A Symbol of Uniqueness: The Cluster Bootstrap for the 3-Loop MHV Heptagon

Georgios Papathanasiou

Laboratoire d'Annecy-le-Vieux
de Physique Théorique & CERN

Humboldt-Universität zu Berlin
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1412.3763 [hep-th] with Drummond & Spradlin
work in progress

Outline

Motivation: Why $\mathcal{N} = 4$ SYM?

Scattering Ampitudes, Wilson Loop OPE and Integrability

The Amplitude Bootstrap and its Cluster Algebra Upgrade
- A Symbol of Uniqueness: The 3-loop MHV Heptagon

Conclusions & Outlook

$\mathcal{N} = 4$ Super Yang Mills Theory & Why Should We Care

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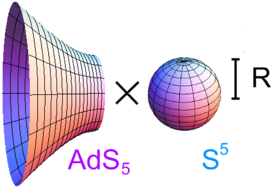

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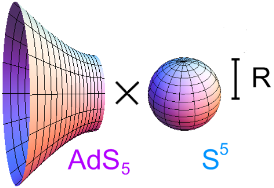

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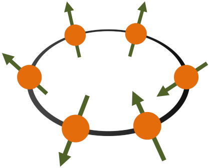

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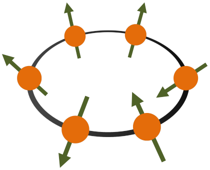

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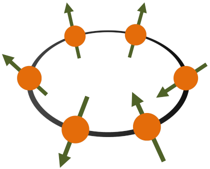

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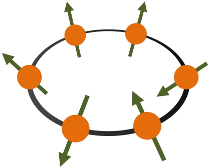

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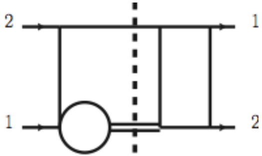

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Then apply to QCD, e.g. $|gg \to Hg|^2$ for N^{3}LO Higgs cross-section!

[Anastasiou,Duhr,Dulat,Herzog,Mistlberger]

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For the gluons $G^\pm$, the gluinos $\Gamma, \bar{\Gamma}$, and the scalars Φ. For n gluons,

$$\begin{aligned} &\mathcal{A}_n^{L\text{–loop}}(\{k_i, h_i, a_i\}) \\ &\quad = \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}})\, A_n^{(L)}(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n})) \\ &\qquad + \text{multitrace terms, subleading by powers of } 1/N^2\,. \end{aligned}$$

$A_n^{(L)}$: color-ordered amplitude, all color factors removed.

Maximally Hellicity Violating (MHV) Amplitudes

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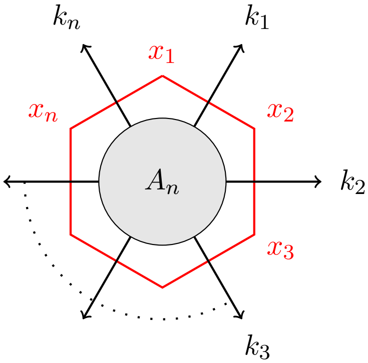
They also have remarkable properties, namely they

- are dual to null polygonal Wilson loops.
  [Alday,Maldacena] [Drummond,Korchemsky,Sokatchev] [Brandhuber,Heslop,Travaglini]



$$k_i \equiv x_{i+1} - x_i \equiv x_{i+1,i}\,,$$

$$k_i^2 = x_{i+1,i}^2 = 0$$

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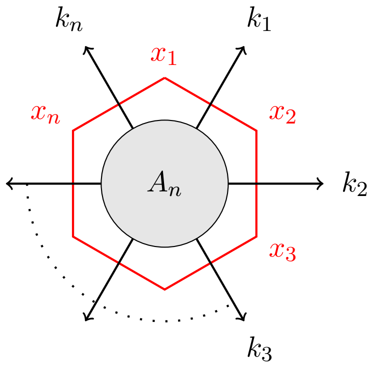

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- exhibit (formally) dual conformal invariance (DCI) under $x_i^\mu \to \frac{x_i^\mu}{x_i^2}$

MHV Scattering Amplitudes

- In reality DCI broken by divergences, (IR in massless $\mathcal{N} = 4$/UV in cusped WL). Breaking controlled by conformal Ward identity. [Drummond,Henn,Korchemsky,Sokatchev]

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- For $n \geq 6$,

$$W_n = W_n^{BDS} e^{R_n(u_1,\ldots,u_m)}$$

where the 'remainder function' R_n is conformally invariant, and thus a function of conformal cross ratios, e.g $u = \frac{x_{46}^2 x_{13}^2}{x_{36}^2 x_{14}^2}$.

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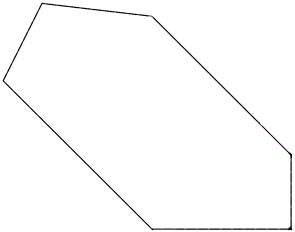
- # of independent u_i: $m = 4n - n - 15 = 3n - 15$

For the moment, focus on $R_6(u_1,u_2,u_3)$.

Nonperturbative Definition via the Collinear Limit

Kinematics



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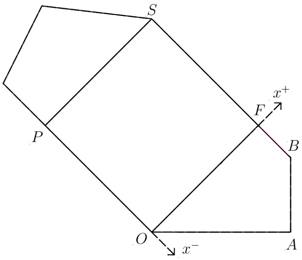

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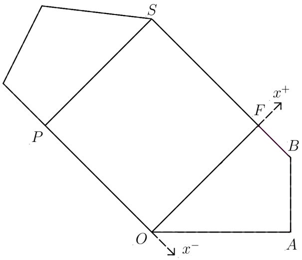

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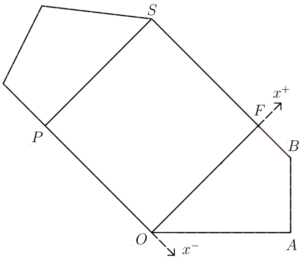

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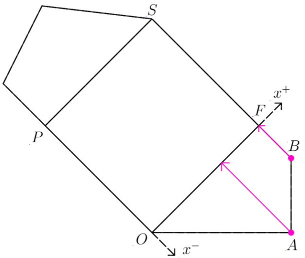
Collinear limit: Act with $e^{-\tau(D-M_{01})}$ on A and B, and take $\tau \to \infty$. Parametrize u_1, u_2, u_3 by group coordinates τ, σ, ϕ.



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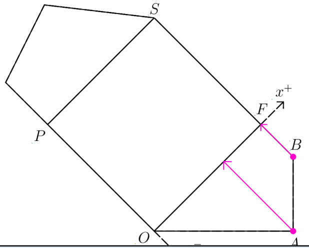

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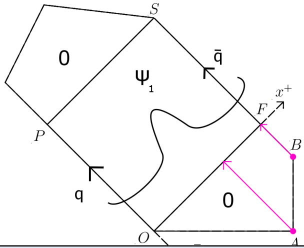

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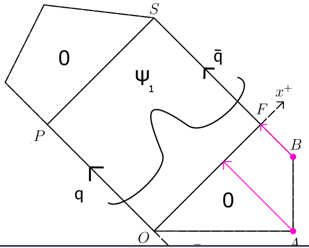

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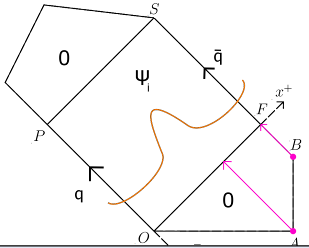

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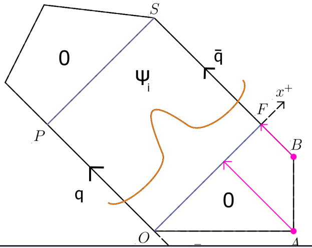

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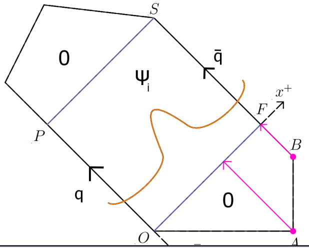

⇒ WL 'Operator Product Expansion' (OPE)

[Alday,Gaiotto,Maldacena,Sever,Vieira]

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Thus, weak coupling WL OPE=expansion in terms $\propto e^{-\tau M}, M = 1, 2 \ldots$

The Proposal of Basso,Sever,Vieira

To complete WL OPE description, also emission/absorption form factors or 'pentagon transitions' $\mathcal{P}(0|\psi_1), \mathcal{P}(\psi_1|0)$ needed.

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Spectral Problem Wisdom

If exact S-matrix within reach, look at many "data points" at weak/strong coupling to extract its general pattern.

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1. What the general class of *functions* that suffices to express $R_n^{(L)}$ is

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> Surprisingly, we found that heptagon bootstrap is more powerful than the hexagon one! Obtained the symbol of $R_7^{(3)}$ from very little input. [Drummond,GP,Spradlin]

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Very convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$, encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

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Empeirical evidence: L-loop amplitudes=GPLs of weight $k = 2L$
[Duhr,Del Duca,Smirnov] [Arkani-Hamed...] [GP]

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The latter is a collection of n ordered *momentum twistors* Z_i on $\mathbb{P}^3$, (an equivalent way to parametrise massless kinematics), modulo dual conformal transformations.

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- $(x_{i+i} - x_i)^2 = 0 \quad \Rightarrow X_i = Z_{i-1}\wedge Z_i$

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix

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Comparing the two matrices,

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$$

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Here, finite number of cluster variables:

$$a_3 = \frac{1+a_2}{a_1}\,, \quad a_4 = \frac{1+a_1+a_2}{a_1 a_2}\,, \quad a_5 = \frac{1+a_1}{a_2}\,, \quad a_6 = a_1\,, \quad a_7 = a_2$$

Cluster algebras (cont'd)

For our purposes, can be described by quivers, where each variable a_k of a cluster corresponds to node k.

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- In this manner, obtain new quiver/cluster where

$$a_k \to a'_k = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

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- Mutate at 1: $1' \leftarrow 2$
- Leads to new cluster $\{a_2, a_3\}$ with $a_3 = a'_1 = \frac{1+a_2}{a_1}$ and so on

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Fundamental assumption of "cluster bootstrap"

Symbol alphabet is made of cluster $\mathcal{A}$-coordinates on $\mathrm{Conf}_n(\mathbb{P}^3)$. For the heptagon, 42 of them.

Heptagon Symbol Letters

Multiply $\mathcal{A}$-coordinates with suitable powers of $\langle i\, i+1\, i+2\, i+3\rangle$ to form conformally invariant cross-ratios,

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}\,, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}\,,$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}\,, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}\,, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

where

$$\langle ijkl\rangle \equiv \langle Z_i Z_j Z_k Z_l\rangle = \det(Z_i Z_j Z_k Z_l)$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle\,,$$

together with a_{ij} obtained from a_{i1} by cyclically relabeling $Z_m \to Z_{m+j-1}$.

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Given a random symbol $\mathcal{S}$ of weight $k > 1$, there does not in general exist any function whose symbol is $\mathcal{S}$. A symbol is said to be **integrable**, (or, to be an **integrable word**) if it satisfies

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Not integrable

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Singularities of generalised polylogarithm functions are encoded in the first entry of their symbols.

First-entry condition: Only $\langle i{-}1\, i\, j{-}1\, j \rangle$ allowed in the first entry of $\mathcal{S}$

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Define a **heptagon symbol**: An integrable symbol with alphabet a_{ij} that obeys first-entry condition.

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Particularly here: Only the 14 letters a_{2j} and a_{3j} may appear in the last symbol entry of R_7.

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS-subtracted n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n-1)$-particle function in any simple collinear limit:

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A function has a well-defined $i+1 \parallel i$ limit only if its symbol is independent of all nine of these letters.

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"Just" linear algebra, however for e.g. 3-loop MHV hexagon A boils down to a size of 63557×15979. Tackled with fraction-free variants of Gaussian elimination that bound the size of intermediate expressions. [Storjohann]

Results

Weight k =	1	2	3	4	5	6
Number of heptagon symbols	7	42	237	1288	6763	?
well-defined in the $7 \parallel 6$ limit	3	15	98	646	?	?
which vanish in the $7 \parallel 6$ limit	0	6	72	572	?	?
well-defined for all $i{+}1 \parallel i$	0	0	0	1	?	?
with MHV last entries	0	1	0	2	1	4
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Comparison with the hexagon case

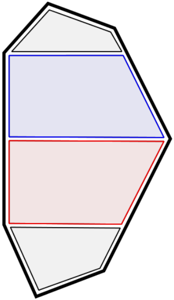
Weight k =	1	2	3	4	5	6
Number of hexagon symbols	3	9	26	75	218	643
well-defined (vanish) in the $6 \parallel 5$ limit	0	2	11	44	155	516
well-defined (vanish) for all $i{+}1 \parallel i$	0	0	2	12	68	307
with MHV last entries	0	3	7	21	62	188
with both of the previous two	0	0	1	4	14	59

Table : Hexagon symbols and their properties.

Surprisingly, heptagon bootstrap more powerful than hexagon one! Fact that $\lim_{7\parallel 6} R_7^{(3)} = R_6^{(3)}$, as well as discrete symmetries such as cyclic $Z_i \to Z_{i+1}$, flip $Z_i \to Z_{n+1-i}$ or parity symmetry **follow for free**, not imposed a priori.

Further check: Heptagon Wilson loop OPE

This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.



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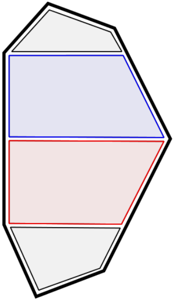

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$$h = e^{i(\phi_1+\phi_2)}\, e^{-\tau_1-\tau_2} \int \frac{du dv}{(2\pi)^2}\, \mu(u)\, P_{FF}(-u|v)\mu(v) \times$$
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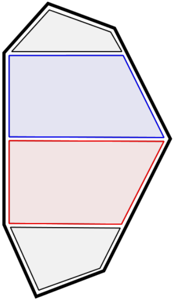

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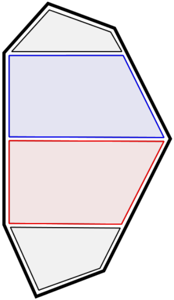

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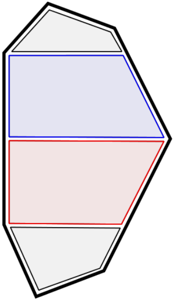

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- The rich interplay between the two approaches

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Ultimately, can the integrability of planar SYM theory, together with a thorough knowledge of the analytic structure of its amplitudes, lead us to the theory's exact S-matrix?

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- Eigenvectors and eigenvalues $\mathcal{D}\mathcal{O} = \Delta\mathcal{O}$, and conventionally we define $\delta\Delta \equiv \Delta - \Delta_0$ as the *anomalous dimension*.

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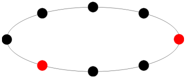
For example, operators made of 2 complex combinations Z, W of the 6 real scalars of SYM can be represented as

$\mathrm{Tr}[Z^4 W Z^2 W]$ $\Leftrightarrow$  $\Leftrightarrow$ $|\downarrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\rangle_{\text{cyclic}}$

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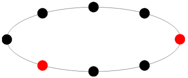
$\mathrm{Tr}[Z^4WZ^2W]$ $\Leftrightarrow$  $\Leftrightarrow$ $|\downarrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\rangle_{\text{cyclic}}$

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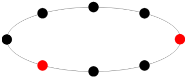
$\mathrm{Tr}[Z^4 W Z^2 W]$ $\Leftrightarrow$  $\Leftrightarrow$ $|\downarrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\rangle_{\mathrm{cyclic}}$

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So the ground state is $\mathrm{Tr}\left(Z^L\right)$ and its excitations are given by spin flips $Z \to W$ or "**magnons**". Can solve by Bethe Ansatz Equations.

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Assuming quantum integrability, possible to obtain equations encoding the all-loop spectrum of scaling dimensions! [Arutyunov,Beisert,Bombardelli,Eden,Fioravanti,Frolov,Gromov,Janik, Kazakov Leurent,Staudacher,Tateo,Vieira,Volin. . .]

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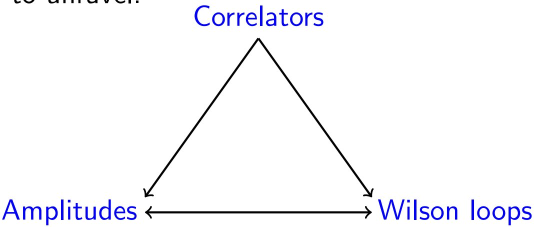

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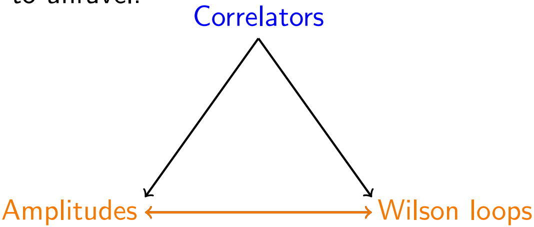

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Clearly, the submatrix N forms basis for the right nullspace of A.

Major complication: For rational matrices like A, standard Gaussian elimination doubles size of entries at each step, leading to runtimes depending exponentially on size of A.

Computing Nullspaces

By Gaussian elimination: Bring A to column echelon form H by transformation U, $A \cdot U = H$,

$$U = (\underbrace{U_1}_{r} | N), \quad H = (\underbrace{H_1}_{r} | \mathbf{0}), \quad r = \operatorname{rank}(A),$$

Clearly, the submatrix N forms basis for the right nullspace of A.

Major complication: For rational matrices like A, standard Gaussian elimination doubles size of entries at each step, leading to runtimes depending exponentially on size of A.

Key idea: Transform A from rational to integer, and use fraction-free variants of Gaussian elimination that bound the size of intermediate expressions by virtue of Hadamard's inequality. [Storjohann]