,\ v_2 = u_1,\ H_{12} = A_{12}\frac{i - u_{12}}{i + u_{12}}. \end{aligned} \tag{7.24}$$

Here we drop the Kronecker deltas $\delta_{v_1,u_1}\delta_{v_2,u_2}$ and $\delta_{v_1,u_2}\delta_{v_2,u_1}$, respectively, in the coefficient H_{12}. Similar to the analogous solutions in the discussion of $Y[\mathfrak{su}(\mathrm{n})]$ and $Y[\mathfrak{su}(1|1)]$, the first solution corresponds to the identity map

$$\mathcal{I}_{12} \propto |\phi^a_1;\phi^b_2\rangle\langle\phi^a_1;\phi^b_2| + |\phi^a_1;\psi^\beta_2\rangle\langle\phi^a_1;\psi^\beta_2| + |\psi^\alpha_1;\phi^b_2\rangle\langle\psi^\alpha_1;\phi^b_2| + |\psi^\alpha_1;\psi^\beta_2\rangle\langle\psi^\alpha_1;\psi^\beta_2|\,, \tag{7.25}$$

where the summation over the indices a, b, α, β is implicit. The second solution is the true 2→2 S-matrix that permutes the Hilbert spaces of the particles as

$$\mathrm{S}_{12}(u_{1,2}) : (2|2)_{u_2} \otimes (2|2)_{u_1} \to (2|2)_{u_1} \otimes (2|2)_{u_2}. \tag{7.26}$$

Its explicit form is given by (7.21) with coefficients

$$\begin{aligned} A_{12} &= -\mathrm{S}^0_{12}\frac{i + u_{12}}{i - u_{12}} = K_{12}, \quad & B_{12} &= -\mathrm{S}^0_{12} = H_{12}, \quad & C_{12} &= 0 = L_{12}, \\ D_{12} &= -\mathrm{S}^0_{12}\frac{u_{12}}{i - u_{12}} = G_{12}, \quad & E_{12} &= -\mathrm{S}^0_{12}\frac{i}{i - u_{12}} = F_{12}. \end{aligned} \tag{7.27}$$

Note that this result is in accordance with the S-matrix of $\mathfrak{su}(1|1)$ given in (6.11) and (6.19). It does not get further restricted by the constraints from $\hat{\mathfrak{L}}^\alpha_\beta$, $\hat{\mathfrak{Q}}^\alpha_a$ and $\hat{\mathfrak{S}}^a_\alpha$. This two-particle S-matrix satisfies the qYBE given by the second equality in (5.45) which we checked by explicit calculation.

7.2.2 Three-Particle S-Matrix

We proceed with the discussion of the three-particle S-matrix S_{123} which is the map

$$\mathrm{S}_{123}(u_{1,2,3}; v_{1,2,3}) : (2|2)_{v_1} \otimes (2|2)_{v_2} \otimes (2|2)_{v_3} \to (2|2)_{u_1} \otimes (2|2)_{u_2} \otimes (2|2)_{u_3} \tag{7.28}$$

with $u_1 > u_2 > u_3$ and $v_1 < v_2 < v_3$.

Solution of the Level-0 Constraint

Once more, the constraint $[\Delta^2\mathfrak{C}, \mathrm{S}_{123}] = 0$ is automatically satisfied because the coproduct $\Delta^2\mathfrak{C}$ is proportional to the identity operator on length 3. The commutators with the generators $\mathfrak{R}^a_b$ and $\mathfrak{L}^\alpha_\beta$ restrict the S-matrix to be a linear combination of 70 operators. We do not print this operator here since it is rather lengthy. This ansatz gets further constrained by the remaining level-0 constraint from $\mathfrak{Q}^\alpha_a$ and we are left with ten degrees of freedom. $\mathfrak{Q}^\alpha_a$ and $\mathfrak{S}^a_\alpha$ set the same constraints on S_{123}.

Solution of the Level-1 Constraint

By imposing the level-1 constraints of the Yangian corresponding to the Lie superalgebra $\mathfrak{su}(2|2)$, we further restrict the form of the coefficients and the outgoing rapidities $v_{1,2,3}$. From $[\Delta^2\hat{\mathfrak{C}}, \mathrm{S}_{123}] = 0$ we find

$$v_1 + v_2 + v_3 = u_1 + u_2 + u_3. \tag{7.29}$$

Demanding its invariance under $\hat{\mathfrak{R}}^a_b$ further constrains S_{123}. As in the previous discussions of three-particle S-matrices, we obtain six solutions which all have the same rapidities in the incoming and outgoing state. For the true 3→3 S-matrix the rapidities in the outgoing state are

$$v_1 = u_3, \qquad v_2 = u_2, \qquad v_3 = u_1. \tag{7.30}$$

Furthermore, four of the ten free coefficients in S_{123} have to vanish and another five are related to the remaining single degree of freedom. Thus, the resulting S-matrix is fixed up to an overall factor and permutes the particles in a scattering event. We checked that it does not get further constrained by the commutation relations with $\hat{\mathfrak{K}}^\alpha_\beta$, $\hat{\mathfrak{Q}}^\alpha_a$ and $\hat{\mathfrak{S}}^a_\alpha$. By comparing the three-particle S-matrix with the product of three two-particle S-matrices as in the qYBE (5.45), we find that it factorizes.

Discussion of the Results

The S-matrices calculated in this section share the features of the S-matrices that are invariant under the undynamic Yangian $Y[\mathfrak{su}(1|1)]$. The list of important features at the end of section 6.2.2 can be inherited. We only enlarge it by the point

- The undynamic $Y[\mathfrak{su}(2|2)]$-invariant two- and three-particle S-matrices reduce to the undynamic $Y[\mathfrak{su}(1|1)]$-invariant S-matrices of section 6.3 by restricting to a single boson and a single fermion.

7.3 Dynamic $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$-Invariant S-Matrices

We now turn to the analysis of the dynamic Yangian constraints for the two- and three-particle S-matrix.

7.3.1 Two-Particle S-Matrix

The $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$-invariant two-particle S-matrix in its dynamic representation is the map

$$\begin{aligned} \mathrm{S}_{12}(u_{1,2}, v_{1,2}) : (2|2)_{v_1, C(v_1), P(v_1), K(v_1)} \otimes (2|2)_{v_2, C(v_2), P(v_2), K(v_2)} \\ \to (2|2)_{u_1, C(u_1), P(u_1), K(u_1)} \otimes (2|2)_{u_2, C(u_2), P(u_2), K(u_2)} \end{aligned} \tag{7.31}$$

with $u_1 > u_2$ and $v_1 < v_2$.

Solution of the Level-0 Constraint

The level-0 constraints (4.18) for the central charges $\mathfrak{C}$, $\mathfrak{P}$ and $\mathfrak{K}$ impose the following three relations including their eigenvalues C, P and K:

$$\begin{aligned} C(u_1) + C(u_2) &= C(v_1) + C(v_2), \\ P(u_1) + P(u_2)U(u_1)^2 &= P(v_1) + P(v_2)U(v_1)^2, \\ K(u_1) + \frac{K(u_2)}{U(u_1)^2} &= K(v_1) + \frac{K(v_2)}{U(v_1)^2}. \end{aligned} \tag{7.32}$$

They constrain the outgoing rapidities v_1 and v_2 and can be analysed best by reparameterizing the coefficients a, b, c, d in (7.4) via

$$a = \sqrt{g}\gamma, \quad b = \sqrt{g}\frac{\alpha}{\gamma}\left(1 - \frac{x^+}{x^-}\right), \quad c = \sqrt{g}\frac{i\gamma}{\alpha x^+}, \quad d = \sqrt{g}\frac{x^+}{i\gamma}\left(1 - \frac{x^-}{x^+}\right) \tag{7.33}$$

with $x^\pm = x^\pm(u), \gamma = \gamma(u)$ and $x^\pm(u) = x(u \pm \frac{i}{2})$. The parameter γ is associated to a relative rescaling between fermions ψ^α and bosons ϕ^a, while the constant α corresponds to a rescaling of the vacuum state $\mathcal{Z}$. The condition (7.7) translates to

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g} \tag{7.34}$$

and the eigenvalues of the central charges using (7.5) and (7.6) are

$$C = -\frac{1}{2} + igx^- - igx^+, \quad P = g\alpha\left(1 - \frac{x^+}{x^-}\right), \quad K = \frac{g}{\alpha}\left(1 - \frac{x^-}{x^+}\right). \tag{7.35}$$

Demanding cocommutativity of the coproduct leads to relations between the braiding factor $\mathcal{U}$ and the central charges $\mathfrak{P}$ and $\mathfrak{K}$, see [11]. These are solved if the eigenvalue U of $\mathcal{U}$ satisfies

$$U = \sqrt{\frac{x^+}{x^-}} \tag{7.36}$$

such that the momentum p and the parameters $x^\pm$ satisfy

$$e^{ip} = \frac{x^+}{x^-} \tag{7.37}$$

similar to (6.36). In these coordinates the rapidity can be expressed in terms of the new coordinates $x^\pm$ as

$$u = \frac{1}{2}\frac{x^+ + x^-}{1 + 1/x^+x^-}. \tag{7.38}$$

These relations can be used to reformulate (7.32) for $g, \alpha \neq 0$ as

$$x_1^+ + x_2^+ - x_1^- - x_2^- = y_1^+ + y_2^+ - y_1^- - y_2^-, \quad \frac{x_1^+ x_2^+}{x_1^- x_2^-} = \frac{y_1^+ y_2^+}{y_1^- y_2^-}, \quad \frac{x_1^- x_2^-}{x_1^+ x_2^+} = \frac{y_1^- y_2^-}{y_1^+ y_2^+} \tag{7.39}$$

with $x_i^\pm = x^\pm(u_i)$ and $y_i^\pm = x^\pm(v_i)$. It is evident that they are solved by

$$\begin{aligned} &(1) \quad v_1 = u_1,\ v_2 = u_2 \\ &(2) \quad v_1 = u_2,\ v_2 = u_1. \end{aligned} \tag{7.40}$$

In order to check whether there are further solutions, we use the solution for $x^\pm = x^\pm(u)$ of (7.34) found by comparison with the discussion in [42] as

$$x(u) = \frac{u}{2g} + \frac{u}{2g}\sqrt{1 - \frac{4g^2}{u^2}}. \tag{7.41}$$

Inserting this into (7.39) and expanding in the coupling constant g relates the incoming and outgoing rapidities at each order of the expansion. Indeed, we find that the solutions (7.40) are the only possible solutions of the level-0 constraints imposed by the central charges. Thus the Yangian constraints corresponding to $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ imply a conservation of rapidities in two-particle scattering events. The first solution in (7.40) is associated to the identity map and the second solution corresponds to the S-matrix we are looking for. Therefore, we concentrate the following analysis on solution (2).

Analysing the constraints $[\Delta\mathfrak{R}^a_b, \mathrm{S}_{12}] = [\Delta\mathfrak{L}^\alpha_\beta, \mathrm{S}_{12}] = 0$ shows that the S-matrix has to be of the form given in (7.21), i.e. we continue with the same ansatz as in the undynamic case. It has ten degrees of freedom. These are reduced by eight if one demands the S-matrix' invariance under $\Delta\mathfrak{Q}^\alpha_b$. Unlike in the analysis of the undynamic Yangian in the previous section, the constraint with $\Delta\mathfrak{S}^a_\beta$ removes another degree of freedom such that the S-matrix is determined up to an overall factor. It arises since we do not assume $P = 0$ and $K = 0$ for each particle anymore. We obtain for the coefficients in (7.21)

$$\begin{aligned}
A_{12} &= \mathrm{S}^0_{12}\frac{x_1^+ - x_2^-}{x_1^- - x_2^+},\\
B_{12} &= \mathrm{S}^0_{12}\frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\left(1 - 2\frac{1 - 1/x_1^- x_2^+}{1 - 1/x_1^+ x_2^+}\frac{x_1^- - x_2^-}{x_1^+ - x_2^-}\right),\\
C_{12} &= \mathrm{S}^0_{12}\frac{2\gamma_1\gamma_2 U_1}{\alpha x_2^+ x_1^+}\frac{1}{1 - 1/x_1^+ x_2^+}\frac{x_1^- - x_2^-}{x_1^- - x_2^+},\\
D_{12} &= \mathrm{S}^0_{12}\frac{1}{U_2}\frac{x_1^+ - x_2^+}{x_1^- - x_2^+},\\
E_{12} &= \mathrm{S}^0_{12}\frac{\gamma_2 U_1}{\gamma_1 U_2}\frac{x_1^+ - x_1^-}{x_1^- - x_2^+},\\
F_{12} &= \mathrm{S}^0_{12}\frac{\gamma_1}{\gamma_2}\frac{x_2^+ - x_2^-}{x_1^- - x_2^+},\\
G_{12} &= \mathrm{S}^0_{12}U_1\frac{x_1^- - x_2^-}{x_1^- - x_2^+}\\
H_{12} &= -\mathrm{S}^0_{12}\frac{U_1}{U_2},\\
K_{12} &= -\mathrm{S}^0_{12}\frac{U_1}{U_2}\left(1 - 2\frac{1 - 1/x_1^+ x_2^-}{1 - 1/x_1^- x_2^-}\frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\right),\\
L_{12} &= -\mathrm{S}^0_{12}\frac{2\alpha(x_2^+ - x_2^-)(x_1^+ - x_1^-)}{\gamma_1\gamma_2 U_2 x_2^- x_1^-}\frac{1}{1 - 1/x_2^- x_1^-}\frac{x_1^+ - x_2^+}{x_1^- - x_2^+}.
\end{aligned} \tag{7.42}$$

This is the well-known result from the literature for the $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$-invariant S-matrix calculated in [10]. We made use of the equivalences (3.5) given in the original paper and which are in this thesis' conventions

$$
\begin{aligned}
\frac{x_2^+ - x_1^+}{1 - 1/x_2^- x_1^-} &= \frac{x_2^- - x_1^-}{1 - 1/x_2^+ x_1^+} \\
\frac{x_1^+ - x_1^- - x_2^+ + x_2^-}{x_2^+ x_1^+ - x_2^- x_1^-} &= \frac{1}{x_2^+ x_2^- x_1^+ x_1^-} \frac{x_2^+ - x_1^+}{1 - 1/x_2^- x_1^-} \\
B_{12}/\mathrm{S}^0_{12} &= -1 + \frac{1}{x_2^+ x_1^+ - x_2^- x_1^-} \frac{x_2^+ x_1^+ - 2x_2^- x_1^+ + x_2^- x_1^-}{1 - 1/x_2^- x_1^-} \frac{x_2^+ - x_1^+}{x_1^- - x_2^+} \\
E_{12}/\mathrm{S}^0_{12} &= \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} - \frac{1}{x_2^+ x_1^+ - x_2^- x_1^-} \frac{x_2^+ x_1^+ - 2x_2^+ x_1^- + x_2^- x_1^-}{1 - 1/x_2^- x_1^-} \frac{x_2^+ - x_1^+}{x_1^- - x_2^+} .
\end{aligned}
\tag{7.43}
$$

Solution of the Level-1 Constraint

The level-0 constraints completely determine the two-particle S-matrix. Thus there remains the question whether this matrix is also Yangian-invariant. Checking the level-1 constraints explicitly using the coproduct structure in (7.17) shows that this matrix is indeed Yangian-symmetric. This was first shown in [11]. By numerical analysis, i.e. insertion of different real and complex values for $x^\pm$ and g, we furthermore checked that the two-particle S-matrix satisfies the qYBE given in (5.45).

It is interesting to check whether S_{12} contains the two-particle S-matrix that is invariant under the undynamic Yangian $Y[\mathfrak{su}(2|2)]$. In order to do so, we have to set all eigenvalues U_i of $\mathcal{U}$ to 1 and expand the remaining ingredients in (7.42) in powers of the coupling constant g. Using $\alpha = \mathcal{O}(g)$, setting the unphysical degrees of freedom γ_1 and γ_2 to 1 and using (7.41), we checked that (7.42) reduces to (7.27). Thus, the limit of the dynamic $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$-invariant two-particle S-matrix yields the $Y[\mathfrak{su}(2|2)]$-invariant S-matrix.

7.3.2 Three-Particle S-Matrix

The $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$-invariant three-particle S-matrix in its dynamic representation is the map

$$
\begin{aligned}
\mathrm{S}_{123}(u_i, v_i) : {} & (2|2)_{v_1,C(v_1),P(v_1),K(v_1)} \otimes (2|2)_{v_2,C(v_2),P(v_2),K(v_2)} \otimes (2|2)_{v_3,C(v_3),P(v_3),K(v_3)} \\
& \to (2|2)_{u_1,C(u_1),P(u_1),K(u_1)} \otimes (2|2)_{u_2,C(u_2),P(u_2),K(u_2)} \otimes (2|2)_{u_3,C(u_3),P(u_3),K(u_3)}
\end{aligned}
\tag{7.44}
$$

with $u_1 > u_2 > u_3$ and $v_1 < v_2 < v_3$.

Solution of the Level-0 Constraint

Similar to the two-particle case, the level-0 constraints of the central charges demand

$$
\begin{aligned}
& C(u_1) + C(u_2) + C(u_3) = C(v_1) + C(v_2) + C(v_3), \\
& P(u_1) + P(u_2)U(u_1)^2 + P(u_3)U(u_1)^2U(u_2)^2 \\
& \qquad\qquad = P(v_1) + P(v_2)U(v_1)^2 + P(v_3)U(v_1)^2U(v_2)^2, \\
& K(u_1) + \frac{K(u_2)}{U(u_1)^2} + \frac{K(u_3)}{U(u_1)^2U(u_2)^2} = K(v_1) + \frac{K(v_2)}{U(v_1)^2} + \frac{K(u_3)}{U(u_1)^2U(u_2)^2} .
\end{aligned}
\tag{7.45}
$$

Once more, we analyse these relations in the reparameterized form

$$x_1^+ + x_2^+ + x_3^+ - x_1^- - x_2^- - x_3^- = y_1^+ + y_2^+ + y_3^+ - y_1^- - y_2^- - y_3^- ,$$
$$\frac{x_1^+ x_2^+ x_3^+}{x_1^- x_2^- x_3^-} = \frac{y_1^+ y_2^+ y_3^+}{y_1^- y_2^- y_3^-}, \quad \frac{x_1^- x_2^- x_3^-}{x_1^+ x_2^+ x_3^+} = \frac{y_1^- y_2^- y_3^-}{y_1^+ y_2^+ y_3^+}. \tag{7.46}$$

Obviously, they are solved if the outgoing rapidities correspond to the incoming rapidities. Out of six possibilities only

$$v_1 = u_3, \quad v_2 = u_2, \quad v_3 = u_1 \tag{7.47}$$

corresponds to a true 3→3 scattering process. In order to check whether there exist more solutions, we use (7.41) and expand in g. The first three orders restrict the outgoing rapidities to be equal to the incoming rapidities. Thus, the level-0 Yangian constraints corresponding to the central charges again imply the conservation of rapidities.

Demanding vanishing commutators $[\Delta^2 \mathfrak{R}^a_b, \mathrm{S}_{123}]$ and $[\Delta^2 \mathfrak{L}^\alpha_\beta, \mathrm{S}_{123}]$ restricts the three-particle S-matrix S_{123} to be of the same form as in the discussion of the undynamic Yangian constraints in the previous section. We do not present this ansatz here since it contains 70 terms. The corresponding 70 degrees of freedom get reduced by the level-0 constraint with $\mathfrak{Q}^\alpha_a$. It restricts 60 coefficients to depend on the remaining ten. The commutator with $\Delta^2 \mathfrak{S}^a_\alpha$ further constrains these. Since the equations turned out to be very long and could not be evaluated analytically via `Mathematica` due to an insufficient memory capacity, we evaluated the results numerically for different values of the $x^\pm$ and g. Doing so, we found that there remain two degrees of freedom after imposing the level-0 constrains on the S-matrix. This is in contrast to the undynamic case where $\mathfrak{S}^a_\alpha$ puts no further constraints on S_{123}, i.e. there remained ten degrees of freedoms. Thus, similar to the two-particle case, we find that the dynamic representation of the Yangian $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$ is more restrictive than the undynamic representation of $Y[\mathfrak{su}(2|2)]$. Note that the number of degrees of freedom for the dynamic case corresponds to the expectations from the discussion of the representation theory of this algebra, see [36, 43]. Here it is shown that the tensor product of three one-particle states denoted by $\langle m = 0, n = 0, \vec{C}\rangle$ with $\mathfrak{su}(2) \times \mathfrak{su}(2)$ Dynkin labels m and n and eigenvalues $\vec{C} = (C, P, K)$ of the central charges decomposes as

$$\langle 0, 0, \vec{C}_1\rangle \otimes \langle 0, 0, \vec{C}_2\rangle \otimes \langle 0, 0, \vec{C}_3\rangle = \{1, 0, \vec{C}_1 + \vec{C}_2 + \vec{C}_3\} \oplus \{0, 1, \vec{C}_1 + \vec{C}_2 + \vec{C}_3\}. \tag{7.48}$$

The bracket $\langle ..\rangle$ denotes a state from a long multiplet and $\{..\}$ from a short multiplet with $C^2 - PK = \frac{1}{4}(n + m + 1)^2$.

Solution of the Level-1 Constraint

Since we found the three-particle $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$-invariant S-matrix only numerically, we proceed with a numerical analysis of the level-1 constraints of $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$. The constraints imposed by the central charges $\hat{\mathfrak{C}}$, $\hat{\mathfrak{K}}$ and $\hat{\mathfrak{P}}$ put no further constraints on the S-matrix. By contrast, the remaining constraints $[\Delta^2 \hat{\mathfrak{R}}^a_b, \mathrm{S}_{123}] = [\Delta^2 \hat{\mathfrak{L}}^\alpha_\beta, \mathrm{S}_{123}] = [\Delta^2 \hat{\mathfrak{Q}}^\alpha_b, \mathrm{S}_{123}] = [\Delta^2 \hat{\mathfrak{S}}^a_\beta, \mathrm{S}_{123}] = 0$ are only fulfilled if the two remaining free coefficients are related to each other. This relation is the same for all the commutators such that we are left with a three-particle S-matrix that is determined up to an overall factor. We compared our numerical result with the product of the three two-particle S-matrices in the qYBE and the equality of both expressions confirms that the three-particle S-matrix factorizes.

Discussion of the Results

Let us summarize the results of this section:

- From the level-0 Yangian constraints imposed by the central charges we found that the outgoing rapidities in two- and three-particle scattering processes correspond to the incoming rapidities.
- We determined the two-particle $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$-invariant S-matrix (7.21) with coefficients (7.42) up to the dressing factor. A discussion of this dressing factor can be found in [33].
- The two-particle S-matrix satisfies the qYBE which we checked numerically.
- We validated that $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$-invariance fixes the three-particle S-matrix up to two degrees of freedom. The level-1 generators corresponding to the Yangian algebra that is associated to this algebra further restricts the S-matrix such that it is fixed up to an overall factor. The resulting matrix factorizes consistently into two-particle S-matrices. This analysis was done numerically.
- The resulting S-matrices do not depend on relative rapidities only. Assuming a conserved set of rapidities, this feature can be conjectured from the discussion at the end of chapter 5 and 6.3.

Chapter 8

Summary and Outlook

Exploiting symmetries simplifies calculations all over physics. In this thesis we focused on symmetries in quantum integrable models and explored their connection to factorization of scattering. First, we discussed a version of the proof in [3] which reveals that factorized scattering is a direct consequence of the existence of a tower of conserved charges in a massive relativistic QFT. We were able to translate it into the language of spin chains and discussed some subtleties in this translation. This proof shows the close connection between the existence of conserved local charges and factorization of scattering in spin chain models.

We then moved on to the connection of Yangian symmetry and factorization of scattering. We used the Yangian corresponding to specific Lie algebras to constrain the dynamics of scattering processes. In particular, we checked whether factorization of the associated S-matrices is a direct consequence of the Yangian. For the Lie (super-)algebras $\mathfrak{su}(\mathrm{n})$, $\mathfrak{su}(1|1)$, $\mathfrak{su}(2|2)$ with undynamic representations we were able to calculate the two- and three-particle S-matrix up to an unknown overall factor. This dressing factor has to be determined by unitarity and crossing relations. We showed that the Yangian-invariant S-matrices preserve the sets of rapidities. This feature is often used as a standard assumption on S-matrices in literature which we confirmed for these specific Yangian-invariant S-matrices. Furthermore, we found that the S-matrices only depend on relative rapidities, i.e. they correspond to models with (quasi-)boost invariance. We were able to understand this feature on the basis of the Yangian constraints. We verified that the three-particle S-matrices factorize into three two-particle S-matrices and satisfy the qYBE as consistency condition.

In the case of the dynamic representations of $Y[\mathfrak{su}(1|1)]$ we were not able to obtain all results we were looking for due to limitations of computational power. In particular, unlike for the two-particle scattering we were not able to confirm that the Yangian constraints restrict three-particle scattering processes in such a way that the rapidities are conserved. Therefore, we assumed that the set of rapidities is conserved, i.e. we looked for solutions that permute the Hilbert spaces of the particles. We found this S-matrix and verified consistent factorization. In contrast to this, we were able to show that the dynamic representation of the Yangian corresponding to $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$ implies conservation of rapidities in two- and three-particle scattering processes. Thus, although $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$ might appear more involved than $Y[\mathfrak{su}(1|1)]$ at first glance, the higher amount of symmetry makes this Yangian more accessible in the analysis. Furthermore, the Yangian constraints fix the two-particle S-matrix up to an overall factor and the result satisfies the qYBE. Unfortunately, we were not able to analyse the three-particle S-matrix analytically but calculated it numerically. Although several numerical values for the parameters of the theory and the particles lead always to the same result, i.e. an S-matrix that is determined up to an overall factor and factorizes consistently into three two-particle S-matrices, an analytic analysis is essential for an indisputable result.

The typical number of terms from the commutators of the general ansatz for the three-particle S-matrix and the $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$-generators is $\mathcal{O}(10^3)$ up to $\mathcal{O}(10^4)$. The calculation of a single commutator takes about 30 hours when parallelizing it on 4 kernels. In order to circumvent these complications that ultimately led to a numerical analysis, it could be interesting to investigate the algebra $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ with respect to its representation theory. Doing so, one might directly come up with an ansatz for S that only contains level-0 invariants. This might give more compact constraints on the prefactors of the S-matrices and thus could facilitate their analysis. In fact, we did that in the discussion of $\mathfrak{su}(\mathrm{n})$ using the symmetric group and Young tableaux, respectively. And indeed, in [43] the authors mention a possibility to construct projection operators on the irreducible representations via a supersymmetric Young tableaux construction. Another approach is the construction of Casimir operators. This is done in [14] for the 2→2 $Y[\mathfrak{su}(1|1)]$-invariant S-matrix. Unfortunately, there is no non-vanishing Killing form for $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$, see [44], such that it is not straightforward to construct Casimirs. Nevertheless, one might circumvent this difficulty by using the exceptional algebra $D(2,1;\varepsilon)$ that was discussed in [44].

Interestingly, the resulting S-matrices in the dynamic representations of $Y[\mathfrak{su}(1|1)]$ and $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$ do not only depend on relative rapidities. Thus, the corresponding models are not invariant under the usual Lorentz (quasi-)boost. In [45] a new boost symmetry of the S-matrices of $\mathcal{N} = 4$ SYM was found. It could be interesting to investigate whether the corresponding generator explains this specific dependence on the rapidities and to understand how it is related to the Yangian generators.

In our discussion of the Yangian constraints we always assumed the conservation of the particle number in a scattering process. This is a characteristic feature of integrable models and it would be interesting to check whether it can be understood from Yangian symmetry. In order to do so, one has to analyse the constraints

$$\begin{aligned}(\Delta^{m_{in}-1}\mathfrak{J})\mathrm{S}_{m_{in}\to m_{out}}&(u_{1,\dots,m_{in}};v_{1,\dots,m_{out}})\\ &-\mathrm{S}_{m_{in}\to m_{out}}(u_{1,\dots,m_{in}};v_{1,\dots,m_{out}})(\Delta^{m_{out}-1}\mathfrak{J})=0,\end{aligned}\tag{8.1}$$

where $\mathfrak{J}$ collectively denotes both the level-0 and level-1 Yangian generators. m_{in} is the number of incoming and m_{out} the number of outgoing particles. Assuming that crossing symmetry holds, one might also transfer the 3→3 S-matrix results to the S-matrix for the scattering process with two incoming and four outgoing particles. Used recursively this might constrain all $m \to n$ S-matrices with $m \neq n$.

Appendix A

Classical Integrability

In this chapter we briefly develop important concepts of classical integrability which motivate the notion of integrability for quantum spin chains. We start by introducing the definition of Liouville integrability for Hamiltonian systems in section A.1 and then move on by reformulating classical integrability in terms of Lax pairs in section A.2. We do so in close analogy to [15], [26] and [46–48].

A.1 Liouville Integrability

Let us introduce Liouville integrability for classical mechanical systems. Its presence is a note-worthy property of a model since it implies solvability of the system's dynamics on the basis of the Hamiltonian, i.e. without investigating the equations of motion. We begin by discussing the notation of a Hamiltonian system. Its motion in d dimensions is described by a trajectory $(p(t), q(t))$ in $2d$-dimensional phase space M spanned by the canonical coordinates

$$(p_i, q_i),\ \ i = 1, 2, ..., d \tag{A.1}$$

with momenta p_i and position variables q_i. The time-evolution $(p(t), q(t))$ is governed by the system's Hamiltonian H and Hamilton's equations of motion

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}. \tag{A.2}$$

We focus on systems whose Hamiltonian is not explicitly time-dependent, i.e. $H = H(p, q)$. Take as an example the one-dimensional harmonic oscillator. It is defined by the Hamiltonian

$$H(p, q) = \tfrac{1}{2}\left(p^2 + \omega^2 q^2\right) \tag{A.3}$$

which gives rise to the equations of motion

$$\dot{p} = -\omega^2 q \qquad \dot{q} = p. \tag{A.4}$$

This set of equations can be solved by

$$p(t) = A\cos(\omega t) + B\sin(\omega t) \qquad q(t) = \tfrac{B}{\omega}\cos(\omega t) - \tfrac{A}{\omega}\sin(\omega t) \tag{A.5}$$

with constants A and B which can be fixed by imposing initial conditions. We will show in the following that this solvability can be predicted by examining the system's Hamiltonian (A.3).

The equations of motion (A.2) imply that the time derivative $\dot{F}$ of any observable $F = F(p, q)$ is given by

$$\dot{F} = \{H, F\} \tag{A.6}$$

where we introduced the Poisson brackets

$$\{f, g\} = \sum_{i=1}^{d} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right). \tag{A.7}$$

Such a function F is called a *conserved quantity* (or *first integral*) if its Poisson bracket with the Hamiltonian vanishes, i.e. $\{H, F\} = 0$. In particular, the Hamiltonian is conserved and the motion takes place on a hypersurface $E = H(p, q)$ in M. This can be easily exemplified by the harmonic oscillator (A.3). Without loss of generality we choose $A = 0$ and the system evolves on the hypersurface

$$(p(t), q(t)) = \tfrac{B}{\omega}(\omega \sin(\omega t), \cos(\omega t)) \tag{A.8}$$

which satisfies $E = H(p, q)$ with $E = B^2$.

A system's dynamics is even more constrained if there exist further first integrals that are independent, i.e. their corresponding gradient vectors are linearly independent. A system is called *Liouville integrable* if it possesses d independent conserved quantities $F_i = F_i(p, q)$, $i = 1, ..., d$ that are in involution, which means they Poisson commute

$$\{F_i, F_j\} = 0 \quad \forall\, i, j \in \{1, ..., d\}. \tag{A.9}$$

One can show that for such a system the motion is restricted to a d-dimensional hypersurface $F_i = F_i(p, q)$, $i = 1, ..., d$, in phase space that is diffeomorphic to a d-dimensional torus.

By this definition every one-dimensional system with time-independent Hamiltonian is integrable since one can always choose the Hamiltonian itself as the conserved quantity F_1. In particular, the previously discussed harmonic oscillator is integrable and the dynamics (A.8) takes place on a one-dimensional torus, i.e. an ellipse. A slightly more involved system is the d-dimensional harmonic oscillator

$$H = \tfrac{1}{2} \sum_{i=1}^{d} \left(p_i^2 + \omega_i^2 q_i^2 \right). \tag{A.10}$$

Choosing $F_i = p_i^2 + \omega_i^2 q_i^2$ for $i = 1, ..., d$ one can easily prove integrability by confirming their conservation ($\{H, F_i\} = 0\ \forall\, i$), the involution property ($\{F_i, F_j\} = 0\ \forall\, i, j$) and their independence ($\vec{\nabla} F_i \cdot \vec{\nabla} F_j = 0\ \forall\, i \neq j$).

The *Liouville theorem* establishes that Liouville integrability of classical systems results in the solvability of its equations of motion. There even exists a procedure known as solution by quadrature that enables us to find the solution of each integrable system by solving a couple of algebraic equations and integrals. Therefore Liouville integrability is a powerful concept enabling us to obtain a deeper understanding of the solvability of a large family of physical systems.

A.2 Lax Pairs and R-Matrices

We will now introduce another approach to classical integrability that is not based on the system's Hamiltonian but on its *Lax pair* (L, M). This object consists of two time-dependent matrices L and M that allow us to rewrite the equations of motion as the *Lax equation*

$$\dot{L} = [L, M] \tag{A.11}$$

with commutator $[.,.]$. Let us illustrate this with the help of the one-dimensional harmonic oscillator. Alternatively to (A.3) we may define this model via the Lax pair

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix} \qquad M = \begin{pmatrix} 0 & \frac{1}{2}\omega \\ -\frac{1}{2}\omega & 0 \end{pmatrix} \tag{A.12}$$

which gives rise to the equations of motion (A.2) by the Lax equation. In general, it might not be easy to find the Lax pair of a physical model. Nevertheless, every integrable system admits one which can be proven by its construction via the so-called action-angle variables. Unfortunately, this procedure is of no real practical use since we first have to solve the system's dynamics before obtaining the Lax pair.

We will now discuss how to transfer the criteria of Liouville integrability to a physical system defined by a Lax pair. The central objects in the previous discussion of integrability were conserved quantities. They can be constructed by tracing over powers of L

$$f_k = \mathrm{tr} L^k \qquad k \in \mathbb{N}. \tag{A.13}$$

These f_k are constant due to the equations of motion and the cyclicity of the trace

$$\dot{f}_k = k \mathrm{tr}\left([L, M] L^{k-1}\right) \equiv 0. \tag{A.14}$$

In order to show that a given Lax pair defines an integrable system, we need to show that we can obtain d conserved quantities f_k that are in involution. First let us suppose L is a diagonalizable square matrix of size D, i.e.

$$L = A \Lambda A^{-1} \qquad \text{with} \quad \Lambda = \mathrm{diag}(l_1, l_2, ..., l_D) \tag{A.15}$$

where the $l_n,\ n = 1, ..., D$, are the eigenvalues of L and A is an invertible matrix. As a result we obtain for the first integrals in (A.13)

$$f_k = \mathrm{tr} \Lambda^k = \sum_{n=1}^{D} l_n^k \tag{A.16}$$

and we realize that the eigenvalues of L are conserved quantities. For a Liouville integrable system d out of these D quantities are independent. For the previously discussed harmonic oscillator (A.12) we obtain the Hamiltonian (A.3) as first integral

$$f_2 = 2(p^2 + \omega^2 q^2) = 4H. \tag{A.17}$$

Further f_k either vanish or linearly depend on f_2.

Now let us find out whether the eigenvalues of L are in involution. In order to do so, an analogue of the Poisson brackets for matrices is needed. For this purpose, we decompose L into its entries L^{ij} being functions on phase space and the basis E_{ij} of some matrix algebra $\mathfrak{g}$ containing the Lax pair with $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$

$$L = L^{ij} E_{ij}. \tag{A.18}$$

Elements defined on the double tensor product $\mathfrak{g} \otimes \mathfrak{g}$ are denoted as

$$\begin{aligned} X_1 &:= X \otimes \mathbb{I} = X^{ij} E_{ij} \otimes \mathbb{I} \qquad & X_2 &:= \mathbb{I} \otimes X = X^{ij}\ \mathbb{I} \otimes E_{ij} \\ X_{12} &:= X^{ij,kl}\ E_{ij} \otimes E_{kl} & X_{21} &:= \mathbb{P}_{12}\ X_{12}\ \mathbb{P}_{12} = X^{ij,kl}\ E_{kl} \otimes E_{ij} \end{aligned} \tag{A.19}$$

with the $D \times D$ identity matrix $\mathbb{I}$ and the permutation operator $\mathbb{P}_{12} = \mathbb{P}_{12}^{-1}$ on spaces 1 and 2. Now we can define the Poisson bracket $\{L_1, L_2\}$ of matrices via the usual Poisson brackets of its entries

$$\{L_1, L_2\} := \{L^{ij}, L^{kl}\}\ E_{ij} \otimes E_{kl}. \tag{A.20}$$

It can be proven (see e.g. [1]) that the eigenvalues l_k of the Lax matrix L are in involution iff there exists a matrix $r_{12} \in \mathfrak{g} \otimes \mathfrak{g}$ on phase-space such that

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{A.21}$$

In the case of the harmonic oscillator one finds that the matrix

$$r_{12} = \frac{1}{q}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \frac{1}{q}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{A.22}$$

satisfies (A.21) with Lax matrix L given in (A.12).

This so-called *r-matrix* has to obey a constraint arising from the usual Jacobi identity for the Poisson bracket $\{\{L_1, L_2\}, L_3\} = \{\{L^i_j, L^k_l\}, L^m_n\}\ E^j_i \otimes E^l_k \otimes E^n_m$. A lot of integrable systems allow for an r-matrix that is constant and antisymmetric $r_{12} = -r_{21}$. Then this constraint reduces to the *classical Yang-Baxter equation* (cYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{A.23}$$

Thus for an integrable system defined by a Lax pair there must exist d linearly independent eigenvalues of L and an r-matrix obeying (A.21) and the cYBE (A.23) for the constant and antisymmetric case.

Appendix B

The Algebraic Bethe Ansatz

We will now discuss a variation of the Coordinate Bethe Ansatz using the framework of the Lax operator introduced in 2.3. We will not investigate it in detail but sketch some results with the help of [15] and [16]. The central idea in this approach is that we can simultaneously find the spectrum of all the conserved quantities by diagonalizing the transfer matrix. We start by considering the monodromy $\mathrm{T}_a(\lambda)$ as a 2×2-matrix in auxiliary space

$$\mathrm{T}_a(\lambda) = \begin{pmatrix} \mathrm{A}(\lambda) & \mathrm{B}(\lambda) \\ \mathrm{C}(\lambda) & \mathrm{D}(\lambda) \end{pmatrix} . \tag{B.1}$$

Examining the operators A, B, C and D in detail and looking at their commutation relations imposed by the RTT-relation suggests to define the reference state by

$$\mathrm{C}(\lambda) \left|0\right\rangle = 0 \tag{B.2}$$

and the excited m-magnon states by

$$\left|\lambda_1, \lambda_2, ..., \lambda_m\right\rangle = \mathrm{B}(\lambda_1)\mathrm{B}(\lambda_2)...\mathrm{B}(\lambda_m) \left|0\right\rangle . \tag{B.3}$$

Acting with the transfer matrix $t(\lambda) = \mathrm{A}(\lambda) + \mathrm{D}(\lambda)$ on these states and using commutation relations between the operators in (B.1) obtained from (2.54), we find that the states (B.3) are indeed the eigenstates of $t(\lambda)$ with eigenvalues[1]

$$\Lambda(\lambda) = (-i)^N(\lambda + \tfrac{i}{2})^N \prod_j \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} + (-i)^N(\lambda - \tfrac{i}{2})^N \prod_j \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j} \tag{B.4}$$

if the λ's satisfy the relation

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^N = \prod_{\substack{k=1 \\ k \neq j}}^{m} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \tag{B.5}$$

which corresponds to the Bethe equations (2.35). Let us compare the results for the momentum operator and Hamiltonian of the Coordinate and Algebraic Bethe Ansatz. The momentum can be found by using (2.47) and inserting the eigenvalue of the transfer matrix

$$k = \sum_j k_j = \sum_j \left(-i \ln \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right) . \tag{B.6}$$

[1] See e.g. [15] for a proof.

Comparing this result with (2.24) we find that the Bethe root λ_j corresponds to the rapidity u_j. The energy computed using (2.51)

$$E = \frac{1}{2}\sum_{j=1}^{m}\frac{1}{\lambda_j^2 + \frac{1}{4}} \tag{B.7}$$

is in agreement with the previous result.

Appendix C

The Yangian Coproducts on Length 3 for $Y[\mathfrak{su}(1|1)]$ and $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$

Here we list the coproduct structure for the Yangian generators of $Y[\mathfrak{su}(1|1)]$ and $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$ on length 3. It can be obtained from the coproducts on length 2 given in sections 6.1 and 7.1 by considering all fermionic gradings $(-1)^F$ and the braiding factor $\mathcal{U}$. For $Y[\mathfrak{su}(1|1)]$ we obtain

$$\begin{aligned}
\Delta^2\mathfrak{C} &= \mathfrak{C} \otimes 1 \otimes 1 + 1 \otimes \mathfrak{C} \otimes 1 + 1 \otimes 1 \otimes \mathfrak{C}, \\
\Delta^2\mathfrak{B} &= \mathfrak{B} \otimes 1 \otimes 1 + 1 \otimes \mathfrak{B} \otimes 1 + 1 \otimes 1 \otimes \mathfrak{B}, \\
\Delta^2\mathfrak{Q} &= \mathfrak{Q} \otimes 1 \otimes 1 + (-1)^F \otimes \mathfrak{Q} \otimes 1 + (-1)^F \otimes (-1)^F \otimes \mathfrak{Q}, \\
\Delta^2\mathfrak{S} &= \mathfrak{S} \otimes 1 \otimes 1 + (-1)^F \otimes \mathfrak{S} \otimes 1 + (-1)^F \otimes (-1)^F \otimes \mathfrak{S}
\end{aligned} \tag{C.1}$$

and

$$\begin{aligned}
\Delta^2\hat{\mathfrak{C}} &= \hat{\mathfrak{C}} \otimes 1 \otimes 1 + 1 \otimes \hat{\mathfrak{C}} \otimes 1 + 1 \otimes 1 \otimes \hat{\mathfrak{C}}, \\
\Delta^2\hat{\mathfrak{B}} &= \hat{\mathfrak{B}} \otimes 1 \otimes 1 + 1 \otimes \hat{\mathfrak{B}} \otimes 1 + 1 \otimes 1 \otimes \mathfrak{B} \\
&\quad - (-1)^F\mathfrak{S} \otimes \mathfrak{Q} \otimes 1 - (-1)^F\mathfrak{S} \otimes (-1)^F \otimes \mathfrak{Q} - 1 \otimes (-1)^F\mathfrak{S} \otimes \mathfrak{Q} \\
&\quad - (-1)^F\mathfrak{Q} \otimes \mathfrak{S} \otimes 1 - (-1)^F\mathfrak{Q} \otimes (-1)^F \otimes \mathfrak{S} - 1 \otimes (-1)^F\mathfrak{Q} \otimes \mathfrak{S}, \\
\Delta^2\hat{\mathfrak{Q}} &= \hat{\mathfrak{Q}} \otimes 1 \otimes 1 + (-1)^F \otimes \hat{\mathfrak{Q}} \otimes 1 + (-1)^F \otimes (-1)^F \otimes \hat{\mathfrak{Q}} \\
&\quad + \tfrac{1}{2}\mathfrak{Q} \otimes \mathfrak{C} \otimes 1 + \tfrac{1}{2}\mathfrak{Q} \otimes 1 \otimes \mathfrak{C} + \tfrac{1}{2}(-1)^F \otimes \mathfrak{Q} \otimes \mathfrak{C} \\
&\quad - \tfrac{1}{2}(-1)^F\mathfrak{C} \otimes \mathfrak{Q} \otimes 1 - \tfrac{1}{2}(-1)^F\mathfrak{C} \otimes (-1)^F \otimes \mathfrak{Q} - \tfrac{1}{2}(-1)^F \otimes (-1)^F\mathfrak{C} \otimes \mathfrak{Q}, \\
\Delta^2\hat{\mathfrak{S}} &= \hat{\mathfrak{S}} \otimes 1 \otimes 1 + (-1)^F \otimes \hat{\mathfrak{S}} \otimes 1 + (-1)^F \otimes (-1)^F \otimes \hat{\mathfrak{S}} \\
&\quad - \tfrac{1}{2}\mathfrak{S} \otimes \mathfrak{C} \otimes 1 - \tfrac{1}{2}\mathfrak{S} \otimes 1 \otimes \mathfrak{C} - \tfrac{1}{2}(-1)^F \otimes \mathfrak{S} \otimes \mathfrak{C} \\
&\quad + \tfrac{1}{2}(-1)^F\mathfrak{C} \otimes \mathfrak{S} \otimes 1 + \tfrac{1}{2}(-1)^F\mathfrak{C} \otimes (-1)^F \otimes \mathfrak{S} + \tfrac{1}{2}(-1)^F \otimes (-1)^F\mathfrak{C} \otimes \mathfrak{S}.
\end{aligned} \tag{C.2}$$

The analogous results for the Yangian corresponding to $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ are

$$\begin{aligned}
\Delta^2\mathfrak{C} &= \mathfrak{C} \otimes 1 \otimes 1 + 1 \otimes \mathfrak{C} \otimes 1 + 1 \otimes 1 \otimes \mathfrak{C}, \\
\Delta^2\mathfrak{P} &= \mathfrak{P} \otimes 1 \otimes 1 + \mathcal{U}^{+2} \otimes \mathfrak{P} \otimes 1 + \mathcal{U}^{+2} \otimes \mathcal{U}^{+2} \otimes \mathfrak{P}, \\
\Delta^2\mathfrak{K} &= \mathfrak{K} \otimes 1 \otimes 1 + \mathcal{U}^{-2} \otimes \mathfrak{K} \otimes 1 + \mathcal{U}^{-2} \otimes \mathcal{U}^{-2} \otimes \mathfrak{K},
\end{aligned}$$

$$
\begin{aligned}
\Delta^2\mathfrak{R}^a_b &= \mathfrak{R}^a_b \otimes 1 \otimes 1 + 1 \otimes \mathfrak{R}^a_b \otimes 1 + 1 \otimes 1 \otimes \mathfrak{R}^a_b,\\
\Delta^2\mathfrak{L}^\alpha_\beta &= \mathfrak{L}^\alpha_\beta \otimes 1 \otimes 1 + 1 \otimes \mathfrak{L}^\alpha_\beta \otimes 1 + 1 \otimes 1 \otimes \mathfrak{L}^\alpha_\beta,\\
\Delta^2\mathfrak{Q}^\alpha_b &= \mathfrak{Q}^\alpha_b \otimes 1 \otimes 1 + \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_b \otimes 1 + \mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_b,\\
\Delta^2\mathfrak{S}^a_\beta &= \mathfrak{S}^a_\beta \otimes 1 \otimes 1 + \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\beta \otimes 1 + \mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\beta.
\end{aligned} \tag{C.3}
$$

and

$$
\begin{aligned}
\Delta^2\hat{\mathfrak{C}} =&\hat{\mathfrak{C}} \otimes 1 \otimes 1 + 1 \otimes \hat{\mathfrak{C}} \otimes 1 + 1 \otimes 1 \otimes \hat{\mathfrak{C}}\\
&+ \tfrac{1}{2}\mathfrak{P}\,\mathcal{U}^{-2} \otimes \mathfrak{K} \otimes 1 + \tfrac{1}{2}\mathfrak{P}\,\mathcal{U}^{-2} \otimes \mathcal{U}^{-2} \otimes \mathfrak{K} + \tfrac{1}{2} \otimes \mathfrak{P}\,\mathcal{U}^{-2} \otimes \mathfrak{K}\\
&- \tfrac{1}{2}\mathfrak{K}\,\mathcal{U}^{+2} \otimes \mathfrak{P} \otimes 1 - \tfrac{1}{2}\mathfrak{K}\,\mathcal{U}^{+2} \otimes \mathcal{U}^{+2} \otimes \mathfrak{P} - \tfrac{1}{2} \otimes \mathfrak{K}\,\mathcal{U}^{+2} \otimes \mathfrak{P},\\
\Delta^2\hat{\mathfrak{P}} =&\hat{\mathfrak{P}} \otimes 1 \otimes 1 + \mathcal{U}^{+2} \otimes \hat{\mathfrak{P}} \otimes 1 + \mathcal{U}^{+2} \otimes \mathcal{U}^{+2} \otimes \hat{\mathfrak{P}}\\
&- \mathfrak{C}\,\mathcal{U}^{+2} \otimes \mathfrak{P} \otimes 1 - \mathfrak{C}\,\mathcal{U}^{+2} \otimes \mathcal{U}^{+2} \otimes \mathfrak{P} - 1 \otimes \mathfrak{C}\,\mathcal{U}^{+2} \otimes \mathfrak{P}\\
&+ \mathfrak{P} \otimes \mathfrak{C} \otimes 1 + \mathfrak{P} \otimes 1 \otimes \mathfrak{C} + \mathcal{U}^{+2} \otimes \mathfrak{P} \otimes \mathfrak{C},\\
\Delta^2\hat{\mathfrak{K}} =&\hat{\mathfrak{K}} \otimes 1 \otimes 1 + \mathcal{U}^{-2} \otimes \hat{\mathfrak{K}} \otimes 1 + \mathcal{U}^{-2} \otimes \mathcal{U}^{-2} \otimes \hat{\mathfrak{K}}\\
&+ \mathfrak{C}\,\mathcal{U}^{-2} \otimes \mathfrak{K} \otimes 1 + \mathfrak{C}\,\mathcal{U}^{-2} \otimes \mathcal{U}^{-2} \otimes \mathfrak{K} + \mathcal{U}^{-2} \otimes \mathfrak{C}\,\mathcal{U}^{-2} \otimes \mathfrak{K}\\
&- \mathfrak{K} \otimes \mathfrak{C} \otimes 1 - \mathfrak{K} \otimes 1 \otimes \mathfrak{C} - \mathcal{U}^{-2} \otimes \mathfrak{K} \otimes \mathfrak{C},\\
\Delta^2\hat{\mathfrak{R}}^a_b =&\hat{\mathfrak{R}}^a_b \otimes 1 \otimes 1 + 1 \otimes \hat{\mathfrak{R}}^a_b \otimes 1 + 1 \otimes 1 \otimes \hat{\mathfrak{R}}^a_b\\
&+ \tfrac{1}{2}\mathfrak{R}^a_c \otimes \mathfrak{R}^c_b \otimes 1 + \tfrac{1}{2}\mathfrak{R}^a_c \otimes 1 \otimes \mathfrak{R}^c_b + \tfrac{1}{2} \otimes \mathfrak{R}^a_c \otimes \mathfrak{R}^c_b\\
&- \tfrac{1}{2}\mathfrak{R}^c_b \otimes \mathfrak{R}^a_c \otimes 1 - \tfrac{1}{2}\mathfrak{R}^c_b \otimes 1 \otimes \mathfrak{R}^a_c - \tfrac{1}{2} \otimes \mathfrak{R}^c_b \otimes \mathfrak{R}^a_c\\
&- \tfrac{1}{2}\mathfrak{S}^a_\gamma\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_b \otimes 1 - \tfrac{1}{2}\mathfrak{S}^a_\gamma\,\mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_b - \tfrac{1}{2} \otimes \mathfrak{S}^a_\gamma\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_b\\
&- \tfrac{1}{2}\mathfrak{Q}^\gamma_b\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\gamma \otimes 1 - \tfrac{1}{2}\mathfrak{Q}^\gamma_b\,\mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\gamma - \tfrac{1}{2} \otimes \mathfrak{Q}^\gamma_b\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\gamma\\
&+ \tfrac{1}{4}\delta^a_b\;\mathfrak{S}^d_\gamma\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_d \otimes 1 + \tfrac{1}{4}\delta^a_b\;\mathfrak{S}^d_\gamma\,\mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_d + \tfrac{1}{4}\delta^a_b \otimes \mathfrak{S}^d_\gamma\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_d\\
&+ \tfrac{1}{4}\delta^a_b\;\mathfrak{Q}^\gamma_d\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^d_\gamma \otimes 1 + \tfrac{1}{4}\delta^a_b\;\mathfrak{Q}^\gamma_d\,\mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^d_\gamma + \tfrac{1}{4}\delta^a_b \otimes \mathfrak{Q}^\gamma_d\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^d_\gamma,\\
\Delta^2\hat{\mathfrak{L}}^\alpha_\beta =&\hat{\mathfrak{L}}^\alpha_\beta \otimes 1 \otimes 1 + 1 \otimes \hat{\mathfrak{L}}^\alpha_\beta \otimes 1 + 1 \otimes 1 \otimes \hat{\mathfrak{L}}^\alpha_\beta\\
&- \tfrac{1}{2}\mathfrak{L}^\alpha_\gamma \otimes \mathfrak{L}^\gamma_\beta \otimes 1 - \tfrac{1}{2}\mathfrak{L}^\alpha_\gamma \otimes 1 \otimes \mathfrak{L}^\gamma_\beta - \tfrac{1}{2} \otimes \mathfrak{L}^\alpha_\gamma \otimes \mathfrak{L}^\gamma_\beta\\
&+ \tfrac{1}{2}\mathfrak{L}^\gamma_\beta \otimes \mathfrak{L}^\alpha_\gamma \otimes 1 + \tfrac{1}{2}\mathfrak{L}^\gamma_\beta \otimes 1 \otimes \mathfrak{L}^\alpha_\gamma + \tfrac{1}{2} \otimes \mathfrak{L}^\gamma_\beta \otimes \mathfrak{L}^\alpha_\gamma\\
&+ \tfrac{1}{2}\mathfrak{Q}^\alpha_c\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\beta \otimes 1 + \tfrac{1}{2}\mathfrak{Q}^\alpha_c\,\mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\beta + \tfrac{1}{2} \otimes \mathfrak{Q}^\alpha_c\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\beta\\
&+ \tfrac{1}{2}\mathfrak{S}^c_\beta\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_c \otimes 1 + \tfrac{1}{2}\mathfrak{S}^c_\beta\,\mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_c + \tfrac{1}{2} \otimes \mathfrak{S}^c_\beta\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_c\\
&- \tfrac{1}{4}\delta^\alpha_\beta\;\mathfrak{Q}^\delta_c\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\delta \otimes 1 - \tfrac{1}{4}\delta^\alpha_\beta\;\mathfrak{Q}^\delta_c\,\mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\delta - \tfrac{1}{4}\delta^\alpha_\beta \otimes \mathfrak{Q}^\delta_c\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\delta\\
&- \tfrac{1}{4}\delta^\alpha_\beta\;\mathfrak{S}^c_\delta\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\delta_c \otimes 1 - \tfrac{1}{4}\delta^\alpha_\beta\;\mathfrak{S}^c_\delta\,\mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\delta_c - \tfrac{1}{4}\delta^\alpha_\beta \otimes \mathfrak{S}^c_\delta\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\delta_c,\\
\Delta^2\hat{\mathfrak{Q}}^\alpha_b =&\hat{\mathfrak{Q}}^\alpha_b \otimes 1 \otimes 1 + \mathcal{U}_F^{+1} \otimes \hat{\mathfrak{Q}}^\alpha_b \otimes 1 + \mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \hat{\mathfrak{Q}}^\alpha_b\\
&- \tfrac{1}{2}\mathfrak{L}^\alpha_\gamma\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_b \otimes 1 - \tfrac{1}{2}\mathfrak{L}^\alpha_\gamma\,\mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_b - \tfrac{1}{2}\mathcal{U}_F^{+1} \otimes \mathfrak{L}^\alpha_\gamma\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_b\\
&+ \tfrac{1}{2}\mathfrak{Q}^\gamma_b \otimes \mathfrak{L}^\alpha_\gamma \otimes 1 + \tfrac{1}{2}\mathfrak{Q}^\gamma_b \otimes 1 \otimes \mathfrak{L}^\alpha_\gamma + \tfrac{1}{2}\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\gamma_b \otimes \mathfrak{L}^\alpha_\gamma\\
&- \tfrac{1}{2}\mathfrak{R}^c_b\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_c \otimes 1 - \tfrac{1}{2}\mathfrak{R}^c_b\,\mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_c - \tfrac{1}{2}\mathcal{U}_F^{+1} \otimes \mathfrak{R}^c_b\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_c\\
&+ \tfrac{1}{2}\mathfrak{Q}^\alpha_c \otimes \mathfrak{R}^c_b \otimes 1 + \tfrac{1}{2}\mathfrak{Q}^\alpha_c \otimes 1 \otimes \mathfrak{R}^c_b + \tfrac{1}{2}\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_c \otimes \mathfrak{R}^c_b\\
&- \tfrac{1}{2}\mathfrak{C}\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_b \otimes 1 - \tfrac{1}{2}\mathfrak{C}\,\mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_b - \tfrac{1}{2}\mathcal{U}_F^{+1} \otimes \mathfrak{C}\,\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_b\\
&+ \tfrac{1}{2}\mathfrak{Q}^\alpha_b \otimes \mathfrak{C} \otimes 1 + \tfrac{1}{2}\mathfrak{Q}^\alpha_b \otimes 1 \otimes \mathfrak{C} + \tfrac{1}{2}\mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\alpha_b \otimes \mathfrak{C}\\
&+ \tfrac{1}{2}\varepsilon^{\alpha\gamma}\varepsilon_{bd}\left(\mathfrak{P}\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^d_\gamma \otimes 1 + \mathfrak{P}\,\mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^d_\gamma + \mathcal{U}_F^{+1} \otimes \mathfrak{P}\,\mathcal{U}_F^{-1} \otimes \mathfrak{S}^d_\gamma\right)\\
&- \tfrac{1}{2}\varepsilon^{\alpha\gamma}\varepsilon_{bd}\left(\mathfrak{S}^d_\gamma\,\mathcal{U}^{+2} \otimes \mathfrak{P} \otimes 1 + \mathfrak{S}^d_\gamma\,\mathcal{U}^{+2} \otimes \mathcal{U}^{+2} \otimes \mathfrak{P} + \mathcal{U}_F^{+1} \otimes \mathfrak{S}^d_\gamma\,\mathcal{U}^{+2} \otimes \mathfrak{P}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Delta^2 \hat{\mathfrak{S}}^a_\beta = & \hat{\mathfrak{S}}^a_\beta \otimes 1 \otimes 1 + \mathcal{U}_F^{-1} \otimes \hat{\mathfrak{S}}^a_\beta \otimes 1 + \mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \hat{\mathfrak{S}}^a_\beta \\
& + \tfrac{1}{2} \mathfrak{R}^a_c \, \mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\beta \otimes 1 + \tfrac{1}{2} \mathfrak{R}^a_c \, \mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\beta + \tfrac{1}{2} \mathcal{U}_F^{-1} \otimes \mathfrak{R}^a_c \, \mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\beta \\
& - \tfrac{1}{2} \mathfrak{S}^c_\beta \otimes \mathfrak{R}^a_c \otimes 1 - \tfrac{1}{2} \mathfrak{S}^c_\beta \otimes 1 \otimes \mathfrak{R}^a_c - \tfrac{1}{2} \mathcal{U}_F^{-1} \otimes \mathfrak{S}^c_\beta \otimes \mathfrak{R}^a_c \\
& + \tfrac{1}{2} \mathfrak{L}^\gamma_\beta \, \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\gamma \otimes 1 + \tfrac{1}{2} \mathfrak{L}^\gamma_\beta \, \mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\gamma + \tfrac{1}{2} \mathcal{U}_F^{-1} \otimes \mathfrak{L}^\gamma_\beta \, \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\gamma \\
& - \tfrac{1}{2} \mathfrak{S}^a_\gamma \otimes \mathfrak{L}^\gamma_\beta \otimes 1 - \tfrac{1}{2} \mathfrak{S}^a_\gamma \otimes 1 \otimes \mathfrak{L}^\gamma_\beta - \tfrac{1}{2} \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\gamma \otimes \mathfrak{L}^\gamma_\beta \\
& + \tfrac{1}{2} \mathfrak{C} \, \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\beta \otimes 1 + \tfrac{1}{2} \mathfrak{C} \, \mathcal{U}_F^{-1} \otimes \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\beta + \tfrac{1}{2} \mathcal{U}_F^{-1} \otimes \mathfrak{C} \, \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\beta \\
& - \tfrac{1}{2} \mathfrak{S}^a_\beta \otimes \mathfrak{C} \otimes 1 - \tfrac{1}{2} \mathfrak{S}^a_\beta \otimes 1 \otimes \mathfrak{C} - \tfrac{1}{2} \mathcal{U}_F^{-1} \otimes \mathfrak{S}^a_\beta \otimes \mathfrak{C} \\
& - \tfrac{1}{2} \varepsilon^{ac} \varepsilon_{\beta\delta} \left(\mathfrak{K} \, \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\delta_c \otimes 1 + \mathfrak{K} \, \mathcal{U}_F^{+1} \otimes \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\delta_c + \mathcal{U}_F^{-1} \otimes \mathfrak{K} \, \mathcal{U}_F^{+1} \otimes \mathfrak{Q}^\delta_c \right) \\
& + \tfrac{1}{2} \varepsilon^{ac} \varepsilon_{\beta\delta} \left(\mathfrak{Q}^\delta_c \, \mathcal{U}^{-2} \otimes \mathfrak{K} \otimes 1 + \mathfrak{Q}^\delta_c \, \mathcal{U}^{-2} \otimes \mathcal{U}^{-2} \otimes \mathfrak{K} + \mathcal{U}_F^{-1} \otimes \mathfrak{Q}^\delta_c \, \mathcal{U}^{-2} \otimes \mathfrak{K} \right) .
\end{aligned}
\tag{C.4}
$$

Appendix D

Implementation in Mathematica

In this appendix we give a brief insight into one possibility to implement and analyse the Yangian constraints (4.18) and (4.19) on S-matrices in the symbolic computation program `Mathematica`. We do so by showing small segments of the complete code. These shall demonstrate the general procedure when carrying out such an analysis. The discussion focuses on the Yangian corresponding to $\mathfrak{su}(\mathrm{n})$ since its implementation is less subtle than for $Y[\mathfrak{su}(1|1)]$ and $Y[\mathfrak{su}(2|2)]$ because it only allows for a single type of particles. Nevertheless, we also show important aspects in the implementation of Yangians corresponding to Lie superalgebras.

D.1 Generators and Commutators

Generators of $\mathfrak{su}(\mathbf{n})$

We begin by introducing a notation for the fundamental representation of the generators $\mathfrak{R}^a_b$ of $\mathfrak{su}(\mathrm{n})$. We translate their bra-ket notation given in equation (5.4) via defining a `Mathematica`-function called `R[..]` with

```
R[a_,b_][c_]:=Sub[{a},{b}]-1/n KD[a,b] Sub[{c},{c}];.        (D.1)
```

Note that `Mathematica` does not have any function implemented that is called `Sub` such that it leaves the above `Sub`s unevaluated. The first argument of `Sub` corresponds to the ket $|\,\rangle$ and the second argument to the bra $\langle\,|$ of the operator in bra-ket notation. The indices `a` and `b` in `R` correspond to the free indices of $\mathfrak{R}^a_b$, `c` is a summation index. For the generators of $\mathfrak{su}(\mathrm{n})$, the indices `a`,`b` and `c` may take the values `1,2,..,n` with `n` being the degree of $\mathfrak{su}(\mathrm{n})$. The function `KD[..]` is the Kronecker-Delta which evaluates to the `Mathematica`-implemented Kronecker-Delta when inserting integers

```
KD[a_Integer,b_Integer]:=KroneckerDelta[a,b];.        (D.2)
```

Note that we do not include the rapidities u inside the `Sub`s in contrast to the bra-ket notation we used in the chapters 5-7. They only occur explicitly in the implementation of coproducts of level-1 generators in the evaluation representation. Furthermore for the dynamic representations of $Y[\mathfrak{su}(1|1)]$ and $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$, we record the rapidity-dependence of the representation parameters and braiding factors.

Multiplication and Commutators

As a next step towards the implementation of Yangian constraints, we build a `Mathematica`-routine that multiplies operators such as (D.1). Note that in bra-ket notation a multiplication of two operators $|a,b\rangle\langle c,d|$ and $|k,l\rangle\langle m,n|$ is simply

$$|a,b\rangle\langle c,d|\ |k,l\rangle\langle m,n| = \delta_{c,k}\delta_{d,l}\,|a,b\rangle\langle m,l|\,. \tag{D.3}$$

Such an operation can be implemented into `Mathematica` via `Multiply[..]` with

```
Multiply[AA_,BB_]:=Expand[AA/.A_Sub:>(BB/.B_Sub:>Multiply12[A,B])];
Multiply12[A_Sub,B_Sub]:=KDs[B[[1]],A[[2]]]Sub[A[[1]],B[[2]]];
KDs[C_,D_]:=Product[KD[C[[i]],D[[i]]],{i,1,Length[C]}];.
```
(D.4)

This routine can multiply linear combinations of operators. Commutators are implemented as

```
Commute[A_,B_]:=Multiply[A,B]-Multiply[B,A];.
```
(D.5)

As a first test we verify that the generators (D.1) for n = 2 span the Lie algebra $\mathfrak{su}(2)$. We explicitly check the relations (5.10) via

```
In[1]:=  Commute[R[1,1][c],R[2,1][d]//UseKD
         -R[2,1][c]
Out[1]=  -Sub[{2},{1}]
Out[2]=  -Sub[{2},{1}]

In[3]:=  Commute[R[1,1][c],R[1,2][d]]//UseKD
         +R[1,2][c]
Out[3]=  Sub[{1},{2}]
Out[4]=  Sub[{1},{2}]

In[5]:=  Commute[R[2,1][c],R[1,2][d]]
         -2R[1,1][c]/.n->2//SumOverIndices[2]
Out[5]=  -Sub[{1},{1}]+Sub[{2},{2}]
Out[6]=  -Sub[{1},{1}]+Sub[{2},{2}]
```
(D.6)

and find that they are satisfied. `UseKD[..]` is a self-implemented routine that simplifies terms by using the rules of Kronecker-Deltas, e.g.

```
In[7]:=  KD[c,1] Sub[{2},{c}]//UseKD
Out[7]=  Sub[{2},{1}].
```
(D.7)

`SumOverIndices[m]` is a method that sums over `1,...,m` for all indices that occur twice in an expression.

Fermionic Generators

In chapters 6 and 7 we analyse algebras including bosons and fermions in their fundamental representation. One can use Latin indices for bosons and Greek indices for fermions to distinguish between both types of particles. Then the fermionic $\mathfrak{su}(2)$ generator $\mathfrak{L}^\alpha_\beta$ can be implemented analogously to (D.1) via

```
L[α_,β_][δ_]:=Sub[{α},{β}]-1/2 KD[α,β] Sub[{δ},{δ}];.
```
(D.8)

Calculating the commutator of the operators in (D.1) and (D.8) using the routine (D.5) yields terms including Kronecker-Deltas of bosonic and fermionic indices, e.g. `KD[α,b]`. The routine `EraseKD[..]`

```
EraseKD[f_]:=f/.KD[a_,b_]:>If[LetterType[a]!=LetterType[b],0,KD[a,b]];
LetterType[a_]:=Module[{s},
  s=ToString[a];
  If[MemberQ[Union[Alphabet["English"],
    ToUpperCase[Alphabet["English"]]],s],1,
   If[MemberQ[Union[Alphabet["Greek"],
     ToUpperCase[Alphabet["Greek"]]],s],2,0]]];
SetAttributes[LetterType,Listable];
```
(D.9)

evaluates them to zero. Doing so, it checks what types of indices the Kronecker-Delta `KD[..]` contains by using the function `LetterType[..]`. If these types match, `EraseKD[..]` returns the original `KD[..]` and otherwise puts it to zero. The method `LetterType` returns `1` for bosonic indices and `2` for fermionic ones, e.g.

```
In[8]:=  LetterType[{a,B,α,Δ}]
Out[8]= {1,1,2,2}.
```
(D.10)

Thus we can check $[\mathfrak{R}^a_b, \mathfrak{L}^\alpha_\beta] = 0$ by running

```
In[9]:=  Commute[R[a,b][c],L[α,β][γ]]//EraseKD
Out[9]= 0.
```
(D.11)

In this framework one can also implement central charges and supersymmetry generators such as $\mathfrak{Q}^\alpha_a$ from (7.4) which is given by

```
Q[α_,a_][d_,δ]_]:=Ca Sub[{α},{a}]+Cb Eps[α,δ] Eps[a,d] Sub[{d},{δ}].
```
(D.12)

Here the constants `Ca, Cb` denote the coefficients a, b in the representation of $\mathfrak{Q}^\alpha_a$. `Eps[a,b]` corresponds to the epsilon symbol ε^{ab}.

D.2 Coproducts of Generators

The coproducts of level-0 and level-1 generators are crucial for the constraints (4.18) and (4.19). Let us show the procedure of implementing these by exemplary looking at the $Y[\mathfrak{su}(2|2)]$-coproduct $\Delta\mathfrak{R}^a_b$ given in (C.3). In `Mathematica` we define it via the function DR[..] as

```
DR[a_,b_][c_,d_,δ_]:=
  Tensorproduct[R[a,b][c],One[d,δ]+
    Tensorproduct[One[d,δ],R[a,b][c]]//Expand;
One[a_,α_]:=Sub[{a},{a}]+Sub[{α},{α}];
Tensorproduct[A_,B_]:=
  A/.Sub[a__]:>
    (B/.Sub[b__]:>Tensorproduct12[Sub[a],Sub[b]]);
Tensorproduct12[Sub[a__],Sub[b__]]:=
  Sub[Join[{a}[[1]],{b}[[1]]],Join[{a}[[2]],{b}[[2]]]];.
```
(D.13)

The indices a and b in DR[..] are the free indices of $\Delta\mathfrak{R}^a_b$, the indices c,d,δ are summation indices. The function One[..] corresponds to the identity operator in (7.16). The routine Tensorproduct[..] builds the tensor product of two linear combinations of operators, e.g.

```
In[10]:=  Tensorproduct[Sub[{a},{a}],Sub[{b},{b}]]
Out[10]=  Sub[{a,b},{a,b}].
```
(D.14)

For the supersymmetric generators of $Y[\mathfrak{su}(2|2)]$ one has to consider the fermionic braiding $\mathcal{U}_F$, see e.g. $\Delta\mathfrak{Q}^\alpha_a$ in (C.3). This operator can be implemented into this Mathematica-code by defining the coproduct DQ[..] of $\mathfrak{Q}^\alpha_a$ as

```
DQ[α_,a_][c_,γ_,d_,δ_]:=
  Tensorproduct[Q[α,a][c,γ],Onef[d,δ]]+
    U Tensorproduct[Onef[c,γ],Q[α,a][d,δ]]//Expand;
Onef[a_,α_]:=Sub[{a},{a}]-Sub[{α},{α}];.
```
(D.15)

It includes the eigenvalue U of $\mathcal{U}$ and Onef[..] corresponding to $(-1)^F$. Note that here the coefficients Ca and Cb and the braiding factor U are treated as constants. For the dynamic representation of $Y[\mathfrak{su}(2|2)]$ one has to make these rapidity-dependent, e.g. by writing Ca[u].

D.3 Yangian Constraints

We proceed with the implementation of the Yangian constraints (4.18) and (4.19).

Yangian Constraints for $Y[\mathfrak{su}(\mathbf{n})]$

We first discuss the general procedure for the Yangian corresponding to $\mathfrak{su}(\mathrm{n})$. As motivated in section 5.2, the level-0 constraint implies that the S-matrix is a linear combination of permutation operators. We obtain it via L0Invariant[..]

```
L0Invariant[m_][a_,l_]:=
  Array[l,m!].Table[Sub[Permutations[Array[a,m]][[i]],Array[a,m]],
    {i,1,m!}].
```
(D.16)

The index `m` denotes the length of the operators, the `a`'s are the summation indices and the `l`'s the coefficients in the linear combination. Let us examine the first two operators obtained from this function. The invariant operator of length `m=1` is

```
In[11]:= L0Invariant[1][a,l]
Out[11]= l[1]Sub[{a[1]},{a[1]}].
```
(D.17)

This operator corresponds to (5.21) and acts on a one-particle state without changing its type and rapidity. For `m=2` we get

```
In[12]:= L0Invariant[2][a,l]
Out[12]= l[1]Sub[{a[1],a[2]},{a[1],a[2]}]+
           l[2]Sub[{a[2],a[1]},{a[1],a[2]}]
```
(D.18)

which again contains the identity and furthermore the permutation of the two excitation types. It corresponds to the operator in (5.25). Using the routine `L0Invariant[..]` for `m=3` yields the `Mathematica` expression corresponding to (5.28).

Implementing the coproduct (5.15) of the level-1 generators $\hat{\mathfrak{R}}^a_b$ is straightforward when using the method `Tensorproduct`. Multiplying this coproduct with the $\mathfrak{su}$(n)-invariant operator in (D.16) in the form (5.19) using `Multiply` gives the level-1 constraint. For different values of a and b this constraint yields a set of equations that restricts the coefficients `l[i]` in the linear combinations from above and the outgoing rapidities `v[i]`. Denoting the outgoing rapidities of a two-particle scattering process by `v[1]` and `v[2]` and the incoming rapidities by `u[1]` and `u[2]`, the constraints on the two-particle S-matrix are solved by

```
Out[13]= {{l[2]->0,v[1]->u[1],v[2]->u[1]},
           {l[2]->-2l[1](u[1]-u[2]),v[1]->u[2],v[2]->u[1]}}.
```
(D.19)

These two solutions correspond to the solutions in (5.30) and (5.31). Similarly, we obtain a set of six solutions for the three-particle S-matrix that are depicted in Table 5.2.

Yangian Constraints for $Y[\mathfrak{su}(1|1)]$ and $Y[\mathfrak{su}(2|2)]$

We now turn to the Yangian constraints corresponding to Lie superalgebras. We calculate the ansatz for the S-matrix that satisfies the level-0 constraints by writing down all possible operators in the theory. Then we demand a vanishing commutator with the generators of the Lie superalgebra. Take as an example the $\mathfrak{su}(1|1)$-invariant operator. We make an ansatz using the function `Ansatz[...]`

```
Ansatz[m_]:=Module[{a1,a2},
  a1=((Sub@@@Tuples[Partition[
        Riffle[Flatten@ToExpression@
          StringSplit[Alphabet["English"][[1;;2m]],","],
         Flatten@ToExpression@StringSplit[
          Alphabet["Greek"][[1;;2m]],","]],2]])/.
     Sub[a__]:>Sub[Partition[{a},n]])/.Sub[{a__}]:>Sub[a];
  a2=Array[l,Length[a1]];
  a1.a2]
```
(D.20)

which evaluates to

```
In[14]:= Ansatz[1]
Out[14]= l[1]Sub[{a},{b}]+l[2]Sub[{a},{β}]+l[3]Sub[{α},{b}]+
          l[4]Sub[{α},{β}]
```
(D.21)

for `m=1`. The `l`'s are the coefficients in the linear combination. Note that one could also use a single variable for the boson ϕ in the fundamental representation and a single variable for the fermion ψ. Nevertheless, the code for $Y[\mathfrak{su}(1|1)]$ serves as the basis of the implementation of the constraints from $Y[\mathfrak{su}(2|2)]$. Therefore, we implemented it more generally and set all bosonic indices to `a` and all fermionic indices to `α` in the end. Commuting `Ansatz[2]` and `Ansatz[3]` with the generators of $Y[\mathfrak{su}(1|1)]$, exploiting that there is only a single boson and a single fermion in the theory and demanding that the results vanish, gives constraints on the `l`'s and rapidities. The results of this analysis are the $Y[\mathfrak{su}(1|1)]$-invariant operators that we discussed in chapter 6.

For the $\mathfrak{su}(2|2)$-invariants this procedure is more involved since both bosonic and fermionic indices may take two values. Inserting for bosonic indices `a=1,2` and for fermionic indices `α=3,4` and analysing the constraints, gives operators containing all values `1,2,3,4`. To obtain a closed form as in (7.21), one has to make a general ansatz only containing indices and compare both by summing over the indices in the ansatz. The results of this analysis for the level-0 constraints with $\mathfrak{R}^a_b$ and $\mathfrak{L}^\alpha_\beta$ at length 2 can be found in (7.21). This operator gets further constrained by the remaining generators of $Y[\mathfrak{su}(2|2)]$ and $Y[\mathfrak{su}(2|2) \ltimes \mathbb{R}^2]$, respectively. Let us illustrate this for the constraint $[\Delta\hat{\mathfrak{C}}, \mathrm{S}_{12}] = 0$ in the undynamic representation. First, we define a routine `DCH[..]` that corresponds to the coproduct of $\hat{\mathfrak{C}}$ given in (7.17)

```
DCH[a_,b_,α_,β_]:=I(u[1]C+ u[2]C)Tensorproduct[One[a,α],One[b,β]];.
```
(D.22)

Then, we calculate both terms in the commutator $[\Delta\hat{\mathfrak{C}}, \mathrm{S}_{12}]$ via

```
S12DCH=
  (Multiply[S120[a,b,α,β],DCH[c,d,γ,δ]]//UseKD)/.
    {u[1]->v[1],u[2]->v[2]};

DCHS12=
  Multiply[DCH[a,b,α,β],S120[c,d,γ,δ]]//UseKD;.
```
(D.23)

The function `S120[..]` corresponds to the $\mathfrak{su}(2|2)$-invariant operator (7.21) with (7.22) that has two degrees of freedom A_{12} and H_{12} which we call `A12` and `H12` here. The `u[1],u[2]` and `v[1],v[2]` denote the rapidities of the two particles in the incoming and outgoing state, respectively. Note that this example corresponds to a calculation in the undynamic representation, i.e. the eigenvalue `C` of $\mathfrak{C}$ does not depend on the rapidity (we write `C` instead of `C[u[1]]` etc.). Manipulating the results via

```
In[15]:= ComSCH=(S12DCH-DCHS12)//Expand//SumOverIndices;

In[16]:= ComSCHlist=(List@@(ComSCH//Collect[#,_Sub]&))/._Sub->1;

In[17]:= Solve[Flatten[{ComSCHlist==0,A12!=0,H12!=0,C!=0}]]
Out[17]= {{v[2]->u[1]+u[2]-v[1]}}                                (D.24)
```

gives the constraint in (7.23). Here we sum over all indices in `S12DCH-DCHS12` using the function `SumOverIndices[..]`. Afterwards we collect the terms involving the same `Sub`s and make a list of their prefactors. One can analyse the constraints from the remaining generators of $Y[\mathfrak{su}(2|2)]$ in a similar manner. The results of this analysis are discussed in chapter 7.

D.4 Quantum Yang-Baxter Equation and Factorization of the S-Matrix

In order to verify consistent factorization one has to build the operators S_{12} and S_{23} in (5.45) that only act on the first two and the last two particles in a three-particle state, respectively. Let us demonstrate how to do this for the $Y[\mathfrak{su}(\mathrm{n})]$-invariant two-particle S-matrix. The result of the above analysis for the two-particle S-matrix is given in Table 5.1 and we defined it in `Mathematica` via

```
In[18]:= S12[a,u[1]-u[2]]
Out[18]= l[2] Sub[{a[2],a[1]},{a[1],a[2]}]-
           (l[2] Sub[{a[1],a[2]},{a[1],a[2]}])/(2(u[1]-u[2])).   (D.25)
```

It contains two summation indices `a[1],a[2]`, an overall factor `l[2]` and the rapidity difference `u[1]-u[2]`. The method `S12on3[..]` enlarges this operator to an operator of length 3 as

```
S12on3[n_,a_,u_]:=
  Module[{b,c},
    S12[b,u]/.Sub[x_,y_]:>Sub[Insert[x,c,n],Insert[y,c,n]]//
      Relabel[a,{},#]&].                                         (D.26)
```

The index `n` denotes the position of the particle that is not scattered, `a` is the summation index and `u` the relative rapidity. The method `Relabel[..]` looks for summation indices and renames them to its first argument. Thus one can simplify terms such as

```
In[19]:= Sub[{a},{a}]-Sub[{b},{b}]//Relabel[{x},{},#]&
Out[19]= 0.                                                       (D.27)
```

The second argument in `Relabel[..]` is reserved for indices that shall not be relabeled. Using this routine we can build the left and right hand side of the qYBE via

```
S23S12S23:=
  (Multiply[Multiply[S12on3[1,a,u[2]-u[3]],
    S12on3[3,b,u[1]-u[3]]],S12on3[1,c,u[1]-u[2]]]//UseKD;
S12S23S12:=
  (Multiply[Multiply[S12on3[3,a,u[1]-u[2]],
    S12on3[1,b,u[1]-u[3]]],S12on3[3,c,u[2]-u[3]]]//UseKD;      (D.28)
```

and verify the qYBE

```
In[20]:= S23S12S23-S12S23S12//Relabel[{x},{},#]&
             //Collect[#,_Sub,Simplify]&
Out[20]= 0.                                                    (D.29)
```

In order to confirm factorization of the three-particle S-matrix S123 we subtract one side of the qYBE as

```
In[21]:= S123-S12S23S12//Relabel[{x},{},#]&
             //Collect[#,_Sub,Simplify]&
Out[21]= 0.                                                    (D.30)
```

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Hilfsmittel

Diese Arbeit wurde mit LaTeXunter Verwendung von Texmaker verfasst. Die Grafiken wurden mit Hilfe des LaTeX-Pakets TikZ erstellt. Alle analytischen wie numerischen Rechnungen wurden in `Mathematica` durchgefürt.

Selbständigkeitserklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und noch nicht für andere Prüfungen eingereicht habe. Sämtliche Quellen einschließlich Internetquellen, die unverändert oder abgewandelt wiedergegeben werden, insbesondere Quellen für Texte, Grafiken, Tabellen und Bilder, sind als solche kenntlich gemacht. Mir ist bekannt, dass bei Verstößen gegen diese Grundsätze ein Verfahren wegen Täuschungsversuchs bzw. Täuschung eingeleitet wird.

Dublin, den 28.10.2017 Anne Spiering