Hierarchy of couplings in the Conformal Standard Model

Hierarchie der Kopplungen im Konformen Standardmodell

MASTERARBEIT

zur Erlangung des akademischen Grades
Master of Science (M. Sc.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät I
Institut für Physik
Humboldt-Universität zu Berlin

von
Julian Miczajka
geboren am 14.06.1992 in Berlin

Betreuung:

1. *Prof. Dr. Jan Plefka*
2. *Prof. Dr. Peter Uwer*

eingereicht am 30. Mai 2017

Contents

List of Figures

List of Tables

Chapter 1
Introduction

Motivation

For quite some time the Standard Model of particle physics has been an incredibly successful description of nature. It has accurately predicted a vast number of observables in scattering processes and has even led to the prediction of new fundamental particles, like the top quark [1] and the Higgs boson [2, 3]. But in spite of all those successes, the Standard Model still has some serious problems.

For one, it is known from astrophysical obervations [4] that only about 5% of the energy content of the universe is made up of Standard Model matter. A vast majority of all existing matter seems to actually be Dark Matter which only interacts very weakly with the fields of the Standard Model. This means that the Standard Model is at least incomplete.
Another problem is the question of naturalness embodied in the hierarchy problem [5]. Roughly speaking it asks the question: Why is the Higgs boson so light? One would assume that the mass of the Higgs boson (and any other scalar particle) would receive corrections from all fields it couples to. This means that if there are heavy degrees of freedom that have been integrated out of the Standard Model, those should contribute to the Higgs mass. An unnaturally fine-tuned cancellation is needed in order to end up with a Higgs mass of the order of 100 GeV, as in the Standard Model.
In the past, both of those problems were adressed by Supersymmetry [6] - a fermionic spacetime symmetry which among other important results leads to the prediction of a superpartner for every fundamental field. The contributions to the Higgs mass correction of pairs of superpartners cancel. One therefore says that Supersymmetry protects the Higgs mass from corrections and allows the hierarchy of the Standard Model without massive fine-tuning.

As pointed out by e.g. Meissner and Nikolai [7], it is also possible to protect the Higgs mass using classical Conformal Symmetry, or rather scale invariance. They observed that the only term in the Standard Model Lagrangian that breaks scale invariance is the Higgs mass term. This is also the term that creates the hierarchy problem. Hence, by removing it and enforcing scale invariance on the classical level, one avoids the problem alltogether. However, the Higgs mass term also plays an important role in the breaking of the Electroweak symmetry and is crucial to the functionality of the Standard Model. Their suggestion is therefore to use the Coleman-Weinberg-mechanism [8], which creates a symmetry breaking effective potential and generates a mass scale from quantum corrections. This idea has led to the development of many variations on the general idea - a classically scale-invariant extension of the Standard Model [9, 10, 11, 12, 13, 14]. There is one feature that they all have in common, namely the introduction of an additional scalar field.

As already realised by Coleman and Weinberg, consistency of the perturbative expansion used to study the effective potential requires a hierarchy of couplings. It is not possible for the quantum corrections of a single interaction to break classical scale invariance. One needs to have at least two interactions that appear at different orders of perturbation theory, such that the quantum corrections of one coupling are of the same order of magnitude as the classical effects of the other.

This hierarchy of couplings is not obeyed in the analysis of [7] and by imposing it, the results are rendered invalid. Therefore, this simplest scale-invariant extension of a single chargeless scalar is unfit to describe an SM-like reality. While this problem is fixed in other, more complicated extensions of the Standard Model, it is not clear what extensions are crucial. Especially in light of the lack of direct evidence for additional particles, it is worthwhile to find this minimal extension.
In this thesis, our aim is therefore to find the minimal perturbatively consistent scale-invariant quantum field theory that allows an embedding of the Standard Model in the sense, that all experimental evidence for the Standard Model can be accomodated.
We especially show that the simplest imaginable extension of a single scalar field as studied in [7] is not sufficient and must at least be accompanied by a gauge interaction.

Outline

This **first chapter** gives a motivation for the subject of the thesis and a general outline.
In the **second chapter**, we briefly summarize some basic features of Quantum Field Theories and the formalism of the effective action. We review the way the Coleman-Weinberg mechanism generates vacuum expectation values in classically scale-invariant field theories. We then go on to describe the field content and the interactions of the Standard Model Lagrangian as far as they are relevant to our discussion.
In the **third chapter**, we study classically scale-invariant extensions of the Standard Model with the aim of describing the minimal extension whose effective potential posesses a minimum for Standard Model parameters. We argue that the Standard Model extended by a single scalar field which only couples to the Higgs field has no stable minimum that fits with experimental data. We therefore introduce a gauge field coupled to this scalar to stabilize the effective potential. While this seems to lead to a locally stable theory, the model still suffers from a similar vacuum instability as the Standard Model. This can for example be solved by an additional 'dormant' scalar field which does not develop a vacuum expectation value.
Finally, the **fourth chapter** gives a summary of our findings and concludes with an outlook on possible future investigations.

Chapter 2
Theoretical Background

To provide the appropriate setting for the main developments of this thesis, this second chapter will give a brief description of the theoretical background.

After some words on notational conventions we will give a very rough introduction to quantum field theory. We point out the places in which the vacuum state of the theory comes into play. Introducing the formalism of the effective action, we relate the vacuum state to the minimum of the effective potential. We then describe how to arrive at an approximation for this effective potential at the first order of perturbation theory.

Using this effective potential we demonstrate how the Coleman-Weinberg mechanism [8] generates scales in classically scale-invariant theories from quantum corrections. We point out that in order to arrive at a result that is perturbatively consistent and gauge-invariant order by order in perturbation theory [15], we need to assume a hierarchy of couplings in which scalar self-couplings appear at higher loop orders than gauge couplings

$$\lambda \sim e^4. \tag{2.1}$$

We then briefly describe the process of renormalization and how to improve effective potentials using the renormalization group equations.

Finally, we describe all relevant parts of the Standard Model Lagrangian. We argue that for the mass generation via the Higgs mechanism it is not crucial how the Higgs field acquires a vacuum expectation value, as long as it gets one.
This might allow an embedding of the Standard Model in a scale-invariant theory in which the vacuum expectation value of the Higgs field is induced by the Coleman-Weinberg mechanism. While there is a multitude of models in which this is accomplished, we want to find the simplest of those models.

The following introduction is based on [16, 17, 18].

2.1 Conventions

We will mostly work in natural units, where

$$\hbar = 1, \qquad c = 1 \tag{2.2}$$

such that all dimensionful quantities are going to be expressed solely in units of energy. We will talk a lot about fields on spacetime, which we often denote as $\phi(x)$, where

$$x = (t, \vec{x}) \tag{2.3}$$

is a coordinate vector in 4-dimensional spacetime. When we adress single coordinates we use greek letters from the middle of the alphabet (μ, ν, ...) as superscript. Indices of vectors are pulled up and down by the 'mostly minus' Minkowski metric

$$x_\mu = \eta_{\mu\nu} x^\nu \tag{2.4}$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.5}$$

In the formula (2.4) and all throughout the thesis, the Einstein summation convention implies that indices that appear twice are automatically summed.
Derivatives with respect to a space time coordinate are given by

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \qquad \Box = \partial_\mu \partial^\mu. \tag{2.6}$$

2.2 Short introduction to Quantum Field Theory

2.2.1 Classical Field Theory

The usual path to a Quantum Field Theory starts at a Classical Field Theory that is then quantised. Classical Field Theories are dynamical systems with infinitely many degrees of freedom. For a field in spacetime, those degrees of freedom are labeled by the continuous spacetime coordinates x^μ.
To give a compact description of a field theories' dynamics, one usually writes down an action functional

$$S[\phi] = \int \mathrm{d}^4x \ \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \tag{2.7}$$

which assigns a real number to every field configuration $\phi(x)$. The function $\mathcal{L}$ is called the Lagrangian of the system and is generally given by the difference between the kinetic and the potential energy

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} - V[\phi]. \tag{2.8}$$

The equation of motion for the field ϕ is then derived by imposing the principle of stationary action: the physical evolution of a classical field is such, that the action of the system is stationary under variations of the field.

$$0 = \frac{\partial S[\phi]}{\partial \phi(x)} := \lim_{\epsilon \to 0} \frac{S[\phi + \epsilon\delta] - S[\phi]}{\epsilon} \tag{2.9}$$

where δ is the Dirac delta function.
Combining (2.7) and (2.9) we find the equation of motion for the classical field, derived from the Lagrangian

$$0 = \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))}. \tag{2.10}$$

In the same way we can talk about theories involving multiple fields or fields with additional discrete indices, e.g. vector fields. (2.9) simply generalizes to give one equation of motion per field.
To give a full definition of a dynamical system of a set classical fields, it is therefore sufficient to write down a Lagrangian $\mathcal{L}$ that contains all the fields and describes their interaction.

2.2.2 Quantisation and particles

Similar to going from Classical to Quantum Mechanics, where position and momentum of a point particle change their physical interpretation drastically, the quantum fields in a Quantum Field Theory are very different from classical fields.
In a Classical Field Theory, the state of a physical system is sufficiently described by giving the function $\phi(x)$, similar to how a classical point particle can be described by its trajectory $x(t)$. In the corresponding quantum theory on the other hand, the state of a system is described by an abstract vector in a Hilbert space

$$|\psi(t)\rangle \in \mathcal{H}_\psi. \tag{2.11}$$

Its time evolution is given by the Schroedinger equation

$$i\frac{\partial|\psi(t)\rangle}{\partial t} = H(t)|\psi(t)\rangle, \tag{2.12}$$

where H is the Hamiltonian of the system, a linear hermitian operator that describes the time development. It has the formal solution

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \tag{2.13}$$

with the time evolution operator

$$U(t,t_0) = T\left[e^{i\int_{t_0}^{t} H(t')\mathrm{d}t'}\right]. \tag{2.14}$$

T is the time ordering operator, that orders operator products according to

$$T\left[A(t_1)B(t_2)\right] = \begin{cases} A(t_1)B(t_2) & t_1 > t_2 \\ B(t_2)A(t_1) & t_1 < t_2 \end{cases}. \tag{2.15}$$

An important object to connect this mathematical framework to experiments is the probability amplitude. Every observable is represented by a hermitian operator A on the Hilbert space, whose eigenvalues a_i are the possible values for a measurement. The probability to find one of those eigenvalues when measuring A is given by

$$p(A = a_i) = |\langle a_i|\psi(t)\rangle|^2. \tag{2.16}$$

The eigenstates of every operator A span the whole Hilbert space.
So far this seems somewhat unconnected to what we said about classical fields, but it is not: Given a Lagrangian $\mathcal{L}$ of a field theory, we define the Hamiltonian density $\mathcal{H}$ by

$$\mathcal{H}(\phi(x),\pi(x)) = \dot{\phi}(x)\pi(x) - \mathcal{L}(\phi(x),\partial_\mu\phi(x)) \tag{2.17}$$

which is related to the Hamiltonian by

$$H(t) = \int \mathrm{d}^3 x \ \mathcal{H}(\phi(x), \pi(x)) \tag{2.18}$$

with the so called conjugate field

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}. \tag{2.19}$$

In Classical Field Theories that are invariant under translations in time, H is the conserved energy of the system. In a step that is called canonical quantisation we now lift the fields to operator valued functions acting on the Hilbert space, which obey the canonical commutator relations

$$\begin{gathered}[\phi(t,\vec{x}), \phi(t,\vec{y})] = 0, \qquad [\pi(t,\vec{x}), \pi(t,\vec{y})] = 0 \\ [\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta(\vec{x}-\vec{y}).\end{gathered} \tag{2.20}$$

All observables are functions of the fields and conjugate fields. To make things more concrete, let us look at on particular example: the quantum field theory of a free scalar boson. It is described by the Lagrangian

$$\mathcal{L} = \partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi(x)^2 \tag{2.21}$$

and its classical equation of motion is the Klein-Gordon-equation

$$\left(\partial_\mu \partial^\mu + m^2\right) \phi(x) = 0. \tag{2.22}$$

This equation is quickly solved using a Fourier decomposition. The general solution reads

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 \sqrt{2\omega_{\vec{k}}}} \left(a(\vec{k}) e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x})} + a^\dagger(\vec{k}) e^{i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x})} \right) \tag{2.23}$$

where the $a(\vec{k})$ and $a^\dagger(\vec{k})$ are arbitrary coefficient functions connected by complex conjugation and

$$\omega_{\vec{k}} = \sqrt{m^2 + \vec{k}^2} \tag{2.24}$$

to satisfy the equation of motion. After quantisation the $a(\vec{k})$ and $a^\dagger(\vec{k})$ become operators as well, obeying

$$\begin{gathered}[a(\vec{k}), a(\vec{k}')] = 0, \qquad [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0 \\ [a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 \delta(\vec{k} - \vec{k}').\end{gathered} \tag{2.25}$$

From those ladder operators we can construct the so called number operator

$$N = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} a^\dagger(\vec{k}) a(\vec{k}), \tag{2.26}$$

whose eigenvalues can be shown to be the natural numbers. As already mentioned, every possible state can be decomposed in eigenstates of this operator.

The eigenvectors of the number operator can actually be explicitly constructed. Given the vacuum state $|0\rangle$ which is annihilated by all $a(\vec{k})$

$$a(\vec{k})|0\rangle = 0 \tag{2.27}$$

we can construct the states

$$|\vec{k}_1, ..., \vec{k}_n\rangle = a^\dagger(\vec{k}_1)...a^\dagger(\vec{k}_n)|0\rangle \tag{2.28}$$

which satisfy

$$N|\vec{k}_1, ..., \vec{k}_n\rangle = n|\vec{k}_1, ..., \vec{k}_n\rangle. \tag{2.29}$$

If we construct the Hamiltonian of this theory then we find

$$H = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\omega_{\vec{k}}\ a^\dagger(\vec{k})a(\vec{k}), \qquad H|\vec{k}_1, ..., \vec{k}_n\rangle = (\omega_{\vec{k}_1} + ... + \omega_{\vec{k}_n})|\vec{k}_1, ..., \vec{k}_n\rangle. \tag{2.30}$$

In a sense this is the only reason why one says that a quantum field theory describes particles - because we can construct the entire Hilbert space of possible states from a basis which contains discrete excitations, all carrying their own portion of energy. Looking at (2.24) we recognize that this theory describes relativistic particles of mass m.

We have only shown this explicitly for the free scalar theory but the above calculation completely generalizes to other free theories. The main statement is: in free theories, there are states with definite numbers of particles, and because the Hamiltonian commutes with the number operator those numbers stay the same over time.
To construct these states we needed to start from a vacuum state $|0\rangle$, which is clearly seen from (2.30) to be the state with the minimal energy. We also see that we find the mass of a scalar particle by taking the second derivative of the potential with respect to the corresponding field

$$m^2 = \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}. \tag{2.31}$$

2.2.3 Interactions and scattering

As soon as we introduce interactions, this changes drastically. An interaction is understood to be a non-linearity in the classical field equation which couples different Fourier modes. This corresponds to terms in the Lagrangian which are powers of the field of a higher order than two.
While you can still follow the same construction as before and define a number operator, this operator will not commute with the Hamiltonian anymore - the number of particles can therefore change over time. This makes it essentially impossible to solve the field equations analytically in general cases.

To still be able to calculate any prediction of the theory at all, one looks at scattering experiments. Assuming that the interaction is only happening in some finite time interval $[t, t']$, we prepare states with definite particle content before this interval

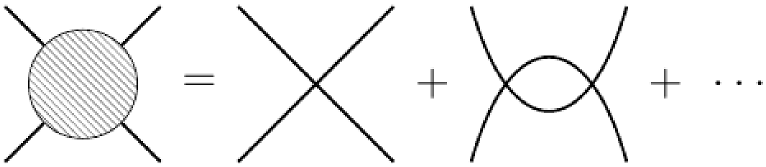


Figure 2.1: Diagrammatic representation of the connected four-point function in a scalar ϕ^4 theory.

and measure the particle content after it. Then, we take the limit to an infinite interval in order to recover the full interacting theory. This means that we are interested in the following matrix element of the S-matrix

$$S_{if} = \langle f|U(\infty, -\infty)|i\rangle, \tag{2.32}$$

with the time evolution operator U and the initial and final states $|i\rangle$ and $|f\rangle$. With the Lehmann-Schimanski-Zimmermann reduction formula, we can relate the elements of this S-matrix to a rather simple set of expectation values

$$\Gamma(p_1, ..., p_n) = \int \prod_{i=1}^{n} \mathrm{d}^4 x_i e^{ip_i \cdot x_i} \langle \Omega|T(\phi(x_1)...\phi(x_n))|\Omega\rangle, \tag{2.33}$$

the so called momentum space n-point functions, where $|\Omega\rangle$ is the vacuum of the interacting theory. It is very important for the construction of this formula that $|\Omega\rangle$ is the eigenstate of the interacting Hamiltonian with the lowest energy. With some effort one can show, that those n-point functions can be calculated using the path integral. Starting from the vacuum-to-vacuum amplitude, given by

$$\langle \Omega|\Omega\rangle = \int [\mathcal{D}\phi]\ e^{i\int \mathrm{d}^4 x\ \mathcal{L}(\phi(x), \partial_\mu \phi(x))} \tag{2.34}$$

all of the position space n-point functions can be derived from the generating functional

$$Z[J] = \int [\mathcal{D}\phi] e^{i\int \mathrm{d}^4 x\ (\mathcal{L}(\phi(x), \partial_\mu \phi(x)) + J(x)\phi(x))} \tag{2.35}$$

by repeated differentiation

$$\langle \Omega|T(\phi(x_1)...\phi(x_n))|\Omega\rangle = \frac{-i^n}{Z[0]} \frac{\partial^n Z[J]}{\partial J(x_1)...\partial J(x_n)}\bigg|_{J=0}. \tag{2.36}$$

The integral in the above expressions is a formal integral over all possible field configurations $\phi(x)$. In the case of a free theory it can be calculated exactly, but for an interacting theory, one needs to use the approach of perturbation theory. By assuming that the interaction is only a small correction to the free theory, one can expand the corresponding term in the path integral and calculate the n-point function in the interacting theory as a series in n-point functions of the free theory. Often the terms of the perturbation series are visualised using Feynman diagrams, as in figure 2.1.
Every line in a diagram represents a free-field two point function, called the propagator. It is essentially the inverse of the differential operator that forms the equation of motion. Every vertex contributes one power of the coupling constant that describes

the strength of the interaction.
Let us for a moment restore $\hbar$ to the expression of $Z[J]$. It then reads

$$Z[J] = \int [\mathcal{D}\phi] e^{\frac{i}{\hbar}\int \mathrm{d}^4x (\mathcal{L}(\phi(x),\partial_\mu\phi(x)) - J(x)\phi(x))} \tag{2.37}$$

This effectively rescales all coupling constants by $\frac{1}{\hbar}$ and all propagators by $\hbar$. It can then be proven inductively, that the overall power p of $\hbar$ related to a diagram in the expansion of the n-point function is given by

$$p = n + L - 1, \tag{2.38}$$

where L is the number of internal loops within the diagram. This shows that in the classical limit where $\hbar \to 0$, the dominant contribution to any n-point function is always coming from the diagrams without any loops, the tree level diagrams.

2.3 The effective action

This opens an interesting possibility: imagine we had an effective action Γ_S, that contained already at tree level all information of a full interacting quantum field theory described by an action S. This effective action Γ_S would define a quantum field theory, whose classical limit is completely equivalent to the original theory. Of course, one way to construct this effective action is by matching all results from the quantum field theory. We define

$$\Gamma_S[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 ... \mathrm{d}^4x_n \; \Gamma(x_1, ..., x_n)\phi(x_1)...\phi(x_n) \tag{2.39}$$

with the n-point functions of the original theory. Clearly, the first term in a perturbative expansion of the path integral involving Γ_S for any n-point function will be $\Gamma(x_1, ..., x_n)$ dressed with some external propagators.
But this definition is not of much use: we still have to do the full perturbative calculation to find the n-point functions of the original theory. Luckily, there is an alternative way.
Starting from the generating function of all connected diagrams

$$W[J] = -i \ln Z[J] \tag{2.40}$$

we define its Legendre transform

$$\Gamma[\phi] = W[J_\phi] - \int \mathrm{d}^4x \; \phi(x) J_\phi(x). \tag{2.41}$$

Here, $J_\phi(x)$ is defined by

$$\left.\frac{\partial W}{\partial J(x)}\right|_{J=J_\phi} = \phi(x). \tag{2.42}$$

Formulating the generating functional for the theory described by this $\Gamma[\phi]$ and taking its classical limit yields

$$\begin{aligned} \lim_{\hbar\to 0}(-i\hbar) \int [\mathcal{D}\phi] e^{\frac{i}{\hbar}\left(\Gamma[\phi] + \int \mathrm{d}^4x \;\; J(x)\phi(x)\right)} &= \left(\Gamma[\phi_J] + \int \mathrm{d}^4x J(x)\phi_J(x)\right)\Big|_{\frac{\partial\Gamma}{\partial\phi}(\phi_J) - J = 0} \\ &= W[J] \end{aligned} \tag{2.43}$$

where we made use of the stationary phase approximation and recognized the right-hand side as the inverse Legendre transformation, bringing us back to the generation function of connected diagrams.
To summarize: the above calculation showed us, that the classical limit of the generating function constructed from Γ is equal to the generating function of connected diagrams constructed from S.
We can use this effective action Γ to find the vacuum of the interacting theory. As we said above, the vacuum should be the state with minimal energy and we have to make sure that our theory is expanded around the true vacuum state. Also, to preserve Lorentz invariance, the vacuum field configuration should be translationally invariant. We therefore expand the effective action around a constant field according to

$$\Gamma[\phi] = \int \mathrm{d}^4x \left(-V_{\text{eff}}(\phi(x)) + \frac{1}{2}(\partial_\mu \phi(x))^2 Z(\phi(x)) + ... \right). \tag{2.44}$$

In terms of those functions, the classical equation of motion (2.10) reads

$$0 = V'_{\text{eff}}(\phi(x)) + \frac{1}{2} Z'(\phi(x))(\partial_\mu \phi(x))^2 + \partial_\mu \partial^\mu \phi(x) Z(\phi(x)) + ... \tag{2.45}$$

Obviously, a constant solution $\phi(x) = \phi$ to this equation must obey

$$0 = V'_{\text{eff}}(\phi). \tag{2.46}$$

In a classical field theory with a Lagrangian of the form

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi - V(\phi). \tag{2.47}$$

the constant field configuration with minimal energy will be the one that minimizes $V(\phi)$. Since $\Gamma[\phi]$ describes a classical field theory that is equivalent to the full quantum field theory described by S, we are therefore led to the conclusion, that we need to find the absolute minimum of $V_{\text{eff}}(\phi)$ in order to find the true vacuum state of the original quantum field theory. V_{eff} is therefore called the effective potential, including all radiative corrections of the quantum field theory.

2.3.1 Towards the one-loop effective potential

To calculate the effective potential let us look at (2.43) again. If we calculate the connected one-point function in the presence of a source $J(x)$ according to

$$\begin{aligned} \langle \Omega | \phi(x) | \Omega \rangle_J &= \frac{\partial W[J]}{\partial J(x)} \\ &= \int \mathrm{d}^4y \left(\frac{\partial \Gamma[\phi]}{\partial \phi(y)} \frac{\partial \phi(y)}{\partial J(x)} \bigg|_{\frac{\partial \Gamma}{\partial \phi}(\phi_J(y)) + J(y) = 0} + J(y) \frac{\partial \phi_J(y)}{\partial J(x)} \right) + \phi_J(x) \\ &= \phi_J(x), \end{aligned} \tag{2.48}$$

we see that it is just the solution to the classical field equation derived from $\Gamma[\phi]$ in the presence of $J(x)$.
Rewriting (2.43) to

$$\begin{aligned} e^{i\Gamma[\phi_J]} &= \int [\mathcal{D}\phi] e^{iS[\phi] + i\int \mathrm{d}^4x (J(x)(\phi(x) - \phi_J(x)))} \bigg|_{\phi_J = \langle \Omega | \phi | \Omega \rangle_J} \\ &= \int [\mathcal{D}\phi] e^{iS[\phi + \phi_J] + i\int \mathrm{d}^4x (J(x)\phi(x))} \bigg|_{\langle \phi \rangle_J = 0} \end{aligned} \tag{2.49}$$

tells us how to find the effective action. The above integral is the vacuum to vacuum amplitude of the action $S[\phi + \phi_J]$ in the presence of a source that cancels all contributions to the one-point function. Of course this means, that the only surviving diagrams contributing to (2.49) are the ones, that don't factorize into one-point functions, i.e. the 1PI diagrams. Those are the diagrams that stay connected when you cut only a single leg.
We therefore write

$$e^{i\Gamma[\hat{\phi}]} = \int_{1\mathrm{PI}} [\mathcal{D}\phi] e^{i\int \mathrm{d}^4x \ S[\phi+\hat{\phi}]}. \tag{2.50}$$

We can treat $\hat{\phi}$ in (2.50) as just another type of field which only appears as an external leg in any diagram. On the other hand, since we calculate a vacuum-to-vacuum amplitude, ϕ only appears as internal legs.

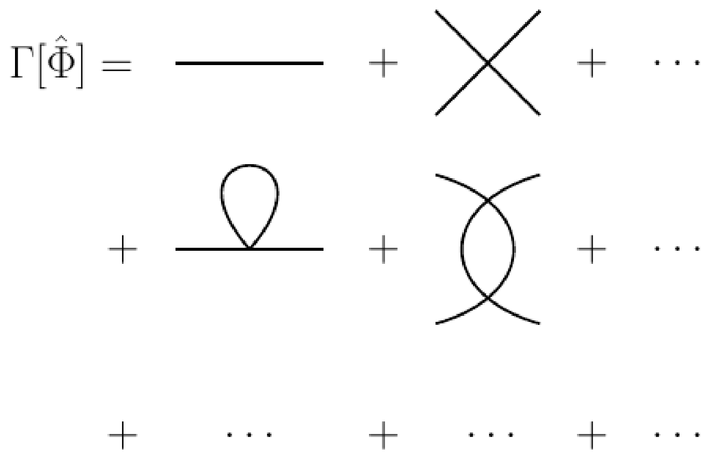


Figure 2.2: Graphical representation of the effective action $\Gamma[\hat{\phi}]$ in massive ϕ^4 theory. The ellipses stand for higher loop orders (vertically) and higher n-point functions (horizontally). All external lines are powers of $\hat{\phi}$.

As can be seen from figure 2.2, the effective action contains not only contributions at all loop-orders but from all n-point functions. Thus, even at the one-loop level we would need to sum an infinite number of diagrams. It becomes very complicated to organize those diagrams efficiently, especially when dealing with multiple fields and different types of interaction vertices. Fortunately, there is a way to calculate the effective potential without any use of diagrams.

To do so, we choose a constant classical field $\hat{\phi}(x) = \hat{\phi}$. The left hand side of (2.50) then becomes

$$e^{i\Gamma[\hat{\phi}]} = e^{iV_4 V_{\mathrm{eff}}(\hat{\phi})} \tag{2.51}$$

with the spacetime volume V_4. Technically this volume is infinite, but we can work in finite volume for now and take the limit later.
We expand the action on the right hand side of (2.50) around the constant background field

$$S[\phi + \hat{\phi}] = S[\hat{\phi}] + S'[\hat{\phi}]\phi + \frac{1}{2}S''[\hat{\phi}]\phi^2 + \frac{1}{3!}S'''[\hat{\phi}]\phi^3 + \dots \, . \tag{2.52}$$

So far everything has been exact, but to actually calculate the effective potential in an interacting theory, we will have to use perturbation theory. In the one-loop approximation, it is clear that only the term with two internal quantum fields from the above expansion will contribute. All the other terms with more powers of the quantum field can not appear in a diagram with only one loop. Also, the term involving only one quantum field does not contribute because all diagrams that incorporate it will be one-particle-reducible.
Thus,

$$e^{-iV_4 V_{\text{eff}}(\hat{\phi})}\Big|_{\text{1-loop}} = \int_{\text{1PI}} [\mathcal{D}\phi] e^{i\int \mathrm{d}^4x \left(-V[\hat{\phi}] - \frac{1}{2}\phi(x)\Sigma(\hat{\phi})\phi(x)\right)} \tag{2.53}$$

where

$$\Sigma(\hat{\phi}) = -\left.\frac{\partial^2 \mathcal{L}}{\partial \phi^2}\right|_{\phi=\hat{\phi}}. \tag{2.54}$$

2.3.2 Performing the Gaussian integral

Before we give the result for the above path integral, let's go back a simpler integral of the same kind:

$$I = \int \left(\prod_{i=1}^{n} \mathrm{d}x_i\right) e^{-\frac{1}{2}\sum_{i,j=1}^{n} x_i S_{ij} x_j} \tag{2.55}$$

with a positive-definite real symmetric $n \times n$-matrix S. This integral is easily solved by a basis transformation that diagonalizes S:

$$\begin{aligned} I &= \int \left(\prod_{i=1}^{n} \mathrm{d}y_i\right) e^{\frac{1}{2}\sum_{i=1}^{n} s_i y_i^2} = \prod_{i=1}^{n} \left(\int \mathrm{d}y_i \; e^{\frac{1}{2}s_i y_i^2}\right) \\ &= \sqrt{\frac{(2\pi)^n}{s_1 s_2 \ldots s_n}} = \sqrt{\frac{(2\pi)^n}{\det(S)}}. \end{aligned} \tag{2.56}$$

In analogy to this finite dimensional result we therefore write down the path integral version of it

$$\begin{aligned} I &= \int [\mathcal{D}\phi] e^{-\frac{1}{2}\int \mathrm{d}^4x \mathrm{d}^4y \;\; \phi(x) M(x,y) \phi(y)} \\ &= \frac{C}{\sqrt{\mathrm{Det} M}} \end{aligned} \tag{2.57}$$

with some infinite constant C and the determinant in the function space that M acts on. Using the identity

$$\ln \mathrm{Det}\ A = \mathrm{Tr} \ln A \tag{2.58}$$

we can find the logarithm of the above expression to be

$$\begin{aligned} \ln I &= -\frac{1}{2}\int \mathrm{d}^4x \langle x|\ln M|x\rangle \\ &= -\frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4p \langle x|p\rangle\langle p|\ln M(p)|x\rangle \\ &= -\frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4} \ln M(p) \\ &= -\frac{V_4}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4} \ln M(p) \end{aligned} \tag{2.59}$$

where it made sense to calculate the trace by inserting the completeness relation in the momentum basis, since M will in general contain spacetime derivatives, which act trivial in momentum space.
Plugging this into the formula (2.53) we arrive at the final expression for the effective potential to one-loop order

$$\boxed{V_{\text{eff}}(\phi) = V(\phi) - \frac{i}{2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \log\det\Sigma(\phi)} \tag{2.60}$$

which we already generalized to the case of more than one quantum field, where Σ is the matrix given by

$$\Sigma(\phi)_{ij} = -\frac{\partial^2 \mathcal{L}}{\partial\phi_i \partial\phi_j}. \tag{2.61}$$

As promised, the dependence on the spacetime volume has cancelled. (2.60) is the main formula we are going to use in chapter 3 to calculate the effective potentials for different models.
The momentum integrals we need to solve are generally of the form

$$I(m) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \ln\left(1 - \frac{m^2}{p^2}\right). \tag{2.62}$$

Since they are divergent, we use the Dimensional Regularization procedure, which treats the integral as an analytic function of the number of spacetime dimensions $D = 4 - \epsilon$. The divergence then appears as a pole in $\frac{1}{\epsilon}$ in the result

$$\begin{aligned} I(m) &= \mu^\epsilon \int \frac{\mathrm{d}^D p}{(2\pi)^D} \ln\left(1 - \frac{m^2}{p^2}\right) \\ &= -\frac{im^4}{16\pi^2}\frac{1}{\epsilon} + \frac{i}{64\pi^2} 2m^4 \left(\ln\frac{m^2}{4\pi\mu^2 e^{-\gamma_E}} - \frac{3}{2}\right). \end{aligned} \tag{2.63}$$

Once we have found an approximation for the effective potential, we apply our arguments that physical particles are excitations around the state with minimal energy. Thus, we minimize the effective potential and find the effective masses of physical particles to be

$$m^2 = \left.\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\phi^2}\right|_{\phi = \langle\phi\rangle}, \tag{2.64}$$

analogous to (2.31), where $\langle\phi\rangle$ is the position of the minimum.

2.3.3 The Coleman-Weinberg effective potential

Let us apply the formalism of the effective potential to a simple example.
We consider the theory of a massless scalar with a quartic interaction, described by the Lagrangian

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\phi^4. \tag{2.65}$$

Obviously, the classical potential $V(\phi) = \lambda\phi^4$ has only the trivial minimum at $\phi = 0$. It seems, as if we expanded the theory around the right vacuum.
But to be sure, we need to calculate the effective potential and find its minima. It is

$$\Sigma = \Box + 3\lambda\hat{\phi}^2, \tag{2.66}$$

which we substitute in the formula for the effective potential

$$\begin{aligned} V_{\text{eff}}(\hat{\phi}) &= \lambda\hat{\phi}^4 - \frac{i}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\log\left(1-\frac{3\lambda\hat{\phi}^2}{p^2}\right) \\ &= \lambda\hat{\phi}^4 - \frac{1}{32\pi^2}\frac{9\lambda^2\hat{\phi}^4}{\epsilon} + \frac{1}{64\pi^2}9\lambda^2\hat{\phi}^4\left(\ln\frac{3\lambda\hat{\phi}^2}{4\pi\mu^2e^{-\gamma_E}} - \frac{3}{2}\right), \end{aligned} \tag{2.67}$$

where we substracted an infinite constant to bring the p^2 to the denominator. After applying the renormalization condition

$$\frac{1}{3!}\frac{\mathrm{d}^4V_{\text{eff}}}{\mathrm{d}\hat{\phi}^4}\bigg|_{\hat{\phi}=M} = \lambda_R \tag{2.68}$$

we arrive at

$$V_{\text{eff}}(\hat{\phi}) = \lambda_R\hat{\phi}^4 + \frac{1}{64\pi^2}9\lambda_R^2\hat{\phi}^4\left(\ln\frac{\phi^2}{M^2} - \frac{25}{6}\right). \tag{2.69}$$

While this expression looks like it might lead to a minimum away from $\hat{\phi} = 0$ because the logarithm becomes negative for small field values, this minimum was argued to be spurious by Coleman and Weinberg [8], when they first studied this model. To understand why this is the case one has to know, that higher orders of perturbation theory will not only introduce higher powers of the coupling constant but also of the combination $\lambda_R \ln\frac{\phi^2}{M^2}$. Thus, in order to have a well behaved perturbation theory, not only λ_R must be small but also $\lambda_R \ln\frac{\phi^2}{M^2}$. Let us therefore assume we had renormalized the model exactly at the position of the minimum, i.e. $M = \langle\phi\rangle$. Then, to be consistent we would need to have

$$0 = \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\hat{\phi}}\bigg|_{\hat{\phi}=\langle\phi\rangle} \tag{2.70}$$

leading to

$$\lambda_R = \frac{33}{16\pi^2}\lambda_R^2. \tag{2.71}$$

In order to arrive at a non-trivial theory we would therefore need to assume that λ_R is from the same order of magnitude or even smaller than λ_R^2. Obviously, this implies that at the minimum of the effective potential perturbation theory is invalid. But since we used perturbation theory to arrive at this result, the whole argument is flawed.
Of course this doesn't mean that the theory is entirely inconsistent. In fact, a more thorough investigation using the tools of the renormalization group shows that the model simply retains the classical minimum at $\phi = 0$ and does not acquire a radiation induced vacuum expectation value.

However, Coleman and Weinberg suggested a way to consistently generate a non-trivial vacuum: by adding a gauge interaction with coupling constant e and by assuming that this gauge interaction is much stronger than the scalar self-interaction, we can restore the validity of perturbation theory.
Since we do a very similar calculation in the main part of the thesis, we just want to cite the result. In the theory with an additional gauge interaction, the effective potential becomes

$$V_{\text{eff}}(\hat{\phi}) = \lambda_R \hat{\phi}^4 + \frac{1}{64\pi^2}\left(3e_R^4 + 9\lambda_R^2\right)\hat{\phi}^4\left(\ln\frac{\hat{\phi}^2}{\langle\phi\rangle^2} - \frac{25}{6}\right) \tag{2.72}$$

and the consistency condition reads

$$\lambda_R = \frac{11}{64\pi^2}\left(e_R^4 + 3\lambda_R^2\right). \tag{2.73}$$

Obviously, the problem can be avoided by assuming

$$\lambda_R = \frac{11}{64\pi^2}e_R^4. \tag{2.74}$$

Then the terms of order λ_R^2 will only appear at higher orders in perturbation theory and there is indeed a minimum of the effective potential that is perturbatively consistent. This minimum sets the scale for the effective masses of both the scalar and the gauge boson.
Restoring the explicit power of $\hbar$, (2.74) reads

$$\lambda_R = \frac{11\hbar}{64\pi^2}e_R^4. \tag{2.75}$$

In other words, the hierarchy of couplings implies, that the scalar interaction itself takes the form of a quantum correction to the classical theory which contains no self-interaction.
It was also pointed out by Andreassen, Frost and Schwartz [15] that this hierarchy assumption is necessary to achieve a result which is gauge invariant at every order of perturbation theory.

When one applies this mechanism to the Standard Model, as we will do in chapter 2, it turns out that In this thesis we want to apply this general idea to the Standard Model: to consistently generate a minimum of an effective potential by radiative corrections we need to assume a hierarchy of couplings, such that quantum effects from one interaction can be from the same order of magnitude as the tree level contributions from another.

2.4 Renormalization improvement

As we have seen in section 2.3.3, the effective potential seems to be infinite when calculated naively. This is no peculiarity of the effective potential but happens in many calculations within Quantum Field Theories. We already anticipated the way out of this when doing the calculation. It turns out that all infinities vanish when

predictions are expressed in terms of measurable quantities only.
This especially excludes the coupling constants as they appear in the Lagrangian. They will only ever appear in predictions including all radiative corrections, so if those corrections are infinite and the results of the measurements are not, then the initial so called naked coupling constants must absorb the infinities. The formal process of this absorbtion is called renormalization.
The renormalized perturbation theory is an approach in which the Lagrangian of some theory is split into a part that depends only on measurable, renormalized quantities and the counterterm Lagrangian, which absorbs the appearing infinities. To make this more clear we will very roughly scetch the renormalization procedure using the example of the scalar ϕ^4 theory given by the 'bare' Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4}\phi_0^4 \tag{2.76}$$

which we decompose as described above

$$\mathcal{L}_0 = \mathcal{L} + \delta\mathcal{L} \tag{2.77}$$

with the renormalized Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 \tag{2.78}$$

and the counterterm Lagrangian

$$\delta\mathcal{L} = \frac{1}{2}\delta_\phi \partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_\lambda}{4}\phi^4 \tag{2.79}$$

where δ_ϕ, δ_m and δ_λ are chosen in such a way as to cancel all divergences order by order in perturbative calculations using $\mathcal{L}_0$. There is a certain freedom in those definitions, because the δ_i are allowed to contain finite terms as well. Different choices for the counter terms are called different renormalization schemes.
If it is possible to cancel all divergences using only a finite amount of counterterms, the corresponding theory is said to be renormalizable.

As we have already seen above, regularization procedures that are used to localize the divergences always introduce a scale, usually called μ. It turns out that the perturbative series is most accurate when the inherent scales of the problem that is discussed, e.g. the mass or momentum scales, are close to this renormalization scale μ. But because this scale was artificially introduced during the regularization, the naked parameters can actually not depend on it and that allows us to find out how the renormalized parameters of the model change for different choices of μ.
Since the careless introduction of scales can actually break the classical scale dependence explicitly [5], we will always use Dimensional Regularization. There, the renormalization scale is only introduced with an evanescent exponent that vanishes in the limit $D \to 4$.

The functions that describe the changes of the coupling constants under changes of the renormalization scale are called the beta functions

$$\beta_\lambda = \mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}. \tag{2.80}$$

Those beta functions allow us to study models for a wide range of scales, even if we only give experimental input at a single scale. We will use them in chapter 3 to study the global stability of local minima of effective potentials.
Since the beta functions for the Standard Model are known to very high order in perturbation theory, we will actually not completely derive any of them for our studies. But because we will make generous use of counterterms to arrive at certain renormalization conditions in chapter 3, let us study the scheme dependence of beta functions.

The most commonly used renormalization scheme is the $\overline{\mathrm{MS}}$-scheme, in which the counterterms are used to cancel only the divergent terms as well as the very often appearing combination $\ln 4\pi - \gamma_E$ with the Euler-Mascheroni constant γ_E. Let us consider the case in which we start from the $\overline{\mathrm{MS}}$-scheme for the above scalar ϕ^4 theory and add an additional finite counter term $\frac{\delta'}{4}\phi^4$. Then, starting from the assumption that λ_0 is independent from the renormalization scale, we find

$$0 = \mu \frac{\mathrm{d}\lambda_0}{\mathrm{d}\mu} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \left(\mu^\epsilon \lambda + \delta_{\overline{\mathrm{MS}}} + \delta' \right), \tag{2.81}$$

where we multiplied the renormalized coupling with an additional factor of μ^ϵ, as is usually done in Dimensional Regularization. $\delta_{\overline{\mathrm{MS}}}$ is a combination of the $\overline{\mathrm{MS}}$ counterterms, which only has a divergent part, i.e.

$$\delta_{\overline{\mathrm{MS}}} = \frac{1}{\epsilon} z_\infty. \tag{2.82}$$

We can then further evaluate (2.81)

$$0 = \epsilon \mu^\epsilon \lambda + \beta_\lambda \left(\mu^\epsilon + \frac{1}{\epsilon} \frac{\mathrm{d}z_\infty}{\mathrm{d}\lambda} + \frac{\partial \delta'}{\partial \lambda} \right) + \frac{\partial \delta'}{\partial \ln \mu}. \tag{2.83}$$

We can solve this at lowest order and take the limit $\epsilon \to 0$, leading to

$$\beta_\lambda = \frac{\mathrm{d}z_\infty}{\mathrm{d}\lambda} - \frac{\partial \delta'}{\partial \ln \mu}. \tag{2.84}$$

Therefore, at least at the one-loop level, there only appears a scheme dependence in the beta functions if we add counterterms with an explicit dependence on the renormalization scale. Let us consider the case in which we substract a term with an additional scale μ' by taking $\delta' = b \ln \frac{\mu^2}{\mu'^2}$. Then, to keep the unrenormalized coupling independent of both of those scales,

$$\begin{aligned} \mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} &= \beta_\lambda^{\overline{\mathrm{MS}}} - b \\ \mu' \frac{\mathrm{d}\lambda}{\mathrm{d}\mu'} &= b \end{aligned} \tag{2.85}$$

and thus the renormalized couplings become functions of two scales $\lambda = \lambda(\mu, \mu')$. We will use this technique of introducing another scale when discussing models containing two scalar fields with different vacuum expectation values in chapter 3.

2.5 The Standard Model

Now that we know the general framework we will work in, let us take a look at the Standard Model of particle physics.
We are going to write down the Lagrangian that describes the Standard Model and then shortly explain every term that appears in it

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{fermions}} + \mathcal{L}_{\text{Higgs}} - V(H). \tag{2.86}$$

2.6 The gauge sector

2.6.1 Gauge invariance

The concept of gauge symmetries has proven extremely fruitful in the study of Quantum Field Theories because it constrains the possible interactions and guarantees the renormalizability of the theory.
Starting from a theory with a global symmetry, say the theory of a complex scalar

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{2.87}$$

we see that this model has a global symmetry of the form

$$\phi(x) \to e^{i\alpha} \phi(x). \tag{2.88}$$

If we gauge this symmetry, that means, if we allow a different transformation at every position and time, then the above Lagrangian won't be invariant anymore. This is because the derivative of ϕ does not transform in a nice way:

$$\begin{aligned} \partial_\mu \phi(x) &\to \partial_\mu \left(e^{i\alpha(x)} \phi(x) \right) \\ &= e^{i\alpha(x)} \left(\partial_\mu + i \partial_\mu \alpha(x) \right) \phi(x). \end{aligned} \tag{2.89}$$

This behaviour can be cured by introducing a gauge field that transforms in a special way under a gauge transformation

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x). \tag{2.90}$$

If we then define the covariant derivative

$$\mathcal{D}_\mu \phi(x) = \left(\partial_\mu - ieA_\mu(x) \right) \phi(x), \tag{2.91}$$

we see that it transforms nicely under gauge transformations

$$\begin{aligned} \mathcal{D}_\mu \phi(x) &\to (\mathcal{D}_\mu - i \partial_\mu \alpha(x)) e^{i\alpha(x)} \phi(x) \\ &= e^{i\alpha(x)} \mathcal{D}_\mu \phi(x). \end{aligned} \tag{2.92}$$

This makes the Lagrangian

$$\mathcal{L} = \left(\mathcal{D}_\mu \phi \right)^* \mathcal{D}^\mu \phi - m^2 \phi^* \phi \tag{2.93}$$

invariant under gauge transformations.
So far the gauge field is non-dynamic. We need to add a term involving its derivatives to the Lagrangian but it should be gauge invariant if we do not want to spoil the symmetry we just imposed. Clearly the term $\partial_\mu A^\nu$ is not gauge invariant and neither is its square. Luckily, there is an easy combination that is gauge invariant

$$\begin{aligned}[\mathcal{D}_\mu, \mathcal{D}_\nu]\phi(x) &= ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\phi(x) \\ &=: ieF_{\mu\nu}\phi(x). \end{aligned} \tag{2.94}$$

Since we know that

$$[\mathcal{D}'_\mu, \mathcal{D}'_\nu]\phi'(x) = e^{i\alpha(x)}[\mathcal{D}_\mu, \mathcal{D}_\nu]\phi(x) \tag{2.95}$$

and because $F_{\mu\nu}$ is no operator anymore we immediately see that

$$F'_{\mu\nu} = F_{\mu\nu}. \tag{2.96}$$

Of course this can also be quickly calculated. We can therefore easily form a term that is both Lorentz and gauge invariant and add it to the Lagrangian

$$\mathcal{L}_{\mathrm{sQED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(\mathcal{D}_\mu\phi\right)^* \mathcal{D}^\mu\phi - m^2\phi^*\phi. \tag{2.97}$$

2.6.2 Non-Abelian gauge symmetry

The model we discussed in the previous section is the simplest case of a gauge invariant theory. It is part of a more general set of theories. While one can in principle study gauge theories with symmetry transformations forming arbitrary Lie groups, we will focus on the case of $SU(N)$, which is the most important one for the Standard Model. Looking at the Lagrangian of N scalar fields of the same mass

$$\mathcal{L} = (\partial_\mu\phi_i)^*\partial^\mu\phi_i - m^2\phi_i^*\phi_i, \tag{2.98}$$

we see that it is not only invariant under the transformation from the previous section for any field separately, but also under a bigger class of transformations.
Indeed,

$$\phi_i \to U_{ij}\phi_j \tag{2.99}$$

leaves the Lagrangian invariant if U is a unitary matrix that obeys

$$U^\dagger = U^{-1}. \tag{2.100}$$

One says that (2.98) possesses a global $SU(N)$ symmetry. To gauge this symmetry, we write the transformation matrix in terms of $N^2 - 1$ hermitian generators T^a as

$$U = \exp(i\alpha_a T^a). \tag{2.101}$$

In the fundamental representation of $SU(N)$, those generators are $N \times N$-matrices. They form an algebra

$$[T^a, T^b] = if_{abc}T^c \tag{2.102}$$

with some structur constants f_{abc} that depend on the symmetry group and the basis of generators.
In the same way as before we allow spacetime dependent coefficients $\alpha(x)$ and can then form gauge covariant derivatives

$$(\mathcal{D}_\mu)_{ij}\phi_j = (\mathbf{1}\partial_\mu - ig\boldsymbol{A}_\mu)_{ij}\phi_j \tag{2.103}$$

where we had to introduce a generator-valued gauge field $\boldsymbol{A}_\mu = A^a_\mu T^a$. We can find the transformation law for the A^a_μ by demanding a simple transformation law for the gauge covariant derivative

$$\begin{aligned}(\mathcal{D}_\mu)_{ij}\,\phi_j(x) \to \left(\mathcal{D}'_\mu\right)_{ij}\phi'_j(x)& \\ \overset{!}{=} U_{ij}\,(\mathcal{D}_\mu)_{jk}\,\phi_k(x).&\end{aligned} \tag{2.104}$$

A quick calculation shows that this implies

$$\boldsymbol{A}'_\mu = U\boldsymbol{A}_\mu U^{-1} - \frac{i}{g}\left(\partial_\mu U\right)U^{-1}. \tag{2.105}$$

As above we can form an object that transforms much nicer than $\boldsymbol{A}_\mu$ by calculating the commutator of two covariant derivatives

$$\begin{aligned}[\mathcal{D}_\mu, \mathcal{D}_\nu]_{ij}\phi_j &= -ig\left(\partial_\mu\boldsymbol{A}_\nu - \partial_\nu\boldsymbol{A}_\mu - ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]\right)_{ij} \\ &=: (\boldsymbol{F}_{\mu\nu})_{ij}\phi_j,\end{aligned} \tag{2.106}$$

with the non-abelian field strength $\boldsymbol{F}_{\mu\nu} = F^a_{\mu\nu}T^a$.
This object now transforms as

$$\boldsymbol{F}_{\mu\nu} \to U\boldsymbol{F}_{\mu\nu}U^-1 \tag{2.107}$$

and therefore we can add $\mathrm{tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu})$ as a gauge invariant kinetic term to the Lagrangian. One usually uses the freedom in choosing a basis of generators to normalize them according to

$$\mathrm{tr}\ T^aT^b = \frac{1}{2}\delta_{ab}. \tag{2.108}$$

With this choice, the full gauge invariant Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + ((\mathcal{D}_\mu)_{ij}\phi_j)^*(\mathcal{D}^\mu)_{ik}\phi_k - m^2\phi^*_i\phi_i. \tag{2.109}$$

While we used different letters for the gauge field and the related field strength in this section, we will use the same letter for both in the following sections. The presence of one or two Lorentz indices will be sufficient to tell them apart.
It is important to notice, that mass terms for the gauge fields of the form $m^2A^\mu A_\mu$ can not be added to the Lagrangian, since they break the gauge symmetry. To explain why we still observe massive gauge bosons in nature, we need the Higgs mechanism which is going to be presented in section 2.7.1 on the scalar sector of the Standard Model.

2.6.3 Gauge group of the Standard Model

In the last section we have seen how to construct gauge invariant Lagrangians, by promoting ordinary derivatives to gauge covariant derivatives.
A physical model can therefore be characterised by listing its gauge symmetries as well as the matter content that transform under some representations of those symmetries. The Standard Model has the gauge group $SU(3) \times SU(2)_L \times U(1)_Y$ and all matter fields transform in fundamental or trivial representations of those groups. They only differ in their charge, coupling them to the gauge fields with differing strength.
The kinetic terms for the gauge sector of the Standard Model thus read

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{2.110}$$

with an 8-component gluon field $G^a_{\mu\nu}$ related to $SU(3)$-symmetry, a 3-component W-Boson field $W^a_{\mu\nu}$ related to the $SU(2)_L$-symmetry and the one-component $U(1)_Y$ B-boson field $B_{\mu\nu}$.
In the later sections we will mostly neglect the influence of the gluons, since we are interested in the one-loop effective potential of the Higgs boson, which does not couple to the gluon field. The influence of the strong interaction will only be reflected by a factor of three when summing over the degrees of freedom of fermions, due to there being three different color charges.

2.7 The scalar sector

The Standard Model contains one scalar field, the Higgs field. It transforms in the fundamental representation of $SU(2)$ and $U(1)_Y$ and trivially under $SU(3)$. Since the fundamental representation of $SU(2)$ is two dimensional, this means that the Higgs field is a complex doublet, containing 4 real fields,

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\psi_1 \\ \phi_2 + i\psi_2\end{pmatrix}. \tag{2.111}$$

The Standard Model Higgs field also has a negative mass squared and a quartic interaction term. All together the Lagrangian reads

$$\mathcal{L}_{\text{scalar}} - V[H] = (\mathcal{D}_\mu H)^\dagger \mathcal{D}_\mu H + \mu^2 H^\dagger H - \lambda(H^\dagger H)^2 \tag{2.112}$$

where the covariant derivative acting on H is given by

$$\mathcal{D}_\mu H = (\partial_\mu - ig_2 W^a_\mu \tau^a - i\frac{g_1}{2}B_\mu)H. \tag{2.113}$$

The τ^a are the generators of $SU(2)$,

$$\tau^a = \frac{\sigma^a}{2}$$
$$\sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma^2 = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \quad \sigma^1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.114}$$

2.7.1 The Higgs mechanism

Looking at the potential of the scalar sector

$$V(H) = \mu^2 H^\dagger H - \lambda (H^\dagger H)^2, \tag{2.115}$$

we realize that $H = 0$ is actually not the minimum and therefore does not correspond to the vacuum. We have argued repeatedly that it is important to expand a quantum field theory around the true vacuum. We therefore need to minimize the potential of this sector and expand the physical Higgs field as excitations around this vacuum.
$V(H)$ has a degenerate set of minima given by

$$H^\dagger H = \frac{\mu^2}{2\lambda}. \tag{2.116}$$

Once we have chosen a minimum, all other minima can be reached by $SU(2) \times U(1)$-transformations. We can therefore choose the minimum to be real and in the second component, as is usually done. Expanded around this minimum the Higgs field then reads

$$H = e^{\frac{2i}{v}\pi^a \tau^a} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix}, \tag{2.117}$$

where $v = \frac{\mu}{\sqrt{\lambda}}$. The excitations around this minimum take the form of scalar fields h and π^a, where h is a gauge singlet while the Goldstone bosons π^a transform according to

$$\pi^a \to \pi^a - \frac{v}{2}\alpha(x). \tag{2.118}$$

We can therefore pick a gauge in which $\pi^a = 0$. This is called the unitary gauge.
If one substitutes the expansion around the minimum into the kinetic terms in the Lagrangian, a lengthy calculation gives

$$\begin{aligned} \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{scalar}} =& \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} \\ & - \frac{1}{2}\left(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu\right)\left(\partial_\mu W^-_\nu - \partial_\nu W^-_\mu\right) \\ & - \frac{1}{2} m_h^2 h^2 + \frac{1}{2} m_Z^2 Z_\mu Z^\mu + m_W^2 W^+_\mu W^-_\mu + (\text{interactions}) \end{aligned} \tag{2.119}$$

with

$$\begin{aligned} Z_\mu = \cos\theta_W W^3_\mu - \sin\theta_W B_\mu, \qquad A_\mu = \sin\theta_W W^3_\mu + \cos\theta_W B_\mu \\ W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \pm i W^2_\mu)^2 \end{aligned} \tag{2.120}$$

as well as

$$m_h = \sqrt{2\lambda} v, \quad m_W = \frac{g_2}{2} v, \quad m_Z = \frac{\sqrt{g_1^2 + g_2^2}}{2} v \tag{2.121}$$

and $\tan\theta_W = \frac{g_1}{g_2}$. As we can see, by expanding the Higgs field around its true vacuum and thereby breaking gauge symmetry explicitly, we have generated mass terms for

some linear combinations of gauge bosons and for the physical Higgs field h. One symmetry stays unbroken, namely the $U(1)$-symmetry of the upper component of the Higgs doublet. The corresponding massless gauge boson is the photon.
While the Higgs mechanism is a simple way to generate masses for the gauge bosons, we see that it can be easily modified: it doesn't matter how the Higgs doublet acquires a non-zero vacuum expectation value. As soon as we have a reason to expand it in the way we did above, we will automatically generate masses for the gauge bosons.

2.8 The fermionic sector

2.8.1 Constructing a fermion Lagrangian

The last piece of the Standard Model Lagrangian is made up of the fermionic sector. A group-theoretic investigation of the Lorentz group, the group of spacetime transformations[1]

$$x^\mu = \Lambda^\mu{}_\nu x^\nu = \left(e^{-i\omega_{\alpha\beta}L^{\alpha\beta}}\right)^\mu{}_\nu x^\nu \tag{2.122}$$

shows, that all of its irreducible representations can be counted according to (j_1, j_2), where $j_{1,2} = 0, \frac{1}{2}, 1, \ldots$.
While the scalar field and the gauge field fall in the $(0,0)$ or $(\frac{1}{2}, \frac{1}{2})$ representation respectively, we need two more representations to describe the entire Standard Model. We define the left- and righthanded Weyl spinors, which are two-component objects transforming under Lorentz transformations according to

$$\psi_{L,R} \to e^{-i\omega_{\mu\nu}S^{\mu\nu}_{L,R}}\psi_{L,R} \tag{2.123}$$

with generators

$$\begin{aligned} S^{ij}_{L,R} &= \frac{1}{2}\epsilon^{ijk}\sigma^k, \\ S^{0k}_{L,R} &= \mp\frac{i}{2}\sigma^k. \end{aligned} \tag{2.124}$$

The σ^i are given in (2.114). Because the boost generators S^{0k} are anti-hermitian, it is clear that combinations like $\psi_L^\dagger\psi_L$ are not Lorentz invariant. One way to define a mass term nevertheless is to combine a left- and a righthanded Weyl spinor to form a Dirac spinor

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \bar{\psi} = \begin{pmatrix} \psi_R^\dagger & \psi_L^\dagger \end{pmatrix}. \tag{2.125}$$

This allows for the Lorentz invariant mass term

$$m\bar{\psi}\psi = m\left(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R\right). \tag{2.126}$$

[1] The Lorentz group is generated by the

$$(L^{\alpha\beta})^\mu{}_\nu = i(\eta^{\alpha\mu}\delta^\beta_\nu - \eta^{\beta\mu}\delta^\alpha_\nu)$$

and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ are a set of parameters describing the transformation.

A kinetic term can be found by realising that $\psi^\dagger_{L,R}\sigma^\mu\psi_{L,R}$ and $\psi^\dagger_{L,R}\bar{\sigma}^\mu\psi_{L,R}$ with

$$\sigma^\mu = (\mathbf{1}, \vec{\sigma}), \qquad \bar{\sigma}^\mu(\mathbf{1}, -\vec{\sigma}) \tag{2.127}$$

transform like vectors. Thus, a proper hermitian kinetic term for the Weyl spinors is $\psi^\dagger_R i\sigma^\mu\partial_\mu\psi_R$ or $\psi^\dagger_L i\bar{\sigma}^\mu\partial_\mu\psi_L$ respectively and the Lagrangian for the Dirac spinor can be written as

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi, \tag{2.128}$$

with the gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{2.129}$$

Oftentimes, left and right handed Weyl spinors are written as projections of four-component spinors with only two non-vanishing components, such that

$$\psi_L = P_L\psi, \qquad \psi_R = P_R\psi \tag{2.130}$$

where

$$P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{2.131}$$

Thus, even though Weyl spinors have only two non-vanishing components, they are sometimes used as four-component objects.

2.8.2 Gauge interaction of fermions

Similar to the scalar theory in section 2.6.1, the theory described by the action (2.128) has a global symmetry

$$\psi \to e^{i\alpha}\psi. \tag{2.132}$$

We can again turn this into a gauge invariant theory by introducing the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - ieA_\mu. \tag{2.133}$$

By adding a kinetic term we arrive at the Lagrangian of Quantum Electrodynamics

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu\mathcal{D}_\mu - m)\psi. \tag{2.134}$$

Again we can also look at models with multiple spinor fields transforming in some representation of $SU(N)$. For those theories the gauge invariant Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \bar{\psi}_i\left(i\gamma^\mu\mathcal{D}_\mu - m\right)_{ij}\psi_j \tag{2.135}$$

with covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - ieA^a_\mu t^a, \tag{2.136}$$

where the t^a are the generators of the gauge group corresponding to the representation the ψ_i transform under.
It is also possible to couple left- or righthanded Weyl fermions to gauge fields using the covariant derivative.

2.8.3 Fermions in the Standard Model

An interesting thing about the Standard Model is that while it contains both left- and righthanded fermions, only the lefthanded ones transform non-trivially under the $SU(2)_L$ symmetry. The kinetic term for the fermions reads

$$\begin{aligned}\mathcal{L}_{\text{kinetic}} =& \bar{L}^i \gamma^\mu (\partial_\mu - i g_2 W^a_\mu \tau^a + \frac{i}{2} g_1 B_\mu) L^i + \bar{Q}^i i \gamma^\mu (\partial_\mu - i g_2 W^a_\mu \tau^a - \frac{i}{6} g_1 B_\mu) Q^i \\ &+ \bar{e}^i_R i \gamma^\mu (\partial_\mu + i g_1 B_\mu) e^i_R + \bar{u}^i_R i \gamma^\mu (\partial_\mu - i \frac{2}{3} g_1 B_\mu) u^i_R + \bar{d}^i_R i \gamma^\mu (\partial_\mu + \frac{i}{3} g_1 B_\mu) d^i_R\end{aligned} \tag{2.137}$$

where L^i and Q^i are the left-handed lepton and quark doublets respectively,

$$\begin{aligned} L^i &= \left\{ \begin{pmatrix} e_L \\ \nu_{e,L} \end{pmatrix}, \begin{pmatrix} \mu_L \\ \nu_{\mu,L} \end{pmatrix}, \begin{pmatrix} \tau_L \\ \nu_{\tau,L} \end{pmatrix} \right\} \\ Q^i &= \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}, \end{aligned} \tag{2.138}$$

whereas e^i_R, u^i_R and d^i_R contain the corresponding righthanded components. In the Standard Model, there are no righthanded neutrinos.
It is clear, that the $SU(2)_L$ symmetry forbids the Dirac mass terms we mentioned above, since a combination like $e^\dagger_L e_R + e^\dagger_R e_L$ simply is not gauge invariant. The solution for this is again to use the Higgs mechanism. Clearly, the combination

$$\mathcal{L}_{\text{Yukawa}} = -y_e \bar{L} H e_R + \text{h.c.} \tag{2.139}$$

will be gauge invariant, since both L and H transform as doublets under $SU(2)$ and also the hypercharges related to $U(1)_Y$ add up in an appropriate way. The h.c. denotes the hermitean conjugate, which is needed to arrive at a real Lagrangian.
After expanding the theory around the true vacuum, the above interaction between the Higgs field and the fermions induces the effective mass term

$$\mathcal{L}_{\text{mass}} = -m_e (\bar{e}_L e_R + \bar{e}_L e_R), \tag{2.140}$$

where $m_e = \frac{y_e}{\sqrt{2}} v$. We see again, that by expanding the Higgs field around the minimum of its potential and thereby breaking gauge invariance, we were able to generate effective mass terms which were formerly forbidden by the gauge symmetry.
While the full mass term of the Standard Model allows for mixing between the flavour eigenstates we wrote down above, we will only be interested in the biggest term in our calculations: the term for the top quark.

2.8.4 Path integral for fermions

When calculating the effective potential for theories involving fermions, we will also need to perform path integrals over fermionic fields. It follows from the spin-statistic-theorem that fermionic fields can not be described by ordinary commuting numbers. Instead one has to use anti-commuting Grassmann numbers

$$\theta_1 \theta_2 = -\theta_2 \theta_1, \qquad \theta_i^2 = 0. \tag{2.141}$$

For those, integration is defined by

$$0 = \int \mathrm{d}\theta, \qquad 1 = \int \mathrm{d}\theta\ \theta. \tag{2.142}$$

These rules can be used to perform the Gaussian integral

$$\begin{aligned}\int \mathrm{d}\bar{\theta}\mathrm{d}\theta e^{-\bar{\theta}A\theta} &= \int \mathrm{d}\bar{\theta}\mathrm{d}\theta(1-\bar{\theta}A\theta) \\ &= A \end{aligned} \tag{2.143}$$

and generalized to a multi-dimensional integral one finds

$$\int \prod_i \left(\mathrm{d}\bar{\theta}_i \mathrm{d}\theta_i\right) e^{-\bar{\theta}_i A_{ij}\theta_j} = \det A. \tag{2.144}$$

for a hermitian matrix A. In calculating the effective potential we encounter integrals of the form

$$I = \int [\mathcal{D}\bar{\psi}\mathcal{D}\psi]\ e^{i\int \mathrm{d}^4x\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)} = \mathrm{Det}(-i\gamma^\mu\partial_\mu + m). \tag{2.145}$$

To calculate the determinant in the infinite dimensional vector space of functions, we again take the logarithm of this expression and calculate the appearing trace in momentum space

$$\ln I = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left(\ln\ (\gamma^\mu p_\mu + m)\right), \tag{2.146}$$

where tr() is the trace in the 4-dimensional space the γ^μ act on. By expanding the logarithm into an infinite sum, we can further evaluate this integral

$$\begin{aligned}\ln I &= \frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\ \left(\ln(\gamma^\mu p_\mu + m) + \ln(-\gamma^\nu p_\nu + m)\right) \\ &= 2\int \frac{\mathrm{d}^4p}{(2\pi)^4}\ln\ (m^2 - p^2), \end{aligned} \tag{2.147}$$

where we used the fact that the integration is symmetric under $p \to -p$ as well as the identities

$$\gamma^\mu p_\mu \gamma^\nu p_\nu = p^2, \qquad \mathrm{tr}\ \gamma^\mu = 0, \qquad \mathrm{tr}\ \mathbb{1} = 4. \tag{2.148}$$

Up to an infinite constant which is independent from m this integral is now equal to (2.63).

2.9 Experimental input

Now that we have described all parts of the Standard Model, we will give some numerical values to the appearing parameters. We will need those when we look for scale-invariant models that allow an embedding of the Standard Model.
The vacuum expectation value of the Higgs field is known from measurements of the Fermi constant [19], which describes the interaction strength in the effective theory of

the weak interaction where the Higgs field has been integrated out. Its numerical value is [20]

$$v = (246.21965 \pm 0.00006)\ \text{GeV}. \tag{2.149}$$

The most precise measurement of the Higgs mass is [21]

$$m_h = (125.1 \pm 0.3)\ \text{GeV}. \tag{2.150}$$

We will also need the coupling constants of the electroweak interaction and the Yukawa coupling of the top quark. They depend on the scale at which predictions are made and since we will be looking at the minimum of the effective potential at the scale of the Higgs vacuum expectation value, we took the values from Butazzo et. al. [22] evaluated at 246.22 GeV:

$$\begin{aligned} g_1(v) &= 0.3590 \pm 0.0005 \\ g_2(v) &= 0.64596 \pm 0.0002 \\ g_3(v) &= 1.14200 \pm 0.003 \\ y_t(v) &= 0.918 \pm 0.006 \end{aligned} \tag{2.151}$$

with the electroweak couplings g_1 and g_2, the strong coupling g_3 and the top Yukawa coupling y_t.
It is important to realize that the quartic coupling of the Higgs field is so far not measured and can only be inferred from the values of m_h and v in the Standard Model.

2.10 Scale-invariant extensions of the Standard Model

As can be seen from the above discussion, there is only one independent dimensionful parameter in the Standard Model. Clearly, if we set the Higgs mass parameter μ to zero, v and m_h will vanish as well.
This makes the Standard Model classically scale-invariant, i.e. its action becomes invariant under transformations of the form

$$\begin{aligned} x &\to \Lambda x \\ H &\to \Lambda^{-1} H \\ A^\mu &\to \Lambda^{-1} A^\mu \\ \psi &\to \Lambda^{-\frac{3}{2}} \psi \end{aligned} \tag{2.152}$$

where A^μ and ψ stand for all gauge fields and fermions respectively. If we alternatively impose this transformation as a classical symmetry, then the Higgs mass term is forbidden. Clearly, without this term there is also no spontanous breaking of gauge symmetry and therefore no effective mass terms for gauge bosons nor fermions at the classical level.
We saw in the section on the Coleman-Weinberg mechanism, that such a classically scale-invariant model can acquire an explicit scale dependence as soon as quantum effects are turned on, thereby inducing a vacuum expectation value for the Higgs field

and triggering Electroweak symmetry breaking. In the next chapter, we want to study this scalefree Standard Model and extensions of it by additional massless fields which keep the classical scale invariance intact and look for the simplest model which allows for a minimum which fits with the experimental values of the couplings from section 2.9.

Chapter 3
Calculation of Effective Potentials in Various Theories

In chapter 2 we described the general method how to calculate the effective potential of a quantum field theory in order to find the true vacuum state. We will now apply this formalism to five different models and discuss the results.
First, we briefly and informally study a model of two massive scalars to convince ourselves of the appearence of the hierarchy problem. To resolve this problem we then turn to scalefree theories involving only massless particle. We try to find a scalefree model that consistently implies a vacuum state with the general properties of the standard model that are known from experiments.

It has been known for a long time that a scalefree version of the Standard Model itself without any further modifications has no stable vacuum because of the influence of the heavy fermions. To develop the formalism we nevertheless study two models which we call the Scalefree Electroweak Theory, containing only the Electroweak Gauge bosons and the Higgs boson, and the Scalefree Standard Model, which additionally contains the top quark. We neglect influences of the other quarks and since we work entirely at one-loop level and the Higgs has no color charge, we can neglect all contributions to the effective potential that come from the strong interaction. They will be of higher loop order.

After realising that the effective potentials of those models do not have stable minima which fit with experimental data, we start to extend the Standard Model by additional fields. First we look at the Conformal Standard Model as suggested by Meissner and Nikolai [7], which adds a new massless scalar field to the Standard Model that is not charged under any of the gauge interactions. We find that after imposing the hierarchy of couplings which is necessary for consistency of the perturbative series, this model also does not allow for a stable vacuum which resembles the Standard Model.
We then propose the Minimal Conformal Standard Model to be an extension of this Conformal Standard Model which includes a new U(1) gauge symmetry under which the new scalar is charged. We show that this model allows for a minimum of the effective potential that resembles the Standard Model. To study if this minimum is also the global minimum of the effective potential, we use the renormalization group equations to evolve the coupling constants up to the Planck scale. We find that this model inherits the vacuum instability of the Standard Model.

To make sure that this is not a general problem of scale-invariant theories, we extend our model by one further massless ‘dormant’ scalar, which doesn’t acquire a vacuum expectation value and can therefore have couplings that don’t obey any hierarchy. This allows for a stabilisation of the minimum of the effective potential.

We perform all calculations using Dimensional Regularisation and first results are always given in the $\overline{\text{MS}}$-renormalization scheme until we apply further renormalization conditions.

3.1 Massive Two-Scalar Model

As mentioned in the introduction, every field interacting with the Higgs gives corrections to the Higgs mass. To see this explicitly in a simple case, we are investigating a model that contains only two massive scalar fields interacting with each other[1].
The Lagrangian for this model is

$$\begin{aligned} \mathcal{L}(\phi_1,\phi_2) &= \mathcal{L}_{\text{kin}}(\phi_1,\phi_2) - V(\phi_1,\phi_2), \\ \mathcal{L}_{\text{kin}}(\phi_1,\phi_2) &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2, \\ V(\phi_1,\phi_2) &= \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}m_2^2\phi_2^2 + \frac{\lambda_1}{4}\phi_1^4 + \frac{\lambda_{12}}{2}\phi_1^2\phi_2^2 + \frac{\lambda_2}{4}\phi_2^4. \end{aligned} \tag{3.1}$$

We restrict ourselves to renormalizable interactions and also enforce the discrete symmetry $\phi_i \to -\phi_i$.
We now calculate the one-loop effective potential, using the functional method outlined in chapter 2. To do this, we expand the quantum fields around homogenous classical field values $\hat{\phi}_i$ and keep only the terms up to quadratic order in the remaining quantum fluctuations. Let us also assume, that λ_2 is so big, that only ϕ_1 can aquire a vacuum expectation value.
In momentum space this leads to

$$\mathcal{L}_{1-\text{loop}}(\hat{\phi}_1+\phi_1,\phi_2) = -V(\hat{\phi}_1,0) - \frac{1}{2}\sum_{i,j}\phi_i\Sigma_{ij}(\hat{\phi}_1,\hat{\phi}_2=0)\phi_j, \tag{3.2}$$

where the matrix Σ_{ij} is given by

$$(\Sigma_{ij}) = \begin{pmatrix} -p^2+m_1^2+3\lambda_1\hat{\phi}_1^2 & 0 \\ 0 & -p^2+m_2^2+\lambda_{12}\hat{\phi}_1^2 \end{pmatrix}. \tag{3.3}$$

The effective potential then reads

$$\begin{aligned} V_{\text{eff}}(\phi_1) =& V(\phi_1,0) - \frac{i}{2}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\ln\det M \\ =& V(\phi_1,0) + \frac{M_1^4}{32\pi^2}\left(\ln\frac{M_1^2}{\mu^2}-\frac{3}{2}\right) + \frac{M_2^4}{32\pi^2}\left(\ln\frac{M_2^2}{\mu^2}-\frac{3}{2}\right) \\ &+ (\text{divergent}), \end{aligned} \tag{3.4}$$

with the mass scales

$$\begin{aligned} M_1^2 &= \left(m_1^2+3\lambda_1\phi_1^2\right) \\ M_2^2 &= \left(m_2^2+3\lambda_{12}\phi_1^2\right). \end{aligned} \tag{3.5}$$

[1]We will use some strong assumptions in order to quickly arrive at analytical results, but we have checked the model with less restrictions numerically and find agreeing results.

We also dropped the hats over the classical field values to simplify the notation. Even in this simple case it is non-trivial to find the exact minimum of the effective potential but luckily it is not necessary for the discussion at hand. We just want to note that the term containing M_2^4 involves the combination $-\frac{3\lambda_{12}}{16\pi^2}m_2^2\phi_1^2$.
If we assume m_2 to be big enough (and λ_1 to be small enough while also choosing $\mu = m_2$) for this term to be the dominant contribution, the effective potential up to slowly varying or constant terms roughly reads

$$V_{\text{eff}}(\phi_1) \approx \frac{\lambda_1}{4}\phi_1^4 + \frac{m_1}{2}\phi_1^2 - \frac{3\lambda_{12}}{16\pi^2}m_2^2\phi_1^2 \tag{3.6}$$

having its minimum at $\lambda_1 v^2 = \frac{3\lambda_{12}}{8\pi^2}m_2^2 - m_1^2$ with a mass for the physical Higgs boson of

$$m_h^2 = \left.\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\phi_1^2}\right|_{\phi_1 \to v} = \frac{3\lambda_{12}}{4\pi^2}m_2^2 - 2m_1^2. \tag{3.7}$$

Now we see, that for m_H to be many magnitues smaller than m_2, m_1 needs to be finely tuned in order to differ only by a tiny fraction from the first term. This is considered highly unnatural and is the basis of the hierarchy problem.
A similar calculation can also be performed for heavy fermions or vector bosons, leading to the same general result: in a theory with heavy degrees of freedom but light scalar bosons coupled to them, an unnatural amount of fine-tuning is needed.
Of course it is possible that the contributions of different heavy fields cancel out to result in a light scalar-boson, but in the absence of a (weakly broken) symmetry this case is also considered unnatural. Supersymmetry provides exactly this: it ensures that the competing contributions from fermions and bosons cancel out exactly and therefore resolves the hierarchy problem.
As already mentioned in the introduction we want to take another route to solve the problem: by enforcing classical scale invariance, all explicit mass terms are forbidden. As we will see this allows the emergence of a hierarchy of scales without fine-tuned cancellations. Our aim in the following sections will therefore be to find a classically scale-invariant model, that incorporates the Standard Model and is perturbatively consistent.

3.2 Scalefree Electroweak Theory

Before we try to incorporate the entire Standard Model, let us start with the Scalefree Electroweak Theory (SFET) which is composed of the Electroweak gauge theory and a massless Higgs doublet. Since the quadratic term of the Higgs potential is the only explicit scale in the SM, this renders the model scale-invariant.
This model was also discussed by Coleman and Weinberg. While we will see that its predictions are inaccurate due to the neglection of fermionic contributions to the effective potential, it has some historical relevance and will help us develop the formalism to investigate the full Standard Model and its extensions.
The SFET is described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}} - V(H) \tag{3.8}$$

where $\mathcal{L}_{\text{kin}}$, $\mathcal{L}_{\text{gf}}$ and $\mathcal{L}_{\text{ghost}}$ are the typical kinetic terms involving the gauge fields $W_{1,2,3}, B$ and the minimally coupled complex scalar doublet H, the gauge fixing and the ghost Lagrangian of the Electroweak Theory and

$$V(H) = \lambda(H^\dagger H)^2 \tag{3.9}$$

is the Higgs potential without the quadratic term.
To be able to apply the functional method from chapter 2, we expand the complex Higgs doublet according to

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\psi_1 \\ \hat{\phi}_2 + \phi_2 + i\psi_2 \end{pmatrix} \tag{3.10}$$

with a real classical field $\hat{\phi}_2$ and real quantum fields ϕ_i, ψ_i. By using this expansion we directly break the Electroweak symmetry but it can be easily restored by replacing $\frac{1}{\sqrt{2}}\hat{\phi}_2 \to |\hat{H}|$ in the final result. Using the above expansion the momentum space Lagrangian reads up to quadratic order in quantum fields

$$\mathcal{L}_{\text{1-loop}}(\phi_1, \psi_1, \phi_2 + \hat{\phi}_2, \psi_2) = -V(\hat{\phi}_2) - \frac{1}{2}\Phi^T \Sigma \Phi. \tag{3.11}$$

Here, Φ denotes the vector formed by all quantum fields

$$\Phi = \begin{pmatrix} W_{1,\mu} & W_{2,\mu} & W_{3,\mu} & B_\mu & \phi_1 & \phi_2 & \psi_1 & \psi_2 \end{pmatrix}^T \tag{3.12}$$

and Σ is the matrix with components

$$\Sigma_{ij} = -\left.\frac{\partial \mathcal{L}}{\partial \Phi_i \partial \Phi_j}\right|_{\Phi = \hat{\Phi}}, \tag{3.13}$$

which is written in block form as

$$\Sigma = \begin{pmatrix} \Delta_{\mu\nu} & M_\nu \\ M_\nu^\dagger & D \end{pmatrix}. \tag{3.14}$$

The entries of M are themselves matrices

$$\begin{aligned}
\Delta_{\mu\nu} &= \begin{pmatrix} \Delta^{(W)}_{\mu\nu} - \frac{1}{4}g_2^2\hat{\phi}_2^2\eta_{\mu\nu} & 0 & 0 & 0 \\ 0 & \Delta^{(W)}_{\mu\nu} - \frac{1}{4}g_2^2\hat{\phi}_2^2\eta_{\mu\nu} & 0 & 0 \\ 0 & 0 & \Delta^{(W)}_{\mu\nu} - \frac{1}{4}g_2^2\hat{\phi}_2^2\eta_{\mu\nu} & \frac{g_1 g_2}{4}\hat{\phi}_2^2\eta_{\mu\nu} \\ 0 & 0 & \frac{g_1 g_2}{4}\hat{\phi}_2^2\eta_{\mu\nu} & \Delta^{(B)}_{\mu\nu} - \frac{1}{4}g_2^2\hat{\phi}_2^2\eta_{\mu\nu} \end{pmatrix} \\
M_\mu &= \begin{pmatrix} 0 & \frac{i}{2}g_2\hat{\phi}_2 p_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{i}{2}g_2\hat{\phi}_2 p_\mu & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2}g_2\hat{\phi}_2 p_\mu & \frac{i}{2}g_1\hat{\phi}_2 p_\mu \end{pmatrix} \\
D &= \begin{pmatrix} -p^2 + \lambda\hat{\phi}_2^2 & 0 & 0 & 0 \\ 0 & -p^2 + 3\lambda\hat{\phi}_2^2 & 0 & 0 \\ 0 & 0 & -p^2 + \lambda\hat{\phi}_2^2 & 0 \\ 0 & 0 & 0 & -p^2 + \lambda\hat{\phi}_2^2 \end{pmatrix}
\end{aligned} \tag{3.15}$$

and

$$\Delta_{\mu\nu}^{(\alpha)} = p^2\eta_{\mu\nu} - \left(1 - \frac{1}{\xi_\alpha}\right) p_\mu p_\nu. \tag{3.16}$$

We find the effective potential using (2.60)

$$V_{\text{eff}}(\phi_2) = V(\phi_2) - \frac{i}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4} \ln\det\Sigma$$

giving the result

$$V_{\text{eff}}(\phi_2) = \frac{\lambda}{4}\phi_2^4 + \frac{1}{64\pi^2}\sum_i n_i m_i^4 \left(\ln\frac{m_i^2}{\mu^2} - a_i\right). \tag{3.17}$$

The m_i are the effective masses of the different fields of the model, represented by the eigenvalues of the mass matrix M at zero momentum. They are given by

$$\begin{aligned}
m_A^2 &= \frac{g_2^2}{4}\phi_2^2 \\
m_B^2 &= \frac{(g_1^2 + g_2^2)}{4}\phi_2^2 \\
m_C^2 &= 3\lambda\phi_2^2 \\
m_{D\pm}^2 &= \frac{1}{2}\left(\lambda \pm \sqrt{\lambda(\lambda - g_2^2\xi_W)}\right)\phi_2^2 \\
m_{E\pm}^2 &= \frac{1}{2}\left(\lambda \pm \sqrt{\lambda(\lambda - g_1^2\xi_B + g_2^2\xi_W)}\right)\phi_2^2.
\end{aligned} \tag{3.18}$$

The n_i count the degrees of freedom belonging to the respective effective mass and the b_i are different for scalar and vector bosons. They are given by

$$\begin{aligned}
&n_A = 6, \qquad n_B = 3, \\
&n_C = 1 = n_E, \quad n_D = 2, \\
&b_A = \frac{5}{6} = b_B, \\
&b_C = \frac{3}{2} = b_D = b_E.
\end{aligned} \tag{3.19}$$

By an appropriate choice of counterterms in the original Lagrangian and by choosing the Landau gauge $\xi_W = \xi_B = 0$ we can bring the effective potential to the form

$$V_{\text{eff}}(\phi_2) = \frac{\lambda}{4}\phi_2^4 + \frac{1}{64\pi^2}\left(\frac{3}{8}g_2^4 + \frac{3}{16}(g_1^2 + g_2^2)^2 + 12\lambda^2\right)\phi_2^4\left(\ln\frac{\phi_2^2}{\mu^2} - \frac{25}{6}\right). \tag{3.20}$$

This corresponds to an on-shell renormalization scheme, in which

$$\begin{aligned}
\left.\frac{\mathrm{d}^2V_{\text{eff}}}{\mathrm{d}\phi_2^2}\right|_{\phi_2=0} &= 0 \\
\frac{1}{3!}\left.\frac{\mathrm{d}^4V_{\text{eff}}}{\mathrm{d}\phi_2^4}\right|_{\phi_2=\mu} &= \lambda
\end{aligned} \tag{3.21}$$

Higher order corrections to the effective potential are not only an expansion in the coupling constants but also in $\log \frac{\phi_2^2}{\mu^2}$. This means that our approximation loses accuracy if μ and ϕ_2 are too different.
Since we are interested in the extremum of the effective potential and since μ is so far an arbitrary scale, we can choose μ to be exactly the position of the extremum $\langle\phi_2\rangle$. Enforcing the consistency condition

$$\left.\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\phi_2}\right|_{\phi=\langle\phi_2\rangle} = 0 \tag{3.22}$$

leads to a condition that the couplings must obey

$$\lambda = \frac{11}{256\pi^2}\left(g_1^4 + 2g_1^2 g_2^2 + 3g_2^4 + 64\lambda^2\right). \tag{3.23}$$

Clearly, if λ were of the same order of magnitude as λ^2 or even smaller, perturbation theory would be highly inaccurate. Therefore, in order to arrive at a result that can be trusted, we need to assume

$$\lambda = \mathcal{O}(g_1^4, g_2^4). \tag{3.24}$$

This means that we can neglect the term proportional to λ^2 and arrive at

$$\lambda = \frac{11}{256\pi^2}\left(g_1^4 + 2g_1^2 g_2^2 + 3g_2^4\right). \tag{3.25}$$

Plugging this back into the effective potential, it takes the compact form

$$\begin{aligned} V_{\text{eff}}(\phi_2) &= \frac{3}{1024\pi^2}\left(g_1^4 + 2g_1^2 g_2^2 + 3g_2^4\right)\phi_2^4\left(\log\frac{\phi_2^2}{\langle\phi_2\rangle^2} - \frac{1}{2}\right) \\ &= \frac{3}{64\pi^2}\left(2m_W^4 + m_Z^4\right)\frac{\phi_2^4}{\langle\phi_2\rangle^4}\left(\log\frac{\phi_2^2}{\langle\phi_2\rangle^2} - \frac{1}{2}\right) \end{aligned} \tag{3.26}$$

where we recognized the masses of the gauge bosons m_W and m_Z, as explained in section 2.6.1.
While the position of the minimum itself is a free parameter in this model, we can predict relationships between effective masses. The effective mass of the Higgs particle is given by

$$\begin{aligned} m_h^2 = \left.\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\phi_2^2}\right|_{\phi_2=\langle\phi_2\rangle} &= \frac{3}{8\pi^2}\left(2m_W^4 + m_Z^4\right)\frac{1}{\langle\phi_2\rangle^2} \\ &= \frac{3g_2^2}{32\pi^2}\frac{2m_W^4 + m_Z^4}{m_W^2}. \end{aligned} \tag{3.27}$$

Substituting the experimental values of the W and Z boson masses we arrive at

$$m_h = 9.7 \text{ GeV}. \tag{3.28}$$

Comparing this result to the experimental value of the Higgs mass [21]

$$m_h^{\text{exp}} = (125.1 \pm 0.3) \text{ GeV} \tag{3.29}$$

shows a big discrepancy. This is no surprise: we neglected contributions of the fermions and since the top quark acquires the biggest effective mass from the Higgs mechanism it also gives the strongest contribution to the Higgs mass.
Nevertheless, this calculation was worth doing - it gave us a general idea how to find estimations for the effective Higgs mass. There will be only slight modifications to the above calculation necessary when dealing with more realistic models.

Let us recapitulate what we have been doing. We first calculated the effective potential of an interacting theory in the loop expansion. We renormalized the theory at the exact position of the minimum, making renormalization improvement unnecessary, since it will only effect the result significantly at field values away from the renormalization scale. We then forced this scale to actually be the minimum of the potential, leading to a condition relating the couplings at that scale.
Looking at this condition (3.23), there are only two self-consistent ways of solving it: we can assume either $\lambda \sim \lambda^2$, rendering perturbation theory invalid, or $\lambda \sim g_i^4$. The second choice corresponds to a resorting of perturbation theory. Instead of the typical loop expansion we perform an expansion in powers of the gauge couplings - even though the λ^2 contribution comes from a one-loop diagram, it is still much smaller than the e^4 contributions and can therefore be neglected.
After imposing this hierarchy on the couplings, we were able to express the Higgs mass predicted by this model in terms of the masses of the electroweak gauge bosons. We will perform the calculations in more complicated models in exactly the same way.

3.3 Scalefree Standard Model

As mentioned in the previous section, we can not neglect the influence of fermionic fields in a realistic model. Since all fermions give identical contributions to the effective potential that differ only in their respective Yukawa coupling, we can collect the influences of all fermions in a single effective Yukawa interaction. Since the top Yukawa coupling is by far the biggest, this effective Yukawa coupling can be reasonably well approximated by the coupling of the top.
The Lagrangian that we will use to study the properties of the Scalefree Standard Model is therefore given by

$$\mathcal{L}_{SFSM} = \mathcal{L}_{SFET} + i\bar{Q}^3\gamma^\mu\partial_\mu Q^3 + i\bar{t}_R\gamma^\mu\partial_\mu t_R + i\bar{b}_R\gamma^\mu\partial_\mu b_R - y_t(\bar{Q}^3\tilde{H}t_R + \text{h.c}), \quad (3.30)$$

where $\mathcal{L}_{SFET}$ is the Lagrangian of the Scalefree Electroweak Theory discussed in the last section and following the conventions from chapter 2

$$Q^3 = \begin{pmatrix} t_L \\ b_L \end{pmatrix} \quad (3.31)$$

and $\tilde{H} = i\sigma_2 H^*$. It makes no difference if we include couplings of the top quark to the gauge bosons, since they will not influence the effective potential in the one-loop approximation.
After expanding the Higgs field around its vacuum expectation value the relevant contribution from the top quark is

$$\mathcal{L}_{\text{1-loop, top}} = -\frac{y_t}{\sqrt{2}}\hat{\phi}_2\bar{\psi}_t\psi_t \quad (3.32)$$

with

$$\psi_t = \begin{pmatrix} t_L \\ t_R \end{pmatrix}. \tag{3.33}$$

Since the quark fields are Grassmann valued, the Gaussian integral will give not an inverse determinant, as argued in section 2.8.4, but the determinant itself, leading to the contribution

$$V_{\text{eff, top}}(\phi_2) = -\frac{3y_t^4}{64\pi^2}\phi_2^4 \left(\ln\frac{\phi_2^2}{\mu^2} - \frac{3}{2}\right). \tag{3.34}$$

The factor of three comes from the fact that there are three colors of quarks, coupling them to the strong interaction.
The rest of the effective potential stays the same as in (3.20). Again we choose μ to be the position of the extremum of the effective potential, in order to achieve the optimal behaviour of the perturbation series. The consistency condition (3.22) now reads

$$\lambda = \frac{11}{256\pi^2}(g_1^4 + 2g_1^2 g_2^2 + 3g_2^4 - 16y_t^4 + 64\lambda^2). \tag{3.35}$$

With the same argument as in the section above we enforce a hierarchy of couplings in order to arrive at a result can be trusted perturbatively. The effective potential is then given by

$$V_{\text{eff}}(\phi_2) = \frac{3}{64\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)\frac{\phi_2^4}{\langle\phi_2\rangle^4}\left(\ln\frac{\phi_2^2}{\langle\phi_2\rangle^2} - \frac{1}{2}\right). \tag{3.36}$$

From this we can extract the effective mass of the Higgs boson and substitute experimental values for the couplings to arrive at

$$\begin{aligned} m_h^2 = \left.\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\phi_2^2}\right|_{\phi_2 = \langle\phi_2\rangle} &= \frac{3}{8\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)\frac{1}{\langle\phi_2\rangle^2} \\ &= \frac{3g_2^2}{32\pi^2}\frac{2m_W^4 + m_Z^4 - 4m_t^4}{m_W^2} \\ &\approx -1300\text{GeV}^2. \end{aligned} \tag{3.37}$$

Since we find a negative value for the Higgs mass squared, this extremum of the effective potential is actually not a minimum but a maximum. While this field value would solve the classical equation of motion derived from the effective action, the solution would be unstable and not represent the vacuum solution, which is defined to be the absolute minimum of the effective potential.
This renders the scalefree Standard Model unfit to describe the real world. It does not however mean that this model is entirely inconsistent - of course it is possible to arrange for a set of mass values in which the extremum we found above is actually a minimum. But those mass values do not fit with the ones that are known from experiments. Alternatively one could start with the Standard Model couplings and an arbitrary scalar coupling λ at the scale of the Higgs vacuum and evolve them using the renormalization group equation until (3.35) is

3.4 The Conformal Standard Model

In this section we are going to apply the developed formalism to the Conformal Standard Model as it was suggested by Meissner and Nicolai [7]. The model is described by the following Lagrangian

$$\mathcal{L}_{\text{CSM}} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{Yuk}} - V(H,S) \tag{3.38}$$

with kinetic terms for the SM fields as well as one new complex scalar S that does not participate in the SM gauge interactions, Yukawa couplings of the SM fermions to the Higgs field H and a scalar potential

$$V(H,S) = \lambda_1 (H^\dagger H)^2 + \lambda_{12}(H^\dagger H)(S^\dagger S) + \lambda_2 (S^\dagger S)^4. \tag{3.39}$$

While there are no explicit mass scales in this model, the portal coupling between the Higgs field and the new scalar can still act as a mass term for the Higgs and trigger the electroweak symmetry breaking, should S develop a vacuum expectation value.
We simply adapt the formalism developed in chapter 2 to the situation of two scalar fields by expanding both of them around classical fields

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\psi_1 \\ \hat{\phi}_2 + \phi_2 + i\psi_2 \end{pmatrix} \qquad\qquad S = \frac{1}{\sqrt{2}} \left(\hat{S}_1 + S_1 + iS_2 \right). \tag{3.40}$$

The Lagrangian containing all one-loop terms is then given by

$$\mathcal{L}_{\text{1-loop}} = -\frac{\lambda_1}{4}\hat{\phi}_2^4 - \frac{\lambda_{12}}{4}\hat{\phi}_2^2\hat{S}_1^2 - \frac{\lambda_2}{4}\hat{S}_1^4 - \frac{1}{2}\Phi^T \underbrace{\begin{pmatrix} D & M_\mu & P \\ M_\mu^\dagger & \bar{\Delta}_{\mu\nu} & 0 \\ P^\dagger & 0 & D_S \end{pmatrix}}_{=:\Sigma} \Phi - \frac{y_t}{\sqrt{2}}\hat{\phi}_2 \bar{\psi}_t \psi_t, \tag{3.41}$$

where

$$D = \begin{pmatrix} \Box + \lambda_1\hat{\phi}_2^2 + \frac{\lambda_{12}}{2}\hat{S}_1^2 & 0 & 0 & 0 \\ 0 & \Box + 3\lambda_1\hat{\phi}_2^2 + \frac{\lambda_{12}}{2}\hat{S}_1^2 & 0 & 0 \\ 0 & 0 & \Box + \lambda_1\hat{\phi}_2^2 + \frac{\lambda_{12}}{2}\hat{S}_1^2 & 0 \\ 0 & 0 & 0 & \Box + \lambda_1\hat{\phi}_2^2 + \frac{\lambda_{12}}{2}\hat{S}_1^2 \end{pmatrix},$$

$$M_\mu = \begin{pmatrix} 0 & -\frac{1}{2}g_2\hat{\phi}_2\partial_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}g_2\hat{\phi}_2\partial_\mu & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}g_2\hat{\phi}_2\partial_\mu & -\frac{1}{2}g_1\hat{\phi}_2\partial_\mu \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ \lambda_{12}\hat{\phi}_2\hat{S}_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\bar{\Delta}_{\mu\nu} = \begin{pmatrix} \Delta^W_{\mu\nu}\frac{g_2^2}{4}\hat{\phi}_2^2 g_{\mu\nu} & 0 & 0 & 0 \\ 0 & \Delta^W_{\mu\nu} - \frac{g_2^2}{4}\hat{\phi}_2^2 g_{\mu\nu} & 0 & 0 \\ 0 & 0 & \Delta^W_{\mu\nu} - \frac{g_2^2}{4}\hat{\phi}_2^2 g_{\mu\nu} & \frac{g_1 g_2}{4}\hat{\phi}_2^2 g_{\mu\nu} \\ 0 & 0 & \frac{g_1 g_2}{4}\hat{\phi}_2^2 g_{\mu\nu} & \Delta^B_{\mu\nu} - \frac{g_1^2}{4}\hat{\phi}_2^2 g_{\mu\nu} \end{pmatrix},$$

$$D_S = \begin{pmatrix} \Box + 3\lambda_2\hat{S}_1^2 + \frac{\lambda_{12}}{2}\hat{\phi}_2^2 & 0 \\ 0 & \Box + \lambda_2\hat{S}_1^2 + \frac{\lambda_{12}}{2}\hat{\phi}_2^2 \end{pmatrix} \tag{3.42}$$

with

$$\Delta_{\mu\nu}^{(\alpha)} = -\left(g_{\mu\nu}\Box - \left(1 - \frac{1}{\xi_\alpha}\right)\partial_\mu\partial_\nu\right) \tag{3.43}$$

and the vector built of all quantum fields

$$\Phi = \begin{pmatrix}\phi_a & \psi_b & W_\mu^i & B_\mu & S_c\end{pmatrix}. \tag{3.44}$$

We now construct the effective potential from the determinant of Σ and find the one-loop contribution to be

$$\begin{aligned} V_{\text{eff,1-loop}} - \mathcal{V}_{\text{divergent}} =& \frac{1}{64\pi^2}\left(6m_A^4\left(\ln\frac{m_A^2}{\mu^2} - \frac{5}{6}\right) + 3m_B^4\left(\ln\frac{m_B^2}{\mu^2} - \frac{5}{6}\right)\right. \\ &+ m_{C+}^4\left(\ln\frac{m_{C+}^2}{\mu^2} - \frac{3}{2}\right) + m_{C-}^4\left(\ln\frac{m_{C-}^2}{\mu^2} - \frac{3}{2}\right) \\ &+ 2m_{D+}^4\left(\ln\frac{m_{D+}^2}{\mu^2} - \frac{3}{2}\right) + 2m_{D-}^4\left(\ln\frac{m_{D-}^2}{\mu^2} - \frac{3}{2}\right) \\ &+ m_{E+}^4\left(\ln\frac{m_{E+}^2}{\mu^2} - \frac{3}{2}\right) + m_{E-}^4\left(\ln\frac{m_{E-}^2}{\mu^2} - \frac{3}{2}\right) \\ &\left.+ m_F^4\left(\ln\frac{m_F^2}{\mu^2} - \frac{3}{2}\right) - 3m_T^4\left(\ln\frac{m_T^2}{\mu^2} - \frac{3}{2}\right)\right) \end{aligned} \tag{3.45}$$

where

$$\begin{aligned} m_A^2 &= \frac{1}{4}g_2^2\phi^2, \qquad\qquad m_B^2 = \frac{1}{4}(g_1^2 + g_2^2)\phi^2, \\ m_{C\pm}^2 &= \frac{1}{4}\left((6\lambda_1 + \lambda_{12})\phi^2 + (6\lambda_2 + \lambda_{12})S^2 \pm \sqrt{\left((6\lambda_1 - \lambda_{12})\,\phi^2 - (6\lambda_2 - \lambda_{12})\,S^2\right)^2 + 4\lambda_{12}^2\phi^2S^2}\right), \\ m_{D\pm}^2 &= \frac{1}{4}\left(2\lambda_1\phi^2 + \lambda_{12}S^2 \pm \sqrt{(2\lambda_1\phi^2 + \lambda_{12}S^2)\,(2\lambda_1\phi^2 + \lambda_{12}S^2 - \xi_W g_2^2\phi^2)}\right), \\ m_{E\pm}^2 &= \frac{1}{4}\left(2\lambda_1\phi^2 + \lambda_{12}S^2 \pm \sqrt{(2\lambda_1\phi^2 + \lambda_{12}S^2)\,(2\lambda_1\phi^2 + \lambda_{12}S^2 - (\xi_B g_1^2 + \xi_W g_2^2)\,\phi^2)}\right), \\ m_F^2 &= \frac{1}{4}\left(2\lambda_2S^2 + \lambda_{12}\phi^2\right), \qquad m_T^2 = \frac{y_t^2}{2}\phi^2. \end{aligned} \tag{3.46}$$

If we would follow the strategy that was developed in the previous sections, we would now use counterterms in order to fulfill the on-shell renormalization conditions

$$\begin{aligned} 0 = \left.\frac{\partial^2 V_{\text{eff}}}{\partial\phi^2}\right|_{\phi=0,S=0} &= \left.\frac{\partial^2 V_{\text{eff}}}{\partial S^2}\right|_{\phi=0,S=0} \\ \lambda_1 =& \frac{1}{3!}\left.\frac{\partial^4 V_{\text{eff}}}{\partial\phi^4}\right|_{\phi=\langle\phi\rangle,S=\langle S\rangle} \\ \lambda_2 =& \frac{1}{3!}\left.\frac{\partial^4 V_{\text{eff}}}{\partial S^4}\right|_{\phi=\langle\phi\rangle,S=\langle S\rangle} \\ \lambda_{12} =& \left.\frac{\partial^4 V_{\text{eff}}}{\partial\phi^2 S^2}\right|_{\phi=\langle\phi\rangle,S=\langle S\rangle} \end{aligned} \tag{3.47}$$

but this is not possible due to the complicated structure of the effective potential. We would need to bring it in a form like

$$V_{\text{eff}} = V + \sum_i c_i m_i^4 \left(\log \frac{m_i^2}{\hat{m}_i^2} - b_i \right) \tag{3.48}$$

where the $\hat{m}_i$ are equal to the field-dependent m_i evaluated at the actual minimum

$$\hat{m}_i = m_i\big|_{\phi=\langle\phi\rangle, S=\langle S\rangle} , \tag{3.49}$$

so that all the logarithmic contributions cancel at the location of the minimum.
This is impossible because of the terms involving square roots, namely $m_{C\pm}^2$, $m_{D\pm}^2$ and $m_{E\pm}^2$, which are not available as counterterms. This fact does not spoil the renormalizability of the theory, since all of those square roots cancel in the divergent piece of the effective potential

$$\begin{aligned}\epsilon\mathcal{V}_{\text{divergent}} = -\frac{1}{64\pi^2} \Big(& 6m_A^4 + 3m_B^4 + m_{C+}^4 + m_{C-}^4 + 2m_{D+}^4 \\ & +2m_{D-}^4 + m_{E+}^4 + m_{E-}^4 + m_F^4 - 4m_t^4 \Big)\end{aligned} \tag{3.50}$$

but it stops us from proceeding in the usual way.

3.4.1 Imposing the hierarchy of couplings

There is a way out of this: instead of deriving that the hierarchy $\lambda \sim g^4$ is the only consistent choice to solve a consistency condition like (3.23) we just impose the hierarchy by hand[2].
This means that inspired by the previous sections, we assume that all scalar couplings are similar in size to the fourth power of the gauge couplings

$$\lambda_i \sim g_j^4. \tag{3.51}$$

This greatly simplifies the effective potential, leaving only the following terms

$$V_{\text{eff}}(\phi, S) = \frac{\lambda_1}{4}\phi^4 + \frac{\lambda_{12}}{4}\phi^2 S^2 + \frac{\lambda_2}{4} S^4 + \frac{3}{64\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)\frac{\phi^4}{\langle\phi\rangle^4}\left(\ln\frac{\phi^2}{\langle\phi\rangle^2} - \frac{25}{6}\right) \tag{3.52}$$

in the on-shell renormalization scheme defined by (3.47). Since all diagrams involving internal S fields have effectively been pushed to higher loop order, there are only logarithmic corrections involving the Higgs field.
To make sure that $(\langle\phi\rangle, \langle S\rangle)$ is actually the minimum of the effective potential, we demand

$$0 = \begin{pmatrix} \frac{\partial V_{\text{eff}}}{\partial \phi} \\ \frac{\partial V_{\text{eff}}}{\partial S} \end{pmatrix}\Bigg|_{\phi=\langle\phi\rangle,\ S=\langle S\rangle} \tag{3.53}$$

[2]An alternative way of arriving at the hierarchy is to compare the size of the different terms appearing in the effective potential. For the terms that involve squares of scalar couplings to be of the same order of magnitude as the tree level potential, the corresponding logarithm would need to be of the order of $\frac{1}{\lambda}$, which is outside of the perturbative regime. Therefore, the quantum contributions that influence the position of the vacuum must come from the gauge couplings. This argument is much more vague though.

leading to

$$\lambda_1\langle\phi\rangle^4 + \frac{\lambda_{12}}{2}\langle\phi\rangle^2\langle S\rangle^2 = \frac{11}{16\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)$$
$$\frac{\lambda_{12}}{2}\langle\phi\rangle^2\langle S\rangle^2 + \lambda_2\langle S\rangle^4 = 0. \tag{3.54}$$

Since this model involves three scalar couplings, it has more freedom than the previous models. We can use the equations (3.54) to constrain two scalar couplings only. We choose to fix λ_{12} and λ_2 and write the effective potential as

$$\begin{aligned} V_{\text{eff}}(\phi, S) =& \frac{\lambda_1}{4}\langle\phi\rangle^4\left(\frac{\phi^2}{\langle\phi\rangle^2} - \frac{S^2}{\langle S\rangle^2}\right)^2 + \frac{3}{64\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)\times \\ &\times\left(\frac{\phi^4}{\langle\phi\rangle^4}\left(\ln\frac{\phi^2}{\langle\phi\rangle^2} - \frac{1}{2}\right) - \frac{11}{3}\left(\frac{\phi^2}{\langle\phi\rangle^2} - \frac{S^2}{\langle S\rangle^2}\right)^2\right). \end{aligned} \tag{3.55}$$

Now that we have two scalar fields with identical quantum numbers, the mass eigenstates will in general contain some mixing of the fields. We have to diagonalize the mass matrix

$$M = \begin{pmatrix} \frac{\partial^2}{\partial\phi^2} & \frac{\partial^2}{\partial\phi\partial S} \\ \frac{\partial^2}{\partial\phi\partial S} & \frac{\partial^2}{\partial S^2} \end{pmatrix} V_{\text{eff}}\big|_{\phi=\langle\phi\rangle, S=\langle S\rangle}. \tag{3.56}$$

It reads explicitly

$$M = \begin{pmatrix} -\frac{8}{3}m_0^2 + 2\lambda_1\langle\phi\rangle^2 & \frac{11}{3}m_0^2\frac{\langle\phi\rangle}{\langle S\rangle} - 2\lambda_1\frac{\langle\phi\rangle^3}{\langle S\rangle} \\ \frac{11}{3}m_0^2\frac{\langle\phi\rangle}{\langle S\rangle} - 2\lambda_1\frac{\langle\phi\rangle^3}{\langle S\rangle} & -\frac{11}{3}m_0^2\frac{\langle\phi\rangle^2}{\langle S\rangle^2} + 2\lambda_1\frac{\langle\phi\rangle^4}{\langle S\rangle^2} \end{pmatrix} \tag{3.57}$$

where m_0^2 is the mass squared of the Higgs boson in the scalefree Standard Model

$$m_0^2 = \frac{3}{8\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)\frac{1}{\langle\phi\rangle^2} \tag{3.58}$$

which we have found to be negative after the SM values for the gauge boson and fermion masses are substituted.
From this matrix we can derive the formulae for the mass eigenvalues

$$\begin{aligned} m_{1/2}^2 =& -\frac{4}{3}m_0^2\left(1 + \frac{11}{8}\frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) + \lambda_1\langle\phi\rangle^2\left(1 + \frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) \\ &\pm\sqrt{\left(\frac{4}{3}m_0^2\left(1 + \frac{11}{8}\frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) - \lambda_1\langle\phi\rangle^2\left(1 + \frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right)\right)^2 + m_0^2\frac{\langle\phi\rangle^2}{\langle S\rangle^2}\left(\frac{11}{3}m_0^2 - 2\lambda_1\langle\phi\rangle^2\right)}. \end{aligned} \tag{3.59}$$

We want to show that as soon as we substitute the SM mass values, one or both of those eigenvalues become negative, indicating again, that the extremum of the effective potential would be a maximum and not a minimum. This can be proven without doing any numerical calculations.

First, let's consider the case in which λ_1 is bigger than zero. Looking at the determinant of the mass matrix

$$m_1^2 m_2^2 = \det M = m_0^2 \frac{\langle\phi\rangle^2}{\langle S\rangle^2}\left(-\frac{11}{3}m_0^2 + 2\langle\phi\rangle^2\lambda_1\right) \tag{3.60}$$

we see that it is negative when λ_1 is positive, since m_0^2 is negative in the SM. This means, that one of the eigenvalues has to be negative in this case, rendering the extremum a saddle point.
Clearly, the determinant is going to turn positive if

$$\lambda_1\langle\phi\rangle^2 < \frac{11}{6}m_0^2. \tag{3.61}$$

If we check the trace of the mass matrix for this case, we find

$$\begin{aligned} m_1^2 + m_2^2 = \text{tr } M &= -\frac{4}{3}m_0^2\left(1 + \frac{11}{8}\frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) + \lambda_1\langle\phi\rangle^2\left(1 + \frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) \\ &< -\frac{4}{3}m_0^2 + \frac{11}{6}m_0^2 = m_0^2 \end{aligned} \tag{3.62}$$

implying again that at least one of the eigenvalues must be negative, since m_0^2 is negative. This completes the proof that independently of the sign of λ_1, at least one of the mass eigenvalues will be negative, implying that the effective potential will be unstable in the corresponding direction. Again, this model can not predict the vacuum state of the Standard Model consistently.
When doing a higher order calculation, one could simply absorb the corrections into the value of m_0^2.

3.4.2 Comments on the Conformal Standard Model

When Meissner and Nicolai studied this model they reported on a set of values that leads to a minimum of the one-loop effective potential which seems to be similar to the Standard Model vacuum [7]. However, they did not enforce the hierarchy of couplings and their result is therefore not viable once higher orders of perturbation theory are considered. This is also reflected by the large logarithms that appear in their numerical solution. They found a minimum for

$$\ln\frac{\langle\phi\rangle^2}{\mu^2} \approx -24, \qquad \lambda_1 = 3.4, \tag{3.63}$$

placing the combination $\lambda \ln\frac{\langle\phi\rangle^2}{\mu^2}$ far outside of the perturbative range. We take this as further evidence that it makes sense to impose the hierarchy of couplings even though we can not strictly prove that it has to hold, as in the previous models.

We also want to comment on the possibility of different hierarchies. We have simply taken all scalar couplings to be surpressed against the gauge couplings. Clearly, λ_1 and λ_2 can't be too big because this would resemble the original case that Coleman and Weinberg discussed in which there are no radiatively generated expectation values. Nevertheless, we could imagine a case in which there is a hierarchy according to

$$\lambda_{1,2} \sim \lambda_{12}^2. \tag{3.64}$$

Notice though that λ_{12} must necessarily be negative in order to induce the negative mass-squared term for the Higgs boson. This would again push all minima of the effective potential out of the perturbative regime, since we had to balance terms according to

$$\lambda_{12} \sim g_i^4 \ln \frac{\phi^2}{\mu^2}. \tag{3.65}$$

Thus, for both $\lambda_{12} \sim g_i$ and $\lambda_{12} \sim g_i^2$ we could only find minima at field values which lead to large logarithms. This leads us to believe that the only consistent choice is the one where all scalar couplings are supressed compared to the gauge couplings.

3.5 The 'Minimal' Conformal Standard Model

In this section we will show that there is actually a modification of the Standard Model with the desired properties: a classically scale-invariant model, whose effective potential has a minimum that allows for effective masses which fit with observations.
Since this is in a sense the simplest version of such a model, we will call it the Minimal Conformal Standard Model (MCSM).
Numerical investigations of the CSM from the previous section for some general choices of the parameters λ_1 and $\langle S \rangle$ show that the direction in field space that corresponds to the physical S-field is generally going to be unstable.
We can improve this behaviour by adding a positive contribution to the effective potential, which can be achieved by an additional gauge field that only couples to S. Since the Lagrangian of the CSM is already invariant under a global $U(1)$-symmetry, acting according to

$$S \to e^{i\alpha} S, \tag{3.66}$$

it makes sense to gauge this symmetry in order to arrive at a minimal amount of new fields. Accordingly, the full Lagrangian for this model is given by

$$\begin{aligned}
\mathcal{L}_{\text{MCSM}} &= \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{Yukawa}} - V(H,S) \\
\mathcal{L}_{\text{kinetic}} &= -\frac{1}{4} \sum_{F \in \{W^a, B, D\}} \text{tr}\,(F_{\mu\nu} F^{\mu\nu}) + (\mathcal{D}_\mu H)^\dagger \mathcal{D}^\mu H \\
&\quad + (\mathcal{D}_\mu S)^\dagger \mathcal{D}^\mu S + \sum_{\psi \in \{Q, L, e_R, \nu_R, u_R, d_R\}} i\bar{\psi}^i \gamma^\mu \mathcal{D}_\mu \psi^i \\
\mathcal{L}_{\text{Yukawa}} &= Y^d_{ij} \bar{Q}^i H d^j_R + Y^u_{ij} \bar{Q}^i \tilde{H} u^j_R + Y^e_{ij} L^i H e^j_R \\
V(H,S) &= \lambda_1 \left(H^\dagger H\right)^2 + \lambda_{12} \left(H^\dagger H\right)\left(S^\dagger S\right) + \lambda_2 \left(S^\dagger S\right)^2
\end{aligned} \tag{3.67}$$

with the same conventions as in chapter 2. Again, terms involving the strong interaction have been left out, as they only appear at higher loop levels.
The covariant derivative couples S to the new gauge field D^μ according to

$$\mathcal{D}_\mu S = \left(\partial_\mu + i g_D D_\mu\right) S, \tag{3.68}$$

inducing cubic and quartic interactions. Because the new gauge field does not interact directly with the fields of the Standard Model we call it a 'dark' photon.

3.5.1 Minimizing the effective potential

As usual we expand the scalar fields around classical field values and arrive at the Lagrangian containing only terms relevant to one loop

$$\mathcal{L}_{\text{1-loop}} = -\frac{\lambda_1}{4}\hat{\phi}_2^4 - \frac{\lambda_{12}}{4}\hat{\phi}_2^2\hat{S}^2 - \frac{\lambda_2}{4}\hat{S}^4 - \frac{1}{2}\Phi^T\Sigma\Phi - \frac{y_t}{\sqrt{2}}\hat{\phi}_2\bar{\psi}_t\psi_t, \tag{3.69}$$

where Φ is the vector formed from all bosonic quantum fields

$$\Phi = \begin{pmatrix}\phi_a & \psi_a & W_\mu^a & B_\mu & S_a & D_\mu\end{pmatrix}^T. \tag{3.70}$$

Here, Σ is the matrix

$$\Sigma = \begin{pmatrix} D_H & M_\nu & P & 0 \\ M_\mu^\dagger & \bar{\Delta}^H_{\mu\nu} & 0 & 0 \\ P^\dagger & 0 & D_S & N_\nu \\ 0 & 0 & N_\mu^\dagger & \bar{\Delta}^D_{\mu\nu}\end{pmatrix} \tag{3.71}$$

with

$$D_H = \begin{pmatrix} \Box + \lambda_1\hat{\phi}_2^2 + \frac{\lambda_{12}}{2}\hat{S}_1^2 & 0 & 0 & 0 \\ 0 & \Box + 3\lambda_1\hat{\phi}_2^2 + \frac{\lambda_{12}}{2}\hat{S}_1^2 & 0 & 0 \\ 0 & 0 & \Box + \lambda_1\hat{\phi}_2^2 + \frac{\lambda_{12}}{2}\hat{S}_1^2 & 0 \\ 0 & 0 & 0 & \Box + \lambda_1\hat{\phi}_2^2 + \frac{\lambda_{12}}{2}\hat{S}_1^2 \end{pmatrix},$$

$$M_\mu = \begin{pmatrix} 0 & -\frac{1}{2}g_2\hat{\phi}_2\partial_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}g_2\hat{\phi}_2\partial_\mu & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}g_2\hat{\phi}_2\partial_\mu & -\frac{1}{2}g_1\hat{\phi}_2\partial_\mu \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ \lambda_{12}\hat{\phi}_2\hat{S}_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\bar{\Delta}^H_{\mu\nu} = \begin{pmatrix} \Delta^W_{\mu\nu} - \frac{g_2^2}{4}\hat{\phi}_2^2 g_{\mu\nu} & 0 & 0 & 0 \\ 0 & \Delta^W_{\mu\nu} - \frac{g_2^2}{4}\hat{\phi}_2^2 g_{\mu\nu} & 0 & 0 \\ 0 & 0 & \Delta^W_{\mu\nu} - \frac{g_2^2}{4}\hat{\phi}_2^2 g_{\mu\nu} & \frac{g_1 g_2}{4}\hat{\phi}_2^2 g_{\mu\nu} \\ 0 & 0 & \frac{g_1 g_2}{4}\hat{\phi}_2^2 g_{\mu\nu} & \Delta^B_{\mu\nu} - \frac{g_1^2}{4}\hat{\phi}_2^2 g_{\mu\nu} \end{pmatrix},$$

$$D_S = \begin{pmatrix} \Box + 3\lambda_2\hat{S}_1^2 + \frac{\lambda_{12}}{2}\hat{\phi}_2^2 & 0 \\ 0 & \Box + \lambda_2\hat{S}_1^2 + \frac{\lambda_{12}}{2}\hat{\phi}_2^2 \end{pmatrix}, \quad N_\mu = \begin{pmatrix} 0 \\ -g_D\hat{S}_1^2\partial_\mu \end{pmatrix}$$

$$\bar{\Delta}^D_{\mu\nu} = \Delta^D_{\mu\nu} - g_D^2\hat{S}_1^2 g_{\mu\nu} \tag{3.72}$$

where $\Delta^{(\alpha)}_{\mu\nu}$ is defined by (3.43). Calculating the determinant of this 26×26 matrix, we find the effective potential

$$V_{\text{eff}}(\phi, S) = \frac{\lambda_1}{4}\phi^4 + \frac{\lambda_{12}}{4}\phi^2 S^2 + \frac{\lambda_2}{4}S^4 + \frac{1}{64\pi^2}\sum_i c_i m_i^4\left(\ln\frac{m_i^2}{\mu^2} + b_i\right) \tag{3.73}$$

with

$$
\begin{aligned}
m_A^2 &= \frac{g_2^2}{4}\phi^2, \qquad m_B^2 = \frac{(g_1^2+g_2^2)}{4}\phi^2, \qquad m_C^2 = g_D^2 S^2, \\
m_{E\pm}^2 &= \frac{1}{4}\left((6\lambda_1+\lambda_{12})\,\phi^2 + (6\lambda_2+\lambda_{12})\,S^2 \pm \sqrt{\left((6\lambda_1-\lambda_{12})\,\phi^2 - (6\lambda_2-\lambda_{12})\,S^2\right)^2 + 4\lambda_{12}^2\phi^2 S^2}\right) \\
m_{F\pm}^2 &= \frac{1}{4}\left(2\lambda_2 S^2 + \lambda_{12}\phi^2 \pm \sqrt{(2\lambda_2 S^2+\lambda_{12}\phi^2)^2 - \xi_D\left(4\lambda_2 g_D^2 S^4 + 2\lambda_{12} g_D^2 \phi^2 S^2\right)}\right), \\
m_{G\pm}^2 &= \frac{1}{4}\left(2\lambda_1\phi^2 + \lambda_{12}S^2 \pm \sqrt{(2\lambda_1\phi^2+\lambda_{12}S^2)^2 - (\xi_B g_1^2 + \xi_W g_2^2)\left(4\lambda_1\phi^4 + 2\lambda_{12}\phi^2 S^2\right)}\right), \\
m_{I\pm}^2 &= \frac{1}{4}\left(2\lambda_1\phi^2 + \lambda_{12}S^2 \pm \sqrt{(2\lambda_1\phi^2+\lambda_{12}S^2)^2 - \xi_W g_2^2\left(4\lambda_1\phi^4 + 2\lambda_{12}\phi^2 S^2\right)}\right), \\
m_T^2 &= \frac{y_t^2}{2}\phi^2.
\end{aligned} \tag{3.74}
$$

as well as

$$
\begin{aligned}
&c_A = 6, \quad c_B = 3, \quad c_C = 3 \\
&c_E = c_F = c_G = 1 = \frac{1}{2}c_I, \quad c_T = -12 \\
&b_A = b_B = b_C = -\frac{5}{6} \\
&b_E = b_F = b_G = b_I = b_T = -\frac{3}{2}.
\end{aligned} \tag{3.75}
$$

As usual we have also dropped the hats and indices of the classical fields ϕ and S for legibility.
Submitting this effective potential to the same hierarchy of couplings

$$\lambda_i \sim g_j^4 \tag{3.76}$$

where the scalar couplings are repressed compared to the gauge couplings, many terms drop out and we are left only with the gauge boson contributions. After using counter terms in order to enforce the renormalization conditions given in (3.47), the effective potential reads

$$
\begin{aligned}
V_{\text{eff}}(\phi, S) = {} & \frac{\lambda_1}{4}\phi^4 + \lambda_{12}\phi^2 S^2 + \frac{\lambda_2}{4}S^4 + \frac{3}{64\pi^2}m_D^4\frac{S^4}{\langle S\rangle^4}\left(\ln\frac{S^2}{\langle S\rangle^2} - \frac{25}{6}\right) \\
& \frac{3}{64\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)\frac{\phi^4}{\langle\phi\rangle^4}\left(\ln\frac{\phi^2}{\langle\phi\rangle^2} - \frac{25}{6}\right).
\end{aligned} \tag{3.77}
$$

Notice that we had to use a counter term of the form

$$\delta_2' S^4 = \frac{3}{64\pi^2}g_D^4 \ln\frac{\mu^2}{\mu'^2} \tag{3.78}$$

to be able to choose scales for both logarithms independently. As discussed in section 2.4, this is going to introduce a second set of beta functions.
Clearly, the term containing radiative corrections by the dark photon with mass m_D gives a positive contribution to the effective mass of the dark scalar S and can therefore potentially turn the saddle point of the Conformal Standard Model into a proper minimum.

To investigate if this is actually the case, we again calculate the consistency condition for $(\langle\phi\rangle, \langle S\rangle)$ to be the minimum of the effective potential

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\phi}\right|_{\phi=\langle\phi\rangle, S=\langle S\rangle} \\ 0 &= \left.\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}S}\right|_{\phi=\langle\phi\rangle, S=\langle S\rangle} \end{aligned} \tag{3.79}$$

leading to

$$\begin{aligned} \lambda_1\langle\phi\rangle^4 + \frac{\lambda_{12}}{2}\langle\phi\rangle^2\langle S\rangle^2 &= \frac{11}{16\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right) \\ \frac{\lambda_{12}}{2}\langle\phi\rangle^2\langle S\rangle^2 + \lambda_2\langle S\rangle^4 &= \frac{11}{16\pi^2}m_D^4. \end{aligned} \tag{3.80}$$

Solving those conditions for λ_{12} and λ_2 and substituting those solutions in the effective potentials gives

$$\begin{aligned} V_{\text{eff}}(\phi, S) = &\frac{\lambda_1}{4}\langle\phi\rangle^4\left(\frac{\phi^2}{\langle\phi\rangle^2} - \frac{S^2}{\langle S\rangle^2}\right)^2 + \frac{3}{64\pi^2}m_D^4\frac{S^4}{\langle S\rangle^4}\left(\ln\frac{S^2}{\langle S\rangle^2} - \frac{1}{2}\right) \\ &+ \frac{3}{64\pi^2}\left(2m_W^4 + m_Z^4 - 4m_t^4\right)\left(\frac{\phi^4}{\langle\phi\rangle^4}\left(\ln\frac{\phi^2}{\langle\phi\rangle^2} - \frac{1}{2}\right) - \frac{11}{3}\left(\frac{\phi^2}{\langle\phi\rangle^2} - \frac{S^2}{\langle S\rangle^2}\right)^2\right) \end{aligned} \tag{3.81}$$

and the mass matrix for the scalar sector reads

$$M^2 = \begin{pmatrix} -\frac{8}{3}m_0^2 + 2\lambda_1\langle\phi\rangle^2 & \left(\frac{11}{3}m_0^2 - 2\lambda_1\langle\phi\rangle\right)\frac{\langle\phi\rangle}{\langle S\rangle} \\ \left(\frac{11}{3}m_0^2 - 2\lambda_1\langle\phi\rangle\right)\frac{\langle\phi\rangle}{\langle S\rangle} & \frac{3g_D^2}{32\pi^2}m_D^2 + \left(-\frac{11}{3}m_0^2 + 2\lambda_1\langle\phi\rangle^2\right)\frac{\langle\phi\rangle^2}{\langle S\rangle^2} \end{pmatrix} \tag{3.82}$$

with eigenvalues

$$\begin{aligned} m_{1/2}^2 = &-\frac{4}{3}m_0^2\left(1 + \frac{11}{8}\frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) + \lambda_1\langle\phi\rangle^2\left(1 + \frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) + \frac{3g_D^2}{64\pi^2}m_D^2 \\ &\mp\left(\left(-\frac{4}{3}m_0^2\left(1 + \frac{11}{8}\frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) + \lambda_1\langle\phi\rangle^2\left(1 + \frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right) + \frac{3g_D^2}{64\pi^2}m_D^2\right)^2\right. \\ &\left.+\left(\frac{3g_D^2}{32\pi^2}m_D^2 + m_0^2\frac{\langle\phi\rangle^2}{\langle S\rangle^2}\right)\left(\frac{11}{3}m_0^2 - 2\lambda_1\langle\phi\rangle^2\right) - \frac{3g_D^2}{32\pi^2}m_0^2m_D^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.83}$$

While this is a rather long formula, we can quickly see one important detail: in the limit of a large mass for the dark photon, we can expand the square root and the mass eigenvalues take the form

$$\begin{aligned} m_h^2 &= -\frac{8}{3}m_0^2 + 2\lambda_1\langle\phi\rangle^2, \\ m_S^2 &= \frac{3g_D^2}{32\pi^2}m_D^2. \end{aligned} \tag{3.84}$$

This clearly shows that as advertised, there is no finetuning problem because in this limit the heavy degrees of freedom in the dark sector do not contribute to the mass of

the Higgs boson.
Note that (3.83) contains three independently adjustable parameters, λ_1, g_D and m_D. m_0 and $\langle\phi\rangle$ are fixed by measurements, while $\langle S\rangle$ is not independent from g_D and m_D. There is one additional experimental constraint: one mass should be equal to the measured Higgs mass [21]

$$m_h = 125.09 \text{ GeV}. \tag{3.85}$$

Because we will not learn much from looking at the analytical expression, we will analyze the situation further using a numerical parameter study.

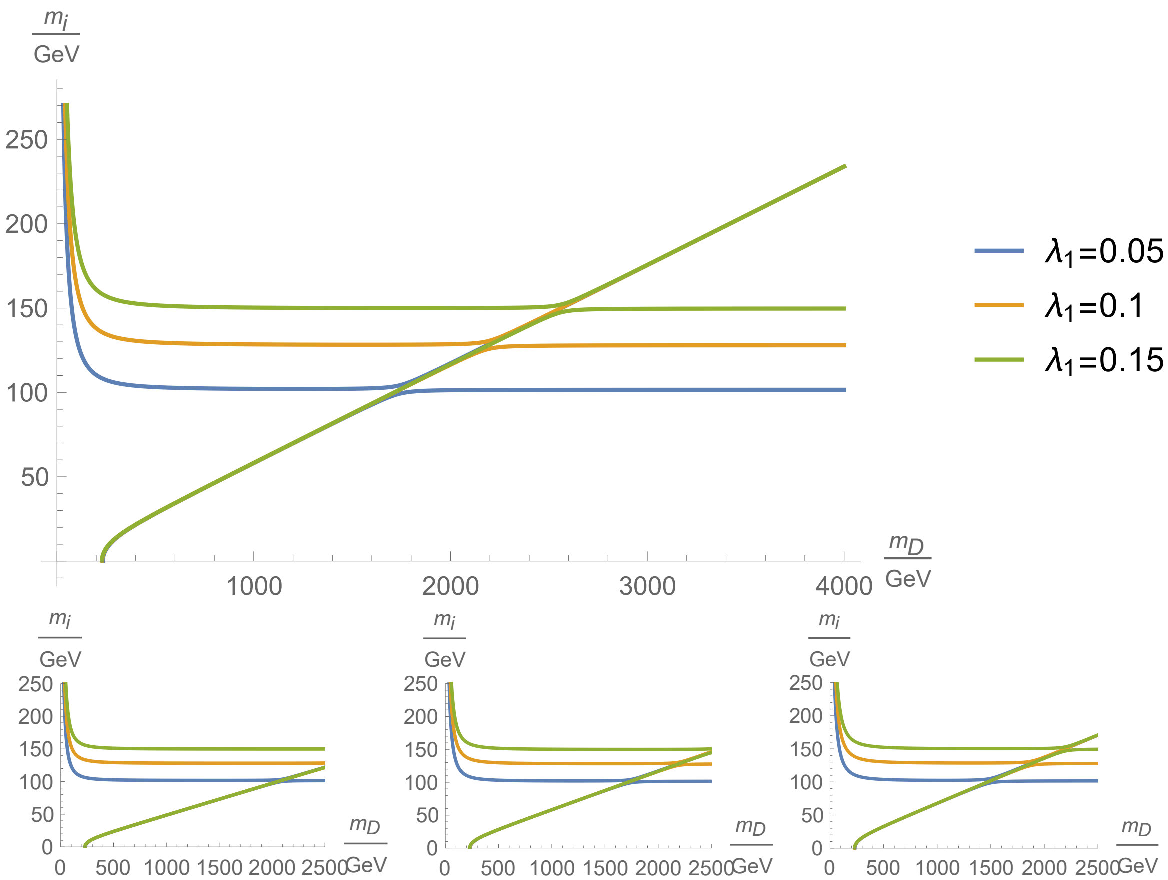
3.5.2 Numerical study of the mass eigenvalues

We want to study the mass eigenvalues of the two scalar fields in the Minimal Conformal Standard Model numerically. The values are given in (3.83) and depend on three parameters: the scalar quartic coupling of the Higgs field λ_1, the gauge coupling of the dark photon g_D and its mass m_D. We plot the mass eigenvalues for different choices of the couplings λ_1 and g_D as functions of m_D.
In order to not disturb the hierarchy of couplings we used to calculate the effective potential in the previous subsection, we will let g_D vary from 0.5 to 0.7 and λ_1 from 0.05 to 0.15.

Looking at figure 3.1, the first thing that catches the eye is that there are wide ranges of m_D in which one of the masses does not change very strongly. We will call those the plateau areas. The heigth of those plateaus depends solely on λ_1 and is hardly influenced by g_D. While m_1 is bounded from above by the plateau, m_2 is bounded from below. It is very interesting that the mass eigenstates seem to change their identity at the point where they nearly meet. This idea will be corroborated once we look at the mixing angle between mass and interaction eigenstates.
The next observation is the fact that the dark photon mass m_D can not be arbitrarily small: there is a minimum mass of about 230 GeV. For lower masses, the effective mass squared of the dark scalar turns negative, rendering the extremum of the effective potential a saddle point. The limit in which m_D goes to zero is equivalent to the limit in which g_D goes to zero, effectively removing the gauge coupling again. This just leads back to the Conformal Standard Model discussed in the previous section, for which we showed that there is no minimum resembling the Standard Model.

We also see that for big parts of the parameter space, one of the two mass eigenvalues is mostly independent from the scalar coupling λ_1 while the other is mostly independent from the dark photon mass m_D. As will be seen when we look at the mixing angle of the two scalar fields, as long as the eigenvalues are not too close, the field whose mass varies with m_D is mostly the dark scalar, while the other field is mostly the Higgs field.
It is clear that independently of the value of λ_1, there is always a value for m_D which sets one of the two masses to the measured Higgs mass. If the plateau is higher than the Higgs mass, m_1 will necessarily pass m_h on its way there. Conversely, for a plateau below the Higgs mass, m_2 will at one point be equal to m_h. We will come back to this observation at the end of this section.
As can be seen from figure 3.1, there is also a particular value of $\lambda_1 \approx 0.09$, where the

Figure 3.1: Mass Eigenvalues of the Minimal Conformal Standard Model depending on the dark photon mass for dark gauge coupling $g_D = 0.6$ (big graphic) as well as for varying values of $g_D \in \{0.5, 0.6, 0.7\}$ and varying values of the quartic Higgs coupling $\lambda_1 \in \{0.05, 0.1, 0.15\}$.



a) $g_D = 0.5$ **b)** $g_D = 0.6$ **c)** $g_D = 0.7$

plateau lies very close to 125 GeV. In this case there is a wide range of possible values for the other mass, both lighter and heavier than the Higgs boson.

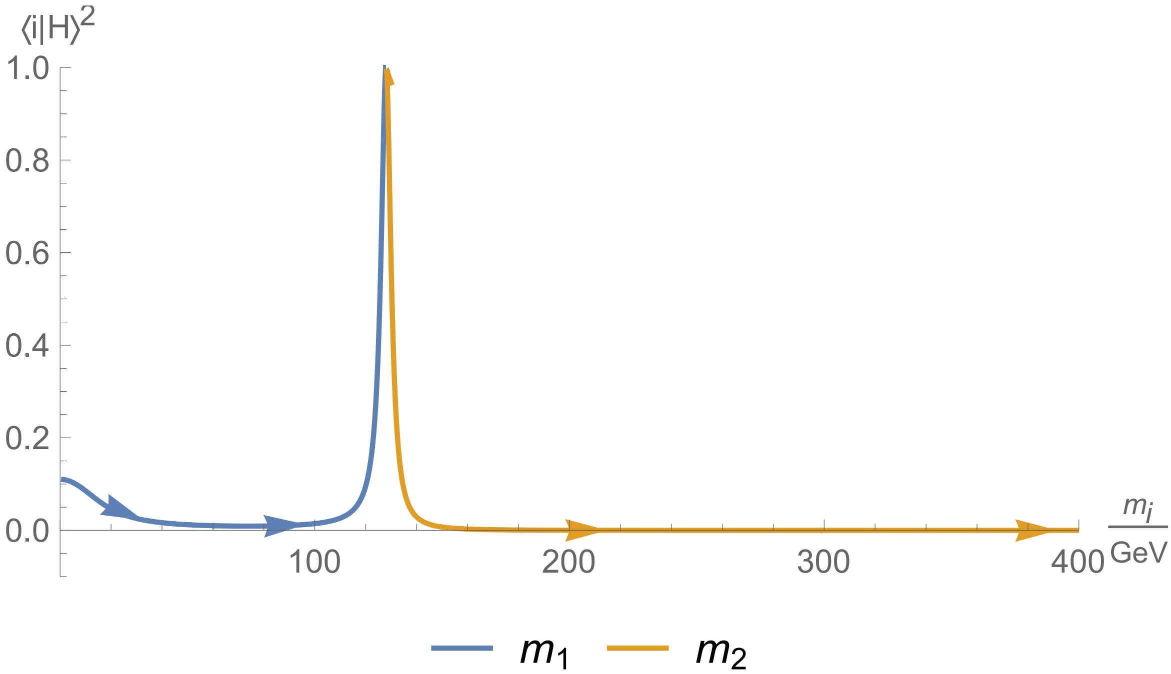
From figure 3.1a-c we learn two things. Firstly, neither the minimum mass of the dark photon nor the value at the plateaus vary much with the gauge coupling g_D. And secondly, what does vary is the slope of the curve corresponding to the dark scalar and therefore the region of maximal mixture between the two fields which is given by the point at which the curves of the two eigenstates are closest to each other. This near-intersection point's position is influenced both by g_D, as it sets the slope of the dark scalar curve, as well as λ_1, because it influences the height of the plateau.

Since the mass matrix is not diagonal in the basis of ϕ and S, there is also some mixing between those interaction eigenstates to form the mass eigenstates. We can explicitly write this as

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi \\ S \end{pmatrix}, \tag{3.86}$$

where φ_i is the field corresponding to the eigenvalue m_i. When we want to know how closely a mass eigenstate resembles one of the interaction eigenstates, it makes sense to look at the corresponding coefficient squared. This is the weight with which interaction eigenstate decay channels contribute to cross sections and decay widths of the mass eigenstates. We therefore give a parametric plot of $(m_1, \cos^2\alpha)$ and $(m_2, \sin^2\alpha)$ as functions of $m_D \in [220, 4000]$ GeV in figure 3.2. This shows, depending on the mass eigenvalue, how closely the respective particle looks like a Higgs particle in experiments.

Figure 3.2: Parametric plot of the squared projection amplitude of the mass eigenstates on the Higgs-like interaction eigenstate for an exemplary choice of parameters $\lambda_1 = 0.1$, $g_D = 0.6$. The arrows indicate the direction of increasing m_D.



The graphic is not completely intuitive. It is best to compare it to figure 3.1. To make figure 3.2 easier to understand we have depicted the direction of increasing m_D on the parametric curves with arrows. The curves start for $m_D = 220$ GeV; the curve for m_1 starts at 0 GeV, while the curve for m_2 starts at 130 GeV. Then, with increasing m_D, m_1 starts to increase as well, while m_2 stays at the plateau. As m_1 approaches 130 GeV, m_2 begins to increase. Finally m_1 plateaus at 130 GeV and m_2 increases without bounds.

As can be seen from the graphic, there is minimal mixing in the plateau areas where one of the mass eigenvalues stays at around 130 GeV. The mass eigenstate with the plateau mass is nearly identical to the Standard Model Higgs boson.

On the other hand there appears maximal mixing at the position where the two eigenmasses are closest and the fields change their identity. Up to the value of the plateau mass, this description is completely general for arbitrary λ_1 and g_D.

To summarize this discussion on mixing: the eigenstate that behaves most like the Standard Model Higgs with respect to interactions is the one with mass around the plateau. Only in the region where both masses are close to the plateau, there exists considerable mixing between the interaction eigenstates.

Now we can go back to matching one of the two mass values to the Higgs mass. We argued that it is possible for arbitrary λ_1 and g_D to find a value for m_D such that one of the mass eigenvalues is equal to 125 GeV. But from the above discussion of the mixing angle we see that this can not work if the value is not close to the plateau, since this will make the corresponding eigenstate predominantly consist of the dark scalar. The only possibility to let the field with mass equal to 125 GeV resemble the Standard Model Higgs experimentally is by letting it be the mass eigenstate at the plateau. This gives us a prediction for the Higgs quartic coupling: If $m_S < m_H$ we find

$$\lambda_1 = (0.089 \pm 0.006) \tag{3.87}$$

where the uncertainty mostly comes from the fact that the plateau left from the meeting of the curves is not as flat as it looks like. It was estimated by varying λ_1 such that the curve of the mass eigenvalues is maximally or minimally 125 GeV respectively. The uncertainty of the experimental mass values for the gauge bosons and the top quark has no significant influence.
For $m_S > m_H$ the plateau is much flatter and a little bit higher and we find

$$\lambda_1 = (0.094 \pm 0.004). \tag{3.88}$$

Both of those values differ significantly from the Standard Model quartic coupling, given by [19]

$$\lambda_1 = \frac{m_h^2}{2v^2} = 0.1292 \pm 0.0006, \tag{3.89}$$

indicating a way to experimentally distinguish this model from the Standard Model.

3.5.3 RGE Improvement and vacuum stability in the MCSM

Now that we have found a consistent way to embed the Standard Model in a scale-invariant theory with a local minimum of the effective potential, it remains the question if this minimum is also global. If we plug arbitrary field values into the formula for the effective potential (3.77), we will have to deal with large logarithms, rendering perturbation theory invalid.
The way out of this is the use of the renormalization group. For regularizing a quantum field theory one always has to introduce an arbitrary renormalization scale μ. Physical predictions should be independent of that scale and this is reflected in the renormalization group equations (RGE), that describe how the parameters of the theory change under changes of μ. We adapt the one-scale beta functions from [23], only modified by

the additional gauge contribution [24] and adapted to our normalization[3]

$$\begin{aligned}
16\pi^2\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} &= 24\lambda_1^2 - 6y_t^4 + \frac{3}{8}\left(g_2^4 + \left(g_1^2+g_2^2\right)^2\right) + \lambda_1\left(12y_t^2 - 9g_2^2 - 3g_1^2\right) + \frac{1}{2}\lambda_{12}^2\\
16\pi^2\frac{\mathrm{d}\lambda_{12}}{\mathrm{d}t} &= 4\lambda_{12}^2 + 12\lambda_1\lambda_{12} - \frac{3}{2}\left(3g_2^2+g_1^2\right)\lambda_{12} + 6y_t^2\lambda_{12} + 6\lambda_2\lambda_{12}\\
16\pi^2\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} &= \frac{1}{2}\lambda_{12}^2 + 20\lambda_2^2 - 6\lambda_2 g_D^2 + 6g_D^4\\
16\pi^2\frac{\mathrm{d}y_t}{\mathrm{d}t} &= y_t\left(\frac{9}{2}y_t^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right)\\
16\pi^2\frac{\mathrm{d}g_i}{\mathrm{d}t} &= b_i g_i^3
\end{aligned} \tag{3.90}$$

with $(b_1, b_2, b_3, b_D) = (41/6, -19/6, -7, 1/3)$ and $t = \ln\mu$. After enforcing the hierarchy of couplings and when considering our argument (2.85) regarding the scale-dependent counter term (3.78), we find two sets of beta functions

$$\begin{aligned}
16\pi^2\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} &= -6y_t^4 + \frac{3}{8}\left(g_2^4 + \left(g_1^2+g_2^2\right)^2\right)\\
16\pi^2\frac{\mathrm{d}y_t}{\mathrm{d}t} &= y_t\left(\frac{9}{2}y_t^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right)\\
16\pi^2\frac{\mathrm{d}g_i}{\mathrm{d}t} &= b_i g_i^3
\end{aligned} \tag{3.91}$$

with the same b_i and t as above, as well as

$$16\pi^2\frac{\mathrm{d}\lambda_2}{\mathrm{d}s} = 6g_D^4 \tag{3.92}$$

with $s = \ln\mu'$ and all other beta functions vanishing at the one-loop level.
The criterion that is usually used to probe the stability of the electroweak vacuum is the sign of the quartic couplings λ_1 and λ_2 [25]. As argued in [23], an additional criterion for stability in the case of $\lambda_{12} < 0$ is $\lambda_1 > \frac{\lambda_{12}^2}{4\lambda_2}$ but those criteria do not differ significantly in the present analysis because of the smallness of λ_{12}. As usual we will only look for the evolution of the couplings up to the Planck scale $M_{\mathrm{Pl}} = 1.22 \times 10^{19}$ GeV, since we can't neglect contributions coming from quantum gravity at higher scales.
Because the system of differential equations (3.90) is highly non-linear, we solve it numerically and discuss the solutions below. There is one thing we can already gain from the equations alone: since the dark sector beta functions are completely decoupled from the Standard Model beta functions, the stability behaviour of the Standrad Model sector won't depend on the initial conditionts of the dark sector. We choose $g_D = 0.5$ at the minimum of the effective potential and $m_D = 1$ TeV. For all the other couplings we take initial values that are equal to the SM-values evaluated at $\mu = 246$ GeV, as given in section 2.9.
As we can see from figure 3.4, the model becomes unstable at around 10^7 GeV. In a way we have inherited this problem from the Standard Model. The instability of the Standard Model vacuum has been studied to great accuracy [25]. There, it was found that the Standard Model allows a stable vacuum at the 2-loop level only for

$$m_h > (129.4 \pm 1.8)\ \text{GeV}. \tag{3.93}$$

[3]Our λ_2 differs from the convention used in [24] by a factor of 4.

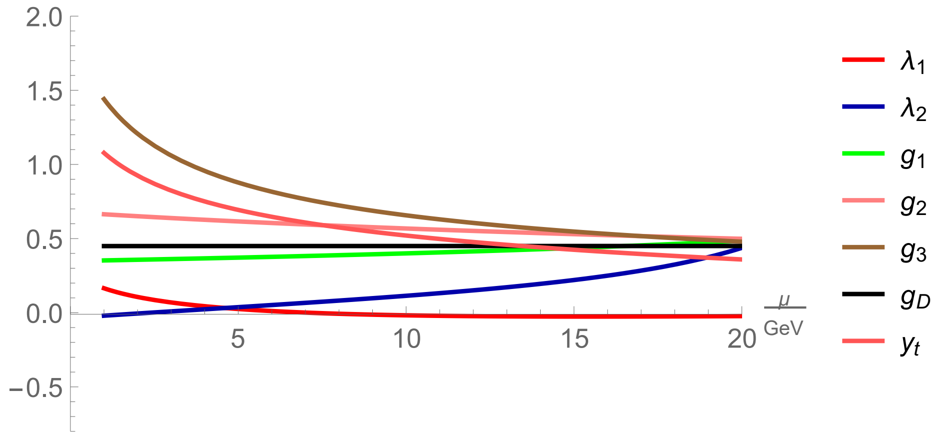


Figure 3.3: Running couplings in the Conformal Standard Model with initial parameters $g_D = 0.5$, $m_D = 1$ TeV, $\lambda = 0.09$.

To find a stability bound for our model, we increase the initial condition for the scalar coupling λ_1 until it stays positive all the way up to the Planck scale.
We find that the scalar coupling remains positive if we tune the initial value higher than $\lambda > (0.110 \pm 0.003)$, with the uncertainty coming mostly from the mass of the top quark. If we go back to our analysis of the Higgs mass with this value for the coupling, we find a stability bound of

$$m_h > (134 \pm 2) \text{ GeV}. \tag{3.94}$$

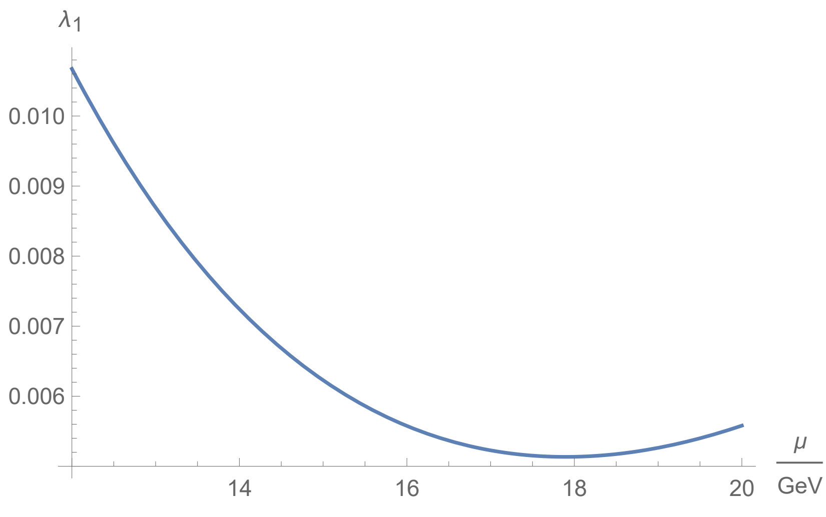


Figure 3.4: Running scalar coupling λ_1 in the Conformal Standard Model with initial parameters $g_D = 0.5$, $m_D = 1$ TeV, $\lambda = 0.11$.

Clearly, our stability bound fits even worse with the experimental value of the Standard Model, which is related to the fact, that the scalar coupling in this scale-invariant model is predicted to have a smaller value for the same value of the Higgs mass. This makes it become negative even faster than in the Standard Model.
In general, additional scalar particles can help cure the Standard Model instability because they give positive contributions to the beta function for the Higgs self-coupling

[23]. But since our use of a multiscale perturbation theory decoupled the beta functions of the dark sector and the Standard Model, this does not happen here.

In order to arrive at an estimation of the theoretical error of the mass bound, we also evaluated the couplings with the full two-loop beta functions of the Standard Model [22] which reduced the mass bound by about 4 GeV. Of course, using the beta functions for our model in the next order of perturbation theory might change this a little bit, but the order of magnitude will be similar.

3.5.4 Comment on multiscale renormalization

We have inserted the second renormalization scale by hand, simply by introducing a counter term depending on it.
There is also a method in which the second scale already appears in the renormalized Lagrangian [26]. This method actually introduces one scale per coupling constant but it turns out that those scales don't track the appearing logarithms in natural ways. In an improved scheme [27] the scales are assigned to the kinetic terms. This produces an effective potential of the same general form as (3.77). While at higher orders of perturbation theory all three methods introduce logarithms of the form $\ln \frac{\mu^2}{\mu'^2}$ to the beta functions, it was argued in [27] that it is always possible to improve the beta functions in a way that resums those logarithms. Since we work entirely at one-loop we don't encounter this problem.
Nevertheless this is something, that should be kept in mind for higher loop order calculations, since then there might arise additional complications with this two-scale approach.

3.6 The 'Next-To-Minimal' Conformal Standard Model

We have seen in the previous section that we were able to find a scale-invariant model that resembled the Standard Model and was at least metastable by adding a complex scalar field interacting with a dark photon. We found that while we were able to turn the saddle point of the effective potential into a true minimum, the model turned out to have another minimum at high fields value, similar to the Standard Model itself.

While one could argue that we set out to solve the hierarchy problem and not the stability of the Standard Model and should not be too disturbed that we did not solve two problems at once, the situation is at least a bit unsettling: inspired by Meissner and Nikolai we argued that the hierarchy problem can be solved by enforcing classical scale invariance and to fix the instability of the Standard Model, the usual argument is to call the instability scale the 'scale of new physics' [28]. However, the physical vacuum does not seem to be unstable and if our theory predicts it to be, then there should rather be some heavy degrees of freedom which do not influence the physics at the scales we are probing it but which change the behaviour of the model at the instability scale in order to render it stable. This is obviously at odds with our assumption of classical scale invariance. If the only way to get rid of the instability were the introduction of massive degrees of freedom, they would explicitly break scale invariance and we could not use it anymore to forbid the bare mass term of the Higgs boson.

Luckily there is a way out of this, again by adding a new field: a so called 'dormant' scalar singlet which couples to the Higgs boson but does not acquire a vacuum expectation value. As we mentioned in section 2.3.3., this scalar field does not need to obey the hierarchy constraint and can therefore give a sizeable contribution to the beta function of the Higgs coupling.
The Lagrangian of this "Next-To-Minimal" Conformal Standard Model reads

$$\begin{aligned}\mathcal{L}_{\text{NMCSM}} = \mathcal{L}_{\text{MCSM}} &+ \frac{1}{2}\partial_\mu U \partial^\mu U \\ &+ \frac{\lambda_{13}}{2}(H^\dagger H)U^2 + \frac{\lambda_3}{4}U^4\end{aligned} \tag{3.95}$$

and to calculate the effective potential, we expand the scalar fields H and S as in (3.40). Since we want to have scalar couplings λ_{13} and λ_3 of sizes comparable to the gauge couplings, we do not expand U around a non-zero classical field. The one-loop Lagrangian is the same as in (3.41) with one extra term

$$\mathcal{L}_{\text{1-loop, NMCSM}} - \mathcal{L}_{\text{1-loop, MCSM}} = \frac{1}{2}U\left(\Box + \frac{\lambda_{13}}{2}\hat{\phi}_2^2\right)U. \tag{3.96}$$

After applying the hierarchy argument to the couplings λ_1, λ_{12} and λ_2, we arrive at the effective potential

$$\begin{aligned}V_{\text{eff}}(\phi, S) = &\frac{\lambda_1}{4}\phi^4 + \lambda_{12}\phi^2 S^2 + \frac{\lambda_2}{4}S^4 + \frac{3}{64\pi^2}m_D^4 \frac{S^4}{\langle S\rangle^4}\left(\ln\frac{S^2}{\langle S\rangle^2} - \frac{25}{6}\right) \\ &+ \frac{1}{64\pi^2}\left(6m_W^4 + 3m_Z^4 - 12m_t^4 + m_U^4\right)\frac{\phi^4}{\langle\phi\rangle^4}\left(\ln\frac{\phi^2}{\langle\phi\rangle^2} - \frac{25}{6}\right)\end{aligned} \tag{3.97}$$

with $m_U^2 = \frac{\lambda_{13}}{2}\langle\phi\rangle^2$. For perturbative values of λ_{13}, i.e. values that do not lead to Landau poles below the Planck scale, this additional contribution does not significantly change our discussion of the mass eigenvalues, so we do not need to repeat it. Our aim was to stabilize the electroweak vacuum and therefore we solve the beta functions for this model, which are identical to the ones given in (3.90), up to

$$16\pi^2\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = (...) + \frac{1}{2}\lambda_{13}^2 \tag{3.98}$$

$$16\pi^2\frac{\mathrm{d}\lambda_{13}}{\mathrm{d}t} = 4\lambda_{13}^2 + 12\lambda_1\lambda_{13} - \frac{3}{2}\left(3g_2^2 + g_1^2\right)\lambda_{13} + 6y_t^2\lambda_{13} + 6\lambda_3\lambda_{13} \tag{3.99}$$

$$16\pi^2\frac{\mathrm{d}\lambda_3}{\mathrm{d}t} = 2\lambda_{13}^2 + 18\lambda_3^2. \tag{3.100}$$

We have studied the evolution of the coupling constants for a range of initial parameters $\lambda_{13} \in [0.38, 0.6]$ and $\lambda_3 \in [0, 0.2]$, while still fixing all other parameters to agree with Standard Model measurements. Our criterion for stability is again the sign of the quartic couplings.

As can be seen from figure 3.5, that is indeed a part of the parameter space in which

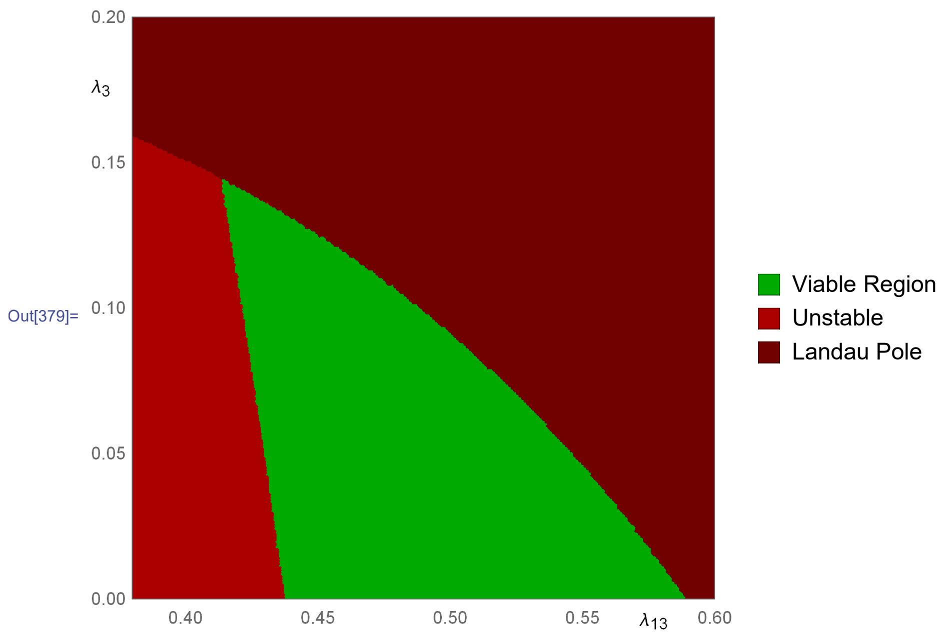


Figure 3.5: Parameter study of the vacuum stability of the Next-To-Minimal Conformal Standard Model. The viable region is the region in which neiter instabilities nor Landau poles appear below the Planck scale.

neither instabilities nor Landau poles appear below the Planck scale.
The allowed range of parameters translates to a range of masses for the scalar U

$$110 \text{ GeV} \lesssim m_U \lesssim 130 \text{ GeV}. \tag{3.101}$$

For those values, this model containing both a scalar S with gauged $U(1)'$-symmetry and a dormant scalar U is able to accomodate a stable SM-like vacuum.

Interaction terms in the Lagrangian that break the Z_2-symmetry $U \to -U$ like $\epsilon_3 U^3$ or $\epsilon_1(H^\dagger H)U$ are forbidden by classical scale invariance because the ϵ_i would introduce explicit scales. Combined with the fact that U does not acquire a vacuum expectation value, this makes U a stable particle. It can only annihilate and produce Standard Model particles via the Higgs portal, making it a viable candidate for cold dark matter.

Chapter 4
Conclusion

One suggestion how to fix the hierarchy problem, which extensions of the Standard Model might face, is to impose classical scale invariance [7]. The vacuum expectation value of the Higgs field might then be generated radiatively by the Coleman-Weinberg mechanism [8]. As was argued by Coleman and Weinberg as well as Andreassen, Frost and Schwartz [15], it is necessary to enforce a hierarchy of couplings to allow for a perturbatively consistent generation of vacuum expectation values at the quantum level.
We have applied this idea of a hierarchy between gauge and scalar couplings to a set of classically scale-invariant theories in order to find a model which resembles the Standard Model. As we have pointed out, this does not work in a scalefree Standard Model without any additional fields since the heavy top quark has a destabilizing effect on the vacuum. Our aim was therefore to find the minimal scalefree extension of the Standard Model for which there actually is a vacuum state which resembles the Standard Model vacuum.

We showed that the so called Conformal Standard Model, which is an extension of the Standard Model by one 'dark' scalar field is not able to accomodate the Standard Model vacuum. The corresponding extremum of the effective potential is necessarily unstable in the direction of the dark scalar, when one enforces the hierarchy of couplings.
By introducing an additional gauge symmetry under which only the dark scalar transforms we were able to locally stabilize the model, such that its effective potential possesses a minimum which fits the Standard Model vacuum. The only necessary condition for this was a lower bound for the mass of the new gauge boson

$$m_D > 220 \text{ GeV}. \tag{4.1}$$

Apart from this we found minima for a wide range of masses for the dark scalar. Since we observed a mixing between the Higgs boson and the dark scalar, a signature of our model would be a clone of the Higgs resonance at the mass of the dark scalar [7].
Because of the mixing between the Higgs and the dark scalar, both the dark photon and scalar will be unstable, though depending on the size of the mixing angle they might have very long lifetimes.

The minimum of the effective potential turned out to be only a local minimum, when we investigated the model further using the renormalization group equations. Similarly to the Standard Model, our 'Minimal Conformal Standard Model' develops an additional minimum at very high field values - this suggests, that it is again only an incomplete model which has to be modified.
To make sure that this is not at odds with our assumption of classical scale invariance, we finally showed that there is a further extension by a dormant scalar which allows for a range of parameters that stabilize the Standard Model vacuum. As such, the model could stay without any modifications all the way up to the Planck scale.

Table 4.1: Comparison of the models we discussed. The abbreviations mean:
HP: Is the hierarchy problem avoided?
EP: Does the effective potential have a minimum that resembles the Standard Model?
VS: Is this minimum the stable vacuum state of the model?
DM: Is there a stable dark matter candidate?

Model	Particle content	HP	EP	VS	DM
SM	Strong and Electroweak gauge bosons 3 Generations of Quarks and Leptons Higgs boson with negative mass squared	✗	✓	✗	✗
SFSM	SM without Higgs mass term	✓	✗	-	✗
CSM	SFSM + scalar singlet	✓	✗	-	✗
MCSM	CSM + dark photon	✓	✓	✗	✗
NMCSM	MCSM + dormant scalar	✓	✓	✓	✓

This 'Next-To-Minimal Conformal Standard Model' only works for a quite narrow mass region for the dormant scalar

$$110 \lesssim m_U \lesssim 130. \tag{4.2}$$

It is very interesting that the dormant scalar is stable and can therefore be considered a candidate for cold dark matter.

In table 4.1, we summarized the models we discussed as well as their main features and problems.

Especially the stability considerations using running couplings might be improved by allowing additional interactions between the dark sector and the Standard Model. In [29], a variation of our model including right-handed neutrinos is discussed. In order to cancel all gauge-anomalies it is then necessary to identify the new gauge symmetry with a mixture of the Standard Model $U(1)_Y$ and a new $U(1)_{B-L}$ symmetry. Another possible extension might be to look at bigger gauge groups. In [30] a similar model with a dark $SU(2)$ gauge group was studied. Both of those modifications also lead to globally stable minima of the effective potential. Since we were looking for the minimally extended model we did not study those additional complexities in this thesis.
However, this is the reason why we called the final model the Next-to-Minimal Conformal Standard Model. We believe that the dark scalar field with a gauge interaction is necessary to arrive at a vacuum state that resembles the Standard Model, while the dormant scalar is only one possible way to stabilize the state.

The next step to check the viability of our model and any extensions of it is to investigate their phenomenology more thoroughly. It is necessary to calculate production rates, lifetimes and cross sections and compare them to experimental data.
Furthermore, it would be interesting to see if our arguments hold up at higher loop order. Especially the multi-scale renormalization theory will not stay decoupled at higher orders and would be worth studying. We have already seen that the two-loop Standard Model beta functions decrease the stability bound for the Higgs mass. It would therefore be possible that already the Minimal Conformal Standard Model becomes stable, once higher order corrections are included.

Bibliography

[1] Abe, F. u. a.: Observation of top quark production in $\bar{p}p$ collisions. In: *Phys. Rev. Lett.* 74 (1995), S. 2626–2631

[2] Aad, G. u. a.: Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. In: *Phys. Lett.* B716 (2012), S. 1–29

[3] Chatrchyan, S. u. a.: Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. In: *Phys. Lett.* B716 (2012), S. 30–61

[4] Clowe, D. ; Bradac, M. ; Gonzalez, A. H. ; Markevitch, M. ; Randall, S. W. ; Jones, C. ; Zaritsky, D.: A direct empirical proof of the existence of dark matter. In: *Astrophys. J.* 648 (2006), S. L109–L113

[5] Bardeen, W. A.: On naturalness in the standard model. In: *FERMILAB-CONF-95-391-T, C95-08-27.3*

[6] Martin, S. P.: A Supersymmetry primer. (1997). – [Adv. Ser. Direct. High Energy Phys.18,1(1998)]

[7] Meissner, K. A. ; Nicolai, H.: Conformal Symmetry and the Standard Model. In: *Phys. Lett.* B648 (2007), S. 312–317

[8] Coleman, S. R. ; Weinberg, E. J.: Radiative Corrections as the Origin of Spontaneous Symmetry Breaking. In: *Phys. Rev.* D7 (1973), S. 1888–1910

[9] Foot, R. ; Kobakhidze, K. L. A.and McDonald M. A.and McDonald ; Volkas, R. R.: A Solution to the hierarchy problem from an almost decoupled hidden sector within a classically scale invariant theory. In: *Phys. Rev.* D77 (2008), S. 035006

[10] Baek, S. ; Ko, P. ; Park, Wa. ; Senaha, E.: Vacuum structure and stability of a singlet fermion dark matter model with a singlet scalar messenger. In: *JHEP* 11 (2012), S. 116

[11] Englert, C. ; Jaeckel, J. ; Khoze, V. V. ; Spannowsky, M.: Emergence of the Electroweak Scale through the Higgs Portal. In: *JHEP* 04 (2013), S. 060

[12] Ghorbani, K. ; Ghorbani, H.: Scalar Dark Matter in Scale Invariant Standard Model. In: *JHEP* 04 (2016), S. 024

[13] Khoze, V.n V. ; Plascencia, A. D.: Dark Matter and Leptogenesis Linked by Classical Scale Invariance. In: *JHEP* 11 (2016), S. 025

[14] Wang, Z. W.: *Aspects of Radiative Symmetry Breaking in the Standard Model and Beyond*, Saskatchewan U., Diss., 2016

[15] Andreassen, A. ; Frost, W. ; Schwartz, M. D.: Consistent Use of Effective Potentials. In: *Phys. Rev.* D91 (2015), Nr. 1, S. 016009

[16] Weinberg, S.: *The Quantum Theory of Fields Vol. 1 and 2.* Cambridge University Press, 1995

[17] Peskin, M.E. ; Schroeder, D.V.: *An Introduction to Quantum Field Theory.* Westview Press, 1995

[18] Schwartz, M. D.: *Quantum Field Theory and the Standard Model.* Cambridge University Press, 2013

[19] Plehn, T. ; Rauch, M.: The quartic higgs coupling at hadron colliders. In: *Phys. Rev.* D72 (2005), S. 053008

[20] Webber, D. M. u. a.: Measurement of the Positive Muon Lifetime and Determination of the Fermi Constant to Part-per-Million Precision. In: *Phys. Rev. Lett.* 106 (2011), S. 041803

[21] Aad, G. u. a.: Combined Measurement of the Higgs Boson Mass in *pp* Collisions at $\sqrt{s} = 7$ and 8 TeV with the ATLAS and CMS Experiments. In: *Phys. Rev. Lett.* 114 (2015), S. 191803

[22] Buttazzo, D. ; Degrassi, G. ; Giardino, P. P. ; Giudice, G. F. ; Sala, F. ; Salvio, A. ; Strumia, A.: Investigating the near-criticality of the Higgs boson. In: *JHEP* 12 (2013), S. 089

[23] Falkowski, A. ; Gross, C. ; Lebedev, Oleg: A second Higgs from the Higgs portal. In: *JHEP* 05 (2015), S. 057

[24] Srednicki, M.: *Quantum field theory.* Cambridge University Press, 2007

[25] Degrassi, G. ; Di Vita, S. ; Elias-Miro, J. ; Espinosa, J. R. ; Giudice, G. F. ; Isidori, G. ; Strumia, A.: Higgs mass and vacuum stability in the Standard Model at NNLO. In: *JHEP* 08 (2012), S. 098

[26] Einhorn, M. B. ; Jones, D. R. T.: A new Renormalization Group Approach to Multiscale Problems. In: *Nucl. Phys.* B230 (1984), S. 261–272

[27] Ford, C. ; Wiesendanger, C.: Multiscale renormalization. In: *Phys. Lett.* B398 (1997), S. 342–346

[28] Eichhorn, A. ; Gies, H. ; Jaeckel, J. ; Plehn, T. ; Scherer, M. M. ; Sondenheimer, R.: The Higgs Mass and the Scale of New Physics. In: *JHEP* 04 (2015), S. 022

[29] Oda, S. ; Okada, N. ; Takahashi, D.: Classically conformal $U(1)'$ extended standard model and Higgs vacuum stability. In: *Phys. Rev.* D92 (2015), Nr. 1, S. 015026

[30] Karam, A. ; Tamvakis, K.: Dark matter and neutrino masses from a scale-invariant multi-Higgs portal. In: *Phys. Rev.* D92 (2015), Nr. 7, S. 075010

Danksagung

Als erstes möchte ich meiner Freundin Rebekka Koch danken, die mich bei jedem Auf und Ab dieser Arbeit begleitet hat. Sie hat mich unermüdlich motiviert und nicht nur einmal eine gute Idee beigetragen, wie es weitergehen könnte.
Außerdem danke ich Prof. Jan Plefka und Prof. Peter Uwer für die Betreuung meiner Arbeit. Die Zusammenarbeit mit Jan Plefka und Florian Loebbert hat mir sehr viel Spaß gemacht und ich hoffe, sie in meiner Promotion vertiefen zu können.
In der Arbeitsgruppe habe ich mich sehr wohl und wilkommen gefühlt und ich danke allen Mitgliedern für die angenehme Atmosphäre und insbesondere meiner Büronachbarin Anne Spiering für Weintrauben, Schokobons und Obstsalate.
Schließlich danke ich meinen Eltern Marion und Bernd Miczajka und meinen Schwestern für den Halt, den sie mir geben, und ihr grenzenloses Vertrauen in mich - auch wenn es meiner Mutter immer noch nicht geheuer ist, dass man in der Physik Näherungen vornimmt.

Selbständigkeitserklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und noch nicht für andere Prüfungen eingereicht habe. Sämtliche Quellen einschließlich Internetquellen, die unverändert oder abgewandelt wiedergegeben werden, insbesondere Quellen für Texte, Grafiken, Tabellen und Bilder, sind als solche kenntlich gemacht. Mir ist bekannt, dass bei Verstößen gegen diese Grundsätze ein Verfahren wegen Täuschungsversuchs bzw. Täuschung eingeleitet wird.

Datum, Ort | Unterschrift