Perturbative quantization of superstring theory in Anti de-Sitter spaces

Integrability in gauge / string dualities

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Abstract

In this thesis we study superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, $\mathrm{AdS}_3 \times \mathrm{S}^3$ and $\mathrm{AdS}_4 \times \mathbb{CP}^3$. A shared feature of each theory is that their corresponding symmetry algebras allows for a decomposition under a $\mathbb{Z}_4$ grading. The grading can be realized through an automorphism which allows for a convenient construction of the string Lagrangians directly in terms of graded components. We adopt a uniform light-cone gauge and expand in a near plane wave limit, or equivalently, an expansion in transverse string coordinates. With a main focus on the two critical string theories, we perform a perturbative quantization up to quartic order in the number of fields. Each string theory is, through holographic descriptions, conjectured to be dual to lower dimensional gauge theories. The conjectures imply that the conformal dimensions of single trace operators in gauge theory should be equal to the energy of string states. What is more, through the use of integrable methods, one can write down a set of Bethe equations whose solutions encode the full spectral problem. One main theme of this thesis is to match the predictions of these equations, written in a language suitable for the light-cone gauge we employ, against explicit string theory calculations. We do this for a large class of string states and the perfect agreement we find lends strong support for the validity of the conjectures.

Zusammenfassung

Um das mikroskopische Verhalten der Gravitation zu beschreiben, ist es n"otig, Quantenfeldtheorie und allgemeine Relativit"atstheorie in einer vereinheitlichten Sprache zu formulieren. Eine M"oglichkeit dieses Problem anzugehen ist es, die Punktteilchen der Quantenfeldtheorie durch fadenf"ormige Strings zu ersetzen. Allerdings erfordert die mathematische Konsistenz, dass sich die String in h"oherdimensionalen Raum-Zeiten bewegen; dies macht es jedoch sehr schwer, physikalische Konsequenzen zu extrahieren. Eine m"ogliche L"osung dieses Problems ist die Verwendung von String-Dualit"aten, welche die Stringtheorie mittels holographischer Beschreibungen mit Eichtheorien auf dem Rand der Raum-Zeit verbinden. Die Dualit"aten sind begr"undete Vermutungen, die die String- und Eichtheorie bei unterschiedlichen Werten der Kopplung gleichsetzen. Nicht zuletzt deshalb ist eine direkte "Uberpr"ufung der Dualit"aten schwierig durchf"uhrbar. Hier hilft jedoch die sehr bemerkenswerte Tatsache, dass eine verborgene Eigenschaft der Vermutungen Integrabilit"at zu sein scheint, welche eine Extrapolation zwischen starker und schwacher Kopplung erm"oglicht. Desweiteren kann das gesamte Spektrum, in gewissen vereinfachenden Grenzf"allen, durch einen kompakten Satz von Bethe-Gleichungen ausgedr"uckt werden. Die Bethe-Gleichungen, welche aus Eichtheorierechnungen hergeleitet und geraten werden, bieten ein exzellentes Hilfsmittel, die vermuteten Dualit"aten zu pr"ufen. Durch das Vergleichen der Vorhersagen der Gleichungen und expliziten Berechnungen in der Stringtheorie erh"alt man starke Argumente f"ur die G"ultigkeit der Vermutung und der angenommenen Integrabilit"at.

Aufgrund der hohen Komplexit"at der Stringtheorien muss man vereinfachende Limites betrachten, um das Spektralproblem zu l"osen. Ein besonders zweckm"a"siger Limes ist die sogenannte *near plane wave* Entwicklung. Diese reduziert sich zu einer Entwicklung in der Anzahl der transversalen Anregungen der Stringkoordinaten. In dieser Dissertation untersuchen wir detialliert die Dynamik und das Spektrum des *near plane wave* Limes von Stringtheorien, die in einer Vielzahl von Eich-/String-Dualit"aten auftreten; besonderes Augenmerk legen wir hierbei auf den AdS$_5\times$S5- und AdS$_4\times\mathbb{CP}_3$-Superstring. Der *near plane wave* Limes ist im Wesentlichen der Grenzwert gro"ser 't Hooft-Kopplung, bei dem man zus"atzlich den Lichtkegelimpuls P_+, welcher zu einer der Lichtkegelkoordinaten konjugiert ist, so skaliert, dass sein Verh"altnis zur Kopplung konstant bleibt. Indem wir perturbative Rechnungen erster und zweiter Ordnung durchf"uhren, zeigen wir, wie die spektralen Informationen f"ur eine gro"se Klasse von Stringzust"anden, sowohl f"ur AdS$_5\times$S^5- als auch AdS$_4\times\mathbb{CP}_3$-Superstrings, erlangt werden k"onnen. Die berechnet Energiekorrekturen werden von uns mit den Vorhersagen der Bethe-Gleichungen vergleichen, wobei wir eine perfekte "Ubereinstimmung finden.

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1 Introduction

Theoretical physics can pride itself with two major achievements in the last century, quantum mechanics and general relativity. Quantum mechanics explains the intricate microscopic behavior of the subatomic world while the theory of general relativity describes the interplay between gravity and space-time itself. General relativity, initially developed by Einstein in the early twenties, describes the complicated interplay between mass and gravity. The discovery has far reaching consequences for our understanding of the world we observe since in order to keep the speed of light at a fixed value, and at the same time incorporate the effects of gravity, the notion of a static space and time has to be abandoned. Space itself is allowed to bend and twist in the vicinity of massive objects. On the other hand, quantum mechanics deals with the microscopic behavior of small and light objects, such as photons and electrons. In quantum mechanics the role of the observer becomes fundamental, and in order to describe the subatomic world one has to make use of a mathematical language based on probabilities which render the exact knowledge of position and energy impossible. The prior world of Newtonian determinism now has to be given up for a richer world of quantum uncertainties where the exact time evolution of a system is unattainable.

Both theories are well fortified in a host of experimental data and it is certain that they, within their limits of validity, are true. Quantum mechanics, or quantum field theories which are a slight generalization to incorporate special relativity[1], is a unified language of all subatomic and electric forces and thus describes everything we believe to know except gravity. Even though gravity is the force we have everyday experience with, its strength is nevertheless drastically weaker than that of the subatomic forces. For example, the force that keeps the Hadronic matter, such as Protons and Neutrons, together is so much stronger than gravity in force so, in comparison, the latter does almost not exist. Nevertheless, there are situations where both descriptions are needed. For example, in the close vicinity of a black hole. A black hole is an object so dense that it has collapsed under its own gravitational potential and everything trapped within it, beyond its so called event horizon, is doomed to remain there for all eternity. Another example is in the early universe where the energy density of space were so high that quantum and gravitational effects were comparable in strength. To describe these two examples one would need a unified language of quantum mechanics and general relativity. This is the question of quantum gravity.

Unfortunately, quantum gravity is very hard. For example, in quantum field theories one has operators that commute on all space-like separated points, reflecting the casual structure of the theory demanded by special relativity. To define a space-like interval,

[1]Special relativity is, as it sounds, a special case of general relativity in the sense that all gravitational effects are ignored but one keeps the speed of light fixed in all inertial frames.

one needs the metric which provides information for how to relate separated points in space. However, in the quantum theory the metric needs to be computed and is thus part of the dynamical problem. What is more, the metric being a dynamical field implies that it fluctuates quantum mechanically and thus it is not clear how to define space-like separation in a well defined way. Of course, one can try to treat the problem through perturbation theory as is the case for most quantum field theoretic calculations anyway, however, since Newtons constant has dimension $(\text{length})^2$, one find by simple power counting that the ultra violet (UV) divergences increases with each loop order in perturbation theory. In order to remove these in a renormalization process one would need to add an ever increasing number of counter terms rendering the process both unphysical and practically impossible.

The infinities that one encounters in the UV are effects at very small distances and one possible remedy could be to introduce some sort of cut off parameter for short distances. For particles there is no geometrical minimal length since they are zero dimensional with no extension in space. However, what if one were to consider something else than particles? For example, an infinite number of particles aligned continuously to form either a closed or an open string. Taking strings as the fundamental building blocks introduces a minimal length scale and thus could be a solution of the problem with UV infinities. This theory of strings, or string theory, will be the main focus of this thesis.

The strings are very small and if one looks at them from distance they resemble point particles, see figure 1.1. Since distance is inversely proportional to energy, the particle description means that only the lightest excitations of the string are of importance. Remarkably, if one goes through with the process of quantization, one of the light vibrational modes of the string is a massless spin two particle which can be identified with the graviton! That is, gravity seem to somehow be incorporated in string theory from the very beginning. What is more, one can also show that the theory is free from UV divergences to the first few orders in perturbation theory and the belief is that this remain true to all orders. Thus, simply replacing particles with strings automatically gives a quantum theory that by itself generates gravity. This, together with the cancellation of anomaly terms, was discovered by Green and Schwarz back in the early eighties and denotes what is called the first string revolution.

Unfortunately, not everything works out as remarkably as the graviton in the spectrum. As it turns out, mathematical consistency demands that the strings oscillate in a ten dimensional space-time, six more than what we are used to! What is more, string theory makes heavy use of the mathematical language developed in theoretical high energy physics and can in one sense be seen as a mathematical generalization of it. This naturally gives a resulting theory with a very self consistent mathematical formulation but also gives a somewhat general theory. That is, since the theory is formulated at such high energies and with the extra dimensions of space, one can by taking various limits almost obtain any consistent four dimensional model[2] rendering the predictive power of the theory very poor. Nevertheless, the theory and the consequences of it have been very well studied over the last three decades and a remarkable host of different

[2] With or with out the observed properties of our world.

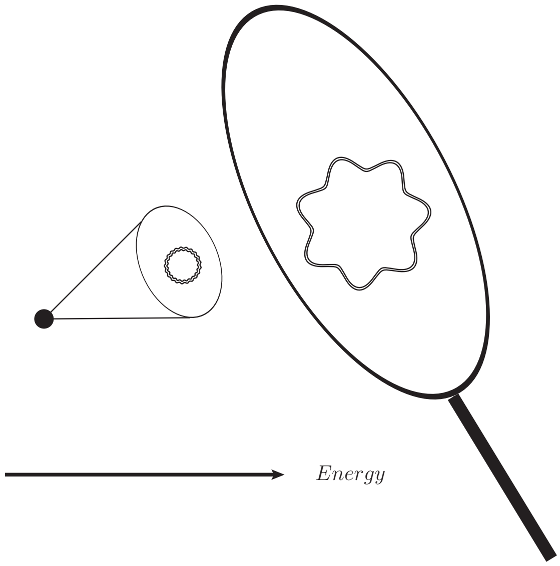

Figure 1.1: Illustration of a closed string under a microscope. At a distance it resembles a particle and only at short distances, with high energy, can one see its true shape.

mathematical structures has emerged. With the simple input of strings, objects such as higher dimensional branes, cosmological models, supersymmetry[3], black hole models, dualities and more emerge.

From a bit more philosophical point of view one can view string theory as an attempt to build a theory on a mathematical language invented to describe particle physics. That is, in the latter part of the twentieth century remarkable progress in high energy physics were made which culminated in the so called Standard Model of particle physics. The Standard Model unites all the three fundamental forces of weak, strong and electromagnetism into an unified mathematical language. This remarkable achievement is based on the formalism of gauge theories which are defined through the symmetries they enjoy. In a technical language the symmetry of the Standard Model can be grouped together in SU(3) $\times$ SU(2) $\times$ U(1), where each esoteric SU(3), SU(2) and U(1) denotes the symmetry of the strong, weak and electro magnetic theories[4]. The remarkable success of the Standard Model leads one to believe that its mathematical language is one of nature and that a more fundamental theory should be described using the same formalism. However, this is an assumption and it might not be true. Even though the success of the standard model in terms of experimental evidence is unquestionable, its mathematical formulation might not be since even the Standard Model is suffering from infinities in the UV. In contrast to perturbative quantum gravity, these can however be removed through

[3] A symmetry relating bosonic and fermionic vibrational modes of the string.

[4] Very loosely speaking, U(N) denotes a symmetry with N^2 symmetry directions and adding the S just means we take one direction away.

a renormalization process. The renormalization process means that one, in a more or less random way, add counter terms with the property that they exactly cancel the divergent expressions. Even though this can be done, and physical quantities can be calculated[5] it could nevertheless signal some sort of inconsistency or incompleteness of the theory. Of course, it could be that the fundamental laws of Nature *has* to be described in this way but then, on the other hand, at some level most physicists share the belief that the laws of nature should exhibit beauty and simplicity, not the opposite. For this reason, taking the wisdom of gauge theory and generalizing into a theory of strings might be the wrong way to approach the problem of quantizing gravity. Thus, even though quantizing gravity is a very important problem of modern physics, it could be that we, as of yet, have not developed the correct tools or understanding for it. Perhaps the correct theory of quantum gravity is not built on gauge theory at all, but some, yet to be discovered, more fundamental mathematical description of nature. Maybe similar breakthroughs as that of general relativity and quantum mechanics are needed before we can construct a theory of quantum gravity. Basically the way to proceed is not known, and without the experimental input (which string theory seem incapable of providing) the right way might be hard to find.

The above might seem like a rather gloomy, and naturally somewhat subjective, picture of string theory as a unifying language of gravity and quantum mechanics. However, there is more to string theory than just an attempt to describe quantum gravity. In the late nineties Juan Maldacena proposed that a ten dimensional string theory in a specific gravitational field, or background, could be identified with a four dimensional quantum field theory **?**. That is, through a holographic description, all the dynamical data of the string theory are equivalent to another, and drastically different, four dimensional particle theory! The proposal, which comes in the form of a conjecture, is rather remarkable since, at first glance, how can a higher dimensional theory of extended objects such as strings describe the same physics as a four dimensional quantum field theory? If it is true, and by now a very large set of independent tests have been performed in favor of the conjecture[6], it leads string theory research in a new direction since the duality is of the character that the string theory is solvable (perturbatively) when the quantum theory is not and vice versa. Thus, instead of thinking of string theory as an attempt to reconcile quantum mechanics with gravity one can adopt a more modest approach and try to solve a strongly coupled quantum field theory exactly[7]. Perhaps this seems somewhat modest in comparison with quantizing gravity but, nevertheless, finding the strong coupling dynamics of a quantum field theory is something highly non trivial. This becomes even more important because, as it turns out, the gauge theory side in one of the correspondences[8] is rather similar to the theory of strong interactions. For the theory of strong interactions perturbative analysis is only possible in the high energy regime, but there are many important, non perturbative, physical effects also such as

[5] At least perturbatively.

[6] Every time the word test, check or confirmation is used in relation to the conjecture it is meant as a theoretical test. Not experimental.

[7] Strongly coupled just means that it can not be treated perturbatively.

[8] By now there are a few and in this thesis we will study two in detail and touch briefly upon a third.

quark confinement and chiral symmetry breaking. At the moment, low energy physics of the strong interactions can only be studied through lattice simulations, but the existence of gauge / string dualities might open up for analytical solutions in the future.

Due to the strong / weak coupling nature of the dualities, it is very hard to prove them rigorously since in order to verify them one needs to calculate the corresponding quantities on each side of the duality to see if they agree. That is, if for example the string theory is open for perturbative calculations then the gauge theory is not and it is, in general, very hard to obtain analytical answers. At first glance this seems like a major set back since how can one verify the correspondence? Remarkably, and very luckily, something completely unexpected enters the game; The appearance of integrable structures! Integrable structures are hidden symmetries that allows one to obtain analytical solutions regardless the value of the coupling. Thus, one can calculate something at weak coupling in, for example, the gauge theory, extrapolate the value to strong coupling and match it against the corresponding string theory calculation. What is more, the emergence of integrability also allows one to write down the spectrum of energies into a very compact set of equations, so called Bethe equations. Thus, from integrability alone, it seems that one can prove large parts of the dualities analytically[9]

In this thesis we will study light-cone string theory on Anti de-Sitter backgrounds. The name light-cone is meant to indicate that one combines two, out of the ten, coordinates in a specific linear combination. This is convenient because one can then align the two internal coordinates of the string, one time and one that parameterize the length of the string, relative to one of the light-cone coordinates which results in a simplified theory. This specific way of combining the coordinates is called light-cone gauge and will be a central theme all through out the thesis. The Anti de-Sitter background is a specific geometry on which the string propagates and it has the characteristic property that parallel lines, in contrast to a sphere for example, tend to diverge when extended. In total we will study three different string theories, each propagating on different background geometries but where parts of it always is of an Anti de-Sitter type.

An especially important theme of this thesis is the verification of the proposed set of Bethe equations. These equations are derived in the gauge theory for small sectors and then conjectured for the full model to all orders in perturbation theory. In order to verify the equations we calculate energies for a large set of string configurations in a strong coupling limit and match these against the predictions coming from the Bethe equations. This is important since it shows that, at least to the relevant order in perturbation theory, integrability is indeed a manifest feature of the theory.

[9]Here a few comments are in order. First of all, integrability is only manifest for certain limits of the problem. Second, not all operators allow for their energies, or spectral parameters, to be encoded in Bethe equations. It is only so called long operators that exhibit this feature and these operators will be the ones studied in this thesis.

1.1 Outline of the thesis

Before we turn to the main text it might be illuminating to summarize and outline the structure of the thesis. The main text is divided into two separate parts. Part I is generally introductory and introduces the concepts of Anti de-Sitter space, gauge theories, gauge / string dualities and light-cone string theory in general. Due to the enormous scope of each subject, most of the text will be rather brief and we will frequently point the reader to relevant references and review articles. Only in the last chapter, on light-cone string theory, will we present the material in full detail since later parts of the thesis uses this as a starting point for explicit calculations. A nice feature is that, even though we will study three different string theories, the construction of the string Lagrangian[10] can be done algebraically without reference to a specific model.

In part II we study three different string theories on $\mathrm{AdS}_3\times\mathrm{S}^3$, $\mathrm{AdS}_5\times\mathrm{S}^5$ and $\mathrm{AdS}_4 \times \mathbb{CP}_{\nvDash 3}$, written in increasing order of complexity. Each theory is highly non linear and in order to extract physical quantities we consider a strong coupling expansion. One central theme in part II is the calculation of energies of large classes of string configurations. As we have mentioned, these energies are also encoded in Bethe equations, and for the two critical string theories, $\mathrm{AdS}_5\times\mathrm{S}^5$ and $\mathrm{AdS}_4 \times \mathbb{CP}_{\nvDash 3}$, we explicitly match them against the Bethe equations.

The outline is as follows; In part I we start out with a short review of Anti de-Sitter space and the gauge theories that appear in the various gauge / string dualities. Having described the field theory part we then turn to present the dualities relating them to the higher dimensional string theories. Once again we will be rather brief and only present the details necessary for the upcoming analysis. One very important ingredient is the emergence of integrable structures and we show how they come about and, especially, how to encode the spectral problem in terms of Bethe equations. These equations are very compact and encode the spectrum of conformal dimensions[11] which we later want to compare against energies of string states.

We then turn to a discussion of light-cone string theory. Since the subject is rather involved we begin this part with a review of the bosonic string. Even though much simpler, the model is nevertheless similar to the full supersymmetric theory. In this section we also provide a few examples of strings in different geometries or backgrounds together with a detailed explanation about light-cone gauge fixing and its physical consequences. After this introductory section we move on to the full string theory with fermions included. This rather lengthy section begins with a review of the symmetry algebras of each theory where an important concept is that they all can be realized in terms of super matrices. Starting from the algebra one can construct group elements which are the fundamental building blocks of the string Lagrangian. We show how to construct the Lagrangian in detail and then turn to discuss its physical consequences where an especially important point is the fixing of a fermionic κ symmetry which allows

[10] A Lagrangian is a very central concept in theoretical physics and is basically the sum of all possible paths a particle, or string, can take. In one sense it can be seen as the definition of a theory.

[11] The conformal dimension is an important observable that can be used to classify the operators of a field theory with a special scaling, or conformal, symmetry.

one to make space-time supersymmetry manifest. The culmination of this chapter is the full string Lagrangian, and its corresponding Hamiltonian, in a notation suitable for a strong coupling expansion.

The second part of the thesis deals with strong coupling expansions of the three string models. This part, which is by far the most technical, is based on the authors research papers **?**, **?** and **?** together with unpublished work on the $AdS_3 \times S^3$ and $AdS_5 \times S^5$ superstring.

The first paper, **?**, written together with A. Hentschel and J. Plefka provides a very detailed check of the validity of the Bethe equations for a string propagating in $AdS_5 \times S^5$. In the paper we calculate energy shifts for a very large class of string configurations and show that they precisely match the predictions of the conjectured Bethe equations.

The second paper, **?**, written by the author alone, deals with the bosonic aspects of a string propagating on $AdS_4 \times \mathbb{CP}_{\not{k}3}$. This theory, vastly more complicated than the $AdS_5 \times S^5$ string, has received a lot of attention lately due to its recent appearance in a gauge / string duality. As was also the case for the $AdS_5 \times S^5$ string, one can conjecture a set of Bethe equations and in the paper it is explicitly verified that they reproduce the string result for a certain set of string configurations.

The third and most technical paper, **?**, which also is written by the author alone, introduces the full supersymmetric $AdS_4 \times \mathbb{CP}_{\not{k}3}$ string. Having established the dynamical theory in a strong coupling regime, two separate calculations are performed. First, a further explicit check of the Bethe equations is provided by matching energies of fermionic operators. Second, a novel feature with the $AdS_4 \times \mathbb{CP}_{\not{k}3}$ string is that the excitations come with different masses. We perform a detailed analysis and show that all the massive bosonic operators are, in fact, composite states of two lighter excitations.

2 AdS / CFT dualities

This chapter will be devoted to the tantalizing dualities relating string theories with conformal field theories (CFT). It is a remarkable fact, discovered and investigated over the last decade, that one can circumvent the complications in a strongly coupled quantum field theory (QFT) by solving a weakly coupled string theory. Since the string theories, which propagate in higher dimensional space-times, at first glance have nothing in common with a lower dimensional QFT we will devote this chapter to provide some basic arguments for why the dualities can be true. All the dualities come in the form of conjectures and to prove them one need to calculate the corresponding quantities on each side of the duality to see if they match. However, since the gauge theory is strongly coupled when the string is weakly coupled and vice versa, it is hard to perform a calculation on both sides simultaneously. This problem can be tackled with the help of hidden integrable structures which we also will review. This will be important because a significant part of the thesis later chapters is devoted to the spectral problem of vibrating strings. The spectral problem can, through the existence of integrability, be reduced to solving a set of so called Bethe equations, and matching these against explicit string theory calculations lends strong support for the validity of the gauge / string correspondences.

In this thesis we study three different string theories appearing in different gauge / string dualities; the $\mathrm{AdS}_5\times\mathrm{S}^5$ string dual to $\mathcal{N}=4$ SYM in four dimensions which will be the main focus of this chapter, the $\mathrm{AdS}_3\times\mathrm{S}^3\times T^4$ string dual to a two dimensional CFT[1] and, finally, the $\mathrm{AdS}_4\times\mathbb{CP}_{\not{k}3}$ string dual to a three dimensional Chern-Simons (CS) theory. In each duality the gauge theory lives on the boundary of the Anti de-Sitter (AdS) space and for that reason we start this chapter by reviewing some basic facts about AdS spaces in general. We will then turn to present some of the characterizing properties of the gauge theories with a focus on the existence of integrable structures. We end the chapter with a short review of how one can map the spectral problem of conformal dimensions to that of solving a one dimensional spin chain Hamiltonian.

2.1 The structure of Anti de-Sitter space

An AdS space in $p+2$ dimensions is defined through the metric and the constraint

$$ds^2=-dx_0^2-dx_{p+2}^2+\sum_{i=1}^{p+1}dx_i^2,\quad x_0^2+x_{p+2}^2-\sum_{i=1}^{p+1}x_i^2=R^2, \tag{2.1}$$

[1]We will solely focus on the non linear six dimensional part and we will not review the two dimensional gauge theory at all, but for the interested reader we point to **?**, **?** and **?**.

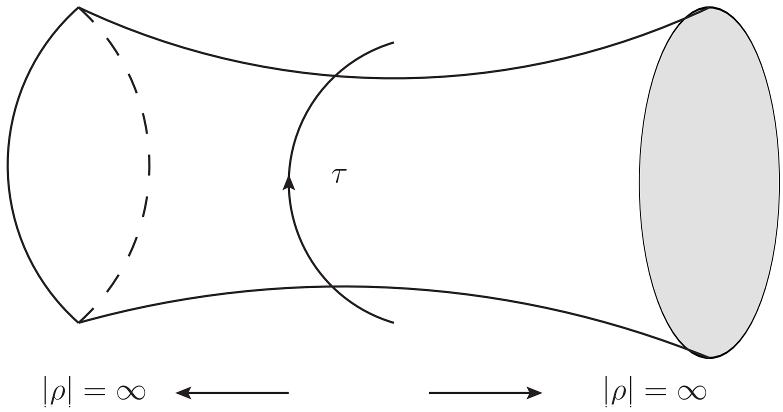

Figure 2.1: The well known picture of AdS_{p+2} embedded in $\mathbb{R}^{2,p+1}$. The time coordinate is compact, and thus the geometry exhibits closed time like curves for the coordinate τ. Usually one considers the universal covering space and decompactify the time coordinate.

which by construction is SO(2,p+1) symmetric. The constraint can be solved through,

$$x_0 = R\cosh\rho\,\cos\tau, \qquad x_{p+2} = R\cosh\rho\,\sin\tau, \tag{2.2}$$
$$x_i = R\sinh\rho\,\Omega_i, \qquad \sum_{i=1}^{p+1}\Omega_i^2 = 1,$$

where the Ω_i parameterize S^p. The choice $\rho \leq 0$ and $\tau \in [0, 2\pi]$ covers the entire hyperboloid once and thus (τ, ρ, Ω_i) are global coordinates, see figure 2.1. Substituting these coordinates in (2.1) gives

$$ds^2 = R^2\big(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega_p^2\big), \tag{2.3}$$

which, from $\rho \sim 0$, can be seen to have the topology of $S^1 \times \mathbb{R}^{p+1}$. The maximal compact subgroups are SO(2)×SO(p+1) where SO(2) generates constant shifts in τ and the SO(p+1) rotates the transverse S^p.

To obtain a casual space one can decompactify the time coordinate to take values on the real line $[-\infty, \infty]$. For a compactified time coordinate, the energy eigenvalues would come in integer values which is not the case for either string or gauge theory. Thus, what one considers is in fact the universal cover of the AdS space, so perhaps a better notation would be CAdS / CFT correspondences, with the C denoting the cover of the AdS.

To study the casual structure of AdS_{p+2}, it is convenient to change to coordinates that map the boundary to a finite value. Introducing

$$\tan\theta = \sinh\rho, \qquad \theta \in [0, 2\pi), \tag{2.4}$$

gives that

$$ds^2 = \frac{R^2}{\cos^2\theta}\big(- d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_p^2\big). \tag{2.5}$$

This has the topology of $\mathbb{R} \times B^{p+1}$ which one can visualize as a solid cylinder. At the boundary $\theta = \pi/2$, which is the spatial infinity of the CAdS, the geometry is that of $\mathbb{R} \times S^p$. Thus, the (conformal) boundary of AdS_{p+2} has the geometry of $\mathbb{R} \times S^p$. This will turn out to be important because, as we will see, one can connect this with the flat Minkowski space in $p+1$ dimensions. If we start with

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{p-1}^2,$$

and introduce

$$t \pm r = \tan[\frac{1}{2}(\tau \pm \theta)] = \tan u_\pm, \tag{2.6}$$

we find

$$ds^2 = \frac{1}{4\cos^2 u_+\, \cos^2 u_-}\big(- d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{p-1}^2\big), \tag{2.7}$$

where the range of the coordinates is $0 < \theta < \pi$ and τ extends over the entire real line. From this we see that flat $p+1$ dimensional Minkowski space is conformally equivalent to $\mathbb{R} \times S^p$, which is nothing else than the conformal boundary of the AdS_{p+2} space. Thus, the conformal boundary of AdS_{p+2} is the same as conformally compactified $p+1$-dimensional Minkowski space! This is one important observation for the existence of dualities relating AdS spaces with boundary gauge theories.

Having established some basic facts about AdS spaces in general, we now turn to review some of the AdS / CFT dualities studied in this thesis. The main focus will be on the simplest, and perhaps most interesting, AdS_5/ CFT_4 duality.

2.2 The AdS$_5$ / CFT$_4$ duality

In **?** it was proposed that a four dimensional gauge theory, namely $\mathcal{N} = 4$ super Yang-Mills (SYM) is dual to type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. In this section we will review some aspects of this duality with a focus on the existence of integrable structures. Integrability, which is basically the answer to the question when something can be solved analytically, has proven to play an extremely important role in classifying the spectrum of observables on both sides of the duality. Since the duality is a strong / weak duality, meaning that when the gauge theory is weakly coupled then the string theory is strongly coupled and vice versa, it is hard to prove the validity of the correspondence. However, integrability often allows one to solve for something perturbatively in a weakly coupled regime and then extrapolate the result, in a well defined way, to strong coupling and, therefore, provides a tool for proving the correspondence. We will not review the remarkable progress in AdS / CFT with the help of integrability, but rather present a consistent whole which will give us enough background information to understand the calculations in later parts of this thesis. For a few renowned papers see **?** **?** **?** and for nice reviews see **?** and **?**.

We start out with reviewing some basic facts about $\mathcal{N}$ =4 SYM.

2.2.1 $\mathcal{N} = 4$ Super Yang-Mills

Four dimensional $\mathcal{N} = 4$ super Yang-Mills is a maximally symmetric conformal theory. It is the boundary theory of the AdS_5 / CFT_4 correspondence and in this section we will outline some of its general features. Since we will be rather brief, we point to **?** and **?** for details and reviews.

The dynamical part of the theory is governed by the fields

$$\mathbb{X} = \{F_{\mu\nu}, \psi_{a\,\alpha}, \dot{\psi}^{a}_{\dot{\alpha}}, \phi^I\}, \tag{2.8}$$

where the field strength given by

$$F_{\mu\nu} = i\, g_{YM}^{-1}[D_\mu, D_\nu] = \partial_{[\mu}\, A_{\nu]} - i\, g_{YM}[A_\mu, A_\nu], \qquad D_\mu = \partial_\mu - i\, g_{YM}\, A_\mu,$$

where g_{YM} is the Yang-Mills coupling. The index notation is as follows; $a = 1, 2, 3, 4$ is a supersymmetry index parameter, $\alpha, \dot{\alpha} = 1, 2$ are $\mathfrak{so}(1,3) = \mathfrak{su}(2) \times \dot{\mathfrak{su}}(2)$ indices, μ is a Lorentz index and I is a SO(6) vector index. The Greek indices correspond to the Lorentz algebra while Latin indices correspond to spinor and vector representations of SO(6).

All fields are assumed to be in the adjoint of the gauge group, U(N), so under a local transformation $g(x) \in U(N)$

$$A_\mu \to g(x)\, A_\mu\, g(x)^{-1} - i\, g_{YM}^{-1}\, \partial_\mu g(x)\, g(x)^{-1}, \qquad \mathbb{X} \to g(x)\, \mathbb{X}\, g(x)^{-1}$$

The action is given by

$$\begin{aligned} S_{\mathcal{N}=4} = \int d^4x\, Tr[\frac{1}{4}F^2 + \frac{1}{2}(D_\mu\, \phi^I)^2 - \frac{g_{YM}^2}{4}([\phi^I, \phi^J])^2 \\ + \dot{\psi}\, \gamma_\mu\, D^\mu\, \psi - i\, g_{YM}(\psi\, \gamma_I\, [\phi^I, \psi] + \dot{\psi}\, \gamma_I\, [\phi^I, \dot{\psi}])], \end{aligned} \tag{2.9}$$

where γ_μ and γ_I are the four and six dimensional Gamma matrices respectively.

The Lagrangian is invariant under a large class of symmetries. The bosonic symmetries are generated by

$$Q_B = \{P_\mu, M_{\mu\nu}, D, K_\mu, T^I\} \tag{2.10}$$

Where P_μ and $M_{\mu\nu}$ are the four dimensional Poincare generators and D and K_μ generates dilatation and special conformal transformations. The dilatations are rigid shifts of the space-time coordinates

$$D: \qquad x_\mu \to \alpha\, x_\mu,$$

for some real constant α. Under these shifts, the classical fields transform as

$$D \cdot X(x^\mu) \to \alpha^{\Delta_0}\, X(\alpha\, x^\mu), \tag{2.11}$$

where Δ_0 is the classical scaling, or mass, dimension for the fields,

$$[F_{\mu\nu}] = 2, \qquad [\phi^I] = 1, \qquad [\psi^a_\alpha] = [\dot{\psi}^a_{\dot{\alpha}}] = \frac{3}{2}.$$

The scaling dimensions are in general not protected quantities and receive corrections in the quantum theory. The special conformal transformations are similar to the dilatations, in the sense of shifting the coordinates, but they act in a more complicated way through

$$K_\mu: \quad x_\mu \to \frac{x_\mu + \alpha_\mu\, x^2}{1 + 2\, x^\nu\, \alpha_\nu + \alpha^2 x^2}.$$

Combining the generators of Poincare, Dilatation and special conformal transformations one forms the conformal group SO(2,4), which as we saw also is the isometry group of AdS_5.

The final set of generators in (2.10) are the generators of the R-symmetry that rotates the six scalars and the supersymmetry index of the fermions. These generators taken together generate the lie algebra of SO(6). The complete set of all bosonic generators form the algebra of SO(2,4)×SO(6).

The bosonic symmetries are augmented by the sixteen (complex) fermionic charges

$$Q_F = \{Q^a_\alpha, \dot{Q}^a_{\dot{\alpha}}, S^a_\alpha, \dot{S}^a_{\dot{\alpha}}\},$$

combining these with the bosonic charges, Q_B, enlarge the conformal group to the projective superconformal group PSU(2,2|4). In section 4.2.1 we will outline the specific representations of PSU(2,2|4) in more detail.

Ordinary Yang-Mills theory, and QCD with zero quark mass, also exhibits classical conformal invariance. However, when going beyond the classical regime the Dilatation symmetry develops an anomaly with the consequence that the beta function, $\beta(g_{YM})$, becomes non zero and breaks the conformal symmetry. Luckily, for the more symmetric $\mathcal{N} = 4$ SYM, supersymmetry guarantees that the beta function remains zero even at the quantum level. This means that the dimensionless coupling, g_{YM}, does not run, i.e. it has no energy dependence, which signals that the conformal SO(2,4) symmetry survives the quantization process, see **?** and **?**.

In a conformal field theory, the most natural observables are correlation functions of local and gauge invariant operators built out of the fields in (2.8). For example,

$$\mathcal{O}_1 = Tr_N(\phi^I\,\phi^J\,D_\mu\psi^a_\alpha), \quad \mathcal{O}_2 = Tr_N(\psi^a_\alpha\,\dot{\psi}^b_{\dot{\alpha}})Tr(F_{\mu\nu}\,\phi^J), \tag{2.12}$$

and so forth. In general, generic correlation functions are constructed by introducing sources $J_i(x)$ for each operator in the exponent of the path integral as

$$\langle\hat{\mathcal{O}}_1(x_1)\,\hat{\mathcal{O}}_2(x_2)...\hat{\mathcal{O}}_n(x_n)\rangle = \frac{\delta^n}{\delta J_1(x_1)\,\delta J_2(x_2)\ldots\delta J_n(x_n)}Z[J], \tag{2.13}$$

with

$$Z[J] = \int d\psi\,d\bar{\psi}\,d\phi\,dA\,\exp\,[i\int d^4x(\mathscr{L} + \sum_{i=1}^{n} J_i(x)\,\hat{\mathcal{O}}_i)], \tag{2.14}$$

which for a general quantum field theory is of a rather involved structure. However, for a field theory with an unbroken conformal symmetry, the form of the correlation functions is severely constrained. Focusing on two point functions, the Poincare invariance demands that

$$\langle\hat{\mathcal{O}}_1(x_1)\,\hat{\mathcal{O}}_2(x_2)\rangle = f_{12}(x_1 - x_2),$$

for an arbitrary scalar function f_{12}. Abbreviating $g_{YM} = g$ for now, the conformal symmetry further restricts the two point function to

$$\langle\hat{\mathcal{O}}_1(x_1)\,\hat{\mathcal{O}}_2(x_2)\rangle = \frac{C_{12}(g)}{|x_1 - x_2|^{2\,\Delta(g)}},$$

where $\Delta(g)$ is the scaling dimension which has an expansion as

$$\Delta(g) = \Delta_0 + \gamma(g), \tag{2.15}$$

where Δ_0 is the classical dimension and $\gamma(g)$ contains the quantum corrections and is called the *anomalous* dimension. Note that there are certain classes of operators, denoted chiral primaries, that do not receive any corrections to their classical scaling dimensions in the quantum theory.

As a brief example, if we pick the simple case $\hat{\mathcal{O}} = Tr_N(\phi^n)$, which naturally have $\Delta_0 = n$, then to leading order

$$\langle\mathcal{O}(x)\,\mathcal{O}(y)\rangle \sim \frac{1}{|x-y|^{2n}}.$$

Also the three point functions get fully constrained by the conformal symmetry to a rather simple form, for details see **?**. However for n-point functions with, $n > 3$, the form is not fully fixed by the conformal symmetry alone and one has to use standard techniques to calculate them.

Beyond tree level, one normally encounters UV divergences in the correlation functions which, following standard field theory methods, can be regularized through

$$\hat{\mathcal{O}}_{ren} = \mathcal{Z}(\Lambda) \cdot \hat{\mathcal{O}}_{bare}, \tag{2.16}$$

where $\mathcal{Z}$ is a mixing matrix and Λ is some cut off parameter. This matrix is essentially given by the quantum part of the Dilatation operator, δD, through

$$\delta D = \mathcal{Z}^{-1} \frac{d\,\mathcal{Z}}{d\,\log\Lambda} \tag{2.17}$$

and as we will see later, this operator will play a very crucial role in finding the exact spectrum of the theory.

To fully classify the theory one need to solve the spectral problem for the operators. The operators are classified according to their Cartan labels, that is, by the conserved U(1) charges, which for $\mathcal{N} = 4$ SYM are,

$$(\Delta(g), S_1, S_2, J_1, J_2, J_3), \tag{2.18}$$

where S_i are conformal spins and J_i are three commuting R charges. Except the scaling dimension, these all correspond to compact subgroups of the bosonic SO(2,4)×SO(6) symmetry and therefore come in integer units.

2.2.2 Planar limit

For QCD, where $N = 3$ and the coupling has an energy dependence, perturbation theory only applies in the UV. On the other hand, low energy physics, which describes important physical processes such as chiral symmetry breaking and quark confinement, is non perturbative and one has to resort to Lattice simulations or other non perturbative methods to calculate physical quantities. However, there exist a special limit, proposed by 't Hooft in 1979, where one take the rank of the gauge group to infinity and let the coupling be small so that

$$N \to \infty, \qquad \lambda = g_{YM}^2\, N = \text{finite}. \tag{2.19}$$

The upshot with this limit is that if one considers a perturbation in $1/N$, one finds that only planar diagram survives, or equivalently, only the single trace correlation functions are relevant. We can understand this if we look at a schematic expansion of the Lagrangian (2.9), which together with a rescaling of all fields $X_i \to \frac{1}{g_{YM}} X_i$, look like

$$\mathscr{L} \sim \frac{1}{g_{YM}^2}\, Tr\big[dX_i\, dX_i + \alpha^{ijk} X_i\, X_j\, X_k + \beta^{ijkl} X_i\, X_j\, X_k\, X_l\big], \tag{2.20}$$

where X_i can be any of the fields in the adjoint. At first glance this seems like a rather odd limit since the coefficient of the Lagrangian diverges, but as we will see, factors of N pops up from the matrix valued fields making actual computations significantly simplified.

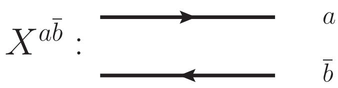

Figure 2.2: Double line notation of fundamental and anti fundamental indices.

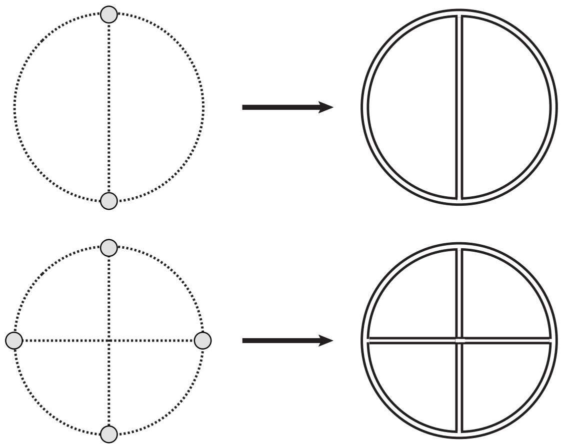

Figure 2.3: Adjoint fields with interaction points on the left and the diagram in 't Hooft double line notation to the right.

A convenient way to rewrite Feynman diagrams involving adjoint fields is to introduce a double line notation. In this notation one substitute the line denoting an adjoint field with a double line corresponding to the fundamental and anti fundamental indices hidden in the adjoint representation. For a SU(N) theory one has

$$\langle X_i^{a\bar{b}} \, X_i^{c\bar{d}} \rangle \sim \delta^{a\bar{d}} \, \delta^{b\bar{c}} - \frac{1}{N} \delta^{a\bar{b}} \, \delta^{c\bar{d}}. \tag{2.21}$$

For each fundamental index one writes an incoming arrow and for each anti fundamental index an outgoing arrow, see figure 2.2. Since we are interested in gauge invariant operators, all indices has to be contracted, and thus each diagram is a closed two dimensional surface as shown in figure 2.3.

From the symbolical expansion (2.20), one can figure out the scaling of a generic diagram. For each vertex we get a factor of N/λ, for each propagator a factor λ/N and for each closed loop a factor of N from the trace. Thus, for a vacuum diagram with E number of propagators (edges), V number of vertices and F loops (faces), we have a coefficient scaling as

$$(\frac{\lambda}{N})^E \, (\frac{N}{\lambda})^V \, N^F = N^\chi \, \lambda^{E-V}, \tag{2.22}$$

where we introduced $\chi = V - E + F$ which is the Euler number for a closed surface. If we add a point at infinity, then the surfaces corresponding to the diagrams become compact and the Euler number can be written as $\chi = 2 - 2g$, where g is the number of handles. For a correlation functions, or a generic physical quantity, it then follows that it admits a perturbative expansion as

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} f_{i,g}\, \lambda^i = Z_0 + \sum_{g=1}^{\infty} N^{2-2g}\, Z_g(\lambda), \tag{2.23}$$

where Z_0 is the purely classical contribution. In the large N limit, the dominating terms are the ones with the minimal number of handles. These diagrams are called planar diagrams, indicating that they can be drawn on a plane. Non planar diagrams are suppressed by additional factors of $1/N^2$ which simplifies the calculations since multi trace operators, as the second one in (2.12), are suppressed.

2.2.3 The duality

Having reviewed some basic facts about AdS spaces and $\mathcal{N} = 4$ SYM we are now in position to present the full duality as presented by Maldacena in **?**,

$$\mathcal{N} = 4, \mathrm{SYM} \quad \leftrightarrow \quad \text{Type IIB string theory on } \mathrm{AdS}_5{\times}\mathrm{S}^5. \tag{2.24}$$

We have already provided a few observations to why this duality could be true. As we saw earlier, the conformal boundary of the AdS space coincides with a conformal compactification of Minkowski space in one dimension lower. Thus, for the AdS_5 case we have the possibility of a flat four dimensional boundary theory. We also learned that the isometry group of the AdS-space, at least for the $d = 4$ case, coincided with the group of conformal symmetries for a four dimensional CFT. Including the compact product space, S^5, we get another SO(6) group which coincides with the R-symmetry group of the gauge theory. Thus, in principle, we can have a four dimensional gauge theory on the boundary of the AdS_5 with the same symmetry group as the full $\mathrm{AdS}_5{\times}\mathrm{S}^5$ product space.

Further hints for the duality can be found if we consider a stack of N D3-banes. D-branes are solutions of supergravity with two distinct kind of excitations. On the D-branes themselves, open strings end which endpoints carry U(N) Chan-Paton factors. In the bulk, the space-time region outside the brane, closed strings propagate. If we consider an energy scale well below $l_s^{-1} \sim \sqrt{\alpha'}^{-1}$, the dynamics are schematically described as

$$S = S_{int} + S_{bulk} + S_{brane}. \tag{2.25}$$

The brane theory is that of $\mathcal{N} = 4$ SYM in four dimension plus higher order derivative terms and the low energy bulk theory is IIB supergravity. The interaction theory, which incorporates effects such as Hawking radiation, contains general interactions between bulk and brane excitations. The leading order contribution of the interaction term can be obtained by covariantizing the world volume theory and it is proportional to $g_s\, \alpha'^2$, where g_s is the string coupling. Therefore, in the low energy limit, the interaction terms can be neglected and the physics is described by two decoupled theories, a $\mathcal{N} = 4$ SYM world volume theory and type IIB supergravity in the bulk.

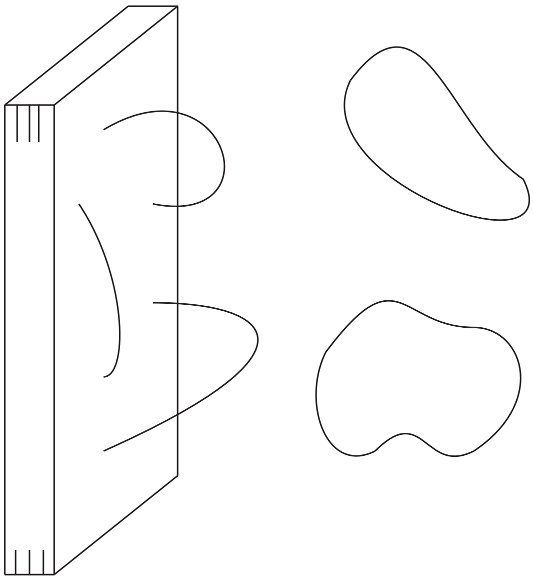

Figure 2.4: A stack of N D3-branes with open string excitations and closed strings in the bulk.

Next we view the same system from a geometrical point of view. Since D-branes carry both charge and mass, they curve the geometry. For N D3-branes, they have supergravity solutions with the line segment **?**

$$ds^2 = h(r)^{-1/2}\eta_{\mu\nu}dx^\mu\, dx^\nu + h(r)^{1/2}(dr^2 + d\Omega_5^2), \tag{2.26}$$

where

$$h(r) = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi\, g_s\, \alpha'^2\, N. \tag{2.27}$$

The string coupling is obtained through the Dilaton field[2] and is, for general Dp-branes, given by

$$e^\phi = g_s\, H(r)^{\frac{3-p}{4}}, \qquad H(r) = 1 + \frac{\alpha}{r^{7-p}}, \tag{2.28}$$

where the constant α depends on both the coupling and the string length. What is important is that for the case $p = 3$, the Dilaton field is constant and the string coupling is just given by the exponent of the Dilaton. This is the string analog of the vanishing β-function in the gauge theory.

Since we have a $h(r)$ dependence in front of dt^2 in (2.26), the energy measured by

[2]Not to be confused with the Dilatation operator in the gauge theory.

observers at separated space-time intervals will differ. Especially, if we consider an object at a fixed position r from the horizon, then the energy difference measured by an observer at infinity is related through the redshift factor as

$$E = h(r)^{-1/2} E_r, \tag{2.29}$$

where E is the energy measured by the observer at infinity. Thus, for objects close to the horizon, the energy measured at infinity will be highly redshifted.

As before we want to study low energy regime of (2.26). From the point of view of the observer at infinity, we can have two distinct type of excitations; Massless long wavelength modes in the bulk and excitations that we bring close to the horizon, $r \sim 0$. These two type of excitations decouple from each other. The wavelength of the massless bulk modes is much larger than the gravitational size of the brane and excitations close to $r = 0$ do not have enough energy to climb the gravitational potential. For the massless modes in the bulk, the dynamics are well described by type IIB supergravity. For the modes close to the branes, the metric becomes

$$ds^2 = \frac{r^2}{R^2}\,\eta_{\mu\nu} dx^\mu\, dx^\nu + \frac{R^2}{r^2}(dr^2 + r^2\, d\Omega_5^2), \tag{2.30}$$

which of course is nothing than the AdS$_5\times$S^5 metric. Thus, we relate string theory on AdS$_5\times$S^5 with type IIB supergravity. Above we related $\mathcal{N} = 4$ SYM to the supergravity, so it is therefore natural to relate the two other theories with each other, leading us the conjecture (2.24).

2.2.4 Matching of parameters

We have now provided quite a few independent arguments for why the duality (2.24) could be true. What remains to be done is to match the various parameters that occur in the different theories. First of all, the rank of the gauge group, N, is related to the flux of the Ramond-Ramond five form field through S^5

$$\int_{S^5} F_5 = N, \tag{2.31}$$

where the five form is the field strength of the brane[3], which in the case of a D3-brane is also self dual.

The coupling constants of the two theories are related as

$$g_s = \frac{g_{YM}^2}{4\pi}, \qquad \frac{R^2}{\alpha'} = \sqrt{g_{YM}^2\, N} = \sqrt{\lambda}, \tag{2.32}$$

and the matching of the couplings is rather easy to understand. The open strings generate gauge potentials on the world volume of the D3-branes where each gauge potential, A_μ, comes with a coupling g_{YM}. From string perturbation theory we know that two

[3]In general, a $p+1$ extended charged object give rise to a $p+3$ field strength.

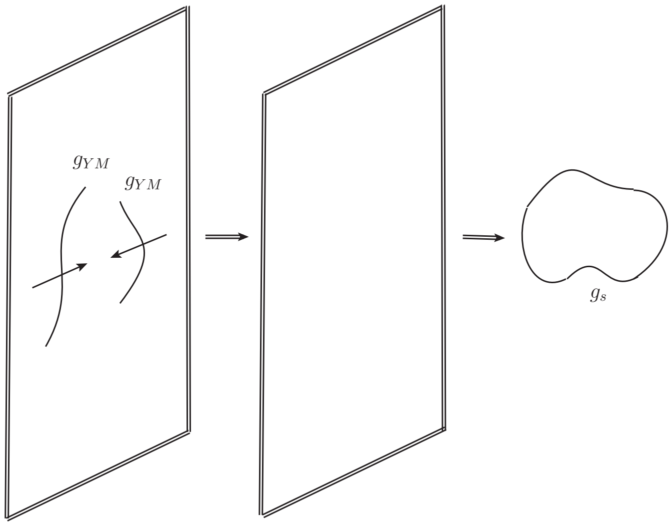

Figure 2.5: Two open strings on the brane joining to form a closed string in the bulk.

open strings can combine into a closed string. Since the coupling constant for the gravity theory in the bulk is g_s, the matching $g_s = g_{YM}^2$ comes very natural, see figure 2.5.

At first glance it might seem we have two unrelated parameters in the string theory, R and α'. However, its only their combination $\frac{R^2}{\alpha'}$ that enters in the string Lagrangian, so effectively, the string tension is given by $\sqrt{\lambda}$. The region of validity for the string theory is when the curvature is much larger than the string length, $\sqrt{\alpha'}$, which boils down to $\sqrt{\lambda} >> 1$. However, for perturbation theory to make sense in the gauge theory, we must have $\sqrt{\lambda} << 1$ and thus (2.24) is a strong / weak coupling duality.

$$\text{SYM}: \qquad \lambda << 1, \qquad \text{AdS}_5\times\text{S}^5: \qquad \lambda >> 1.$$

In the 't Hooft limit, where $N \to \infty$ with λ fixed, we see that $g_s \sim \frac{1}{N}$, so planar $\mathcal{N} = 4$ SYM correspond to free $\text{AdS}_5\times\text{S}^5$ string theory. In this thesis we will exclusively discuss free, or planar limit, strings.

2.2.5 Observables

From the arguments above it almost seem like the duality is proven. However, all arguments provided were for low energy dynamics and nothing at all in the discussions related to higher order effects such as quantum loops etc. A priori there is nothing that says that the two theories should be equal beyond the classical low energy regime. However, the conjecture relates *full* $\mathcal{N} = 4$ SYM and type IIB $\text{AdS}_5\times\text{S}^5$ super string theory for *all* values of g_{YM} and N. Naturally, since the string theory is tractable for large values of the 't Hooft coupling while the gauge theory is reliable for small values, the conjecture is very hard to prove.

As we saw in (2.18), the correlation functions of the gauge theory are classified according to their scaling dimension and Cartan labels of PSU(2,2|4). In the string theory, we have two commuting spins, S_1 and S_2 from the AdS_5 space and three commuting J_i charges of the S^5. There is also a sixth charge, E, related to the constant shift of the global AdS time coordinate. Thus, for the $AdS_5 \times S^5$ string, each string mode is labelled by

$$(E, S_1, S_2, J_1, J_2, J_3),$$

where the Cartan labels take integer values. Thus, one proof, at least in the sense of theoretical physics, would be to match the spectrum of conformal dimensions with the energies of string states as,

$$\Delta(g_{YM}, N) = E(g_s, \frac{R^2}{\alpha'}). \tag{2.33}$$

However, with the current level of computational technology, it is not very feasible that one can tackle the problem beyond the planar limit. Thus, in practise, what usually is matched on both sides of the correspondence is

$$\Delta(g_{YM}, \infty) = E(0, \sqrt{\lambda}). \tag{2.34}$$

Since the validity of each side of the correspondence is for different values of the coupling, matching the calculated observables seem very hard. However, as it turns out, there are integrable structures hiding in the planar theories which make the problem feasible.

2.3 The AdS_4 / CFT_3 duality

In this section we will very briefly outline a new tantalizing gauge / string correspondence. The original incarnation of a AdS / CFT duality, namely AdS_5 / CFT_4 was presented by Maldacena in '97 and since it relates a four dimensional gauge theory with strings, it naturally stirred a lot of research interest. Since then, a spectacular host of results have been obtained, where perhaps the most remarkable is, albeit somewhat conjectural, a complete solution for the large N asymptotic spectrum **?**.

However, the story does not end with AdS_5 / CFT_4. In **?** Aharony, Bergman, Jafferis and, once again, Maldacena (ABJM) proposed that, following **?????????**, the effective world volume theory of a stack of N M2 branes, with N large, could be identified with a three dimensional superconformal SU(N) $\times$ SU(N) Chern-Simons (CS) theory with M theory on $AdS_4 \times S^7/\mathbb{Z}_k$ as a gravitational dual. On the gauge theory side, the parameter k enters as the level[4] and it take opposite values for the two SU(N) groups.

In contrast to $\mathcal{N} = 4$ SYM with manifest SO(6) R-symmetry, the CS theory only has a manifest SU(2)$\times$SU(2)$\times$U(1) R-symmetry which, however, is enhanced to SO(6) for $k > 2$. The theory also has two sets of scalar fields, each transforming under one of the

[4]The level of a CS theory can be thought of as the inverse of the gauge coupling constant, see equation (2.38).

SU(2)'s where both are in the bi-fundamental representation of SU(N). Denoting the two sets of scalars as A_a and $B_{\dot{a}}$, then A_a transform in $(N, \bar{N})$ while $B_{\dot{a}}$ transform in $(\bar{N}, N)$.

It is convenient to group the scalars into SU(4) $\simeq$ SO(6) multiplets, Y^A, as

$$Y^A = (A_1, A_2, B_{\dot{1}}^{\dagger}, B_{\dot{2}}^{\dagger}), \tag{2.35}$$

which allows one to easily construct single trace gauge invariants as

$$\mathcal{O} = Tr\,(Y^{A_1}\,Y^{\dagger}_{B_1}\,Y^{A_2}\,Y^{\dagger}_{B_2}\ldots Y^{A_n}\,Y^{\dagger}_{B_n})\chi^{B_1\,\ldots\,B_n}_{A_1\,\ldots\,A_n}. \tag{2.36}$$

If the matrix $\chi^{B_1\,\ldots\,B_n}_{A_1\,\ldots\,A_n}$ is traceless and symmetric in all indices, then $\mathcal{O}$ is a chiral primary operator and do not receive corrections to the anomalous dimension **?**. If the operator is not a chiral primary, the anomalous dimension receives quantum corrections and, in contrast to $\mathcal{N}=4$ SYM, these additional contributions starts at second loop order in perturbation theory. Of course, these operators will in general suffer from UV divergences which can be renormalized as in (2.16) through

$$\mathcal{O}_{ren} = \mathcal{Z}\cdot\mathcal{O}_{bare}, \tag{2.37}$$

where Λ is some cut of parameter and the mixing matrix $\mathcal{Z}$ can be determined through the quantum part of the Dilatation operator.

The action of $\mathcal{N}=6$ superconformal CS is given by

$$\begin{aligned} S &= \\ &\frac{k}{4\pi}\int d^3x\ \mathrm{Tr}\Big[\varepsilon^{\mu\nu\lambda}\Big(A_\mu\partial_\nu A_\lambda + \frac{2}{3}\,A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2}{3}\,\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda \\ &+ D_\mu Y^{\dagger}_A D^\mu Y^A + \frac{1}{12}\,Y^A Y^{\dagger}_A Y^B Y^{\dagger}_B Y^C Y^{\dagger}_C + \frac{1}{12}\,Y^A Y^{\dagger} Y_B Y^B Y^{\dagger}_C Y^C Y^{\dagger}_A \\ &- \frac{1}{2}\,Y^A Y^{\dagger}_A Y^B Y^{\dagger}_C Y^C Y^{\dagger}_B + \frac{1}{3}\,Y^A Y^{\dagger}_B Y^C Y^{\dagger}_A Y^B Y^{\dagger}_C + \text{fermions}\Big], \end{aligned} \tag{2.38}$$

where $D_\mu Y^A = \partial_\mu Y^A - A_\mu\,Y - Y\,A_\mu$. A characteristic of a CS theory is that there are no kinetic terms for the gauge fields as $\partial A\cdot\partial A$. We will not present the fermionic part of the action, but note that it has fermionic interaction terms of the type $Y^2\,\Psi^2$. The action, including the fermions, is invariant under OSP$(2,2|6)$ which even part coincides with SO(2,3)×SO(6).

As was the case for $\mathcal{N}=4$ SYM, one can introduce a 't Hooft coupling,

$$\lambda = \frac{N}{k}, \tag{2.39}$$

kept fixed in the large N and k limit.

As can be seen from the above, the action (2.38) is rather more involved than its four dimensional SYM counterpart in (2.9). Nevertheless, as we will show in the upcoming, integrability seem to survive to the quantum level, and the problem of obtaining the

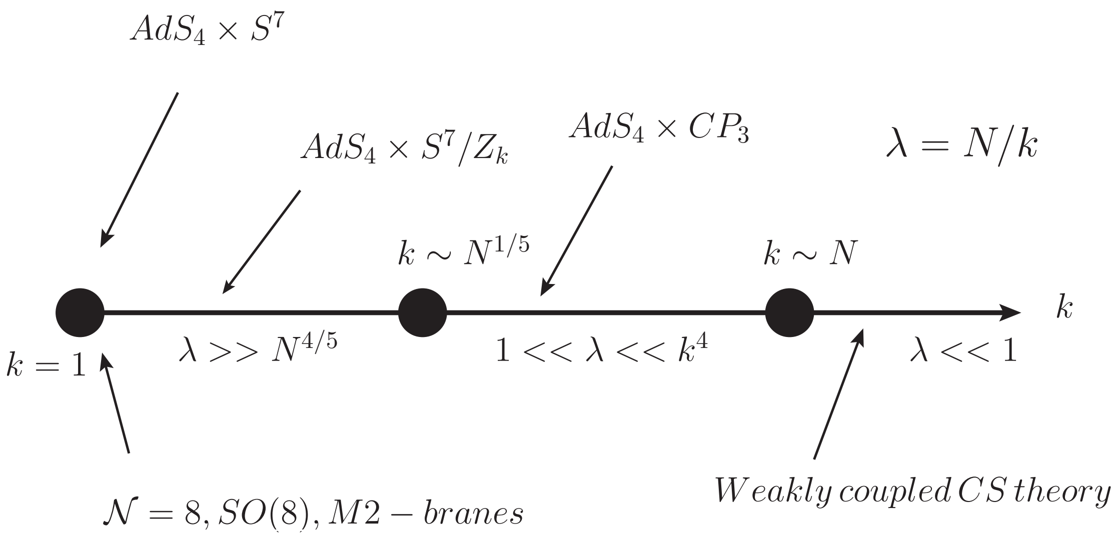

Figure 2.6: The parameter space of the level k in relation to the rank N of the gauge group.

spectrum of conformal dimensions can be mapped to that of diagonalizing a spin-chain Hamiltonian.

We should mention that since the 't Hooft coupling is a ratio between two parameters the physics are best described with different theories depending on the values of N and k. What we will mostly be concerned with is a large λ in the interval $1 << \lambda << k^4$. For these values of λ the M-theory can be described by type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\mathbb{F}}3}$, thus

$$\mathcal{N} = 6, \mathrm{D}{=}3, \text{ CS theory} \quad \leftrightarrow \quad \text{Type IIA super string theory on } \mathrm{AdS}_4 \times \mathbb{CP}_{\not{\mathbb{F}}3},$$

with the identification

$$\sqrt{\lambda} = \frac{R^2}{\alpha'}, \tag{2.40}$$

where R is the radius of the AdS space. We will not review the reduction here, but see **?** for details. For some other ranges of N and k, see figure 2.6.

One way to prove the correspondence is to match the observables of each respective theory. As before, the set of observables are as those in (2.34) and in later chapters of this thesis we will spend some considerable effort on matching a large set of these observables.

3 Integrability in AdS / CFT dualities

One of the remarkable facts with the AdS / CFT correspondence is that both the string and gauge theory seem to contain hidden integrable structures. What is integrability? Integrability is the existence of constants of motion along a particle, or system of particles, trajectories that allows the dynamics to be solved analytically.

If we consider a classical system, then we can be a little more precise when defining integrability. If we have a $2N$ dimensional phase-space, then the dynamics can be solved for through

$$\dot{q}_i = \{H(q,p), q_i\}, \qquad \dot{p}_i = \{H(q,p), p_i\}, \tag{3.1}$$

with $i = 1, 2..., N$ and the Poisson bracket is given by

$$\{A, B\} = \sum_{i=1}^{N} \Big(\frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \Big). \tag{3.2}$$

To have the complete dynamical solutions of the system, one need to solve for all q_i and p_i with initial data $q_i(0)$ and $p_i(0)$. Since these are coupled nonlinear differential equations it is in general hard to obtain the analytical solutions. Generally, in situations where one can obtain analytical solutions, it is usually closely connected to conserved constants of motions. Or equivalently, when the problem contains enough symmetry. Through the existence of conserved charges, we define integrability for a classical system as

Definition. A (classical) Hamiltonian system of 2N degrees of freedom is called integrable iff there exist N independent constants of motion, Q_i, such that

$$\{Q_i, Q_j\} = 0, \qquad \{H, Q_i\} = 0, \quad \forall\ i, j. \tag{3.3}$$

To obtain the spectrum for a dynamical system with enough isometries, one can fall back on a famous theorem by Liouville which states that given (3.3), then the system can be solved analytically through the methods of quadratures.

When moving from a classical to a quantum system, the standard procedure is to promote the fields to operators and exchange the phase-space by a Hilbert space as

$$q_i \to \hat{q}_i, \quad p_i \to \hat{p}_i, \quad \{,\} \to i\,[,].$$

A natural extension of the definition for quantum integrability would be to take the definition above and promote the charges to N commuting operators. However, for a

	J	J_1	J_2
X	0	1	0
Y	0	0	1
Z	1	0	0

Table 3.1: The U(1) charge of the complex combinations of the scalars ϕ^I. Note that we relabelled the charges to J, J_1 and J_2 compared J_i as in (2.18).

quantum system, this is sadly not enough because as one can show, see for example **?**, the commuting operators can be algebraically dependent. However, having N commuting charges is nevertheless a necessary, but not sufficient, condition for integrability.

For the quantum theory, where one can not resort to the theorem of Liouville, one has to use other methods. One such method, and one we will discuss extensively in this thesis, is the so called Bethe ansatz **?**. The Bethe ansatz, named after Hans Bethe who used it to solve the Ferro magnet problem in 1931, maps the spectral problem to that of an integrable spin chain, or equivalently a one dimensional lattice model.

3.1 Integrability in $\mathcal{N} = 4$ SYM

We will review some basic facts about integrability in field theory by focusing mainly on $\mathcal{N} = 4$ SYM and for simplicity, mostly focus on certain subsectors of the theory.

First we introduce the following complex combinations of the scalars, ϕ^I, to make the action of the three U(1) subgroups of SO(6) manifest, see tabular 3.1,

$$X = \phi_1 + i\,\phi_2, \quad Y = \phi_3 + i\,\phi_4, \quad Z = \phi_5 + i\,\phi_6, \tag{3.4}$$

We will pick the Z fields as the building blocks of a reference operator[1]

$$\mathcal{O}_J \sim Tr_N\, Z^J, \tag{3.5}$$

which is a chiral primary operator so its scalar dimension does not receive any quantum corrections. Thus, to all orders in g_{YM}, the scalar dimension for $\mathcal{O}_J$ is merely $\Delta = J$. The idea now is to consider other fields as excitations on this reference state. If we for simplicity only focus on a SU(2) subsector constituted of Y and Z fields, then

$$\mathcal{O}_{ZY} \sim Tr_N\, Z\, Z\, ...\, Y\, ...\, Z\, ...Y\, ...\, Z\, Y\, ...Z, \tag{3.6}$$

and so on. If we take J number of Z fields and M number of Y fields, then the classical dimension is simply $\Delta_0^{ZJ} = J + M$.

We are interested in the spectrum of anomalous dimensions which are the eigenvalues of the Dilatation operator, $D = i\,\Delta_0 + \delta D$. The eigenvalues enters through the correlation

[1]We can of course pick any of the three scalars X, Y or Z as the reference state.

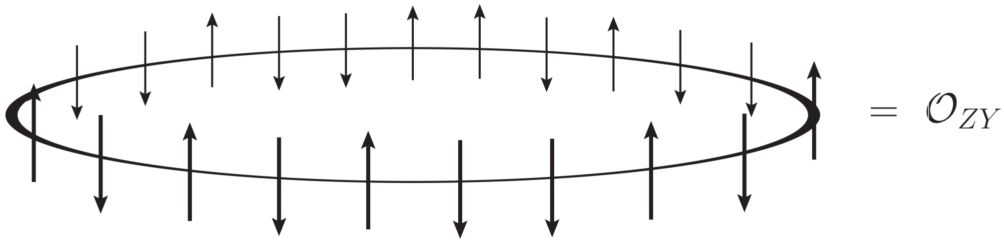

Figure 3.1: Spin chain picture of the SU(2) operator $\mathcal{O}_{ZY}$.

functions as

$$\langle \mathcal{O}^I_{ren}(x), \mathcal{O}^J_{ren}(y) \rangle = \frac{C_{ZY}(\lambda)}{|x-y|^{D_{IJ}}}, \tag{3.7}$$

which in general give quite complicated expressions since the renormalization procedure (2.16) will induce mixing with other operators from the same subsector,

$$\mathcal{O}^I_{ZY,ren} = \mathcal{Z}_{IJ} \cdot \mathcal{O}^J_{ZY,bare}$$

where the indices I, J runs over all possible SU(2) states. To find the complete spectrum is in general very hard, but however, one can understand this problem in an alternate and much simpler way. In the seminal paper **?** from 2002, Minahan and Zarembo showed that the one-loop piece of the Dilatation operator could be understood as the Hamiltonian of a one dimensional spin chain[2]. The spin chain picture emerges rather naturally if we associate each of the Z and Y with down and up spins

$$Z = \downarrow, \qquad Y = \uparrow, \tag{3.8}$$

so the state (3.6) is written as

$$\mathcal{O}_{ZY} = Tr_N \downarrow\downarrow \dots \uparrow \dots \downarrow \dots \uparrow \dots \downarrow\uparrow \dots \downarrow,$$

since the trace is cyclic, we can associate the above with a closed spin chain of length $L = J + M$ as in figure 3.1. As it turns out, the one-loop piece of the Dilatation operator then act as **?**

$$\delta D = \frac{\lambda}{8\pi^2} \hat{H} + \mathcal{O}(\lambda^2) \tag{3.9}$$

where $\hat{H}$ is a spin chain Hamiltonian given by

$$\hat{H} = \sum_{l=1}^{L} (\mathbb{1}_{l,l+1} - \mathbb{P}_{l,l+1}), \tag{3.10}$$

[2]For a nice review see **?**.

with $\mathbb{P}$ a permutation operator

$$\mathbb{P}_{l,l+1} \cdot |\ldots \uparrow_l \downarrow_{l+1} \ldots\rangle = |\ldots \downarrow_l \uparrow_{l+1} \ldots\rangle. \tag{3.11}$$

Thus, in this language, the problem is reduced to diagonalizing a spin chain Hamiltonian. Somewhat surprising, the chain that we consider with a Z vacuum and Y impurity spin flips is in fact nothing else than the $\text{XXX}_{\frac{1}{2}}$ spin chain, or the Ferro magnet, diagonalized by Bethe in 1931 **?**. Below we will briefly outline the method.

3.1.1 The Bethe equations

The state consisting of only downspins we will denote the vacuum and since it is protected, it has trivially $\hat{H} \cdot |\downarrow \ldots \downarrow\rangle = 0$. For a state with M spin flips at positions y_i we write $|y_1\, y_2 \ldots y_M\rangle$ where $y_1 < y_2 < \ldots < y_M$. So for example, $|\uparrow\downarrow\uparrow\rangle = |1\,3\rangle_{L=3}$ and so on. First we consider a state with just one spin flip which is almost trivially diagonalized by a plane wave ansatz

$$|\psi(p_1)\rangle = \sum_{y=1}^{L} e^{i\,p_1\,y}|y\rangle, \tag{3.12}$$

which using (3.9) and (3.10) gives

$$\begin{aligned} &D \cdot |\psi(p_1)\rangle = \\ &(L + \frac{\lambda}{8\pi^2}(2 - e^{i\,p_1} - e^{-i\,p_1}))|\psi(p_1)\rangle = (L + 4\,\frac{\lambda}{8\pi^2}\sin^2\frac{p_1}{2})|\psi(p_1)\rangle, \end{aligned} \tag{3.13}$$

so $|\psi(p_1)\rangle$ is an eigenstate of the Dilatation operator. The periodicity condition $|y + L\rangle = |y\rangle$ implies that the momentum of the magnons, which is just another fancy word for the spin flips, is quantized, $p_1 = \frac{2\pi n}{L}$ for $n \in \mathbb{Z}$.

Next we make an ansatz for the two magnon state as

$$|\psi(p_1, p_2)\rangle = \sum_{1 \le y_1 < y_2 \le L} \psi(y_1, y_2)|y_1, y_2\rangle, \tag{3.14}$$

since we want it to be an eigenstate of the one loop Dilatation Hamiltonian $\hat{H}$ in δD, it has to satisfy

$$\hat{H} \cdot |\psi(p_1, p_2)\rangle = E_1\,|\psi(p_1, p_2)\rangle, \tag{3.15}$$

which, depending on if the excitations lie next to each other or not, leads to two sets of equations

$$\begin{aligned} y_2 > y_1 + 1, \qquad & E_1\,\psi(y_1, y_2) = 2\psi(y_1, y_2) - \psi(y_1 + 1, y_2) \\ & - \psi(y_1 - 1, y_2) + 2\psi(y_1, y_2) - \psi(y_1, y_2 + 1) - \psi(y_1, y_2 - 1), \end{aligned} \tag{3.16}$$

together with

$$y_2 = y_1 + 1, \qquad E_1\,\psi(y_1, y_2) = 2\psi(y_1, y_2) - \psi(y_1, y_2 - 1) \\ - \psi(y_1 - 1, y_2). \tag{3.17}$$

This is the equation that was solved by Bethe in **?** using a superposition of an incoming and outgoing plane wave,

$$\psi(y_1, y_2) = e^{i(p_1\,y_1 + p_2\,y_2)} + S(p_2, p_1)\,e^{i(p_2\,y_1 + p_1\,y_2)}, \tag{3.18}$$

where $S(p_2, p_1)$ is a two particle S-matrix. The first equation, (3.16), leaves $S(p_2, p_1)$ arbitrary **?** and give that the energy is just the sum of two one-magnon excitations

$$E_1 = 4(\,\sin^2\frac{p_1}{2} + \sin^2\frac{p_2}{2}). \tag{3.19}$$

The second equation, (3.17), determines the S-matrix to be **?**

$$S(p_1, p_2) = \frac{\phi(p_1) - \phi(p_2) + i}{\phi(p_1) - \phi(p_2) - i}, \qquad \phi(p_k) = \frac{1}{2}\cot\frac{p_k}{2}, \tag{3.20}$$

which satisfy

$$S(p_1, p_2)^{-1} = S(p_2, p_1). \tag{3.21}$$

For a finite length spin chain, the imposing of periodic boundary conditions

$$\psi(y_1, y_2) = \psi(y_2, y_1 + L), \tag{3.22}$$

leads to a set of Bethe equations

$$e^{i\,p_1\,L} = S(p_1, p_2), \qquad e^{i\,p_2\,L} = S(p_2, p_1), \tag{3.23}$$

which from (3.21) are augmented with

$$p_1 + p_2 = 2\pi\,m \tag{3.24}$$

for an arbitrary integer m.

Thus, for the two excitation spin chain of length L, one obtains the one loop anomalous dimension by solving the algebraic equation (3.23) and using the solution for the quasi momenta p_k in (3.19). This gives the spectrum for an operator with two Y fields and an arbitrary number of Z fields **? ?**.

Now the full machinery of integrability kicks in; The information above is all that is needed to solve the full N-body problem! This phenomena, denoted factorized scattering, implies that the multi-particle scattering factorizes into a sequence of two-particle interactions. Thus, for an arbitrary SU(2) spin chain, with M number of spin flips, the

Bethe equations are given by[3] ?

$$e^{ip_k L} = \prod_{j\neq k}^{M} S(p_k, p_j) = \prod_{j\neq k}^{M} \frac{\phi(p_k) - \phi(p_j) + i}{\phi(p_k) - \phi(p_j) - i}, \qquad \prod_{j=1}^{M} e^{ip_j L} = 1, \tag{3.25}$$

where $\phi(p_k) = \frac{1}{2} \cot \frac{p_k}{2}$ and the total one-loop energy is

$$E_1 = 4 \sum_{j=1}^{M} \sin^2 \frac{p_j}{2}, \tag{3.26}$$

where, for physical operators, the momentas need to satisfy

$$\sum_{j=1}^{M} p_j = 0. \tag{3.27}$$

Let us pause and ponder what we have established so far. Through the power of integrability, we have been able to map the full spectral problem of an arbitrary operator consisting of Z and Y fields into a compact set of Bethe equations (3.25) which solutions in (3.26), augmented with (3.27), give us the anomalous dimension for any given SU(2) operator. For readers acquainted with the corresponding field theoretic calculation using ordinary methods, the above set of compact equations is truly remarkable.

So far we have only considered states in a closed SU(2) sector. However, by using the method of nested Bethe ansatz, the SU(2) equations can be extended to the full PSU(2, 2|4) supergroup. The nested Bethe ansatz works through the introduction of extra, auxiliary, spin chains that enlarge the original set of equations. Then by solving each equation in turn, one can determine the spectrum for any given operator constructed from the fields in (2.8). For a nice review, see ?.

What is more, by now there is a compelling amount of evidence that integrability extends to the full quantum theory, or at least in the limit of a very long spin chain which allows one to define asymptotic states in the scattering theory. Over the last years an outstanding research effort has been directed toward finding all loop Bethe equations and, remarkably, the problem now seem to be fully solved, see ? ? ? and ?. We will not provide a review nor present the full set of equations here[4], but only comment on the all loop generalization of the already described SU(2) sector. As it turns out, the supersymmetry puts severe constraints on the form of the Bethe equations and generalizing the results of ?, it was shown in ? that the Bethe equations and the dispersion relation could be fully determined, up to a scalar phase factor, by symmetry arguments alone. As for the one loop case, the energy is just a sum over the individual

[3] Since we are dealing with an infinite dimensional system, one might wonder where the corresponding tower of commuting charges, related to the integrability of the model, are hiding. As can be shown, see for example ?, these can be expressed through various combinations of the rapidity functions $\phi(p_k)$.

[4] They will, however, be presented in later chapters of this thesis when we match the equations against explicit string theory calculations.

magnon energies, where each contribute as

$$E(p_k) = \frac{\lambda}{8\pi^2}\left(\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_k}{2}}-1\right), \tag{3.28}$$

which leading order piece precisely coincides with the individual parts of (3.26). The two body S-matrix entering in the Bethe equations now take the form

$$S(p_k,p_j) = \frac{u(p_k)-u(p_j)+i}{u(p_k)-u(p_j)-i} \times S_0^2(p_k,p_j), \tag{3.29}$$

where the rapidity functions, $u(p_k)$, now includes higher loop corrections

$$u(p_k) = \frac{1}{2}\cot\frac{p_k}{2}\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_k}{2}} = \phi(p_k)+\mathcal{O}(\lambda). \tag{3.30}$$

The function $S_0^2(p_k,p_j)$ is the scalar phase, or dressing phase, not determined by the symmetry algebra alone. It starts contributing at four loop order in perturbation theory and it will be presented later[5] in section 5.3. It should be mentioned that the dressing factor, or phase, is rather involved and at the moment its full form is only conjectural, see **?**. However, by now it is probably safe to say that the conjecture is correct since it has passed a large number of independent tests, see **?** and references therein for details. Putting it all together, the all loop Bethe equations for the SU(2) sector reads

$$e^{ip_kL} = \prod_{i\neq k}^{M}\frac{u(p_k)-u(p_j)+i}{u(p_k)-u(p_j)-i}\times S_0^2(p_k,p_j), \qquad \prod_{j=1}^{M} e^{ip_jL} = 1. \tag{3.31}$$

As a concluding remark we would like to comment on the range of validity for the above SU(2) equations. As we remember, the Dilatation operator could be identified with a nearest neighbor spin-chain at one loop. At two loop the interactions reach the next to nearest neighbor and at three loops the third and so forth. Therefore, if one considers a finite length chain, then at some point the range of interaction will extend beyond the length of the spin-chain, i.e., one need to consider some sort of self interaction. These self interactions are called wrapping effects, starting at order $\mathcal{O}(\lambda^L)$, and it has been shown **?** that the effects from these are not incorporated in the asymptotic equations (3.25).

An especially suitable sector of the theory, which is manifestly free from the wrapping effects, is a so called BMN sector **?**. The BMN sector is constituted of states with $J >> M$, so the length of the spin-chain is much greater than the number of impurities. This sector, corresponding to a plane wave string configuration, is what we solely will focus on in the upcoming analysis in later chapters of this thesis.

[5] For the impatient reader, jump to equation (5.49).

3.2 Integrability in ABJM theory

Remarkably it seems that quantum integrability is a quantum property also in the AdS_4 / CFT_3 correspondence **?**. What is more, it seems to manifest itself in ways that are surprisingly similar to the well studied AdS / CFT case. Not only can one map the Dilatation operator to an integrable spin-chain Hamiltonian, but one can also diagonalize it in terms of Bethe equations that look almost identical to the $PSU(2,2|4)$ case.

The mixing operator in (2.37), or equivalently the quantum part of the Dilatation operator, acts on a Hilbert space of the form $(V \otimes \bar{V})^{\otimes L}$, where $V, \bar{V}$ is the $\mathbf{4}$ or $\bar{\mathbf{4}}$ of SU(4) and L is the length of the operator. The Dilatation operator can be identified with the Hamiltonian for a length $2L$ spin-chain through **?**

$$\delta D = \frac{\lambda^2}{4} \sum_{l=1}^{2L} \hat{H}_{l,l+1,l+2}, \tag{3.32}$$

where the spin-chain states, or spin flips, are the physical fields of the theory, see (2.38). Using (2.38) one can deduce that the spin-chain Hamiltonian equals **?**

$$\delta D = \frac{\lambda^2}{2} \sum_{l=1}^{2L} \Big(2 - 2\, P_{l,l+2} + P_{l,l+2}\, K_{l,l+1} + K_{l,l+1}\, P_{l,l+2}\Big), \tag{3.33}$$

where P stand for permutation and K for trace. An odd feature with this Hamiltonian is that it exhibits no sole nearest neighbor interactions and thus have an interacting theory starting at two loops in perturbation theory.

Through a Bethe ansatz one can construct a set of Bethe equations that diagonalize the Hamiltonian and the derivation is very similar to $\mathcal{N} = 4$ SYM **?** so we will not present the derivation here but just mention a few key facts. First of all, one identify a chiral primary operator using (2.35) for the ground state, which we take to be

$$|0\rangle = Tr\,(Y^1\, Y_4^\dagger)^L. \tag{3.34}$$

For impurities Y^2 and $Y_3^\dagger$ one has a closed SU(2)×SU(2) subsector[6] consisting of two decoupled Heinsenberg SU(2) spin chains, which to leading order in perturbation theory, is only related through the momentum constraint.

Rather remarkably, as was the case for the $PSU(2,2|4)$ equations, one can guess the (asymptotic) all loop Bethe equations from the leading order ones. In **?**, closely following

[6] Later we will construct a similar closed subsector out of fermionic excitations.

?, a proposal was made which for the two scalar excitations above reads

$$e^{ip_k L} = \prod_{k\neq j}^{M} S(p_k, p_j) \prod_{j=1}^{M} S_0(p_k, p_j) \prod_{j=1}^{N} S_0(p_k, q_j) \tag{3.35}$$
$$e^{iq_k L} = \prod_{k\neq j}^{N} S(q_k, q_j) \prod_{j=1}^{N} S_0(q_k, p_j) \prod_{j=1}^{M} S_0(q_k, p_j), \qquad \prod_{j,l=1}^{M,N} e^{i(p_j+q_l)L} = 1,$$

where M and N counts the number of Y^2 and $Y_3^\dagger$ excitations and the S-matrix and $S_0(p_k, p_j)$ is of the same structure as in (3.29)

$$S(p_k, p_j) = \frac{u(p_k) - u(p_j) + i}{u(p_k) - u(p_j) - i}. \tag{3.36}$$

The rapidity functions in (3.35) differ compared to (3.30)

$$u(p_k) = \cot\frac{p_k}{2}\sqrt{\frac{1}{4} + 4\,h(\lambda)^2 \sin^2\frac{p_k}{2}}, \tag{3.37}$$

in the sense that there is an undetermined scaling function $h(\lambda)$ which interpolates between λ for small values of the 't Hooft coupling and $\sqrt{\lambda/2}$ for large values ??. Its full form is currently unknown, but for some perturbative results see ?, ? and ?.

If one put either M or N to zero, then somewhat surprisingly, (3.35) is very similar to the SU(2) equation in (3.31). The only difference lies in the form of the interpolating function $h(\lambda)$ (which is constant in the former case) and the phase factor. The structural form of the phase factor is the same in both cases, but as can be seen from (3.29) they enter with different powers ?, linear for the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ CS and squared for the SYM SU(2). This also generalizes to the complete all loop asymptotic $\mathrm{OSP}(2,2|6)$ equations.

As for SYM, the momentum variables of the rapidities, p_k and q_k, has to satisfy a momentum constraint

$$\sum_{i=1}^{M} p_k + \sum_{j=1}^{N} q_j = 0, \tag{3.38}$$

which couples the two SU(2)'s. Also, and as before, the total energy is just the sum of each separate magnon energy as

$$E = \sum_{j=1}^{M}\sqrt{\frac{1}{4} + 4\,h(\lambda)^2\sin^2\frac{p_j}{2}} + \sum_{j=1}^{N}\sqrt{\frac{1}{4} + 4\,h(\lambda)^2\sin^2\frac{q_j}{2}}. \tag{3.39}$$

As a summary, we presented a set of conjectured all loop Bethe equations which describes the spectrum of conformal dimensions for a closed subsector of $\mathrm{OSP}(2,2|6)$. The equations are very similar to that of $\mathcal{N}=4$ SYM, which is rather remarkable since the action (2.38) is significantly more complicated than the four dimensional action in (2.9). Later in this thesis, we will explicitly match the energies from the conjectured all loop equations, or a generalization of them, against string theory calculations.

4 Light-cone String theory

As was outlined in the introductional chapter, the research presented in this thesis mainly concern different aspects of the string theories appearing in various gauge / string dualities. The theories studied are of a rather different nature, but nevertheless the construction exhibits many shared features. For this reason we will outline in some detail how to do this construction generally for each of the three cases. As far as possible we will try to present the discussion in general and only when absolutely necessary discuss each theory separately. Before tackling the full supersymmetric strings, lets start with something simpler.

4.1 Warm up and introduction - Bosonic string theory

String theory is hard. Very hard. It is a theory which mixes a wide array of different disciplines both from physics and advanced topics in modern mathematics; ranging from Einstein's general relativity and theoretical particle physics to representation theory and differential geometry. Thus, as a warm up, it might be wise to start with a simpler model than the full supersymmetric theories appearing in the various gauge / string dualities. For this reason we choose to embark on our journey in the world of strings with a thorough description of the bosonic string propagating on a smoothly curved background. Even though a much simpler model, it nevertheless shares many features with the full supersymmetric theory.

The starting point of our analysis will be the string action which is essentially just an integral over the area, denoted the worldsheet, swept out by an open or closed string when propagating in space-time. To parameterize the worldsheet we introduce two coordinates τ, σ where the first is a time coordinate and the second a length parameter of the string. The embedding of the worldsheet into space-time is done by the embedding functions, or string coordinates, $x^M(\tau, \sigma)$, where $M = 0, 1, .., D$, see figure 4.1.

By pulling back the space-time metric to the worldsheet and taking the square root, one gets the Nambu-Goto action,

$$S = \frac{1}{\alpha'} \int d\tau\, d\sigma \sqrt{-det(\partial_\alpha x \cdot \partial_\beta x)}, \tag{4.1}$$

where σ takes values in some finite interval, $\sigma \in [-r, r]$ and the scalar product is with the background metric G_{MN}. Throughout the thesis we will use Greek letters for worldsheet indices. The parameter $\alpha' \sim l_s^2$ defines the energy, or equivalently, the length scale of the theory.

The action above is rather cumbersome due to its square root structure, and a nice

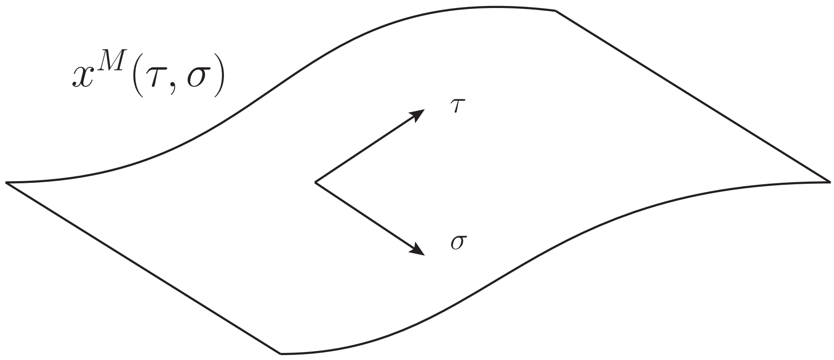

Figure 4.1: The string worldsheet of an open string

way to avoid this difficulty is to introduce an auxiliary worldsheet metric, $h^{\alpha\beta}$, which allows us to rewrite the action in a simpler form

$$S = \int d\tau\, \mathscr{L} = -\frac{g}{2} \int d\tau\, d\sigma \sqrt{-h}\, h^{\alpha\beta}\, G_{MN}\, \partial_\alpha\, x^M\, \partial_\beta\, x^N, \tag{4.2}$$

where we also introduced a string coupling constant $g \sim \alpha'^{-1}$. This action will be the starting point for the bosonic analysis of this chapter. First of all, and perhaps most important, what symmetries does it possess?

Space-time diffeomorphisms: The action is invariant under the full symmetry of the background metric G_{MN}. For example, for the case of flat space, the string is invariant under global Poincare transformations as

$$\delta x^M = a^M_N\, x^N + b^M, \qquad \delta h^{\alpha\beta} = 0.$$

Worldsheet diffeomorphisms: Since the worldsheet coordinates are arbitrary, the string is invariant under general two dimensional coordinate transformations,

$$\sigma^\alpha \to f^\alpha(\sigma), \qquad h_{\alpha\beta} \to \frac{\partial f^\rho}{\partial \sigma^\alpha} \frac{\partial f^\gamma}{\partial \sigma^\beta}\, h_{\rho\gamma}.$$

Weyl transformations: In two dimensions, scale transformations of the worldsheet metric as,

$$h^{\alpha\beta} \to e^{\Lambda(\tau,\sigma)}\, h^{\alpha\beta}, \qquad \delta x^M = 0,$$

leaves the combination $\sqrt{-h}\, h^{\alpha\beta}$ invariant. For this reason it is convenient to introduce the notation $\gamma^{\alpha\beta} = \sqrt{-h}\, h^{\alpha\beta}$, which obeys $\det \gamma = -1$ due to the Weyl symmetry. A consequence of the Weyl symmetry is that the stress energy tensor is traceless.

4.1.1 Gauge fixing

Since $h^{\alpha\beta}$, or equivalently $\gamma^{\alpha\beta}$, has no physical origin, the action has to be supplemented with the Virasoro constraints,

$$\frac{\delta S}{\delta\gamma^{\alpha\beta}} = 0 \quad \rightarrow \quad T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\gamma_{\alpha\beta}\,\partial_\gamma X \cdot \partial^\gamma X = 0. \tag{4.3}$$

Since the action is rather involved, it is convenient to gauge fix some of the worldsheet symmetries. Perhaps the most intuitive gauge is to use the reparameterization invariance to fix

$$h_{\alpha\beta} \rightarrow e^{\Lambda(\tau,\sigma)}\,\eta_{\alpha\beta},$$

and then remove the scale factor through a Weyl transformation, which gives

$$S = -\frac{g}{2}\int d\sigma^+\,d\sigma^-\,\partial_+ x \cdot \partial_- x, \qquad T_{\pm\pm} = 0 \tag{4.4}$$

where we introduced conformal worldsheet coordinates $\sigma^\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma)$.

This gauge, which hold for any sensible background metric G_{MN}, is called the conformal gauge. However, even after the conformal gauge, the theory is still invariant under right / left moving conformal transformations

$$\sigma^\pm \rightarrow f^\pm(\sigma^\pm).$$

This remaining symmetry can be fixed in various ways and the approach employed in this thesis is a gauge where one combines two of the space-time coordinates x^0 and x^{D-1} into a light-cone pair as

$$x^\pm = x^0 \pm x^{D-1}$$

and then use the residual symmetry to fix

$$x^+ \sim \tau. \tag{4.5}$$

However, its only for very specific sets of backgrounds that this gauge is consistent with the conformal gauge. Basically the background space-time has to be of a product form $R^{1,1} \times \mathcal{M}^{d-2}$ for the gauge to be admissible **?** **?**. In general, and if one insist on a light-cone gauge, one has to add corrections to the worldsheet metric so that the gauge is consistent with the equation of motion for the light-cone coordinates, see **?** and **?**.

We can fix the light-cone gauge without reference to the worldsheet metric if we work in a first order formalism. The velocities can be expressed in terms of conjugate variables if we calculate the momentas of x^M (with respect to τ)

$$p_M = \gamma^{0\,\alpha}\,G_{MN}\,\partial_\alpha\,x^N. \tag{4.6}$$

This allows us to write the phase space Lagrangian as

$$\mathscr{L} = -g\big(\, p \cdot \dot{x} - \mathcal{H}(p, x, x')\big). \tag{4.7}$$

where the Hamiltonian is just a sum of two constraints

$$\mathcal{H} = \frac{1}{2\,\gamma^{00}}\big(p \cdot p + x' \cdot x'\big) - \frac{\gamma^{01}}{\gamma^{00}}\, x' \cdot p, \tag{4.8}$$

so the worldsheet metric only enters as Lagrange multipliers.

If we now impose the light-cone gauge together with a subsidiary uniform gauge as **?**

$$x^{+} = \tau, \qquad p_{+} = \text{Constant}, \tag{4.9}$$

where p_+ is the conjugate momentum density to $\dot{x}^-$, we find that up to a total derivative

$$\mathscr{L} = g\big(p_a\, \dot{x}^a + p_-\big), \tag{4.10}$$

where the lower case Latin indices denotes transverse directions. This gauge is denoted uniform light-cone gauge in the literature since p_+ is uniformly spread out over the string (i.e., independent of σ). In this gauge the conjugate momenta to $\dot{x}^+$ correspond to the gauge fixed Hamiltonian, $-p_-$. That is, $-p_-$ is the phase space function that generates τ translations in the transverse coordinates x^a and p_b.

Since the light-cone gauge eliminates one space-time coordinate and the second Hamiltonian constraint allow us to express x^- in terms of transverse coordinates, the gauge fixed string exhibits $D-2$ (bosonic) degrees of freedom and the physics is described in terms of transverse vibrations only.

In the light-cone gauge, the two Hamiltonian constraints in (4.8) turn into

$$\begin{aligned} C_1 : &\qquad p_+\, x^- + p_a\, x'^a = 0, \\ C_2 : &\qquad p \cdot p + G_{--}(x'^-)^2 + G_{a\,b}\, x'^a\, x'^b = 0. \end{aligned} \tag{4.11}$$

The first constraint, C_1, allows us to express the light-cone coordinate x'^- in terms of transverse fields and integrating the constraint gives the so-called level matching condition which enforces that the mode numbers of string oscillators sums up to zero. The second constraint, C_2, gives an algebraic equation for the light-cone Hamiltonian $-p_-$,

$$\begin{aligned} p_- = & \\ & -\frac{p_+\, G^{+-}}{G^{--}} \pm \frac{1}{G^{--}} \sqrt{\big(p_+\, G^{+-}\big)^2 - G^{--}\big(G^{a\,b}\, p_a\, p_b + G^{++}\, p_+^2 + x' \cdot x'\big)}, \end{aligned} \tag{4.12}$$

where we need to pick the minus solution to have the energy spectrum bounded from below.

For a general curved background metric, the light-cone Hamiltonian above is highly non trivial. To extract any sensible results from it one need to consider simple backgrounds or various simplifying limits. To illustrate the procedure we give provide simple examples below.

Example one - Flat space

For a flat target space where $G_{MN} = \eta_{MN}$, we have the almost trivial result

$$p_- = -\frac{1}{4p_+}(p_a^2 - (x'^a)^2). \tag{4.13}$$

In the conventions we use, it is convenient to put $p_+ = 2$ and rescale all the fields with $1/\sqrt{g}$ which gives

$$\mathscr{L} = p_a\,\dot{x}^a - \frac{1}{2}(p_a^2 + (x'^a)^2), \tag{4.14}$$

which describes a free theory of $D-2$ massless worldsheet scalars, exact for all values of the coupling g.

Example two - Bosonic AdS$_5\times$S^5

A more interesting example is to take the background metric to be AdS$_5\times$S^5, which line segment is just a sum of (2.3), with $p = 3$, and the S^5 metric

$$ds^2 = R^2\Big(-\cosh^2\rho\,dt^2 + d\rho^2 + \sinh^2\rho\,d\Omega_3^2 + d\theta^2 + \sin^2\theta\,d\widetilde{\Omega}_3^2\Big). \tag{4.15}$$

Since we want to expand the theory around a light-like geodesic defined by $t = \phi$ and $\rho = \theta = 0$, it is convenient to introduce the coordinates

$$\cosh\rho = \frac{1+\frac{1}{4}z^2}{1-\frac{1}{4}z^2}, \qquad \cos\theta = \frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2},$$

where $z^2 = z_1^2 + z_2^2 + z_3^2 + z_4^2$ and $y^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2$. This gives the metric

$$\frac{1}{R^2}\,ds^2 = -\Big(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\Big)^2 dt^2 + \Big(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\Big)^2 d\phi^2 + \frac{dz_i^2}{(1-\frac{z^2}{4})^2} + \frac{dy_a^2}{(1+\frac{y^2}{4})^2}, \tag{4.16}$$

where R is the radius of the AdS$_5$ and S^5 space.

On the AdS space, t is the time coordinate while ϕ is an angle coordinate on the S^5. Both these coordinates are invariant under constant shifts, giving rise to two conserved charges E and J,

$$E = -\int_{-r}^{r} d\sigma\,p_t, \qquad J = \int_{-r}^{r} d\sigma\,p_\phi, \tag{4.17}$$

where r is the radius of the worldsheet. Additionally, each space has four transverse z_i and y_a directions invariant under SO(4) $\times$ SO(4) rotations. For the upcoming gauge fixing, it is convenient to combine the time and angle coordinate into a light-cone pair

$$x^{\pm} = \phi \pm t \qquad P_{\pm} = \pm E + J. \tag{4.18}$$

As can be seen from the form of (4.16), the full light-cone Lagrangian will be more complicated than the flat space case considered above. What is more, in general the $AdS_5 \times S^5$ background is not consistent with the gauge $\gamma^{\alpha\beta} = \eta^{\alpha\beta}$ and $x^+ = \tau$. At quadratic level they are compatible but beyond leading order, one finds that the gauge needs to be modified. If one insist on a second order formalism, then depending on taste, one needs to choose one of the two gauges and then add perturbative corrections to the other. The form of the corrections are found by demanding that $p_+ =$ constant remains consistent with the equations of motion for $x^{\pm}$.

However, if we stick to the first order formalism, then we do not need to worry about the worldsheet metric at all. Doing that and imposing the uniform light-cone gauge (4.9) and, for simplicity, only keeping the leading order part, in a $g, P_+ \to \infty$ expansion, gives

$$\frac{1}{g} p_- = -\frac{1}{p_+} p_m^2 - \frac{p_+}{4} x_m^2 - \frac{1}{p_+} (x'_m)^2 + ..., \tag{4.19}$$

where the m index runs over both transverse AdS and S^5 coordinates. The corresponding Lagrangian is

$$\mathscr{L} = g\big(p_m \dot{x}^m - \frac{1}{p_+} p_m^2 - \frac{p_+}{4} x_m^2 - \frac{1}{p_+} (x'_m)^2\big) + \mathcal{O}(\chi^4). \tag{4.20}$$

If we scale $p_m \to \sqrt{\frac{p_+}{2}}$ and $x_m \to \sqrt{\frac{2}{p_+}}$ and write the full space-time Lagrangian we find

$$L = g \int d\sigma \big(p_m \dot{x}^m - \frac{1}{2} p_m^2 - \frac{1}{2} x_m^2 - \frac{2}{p_+^2} (x'_m)^2\big) + \mathcal{O}(\chi^4). \tag{4.21}$$

To put this in a canonical form we need to rescale $\sigma \to \frac{1}{g}\sigma$ which, together with an effective coupling constant $\lambda' = \frac{4g^2}{p_+^2}$, gives

$$L = \int d\sigma \big(p_m \dot{x}^m - \frac{1}{2} p_m^2 - \frac{1}{2} x_m^2 - \frac{\lambda'}{2} (x'_m)^2\big) + \mathcal{O}(\chi^4), \tag{4.22}$$

so we see that the leading order quadratic fluctuations describe, in contrast to the flat space case, a free theory of 8 massive coordinates.

From cylinder to plane

The above expansion looks rather complicated. Not only did we scale the transverse phase space fluctuations differently but we also had to perform a rescaling of the length parameter of the string to obtain a canonical leading order piece. However, the expansion can be understood in simpler terms. The expansion, denoted plane wave or BMN

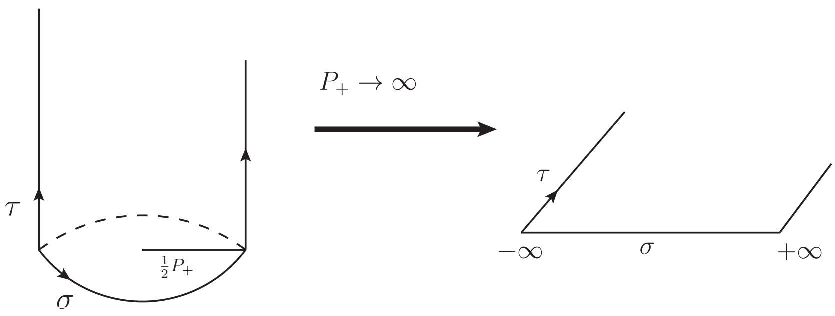

Figure 4.2: Decompactification of the string worldsheet

expansion **?**, can be defined through

$$g, P_+ \to \infty, \qquad g/P_+ = \text{Constant}, \tag{4.23}$$

and by direct computation the relationship between P_+ and g are given from

$$P_+ = g \int_{-r}^{r} d\sigma \, p_+, \tag{4.24}$$

which gives

$$r = \frac{P_+}{2\, g\, p_+},$$

so we see that the radius of the worldsheet gets related to p_+ and g. From (4.21) we saw that in order to obtain a canonical quadratic theory we had to scale σ with $1/g$ implying

$$r = \frac{P_+}{2\, p_+}, \tag{4.25}$$

which has the effect that P_+ becomes infinite. For the special case of $p_+ = 1$ see figure 4.2.

As we mentioned, the expansion is rather complicated, and from a computational point of view, also rather cumbersome. However, one can express the expansion in terms of the momentum density p_+ and the coupling g alone by sending,

$$\chi \to \frac{\chi}{\sqrt{g}}, \qquad g \to \infty, \qquad \sigma \in \{-\infty, \infty\}, \qquad p_+ = \text{Constant}, \tag{4.26}$$

where χ denotes all of the transverse coordinates, including the momenta variables which is equivalent to (4.23) with $\lambda' = 1$[1].

With this we end our exposition of bosonic string theory. Hopefully the reader (who did not have it before) have acquired a rudimentary feeling for how the gauge fixing and

[1]One can always choose p_+ so that λ' is eliminated and is thus not a fundamental parameter.

light-cone Hamiltonian is obtained. In the upcoming chapters we will introduce fermions which complicate things drastically, but nevertheless, gauge fixing and Hamiltonian constraints are obtained and solved in more or less the same fashion.

4.2 Full story - Supersymmetric theory and its properties

In the previous section we introduced the reader to some general aspects of bosonic strings on various background manifolds. Even though we restricted the exhibition to bosonic strings only, we discussed several general features that behave similarly, but naturally more involved, in the full supersymmetric theory.

In this thesis we will concern ourself with three different supersymmetric string theories, each one describing the bulk theory of a specific AdS / CFT correspondence,

- Type IIB superstring on $\mathrm{AdS}_5\times\mathrm{S}^5$
- Type IIA superstring on $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{k}3}$
- Type IIB superstring on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$

In this section we will outline how to construct each of these theories in detail. Even though each string theory is rather different, it turns out that the construction is very similar in each case.

Historically, one of the first non trivial and supersymmetric string theories studied were the $\mathrm{AdS}_5\times\mathrm{S}^5$string **?**. Not long after the same authors considered the construction of the supersymmetric $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ string using the same approach **?**. For the $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{k}3}$ string, it is only in later years it has enjoyed an interest due to the recent incarnation of the AdS / CFT duality **?**. Using modern approaches, which we will outline in detail in the upcoming, it has been fully constructed in **?** and **?**.

The outline of this rather lengthy section is as follows; We start out by reviewing some basic facts about the superalgebras, and especially their matrix realizations, that occur in each string theory. Each algebra allows for a decomposition under a $\mathbb{Z}_4$ grading, and by constructing a flat, we show how to obtain the string Lagrangian directly in terms of the graded components of the current. The superspace that the strings propagate on is of a quotient manifold type G/H, where G is the global isometry group and H local Lorentz transformations / rotations, see table 4.1. Having obtained the string Lagrangian, we then turn to a discussion of some of its properties such as classical integrability, bosonic and fermionic gauge fixing and conserved charges[2]. Actual calculations of physical properties will be postponed to the last part of this thesis.

4.2.1 Matrix realization of superalgebras

One of the most beautiful ways to describe a physical theory is through the use of its symmetries. The string theories we will consider exhibits a large degree of symmetry, and as it turns out, the construction of the theories can be done directly from their symmetry

[2]For references see, **?**, **?**, **?**, **?**, **?** and for a beautiful review see **?**.

	G	H
$AdS_5 \times S^5$	PSU(2,2|4)	SO(1,4)×SO(5)
$AdS_4 \times \mathbb{CP}_{\not{F}3}$	OSP(2,2|6)	SO(1,3)×U(3)
$AdS_3 \times S^3 \times T^4$	$PSU(1,1\|2)^2 \times$ sP(T^4)	SO(1,2)×SO(3)

Table 4.1: The group entering in the various coset models. Note that the critical $AdS_3 \times S^3 \times T^4$ string do not allow for a simple coset construction, so the H here only denotes the six dimensional $AdS_3 \times S^3$ part.

groups alone, see table 4.1. For this reason we will review each of the symmetry algebras in some detail utilizing a matrix representation of them. The review of the two algebras from the ten dimensional strings closely follow the notation and outline of **?** and **?** separately.

The $\mathfrak{psu}(2,2|4)$ algebra

All the super algebras under consideration, can be expressed in terms of super matrices. If we introduce

$$M = \begin{pmatrix} X_{4\times4} & \theta_{4\times4} \\ \eta_{4\times4} & Y_{4\times4} \end{pmatrix}, \qquad M \in \mathfrak{g} = \mathfrak{su}(2,2|4),$$

where the block matrices X and Y have even matrix elements and satisfy $Str\, M = Tr\, X - Tr\, Y = 0$, while the off diagonal blocks θ and η are odd in the sense of having Grassmannian entries, and impose the constraint

$$M^\dagger \mathcal{H} + \mathcal{H}\, M = 0, \tag{4.27}$$

where $\mathcal{H}$ and Σ are Hermitian and of the form

$$\mathcal{H} = \begin{pmatrix} \Sigma & 0 \\ 0 & \not{F}_4 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \not{F}_2 & 0 \\ 0 & -\not{F}_2 \end{pmatrix},$$

then we single out the superalgebra $\mathfrak{su}(2,2|4)$ where the projective algebra in 4.1 is given by $\mathfrak{psu}(2,2|4) \oplus \mathfrak{u}(1) = \mathfrak{su}(2,2|4)$. Note that the projective algebra have no realization in terms of super matrices since the $\mathfrak{u}(1)$ is central and commute with everything in (4.27). Later we will gauge away this extra $\mathfrak{u}(1)$ by demanding that the current, from where we construct the string Lagrangian, is traceless.

From (4.27) we find that the blocks in M satisfy

$$X^\dagger = -\Sigma\, X\, \Sigma, \quad Y^\dagger = -Y, \quad \eta = -\theta^\dagger\, \Sigma, \tag{4.28}$$

so the bosonic part of M is[3]

$$\mathfrak{su}(2,2) \oplus \mathfrak{su}(4). \tag{4.29}$$

[3] Up to an irrelevant $\mathfrak{u}(1)$.

The odd part consist of 32 complex fermions which gets reduced by a factor of half through a Majorana like condition.

As advocated, the algebra can be endowed with a $\mathbb{Z}_4$ structure

$$M = M^{(0)} + M^{(2)} + M^{(1)} + M^{(3)}, \tag{4.30}$$

which can be realized through an automorphism of the form

$$\Omega : \quad M \to \Omega(M), \qquad \Omega(M) = -\Upsilon\, M^{st}\, \Upsilon^{-1}, \tag{4.31}$$

where Υ is a constant matrix

$$\Upsilon = \begin{pmatrix} K_4 & 0 \\ 0 & K_4 \end{pmatrix}, \qquad K_4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with $K_4^2 = -\mathbb{1}_4$ and the supertranspose is defined as

$$M^{st} = \begin{pmatrix} X^t & -\eta^t \\ \theta^t & Y^t \end{pmatrix}.$$

Each graded component of M can be decomposed as

$$M^{(k)} = \frac{1}{4}(M + i^{3k}\Omega(M) + i^{2k}\Omega^2(M) + i^k\Omega^3(M)), \tag{4.32}$$

where each $M^{(k)}$ is an eigenstate of Ω

$$\Omega(M^{(k)}) = i^k M^{(k)}. \tag{4.33}$$

In matrix form, we can express the even part of M as

$$M^{(0)} = \frac{1}{2}\begin{pmatrix} X - K_4\, X^t\, K_4^{-1} & 0 \\ 0 & Y - K_4\, Y^t\, K_4^{-1} \end{pmatrix},$$

$$M^{(2)} = \frac{1}{2}\begin{pmatrix} X + K_4\, X^t\, K_4^{-1} & 0 \\ 0 & Y + K_4\, Y^t\, K_4^{-1} \end{pmatrix}.$$

Since each string theory is defined on a quotient manifold G/H we need to find a way to isolate H, or $\mathfrak{h}$, from G. For the $\mathrm{AdS}_5\times\mathrm{S}^5$ string, the quotient is

$$G/H = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)},$$

and the zero graded projection, $M^{(0)}$, of M coincides with $\mathfrak{h}$ which we want to mod out. If we expand X and Y in a basis of real γ-matrices as $X = x\cdot\gamma$ and $Y = y\cdot\gamma$, then from the expansion of $M^{(0)}$ above we see that an orthogonal basis is one with the property

$$(\gamma^i)^t = K_4\,\gamma^i\,K_4^{-1}. \tag{4.34}$$

The SO(5) γ-matrices are defined as,

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma^0 = -\gamma^1\,\gamma^2\,\gamma^3\,\gamma^4 = \Sigma. \tag{4.35}$$

In this basis a general algebra element in $M^{(2)}$ can be written as

$$M^{(2)} = i\,x^+\Sigma_+ + i\,x^-\Sigma_- + \begin{pmatrix} z_i\,\gamma^i & 0 \\ 0 & i\,y_i\,\gamma^i \end{pmatrix} + i\,\mathbb{1}_8, \tag{4.36}$$

and where we introduced a light-cone basis for the pair $x^\pm$ as

$$\Sigma_\pm = \begin{pmatrix} \pm\Sigma & 0 \\ 0 & \Sigma \end{pmatrix}. \tag{4.37}$$

With this we have found a nice parametrization of the algebra elements that will be the basic building blocks of the string Lagrangian.

The $\mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2)$ algebra

In this section we focus on the isometry algebra for the non critical $AdS_3{\times}S^3$ string. For a 4×4 supermatrix

$$m = \begin{pmatrix} x & \theta_2 \\ \eta_2 & y \end{pmatrix},$$

the $\mathfrak{su}(1,1|2)$ algebra is singled out by the conditions

$$h\,m + m^\dagger\,h = 0, \qquad \mathrm{Tr}\,x - \mathrm{Tr}\,y = 0, \tag{4.38}$$

where

$$h = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This implies the conjugation rules,

$$x^\dagger = -\sigma_3\,x\,\sigma_3, \quad \eta = -\theta^\dagger\sigma_3, \quad y^\dagger = -y. \tag{4.39}$$

The full isometry algebra $\mathfrak{su}(1,1|2) \oplus \widetilde{\mathfrak{su}}(1,1|2)$ algebra can be combined into an 8×8 supermatrix as[4],

$$M = \begin{pmatrix} x & 0 & \theta_2 & 0 \\ 0 & \tilde{x} & 0 & \tilde{\theta}_2 \\ \eta_2 & 0 & y & 0 \\ 0 & \tilde{\eta}_2 & 0 & \tilde{y} \end{pmatrix} = \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix}.$$

obeying the conditions[5]

$$H\,M + M^\dagger\,H = 0, \qquad \operatorname{Tr} X - \operatorname{Tr} Y = 0, \tag{4.40}$$

where

$$H = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & \mathbb{1}_2 & 0 \\ 0 & 0 & 0 & \mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}.$$

The 4×4 version of (4.39) then becomes

$$X^\dagger = -\Sigma\,X\,\Sigma, \qquad Y^\dagger = -Y, \qquad \eta = -\theta^\dagger\,\Sigma, \tag{4.41}$$

showing that X and Y describe the bosonic isometry groups of $\mathrm{AdS}_3 \times \mathrm{S}^3$

$$X = \mathfrak{su}(1,1) \oplus \mathfrak{su}(1,1) \simeq \mathfrak{so}(2,2), \qquad Y = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \simeq \mathfrak{so}(4).$$

Each of the copies of the $\mathfrak{su}(1,1|2)$ algebras allow for a $\mathbb{Z}_4$ grading and it is convenient to realize the grading in a way that mixes the two copies of $\mathfrak{su}(1,1|2)$. This can be done with the automorphism

$$\Omega(M) = -\Upsilon\,M^{st}\,\Upsilon^{-1}, \tag{4.42}$$

with $\Upsilon^2 = -\mathbb{1}_8$ and

$$\Upsilon = \begin{pmatrix} \tilde{K}_4 & 0 \\ 0 & \tilde{K}_4 \end{pmatrix}, \qquad \tilde{K}_4 = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix},$$

which is similar but not identical to the automorphism that realized the $\mathbb{Z}_4$ grading of the $\mathfrak{su}(2,2|4)$ algebra. Note that Υ takes values in $\mathfrak{sl}(4|4)$ and not $\mathfrak{su}(1,1|2) \oplus \mathfrak{su}(1,1|2)$. The automorphism (4.42) flips the two copies of $\mathfrak{su}(1,1|2)$ as can be seen from

$$-\Upsilon\,M^{st}\,\Upsilon^{-1} = \begin{pmatrix} -\tilde{x}^t & 0 & \tilde{\eta}^t & 0 \\ 0 & -x^t & 0 & \eta^t \\ -\tilde{\theta}^t & 0 & -\tilde{y}^t & 0 \\ 0 & -\theta^t & 0 & -y^t \end{pmatrix} \in \mathfrak{su}(1,1|2) \oplus \mathfrak{su}(1,1|2).$$

[4]Naturally we can write the direct sum in many different, equivalent, ways.
[5]In fact, since x and y takes values from $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)$ respectively, they are both separately traceless (and similar for $\tilde{x}$ and $\tilde{y}$).

As in (4.32), each component of M is an eigenstate of Ω

$$\Omega(M^{(k)}) = i^k\, M^{(k)}, \tag{4.43}$$

and can be decomposed as in (4.30).

As earlier we introduce γ_i matrices as a basis for $M^{(2)}$,

$$\Sigma = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

which all obeys $\tilde{K}_4\, \gamma_a^t\, \tilde{K}_4^{-1} = \gamma_i$. It was to obtain this feature that we picked an automorphism which mixes the two copies of $\mathfrak{su}(1,1|2)$. A generic element in $M^{(2)}$ can now be written as

$$M^{(2)} = i\, x^+ \Sigma_+ + i\, x^- \Sigma_- + \begin{pmatrix} \gamma_a\, z_a & 0 \\ 0 & i\, \gamma_s\, y_s \end{pmatrix} + i\, \mathbb{1}_8, \tag{4.44}$$

where as for the ten dimensional case, $\Sigma_\pm = \pm\Sigma \oplus \Sigma$.

The $\mathfrak{osp}(2,2|6)$ algebra

Now we turn to the isometry algebra of the $\mathrm{AdS}_4 \times \mathbb{CP}_{\mathbb{1}3}$ string. The basic building blocks are 10×10 matrices as

$$M = \begin{pmatrix} X_{4\times4} & \theta_{4\times6} \\ \eta_{6\times4} & Y_{6\times6} \end{pmatrix}$$

where as before X and Y are even matrices whereas θ and η are Grassmannian odd.

The super algebra $\mathfrak{osp}(2,2|6)$ is singled out through,

$$M^{st} \begin{pmatrix} C_4 & 0 \\ 0 & \mathbb{1}_6 \end{pmatrix} + \begin{pmatrix} C_4 & 0 \\ 0 & \mathbb{1}_6 \end{pmatrix} M = 0$$

$$M^\dagger \begin{pmatrix} \Gamma^0 & 0 \\ 0 & -\mathbb{1}_{6\times6} \end{pmatrix} + \begin{pmatrix} \Gamma^0 & 0 \\ 0 & -\mathbb{1}_{6\times6} \end{pmatrix} M = 0$$

where the charge conjugation matrix satisfies $C_4^2 = -\mathbb{1}_{4\times4}$ and Γ_0 is one of the AdS_4 Γ-matrices,

$$\Gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \Gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad C_4 = i\,\Gamma^0\,\Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

satisfying $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ with signature (+,-,-,-).

The above reality and transposition rules imply

$$X^t = -C_4\,X\,C_4^{-1} \qquad Y^t = -Y, \qquad \eta = -\theta^t\,C_4, \qquad \theta^* = \Gamma^0\,C_4\,\theta, \tag{4.45}$$

so the even X and Y block correspond to $\mathfrak{usp}(2,2)$ and $\mathfrak{so}(6)$ of the AdS_4 and $\mathbb{CP}_{\not\vDash 3}$ respectively. The odd blocks are related by conjugation and constitute 24 real spinor variables. The reality condition on the fermionic block θ relates[6]

$$\theta_{4,i} = \bar{\theta}_{1,i}, \qquad \theta_{3,i} = -\bar{\theta}_{2,i}. \tag{4.46}$$

For a critical string theory one would expect 32 real fermions, i.e. eight more than in the present case. However, later we will show how one can perform a partial fermionic gauge fixing that leaves the spectrum with the expected fermionic degrees of freedoms.

As for the other super algebras, $\mathfrak{osp}(2,2|6)$ admits a $\mathbb{Z}_4$ decomposition as in (4.30) and we want to construct an automorphism such that its stationary point coincides with $\mathfrak{so}(1,3) \oplus \mathfrak{u}(3)$. This can be done with the two matrices K_4 and K_6, where K_6 is given by

$$K_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

which, as K_4, satisfy $K_6^2 = -\not\Vdash$. These two matrices together with the charge conjugation matrix allows us to define an automorphism as **?**

$$\Omega(M) = \begin{pmatrix} K_4 C_4 & 0 \\ 0 & -K_6 \end{pmatrix} M \begin{pmatrix} K_4 C_4 & 0 \\ 0 & -K_6 \end{pmatrix}^{-1} = \Upsilon\,M\,\Upsilon^{-1},$$

which can be used to construct the different $\mathbb{Z}_4$ components as in (4.32), where as before each component $M^{(k)}$ is an eigenstate of Ω,

$$\Omega(M^{(k)}) = i^k M^{(k)}. \tag{4.47}$$

The stationary subalgebra, $M^{(0)}$, coincides with $\mathfrak{h} = \mathfrak{so}(1,3) \oplus \mathfrak{u}(3)$ which is the part of $\mathfrak{osp}(2,2|6)$ we want to divide out.

The orthogonal complement $M^{(2)}$ is spanned by matrices satisfying $\Upsilon\,M\,\Upsilon^{-1} = -M$,

[6] To avoid cluttering the notation to much, we denote conjugated objects with bar.

which boils down to the conditions

$$\{X, \Gamma^5\} = 0, \qquad \{Y, K_6\} = 0. \tag{4.48}$$

These two equations can be solved by

$$X = x_\mu \Gamma^\mu, \qquad Y = y_i T_i, \tag{4.49}$$

where the six T_i matrices are generators of $\mathfrak{so}(6)$ along $\mathbb{CP}_{\not{3}}$ and are given by

$$\begin{aligned} T_1 &= E_{13} - E_{31} - E_{24} + E_{42}, \qquad & T_2 &= E_{14} - E_{41} + E_{23} - E_{32}, \\ T_3 &= E_{15} - E_{51} - E_{26} + E_{62}, \qquad & T_2 &= E_{16} - E_{61} + E_{25} - E_{52}, \\ T_5 &= E_{35} - E_{53} - E_{46} + E_{64}, \qquad & T_2 &= E_{36} - E_{63} + E_{45} - E_{54}, \end{aligned} \tag{4.50}$$

where E_{ij} is the 6×6 matrix with all elements zero except the i, j'th component which is unity. The normalization is as follows,

$$Tr(T_i\, T_j) = -4\, \delta_{ij}. \tag{4.51}$$

The T_i matrices satisfy the following important properties,

$$\{T_1, T_2\} = 0, \qquad \{T_3, T_4\} = 0, \qquad \{T_5, T_6\} = 0. \tag{4.52}$$

In the text we frequently make use of the complex combinations,

$$\tau_1 = \frac{1}{2}(T_1 - i\, T_2), \qquad \tau_2 = \frac{1}{2}(T_3 - i\, T_4), \tag{4.53}$$

and $\bar{\tau}_i$ for conjugated combinations.

The first solution in (4.48) parameterize SO(3,2)/SO(1,3) and the second parameterize SO(6)/U(3). As in the previous cases, we can write $M^{(2)}$ as

$$M^{(2)} = x^+ \Sigma_+ + x^- \Sigma_- + \left(\begin{array}{cc} x_i \Gamma^i & 0 \\ 0 & y\, T_5 + \omega_a\, \tau_a + \bar{\omega}_a\, \bar{\tau}_a \end{array} \right), \tag{4.54}$$

where $i = 1, 2, 3$ and the light-cone basis is given by

$$\Sigma_\pm = \left(\begin{array}{cc} \pm\Gamma_0 & 0 \\ 0 & -i\, T_6 \end{array} \right).$$

Note that we have a natural splitting of the transverse $\mathbb{CP}_3$ coordinates. We have two complex ω_i and one real coordinate y. Later in this thesis we will spend some time investigating the physical meaning of this coordinate. But for now, just note that the transverse directions are not uniform as in the AdS_5 and AdS_3 case.

With this we conclude the short summary and review of the various super algebras. As have been seen, the AdS_5 and AdS_3 strings are rather similar while the symmetry algebra of the AdS_4 string is a little bit more involved.

4.2.2 String Lagrangian in terms of a flat current

In the previous section we reviewed some general properties of the super algebras that occur in the various string theories under consideration. To promote the algebra elements to group elements, we basically use the exponential map, $\mathfrak{g} \to \exp \mathfrak{g}$. Even though the form of the group element will vary slightly in each separate string theory, its general structure will be the same.

If we take a $g \in G$ we can build the following current

$$\mathcal{A}_\alpha = \mathcal{A}_\alpha^{(0)} + \mathcal{A}_\alpha^{(2)} + \mathcal{A}_\alpha^{(1)} + \mathcal{A}_\alpha^{(3)} = -g^{-1}\,\partial_\alpha\, g, \tag{4.55}$$

which basically is the pullback of a group element to its respective (super) lie-algebra. For clarity, we also made the $\mathbb{Z}_4$ decomposition of the current explicit. Almost by direct inspection we see that the current satisfy the following flatness condition

$$\partial_\alpha\, \mathcal{A}_\beta - \partial_\beta\, \mathcal{A}_\alpha - [\mathcal{A}_\alpha, \mathcal{A}_\beta] = 0. \tag{4.56}$$

By using the different graded components of the current we will later construct the string action. Before we present it, let us go through the properties it should fulfil. First of all, it should naturally be invariant under global transformations from G. However, since the string propagates on the super manifold G/H, which is only defined up to a H rotation, the action needs to be invariant under local H transformations. Following **?** we introduce, or to be more precise, we postulate the string Lagrangian

$$\mathscr{L} = -\frac{g}{2}\gamma^{\alpha\,\beta}\, Str\, \mathcal{A}_\alpha^{(2)}\, \mathcal{A}_\beta^{(2)} - \kappa\, \frac{g}{2}\epsilon^{\alpha\,\beta}\, Str\, \mathcal{A}_\alpha^{(1)}\, \mathcal{A}_\beta^{(3)}, \tag{4.57}$$

where the constant in front of the WZ term satisfy $\kappa^2 = 1$ and $\epsilon^{01} = 1$.

As before, the Lagrangian needs to be augmented with the vanishing of the stress energy tensor. The super symmetric equivalence of (4.3) is

$$T_{\alpha\beta} = Str\, \mathcal{A}_\alpha^{(2)}\, \mathcal{A}_\beta^{(2)} - \frac{1}{2}\gamma_{\alpha\beta}\,\gamma^{\delta\rho}\, Str\, \mathcal{A}_\delta^{(2)}\, \mathcal{A}_\rho^{(2)} = 0. \tag{4.58}$$

Local transformations from $h \in H$ acts from the right on the group element as

$$g \to g \cdot h, \tag{4.59}$$

which from (4.55) gives that

$$\mathcal{A}^{(0)} \to h^{-1}\,\mathcal{A}^{(0)}\,h - h^{-1}\,dh, \qquad \mathcal{A}^{(k)} \to h^{-1}\,\mathcal{A}^{(k)}\,h, \quad k \neq 0, \tag{4.60}$$

which shows that (4.57) is invariant under local transformations from H. Global transformations from G act on the group element from the left as

$$G \cdot g \to g' \cdot h, \tag{4.61}$$

where h is a compensating transformation from H, which, using (4.60), leaves the string action (4.57) invariant.

Under a shift $\delta\mathcal{A}$ the Lagrangian density transform as

$$\delta\mathscr{L} = -g\,\gamma^{\alpha\beta} Str\,\delta\mathcal{A}^{(2)}_{\alpha}\,\mathcal{A}^{(2)}_{\beta} - \kappa\frac{g}{2}\epsilon^{\alpha\beta} Str(\delta\mathcal{A}^{(1)}_{\alpha}\,\mathcal{A}^{(3)}_{\beta} + \mathcal{A}^{(1)}_{\alpha}\,\delta\mathcal{A}^{(3)}_{\beta}), \tag{4.62}$$

using the identity

$$Str\,M_1\,\Omega^{4-k}(M_2) = Str\,\Omega^k(M_1)\,M_2,$$

the variation can be written as

$$\delta\mathscr{L} = -Str\,\delta\mathcal{A}_{\alpha}\,S^{\alpha} = -(g^{-1}\delta g\,\mathcal{A}_{\alpha} + g^{-1}\,\delta(\partial_{\alpha}\,g))S^{\alpha}, \tag{4.63}$$

with S^{α} given by

$$S^{\alpha} = g(\gamma^{\alpha\beta}\mathcal{A}^{(2)}_{\beta} - \frac{\kappa}{2}\epsilon^{\alpha\beta}(\mathcal{A}^{(1)}_{\beta} - \mathcal{A}^{(3)}_{\beta})). \tag{4.64}$$

Up to a total derivative (4.63) can be rewritten as

$$\delta\mathscr{L} = -Str(g^{-1}\delta\,g(\partial_{\alpha}\,S^{\alpha} - [\mathcal{A}_{\alpha}, S^{\alpha}])),$$

which gives the following equations of motion[7]

$$\partial_{\alpha}\,S^{\alpha} - [\mathcal{A}_{\alpha}, S^{\alpha}] = 0. \tag{4.65}$$

From this we can construct the following current

$$J^{\alpha} = g\,S^{\alpha}\,g^{-1}, \tag{4.66}$$

which is conserved due to the equations of motion (4.65)

$$\partial_{\alpha}\,J^{\alpha} = g(\partial_{\alpha}\,S^{\alpha} - [\mathcal{A}_{\alpha}, S^{\alpha}])g^{-1} = 0.$$

The J^{α} is the conserved current from the global G symmetry with corresponding Noether charge

$$Q = \int_{-r}^{r} d\sigma\,J^0. \tag{4.67}$$

To single out the specific charges corresponding to boosts, rotations and suchlike one

[7] For the projective algebras this equation is defined up to an U(1) which divided away naturally restricts to $\mathfrak{psu}(2,2|4)$ or $\mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2)$.

multiply the above with an appropriate basis element $\mathcal{M} \in \mathfrak{g}$ and take the super trace

$$Q_{\mathcal{M}} = Str\, Q\, \mathcal{M}. \tag{4.68}$$

The Poisson bracket between two charges can conveniently be written as

$$\{Q_1, Q_2\} = (-1)^{\eta_1\, \eta_2}\, Str\, Q\, [\mathcal{M}_1, \mathcal{M}_2]_{\pm} + \mathcal{C}(\mathcal{M}_1, \mathcal{M}_2), \tag{4.69}$$

where η_i is the parity of the super matrices and the $\pm$ denotes that the commutator is graded. The function $\mathcal{C}(\mathcal{M}_1, \mathcal{M}_2)$ is a possible central extension which for purely bosonic generators $\mathcal{M}_i$ is zero.

We have now established the full string Lagrangian invariant under global G and local H transformations. One can also show that the Lagrangian is invariant under parity and time transversal. Since this analysis is not directly relevant for the theme of this thesis we point the interested reader to **?**.

4.2.3 Fermionic local symmetry

The Lagrangian is also invariant under another hidden[8] local symmetry denoted κ symmetry **?**. This is a fermionic symmetry, in the sense of having a Grassmann valued transformation parameter, which was first discovered for the flat super string. This symmetry can be used to reduce the number of fermionic degrees of freedom which is important since the covariant action, i.e. non gauge fixed, generally exhibits a mismatch in the number of fermionic and bosonic coordinates.

The global action of G on the group element g were realized through multiplication from the left. In contrast, a local fermionic κ symmetry transformation can be realized through multiplication from the right as **?**

$$g \to g \cdot e^{\chi}, \tag{4.70}$$

where χ is the fermionic transformation parameter. Under this transformation the current transform as

$$\delta \mathcal{A}_\alpha = \partial_\alpha\, \chi + [\mathcal{A}_\alpha, \chi] + \mathcal{O}(\chi^2). \tag{4.71}$$

For general χ this transformation does not leave the Lagrangian (4.57) invariant; It is only for a specific choice of transformation parameter that the variation is a symmetry and in the below we will outline how to find the precise form of the transformation. With the natural assumption that $\chi = \chi^{(1)} + \chi^{(3)}$, the $\mathbb{Z}_4$ decomposition of the variation

[8] Hidden in the sense of being far from obvious.

(4.71) is

$$\begin{aligned} &\delta_\chi \mathcal{A}^{(1)} = -d\chi^{(1)} + [\mathcal{A}^{(0)}, \chi^{(1)}] + [\mathcal{A}^{(2)}, \chi^{(3)}], \\ &\delta_\chi \mathcal{A}^{(3)} = -d\chi^{(3)} + [\mathcal{A}^{(0)}, \chi^{(3)}] + [\mathcal{A}^{(2)}, \chi^{(1)}], \\ &\delta_\chi \mathcal{A}^{(0)} = [\mathcal{A}^{(3)}, \chi^{(1)}] + [\mathcal{A}^{(1)}, \chi^{(3)}], \quad \delta_\chi \mathcal{A}^{(2)} = [\mathcal{A}^{(3)}, \chi^{(3)}] + [\mathcal{A}^{(1)}, \chi^{(1)}], \end{aligned} \tag{4.72}$$

which, together with the flatness condition (4.56), can be used to deduce that the variation of (4.57) becomes

$$\begin{aligned} &\delta_\chi \mathscr{L} = \\ &-\frac{g}{2}(\delta\gamma^{\alpha\beta} Str\, \mathcal{A}^{(2)}_\alpha \mathcal{A}^{(2)}_\beta - 4Str(P^{\alpha\beta}_+[\mathcal{A}^{(1)}_\beta, \mathcal{A}^{(2)}_\alpha]\chi^{(1)} + P^{\alpha\beta}_-[\mathcal{A}^{(3)}_\beta, \mathcal{A}^{(2)}_\alpha]\chi^{(3)}), \end{aligned} \tag{4.73}$$

where we also introduced the projection operators

$$P^{\alpha\beta}_\pm = \frac{1}{2}(\gamma^{\alpha\beta} \pm \kappa\epsilon^{\alpha\beta}), \quad P^{\alpha\beta}_\pm P^{\gamma}_{\pm\,\beta} = P^{\alpha\gamma}_\pm, \quad P^{\alpha\beta}_\pm P^{\gamma}_{\mp\,\beta} = 0. \tag{4.74}$$

A worldsheet vector can be projected with these through

$$V^\alpha_\pm = P^{\alpha\beta}_\pm V_\beta,$$

which shows that the projected components of $\mathcal{A}_\pm$ are related as

$$\mathcal{A}_{\tau,\pm} = -\frac{\gamma^{\tau\sigma} \mp \kappa}{\gamma^{\tau\tau}} \mathcal{A}_{\sigma,\pm}. \tag{4.75}$$

Up till this point the derivation of κ symmetry is identical for all the three string models. However, we will make an ansatz for the transformation parameter which form differs in the various models. For this reason we will go through the derivation for the different models separately.

Deriving κ symmetry for the AdS$_5\times$S^5 and AdS$_3\times$S^3 string

For the AdS$_5$ and AdS$_3$ string the derivation of κ symmetry is almost identical and we will start out with describing these two theories. As before, we closely follow the outline in **?**.

As mentioned, it is only for a specific choice of χ that the variation (4.73) leaves the Lagrangian invariant. An appropriate ansatz for the transformation parameter is[9] **?**

$$\chi^{(1)} = \{\mathcal{A}^{(2)}_{\alpha,-}, \kappa^{(3),\alpha}_+\}, \qquad \chi^{(3)} = \{\mathcal{A}^{(2)}_{\alpha,+}, \kappa^{(1),\alpha}_-\}, \tag{4.76}$$

where we introduced new independent transformation parameters $\kappa^{(i),\alpha}_\pm$. For χ to take values in $\mathfrak{su}(2,2|4)$ or $\mathfrak{su}(1,1|2)\oplus\mathfrak{su}(1,1|2)$, depending on the theory, the new parameters

[9]Note that we use a different convention compared to **?**. Our convention is chosen so that to highlight the explicit grading of the terms in χ.

need to satisfy the 'conjugated' reality conditions

$$\mathcal{H}\,\kappa^{(i)} - (\kappa^{(i)})^{\dagger}\mathcal{H} = 0,$$

i.e. with a relative sign flip compared to (4.27) and (4.40).

The two components $\mathcal{A}_{\pm}$ can, as explained earlier, be parameterized as

$$\mathcal{A}_{\pm} = \begin{pmatrix} x^{\mu}_{\pm}\,\gamma^{\mu} & 0 \\ 0 & i\,y^{\mu}_{\pm}\,\gamma^{\mu} \end{pmatrix},$$

where $\mu = 0,1,2,3,4$ for AdS$_5$ and $\mu = 0,1,2$ for AdS$_3$. Note that $x^0_{\pm}$ is purely imaginary while all other coordinates are real. Using this we find

$$\mathcal{A}^{(2)}_{\alpha,\pm}\,\mathcal{A}^{(2)}_{\alpha,\pm} = \frac{1}{8}\Upsilon\,Str\,\mathcal{A}^{(2)}_{\alpha,\pm}\,\mathcal{A}^{(2)}_{\alpha,\pm} + c_{\alpha\beta}\not{1}_8, \tag{4.77}$$

where $c_{\alpha\beta}$ is a smooth function of the coordinates $x^{\mu}_{\pm}$ and $y^{\mu}_{\pm}$. With this (4.73) becomes

$$\delta_{\chi}\mathscr{L} = -\frac{g}{2}\big(\delta\gamma^{\alpha\beta}Str\,\mathcal{A}^{(2)}_{\alpha}\,\mathcal{A}^{(2)}_{\beta} - \frac{1}{2}Str\,\mathcal{A}^{(2)}_{\alpha,-}\,\mathcal{A}^{(2)}_{\beta,-}\,Str\big(\Upsilon[\kappa^{(3),\beta}_{+},\mathcal{A}^{(1),\alpha}_{+}]\big)$$
$$-\frac{1}{2}Str\,\mathcal{A}^{(2)}_{\alpha,+}\,\mathcal{A}^{(2)}_{\beta,+}\,Str\big(\Upsilon[\kappa^{(1),\beta}_{-},\mathcal{A}^{(3),\alpha}_{-}]\big)\big),$$

where the unspecified function $c_{\alpha\beta}$ does not contribute due to the properties of the super trace. By using the identity $P^{\alpha\gamma}_{\pm}\,P^{\beta\delta}_{\pm} = P^{\beta\gamma}_{\pm}P^{\alpha\delta}_{\pm}$, one can show that for the variation to be a symmetry, one has to assume the following variation of the worldsheet metric

$$\delta\gamma^{\alpha\beta} = \frac{1}{2}Tr\Big([\kappa^{3,\alpha}_{+},\mathcal{A}^{1,\beta}_{+}] + [\kappa^{1,\alpha}_{-},\mathcal{A}^{3,\beta}_{-}]\Big). \tag{4.78}$$

By using the reality conditions for $\mathcal{A}_{\alpha}$ and κ one can show that $\delta\gamma^{\alpha\beta}$ is purely real and satisfy $\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} = 0$, where the latter need to hold for the classical Weyl scaling to be manifest. The above expression, through the orthogonality of the projection operators, also imply that the κ parameter in front of the WZ term has to fulfil, $\kappa^2 = 1$.

Now, the interesting question to ask is naturally - how many fermionic degrees of freedom does the κ symmetry allow us to remove? We do not loose any generality by assuming that the transverse bosonic fluctuations are suppressed so

$$\mathcal{A}^{(2)} = \begin{pmatrix} i\,x^0\,\Sigma & 0 \\ 0 & i\,y^0\,\Sigma \end{pmatrix}. \tag{4.79}$$

If we go on shell, in the sense of solving the Virasoro constraint in (4.58) with $\gamma^{\alpha\beta} = \eta^{\alpha\beta}$, we find $(x^0)^2 = (y^0)^2$, so

$$\mathcal{A}^{(2)} = i\,x^0\,\Sigma_{\pm}, \tag{4.80}$$

where the plus / minus is specified by which solution of $\pm x^0$ one picks. Thus, from

(4.76) we find that

$$\epsilon^{(1)} = i\,x^0\,\{\Sigma_\pm, \hat{\kappa}^{(1)}\}, \qquad \epsilon^{(3)} = i\,x^0\,\{\Sigma_\pm, \hat{\kappa}^{(3)}\}, \tag{4.81}$$

where $\hat{\kappa}^{(i)}$ is a linear combination of $\chi^{(i)}$'s. From this, and writing $M^{(1)} + M^{(3)} = \eta$, one find that a suitable gauge can be imposed as

$$\{\eta, \Sigma_+\} = 0, \tag{4.82}$$

which eliminates half of the fermionic degrees of freedom for the AdS_5 and AdS_3 string. For the ten dimensional string, this gauge was first imposed in **?**. It is worth noting that this is the first time it is shown that a similar gauge choice can be done for the non critical AdS_3 string.

The form of the gauge fixed fermionic block matrices will be presented later when we investigate the physical properties of each string theory.

Deriving κ symmetry for the $AdS_4 \times \mathbb{CP}_{\nVdash 3}$ string

The derivation of κ symmetry invariance for the $AdS_4 \times \mathbb{CP}_{\nVdash 3}$ string is very similar to the other two theories outlined above and for that reason we will be rather brief and point the interested reader to **?**.

The ansatz for the κ symmetry is however a bit more complicated

$$\begin{aligned} &\chi^{(1)} = \\ &\mathcal{A}^{(2)}_{\alpha,-}\,\mathcal{A}^{(2)}_{\beta,-}\,\kappa^{\alpha\beta}_{++} + \kappa^{\alpha\beta}_{++}\,\mathcal{A}^{(2)}_{\alpha,-}\,\mathcal{A}^{(2)}_{\beta,-} + \mathcal{A}^{(2)}_{\alpha,-}\,\kappa^{\alpha\beta}_{++}\,\mathcal{A}^{(2)}_{\beta,-} - \frac{1}{8} Str(\hat{\Sigma}\,\mathcal{A}^{(2)}_{\alpha,-}\,\mathcal{A}^{(2)}_{\beta,-})\,\kappa^{\alpha\beta}_{++}, \\ &\chi^{(3)} = \\ &\mathcal{A}^{(2)}_{\alpha,+}\,\mathcal{A}^{(2)}_{\beta,+}\,\kappa^{\alpha\beta}_{--} + \kappa^{\alpha\beta}_{--}\,\mathcal{A}^{(2)}_{\alpha,+}\,\mathcal{A}^{(2)}_{\beta,+} + \mathcal{A}^{(2)}_{\alpha,+}\,\kappa^{\alpha\beta}_{--}\,\mathcal{A}^{(2)}_{\beta,+} - \frac{1}{8} Str(\hat{\Sigma}\,\mathcal{A}^{(2)}_{\alpha,+}\,\mathcal{A}^{(2)}_{\beta,+})\,\kappa^{\alpha\beta}_{--}, \end{aligned} \tag{4.83}$$

where $\hat{\Sigma}$ =Diag$(\nVdash_4, -\nVdash_4)$ and the undetermined parameters $\kappa^{\alpha\beta}_{\pm\pm}$ takes values in $\mathfrak{osp}(2,2|6)$. Using various identities of the super algebra, see the appendix of **?** for details, one can show that together with the metric variation

$$\delta\gamma^{\alpha\beta} = \frac{1}{2} Str\,\hat{\Sigma}\Big(\mathcal{A}^{(2)}_{\delta,-}[\kappa^{\alpha\beta}_{++}, \mathcal{A}^{1,\delta}_{+}] + \mathcal{A}^{(2)}_{\delta,+}[\kappa^{\alpha\beta}_{--}, \mathcal{A}^{3,\delta}_{-}]\Big), \tag{4.84}$$

the action (4.57) is invariant under the transformation. As before, the parameter in front of the WZ term is forced to satisfy $\kappa^2 = 1$.

Unfortunately one can not impose as nice a κ gauge as in (4.82). However, one can impose something similar. For a general η one can fix

$$\{\Sigma_+, \eta\} = \eta_{g.f}, \tag{4.85}$$

where $\eta_{g.f}$ is a kappa gauge fixed fermionic matrix. In general, the gauge is capable of removing four complex fermions leaving us with the desired eight complex. However, it should be noted that if one considers strings moving only in the AdS_4 space, then, somewhat surprisingly, the ansatz (4.83) is in fact zero. In general this feature is not properly understood at the moment, but it is probably related to the coset formalism somehow. For more details, see the discussion at the end of section 3 of **?**.

The exact form of $\eta_{g.f}$ will be presented when we consider the $\mathrm{AdS}_4 \times \mathbb{CP}_{\nVdash 3}$ string in full detail.

4.2.4 Parametrization of the group element

To be order to extract the physics out of the string theories one naturally needs to specify the form of the group element $g \in G$ from which we constructed the current (4.55). Naturally, one can choose this representation in many different ways and we will use one which is especially convenient in the light-cone language. Since each theory is different, the exact form of g vary but, nevertheless, the general form of the element we will choose is the same

$$g = \Lambda(x^+, x^-)\, f(\eta)\, G_t \in G. \tag{4.86}$$

To the left we have a function only dependent on the light-cone pair $x^{\pm}$ and their respective basis elements as

$$\Lambda(x^+, x^-) = \exp[\frac{i}{2}(x^+\Sigma_+ + x^-\Sigma_-)], \tag{4.87}$$

whereas in the middle we sandwich the fermionic dependence of g through

$$f(\eta) = \eta + \sqrt{\nVdash + \eta^2}, \tag{4.88}$$

where $\eta = M^{(1)} + M^{(3)}$. To the far right we have the dependence on the transverse bosonic coordinates, which differs in each theory. For example, if we restrict to the purely bosonic case, i.e. $f(\eta) = \nVdash$, and consider the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string, then we choose a representation of G_t so that $Str\,\mathcal{A}^{(2)}_\alpha\,\mathcal{A}^{(2)}_\beta$ coincides with the line segment for the background (4.16).

Using (4.86) we see that the current splits up as

$$\mathcal{A}_\alpha = -G_t^{-1}(f(\eta)^{-1}\,\partial_\alpha\,\Lambda\, f(\eta) + f(\eta)^{-1}\,\partial_\alpha\, f(\eta))G_t - G_t^{-1}\,\partial_\alpha\, G_t, \tag{4.89}$$

where we used that $[d\Lambda, \Lambda] = 0$. For later use, it will also be convenient to split the current up in even and odd parts respectively

$$
\begin{aligned}
&\mathcal{A}_\alpha^{even} = -G_t^{-1}\,\partial_\alpha G_t \\
&- G_t^{-1}\Big(-\eta\,\partial_\alpha\,\Lambda\,\eta + \sqrt{\nVdash+\eta^2}\,\partial_\alpha\,\Lambda\,\sqrt{\nVdash+\eta^2} - \eta\,\partial_\alpha\,\eta + \sqrt{\nVdash+\eta^2}\,\partial_\alpha\,\sqrt{\nVdash+\eta^2}\Big)G_t, \\
&\mathcal{A}_\alpha^{odd} = \\
&- G_t^{-1}\Big(\eta\,\partial_\alpha\,\Lambda\,\sqrt{\nVdash+\eta^2} - \sqrt{\nVdash+\eta^2}\,\partial_\alpha\,\Lambda\,\eta + \eta\,\partial_\alpha\,\sqrt{\nVdash+\eta^2} - \sqrt{\nVdash+\eta^2}\,\partial_\alpha\,\eta\Big)G_t.
\end{aligned}
\tag{4.90}
$$

A desirable feature with the group parametrization (4.86) is that the kinetic term of the bosons enter without the fermions which, when we later introduce a first order formalism, will turn out to be rather convenient.

From the form of (4.86), it should be clear that shifts in $x^\pm$ are generated by $\Sigma_\pm$ as

$$
\alpha_+\,\Sigma_+\cdot g = \Lambda(x^+ + \alpha_+, x^-)\,f(\eta)\,G_t, \qquad \alpha_-\,\Sigma_-\cdot g = \Lambda(x^+, x^- + \alpha_-)\,f(\eta)\,G_t,
$$

thus, using (4.68), the conserved charges $P_\pm$ should be given by

$$
P_\pm \sim \int_{-r}^{r} d\sigma\, Str\, J^0\,\Sigma_\mp, \tag{4.91}
$$

where the normalization in front can be fixed by direct inspection of the quadratic theory. A desirable feature of the group element we work with is that the transverse fields, bosonic as well as fermionic, are uncharged under the two U(1) shifts $\Sigma_\pm$. Or, in other words, the separate fields are not charged under E and J.

The time evolution is naturally generated by the Hamiltonian through

$$
\frac{dQ_\mathcal{M}}{dt} = \frac{\partial Q_\mathcal{M}}{\partial t} + \{H, Q_\mathcal{M}\},
$$

so conserved charges Poisson commute with the Hamiltonian. After gauge fixing, the subalgebra that commutes with the light-cone Hamiltonian is given by

$$
\mathcal{J}: \qquad [\mathcal{M}, \Sigma_+] = 0, \tag{4.92}
$$

which is the algebra that remains after the gauge fixing procedure. We should also investigate how the fields transform under the bosonic part of $\mathcal{J}$, which is given by

$$
\mathcal{J}_B: \qquad [\mathcal{M}, \Sigma_\pm] = 0. \tag{4.93}
$$

Since any element $\mathfrak{g}_B \in \mathcal{J}_B$ commutes with $\Sigma_\pm$ it is easy to see that a corresponding group element acts as

$$
G_B\cdot g = \Lambda(x^+, x^-)\cdot G_B\,f(\eta)\,G_B^{-1}\cdot G_B\,G_t\,G_B^{-1}\cdot G_B, \tag{4.94}
$$

so the element G_B itself acts as the compensating transformation from H. This implies

G	J	J_B
PSU(2,2|4)	$SU(2\|2)^2$	$SU(2)^4$
OSP(2,2|6)	SU(2|2)×U(1)	$SU(2)^2$×U(1)
$PSU(1,1\|2)^2$	$SU(1\|1)^2$	$U(1)^4$

Table 4.2: The subalgebras that commute with Σ_+, J, and the subalgebras that commute with both $\Sigma_\pm$, J_B, for each supergroup under consideration.

that the fields transform in the adjoint with respect to the bosonic symmetries as

$$f(\eta) \to G_B\, f(\eta)\, G_B^{-1}, \qquad G_t \to G_B\, G_t\, G_B^{-1}, \qquad G_B \in H \tag{4.95}$$

Later, when we introduce the exact form of each G_t and η, it is very convenient to label the fields so that they transform covariantly under the bosonic symmetries.

In table 4.2, each subgroup J and J_B is presented for three supergroups PSU(2,2|4), OSP(2,2|6) and PSU$(1,1|2)^2$.

4.3 Gauge fixed theory

As for the bosonic theory, the supersymmetric Lagrangian (4.57) exhibits worldsheet Weyl and diffeomorphism invariance. In the bosonic theory we went to a first order formalism and gauge fixed these non perturbatively, i.e. without reference to the worldsheet metric. A natural question to ask is if there exist an equivalent first order formalism for the Lagrangian expressed in terms of super currents? Naturally one can invert all the velocities by hand and reexpress them in terms of phase space variables, but judging from the complexity of (4.57) this seems rather cumbersome. A better way is to introduce an auxiliary fields, π, as

$$\pi = \pi_+\Sigma_+ + \pi_-\Sigma_- + \pi_t, \tag{4.96}$$

where π_t is expressed in an basis over all the transverse directions, i.e, eight or four independent components depending on which theory we consider. From the automorphism that realize the $\mathbb{Z}_4$ grading, it is an easy exercise to verify that

$$Str\, \pi\, \mathcal{A}_\alpha^{(2)} = Str\, \pi\, \mathcal{A}_\alpha^{even}, \tag{4.97}$$

which in the upcoming analysis simplify a few expressions.

The idea now is to eliminate the quadratic $\mathcal{A}^{(2)}$ dependence in the kinetic term of (4.57) with something linear in $\mathcal{A}_0^{(2)}$ and π. This can be done with the Lagrangian

$$\mathscr{L} = -g\, Str\Big(\pi\, \mathcal{A}_0^{(2)} + \frac{\gamma^{01}}{\gamma^{00}}\pi\, \mathcal{A}_1^{(2)} - \frac{1}{2\gamma^{00}}(\pi^2 + (\mathcal{A}_1^{(2)})^2) + W.Z\Big), \tag{4.98}$$

where the WZ term is identical to the one in (4.57). The above action is classically equivalent to the second order Lagrangian which can be seen from the equations of motion for π

$$\pi = \gamma^{0\alpha} \mathcal{A}^{(2)}_\alpha . \tag{4.99}$$

The nice feature with (4.98) is that the worldsheet metric only enters as Lagrange multipliers giving rise to

$$C_1 : \quad Str\, \pi\, \mathcal{A}^{(2)}_1 = 0, \qquad C_2 : \quad Str\, (\pi^2 + (\mathcal{A}^{(2)}_1)^2) = 0, \tag{4.100}$$

which of course is nothing else than the super current version of (4.11).

The on shell theory has no dependence on the metric components so imposing the uniform light-cone gauge is straight forward. However, as compared to the bosonic section, we choose to be a little bit more general now and impose

$$x^+ = (1-a)\tau + a\,\phi, \qquad p_+ = (1-a)p_\phi - a\,p_t, \tag{4.101}$$

where a is a parameter that parameterize there most general uniform light-cone gauge. The choice $a = \frac{1}{2}$ give standard light-cone gauge as used in (4.5).

The components of π in (4.96) are unknown so to specify the physical theory, we need to solve for these. For the transverse, π_t we know that

$$Str\, \pi\, G_t^{-1}\, \partial_0\, G_t = p_m\, \dot{x}^m, \tag{4.102}$$

which gives a perturbative, or exact depending on theory, solution for the transverse components of π. The solution for π_+ is generally a bit more complicated and it also involves π_-,

$$P_+ = \text{Constant} \rightarrow \pi_+ = f(\pi_-, \pi_t, x^m, \eta, P_+),$$

where the function f is at maximum linear in π_- and allows for a perturbative expansion in the coupling. The form of π_+ will be presented when we study the expansion of each theory in general.

To solve for the last component of π we use C_2 in (4.100)

$$\pi_+\, \pi_-\, Str\, \Sigma_+\, \Sigma_- + Str\, \pi_t^2 + Str\, (\mathcal{A}^{(2)}_1)^2 = 0,$$

which is quadratic in π_- and thus give a solution similar to the bosonic one in (4.12).

Thus, as a quick summary, the gauge fixing is simplified by the introduction of an auxiliary field, π, which can be thought of as a super matrix equivalence of the normal first order phase-space procedure. As was also the case for the bosonic string, the (matrix) first order formalism allows us to fix a light-cone gauge without reference to the worldsheet metric. The auxiliary field comes in terms of unknown components which one can reexpress in terms of physical variables through the conjugate momentas and (4.100). Doing this, and enforcing the κ gauge, leaves us with the full, and exact, gauge fixed string Lagrangian.

4.4 Summary and outlook

Hopefully we have managed to convey a general picture for how the construction of the string Lagrangian works. We started out by presenting a rather detailed discussion about the various symmetry algebras and how to construct group elements from them. A lot of the construction follows similar lines for the various models which is rather surprising since the symmetry group in each specific case is rather different. By using the group element we constructed a flat current whose components constituted the string Lagrangian. A crucial ingredient were the existence of a $\mathbb{Z}_4$ grading which could be used to isolate the relevant parts of the current that entered the Lagrangian.

After we obtained the Lagrangian we investigated its properties under local and global symmetries. The global symmetries, which we denoted G, act through left multiplication while the local H and κ symmetries act through right multiplication. Gauge fixing the κ symmetry had the important effect that it allowed us to remove some of the fermionic coordinates. As we will later see, this makes the string manifestly space-time supersymmetric since the bosonic and fermionic degrees of freedom match.

Having explained the symmetries of the string, we turned to a discussion about the gauge fixing procedure. We introduced an auxiliary matrix field which allowed us to remove the explicit dependence of the worldsheet metric. As was also the case for the bosonic string, this allowed us to impose an uniform light-cone gauge in an convenient way. We fixed the gauge by aligning the worldsheet time coordinate along one of the light-cone directions together with fixing the conjugated light-cone momentum to be distributed uniformly along the string.

In the upcoming section of this thesis we will investigate each of the string theories in some considerable detail. In all the cases we will consider strong coupling expansions, or close cousins of them, and investigate the physical properties of the resulting theories.

5 The AdS$_5\times$S^5 string at strong coupling

In the last section we reviewed how to construct the string Lagrangian directly in terms of the graded components of the current. The only explicit choice of representation we made was that the current was of the form $\mathcal{A} = -g^{-1}\,dg$ and that the group element were built out of different components as

$$G = \Lambda(x^+, x^-)\, f(\eta)\, G_t,$$

where the transverse bosonic part, G_t, was left unspecified. In the upcoming we will pick an explicit representation for G_t, and by rescaling the fields appropriately, show how the theory can be put in a form suitable for a large coupling expansion. The strongly coupled AdS$_5\times$S^{5}string allows for an even expansion in number of fields and we will expand the theory up to quartic order in fields. The main focus will be on the light-cone Hamiltonian, which will be used to calculate energy corrections to a large class of string configurations.

Through the AdS / CFT correspondence, energies of string states should be dual to conformal dimensions of the single trace operators on the gauge theory side. As we described earlier, the spectrum of conformal dimensions can be mapped to an abstract spin-chain **?**, and **?** for a review. The problem is then reduced to solving a set of Bethe equations, whose solution encodes the spectrum of conformal dimensions. Remarkably, in **?**, a set of all loop asymptotic Bethe equations (ABE) were proposed, encoding the conformal dimensions of all possible single trace operators.

We will rewrite the ABE in a language suitable for a large coupling, or equivalently large light-cone momentum, expansion. This allows us to extract predictions for the string energies which we will explicitly compare with the diagonalization of the light-cone Hamiltonian. Since the string oscillators come in $8_B{+}8_B$ modes, the diagonalization of the full string Hamiltonian is naturally very involved, so by necessity, we will restrict to various subsectors of the theory. This analysis, mainly based on **?**, is rather involved and the bulk of this section will be devoted to this study.

The last topic we will touch upon for the strongly coupled AdS$_5\times$S^{5}string is a so called near flat space limit. This limit, originally presented in **?**, is a very close cousins of the BMN limit with the novel feature that the left and right moving worldsheet sectors are scaled differently. The resulting theory is still quartic but nevertheless significantly simpler than the full near BMN theory. We will show how one can obtain the near flat space model directly from the near BMN theory.

The outline is as follows; We start out with the string Lagrangian, with a focus on the light-cone Hamiltonian, and show how to obtain the strongly coupled quartic theory. Having established this we turn our attention to the ABE and rewrite these set of equations in a light-cone language. We then start a rather involved analysis where we compare the ABE predictions with explicit string theory calculations, where the comparison is done for a very large class of string oscillators constituted of both bosonic and fermionic operators.

We end the section with a short summary of the near flat space model and then turn to show how it can be obtained directly from the near BMN Hamiltonian.

5.1 Parametrization of the $\mathrm{AdS}_5\times$ S^5 super string

The first thing we need to do is to pick a suitable representation of the group element $G \in \mathrm{PSU}(2,2|4)$ which is represented by an 8×8 super matrix with both odd and even matrix entries.

It is convenient to choose the even part of G so that the bososnic Lagrangian coincides with (4.16). Using (4.36) we can construct

$$G_t = \left(\begin{array}{cc} \frac{1}{\sqrt{1-\frac{1}{4}z^2}}(\not{\mathbb{K}}_4 + \frac{1}{2}z_i\gamma_i) & 0 \\ 0 & \frac{1}{\sqrt{1+\frac{1}{4}y^2}}(\not{\mathbb{K}}_4 + \frac{i}{2}y_i\gamma_i) \end{array} \right), \tag{5.1}$$

where $i = 1,2,3,4$, which together with (4.87) gives that

$$Str\, \mathcal{A}^{(2)}_\alpha\, \mathcal{A}^{(2)}_\beta|_{even} = G_{MN}\, \partial_\alpha x^M\, \partial_\beta x^N,$$

where G_{MN} is the $\mathrm{AdS}_5\times\mathrm{S}^5$ metric (4.16).

The fermionic part $f(\eta)$ are as before given by

$$f(\eta) = \eta + \sqrt{\not{\mathbb{K}}_8 + \eta^2}, \qquad f^{-1}(\eta) = -\eta + \sqrt{\not{\mathbb{K}}_8 + \eta^2}$$

which due to the kappa symmetry gauge (4.82) satisfy the following important identity

$$\Sigma_+\, f(\eta) = f^{-1}(\eta)\, \Sigma_+, \qquad [f(\eta), \Sigma_-] = 0. \tag{5.2}$$

The kappa gauge amounts to reduce the number of fermionic degrees of freedom by one half. Remember that the full fermionic matrix η is constituted of two 4×4 off diagonal blocks $\theta_{4\times 4}$ and $\eta_{4\times 4} = -\theta^\dagger_{4\times 4}\,\Sigma$. Before the gauge fixing $\theta_{4\times 4}$ have a general form

$$\theta_{4\times 4} = \left(\begin{array}{cc} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{array} \right),$$

where each θ_i is a 2×2 matrix. The kappa gauge (4.82) boils down to $\{\theta_{4\times 4}, \Sigma\} = 0$ which gives

$$\theta_{g.f} = \left(\begin{array}{cc} 0 & \theta_2 \\ \theta_3 & 0 \end{array} \right) = \mathcal{P}_+\, \eta_a\, \Gamma_a + \mathcal{P}_-\, (\theta_a \Gamma_a)^\dagger, \tag{5.3}$$

where we introduced the projection operators $\mathcal{P}_\pm=\frac{1}{2}(\nVdash_4\pm\Sigma)$ and

$$\Gamma_1=\frac{1}{2}(\gamma_2-i\,\gamma_1),\qquad \Gamma_2=\frac{1}{2}(\gamma_4-i\,\gamma_3),\qquad \Gamma_3=\Gamma_2^\dagger,\qquad \Gamma_4=\Gamma_1^\dagger. \tag{5.4}$$

For PSU$(2,2|4)$, the subgroup that commutes with $\Sigma_\pm$ coincides with G_B=SU$(2)^4$ and it is an easy task to show that the kappa gauge above is compatible with the transformation (4.95).

With the kappa gauge imposed, the even and odd part of the current (4.90) becomes

$$\begin{aligned}
&\mathcal{A}_{even}= \\
&-G_t^{-1}\big(\frac{i}{2}(dx^+ + (\frac{1}{2}-a)dx^-)+\Sigma_+(\nVdash+\eta^2)+\frac{i}{2}dx^-\Sigma_-\big)G_t \\
&-G_t^{-1}\big(\sqrt{\nVdash+\eta^2}\,d\sqrt{\nVdash+\eta^2}-\eta\,d\eta\big)G_t-G_t^{-1}\,dG_t, \\
&\mathcal{A}_{odd}= \\
&-G_t^{-1}\big(i(dx^+ + (\frac{1}{2}-a)dx^-)\Sigma_+\,\eta\,\sqrt{\nVdash+\eta^2}+\sqrt{\nVdash+\eta^2}\,d\eta-\eta\,d\sqrt{\nVdash+\eta^2}\big)G_t,
\end{aligned} \tag{5.5}$$

where we see that for the choice $a=\frac{1}{2}$, the odd part of the current is independent of the light-cone coordinate x^-. From now one this is the gauge we will choose. Also note the pleasant feature that the light-cone coordinates only enters with derivatives.

It is convenient to normalize the auxiliary field π in (4.96) as

$$\pi=\frac{i}{2}\pi_+\,\Sigma_+ +\frac{i}{4}\pi_-\,\Sigma_- +\frac{1}{2}\pi_M\,\Sigma_M, \tag{5.6}$$

where the transverse part is found by demanding that, using (4.98) and (5.5),

$$g\,Str\,\pi\,G_t^{-1}\,\partial_0\,G_t=g\,p_m\,\dot{x}^m, \tag{5.7}$$

which gives

$$\pi_M=\begin{pmatrix} p_a(1-\frac{1}{4}z^2) & 0 \\ 0 & p_s(1+\frac{1}{4}y^2)\end{pmatrix}, \tag{5.8}$$

with a denoting transverse AdS$_5$ index and s denoting transverse S^5 index.

The Lagrangian depends on two unknown variables π_+ and π_-. The first one is solved for through the gauge constraint p_+=constant and the second through the quadratic constraint

$$Str(\pi^2+(\mathcal{A}_1^{(2)})^2)=0.$$

For π_+ we get using (4.98), (4.97) and (5.5)

$$p_+=g\frac{i}{2}Str\pi\,G_t^{-1}\,\Sigma_-\,G_t\Rightarrow\pi_+=\frac{1}{G_+}(p_+ +\frac{1}{2}G_-\pi_-), \tag{5.9}$$

where we introduced

$$G_{\pm} = \frac{1}{2}\Big(\frac{1+\frac{1}{4}z^2}{1-\frac{1}{4}z^2} \pm \frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}\Big). \tag{5.10}$$

Using this in the quadratic constraint above, we see that we will get a polynomial of degree two for π_-,

$$\frac{2}{G_+}(p_+ + G_-\,\pi_-)\pi_- + \frac{1}{4}Str\big(\pi_M\,\Sigma_M\big)^2 + Str\big(\mathcal{A}^{(2)}_1\big)^2 = 0. \tag{5.11}$$

Naturally, only one of the two solutions is admissible and one picks the one which bounds the spectrum of the Hamiltonian from below.

It is convenient to split up the Lagrangian according to its Hamiltonian, or p_-, and kinetic term as[1]

$$\mathscr{L} = \mathscr{L}_{kin} + p_-, \tag{5.12}$$

where $\mathscr{L}_{kin}$ contain all ∂_0 derivatives. The two parts are given by

$$\begin{aligned}
&\mathscr{L}_{kin} = \qquad\qquad (5.13)\\
&g\,p_m\,\dot{x}^m + g\,Str\,\pi\,G_t^{-1}\big(\sqrt{\nVdash+\eta^2}\,\partial_0\,\sqrt{\nVdash+\eta^2} - \eta\,\partial_0\,\eta\big)G_t\\
&-g\frac{i}{2}\kappa\,Str\,G_t^{-1}\big(\sqrt{\nVdash+\eta^2}\,\partial_0\,\eta - \eta\,\partial_0\,\sqrt{\nVdash+\eta^2}\big)G_t\\
&\times\Upsilon\Big(G_t^{-1}\big(\sqrt{\nVdash+\eta^2}\,\eta' - \eta\,\partial_1\sqrt{\nVdash+\eta^2}\big)G_t\Big)^{st}\Upsilon,\\
&p_- = g\frac{i}{2}Str\,\pi\,G_t^{-1}\,\Sigma_+(\nVdash+2\eta^2)G_t - g\,\frac{i}{2}\kappa\,Str\,G_t^{-1}\big(i\,\Sigma_+\,\eta\,\sqrt{\nVdash+\eta^2}\big)G_t\\
&\times\Upsilon\Big(G_t^{-1}\big(\sqrt{\nVdash+\eta^2}\,\eta' - \eta\,\partial_1\sqrt{\nVdash+\eta^2}\big)G_t\Big)^{st}\Upsilon,
\end{aligned}$$

where we made use of the identity

$$\epsilon^{\alpha\beta}Str\,\mathcal{A}^{(1)}_{\alpha}\,\mathcal{A}^{(3)}_{\beta} = i\,Str\,\mathcal{A}^{odd}_0\,\Upsilon(\mathcal{A}^{odd}_1)^{st}\Upsilon, \tag{5.14}$$

for the WZ contributions. As can be seen, the above expressions are rather involved and some comments are in order. First of all, one need to substitute the expressions for $\pi_{\pm}$ and (5.8) in π. Having done that, one have in principle the full gauge-fixed $\mathrm{AdS}_5 \times \mathrm{S}^5$ Lagrangian. Secondly, the Lagrangian is very involved and one need to consider some sort of simplifying limit which will be the study of the next section.

[1] Where we neglect the total derivative $p_+\,\dot{x}^-$ term.

5.2 Strong coupling expansion

We are now in position to expand the Lagrangian utilizing a strong coupling expansion. For example, scaling the fields as

$$x^m \to \frac{x^m}{\sqrt{g}}, \qquad p_m \to \frac{x_m}{\sqrt{g}}, \qquad \eta \to \frac{\eta}{\sqrt{g}}, \qquad g, r \to \infty, \qquad p_+ = 1 \tag{5.15}$$

where r is the radius of the string worldsheet, gives an adequate expansion scheme. Or we could consider an equivalent expansion by substituting g in favor for P_+ as

$$\begin{aligned} & x_m \to \sqrt{\frac{2}{P_+}}\, x^m, \qquad p_m \to \sqrt{\frac{P_+}{2}}\, p_m, \qquad \eta \to \sqrt{\frac{2}{P_+}}\, \eta, \\ & \widetilde{\lambda} = \frac{4\,g}{P_+^2} = \text{fix}, \qquad \sigma \to \frac{1}{g}\sigma, \qquad P_+ \to \infty. \end{aligned} \tag{5.16}$$

Both these expansion schemes are, as explained in section 4.1, equivalent and in practice boils down to an expansion in even powers of the transverse fields **?**. The rescaling of the string length parameter in the second expansion, was done to eliminate the factor of g in front of the Lagrangian. This allows us to establish a direct connection to the work **?** which will be the starting point in the next section.

Using either of the two expansion parameters above, it is straight forward, but notoriously tedious, to expand the Lagrangian to any order in fields. However, the leading order quadratic piece is rather easily obtained. If we introduce a complex combination of the bosons, so that the invariance under four $\mathrm{U}(1) \subset \mathrm{SU}(2)^4$ becomes manifest, as

$$\begin{aligned} & X_1 = x_2 + i x_1, \qquad X_2 = x_4 + i x_3, \qquad X_4 = X_1^\dagger, \qquad X_3 = X_2^\dagger, \\ & P_1^x = \frac{1}{2}(p_2^x + i p_1^x), \qquad P_2^x = \frac{1}{2}(p_4^x + i p_3^x), \qquad P_4^x = (P_1^x)^\dagger, \qquad P_3^x = (P_2^x)^\dagger, \end{aligned} \tag{5.17}$$

where x is either Z or Y, then the quadratic Lagrangian is given by

$$\begin{aligned} \mathscr{L}_2 &= P^z \cdot \dot{Z}^\dagger + (P^z)^\dagger \cdot \dot{Z} + P^y \cdot \dot{Y}^\dagger + (P^y)^\dagger \cdot \dot{Y} + i(\eta_a^\dagger\, \dot{\eta}_a + \theta_a^\dagger\, \dot{\theta}_a) \\ &- ((P^z)^\dagger \cdot P^z + (P^y)^\dagger \cdot P^y + \frac{1}{4} Z^\dagger \cdot Z + \frac{1}{4} Y^\dagger \cdot Y + \frac{\widetilde{\lambda}}{4} Z'^\dagger \cdot Z' + \frac{\widetilde{\lambda}}{4} Y^\dagger \cdot Y') \\ &- \frac{1}{2} \sum_{a=1}^{2} (\eta_a^\dagger\, \eta_{5-a} + \theta_a^\dagger\, \theta_{5-a} + \frac{\kappa\sqrt{\widetilde{\lambda}}}{2}(\eta_a\, \eta'_{5-a} + \theta_a\, \theta'_{5-a} - \eta_a^\dagger\, \eta'^\dagger_{5-a} - \theta_a^\dagger\, \theta'^\dagger_{5-a})), \end{aligned} \tag{5.18}$$

which is a free theory consisting of $8_B + 8_F$ massive excitations.

We also need to consider the constraint C_1 in (4.100), which to quadratic order gives

$$\Delta x'^- = \int d\sigma (p_m\, x'^m - \frac{i}{2} Str\, \Sigma_+ \eta\, \eta') = 0, \tag{5.19}$$

this is the so called level matching constraint enforces that the worldsheet momentum is zero.

Having established the quadratic theory, we naturally want to extend the analysis to higher orders in number of fields. However, we run into a problem immediately by noticing that the kinetic term in (5.13) contains higher order fermionic derivatives. Since we are about to calculate energy shifts of string configurations, the presence of higher order kinetic terms is very unpleasant since they induce corrections to the Poisson structure of the theory. A simple Poisson structure, which is promoted to commutation relations in the quantum theory, is essential when evaluating the matrix elements of the perturbation Hamiltonian. Having to deal with higher order corrections severely involves the already involved computations, see for example **?**. Luckily, one can avoid this complication by performing a shift of the fermions. If we focus on the fermionic part of the quadratic Lagrangian, written in terms of the matrix η, we have

$$\mathscr{L}^2_F = \frac{i}{2} Str\, \Sigma_+ \,\dot{\eta}\, \eta - \frac{1}{2} Str\, \eta\, \eta + \frac{\kappa}{2} Str\, \Sigma_+ \,\eta\, \Upsilon\, \eta'^{st}\, \Upsilon. \tag{5.20}$$

The idea is now to shift the fermions as

$$\eta \to \eta + \Phi, \tag{5.21}$$

where Φ is cubic in fields, so that

$$\frac{i}{2} Str\, \Sigma_+ (\dot{\eta}\, \Phi - \Phi\, \dot{\eta}) = -\mathscr{L}^4_{kin}, \tag{5.22}$$

and thus removes the higher order terms involving derivatives of the fermionic coordinates. However, as is clear from the remaining quadratic terms, this shifts induces additional quartic terms through

$$\mathscr{L}^4_{add} = -Str\, \Phi\, \eta + \kappa\, Str\, \Sigma_+ \,\Phi\, \Upsilon\, \eta'^{st}\, \Upsilon. \tag{5.23}$$

We will not specify the exact form of these terms here but merely state that they actually simplify the original quartic Hamiltonian in (5.13), for details see **?**.

With the fermionic shift we now have a quadratic kinetic theory and the quartic theory is fully governed by the light-cone Hamiltonian. Before we present it however, lets introduce a field decomposition in terms of oscillators that diagonalizes the quadratic Lagrangian.

For the eight complex bosonic fields, following the notation of **?**, we use the following decompositions

$$\begin{aligned}
Z_a(\tau,\sigma) =& \sum_n e^{in\sigma} Z_{a,n}(\tau) & P^z_a(\tau,\sigma) =& \sum_n e^{in\sigma} P^z_{a,n}(\tau) \\
Z_{a,n} =& \frac{1}{i\sqrt{\omega_n}}(\beta^+_{a,n} - \beta^-_{5-a,-n}) & P^z_{a,n} =& \frac{\sqrt{\omega_n}}{2}(\beta^+_{a,n} + \beta^-_{5-a,-n}) \\
Y_a(\tau,\sigma) =& \sum_n e^{in\sigma} Y_{a,n}(\tau) & P^y_a(\tau,\sigma) =& \sum_n e^{in\sigma} P^y_{a,n}(\tau) \\
Y_{a,n} =& \frac{1}{i\sqrt{\omega_n}}(\alpha^+_{a,n} - \alpha^-_{5-a,-n}) & P^y_{a,n} =& \frac{\sqrt{\omega_n}}{2}(\alpha^+_{a,n} + \alpha^-_{5-a,-n})\,,
\end{aligned} \tag{5.24}$$

where the frequency ω_n is defined as

$$\omega_n = \sqrt{1 + \tilde{\lambda}\, n^2}\,. \tag{5.25}$$

The decomposition has been chosen so that the creation and annihilation operators obey canonical commutation relations

$$[\alpha^-_{a,n}, \alpha^+_{b,m}] = \delta_{a,b}\, \delta_{n,m} = [\beta^-_{a,n}, \beta^+_{b,m}], \tag{5.26}$$

where $a \in \{1,2,3,4\}$ is the flavor index and n,m are the mode numbers which, from (5.19), are subject to the level matching condition

$$\sum_{j=1}^{K_4} m_j = 0\,, \tag{5.27}$$

where K_4 denotes the total number of excitations. The mode decompositions for the fermions are

$$\begin{aligned}
\eta(\tau,\sigma) =& \sum_n e^{in\sigma} \eta_n(\tau) & \theta(\tau,\sigma) =& \sum_n e^{in\sigma} \theta_n(\tau) \\
\eta_n =& f_n \eta^-_{-n} + i g_n \eta^+_n & \theta_n =& f_n \theta^-_{-n} + i g_n \theta^+_n
\end{aligned} \tag{5.28}$$

$$\text{with} \qquad \eta^-_k = \eta^-_{a,k}\Gamma_{5-a}\,, \quad \eta^+_k = \eta^+_{a,k}\Gamma_a\,, \quad \theta^-_k = \eta^-_{a,k}\Gamma_{5-a}\,, \quad \theta^+_k = \eta^+_{a,k}\Gamma_a\,. \tag{5.29}$$

The functions f_m and g_m above are defined as

$$f_m = \sqrt{\frac{1}{2}(1 + \frac{1}{\omega_m})}, \quad g_m = \frac{\kappa\sqrt{\tilde{\lambda}} m}{1 + \omega_m} f_m. \tag{5.30}$$

The anti-commutators between the fermionic mode operators are then

$$\{\eta^-_{a,n}, \eta^+_{b,m}\} = \delta_{a,b}\, \delta_{n,m} = \{\theta^-_{a,n}, \theta^+_{b,m}\}\,. \tag{5.31}$$

Using this oscillator representation, the leading order Hamiltonian becomes

$$\mathcal{H}_2 = \sum_n \omega_n (\theta^+_{a,n}\theta^-_{a,n} + \eta^+_{a,n}\eta^-_{a,n} + \beta^+_{a,n}\beta^-_{a,n} + \alpha^+_{a,n}\alpha^-_{a,n}) \,. \tag{5.32}$$

It is a tedious but straight forward task to derive the quartic Hamiltonian from (5.13) and it is given by[2] **?**

$$\mathcal{H}_4 = \mathcal{H}_{bb} + \mathcal{H}_{bf} + \mathcal{H}_{ff}(\theta) - \mathcal{H}_{ff}(\eta), \tag{5.33}$$

with

$$\mathcal{H}_{bb} = \frac{\widetilde{\lambda}}{4}(Y'_{5-a}Y'_a Z_{5-b}Z_b - Y_{5-a}Y_a Z'_{5-b}Z'_b + Z'_{5-a}Z'_a Z_{5-b}Z_b - Y'_{5-a}Y'_a Y_{5-b}Y_b) \tag{5.34}$$

$$\begin{aligned}
\mathcal{H}_{bf} = \frac{\widetilde{\lambda}}{4}\mathrm{tr}\Big[\; & (Z_{5-a}Z_a - Y_{5-a}Y_a)(\eta'^\dagger\eta' + \theta'^\dagger\theta') \\
& - Z'_a Z_b [\Gamma_a, \Gamma_b] \left(\mathcal{P}_+(\eta\eta'^\dagger - \eta'\eta^\dagger) - \mathcal{P}_-(\theta^\dagger\theta' - \theta'^\dagger\theta)\right) \\
& + Y'_a Y'_b [\Gamma_a, \Gamma_b] \left(-\mathcal{P}_-(\eta^\dagger\eta' - \eta'^\dagger\eta) - \mathcal{P}_+(\theta\theta'^\dagger - \theta'\theta^\dagger)\right) \\
& - \frac{i\kappa}{\sqrt{\widetilde{\lambda}}}(Z_a P^z_b)'[\Gamma_a, \Gamma_b] \left(\mathcal{P}_+(\eta^\dagger\eta^\dagger + \eta\eta) + \mathcal{P}_-(\theta^\dagger\theta^\dagger + \theta\theta)\right) \\
& + \frac{i\kappa}{\sqrt{\widetilde{\lambda}}}(Y_a P^y_b)'[\Gamma_a, \Gamma_b] \left(\mathcal{P}_-(\eta^\dagger\eta^\dagger + \eta\eta) + \mathcal{P}_+(\theta^\dagger\theta^\dagger + \theta\theta)\right) \\
& + 8i Z_a Y_b \left(-\mathcal{P}_-\Gamma_a\eta'\Gamma_b\theta' + \mathcal{P}_+\Gamma_a\theta'^\dagger\Gamma_b\eta'^\dagger\right) \; \Big]
\end{aligned} \tag{5.35}$$

$$\mathcal{H}_{ff}(\eta) = \frac{\widetilde{\lambda}}{4}\mathrm{tr}\Big[\Gamma_5 \left(\eta'^\dagger\eta\eta'^\dagger\eta + \eta^\dagger\eta'\eta^\dagger\eta' + \eta'^\dagger\eta^\dagger\eta'^\dagger\eta^\dagger + \eta'\eta\eta'\eta\right)\Big] \,. \tag{5.36}$$

This is the Hamiltonian for which we will determine the energy shifts δP_- of the free, degenerate eigenstates $|\psi_{0,n}\rangle$ with $\mathcal{H}_2 |\psi_{0,n}\rangle = -(P_-)_0 |\psi_{0,n}\rangle$ by diagonalizing the matrix $\langle\psi_{0,n}| \mathcal{H}_4 |\psi_{0,m}\rangle$. These will then be compared to the energies resulting from the proposed light-cone Bethe equations. Due to the complexity of the Hamiltonian it is often hard to obtain analytical results for these energy shifts in larger sectors with more than a few number of excitations. We will then have to resort to numerical considerations.

5.3 The light-cone Bethe equations

In an inspiring paper **?** the long range gauge and string theory Bethe equations were proposed for the full $\mathfrak{psu}(2,2|4)$ sector, generalizing the equations for the SU(2) sector in (3.23). This proposal was based on a coordinate space, nested Bethe ansatz of the smaller $\mathfrak{su}(1,1|2)$ sector, a construction later on **?** generalized to $\mathfrak{su}(2|3)$.

[2] We have removed some quartic bosonic non derivative term through a canonical transformation. For details, see either **?** or **?**.

We shall start our analysis from the full set of $\mathfrak{psu}(2,2|4)$ Bethe equations proposed in **?** in table 5 and adapt them to a language suitable for the light-cone gauge and large P_+ expansion. This will set the basis for the subsequent comparison to the explicit diagonalization of the worldsheet Hamiltonian (6.10).

The proposed set of Bethe equations for the spectral parameters $x_{i,k}$ of Beisert and Staudacher **?** for the full model can be brought into the form

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,k}^+}{x_{4,k}^-} \tag{5.37}$$

$$1 = \prod_{j=1 j\neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i\eta_1}{u_{2,k} - u_{2,j} + i\eta_1} \prod_{j=1}^{K_3+K_1} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}\eta_1}{u_{2,k} - u_{3,j} - \frac{i}{2}\eta_1} \tag{5.38}$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}\eta_1}{u_{3,k} - u_{2,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{x_{4,j}^{+\eta_1} - x_{3,k}}{x_{4,j}^{-\eta_1} - x_{3,k}} \tag{5.39}$$

$$\begin{aligned} 1 = \Big(\frac{x_{4,k}^-}{x_{4,k}^+}\Big)^{L-\eta_1 K_1 - \eta_2 K_7} & \prod_{j=1 j\neq k}^{K_4} \Big(\frac{x_{4,k}^{+\eta_1} - x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2} - x_{4,j}^{+\eta_2}} \frac{1 - g^2/(x_{4,k}^+ x_{4,j}^-)}{1 - g^2/(x_{4,k}^- x_{4,j}^+)} S_0^2\Big) \\ & \times \prod_{j=1}^{K_3+K_1} \frac{x_{4,k}^{-\eta_1} - x_{3,j}}{x_{4,k}^{+\eta_1} - x_{3,j}} \prod_{j=1}^{K_5+K_7} \frac{x_{4,k}^{-\eta_2} - x_{5,j}}{x_{4,k}^{+\eta_2} - x_{5,j}} \end{aligned} \tag{5.40}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}\eta_2}{u_{5,k} - u_{6,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{x_{4,j}^{+\eta_2} - x_{5,k}}{x_{4,j}^{-\eta_2} - x_{5,k}} \tag{5.41}$$

$$1 = \prod_{j=1 j\neq k}^{K_6} \frac{u_{6,k} - u_{6,j} - i\eta_2}{u_{6,k} - u_{6,j} + i\eta_2} \prod_{j=1}^{K_5+K_7} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}\eta_2}{u_{6,k} - u_{5,j} - \frac{i}{2}\eta_2} . \tag{5.42}$$

In the above the variables $u_{i,k}$ are defined by $u_{i,k} = x_{i,k} + g^2 \frac{1}{x_{i,k}}$ and the Bethe roots $x_{n,k}$ come with the multiplicities

$$\begin{aligned} & x_{2,k} : k = 1, \ldots, K_2 \qquad x_{3,k} : k = 1, \ldots, (K_1 + K_3) \qquad x_{4,k}^{\pm} : k = 1, \ldots K_4 \\ & x_{5,k} : k = 1, \ldots, (K_5 + K_7) \qquad x_{6,k} : k = 1, \ldots, K_6 \end{aligned} \tag{5.43}$$

Moreover the spectral parameters $x_{4,k}^{\pm}$ are related to the magnon momenta p_k via

$$x_{4,k}^{\pm} = \frac{1}{4}(\cot\frac{p_k}{2} \pm i)\Big(1 + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_k}{2}}\Big), \tag{5.44}$$

which relates to the coordinates used in (3.30) as

$$x^{\pm} = x(u(p_k) \pm \frac{i}{2}), \tag{5.45}$$

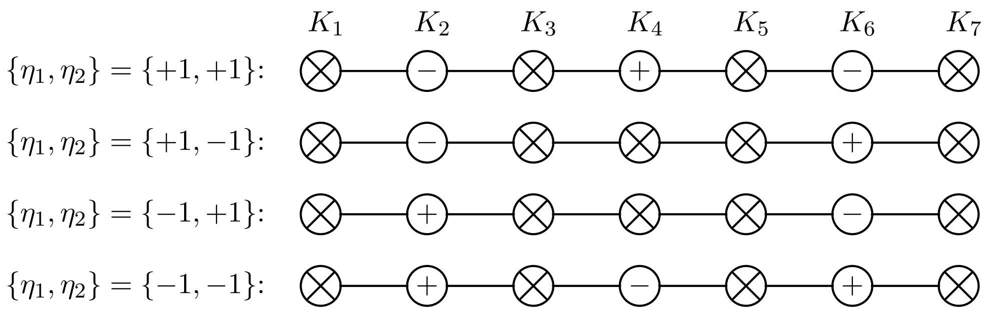

Figure 5.1: Four different choices of Dynkin diagrams of $\mathfrak{su}(2,2|4)$ specified by the grading η_1 and η_2. The signs in the white nodes indicate the sign of the diagonal elements of the Cartan matrix **?**.

where

$$x(u) = \frac{1}{2}u(1 + \sqrt{1 - \frac{2g^2}{u^2}}). \tag{5.46}$$

In the above we have also rescaled the coupling as

$$g := \frac{g}{4\pi} = \frac{\sqrt{\lambda}P_+}{8\pi}\,. \tag{5.47}$$

Note that we have chosen to write down the Bethe equations in a more compact “dynamically” transformed language. In order to convert (5.37)-(5.42) to the form found in table 5 of Beisert and Staudacher **?** one introduces the K_1 resp. K_7 roots $x_{1,k}$ and $x_{7,k}$ by splitting off the ‘upper’ $x_{3,k}$ and $x_{5,k}$ roots via

$$x_{1,k} := g^2/x_{3,K_3+k} \quad k = 1, \ldots K_1 \qquad x_{7,k} := g^2/x_{5,K_5+k} \quad k = 1, \ldots K_7\,. \tag{5.48}$$

This coordinate renaming unfolds the equations associated to the fermionic roots (5.38) and (5.41) into two structurally new sets of K_1 and K_7 equations and removes the K_1 and K_7 dependent exponent in the central equation (5.40).

The first equation (5.37) of the form we will be using is the cyclicity constraint on the total momentum of the spin chain. The following $K_2 + (K_1 + K_3) + K_4 + (K_5 + K_7) + K_6$ equations in (5.38)-(5.42) determine the sets of Bethe roots $\{x_{2,k}, x_{3,k}, x^{\pm}_{4,k}, x_{5,k}, x_{6,k}\}$. Let us stress once more that it is only the combinations $(K_1 + K_3)$ and $(K_5 + K_7)$ which enter in the Bethe equations. Moreover the gradings η_1 and η_2 take the values ± 1 corresponding to four different choices of Dynkin diagrams for $\mathfrak{psu}(2,2|4)$ as discussed in **?** see figure 1.

These four different choices of diagrams can be traced back to the derivation of the nested Bethe ansatz in the $\mathfrak{su}(1,1|2)$ sector in the gauge theory spin chain language. In this sector there are four distinct excitations placed on a vacuum of Z fields. These four excitations are the two bosonic Y and DZ fields and the two fermionic $\mathcal{U}$ and $\dot{\mathcal{U}}$ fields. In the nested Bethe ansatz **?** one selects one out of these four excitations as a second effective vacuum of a shorter spin chain, after having eliminated all the sites Z from the original chain. Depending on this choice η_1, η_2 take the values ± 1.

Finally, the undetermined function S_0^2 in (5.40) is the famous scalar dressing factor which is conjectured to take the form $S_0^2 = S_0^2(x_{4,k}, x_{4,j}) = e^{2i\theta(x_{4,k},x_{4,j})}$ **?**, where

$$\theta(x_{4,k}, x_{4,j}) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(g) \left[q_r(x_{4,k}^{\pm})\, q_s(x_{4,j}^{\pm}) - q_r(x_{4,j}^{\pm})\, q_s(x_{4,k}^{\pm}) \right] \tag{5.49}$$

with the local conserved charge densities

$$q_r(x^{\pm}) = \frac{i}{r-1}\, g^{r-1} \left[\left(\frac{1}{x^+}\right)^{r-1} - \left(\frac{1}{x^-}\right)^{r-1} \right] \tag{5.50}$$

and to leading order

$$c_{r,s}(g) = g \left[\delta_{r+1,s} + \mathcal{O}(1/g) \right]. \tag{5.51}$$

In this thesis, we shall only be interested in this leading order contribution, the AFS phase **?**, where the phase factor may be summed **?** to yield

$$\begin{aligned} \theta_{kj} &= (x_j^+ - x_k^+)\, F(x_k^+ x_j^+) + (x_j^- - x_k^-)\, F(x_k^- x_j^-) \\ &\quad - (x_j^+ - x_k^-)\, F(x_k^- x_j^+) - (x_j^- - x_k^+)\, F(x_k^+ x_j^-)\,, \end{aligned} \tag{5.52}$$

with

$$F(a) = (1 - \frac{g^2}{a})\, \log(1 - \frac{g^2}{a})\,. \tag{5.53}$$

The string oscillator excitations are characterized by the values of four $U(1)$ charges (S_+, S_-, J_+, J_-) as introduced in **?**. They are related to the two spins $\{S_1, S_2\}$ on AdS_5 and two angular momenta $\{J_1, J_2\}$ on the S_5 via $S_\pm = S_1 \pm S_2$ and $J_\pm = J_1 \pm J_2$. The relationship between these and the excitation numbers $\{K_i\}$ in the Bethe equations are[3]

$$\begin{aligned} S_+ &= \eta_2\,(K_5 + K_7) - (1 + \eta_2)\, K_6 + \frac{1}{2}(1 - \eta_2)\, K_4, \\ S_- &= \eta_1\,(K_1 + K_3) - (1 + \eta_1)\, K_2 + \frac{1}{2}(1 - \eta_1)\, K_4, \\ J_+ &= -\eta_2\,(K_5 + K_7) - (1 - \eta_2)\, K_6 + \frac{1}{2}(1 + \eta_2)\, K_4, \\ J_- &= -\eta_1\,(K_1 + K_3) - (1 - \eta_1)\, K_2 + \frac{1}{2}(1 + \eta_1)\, K_4. \end{aligned}$$

Using these together with the (S_+, S_-, J_+, J_-) charge values for the string oscillators of table 1 (see also **?**) we can construct the excitation pattern for each oscillator, see table 5.1. For example, the excitations in the $\mathfrak{su}(1,1|2)$ sector correspond to the following

[3] To make a connection to **?**, we have $J_- = q_1, J_+ = q_2, S_- = s_1$ and $S_+ = s_2$. The two other charges, p and r are functions of the length of the spin chain, so in the large P_+ limit these are infinite.

	$\mathbf{K}_1 + \mathbf{K}_3$	$\mathbf{K}_2$	$\mathbf{K}_4$	$\mathbf{K}_6$	$\mathbf{K}_5 + \mathbf{K}_7$	S_+	S_-	J_+	J_-
α_1^+	$0+\frac{1}{2}(1-\eta_1)$	0	1	0	$\frac{1}{2}(1-\eta_2)+0$	0	0	1	1
α_2^+	$\frac{1}{2}(1+\eta_1)+1$	1	1	0	$\frac{1}{2}(1-\eta_2)+0$	0	0	1	-1
α_3^+	$0+\frac{1}{2}(1-\eta_1)$	0	1	1	$1+\frac{1}{2}(1+\eta_2)$	0	0	-1	1
α_4^+	$\frac{1}{2}(1+\eta_1)+1$	1	1	1	$1+\frac{1}{2}(1+\eta_2)$	0	0	-1	-1
β_1^+	$0+\frac{1}{2}(1+\eta_1)$	0	1	0	$\frac{1}{2}(1+\eta_2)+0$	1	1	0	0
β_2^+	$\frac{1}{2}(1-\eta_1)+1$	1	1	0	$\frac{1}{2}(1+\eta_2)+0$	1	-1	0	0
β_3^+	$0+\frac{1}{2}(1+\eta_1)$	0	1	1	$1+\frac{1}{2}(1-\eta_2)$	-1	1	0	0
β_4^+	$\frac{1}{2}(1-\eta_1)+1$	1	1	1	$1+\frac{1}{2}(1-\eta_2)$	-1	-1	0	0
θ_1^+	$0+\frac{1}{2}(1+\eta_1)$	0	1	0	$\frac{1}{2}(1-\eta_2)+0$	0	1	1	0
θ_2^+	$\frac{1}{2}(1-\eta_1)+1$	1	1	0	$\frac{1}{2}(1-\eta_2)+0$	0	-1	1	0
θ_3^+	$0+\frac{1}{2}(1+\eta_1)$	0	1	1	$1+\frac{1}{2}(1+\eta_2)$	0	1	-1	0
θ_4^+	$\frac{1}{2}(1-\eta_1)+1$	1	1	1	$1+\frac{1}{2}(1+\eta_2)$	0	-1	-1	0
η_1^+	$0+\frac{1}{2}(1-\eta_1)$	0	1	0	$\frac{1}{2}(1+\eta_2)+0$	1	0	0	1
η_2^+	$\frac{1}{2}(1+\eta_1)+1$	1	1	0	$\frac{1}{2}(1+\eta_2)+0$	1	0	0	-1
η_3^+	$0+\frac{1}{2}(1-\eta_1)$	0	1	1	$1+\frac{1}{2}(1-\eta_2)$	-1	0	0	1
η_4^+	$\frac{1}{2}(1+\eta_1)+1$	1	1	1	$1+\frac{1}{2}(1-\eta_2)$	-1	0	0	-1

Table 5.1: The translation scheme of string oscillator excitations to the Dynkin node excitation numbers of the Bethe equations. We have also listed the space-time $U(1)$ charges $J_\pm$ and $S_\pm$ of the string oscillators. From this table we easily see which operators represent the middle node for the different choices of gradings. That is, $(\eta_1, \eta_1) = (+,+) : \alpha_1^+$, $(-,+) : \theta_1^+$, $(+,-) : \eta_1^+$ and $(-,-) : \beta_1^+$.

string oscillators,

$$Y \doteq \alpha_1^+, \quad DZ \doteq \beta_1^+, \quad \mathcal{U} \doteq \theta_1^+, \quad \dot{\mathcal{U}} \doteq \eta_1^+. \tag{5.54}$$

These are the four fields which are picked out as a new vacuum in the smaller spin chains by specifying the values[4] of the gradings η_1 and η_2. The vacuum of Z fields corresponds to the string ground state $|0\rangle$ with charge J.

Let us stress that in the dictionary of table 5.1 a single string oscillator excitation does not corresponds to a single Dynkin node excitation, but rather to a five component excitation vector, with uniform $K_4 = 1$ entry. This is how the naive mismatch of 16 string oscillators versus 7 (or better 4) Dynkin node excitations is resolved: One should think of a string oscillator as being indexed by the space-time charge vector (S_+, S_-, J_+, J_-) *or* by the Dynkin vector $(K_1 + K_3, K_2, K_6, K_5 + K_7)$. These two labelling are equivalent and the one-to-one map between them is given in (5.54).

There are several things we need to do in order to translate the Bethe equations (5.37)-

[4] The field that is picked as the second vacuum in the nested Bethe ansatz only excites the middle node of the Dynkin diagram, so one immediately sees from the table which combinations of the gradings correspond to which choice of vacuum.

(5.42) into their light-cone form in order to make a direct comparison to uniform light-cone gauged, near plane-wave string theory. First of all, since the light-cone Hamiltonian is expanded in the large P_+ limit we need to express L in (5.40) in terms of the light-cone momenta. This can be done by using the expression for the eigenvalues of the dilatation operator and the J charge of S^5 **?**,

$$\begin{aligned} J &= L + \frac{1}{2}\eta_1(K_3 - K_1) - \frac{1}{4}(2 + \eta_1 + \eta_2)K_4 + \frac{1}{2}\eta_2(K_5 - K_7), \\ D &= L + \frac{1}{2}\eta_1(K_3 - K_1) + \frac{1}{4}(2 - \eta_1 - \eta_2)K_4 + \frac{1}{2}\eta_2(K_5 - K_7) + \delta D, \end{aligned} \tag{5.55}$$

where the anomalous dimension δD reads

$$\delta D = 2g^2 \sum_{j=1}^{K_4} \Big(\frac{i}{x_{4,j}^+} - \frac{i}{x_{4,j}^-} \Big) , \tag{5.56}$$

Using (5.55) we can write the light-cone momenta and energy as,

$$\begin{aligned} P_+ &= D + J \\ &= 2L + \eta_1(K_3 - K_1) - \frac{1}{2}(\eta_1 + \eta_2)K_4 + \eta_2(K_5 - K_7) + \delta D \\ P_- &= J - D = -K_4 - \delta D \, . \end{aligned} \tag{5.57}$$

Hence we see that the large P_+ limit discussed in the previous section corresponds to an infinitely long chain with a finite number of excitations. Using this, the central K_4 Bethe equations (5.40) become

$$\begin{aligned} &\Big(\frac{x_{4,k}^+}{x_{4,k}^-}\Big)^{\frac{1}{2}P_+} = \Big(\frac{x_{4,k}^-}{x_{4,k}^+}\Big)^{\frac{1}{2}(\frac{1}{2}(\eta_1+\eta_2)K_4 - \eta_1(K_1+K_3) - \eta_2(K_5+K_7) - \delta D)} \\ &\times \prod_{j=1 j\neq k}^{K_4} \Big(\frac{x_{4,k}^{+\eta_1} - x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2} - x_{4,j}^{+\eta_2}} \frac{1 - g^2/(x_{4,k}^+ x_{4,j}^-)}{1 - g^2/(x_{4,k}^- x_{4,j}^+)} S_0^2 \Big) \prod_{j=1}^{K_3+K_1} \frac{x_{4,k}^{-\eta_1} - x_{3,j}}{x_{4,k}^{+\eta_1} - x_{3,j}} \prod_{j=1}^{K_5+K_7} \frac{x_{4,k}^{-\eta_2} - x_{5,j}}{x_{4,k}^{+\eta_2} - x_{5,j}} . \end{aligned} \tag{5.58}$$

We want to compare the spectrum up to $\mathcal{O}(\frac{1}{P_+^2})$ and to this order a nice thing happens. As a matter of fact, one can show using only the leading AFS piece of (5.51) that

$$\Big(\frac{x_{4,k}^-}{x_{4,k}^+}\Big)^{-\frac{1}{2}\delta D} \prod_{j=1 j\neq k}^{K_4} \Big(\frac{1 - g^2/(x_{4,k}^+ x_{4,j}^-)}{1 - g^2/(x_{4,k}^- x_{4,j}^+)} S_0^2 \Big) = 1 + \mathcal{O}(\frac{1}{P_+^3}) \tag{5.59}$$

holds, once one inserts the large P_+ expansion of p_k (to be established in (5.61) and (5.63)) as well as the relevant leading AFS contribution to the dressing factor S_0 of (5.51). Curiously enough, not only the $1/P_+$ contribution, but also the $1/P_+^2$ term vanishes in this expansion – the $1/P_+^3$ term is nonvanishing though. Therefore, to the order we are interested in, the light-cone Bethe equations are given by the previous equations of (5.37)-(5.42) with the central node K_4 Bethe equations (5.40) exchanged by the simpler dressing factor free form

$$\Big(\frac{x^+_{4,k}}{x^-_{4,k}}\Big)^{\frac{1}{2}P_+} = \Big(\frac{x^-_{4,k}}{x^+_{4,k}}\Big)^{\frac{1}{2}(\frac{1}{2}(\eta_1+\eta_2)K_4-\eta_1(K_1+K_3)-\eta_2(K_5+K_7))}$$
$$\times \prod_{j=1 j\neq k}^{K_4} \frac{x^{+\eta_1}_{4,k}-x^{-\eta_1}_{4,j}}{x^{-\eta_2}_{4,k}-x^{+\eta_2}_{4,j}} \prod_{j=1}^{K_3+K_1} \frac{x^{-\eta_1}_{4,k}-x_{3,j}}{x^{+\eta_1}_{4,k}-x_{3,j}} \prod_{j=1}^{K_5+K_7} \frac{x^{-\eta_2}_{4,k}-x_{5,j}}{x^{+\eta_2}_{4,k}-x_{5,j}} + \mathcal{O}(\frac{1}{P_+^2})\,, \tag{5.60}$$

Putting all $K_j = 0$, for $j \neq 4$, we indeed reproduce the results for the rank one subsectors presented in **?**. This explains the simple form of the equations established there.

5.3.1 Large P_+ expansion

We will now explicitly expand the Bethe equations in the large P_+ limit. The mode numbers of the string oscillators will enter in the equations as the zero mode of the magnon momenta p_k. However, depending on if we are looking at a state with confluent mode numbers or not, the procedure is somewhat different. We will begin with the simpler case where all mode numbers are distinct.

Non-confluent mode numbers

For distinct mode numbers one assumes an expansion of p_k as **??**

$$p_k = \frac{p^0_k}{P_+} + \frac{p^1_k}{P_+^2}. \tag{5.61}$$

Determining the analogous expansion of $x^\pm_{4,k}$

$$x^\pm_{4,k} = P_+\, x^0_{4,k} + x^{1,\pm}_{4,k} + \dots\,, \tag{5.62}$$

where

$$x^0_{4,k} = \frac{1+\omega_k}{2p^0_k}, \quad x^{1,\pm}_{4,k} = \frac{1}{4}(1+\omega_k)\Big(\pm i - \frac{2p^1_k}{(p^0_k)^2\,\omega_k}\Big)\,, \tag{5.63}$$

and $\omega_k = \sqrt{1+\tilde\lambda\frac{(p^0_k)^2}{16\pi^2}}$. Consistency then implies that the spectral parameters $x_{3,k}$ and $x_{5,k}$ have the expansion[5]

$$x_{3,k} = P_+\, x^0_{3,k} + x^1_{3,k} + \dots, \qquad x_{5,k} = P_+\, x^0_{5,k} + x^1_{5,k} + \dots\,. \tag{5.64}$$

[5] The expansion of $x_{3,k}$ and $x_{5,k}$ remains the same in the case of confluent mode numbers, while the expansion of $x^\pm_{4,k}$ differs.

Taking the logarithm of (5.60) and expanding we find that the momentum at leading order p_k^0 in (5.61) satisfy

$$p_k^0 = 4\pi m_k, \qquad m_k \in \mathbb{Z}, \tag{5.65}$$

the integer here is what will correspond to the mode numbers of the string oscillators. Expanding (5.60) to the next order we find that the p_k^1 should satisfy

$$\begin{aligned} p_k^1 = &\ \frac{1}{2}(\eta_1+\eta_2)\sum_{j=1 j\neq k}^{K_4}\frac{2+\omega_k+\omega_j}{x_{4,k}^0-x_{4,j}^0}-\eta_1\sum_{j=1}^{K_1+K_3}\frac{1+\omega_k}{x_{4,k}^0-x_{3,j}^0} \\ &-\eta_2\sum_{j=1}^{K_5+K_7}\frac{1+\omega_k}{x_{4,k}^0-x_{5,j}^0}-(\frac{1}{2}(\eta_1+\eta_2)K_4-\eta_1(K_1+K_3)-\eta_2(K_5+K_7))p_k^0. \end{aligned} \tag{5.66}$$

We also want to expand the light-cone energy (5.57), using (5.56) and (5.44) we find

$$P_- = -\sum_{k=1}^{K_4}\omega_k + \delta P_-, \tag{5.67}$$

where the energy shift, δP_-, is given by

$$\delta P_- = -\frac{\tilde{\lambda}}{P_+}\frac{1}{16\pi^2}\sum_{k=1}^{K_4}\frac{p_k^0 p_k^1}{\omega_k}. \tag{5.68}$$

Confluent mode numbers

For the case of confluent mode numbers we run into trouble because of the zero denominator in (5.66), which is caused by the term

$$\prod_{j=1 j\neq k}^{K_4}\frac{x_{4,k}^{+\eta_1}-x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2}-x_{4,j}^{+\eta_2}} \tag{5.69}$$

of (5.60). One could try to only look at the case with the gradings chosen so that $\pm\eta_1 = \mp\eta_2$. However, this would mean that we pick a fermionic vacuum in the nested Bethe ansatz and since the rapidities $x_{4,k}$ are degenerate, we end up with zero. So for the case of confluent mode numbers we are forced to pick $\eta_1 = \eta_2$.

The way to proceed is to assume an expansion of p_k as **?**,

$$p_k = \frac{p_k^0}{P_+} + \frac{p_{k,l_k}^1}{P_+^{3/2}} + \frac{p_{k,l_k}^2}{P_+^2}. \tag{5.70}$$

Where we, following **?**, denote the multiplicity of the degeneracy as ν_k so $\sum_{k=1}^{K_4'}\nu_k = K_4$ and $\sum_{k=1}^{K_4'}\nu_k m_k = 0$, where K_4' is the number of distinct mode numbers. The first order term in (5.70) is degenerate for confluent mode numbers while for the higher order terms the degeneracy might be lifted ($l_k \in \{1, 2, ..., \nu_k\}$).

Using (5.70) the energy shift will decompose as

$$\delta P_- = \sum_{k=1}^{K_4'} \sum_{l_k=1}^{\nu_k} \delta P_{-,k,l_k}. \tag{5.71}$$

The contribution from mode numbers m_j with $\nu_j = 1$ look the same as in (5.68) while modes m_k with $\nu_k > 1$ will have contribution from p^1_{k,l_k}. Using (5.70) and expanding (5.69) we find that p^1_{k,l_k} satisfy a Stieltjes equation **?** of the form **?**

$$p^1_{k,l_k} = -2(\eta_1 + \eta_2)(p^0_k)^2\omega_k \sum_{\mu_k=1 \mu_k \neq l_k}^{\nu_k} \frac{1}{p^1_{k,l_k} - p^1_{k,\mu_k}}. \tag{5.72}$$

It is useful to note that $\sum_{l_k=1}^{\nu_k} p^1_{k,l_k} = 0$. The momenta p^1_{k,l_k} can be written as

$$(p^1_{k,l_k})^2 = -2\,(\eta_1 + \eta_2)\,(p^0_k)^2\,\omega_k\, h^2_{\nu_k,l_k} \qquad \text{with} \quad l_k = 1, ..., \nu_k \tag{5.73}$$

where h_{ν_k,l_k} are the ν_k roots of Hermite polynomials of degree ν_k. However, the explicit solutions h_{ν_k,l_k} are not needed since when summing over k the following property applies

$$\sum_{l_k=1}^{\nu_k} (h_{\nu_k,l_k})^2 = \frac{\nu_k(\nu_k - 1)}{2}. \tag{5.74}$$

The expansion for the second order contribution p^2_{k,l_k} in (5.70) is considerably more complicated, we therefore refer only to its general structure

$$p^2_{k,l_k} = \tilde{p}^2_k + \sum_{\mu_k=1 \mu_k \neq l_k}^{\nu_k} f_k(\mu_k, l_k)\,. \tag{5.75}$$

We split p^2_{k,l_k} into a part not depending on l_k, which is equivalent to p^1_k given in (5.66): $\tilde{p}^2_k \equiv p^1_k$. The function f_k has the property $f_k(\mu_k, l_k) = -f_k(l_k, \mu_k)$ and thus the second term drops out when summed over l_k. The final expression for the energy shift becomes then

$$\begin{aligned}\delta P_- &= -\frac{1}{P_+}\frac{\tilde{\lambda}}{16\pi^2} \sum_{k=1}^{K_4'} \sum_{l_k=1}^{\nu_k} \frac{\frac{1}{2}(p^1_{k,l_k})^2 + p^0_k \omega^2_k p^2_{k,l_k}}{\omega^3_k} \\ &= -\frac{1}{P_+}\frac{\tilde{\lambda}}{32\pi^2} \sum_{k=1}^{K_4'} \nu_k p^0_k \Big(\frac{2\tilde{p}^2_k\omega_k - (\eta_1 + \eta_2)p^0_k(\nu_k - 1)}{\omega^2_k}\Big).\end{aligned} \tag{5.76}$$

Bethe equations for the smaller spin chains

To be able to solve for p_k^1 it is clear from the form of (5.66) that we need the values of the Bethe roots $x_{3,k}$ and $x_{5,k}$ at leading order in P_+. Note that the variables u_k scale as $u_k = P_+ u_k^0 + u_k^1 + \ldots$. Expanding (5.38), (5.39), (5.41) and (5.42) yields

$$
\begin{aligned}
0 &= \sum_{j=1 j\neq k}^{K_2} \frac{2}{u_{2,j}^0 - u_{2,k}^0} + \sum_{j=1}^{K_1+K_3} \frac{1}{u_{2,k}^0 - (x_{3,j}^0 + \frac{\tilde{\lambda}}{64\pi^2}\frac{1}{x_{3,j}^0})}\,, \\
0 &= \eta_1 \sum_{j=1}^{K_2} \frac{1}{x_{3,k}^0 + \frac{\tilde{\lambda}}{64\pi^2}\frac{1}{x_{3,k}^0} - u_{2,j}^0} + \frac{1}{2}\sum_{j=1}^{K_4} \frac{1+\omega_j}{x_{4,j}^0 - x_{3,k}^0}\,, \\
0 &= \eta_2 \sum_{j=1}^{K_6} \frac{1}{x_{5,k}^0 + \frac{\tilde{\lambda}}{64\pi^2}\frac{1}{x_{5,k}^0} - u_{6,j}^0} + \frac{1}{2}\sum_{j=1}^{K_4} \frac{1+\omega_j}{x_{4,j}^0 - x_{5,k}^0}\,, \\
0 &= \sum_{j=1 j\neq k}^{K_6} \frac{2}{u_{6,j}^0 - u_{6,k}^0} + \sum_{j=1}^{K_5+K_7} \frac{1}{u_{6,k}^0 - (x_{5,j}^0 + \frac{\tilde{\lambda}}{64\pi^2}\frac{1}{x_{5,j}^0})}\,,
\end{aligned}
\tag{5.77}
$$

which determine the $x_{2,k}^0$, $x_{3,k}^0$, $x_{5,k}^0$ and $x_{6,k}^0$ in terms of $x_{4,k}^0$. Note that the two sets of the first two and the last two equations are decoupled and identical in structure.

Let us briefly discuss how one goes about solving these equations for a given excitation sector. First one needs to commit oneself to a specific grading by specifying the numbers $\eta_{1,2} = \pm 1$. Then one reads off the values for $\{K_i\}$ in table 5.1 corresponding to the excitation pattern in question. The four different choices of gradings can be grouped into two classes, one with fermionic middle node, $\eta_1 = -\eta_2$, and one with bosonic middle node, $\eta_1 = \eta_2$ in the associated Dynkin diagram. The difference between the two is important in the case of confluent mode numbers. The K_3 and K_5 (and for $\eta_1 = -\eta_2$, also K_4) are fermionic nodes which means that the solutions for $x_{3,k}^0$ and similarly for $x_{5,k}^0$ for different values of k are not allowed to be degenerate by the Pauli principle.

Consider for example the $\mathfrak{su}(1,1|2)$ sector containing only nonvanishing values for $\{K_3, K_4, K_5\}$. Then, due to $K_2 = 0 = K_6$, the equations (5.77) condense to two identical, degree K_4 polynomial equations for $x_{3,k}^0$ and $x_{5,k}^0$ yielding K_4 solutions, including the degenerate solution $\{x_{3/5,k}^0 \to \infty\}$. These K_4 solutions are then used once on each node K_3 and K_5, each generating $\frac{K_4\,(K_4-1)\times\ldots\times(K_4-K_j)}{K_j!}$ (with $j = 3, 5$) number of solutions. For a bosonic node, however, we may pick the same solution repeatedly.

Having distributed the solutions for $x_{3,k}^0$ and $x_{5,k}^0$ one then determines p_k^1 from (5.66) and finally solves for the energy shift using (5.68) or (5.76). The obtained value is what we then compare with a direct diagonalization of the string Hamiltonian.

5.3.2 Comparing the Bethe equations with string theory

We have calculated the energy shifts (both analytically and numerically) for a large number of states. The numerical results will be presented in appendix 2, while here in the main text we shall focus on the analytical results. On the string theory side one studies the Hamiltonian in first order degenerate perturbation theory, which in practice demands the diagonalization of the Hamiltonian in the relevant subsectors. In the near plane-wave limit, this was first done in **?** using a different gauge.

General structure of solutions

We will present analytical results for three different sectors, $\mathfrak{su}(1|2)$, $\mathfrak{su}(1,1|2)$ and $\mathfrak{su}(2|3)$. The operators in each sector are

$$\mathfrak{su}(1|2): \quad \{\alpha_1^+, \theta_1^+\}, \quad \mathfrak{su}(1,1|2): \quad \{\alpha_1^+, \beta_1^+, \theta_1^+, \eta_1^+\}, \quad \mathfrak{su}(2|3): \quad \{\alpha_1^+, \alpha_2^+, \theta_1^+, \theta_2^+\}.$$

As we can see there is a mixing between the sectors, the $\mathfrak{su}(1|2)$ is contained within the larger $\mathfrak{su}(2|3)$ sector and in $\mathfrak{su}(1,1|2)$, but the latter is not a part of $\mathfrak{su}(2|3)$. When calculating the energy shifts, things are straightforward for the first two sectors, $\mathfrak{su}(1|2)$ and $\mathfrak{su}(1,1|2)$. The excited nodes are K_3, K_4 and K_5 and for these excitation numbers (5.77) is significantly simplified since there are no $u_{2,k}$ roots. Each $x_{3,k}$ and $x_{5,k}$ satisfy a $K_2 - \nu$ degree polynomial equation, where ν is the number of confluent mode numbers, which is the same for each value of k. However, this is not the case for the $\mathfrak{su}(2|3)$ sector where we have nonvanishing K_2 excitations and a resulting set of coupled polynomial equations for the $x_{2,k}$ and $x_{3,k}$ following from (5.77)

The $\mathfrak{su}(1|2)$ sector

As stated, this sector is spanned by the oscillators α_1^+ and θ_1^+. The contributing parts from the string Hamiltonian are $\mathcal{H}_{bb}$ and $\mathcal{H}_{bf}$. The explicit expression for the effective $\mathfrak{su}(1|2)$ Hamiltonian can be found in (8). Let us count the number of solutions for the grading $\eta_1 = \eta_2 = 1$. Then the only excited nodes of the Dynkin diagram in this sector are K_4 and K_3, so the polynomials in (5.77) give $K_4 - \nu$ solutions[6]. Two of these solutions are always 0 and ∞ while the other $K_4 - 2 - \nu$ are non-trivial. Before we perform the actual computation let us count the number of solutions. Say we have a total of K_3 θ_1^+ oscillators and $K_4 - K_3$ α_1^+ oscillators, then this state will yield $\frac{(K_4-\nu)\times(K_4-\nu-1)\times\ldots\times(K_4-\nu-K_3+1)}{K_3!}$ number of solutions. So, for all possible combinations of a general K_4 impurity state the number of solutions are

$$\sum_{K_3=0}^{K_4-\nu} \binom{K_4-\nu}{K_3} = 2^{K_4-\nu}. \tag{5.78}$$

Since the worldsheet Hamiltonian is a $2^{K_4-\nu} \times 2^{K_4-\nu}$ matrix, the number of solutions matches.

[6] The number of confluent mode numbers must satisfy, $\nu \leq K_4 - K_3 + 1$ since we cannot have fermionic excitations of the same flavor with confluent mode numbers.

Two impurities: For the two impurity sector the perturbative string Hamiltonian is a 4×4 matrix, but we are only interested in a 2×2 submatrix since the other part falls into the rank one sectors $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$. The relevant matrix elements, with mode numbers $\{q, -q\}$, are

$$\left(\begin{array}{c|c|c} & \alpha^+_{1,q}\theta^+_{1,-q}|0\rangle & \alpha^+_{1,-q}\theta^+_{1,q}|0\rangle \\ \hline \langle 0|\alpha^-_{1,q}\theta^-_{1,-q} & \mathcal{H}_{bf} & \mathcal{H}_{bf} \\ \hline \langle 0|\alpha^-_{1,-q}\theta^-_{1,q} & \mathcal{H}_{bf} & \mathcal{H}_{bf} \end{array}\right)$$

The energy shifts are the non-zero values in (10). Now, the interesting question is of course if we can reproduce this result from the Bethe equations. For the two impurity state $\alpha^+\theta^+|0\rangle$ it is easiest to work with the gradings[7] $\eta_1 = -1$ and $\eta_2 = 1$ where we have $K_4 = 2$ and $K_3 = 1$. From (5.77) wee see that the only solutions for $x_{3,k}$ are 0 and ∞. Since we have two roots, and one K_3 excitation we get two solutions for p^1_k. Solving (5.66) gives $p^1_k = \pm p^0_k$. Plugging these into (5.68) gives

$$\delta P_- = \pm \frac{\tilde{\lambda}}{P_+} \sum_{j=1}^{2} \frac{q_j^2}{\omega_{q_j}} = \pm 2 \frac{\tilde{\lambda}}{P_+} \frac{q^2}{\omega_q} =: \kappa_2, \tag{5.79}$$

which equals the non-zero values in (10).

Three impurities, distinct mode numbers: The full perturbative string Hamiltonian is a 8×8 matrix but the relevant $\mathfrak{su}(1|2)$ part splits up into two independent submatrices coming from the Fermi-Fermi matrix elements $\langle 0|\alpha^-_1\alpha^-_1\theta^-_1(\mathcal{H}_{bb} + \mathcal{H}_{bf})\theta^+_1\alpha^+_1\alpha^+_1|0\rangle$ and the Bose-Bose elements
$\langle 0|\alpha^-_1\theta^-_1\theta^-_1(\mathcal{H}_{bf})\theta^+_1\theta^+_1\alpha^+_1|0\rangle$. Schematically written we have,

$$\left(\begin{array}{c|c|c} & \alpha^+_1\alpha^+_1\theta^+_1|0\rangle & \alpha^+_1\theta^+_1\theta^+_1|0\rangle \\ \hline \langle 0|\theta^-_1\alpha^-_1\alpha^-_1 & (\mathcal{H}_{bb} + \mathcal{H}_{bf})^{3\times 3} & 0_{3\times 3} \\ \hline \langle 0|\theta^-_1\theta^-_1\alpha^-_1 & 0_{3\times 3} & \mathcal{H}^{3\times 3}_{bf} \end{array}\right) \tag{5.80}$$

The eigenvalues of the Bose-Bose submatrix, the bottom right, is given in (11). To reproduce these shifts from the Bethe equations we once again choose $\eta_1 = -1$ and $\eta_2 = 1$ so $K_4 = 3$ and $K_3 = 1$. Solving (5.77) give, as before, $x^0_{3,k} = \{0, \infty\}$ together with a novel third solution

$$y = \frac{(2 + \omega_{q_1} + \omega_{q_2})\, x^0_{4,3} + (2 + \omega_{q_2} + \omega_{q_3})\, x^0_{4,1} + (2 + \omega_{q_1} + \omega_{q_3})\, x^0_{4,2}}{3 + \omega_{q_1} + \omega_{q_2} + \omega_{q_3}} . \tag{5.81}$$

The first two solutions, 0 and ∞, give as before $p^1_k = \pm p^0_k$. For generic values of K_4, and with $K_3 = 1$, these two solutions will always appear. Using the third solution in (5.66)

[7] All choices of gradings of course give the same result, however, the calculation will be more or less complicated depending on the choice.

yields

$$p_k^1 = \frac{1+\omega_k}{x_{4,k}^0 - y} - p_k^0. \tag{5.82}$$

Plugging this into (5.68), together with some algebra, gives the three solutions

$$\delta P_- = \Big\{ \pm \frac{\tilde{\lambda}}{P_+} \sum_{j=1}^{3} \frac{q_j^2}{\omega_{q_j}}, \ \frac{\tilde{\lambda}}{P_+ \omega_{q_1}\omega_{q_2}\omega_{q_3}} \sum_{j=1}^{3} q_j^2 \omega_{q_j} \Big\} =: \Lambda_3 \,, \tag{5.83}$$

which agrees with the string result obtained in (11).

Let us now focus on the Fermi-Fermi matrix elements, the upper left 3×3 block of (5.80). First, (5.77) give the same three solutions as before, namely $\{0, \infty, y\}$ with the same y as in (5.81). Since $K_3 = 2$ we now, for each p_k^1, use two of the solutions for $x_{3,k}^0$

$$p_k^1 = (1 + \omega_{p_k^0})\Big(\frac{1}{x_{4,k}^0 - x_{3,1}^0} + \frac{1}{x_{4,k}^0 - x_{3,2}^0}\Big) - 2p_k^0. \tag{5.84}$$

The three possible distributions of the roots, $\{0, \infty\}, \{0, y\}$ and $\{y, \infty\}$, give the three solutions

$$\delta P_- = \Big\{0, \ -\frac{\tilde{\lambda}}{P_+}\frac{1}{16\pi^2}\sum_{j=1}^{K_4} \frac{p_k^0}{\omega_k}\Big((\frac{1+\omega_k}{x_{4,k}^0 - y} - p_k^0) \pm p_k^0 \Big) \Big\} =: \Omega_3 \tag{5.85}$$

With a little bit of work one can show that these match the eigenvalues from the string Hamiltonian in (12).

Three impurities, confluent mode numbers: For three impurities, with mode numbers $\{q, q, -2q\}$, the only state that does not fall into the already checked rank one sectors **?** are $\alpha_1^+\alpha_1^+\theta_1^+|0\rangle$ and $\alpha_1^+\theta_1^+\theta_1^+|0\rangle$. For the former, we get from (5.66) (with grading $\eta_1 = \eta_2 = 1$)

$$\tilde{p}_q^2 = -2p_q^0 + \frac{2\omega_q + \omega_{2q}}{x_{4,q}^0 - x_{4,2q}^0} - \frac{1+\omega_q}{x_{4,q}^0 - x_3^0}, \quad \tilde{p}_{2q}^2 = -2p_{2q}^0 + 2\frac{2\omega_q + \omega_{2q}}{x_{4,2q}^0 - x_{4,q}^0} - \frac{1+\omega_{2q}}{x_{4,2q}^0 - x_3^0}.$$

The polynomials in (5.77) give two solutions $\{0, \infty\}$ for $x_{3,k}^0$. Using these in (5.76), together with some algebra, gives

$$\delta P_- = \frac{2q^2\tilde{\lambda}}{P_+\omega_q^2\omega_{2q}}\Big\{\frac{3\omega_{2q} + (2\omega_q + \omega_{2q})(4\omega_q(1+\omega_q) + \omega_{2q})}{3 + 2\omega_q + \omega_{2q}}, \\ -\frac{4\omega_q^2 - (3 - 4\omega_q^2)\omega_{2q} - (1 - 2\omega_q)\omega_{2q}^2}{3 + 2\omega_q + \omega_{2q}}\Big\}. \tag{5.86}$$

It is not immediately apparent that this equals the string Hamiltonian result (14) but after some work one can show that these two solutions are equal.

For the second state, $\alpha_1^+\theta_1^+\theta_1^+|0\rangle$, we have $K_3 = 2$ and the two roots $\{0,\infty\}$ for $x_{3,k}^0$ can only be distributed in one way. By doing analogously as above and using (5.66) in (5.76), we find

$$\delta P_- = \frac{2q^2\tilde{\lambda}}{P_+}\frac{(\omega_q + \omega_{2q})}{\omega_q\omega_{2q}}, \tag{5.87}$$

which reproduces the string Hamiltonian result of (13).

The $\mathfrak{su}(1,1|2)$ sector

Now we turn to the larger $\mathfrak{su}(1,1|2)$ sector. The procedure is the same as above but now both sides of the Dynkin diagram gets excited and a general state has the three middle nodes K_3, K_4 and K_5 excited. We are allowed to pick the same solution, on the K_3 and K_5 node, but as before we must put distinct solutions on the fermionic nodes. In this sector a new feature appears: The states $\alpha_1^+\beta_1^+$ and $\theta_1^+\eta_1^+$ are allowed to mix. Also, in the case of confluent mode numbers, it turns out that we have to make use of different gradings on some states to generate all the solutions from the string Hamiltonian.

Let us first investigate if the number of solutions from the string Hamiltonian and the Bethe equations match. A general $\mathfrak{su}(1,1|2)$ state with K_4 excitations and *distinct* mode numbers will yield a $2^{2K_4} \times 2^{2K_4}$ matrix and thus 2^{2K_4} energy shifts. The total number of solutions from the Bethe equations are just the square of (5.78), with $\nu = 0$, which equals the number of eigenvalues from the perturbative string Hamiltonian (15).

Two impurities: The Hamiltonian is a 16×16 matrix but it is only a 13×13 part which lies outside the already calculated $\mathfrak{su}(1|2)$ sector. There are seven different independent submatrices where the largest is a 4×4 matrix and is generated by the base kets $\alpha_1^+\beta_1^+|0\rangle$ and $\theta_1^+\eta_1^+|0\rangle$. There are three 2×2 submatrices, $\alpha_1^+\eta_1^+|0\rangle, \beta_1^+\theta_1^+|0\rangle$ and $\beta_1^+\eta_1^+|0\rangle$. And three are one valued $\beta_1^+\beta_1^+|0\rangle, \eta_1^+\eta_1^+|0\rangle$ and $\theta_1^+\theta_1^+|0\rangle$, these will give the same results as presented in **?** so these we will ignore. The only part with mixing is the subpart generated by $\alpha_1^+\beta_1^+|0\rangle$ and $\theta_1^+\eta_1^+|0\rangle$. To calculate the energy shifts we start by solving (5.77) and, as before, the two solutions are $\{0,\infty\}$. With $\eta_1 = -1$ and $\eta_2 = 1$, so $K_4 = 3$ and $K_5 = K_3 = 1$, we have

$$p_k^1 = (1+\omega_k)\Big(\frac{1}{x_{4,k}^0 - x_{3,k}^0} - \frac{1}{x_{4,k}^0 - x_{5,k}^0}\Big). \tag{5.88}$$

Whenever we pick the same solution for $x_{3,k}^0$ and $x_{5,k}^0$ we get zero and since we can do this in two ways we get two zero solutions. The other two solutions are obtained by setting $\{x_{3,k}^0, x_{5,k}^0\} = \{0,\infty\}$ and $\{\infty, 0\}$ which gives $p_k^1 = \pm 2p_k^0$. Using this in (5.68) gives

$$\delta P_- = (0, 0, \pm\frac{2\tilde{\lambda}}{P_+}\sum_{j=1}^{2}\frac{q_j^2}{\omega_{q_j}}), \tag{5.89}$$

$\{\eta_1, \eta_2\}$	$\{K_1 + K_3, K_4, K_5 + K_7\}$	$\{S_+, S_-, J_+, J_-\}$	δP_-
$\{-,+\}$	$\{2,3,0\}$	$\{0,1,3,2\}_{\alpha_1^+\alpha_1^+\theta_1^+}$	Ω_3
$\{+,-\}$	$\{0,3,2\}$	$\{1,0,2,3\}_{\alpha_1^+\alpha_1^+\eta_1^+}$	$-\Omega_3$
$\{-,+\}$	$\{0,3,2\}$	$\{2,3,1,0\}_{\beta_1^+\beta_1^+\theta_1^+}$	Ω_3
$\{+,-\}$	$\{2,3,0\}$	$\{3,2,0,1\}_{\beta_1^+\beta_1^+\eta_1^+}$	$-\Omega_3$
$\{-,+\}$	$\{1,3,0\}$	$\{0,2,3,1\}_{\theta_1^+\theta_1^+\alpha_1^+}$	Λ_3
$\{-,+\}$	$\{0,3,1\}$	$\{1,3,2,0\}_{\theta_1^+\theta_1^+\beta_1^+}$	$-\Lambda_3$
$\{+,-\}$	$\{0,3,1\}$	$\{2,0,1,3\}_{\eta_1^+\eta_1^+\alpha_1^+}$	Λ_3
$\{+,-\}$	$\{1,3,0\}$	$\{3,1,0,2\}_{\eta_1^+\eta_1^+\beta_1^+}$	$-\Lambda_3$

Table 5.2: The states reproducing the 3×3 submatrices of the string Hamiltonian. Ω_3 and Λ_3, where the subscript indicate the number of solutions as given in (5.85) for Ω_3 and (5.83) for Λ_3.

which is in agreement with the string Hamiltonian result in (16).

For the three parts $\alpha^+\eta^+|0\rangle, \beta^+\theta^+|0\rangle$ and $\beta^+\eta^+|0\rangle$, we see that solving for the first state is analogous to the discussion after (5.79) but with $\eta_1 = 1$ and $\eta_2 = -1$. For the two other, the procedure will again be identical if we choose the opposite gradings. That is, for $\beta^+\theta^+|0\rangle$ we pick $\eta_1 = 1$ and $\eta_2 = -1$, while for $\beta^+\eta^+|0\rangle$ we choose $\eta_1 = -1$ and $\eta_2 = 1$ which give the same set of solution for all three states

$$\delta P_- = \pm \frac{2\tilde{\lambda}}{P_+}\frac{q^2}{\omega_q}, \tag{5.90}$$

which is in agreement with (17).

Three impurities, distinct mode numbers: The full perturbative string Hamiltonian will now be a 64×64 matrix with non trivial 3×3 and 9×9 subsectors. Since the logic of solving the Bethe equation should be clear by now, we only present the obtained results in tabular form. Also, to make the comparison with the string Hamiltonian more transparent, we now also label the states by their charges $\{S_+, S_-, J_+, J_-\}$. The energy shifts for the 3×3 parts are given in table 5.2 and for the larger 9×9 subparts in table 5.3. For the larger sectors we have a mixing between states of different boson and fermion number.

The functions Ω_9 and Λ_9 in table 5.3 depend on the mode numbers $\{q_1, q_2, q_3\}$ and are given by

$$\Omega_9 = \frac{\tilde{\lambda}}{P_+}\frac{1}{16\pi^2}\sum_{k=1}^{3}\frac{p^0_{q_k}}{\omega_{q_k}}\Big(\sum_{j=1, j\neq k}^{3}\frac{2+\omega_{q_k}+\omega_{q_j}}{x^0_{4,q_k}-x^0_{4,q_j}} - \frac{1+\omega_{q_k}}{x^0_{4,q_k}-x^0_3} - \frac{1+\omega_{q_k}}{x^0_{4,q_k}-x^0_5}) - p^0_{q_k}\Big) \tag{5.91}$$

$$\Lambda_9 = -\frac{\tilde{\lambda}}{P_+}\frac{1}{16\pi^2}\sum_{k=1}^{3}\frac{p^0_{q_k}}{\omega_{q_k}}\Big(\frac{1+\omega_{q_k}}{x^0_{4,q_k}-x^0_3} - \frac{1+\omega_{q_k}}{x^0_{4,q_k}-x^0_5}\Big). \tag{5.92}$$

$\{\eta_1, \eta_2\}$	$\{K_1 + K_3, K_4, K_5 + K_7\}$	$\{S_+, S_-, J_+, J_-\}$	δP_-
$\{+,+\}$	$\{1,3,1\}$	$\{1,1,2,2\}_{(\alpha_1^+\alpha_1^+\beta_1^+),(\alpha_1^+\theta_1^+\eta_1^+)}$	Ω_9
$\{-,-\}$	$\{1,3,1\}$	$\{2,2,1,1\}_{(\alpha_1^+\beta_1^+\beta_1^+),(\beta_1^+\theta_1^+\eta_1^+)}$	$-\Omega_9$
$\{-,+\}$	$\{1,3,1\}$	$\{1,2,2,1\}_{(\alpha_1^+\beta_1^+\theta_1^+),(\theta_1^+\theta_1^+\eta_1^+)}$	Λ_9
$\{+,-\}$	$\{1,3,1\}$	$\{2,1,1,2\}_{(\alpha_1^+\beta_1^+\eta_1^+,(\theta_1^+\eta_1^+\eta_1^+)}$	$-\Lambda_9$

Table 5.3: The states reproducing the 9×9 submatrices of the string Hamiltonian. Ω_9 and Λ_9, where the subscript indicate the number of solutions, is given by (5.91) and (5.92).

$\{\eta_1, \eta_2\}$	$\{K_1 + K_3, K_4, K_5 + K_7\}$	$\{S_+, S_-, J_+, J_-\}$	δP_-
$\{+,+\}$	$\{1,3,0\}$	$\{0,1,3,2\}_{\alpha_1^+\alpha_1^+\theta_1^+}$	$\tilde{\Omega}_2$
$\{+,+\}$	$\{0,3,1\}$	$\{1,0,2,3\}_{\alpha_1^+\alpha_1^+\eta_1^+}$	$\tilde{\Omega}_2$
$\{-,-\}$	$\{0,3,1\}$	$\{2,3,1,0\}_{\beta_1^+\beta_1^+\theta_1^+}$	$-\tilde{\Omega}_2$
$\{-,-\}$	$\{1,3,0\}$	$\{3,2,0,1\}_{\beta_1^+\beta_1^+\eta_1^+}$	$-\tilde{\Omega}_2$
$\{+,+\}$	$\{2,3,0\}$	$\{0,2,3,1\}_{\theta_1^+\theta_1^+\alpha_1^+}$	$\tilde{\Lambda}_1$
$\{-,-\}$	$\{0,3,2\}$	$\{1,3,2,0\}_{\theta_1^+\theta_1^+\beta_1^+}$	$-\tilde{\Lambda}_1$
$\{+,+\}$	$\{0,3,2\}$	$\{2,0,1,3\}_{\eta_1^+\eta_1^+\alpha_1^+}$	$\tilde{\Lambda}_1$
$\{-,-\}$	$\{2,3,0\}$	$\{3,1,0,2\}_{\eta_1^+\eta_1^+\beta_1^+}$	$-\tilde{\Lambda}_1$

Table 5.4: The states reproducing the 2×2 submatrices for *confluent* mode numbers of the string Hamiltonian. $\tilde{\Omega}_2$ and $\tilde{\Lambda}_2$, where the subscript indicate the number of solutions, is given by (5.86) and (5.87)

To obtain the nine solutions for Ω_9 and Λ_9 one has to insert one of the three roots $\{0, \infty, y\}$ for each x_3^0 and x_5^0. We have not managed to match these results with the perturbative string Hamiltonian (15) analytically, but tested the agreement extensively numerically. The details of the numerical tests can be found in Appendix 2.

Three impurities, confluent mode numbers: We will now look at three impurities with confluent mode numbers, $\{q, q, -2q\}$. With two distinct mode numbers we see from (5.77) that we have the two standard solutions $\{0, \infty\}$ for $x_{3,k}^0$ and $x_{5,k}^0$. The sectors exhibiting mixing, i.e. the states that span the 9×9 subparts of the previous section, now exhibit a new feature. The gradings are no longer equivalent and we will be forced to use both to generate all the desired solutions. The simpler states, that do not exhibit this feature, are presented in table5.4 and the states where different gradings had to be used are presented in table 5.5. The energy shifts Γ_4 and $\tilde{\Gamma}_1$ appearing in

$\{\eta_1,\eta_2\}$	$\{K_1+K_3,K_4,K_5+K_7\}$	$\{S_+,S_-,J_+,J_-\}$	δP_-
$\{+,+\}$	$\{1,3,1\}$	$\{1,1,2,2\}_{(\alpha_1^+\alpha_1^+\beta_1^+),(\alpha_1^+\theta_1^+\eta_1^+)}$	Γ_4
$\{-,-\}$	$\{2,3,2\}$	$\{1,1,2,2\}_{(\alpha_1^+\alpha_1^+\beta_1^+),(\alpha_1^+\theta_1^+\eta_1^+)}$	$\tilde\Gamma_1$
$\{-,-\}$	$\{1,3,1\}$	$\{2,2,1,1\}_{(\alpha_1^+\beta_1^+\beta_1^+),(\beta_1^+\theta_1^+\eta_1^+)}$	$-\Gamma_4$
$\{+,+\}$	$\{2,3,2\}$	$\{2,2,1,1\}_{(\alpha_1^+\beta_1^+\beta_1^+),(\beta_1^+\theta_1^+\eta_1^+)}$	$-\tilde\Gamma_1$
$\{+,+\}$	$\{2,3,1\}$	$\{1,2,2,1\}_{(\alpha_1^+\beta_1^+\theta_1^+),(\theta_1^+\theta_1^+\eta_1^+)}$	$\tilde\Omega_2$
$\{-,-\}$	$\{1,3,2\}$	$\{1,2,2,1\}_{(\alpha_1^+\beta_1^+\theta_1^+),(\theta_1^+\theta_1^+\eta_1^+)}$	$-\tilde\Omega_2$
$\{-,-\}$	$\{2,3,1\}$	$\{2,1,1,2\}_{(\alpha_1^+\beta_1^+\eta_1^+,(\theta_1^+\eta_1^+\eta_1^+)}$	$-\tilde\Omega_2$
$\{+,+\}$	$\{1,3,2\}$	$\{2,1,1,2\}_{(\alpha_1^+\beta_1^+\eta_1^+,(\theta_1^+\eta_1^+\eta_1^+)}$	$\tilde\Omega_2$

Table 5.5: The states reproducing the larger submatrices, with *confluent* mode numbers, of the string Hamiltonian. The functions Γ_4 and $\tilde\Gamma_1$ are given in (5.93) and $\tilde\Omega_2$ is given in (5.86).

$\{S_+,S_-,J_+,J_-\}$	State pattern			Number of solutions
$\{2,2,2,2\}$	$\theta_1^+\theta_1^+\eta_1^+\eta_1^+\|0\rangle$,	$\theta_1^+\eta_1^+\beta_1^+\alpha_1^+\|0\rangle$,	$\beta_1^+\beta_1^+\alpha_1^+\alpha_1^+\|0\rangle$	36 energy shifts
$\{2,2,3,3\}$	$\theta_1^+\theta_1^+\eta_1^+\eta_1^+\alpha_1^+\|0\rangle$,	$\theta_1^+\eta_1^+\beta_1^+\alpha_1^+\alpha_1^+\|0\rangle$,	$\beta_1^+\beta_1^+\alpha_1^+\alpha_1^+\alpha_1^+\|0\rangle$	100 energy shifts

Table 5.6: Checked 4 and 5 impurity states of $\mathfrak{su}(1,1|2)$.

table 5.5 are given by

$$
\begin{aligned}
\tilde\Gamma_1 &= \frac{2q^2\tilde\lambda}{P_+\omega_q^2\omega_{2q}}\Big(\frac{1}{\omega_q}+\frac{1}{\omega_{2q}}\Big)\,,\\
\Gamma_4 &= -\frac{2q^2\tilde\lambda}{P_+\omega_q^2\omega_{2q}}\Big\{(\frac{1}{\omega_q}+\frac{1}{\omega_{2q}}),(\frac{1}{\omega_q}+\frac{1}{\omega_{2q}}),\frac{3\omega_{2q}+(2\omega_q+\omega_{2q})(\omega_{2q}+\omega_q(7+6\omega_q+\omega_{2q}))}{3+2\omega_q+\omega_{2q}},\\
&\quad \frac{3\omega_{2q}-(2\omega_q+\omega_{2q})(\omega_q(5+2\omega_q+3\omega_{2q})-\omega_{2q})}{3+2\omega_q+\omega_{2q}}\Big\}.
\end{aligned} \tag{5.93}
$$

Again, for the comparison to the eigenvalues of the string Hamiltonian in this subsector we had to resort to numerical verifications, see Appendix 2 for details.

Higher impurities: In going beyond three impurities numerical calculations on both sides, the Bethe equations and the string Hamiltonian, have been performed for a number of four and five impurity states. All numerical energy shifts match precisely, the tested configurations are listed in table 5.6.

The $\mathfrak{su}(2|3)$ sector

Now things become more complex. The polynomials (5.77) for a general state are highly non-linear, coupled and involve several variables. For this reason we will not be as thorough in our testing for the higher impurity cases as in the previous sections. The oscillators in this sector are $\alpha_1^+, \alpha_2^+, \theta_1^+$ and θ_2^+ where there is a mixing between $\alpha_1^+\alpha_2^+\,|0\rangle$ and $\theta_1^+\theta_2^+\,|0\rangle$. The string Hamiltonian is given in (18).

Two impurities: The $\mathfrak{su}(2|3)$ two impurity sector of the perturbative string Hamiltonian (18) will be a 12×12 matrix. Let us begin with the largest subpart, the one with mixing between $\alpha_1^+\alpha_2^+|0\rangle$ and $\theta_1^+\theta_2^+|0\rangle$. The excitation numbers, with grading $\eta_1 = \eta_2 = 1$, for $\alpha_1^+\alpha_2^+|0\rangle$ are $K_1 = K_2 = K_3 = 1$ and $K_4 = 2$ while for $\theta_1^+\theta_2^+|0\rangle$ we have $K_2 = 1$ and $K_3 = K_4 = 2$. Here the dynamically transformed version of the Bethe equations is advantageous, as it makes explicit that the relevant combination $K_1 + K_3 = 2$ is the same for these two states. This is how the Bethe equations take care of the mixing. Solving for u_2^0 in (5.77), and using $u_{3,k}^0 = x_{3,k}^0 + \frac{\tilde{\lambda}}{64\pi^2}\frac{1}{x_{3,k}^0}$, gives

$$u_2^0 = \frac{1}{2}(x_{3,1}^0 + x_{3,2}^0 + \frac{\tilde{\lambda}}{64\pi^2}(\frac{1}{x_{3,1}^0} + \frac{1}{x_{3,2}^0})).$$

Plugging this into the second line of (5.77) gives

$$\begin{aligned} \frac{1}{x_{3,1}^0 - x_{3,2}^0 + \frac{\tilde{\lambda}}{64\pi^2}(\frac{1}{x_{3,1}^0} - \frac{1}{x_{3,2}^0})} + \sum_{j=1}^{2} \frac{1+\omega_j}{x_{4,j}^0 - x_{3,1}^0} &= 0, \\ \frac{1}{x_{3,2}^0 - x_{3,1}^0 + \frac{\tilde{\lambda}}{64\pi^2}(\frac{1}{x_{3,2}^0} - \frac{1}{x_{3,1}^0})} + \sum_{j=1}^{2} \frac{1+\omega_j}{x_{4,j}^0 - x_{3,2}^0} &= 0. \end{aligned} \tag{5.94}$$

We can add these two equations above and see that four solutions are $(x_{3,1}^0, x_{3,2}^0) = (0,0), (0,\infty), (\infty,0)$ and (∞,∞). This may at first glance seem strange since the seemingly equivalent state $\theta_1^+\theta_2^+|0\rangle$ only has the K_2 and K_3 node excited, implying that we can not pick the same solution twice for $x_{3,k}^0$ since K_3 is fermionic. However, the correct state to use is the $\alpha_1^+\alpha_2^+|0\rangle$ state. Here two different fermionic nodes K_1 and K_3 are excited and because of this we can use the same solutions on both nodes simultaneously.

Let us now turn to the calculation of the energy shifts for the these four states. We use the solutions from (5.94) in (5.66) and plug this into (5.68) which gives

$$\delta P_- = \{0, 0, \pm\frac{\tilde{\lambda}}{P_+}\frac{4q^2}{\omega_q}\} =: \chi_4, \tag{5.95}$$

which is in perfect agreement with (19). The energy shifts for the other states follows immediately and we present the results in table 5.7. From this table we see that all the energy shifts from (18), presented in (20) and (19), are reproduced.

Higher impurities: Due to the non linearity of the polynomials relating the Bethe roots we will only present results for excitations with $K_2 = K_3 = 1$, corresponding to states of the form $\alpha_1^+ \ldots \alpha_1^+\theta_2^+|0\rangle$ with space-time charge vector $\{S_+, S_-, J_+, J_-\} =$

$\{\eta_1, \eta_2\}$	$\{K_1 + K_3, K_2, K_4\}$	$\{S_+, S_-, J_+, J_-\}$	δP_-
$\{+, +\}$	$\{2, 1, 2\}$	$\{0, 0, 2, 0\}_{(\alpha_1^+ \alpha_2^+),(\theta_1^+ \theta_2^+)}$	χ_4
$\{-, +\}$	$\{1, 0, 2\}$	$\{0, 1, 2, 1\}_{\alpha_1^+ \theta_1^+}$	κ_2
$\{-, +\}$	$\{1, 0, 2\}$	$\{0, -1, 2, -1\}_{\alpha_2^+ \theta_2^+}$	κ_2
$\{+, +\}$	$\{1, 1, 2\}$	$\{0, -1, 2, 1\}_{\alpha_1^+ \theta_2^+}$	κ_2
$\{+, +\}$	$\{1, 1, 2\}$	$\{0, 1, 2, -1\}_{\alpha_2^+ \theta_1^+}$	κ_2

Table 5.7: The two impurity states that fall into to the rank ≥ 1 sectors for $\mathfrak{su}(2|3)$. Here χ_4 is given by (5.95) and κ_2 is given by (5.79). For two of the states we have permutated the space-time indices.

$\{\eta_1, \eta_2\}$	$\{K_1 + K_3, K_2, K_4\}$	$\{S_+, S_-, J_+, J_-\}$	δP_-
$\{+, +\}$	$\{1, 1, K_4\}$	$\{0, -1, K_4, K_4 - 1\}_{(\alpha_1^+ \ldots \alpha_1^+ \theta_2^+)}$	Λ_{K_4}

Table 5.8: Higher impurity states from the $\mathfrak{su}(2|3)$ sector for states of the form $\alpha_1^+ \ldots \alpha_1^+ \theta_2^+ |0\rangle$. The function Λ_{K_4}, where K_4 indicates the number of solutions, is given in (5.97).

$\{0, -1, K_4, K_4 - 1\}$. From the first line in (5.77) we see that

$$\frac{1}{u_2^0 - (x_3^0 + \frac{\tilde{\lambda}}{64\pi^2} \frac{1}{x_3^0})} = 0,$$

and using this in the second line implies that the equation for x_3^0 reduces to the familiar form

$$\sum_{j=1}^{K_4} \frac{1 + \omega_j}{x_{4,j}^0 - x_3^0} = 0. \tag{5.96}$$

Thus, the energy shift for this state is the same as for the $\alpha_1^+ \ldots \alpha_1^+ \theta_1^+ |0\rangle$ states. For $K_4 = 3$, the energy shift is presented in (5.83). For $K_4 - 1$ number of α_1^+ excitations and one θ_2^+ excitation, the energy shift, with gradings $\{+, +\}$, is given by

$$\Lambda_{K_4} = \frac{1}{16\pi^2} \sum_{k=1}^{K_4} \frac{p_k^0}{\omega_k} \Big(\sum_{j=1 j \neq k}^{K_4} \frac{2 + \omega_j + \omega_k}{x_{4,k}^0 - x_{4,j}^0} - \frac{1 + \omega_k}{x_{4,k}^0 - x_3^0} - p_k^0 (K_4 - 1) \Big). \tag{5.97}$$

This prediction we have verified numerically for $K_4 \leq 6$ with the energy shifts obtained by diagonalization of the string Hamiltonian (18).

5.3.3 Summary of results

In the last sections we have explored the quantum integrability of the $AdS_5 \times S^5$ superstring by confronting the conjectured set of Bethe equations with an explicit diagonalization of the light-cone gauged string Hamiltonian.

For this we have presented the Bethe equations for the most general excitation pattern of the uniform light-cone gauged $AdS_5 \times S^5$ superstring in the near plane-wave limit. Moreover, it was demonstrated how excited string states may be translated to distributions of spectral parameters in the Bethe equations as given in table 1. Using this we have explicitly compared the predictions from the light-cone Bethe equations with direct diagonalization of the string Hamiltonian in perturbation theory at leading order in $1/P_+$. For operators from the non dynamical sectors, we have verified the spectrum for a large number of states giving us a strong confidence in the validity of the light-cone Bethe equations for these classes of operators. For a generic $\mathfrak{su}(1,1|2)$ state, it is much easier to calculate the energy shifts using the Bethe equations. The characteristic polynomial from the perturbative string Hamiltonian is of degree 2^{2K_4} whereas the polynomials needed to be solved in the Bethe equations (5.77) are of degree $K_4 - 2$. Still, one generically deals with polynomials of a high degree, making it hard to explicitly find analytical results for states with large total excitation number K_4.

When it comes to the dynamical sector $\mathfrak{su}(2|2)$, a direct comparison is much more difficult due to the non linearity and coupled structure of the Bethe equations in (5.77). Here analytical results were established only for the two impurity case. Nevertheless, tests up to impurity number six could be performed numerically.

5.4 The near flat space limit

In the last sections we considered the near BMN theory of the $\mathrm{AdS}_5\times\mathrm{S}^5$ string, and its corresponding light-cone Bethe equations, in detail. As is probably clear by now, the theory is rather involved and it would be nice if one could find some sort of simpler, but still non trivial, theory. One such limit is the so called near flat space limit (NFS) introduced by Maldacena and Swanson in **?** which resembles the BMN limit in the sense that it is a large radius expansion, or, equivalently, a strong coupling expansion. However, in contrast to the BMN expansion, the expansion scheme is now such that the left and right moving sector of the theory is scaled differently.

5.4.1 Lightning review of the Maldacena Swanson approach

We start out by describing the original approach of **?** for the bosonic case. The light-cone and transverse bosonic coordinates are defined and scaled as

$$x^+ = \sqrt{g}\,\sigma^+ + \frac{1}{\sqrt{g}}U(\sigma^+,\sigma^-), \qquad x^- = \frac{1}{\sqrt{g}}V(\sigma^+,\sigma^-), \qquad \frac{x_m}{\sqrt{g}}, \tag{5.98}$$

where m runs over the $4+4$ transverse degrees of freedom and U and V are fluctuations of the light-cone coordinates.

The limit we will take treats the left and right moving sector differently

$$\sigma^\pm \to g^{\pm 1/2}\,\sigma^\pm, \qquad g\to\infty \tag{5.99}$$

where $\sigma^{\pm} = \sigma^0 \pm \sigma^1$.

Using the parametrization (4.16) of the background metric together with the conformal gauge, gives that the Lagrangian expands to

$$\begin{aligned} -\mathscr{L} &= g\,\partial_- V + \partial_+ z\,\partial_- z + \partial_+ y\,\partial_- y \\ &+ \frac{1}{2}(z^2 - y^2)\partial_- V - \frac{1}{2}(z^2 + y^2)\partial_- U + \frac{1}{2}(\partial_+ U \partial_- V + \partial_- U \partial_+ V) + \mathcal{O}(g^{-1}). \end{aligned} \tag{5.100}$$

Neglecting the total derivative term, we see that we have a quartic theory without any coupling dependence and which is invariant under right moving conformal transformations

$$\sigma^- \to f(\sigma^-).$$

In the conformal gauge, the two Virasoro constraints can be written as $T_{\pm\pm} = 0$, which to leading order equals

$$\begin{aligned} T_{--} &= \partial_- U \partial_- V + (\partial_- z)^2 + (\partial_- y)^2 = 0, \\ T_{++} &= \quad 2\partial_+ V - (z^2 + y^2) = 0. \end{aligned} \tag{5.101}$$

It is tempting to fix the NFS analogue of light-cone gauge, $x^+ = \sigma^0$, by choosing $U = \sigma^-$ so that

$$x^+ = \sqrt{g}\,\sigma^+ + \frac{1}{\sqrt{g}}\,\sigma^-.$$

However, by investigating the equations of motion for U and V, one find that this gauge is not consistent with the conformal gauge, which was also the case for the near BMN string. This complication was cleverly avoided in **?** by introducing new worldsheet coordinates. Which coordinates to choose can be understood by looking at

$$\partial_- \Big(\partial_+ U + \frac{1}{2}(z^2 - y^2) \Big) = 0, \tag{5.102}$$

which comes from varying the action with respect to V. The trick now is to implement a gauge which automatically solves the above. Thus, if one introduces

$$\tilde{\sigma}^+ = \sigma^+, \quad \tilde{\sigma}^- = U, \tag{5.103}$$

which imply that the worldsheet derivatives transform as

$$\partial_+ = \tilde{\partial}_+ - \frac{1}{2}(z^2 - y^2)\tilde{\partial}_-, \quad \partial_- = \partial_- U \tilde{\partial}_-, \tag{5.104}$$

we see that (5.102) is satisfied.

In the new worldsheet coordinates, the Virasoro constraints becomes

$$T_{--} = \tilde{\partial}_- V + (\tilde{\partial}_- z)^2 + (\tilde{\partial}_- y)^2 = 0, \qquad (5.105)$$
$$T_{++} = 2\tilde{\partial}_+ V - (z^2 - y^2)\tilde{\partial}_- V - (z^2 + y^2) = 0,$$

which allows us to express $\tilde{\partial}_- V$ in terms of the transverse coordinates. Using this, the gauge fixed version of (5.100) is now easily obtained

$$\mathscr{L} \sim \qquad (5.106)$$
$$\tilde{\partial}_+ x_m \tilde{\partial}_- x_m - \frac{1}{2}(z^2 - y^2)\big((\tilde{\partial}_- z)^2 + (\tilde{\partial}_- y)^2\big) - \frac{1}{2}(z^2 + y^2).$$

Thus we have an interacting theory with only right moving derivatives.

Naturally, one is interested in the full theory, including not only the bosonic but also the fermionic interactions. We will not describe the procedure of **?** for the fermions explicitly, but only comment on the general structure. First, the fermions are split up into their respective left and right moving components, $\eta_\pm$, where each sector scale as

$$\eta_\pm \to g^{\mp 1/4} \frac{\eta_\pm}{\sqrt{g}}. \qquad (5.107)$$

The gauge (5.103) is the same also when the fermions are included, and expanding the Lagrangian one finds quartic terms of the form

$$\eta_- \, \partial_- \eta_- \, f_{ab} \, x^a \, x^b, \qquad \eta_-^2 \, g_{ab} \, x^a \, x^b, \qquad \eta_-^4,$$

where the components f_{ab} and g_{ab} are constant. Thus we see that the higher order theory is fully governed by the left moving excitations alone. The action supposedly posses the full SU(2|2)$^2\times$R^2 symmetry and it has been used to study higher loop effects and factorization properties in **? ?** and **?**.

5.4.2 From BMN to NFS

In the above we shortly outlined the procedure of **?**. However, if one investigates the scalings of the physical parameters, one finds that $P_+ \sim g$ as for the BMN scaling. Thus, since the physical parameters are scaled in the same way, it should be possible to go directly from the near BMN model, which includes the full excitation pattern of the strongly coupled string, to the NFS Lagrangian. The up shoot is rather clear; one simply starts with the first order near BMN Lagrangian, inverts the bosonic momentas, scale the fermions appropriately and perform the limit (5.99) and (5.107).

For this analysis, it is very convenient to introduce an alternative parametrization of η and G_t to the one presented in (5.1) and (5.3). The bosonic subgroup that leaves the light-cone Hamiltonian is, as we remember, G_B=SU(2)^4 and it is very useful to introduce a notation covariant under these transformations. A general element $G \in G_B$ can be represented as a block diagonal matrix consited of independent SU(2)'s

$$G = \begin{pmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{pmatrix}. \tag{5.108}$$

If closely follow **?** and denote indices corresponding to the first two SU(2)'s with normal and dotted Greek indices, taking values in the set $\{3,4\}$ and $\{\dot{3},\dot{4}\}$, and the second two SU(2) copies with normal and dotted Latin indices taking values in $\{1,2\}$ and $\{\dot{1},\dot{2}\}$ then if we introduce

$$\mathbb{X} = \left(\begin{array}{cccc|cccc} 0 & 0 & Z^{3\dot{4}} & -Z^{3\dot{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & Z^{4\dot{4}} & -Z^{4\dot{3}} & 0 & 0 & 0 & 0 \\ -Z^{4\dot{3}} & Z^{3\dot{3}} & 0 & 0 & 0 & 0 & 0 & 0 \\ -Z^{4\dot{4}} & Z^{3\dot{4}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & iY^{1\dot{2}} & -iY^{1\dot{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & iY^{2\dot{2}} & -iY^{2\dot{1}} \\ 0 & 0 & 0 & 0 & -iY^{2\dot{1}} & iY^{1\dot{1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -iY^{2\dot{2}} & iY^{1\dot{2}} & 0 & 0 \end{array}\right), \tag{5.109}$$

then, see **?** for details, we find that

$$Z'^{\alpha\dot{\alpha}} = g^{\alpha}_{\beta}\, g^{\dot{\alpha}}_{\dot{\beta}} Z^{\beta\dot{\beta}}, \qquad Y'^{a\dot{a}} = g^{a}_{b}\, g^{\dot{a}}_{\dot{b}} Y^{b\dot{b}}. \tag{5.110}$$

Similarly we can introduce a covariant notation for the kappa gauge fixed fermions

$$\eta = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \eta^{3\dot{2}} & -\eta^{3\dot{1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta^{4\dot{2}} & -\eta^{4\dot{1}} \\ 0 & 0 & 0 & 0 & \theta^{\dagger}_{1\dot{4}} & \theta^{\dagger}_{2\dot{4}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta^{\dagger}_{1\dot{3}} & -\theta^{\dagger}_{2\dot{3}} & 0 & 0 \\ \hline 0 & 0 & \theta^{1\dot{4}} & -\theta^{1\dot{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^{2\dot{4}} & -\theta^{2\dot{3}} & 0 & 0 & 0 & 0 \\ -\eta^{\dagger}_{3\dot{2}} & -\eta^{\dagger}_{4\dot{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta^{\dagger}_{3\dot{1}} & \eta^{\dagger}_{4\dot{1}} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{5.111}$$

Conjugation flips the indices through

$$(\eta^{\alpha\dot{a}})^{\star} = \eta^{\dagger}_{\alpha\dot{a}}, \quad (\theta^{\dot{a}\alpha})^{\star} = \eta^{\dagger}_{\dot{a}\alpha}, \qquad (Z^{\alpha\dot{\beta}})^{\star} = Z_{\alpha\dot{\beta}}, \quad (Y^{a\dot{b}})^{\star} = Y_{a\dot{b}}, \tag{5.112}$$

and we raise and lower indices using ϵ-tensors as

$$X^{A\dot{B}} = \epsilon^{AC}\epsilon^{\dot{B}\dot{D}} X_{C\dot{D}}, \tag{5.113}$$

for generic field X and indices $A, \dot{B}$. Thus, in total we have $2_B + 2_F$ fields transforming in a bi fundamental representation of SU(2)×SU(2).

Utilizing the new covariant notation, the group element incorporating the transverse bosons, G_t, in (5.1) can now be represented through

$$G_t = g(\mathbb{X}) = \sqrt{\frac{\mathbb{1} + \mathbb{X}}{\mathbb{1} - \mathbb{X}}}, \tag{5.114}$$

and the fermionic element $f(\eta) = \eta + \sqrt{\mathbb{1} + \eta^2}$ remains unchanged.

In this section we will use the expansion scheme (5.15), as compared to the earlier section where we used (5.16). We apologize for this inconvenience and the reason we choose to do it because the scalings of the NFS fermions are, as we shortly will see, rather intricate and the notation is much simpler if we eliminate P_+ in favor for g.

Expanding in inverse powers of g, gives to leading order the quadratic BMN Lagrangian as

$$\begin{aligned}\mathscr{L}_2 &= P_{\alpha\dot{\alpha}}\, \dot{Z}^{\alpha\dot{\alpha}} + P_{a\dot{a}}\, \dot{Y}^{a\dot{a}} + i\eta^\dagger_{\alpha\dot{a}}\, \dot{\eta}^{\alpha\dot{a}} + i\theta^\dagger_{a\dot{\alpha}}\, \dot{\theta}^{a\dot{\alpha}} \\ &- \frac{1}{4}P_{\alpha\dot{\alpha}}\, P^{\alpha\dot{\alpha}} - \frac{1}{4}P_{a\dot{a}}\, P^{a\dot{a}} - Z'_{\alpha\dot{\alpha}}\, Z'^{\alpha\dot{\alpha}} - Y'_{a\dot{a}}\, Y'^{a\dot{a}} - Z_{\alpha\dot{\alpha}}\, Z^{\alpha\dot{\alpha}} - Y_{a\dot{a}}\, Y^{a\dot{a}} \\ &- \eta^\dagger_{\alpha\dot{a}}\, \eta^{\alpha\dot{a}} - \theta^\dagger_{a\dot{\alpha}}\, \theta^{a\dot{\alpha}} - \frac{\kappa}{2}(\eta^{\dagger\alpha\dot{a}}\eta'^{\dagger}_{\alpha\dot{a}} + \theta^{\dagger a\dot{\alpha}}\theta'^{\dagger}_{a\dot{\alpha}} - \eta^{\alpha\dot{a}}\eta'_{\alpha\dot{a}} - \theta^{a\dot{\alpha}}\theta'_{a\dot{\alpha}}).\end{aligned} \tag{5.115}$$

For simplicity we will not present the higher order contributions here, note however that we do not perform the fermionic shift (5.21) since this will complicate the final expressions[8]. However, note that we do perform the canonical transformation so all higher non derivative terms are removed.

To implement the worldsheet scalings (5.99) we should invert the momentum variables $P_{\alpha\dot{\alpha}}$ and $P_{a\dot{a}}$ in favor of the velocities. This is easily done using the equations of motion for each respective conjugate variable which to leading order equals

$$P_{\alpha\dot{\alpha}} = 2\dot{Z}_{\alpha\dot{\alpha}} + \mathcal{O}(g^{-1}), \qquad P_{a\dot{a}} = 2\dot{Y}_{a\dot{a}} + \mathcal{O}(g^{-1}). \tag{5.116}$$

Having expressed the momentum variables in terms of velocities, we perform the shift (5.99) which gives us to bosonic part of the NFS model. However, it remains to figure out how to scale the fermions appropriately. As it turns out, the combinations $\eta_\pm$ will roughly correspond to the respective graded components of η as

$$\eta_+ \sim \eta^{(1)}, \qquad \eta_- \sim \eta^{(3)}. \tag{5.117}$$

If one implement this directly in the action (5.13), one finds, up to a fermionic shift, a Lagrangian that resembles the full action of **?**. However, equating $\eta_\pm$ directly with $\eta^{(1)}$ and $\eta^{(3)}$, breaks the bosonic $\mathrm{SU}(2)^4$ invariance of the theory.

Luckily, we can introduce a linear combination of η so the bosonic symmetry is left

[8]This is easy to understand since we split up the derivatives ∂_α in right and left moving parts and some of the derivative terms in the kinetic and the p_- term of (5.13) tend to cancel among themselves.

manifest. The right combinations can be found if we first split up the matrix elements of η in real and complex part as

$$\eta^{\alpha\dot{k}} = \frac{1}{\sqrt{2}}(a^{\alpha\dot{k}} + i\, b^{\alpha\dot{k}}), \qquad \theta^{\dot{k}\alpha} = \frac{1}{\sqrt{2}}(c^{\dot{k}\alpha} + i\, d^{\dot{k}\alpha}). \tag{5.118}$$

The automorphism (4.30), implies that the graded components of η are

$$\begin{aligned} \eta_{(1)}^{\alpha\dot{k}} &= \frac{1}{\sqrt{2}}(\frac{1}{2}+\frac{i}{2})(a^{\alpha\dot{k}} + b^{\alpha\dot{k}}), \quad \theta_{(1)}^{k\dot{\alpha}} = \frac{1}{\sqrt{2}}(\frac{1}{2}+\frac{i}{2})(c^{k\dot{\alpha}} + d^{k\dot{\alpha}}), \\ \eta_{(3)}^{\alpha\dot{k}} &= \frac{1}{\sqrt{2}}(\frac{1}{2}-\frac{i}{2})(a^{\alpha\dot{k}} - b^{\alpha\dot{k}}), \quad \theta_{(3)}^{k\dot{\alpha}} = \frac{1}{\sqrt{2}}(\frac{1}{2}-\frac{i}{2})(c^{k\dot{\alpha}} - d^{k\dot{\alpha}}), \end{aligned} \tag{5.119}$$

then the fermions we want to scale are just the linear combinations of the real and complex variables as

$$\eta_{\pm}^{\alpha\dot{k}} = \frac{1}{\sqrt{2}}(a^{\alpha\dot{k}} \pm b^{\alpha\dot{k}}), \qquad \theta_{\pm}^{\dot{k}\alpha} = \frac{1}{\sqrt{2}}(c^{\dot{k}\alpha} \pm d^{\dot{k}\alpha}). \tag{5.120}$$

Since we will go from the near BMN to the NFS Lagrangian, where the original fermions are already suppressed with a factor of $1/\sqrt{g}$, the scalings of the new fermionic parameters should be

$$\eta_{\pm}^{\alpha\dot{\alpha}} \to g^{\mp 1/4}\eta_{\pm}^{\alpha\dot{\alpha}}, \qquad \theta_{\pm}^{a\dot{\alpha}} \to g^{\mp 1/4}\theta_{\pm}^{a\dot{\alpha}}. \tag{5.121}$$

Having established the fermionic scalings and using the inverted bosonic coordinates, it is straightforward to obtain the NFS Lagrangian from the gauge fixed Lagrangian (5.13).

Splitting up the contributions with respect to boson / fermion field content as $\mathscr{L}_{NFS} = \mathscr{L}_{BB} + \mathscr{L}_{BF} + \mathscr{L}_{FF}$ and picking $\kappa = 1$, we find

$$\begin{aligned} \mathscr{L}_{FF} &= i\,\eta_{+\alpha\dot{a}}\partial_-\eta_+^{\alpha\dot{a}} + i\,\theta_{+a\dot{\alpha}}\partial_-\theta_+^{a\dot{\alpha}} + i\,\eta_{-\alpha\dot{a}}\partial_+\eta_-^{\alpha\dot{a}} + i\,\theta_{-a\dot{\alpha}}\partial_+\theta_-^{a\dot{\alpha}} \\ &- i\,\eta_{-\alpha\dot{a}}\,\eta_+^{\alpha\dot{a}} - i\,\theta_{-a\dot{\alpha}}\,\theta_+^{a\dot{\alpha}} + \frac{1}{2}(\eta_{-\alpha\dot{a}}\eta_{-\gamma\dot{c}}\,\eta_+^{\gamma\dot{a}}\partial_-\eta_-^{\alpha\dot{c}} + \theta_{-a\dot{\alpha}}\theta_{-c\dot{\gamma}}\,\theta_+^{a\dot{\gamma}}\partial_-\theta_-^{c\dot{\alpha}}), \\ \mathscr{L}_{BB} &= 4\partial_+ Z_{\alpha\dot{\beta}}\partial_- Z^{\alpha\dot{\beta}} + 4\partial_+ Y_{a\dot{b}}\partial_- Y^{a\dot{b}} - Z_{\alpha\dot{\beta}}Z^{\alpha\dot{\beta}} - Y_{a\dot{b}}Y^{a\dot{b}} \\ &+ \frac{1}{8}(Y_{a\dot{b}}Y^{a\dot{b}} - Z_{\alpha\dot{\beta}}Z^{\alpha\dot{\beta}})(\partial_- Z_{\gamma\dot{\epsilon}}\partial_- Z^{\gamma\dot{\epsilon}} + \partial_- Y_{c\dot{d}}\partial_- Y^{c\dot{d}}), \\ \mathscr{L}_{BF} &= \frac{i}{2}(\theta_{-a\dot{\beta}}\partial_-\theta_-^{a\dot{\beta}} + \eta_{-\beta\dot{a}}\partial_-\eta_-^{\beta\dot{a}})(Y_{b\dot{c}}Y^{b\dot{c}} - Z_{\gamma\dot{\alpha}}Z^{\gamma\dot{\alpha}}) + i\,\theta_{-a\dot{\beta}}\,\theta_-^{b\dot{\beta}}\,Y_{b\dot{c}}\,\partial_- Y^{a\dot{c}} \\ &+ i\,\eta_{-\beta\dot{a}}\,\eta_-^{\beta\dot{b}}\,Y_{c\dot{b}}\,\partial_- Y^{c\dot{a}} - i\,\theta_{-a\dot{\alpha}}\,\theta_-^{a\dot{\beta}}\,Z_{\gamma\dot{\beta}}\,\partial_- Z^{\gamma\dot{\alpha}} - i\,\eta_{-\alpha\dot{a}}\,\eta_-^{\beta\dot{a}}\,Z_{\beta\dot{\gamma}}\,\partial_- Z^{\alpha\dot{\gamma}} \\ &+ 2i\,\theta_{-a\dot{\gamma}}\,\partial_-\eta_-^{\alpha\dot{c}}\,Y^a_{\dot{c}}\,Z^{\dot{\gamma}}_{\alpha} - 2i\,\eta_{-\gamma\dot{a}}\,\partial_-\theta^{c\dot{\alpha}}\,Y^{\dot{a}}_c\,Z^{\gamma}_{\dot{\alpha}}. \end{aligned} \tag{5.122}$$

Naturally some comments are in order. First of all, the action is obviously invariant under the bosonic $\mathrm{SU}(2)^4$ symmetry due to the covariant notation. It was to achieve this that we had to pick such a complicated combination in (5.120). We also see that the action is considerably simpler than the full near BMN action and except for two of the quartic $\mathscr{L}_{FF}$ terms, the higher order interactions containing fermionic terms only depend on η_- and θ_-. Except for these two term, the action is structurally the same as the one presented in ?. To establish a precise connection with ?, we first note that the quartic pure fermion terms can be written as

$$\frac{1}{2}(\eta_{-\alpha\dot{a}}\eta_{-\gamma\dot{c}}\,\eta_+^{\gamma\dot{a}}\partial_-\eta_-^{\alpha\dot{c}} + \theta_{-a\dot{\alpha}}\theta_{-c\dot{\gamma}}\,\theta_+^{a\dot{\gamma}}\partial_-\theta_-^{c\dot{\alpha}}) =$$
$$-\frac{1}{6}\,\partial_-(\eta_{-\alpha\dot{a}}\eta_{-\gamma\dot{c}}\,\eta_-^{\alpha\dot{c}})\eta_+^{\gamma\dot{a}} - \frac{1}{6}\,\partial_-(\theta_{-a\dot{\alpha}}\theta_{-c\dot{\gamma}}\,\eta_-^{c\dot{\alpha}})\theta_+^{a\dot{\gamma}}.$$

Thus, if we were to shift the η_+ and θ_+ variables as

$$\eta_{+\alpha\dot{a}} \to \eta_{+\alpha\dot{a}} + \frac{i}{12}\,\eta_{-\gamma\dot{a}}\eta_{-\alpha\dot{c}}\,\eta_-^{\gamma\dot{c}}, \quad \theta_{+a\dot{\alpha}} \to \theta_{+a\dot{\alpha}} + \frac{i}{12}\,\theta_{-a\dot{\gamma}}\theta_{-c\dot{\alpha}}\,\eta_-^{c\dot{\gamma}}, \tag{5.123}$$

we can, up to a total derivative, remove the quartic terms involving η_+ and θ_+. Of course, this induces additional quartic fermion interactions through the quadratic mass terms and the shifted $\mathscr{L}_{FF}$ equals

$$\begin{aligned}&\mathscr{L}_{FF}^{shifted} = \\ &i\,\eta_{+\alpha\dot{a}}\partial_-\eta_+^{\alpha\dot{a}} + i\,\theta_{+a\dot{\alpha}}\partial_-\theta_+^{a\dot{\alpha}} + i\,\eta_{-\alpha\dot{a}}\partial_+\eta_-^{\alpha\dot{a}} + i\,\theta_{-a\dot{\alpha}}\partial_+\theta_-^{a\dot{\alpha}} \\ &-i\,\eta_{-\alpha\dot{a}}\,\eta_+^{\alpha\dot{a}} - i\,\theta_{-a\dot{\alpha}}\,\theta_+^{a\dot{\alpha}} + \frac{1}{12}(\eta_{-\alpha\dot{a}}\eta_{-\gamma}^{\dot{a}}\eta_{-\dot{c}}^{\alpha}\,\eta_-^{\gamma\dot{c}} + \theta_{-a\dot{\alpha}}\theta_{-\dot{\gamma}}^{a}\theta_{-c}^{\dot{\alpha}}\theta_-^{c\dot{\gamma}}),\end{aligned} \tag{5.124}$$

which is of the same form as the Lagrangian found in ?.

It is not very surprising that we find a quartic fermionic theory different than the one found in ?. There the quartic fermions only involve η_- and θ_- without any derivatives. However, since we have just shown how to get the NFS theory from the near BMN string, whose quadratic part is known to incorporate the full supergravity dynamics, it is rather odd to have a non derivative higher order term. The supergravity limit can loosely speaking be defined as the $\sigma \to 0$ limit, which for the NFS limit implies $\partial_- \sim g^{-1}\,\partial_+$ and thus kills all higher order terms in (5.122). However, in the coordinates of ? the $\mathscr{L}_{FF}^4 \sim \eta_-^4$ term survives. To remove this term one would need to investigate the first and second order constraints, or equivalently shift the fermions through an unitary transformations, to obtain the correct particle limit, see ?, ? and ?. This shift would probably introduce a term as the one we found in (5.122). However, from a computational point of view the two Lagrangians are equivalent and, perhaps, it is aesthetically more pleasing to have a higher order Lagrangian that does not mix in the left moving fermions.

Before we close this section we would like to point out another scaling of the fermions that give a similar, but not equivalent[9], theory. We now associate a, b, c and d in (5.118)

[9] Equivalent in the sense of containing the same symbolical expressions. It could be so that the physical observables coincide.

with $\eta_\pm$ and $\theta_\pm$ directly as

$$\eta_\pm^{\alpha\dot{k}} = \frac{1}{\sqrt{2}}(a^{\alpha\dot{k}} \pm b^{\alpha\dot{k}}), \qquad \theta_\pm^{\dot{k}\alpha} = \frac{1}{\sqrt{2}}(c^{\dot{k}\alpha} \pm d^{\dot{k}\alpha}). \tag{5.125}$$

Implementing the scalings (5.121) gives a quadratic theory identical to (5.122) and the pure bosonic interaction terms naturally remains the same. However, now the higher order BF and FF terms are symmetric in $\eta_\pm$ and $\theta_\pm$, that is, the action is symmetric under the exchange

$$\eta_\pm \to \eta_\mp, \qquad \theta_\pm \to \theta_\mp.$$

We will not present the full Lagrangian here, but since it only contains right moving derivatives, it is still simpler than the full near BMN Lagrangian. Also, somewhat surprisingly, one can perform a fermionic shift and reexpress the quartic fermion terms purely in terms of η_- and θ_-[10]. However, one still have mixing between left and right moving fermions in the BF part of the Lagrangian, and it seems that these terms can not be removed through a fermionic shift. In this sense, this scaling of the fermions seem to give a similar, but not identical theory. It is unclear to us what the physical content of this theory is.

[10]Or equivalently, solely in terms of η_+ and θ_+.

6 The $\mathrm{AdS}_3\times$ S^3 string at strong coupling

We now turn to the non critical $\mathrm{AdS}_3\times\mathrm{S}^3$ superstring which, in many ways, is very similar to the $\mathrm{AdS}_5\times\mathrm{S}^5$ string. The main difference lie in the form of the symmetry algebra which now is of a direct product type, $\mathrm{PSU}(1,1|2)^2$. Even though different, it turns out that one is able to construct a group element, as in (4.86), with a transverse bosonic part almost identical to the ten dimensional case. Also, the fermionic matrix, η, now consist of eight complex fermions, reduced by half through the κ gauge, and one can decompose the fermions in a way similar to the η matrix of the $\mathrm{AdS}_5\times\mathrm{S}^5$ string. This allows one to immediately take expressions, as for example the near BMN Hamiltonian in (5.34)-(5.36), and truncate them to the six dimensional string.

As we mentioned in part one, we will only study the non critical string. The reason for this is that the $\mathrm{AdS}_3\times\mathrm{S}^3$ string with the compact T^4 factor included, do not allow for a simple coset construction[1]. To quadratic order, one can add the T^4 factor by hand, but beyond leading order a non-trivial mixing between the six and four dimensional parts occur[2].

The presentation in this chapter should be viewed as an investigation of a non critical (and highly non trivial) string theory and not a check of the AdS_3 / CFT_2 duality. For this gauge / string correspondence a D1-D5 brane system wraps the T^4 factor and this factor naturally needs to be included in the full analysis.

The outline of this chapter is as follows; We start the exposition with a construction of the group element and how to obtain the quartic near BMN Hamiltonian. We rely heavily on the former chapter, so for details, please refer to the main text there, especially section 5.1 and 5.2. We then turn to an investigation of the symmetry algebra, which after gauge fixing is $\mathrm{SU}(1|1)^2$. The form of the generators are in direct analogue to the $\mathrm{PSU}(2,2|4)$ case, and from **?** it becomes more or less obvious that also the non critical $\mathrm{SU}(1|1)^2$ gets centrally extended in a similar way.

6.1 Parametrization

As in the $\mathrm{AdS}_5\times\mathrm{S}^5$ case we start out by building the group element, which as in (4.86), is of the form

$$G=\Lambda(t,\phi)\,f(\eta)\,g(x). \tag{6.1}$$

[1] However, one can go from the $\mathrm{AdS}^3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ string and construct a truncation such that the resulting theory coincide with $\mathrm{AdS}_3\times\mathrm{S}^3\times T^4$, see **?**.

[2] Also, it is not clear how to fix the κ gauge in a consistent way. At least not when starting directly from the $\mathrm{AdS}_3\times\mathrm{S}^3\times T^4$ string, see **?** for a discussion regarding this issue.

The transverse part is basically just the truncation of two transverse coordinates in (5.1)

$$g(x) = \begin{pmatrix} g_a(z) & 0 \\ 0 & g_s(y) \end{pmatrix}$$

with

$$g_a(z) = (1 - \frac{1}{4}z^2)^{-1/2}(1 + \frac{1}{2}z_a\gamma^a), \quad g_s(y) = (1 + \frac{1}{4}y^2)^{-1/2}(1 + \frac{i}{2}y_a\gamma^a), \tag{6.2}$$

where the γ matrices are those used in (4.44).

The light-cone coordinates enter as

$$\Lambda(t,\phi) = \exp(\frac{i}{2}(x_+\Sigma_+ + x_-\Sigma_-)) \tag{6.3}$$

where

$$\Sigma_\pm = \begin{pmatrix} \pm\Sigma & 0 \\ 0 & \Sigma \end{pmatrix},$$

and the fermionic contributions are incorporated through

$$f(\eta) = \eta + \sqrt{1+\eta^2}. \tag{6.4}$$

As described in (4.82), the κ gauge can be defined as

$$\{\Sigma_+, \eta\} = 0, \tag{6.5}$$

which boils down to

$$\{\Sigma, \theta_{4\times4}\} = 0, \qquad \{\Sigma, \eta_{4\times4}\} = 0, \tag{6.6}$$

and reduces the number of fermions from eight to four complex ones. In matrix notation, $\theta^{g.f}_{4\times4}$ equals

$$\theta^{g.f}_{4\times4} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ \theta_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & \theta_{43} & 0 \end{pmatrix}.$$

Introducing a complex combination of the Γ-matrices

$$\Gamma = \frac{1}{2}(\gamma_2 - i\gamma_1), \quad \Gamma^\dagger = \frac{1}{2}(\gamma_2 + i\gamma_1), \tag{6.7}$$

and noting that η is of the form

$$\eta = \begin{pmatrix} 0 & \theta_{4\times4} \\ \eta_{4\times4} & 0 \end{pmatrix},$$

allows us to expand each matrix in a basis of Γ's as

$$\theta_{4\times 4} = \mathcal{P}_+(\theta_1^\dagger + \theta_2) + \mathcal{P}_-(\eta_1 + \eta_2^\dagger), \quad \eta_{4\times 4} = \mathcal{P}_+\,\Sigma(\theta_1 + \theta_2^\dagger) + \mathcal{P}_-\,\Sigma(\eta_1^\dagger + \eta_2), \tag{6.8}$$

where we introduced the projection operators

$$\mathcal{P}_+ = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{P}_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix},$$

together with the notation $\theta_i = \theta_i\,\Gamma, \eta_i = \eta_i\,\Gamma$. In this notation, θ_i belongs to the first and η_i to the second copy of $\mathfrak{su}(1,1|2)$. It is also convenient to introduce the notation

$$\theta_i^\dagger = \theta^{i,\dagger}, \qquad \eta_i^\dagger = \eta^{i,\dagger}, \tag{6.9}$$

so the block matrices can be written as

$$\theta_{4\times 4} = \begin{pmatrix} 0 & i\,\theta^{1,\dagger} & 0 & 0 \\ -i\,\theta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\,\eta_1 \\ 0 & 0 & i\,\eta^{2,\dagger} & 0 \end{pmatrix}, \quad \eta_{4\times 4} = \begin{pmatrix} 0 & i\,\theta^{2,\dagger} & 0 & 0 \\ i\,\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\,\eta_2 \\ 0 & 0 & -i\,\eta^{1,\dagger} & 0 \end{pmatrix}.$$

We introduced the projection operators $\mathcal{P}_\pm$ above so that we can establish a direct connection with the quartic AdS$_5\times$S^5 Hamiltonian in (5.34), (5.35) and (5.36). Using that expression, we find that the quartic AdS$_3\times$S^3 Hamiltonian equals

$$\begin{aligned}\mathcal{H}_4 = \frac{\widetilde{\lambda}}{2\,P_+}\Big[&2\,(y'^2\,z^2 - z'^2\,y^2 + z'^2\,z^2 - y'^2\,y^2) \\ &+ Str\Big((z^2-y^2)\,\eta'^2 + \frac{1}{2}\,x'_m\,x_n\,[\Sigma_m,\Sigma_n]\,[\eta,\eta'] - 2\,x_m\,x_n\,\Sigma_m\,\eta'\,\Sigma_n\,\eta'\Big) \\ &+ i\,\kappa\,\frac{\sqrt{\widetilde{\lambda}}}{4}\,x_m\,p_n\,Str\Big([\Sigma_m,\Sigma_n]\,[\Upsilon\,\eta^{st}\,\Upsilon,\eta]'\Big)\Big],\end{aligned} \tag{6.10}$$

where the quartic fermionic dependence vanishes due to the simple form of the κ fixed η matrix.

6.2 Transverse U(1) charges

There are in total four U(1) charges of SU$(1,1|1)^2$ and two survives when restricting to the projective groups. The two surviving U(1) charges correspond to rotations in the z_1, z_2 and y_1, y_2 plane. Or, for the complex combinations

$$Z = z_2 + i\,z_1, \quad Y = y_2 + i\,y_1, \qquad P_z = \frac{1}{2}(p_2^z + i\,p_1^z), \quad P_y = \frac{1}{2}(p_2^y + i\,p_1^y), \tag{6.11}$$

they correspond to constant complex shifts of Z and Y. As we described in section 4.2.2, a transformation on the group element (6.1) acts from the left

$$\mathfrak{g}\,G = G'\,\mathfrak{h}, \tag{6.12}$$

where $\mathfrak{h}$ is a compensating transformation from $\mathfrak{so}(1,2)\times\mathfrak{so}(3)$. Since we want to find the elements that generate shifts in the transverse fields but leaves the light-cone directions invariant, the transformations should take values in $\mathcal{J}_B$. For a $\mathfrak{g}\in\mathcal{J}_B$, its action on G is

$$\mathfrak{g}\,G = \Lambda(t,\phi)\,\mathfrak{g}\,f(\eta)\,\mathfrak{g}^{-1}\,\mathfrak{g}\,g\,\mathfrak{g}^{-1}\,\mathfrak{g}. \tag{6.13}$$

If $\mathfrak{g}$ obeys the property, $-\Upsilon\,\mathfrak{g}^{st}\,\Upsilon^{-1} = \mathfrak{g}$, then $\mathfrak{g}$ itself is the compensating transformation from $\mathfrak{so}(1,2)\times\mathfrak{so}(3)$ and $f(\eta)$ and g_t transform in the adjoint of $\mathfrak{g}$.

For $\mathfrak{g}\in M^{(0)}$, one finds from (4.39) that $x=-\tilde{x}$ and $y=-\tilde{y}$ so these transformations constitute a $\mathfrak{u}(1)\oplus\mathfrak{u}(1)$ subgroup of $\mathcal{J}_B$ and these are the charges that generate shifts in the dynamical variables.

To figure out how the fields transform under the U(1)'s, we vary the group element with respect to the shifts (which we denote ϕ from now)

$$\delta_\phi\,G = \delta\Lambda\,f\,g + \Lambda\,\delta f\,g + \Lambda\,f\,\delta g = [\phi,\Lambda]\,f\,g + \Lambda\,[\phi,f]\,g + \Lambda\,f(\phi\,g - g\,\phi_c) + \mathcal{O}(\phi^2)$$

where ϕ_c is the compensating SO(1,2)×SO(3) transformation. Thus,

$$\delta\Lambda = [\phi,\Lambda], \quad \delta f = [\phi,f], \quad \delta g = \phi\,g - g\,\phi_c. \tag{6.14}$$

Since g by construction belongs to $M^{(2)}$ it satisfies, $\delta g = \Upsilon\,\delta g^{st}\,\Upsilon^{-1}$ which, together with the fact that $\phi_c\in M^{(0)}$, gives

$$\phi\,g - g\,\mathcal{K}\,\phi^{st}\,\mathcal{K}^{-1} = \phi_c\,g + g\,\phi_c. \tag{6.15}$$

Close to the identity we know that ϕ_c should take values in $M^{(0)}$ which implies that $\phi=\phi_c$ so

$$\delta_\phi\,g = [\phi,g]. \tag{6.16}$$

One suitable representation for ϕ is

$$\phi = \begin{pmatrix} \alpha_S\,\Phi & 0 \\ 0 & \beta_J\,\Phi \end{pmatrix}, \qquad \Phi = \frac{i}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad -\Upsilon\,\phi^{st}\,\Upsilon^{-1} = \phi, \tag{6.17}$$

so that $e^{\phi}\, g\, e^{-\phi}$ generates the shifts

$$\begin{array}{ccccccc} Z & \to & e^{i\alpha}\, Z, & & Z^{\dagger} & \to & e^{-i\alpha}\, Z^{\dagger}, \\ Y & \to & e^{i\beta}\, Y, & & Y^{\dagger} & \to & e^{-i\beta}\, Y^{\dagger}. \end{array} \tag{6.18}$$

From this we find the charge for the bosonic fields under the U(1) transformations. Of course, how the bosonic fields transform is trivial and we do not need to construct the specific matrix form of the charges. However, when we now turn to the construction of all the charges of the symmetry algebra, then ϕ will turn out to be a convenient building block.

6.3 Symmetry algebra

In general, the $\mathfrak{su}(1,1|2) \oplus \mathfrak{su}(1,1|2)$ algebra consist of charges of the form

$$\mathfrak{su}(1,1|2) \oplus \mathfrak{su}(1,1|2) = \begin{pmatrix} \mathbb{L} & 0 & \mathbb{Q} & 0 \\ 0 & \dot{\mathbb{L}} & 0 & \dot{\mathbb{Q}} \\ \overline{\mathbb{Q}} & 0 & \mathbb{R} & 0 \\ 0 & \dot{\overline{\mathbb{Q}}} & 0 & \dot{\mathbb{R}} \end{pmatrix}, \tag{6.19}$$

where $\overline{\mathbb{Q}} = -\mathbb{Q}^{\dagger}\sigma_3$, and similar for dotted ones. We are interested in the effective symmetry algebra that leaves the light-cone Hamiltonian invariant which is defined by

$$\mathcal{J} \quad : \quad \mathfrak{g} \in \mathfrak{su}(1,1|2) \oplus \mathfrak{su}(1,1|2), \quad [\mathfrak{g}, \Sigma_{+}] = 0, \tag{6.20}$$

implying that $\mathcal{J}$ is spanned by matrices of the form

$$\widetilde{M} = \begin{pmatrix} x_{11} & 0 & 0 & 0 & x_{15} & 0 & 0 & 0 \\ 0 & -x_{11} & 0 & 0 & 0 & x_{26} & 0 & 0 \\ 0 & 0 & \tilde{x}_{33} & 0 & 0 & 0 & \tilde{x}_{37} & 0 \\ 0 & 0 & 0 & -\tilde{x}_{33} & 0 & 0 & 0 & \tilde{x}_{48} \\ -x_{15}^{\dagger} & 0 & 0 & 0 & y_{44} & 0 & 0 & 0 \\ 0 & x_{26}^{\dagger} & 0 & 0 & 0 & -y_{44} & 0 & 0 \\ 0 & 0 & -\tilde{x}_{37}^{\dagger} & 0 & 0 & 0 & \tilde{y}_{55} & 0 \\ 0 & 0 & 0 & \tilde{x}_{48}^{\dagger} & 0 & 0 & 0 & -\tilde{y}_{55} \end{pmatrix}, \tag{6.21}$$

with purely imaginary diagonal elements. It is easy to see that $\widetilde{M}$ takes values in $\mathfrak{su}(1|1) \oplus \mathfrak{su}(1|1)$ with bosonic part $\mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) = \mathcal{J}_B$.

As we explained earlier, the charges can be derived from the equations of motions, see

(4.67), and are given by a super matrix

$$Q = \int d\sigma\, Str\Big(\Lambda(t,\phi)\, f(\eta)\, g\Big(\pi - \kappa\, \frac{i}{2}\sqrt{\widetilde{\lambda}}\, g\, \mathcal{K}\, \partial_1 \eta^{st}\, \mathcal{K}^{-1}\, g^{-1}\Big) g^{-1}\, f^{-1}(\eta)\, \Lambda(t,\phi)^{-1}\Big)\mathcal{M}, \tag{6.22}$$

where $\mathcal{M}$ is some constant matrix in $\mathfrak{su}(1|1)\oplus\mathfrak{su}(1|1)$ that single out the specific charges corresponding to shifts and supersymmetry transformations.

For $\mathcal{M} = \Sigma_{\pm}$ we have

$$\mathcal{H} = -\frac{i}{2}\, Str\, Q\, \Sigma_{+}, \qquad P_{+} = \frac{i}{2}\, Str\, Q\, \Sigma_{-}, \tag{6.23}$$

where by construction the Hamiltonian is central and to leading order its given by

$$\begin{aligned} \mathcal{H}_2 = \frac{1}{2}\int d\sigma \Big\{ & 4(P_z^{\dagger}\, P_z + P_y^{\dagger}\, P_y) + Z^{\dagger}\, Z + Y^{\dagger}\, Y + \widetilde{\lambda}(Z'^{\dagger}\, Z' + Y'^{\dagger}\, Y') \\ & + \eta^{\alpha,\dagger}\, \eta_{\alpha} - \theta^{\alpha,\dagger}\, \theta_{\alpha} + \kappa\sqrt{\widetilde{\lambda}}(\eta^{\alpha,\dagger}\, \theta'_{\alpha} + \theta'^{\alpha,\dagger}\, \eta_{\alpha} - \eta'^{\alpha,\dagger}\, \theta_{\alpha} - \theta^{\alpha,\dagger}\, \eta'_{\alpha})\Big\}, \end{aligned} \tag{6.24}$$

where we also introduced the convenient notation

$$\overline{Q}^{\alpha} = -\sigma_3^{\alpha\beta} Q^{\dagger}_{\beta}, \qquad \dot{\overline{Q}}^{\alpha} = -\sigma_3^{\alpha\beta}\, \dot{Q}^{\dagger}_{\beta}, \tag{6.25}$$

which also implies that the charge matrix in (6.22) equals

$$\overline{\mathbb{Q}} = \begin{pmatrix} \overline{Q}^1 & 0 \\ 0 & Q_2 \end{pmatrix}, \qquad \mathbb{Q} = \begin{pmatrix} Q_1 & 0 \\ 0 & \overline{Q}^2 \end{pmatrix}, \qquad \dot{\overline{\mathbb{Q}}} = \begin{pmatrix} \dot{Q}_1 & 0 \\ 0 & \dot{\overline{Q}}^2 \end{pmatrix}, \qquad \mathbb{Q} = \begin{pmatrix} \dot{\overline{Q}}^1 & 0 \\ 0 & Q_2 \end{pmatrix}.$$

Only focusing of the undotted algebra[3], we find that

$$\{Q_{\alpha}\, \overline{Q}^{\beta}\}_{P.B} = -i(\mathbb{L} + \mathbb{R})\delta_{\alpha}^{\beta}, \qquad \{\dot{Q}_{\alpha}\, \dot{\overline{Q}}^{\beta}\}_{P.B} = i(\dot{\mathbb{L}} + \dot{\mathbb{R}})\delta_{\alpha}^{\beta}, \tag{6.26}$$

and

$$\begin{aligned} &[\mathbb{L} + \mathbb{R}, Q_{\alpha}]_{P.B} = [\mathbb{L} + \mathbb{R}, \overline{Q}^{\alpha}]_{P.B} = 0, \\ &[\dot{\mathbb{L}} + \dot{\mathbb{R}}, \dot{Q}_{\alpha}] = [\dot{\mathbb{L}} + \dot{\mathbb{R}}, \dot{\overline{Q}}^{\alpha}] = 0. \end{aligned} \tag{6.27}$$

The bosonic charges are expressed as

$$\begin{aligned} \mathbb{L} &= P_{+}^{2} \otimes (\mathcal{P}_{+}\Phi) \qquad & \mathbb{R} &= P_{-}^{2} \otimes (\mathcal{P}_{+}\Phi), \\ \dot{\mathbb{L}} &= P_{+}^{2} \otimes (\mathcal{P}_{-}\Phi) \qquad & \dot{\mathbb{R}} &= P_{-}^{2} \otimes (\mathcal{P}_{-}\Phi), \end{aligned} \tag{6.28}$$

[3] The other copy follows trivially.

with Φ from (6.17) and $P^2_\pm$ are projection operators

$$P^2_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P^2_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that the bosonic charges corresponding to the shifts commute with everything so they are central.

With all this, and treating the x^- vertex as a static object, we find that the quadratic supercharges equal

$$Q_\alpha = \tag{6.29}$$
$$\frac{1}{2}\int d\sigma\, e^{i\,dx_-}\Big(\epsilon_{\alpha\beta}(2\,P_z + i\,Z)\,\theta^{\beta,\dagger} + i(2\,P_y^\dagger - i\,Y^\dagger)\,\theta_\alpha - \kappa\sqrt{\widetilde{\lambda}}(i\,\epsilon_{\alpha\beta}\,Z\,\eta'^{\beta,\dagger} + Y^\dagger\,\eta'_\alpha)\Big),$$

$$\overline{Q}^\alpha =$$
$$-\frac{1}{2}\int d\sigma\, e^{-i\,dx_-}\Big(\epsilon^{\alpha\beta}(2\,P_z^\dagger - i\,Z^\dagger)\,\theta_\beta - i(2\,P_y + i\,Y)\,\theta^{\alpha,\dagger} + \kappa\sqrt{\widetilde{\lambda}}(i\,\epsilon^{\alpha\beta}\,Z^\dagger\,\eta'_\beta - Y\,\eta'^{\alpha,\dagger})\Big),$$

$$\dot{Q}_\alpha =$$
$$-\frac{1}{2}\int d\sigma\, e^{i\,dx_-}\Big(\epsilon_{\alpha\beta}(2\,P_z - i\,Z)\eta^{\beta,\dagger} - i(2\,P_y^\dagger + i\,Y^\dagger)\eta_\alpha + \kappa\sqrt{\widetilde{\lambda}}(i\,\epsilon_{\alpha\beta}\,Z\,\theta'^{\beta,\dagger} - Y^\dagger\,\theta'_\alpha)\Big),$$

$$\dot{\overline{Q}}^\alpha =$$
$$\frac{1}{2}\int d\sigma\, e^{-i\,dx_-}\Big(\epsilon^{\alpha\beta}(2\,P_z^\dagger + i\,Z^\dagger)\eta_\beta + i(2\,P_y - i\,Y)\eta^{\alpha,\dagger} - \kappa\sqrt{\widetilde{\lambda}}(i\,\epsilon^{\alpha\beta}\,Z^\dagger\,\theta'_\beta + Y\,\theta'^{\alpha,\dagger})\Big).$$

For the bosonic charges, we combine them into the combinations $\mathbb{L} + \dot{\mathbb{L}}$ and $\mathbb{R} + \dot{\mathbb{R}}$ since these are the combinations that generate the complex shifts (6.18). Up to a total derivative the combinations equal

$$\begin{aligned} \mathbb{L} + \dot{\mathbb{L}} &= \int d\,\sigma\Big(i(P_z^\dagger\,Z - Z^\dagger\,P_z) - \frac{1}{2}\eta^{\alpha,\dagger}\,\eta_\alpha - \frac{1}{2}\theta^{\alpha,\dagger}\,\theta_\alpha\Big), \\ \mathbb{R} + \dot{\mathbb{R}} &= \int d\,\sigma\Big(i(P_y^\dagger\,Y - Y^\dagger\,P_y) - \frac{1}{2}\eta^{\alpha,\dagger}\,\eta_\alpha - \frac{1}{2}\theta^{\alpha,\dagger}\,\theta_\alpha\Big). \end{aligned} \tag{6.30}$$

Now, if we postulate

$$\begin{aligned} &[P_z(\sigma), Z^\dagger(\sigma')] = -i\,\delta(\sigma - \sigma'), \qquad [P_y(\sigma), Y^\dagger(\sigma')] = -i\,\delta(\sigma - \sigma'), \\ &\{\eta_\alpha(\sigma), \eta^{\beta,\dagger}(\sigma')\} = \delta_\alpha^\beta\,\delta(\sigma - \sigma'), \qquad \{\theta_\alpha(\sigma), \theta^{\beta,\dagger}(\sigma')\} = \delta_\alpha^\beta\,\delta(\sigma - \sigma'), \end{aligned} \tag{6.31}$$

then one finds that the charges in (6.29) satisfy the anti commutation relations

$$\{Q_\alpha \overline{Q}^\beta\} = -(\mathbb{L}+\mathbb{R})\delta^\beta_\alpha, \qquad \{\dot{Q}_\alpha \dot{\overline{Q}}^\beta\} = (\dot{\mathbb{L}}+\dot{\mathbb{R}})\delta^\beta_\alpha, \tag{6.32}$$

which indeed is the $\mathfrak{su}(1|1)\oplus\mathfrak{su}(1|1)$ algebra. If we combine the bosonic charges, we find that they correspond to the Hamiltonian

$$\mathbb{L}+\mathbb{R}-\dot{\mathbb{L}}-\dot{\mathbb{R}} = \mathbb{H}. \tag{6.33}$$

Even though we have not performed the calculation in detail, the commutators between the supercharges, $\{\overline{\mathbb{Q}},\overline{\mathbb{Q}}\}$ and $\{\mathbb{Q},\mathbb{Q}\}$, should extend the quantum algebra with two central charges proportional to the level matching constraint, see **?** for a detailed discussion.

6.4 Outlook

A very interesting continuation of this work would be to calculate the central extensions in more detail. Even though the outline is very similar to **?**, where the central extension of the AdS$_5\times$S^5 string was derived, one should still do it in detail. Also, in all of the analysis we ignored the T^4 factor since its inclusion severely complicates the coset construction. it would be very interesting to investigate the symmetry algebra with this factor included. Rather recently in **?** it was shown that one can treat the AdS$_3\times$S^3 $\times T^4$ string as a reduction from a coset model with an exceptional superalgebra as G. Using this as a starting point, one should be able to investigate the gauge fixed symmetry algebra in detail.

7 The $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\kern1pt}3}$ string at strong coupling

In this chapter we will describe the strong coupling dynamics of the $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\kern1pt}3}$ string. The discussion will be similar to section 5.2 but is much more involved since, rather strangely perhaps, the $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\kern1pt}3}$ is significantly more complicated than its $\mathrm{AdS}_5 \times \mathrm{S}^5$ cousin. This is a bit surprising since we saw earlier that the symmetry group that leaves the light-cone Hamiltonian invariant, namely $\mathrm{SU}(2|2) \times \mathrm{U}(1)$, is rather similar to $\mathrm{SU}(2|2)^2$ of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string. However, one reason for the more complicated structure can be found in the presence of cubic terms in the near BMN Lagrangian. Since we will present a similar analysis as before, namely comparing energy shifts with light-cone Bethe equations, it is crucial that we have a canonical Lagrangian both for bosons and fermions which demands that we shift the fermions in an appropriate way. Due to the cubic terms in the Lagrangian, this shift, in contrast to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case, severely complicates the canonical theory. Nevertheless, it can be performed which allows one to perform a perturbative analysis of the energy levels of string configurations.

We will start out this chapter by constructing the $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\kern1pt}3}$ super string along lines similar to section 5.2 with an emphasis on the light-cone Hamiltonian. The starting point will be the supercoset model presented by Arutyunov and Frolov in **?**, as described in section 4.2.1, from which we derive the quartic string Lagrangian. As we mentioned above, the situation becomes rather complicated due to the non canonical structure of the fermions. We then calculate energy shifts for a large set of both bosonic and fermionic string states following **?** and **?**. These shifts we match against a conjectured set of asymptotic Bethe equations and find precise agreement.

7.1 Introduction and background

Recently strings on $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\kern1pt}3}$ have enjoyed an increased interest due to the AdS_4 / CFT_3 duality proposed in **?**, **?**, . The conjecture, nowadays dubbed ABJM duality in the literature, states that a three dimensional $\mathcal{N} = 6$ and SU(N) Chern Simons theory living on the boundary of AdS_4 are in certain limits dual to type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\kern1pt}3}$.

The duality exhibits many shared features with the well studied AdS_5 / CFT_4 correspondence, where perhaps the most striking similarity is the emergence of integrable structures **?**, **?**. On the gauge theory side, integrability was demonstrated for the two loop Hamiltonian[1] in **?**. Quickly after, the algebraic curve encoding all the classical solu-

[1]The one loop piece vanishes trivially.

tions at strong and weak coupling together with the all loop asymptotic Bethe equations were put forward in **?**, **?**, **?**. There after, and under the assumptions of a SU(2|2)×U(1) symmetry, the exact S matrix were proposed in **?**. Following these findings, a host of various checks and higher order calculations have been performed **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**.

That all this has been achieved with such a rapid progress is remarkable since in both dualities the full dynamics can be constructed from symmetry arguments alone. For ABJM, the symmetry group is OSP(2, 2|6), which differs quite much from the well known PSU(2, 2|4) of AdS_5 / CFT_4. Nevertheless, planar integrability, all loop asymptotic Bethe equations, SU(2|2) scattering and central extension occur in similar ways in both dualities.

In this chapter we will perform a detailed study of the $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\not{\ }}3}$ string. Starting from the symmetry group we derive the full Lagrangian in a uniform light-cone gauge following similar procedures as those outline in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ section.

As has been demonstrated by Bykov in **?**, the symmetry of the gauge fixed string reduces from OSP(2, 2|6) to a centrally extended SU(2|2)×U(1). This is rather similar to the superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ which, as we saw in table 4.2 have a centrally extended $\mathrm{SU}(2|2)^2$ algebra **?**. Even though the gauge fixed subalgebras are rather similar, we find that the general structure of the type IIA superstring is considerably more involved than its $\mathrm{AdS}_5 \times \mathrm{S}^5$ cousin.

After we have established the exact string Lagrangian, covariant under SU(2|2)×U(1), we turn to a perturbative expansion in the string coupling g. Taking the coupling large we derive the full Hamiltonian up to quartic order in number of fields.

To avoid the rather severe complications of gauge fixing the worldsheet metric, we, as in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case, work in a first order formalism. This has the upshot that the metric components only enters as Lagrange multipliers. However, the theory exhibits higher order fermionic worldsheet time derivatives and to preserve a canonical Poisson structure we need to shift the fermions in a appropriate way. Unfortunately, and in contrast to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string, this shift adds a 'self interacting' term which is very hard to remove. Not only is the structure complicated, but it also introduces corrections to the bosonic momentas. The way we approach this problem is to only present the canonical Hamiltonian for pure boson / fermion fields. For the reader interested in the full dynamics of the theory, we present the full Hamiltonian, prior to the shift, in the appendix.

Having established the first order theory to quartic order, we calculate energy corrections to a certain set of bosonic and fermionic string states. Even though the general structure of relevant parts of the Hamiltonian is rather involved, we find that the energy shifts takes a remarkably simple form. As we did for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string, we then match the energy shifts with the predictions coming from a conjectured set of Bethe equations proposed in **?**, and rewritten in a light-cone language in **?** and **?**.

After this we turn to investigate the role of the massive modes of the theory. At the quadratic level the string oscillators come in $4_F + 4_B$ heavy and light modes respectively. From the point of view of the conjectured exact scattering theory **?**, the fundamental excitations in the S matrix are the light modes, leaving us with a miss match between the degrees of freedom.

In **?** Zarembo calculated the loop corrections for a massive bosonic mode. There it was found that when quantum corrections are taken into account, the analytic properties of the propagator changes. What happens is that the pole gets shifted onto the branch cut and vanishes. Therefore the heavy mode is not fundamental but rather a composite continuum state of two light particles.

We continue this line of research by showing that exactly the same thing happens with the remaining massive bosons. Even though we do not calculate it explicitly, we also provide some general arguments for why the same thing should happen with the remaining massive fermionic coordinates.

This chapter is organized as follows; We start out in section two by presenting some general facts about the (super)matrix representation of the $\mathfrak{osp}(2,2|6)$ algebra. Then by making use of the $\mathbb{Z}_4$ grading of the algebra, we construct the exact string Lagrangian in a convenient kappa and light-cone gauge. In section three we expand the derived theory in a strong coupling limit, equivalent to a near plane-wave expansion, to quartic order. We find that the theory exhibits higher order time derivatives of the fermions, and thus naively introduces a complicated Poisson structure. To tackle this problem, we follow **?** and introduce a fermionic shift with the property that it removes the higher order kinetic terms. Sadly, this shift comes with the price of adding additional cubic and quartic terms to the interacting Hamiltonian. In section four we turn to a perturbative analysis of the string spectrum by calculating energy shifts for fermionic states. These we then match with a set of uniform light-cone Bethe equations, finding perfect agreement. The last analysis we perform is to calculate loop diagrams for the bosonic heavy modes in section six. We show that all the massive bosonic modes dissolve into a two particle continuum, and therefore, do not appear as fundamental excitations of the scattering theory.

7.2 Group parameterization and string Lagrangian

There are many ways to parameterize OSP(2,2|6) and they are all related through non linear field transformations. In this thesis we will use a particulary suitable representation that allows us to fix the bosonic and fermionic worldsheet symmetries in a convenient way **?**.

The starting point is a group element of the form (4.86)

$$G = \Lambda(x^+, x^-)\, f(\eta)\, G_t, \tag{7.1}$$

where the different components are given by

$$\Lambda(x^+, x^-) = \exp\frac{i}{2}(x^+\Sigma_+ + x^-\Sigma_-), \qquad G_t = G_y\, G_{AdS}\, G_{CP},$$
$$f(\eta) = \eta + \sqrt{\nVdash + \eta^2}.$$

As before, $x^\pm = \phi \pm t$ are a light-cone pair constituted of the time and angle coordinate of AdS$_4$ and $\mathbb{CP}_{\nVdash 3}$ and $\Sigma_\pm$ is the corresponding basis element, $\Sigma_\pm = \pm\Gamma_0 \oplus -i\,T_6$, see section 4.2.1.

The transverse bosonic degrees of freedom are described by G_t and differs somewhat from the prior cases,

$$G_t = \begin{pmatrix} G_{AdS} & 0 \\ 0 & G_y \, G_{CP} \end{pmatrix}.$$

The AdS$_4$ part is parameterized by three transverse coordinates, z_i

$$G_{Ads} = \frac{\nVdash + \frac{i}{2} z_i \Gamma^i}{\sqrt{1 - \frac{z_i^2}{4}}}. \tag{7.2}$$

and the G_y element is described by a single real coordinate, y, of the $\mathbb{CP}_{\nVdash 3}$,

$$G_y = e^{y \, T_5}, \tag{7.3}$$

which is a function of $\cos(y)$ and $\sin(y)$. For the upcoming perturbative analysis it is convenient to relabel the trigonometric functions as

$$\sin(y) \to \frac{1}{2} y, \qquad \cos(y) \to \sqrt{1 - \frac{1}{4} y^2}.$$

The last component of G_t is parameterized by two complex coordinates ω_i (and its conjugate $\bar{\omega}_i$)

$$\begin{aligned} G_{CP} &= \\ \nVdash &+ \frac{1}{\sqrt{1 + \frac{1}{4}|w|^2}} (W + \bar{W}) + 4 \, \frac{\sqrt{1 + \frac{1}{4}|w|^2} - 1}{|w|^2 \sqrt{1 + \frac{1}{4}|w|^2}} (W \cdot \bar{W} + \bar{W} \cdot W), \end{aligned} \tag{7.4}$$

where $W = \frac{1}{2} \omega_i \, \tau_i$ and $|w|^2 = \omega_i \, \bar{\omega}_i$.

For the auxiliary field π that parameterize the first order Lagrangian (4.98), we introduce a basis decomposition as in (4.96)

$$\pi = \pi_+ \Sigma_+ + \pi_- \Sigma_- + \pi_t, \tag{7.5}$$

where

$$\pi_t = \begin{pmatrix} \pi_i^{(z)} \, \Gamma_i & 0 \\ 0 & \pi^{(y)} \, T_5 + \pi_i^{(\omega)} \, \tau_i + \bar{\pi}_i^{(\bar{\omega})} \, \bar{\tau}_i \end{pmatrix}$$

Remember that the components of π does not directly correspond to the conjugate momentas of the bosonic fields. In order to obtain the physical Hamiltonian, one have to solve for these components and use the solutions in the Lagrangian (4.98).

7.2.1 Gauge fixing and field content

As before we use the bosonic symmetries to fix a uniform light-cone gauge as

$$x^+ = \sigma^0 = \tau, \qquad p_+ = \text{Constant}, \tag{7.6}$$

The super group OSP(2,2|6) contains 24 real fermions whereas supersymmetry demands that the number of fermionic and bosonic excitations should be equal. At first glance, this looks like a problem since common lore has is that kappa symmetry removes half of the fermions, which in our case would leave us with to few fermions for supersymmetry to be manifest. However, as we saw in section 4.2.3, the kappa symmetry for strings in $\text{AdS}_4 \times \mathbb{CP}_{\not\Bbbk 3}$ is partial and only allows for eight real fermions to be removed **?**. Therefore the kappa fixed model has equal number of fermionic and bosonic excitations.

There are many ways to impose the kappa symmetry. In this thesis we will use an especially convenient gauge introduced by Bykov which is compatible with the bosonic part of the subgroup that commutes with the gauge-fixed string Hamiltonian[2] **?**.

As was explained in **?**, a kappa gauge that transform covariantly under G_B can be constructed by first enforcing

$$\theta_{1,5} = i\,\theta_{1,4}, \quad \theta_{1,6} = i\,\theta_{1,3}, \quad \theta_{2,5} = i\,\theta_{2,4}, \quad \theta_{2,6} = i\,\theta_{2,3}, \tag{7.7}$$

which removes four complex fermions and thus leave us with a total of sixteen real ones as desired[3]. As it stands, the gauge (7.7), does not transform covariantly under the bosonic symmetries. However, if we augment the gauge with the following linear combinations of the spinor components[4]

$$\begin{aligned}
&\theta_{1,1} = \kappa^{+\,1} - \kappa_{+\,2}, \quad \theta_{1,2} = -i(\kappa^{+\,1} + \kappa_{-\,2}), \quad \theta_{2,1} = \kappa^{+\,2} + \kappa_{-\,1},\\
&\theta_{2,2} = -i(\kappa^{+\,2} - \kappa_{-\,1}), \quad \theta_{1,3} = \frac{1}{2}(s^1_{\dot 1} - s^1_{\dot 2}), \quad \theta_{1,4} = -\frac{i}{2}(s^1_{\dot 1} + s^1_{\dot 2}),\\
&\theta_{2,3} = \frac{1}{2}(s^2_{\dot 1} - s^2_{\dot 2}), \quad \theta_{2,4} = -\frac{i}{2}(s^2_{\dot 1} + s^2_{\dot 2}),
\end{aligned}$$

then the new variables transform under G_B as

$$\kappa^{+,a} \to e^{i\alpha}\, g^a_b\, \kappa^{+\,b}, \quad \kappa_{-\,a} \to e^{-i\alpha}\, g^b_a\, \kappa_{-\,b}, \quad s^a_{\dot b} \to g^a_b\, g^{\dot a}_{\dot b}\, s^b_{\dot a}, \tag{7.8}$$

where $g^a_b \in \text{SU}(2)_{AdS}$, $g^{\dot a}_{\dot b} \in \text{SU}(2)_{CP}$ and $e^{\pm i\phi} \in \text{U}(1)$. Thus, in our notation, undotted indices correspond to the SU(2) from the AdS space and dotted ones correspond to the

[2] For another covariant kappa gauge, see **?**.

[3] One can also think about the kappa gauge in the following way; if we anticommute a generic, non kappa gauge fixed odd matrix, with Σ_+, one find that the resulting object has the form of a kappa gauge fixed matrix. In one sense this can be seen as a defining property of the gauge. This is very similar to the kappa gauge imposed in (4.82) where the gauge fixing was defined through a commutation relation between a light-cone basis element and η.

[4] Note that the fermions denoted with $\kappa^\pm$ has no relation with the constant κ in front of the WZ term in the Lagrangian. Also note that the $\pm$ denotes U(1) charge and should not be confused as sign of the SU(2) index.

SU(2) from $\mathbb{CP}_{\not{\,}3}$. In this notation it becomes clear that we have two set of spinors, $\kappa^{\pm}$, with opposite U(1) charge transforming under the AdS SU(2)[5]. There is also a spinor, $s^a_{\dot{b}}$, uncharged under the U(1) but in a bifundamental representation of the two SU(2)'s.

We should also classify how the bosonic fields transform. Clearly, the z_i coordinates only transform under the SU(2) from the AdS space. The singlet y does not transform at all, neither under any SU(2) or the U(1). The only bosonic fields charged under the U(1) are the complex ω_i and $\bar{\omega}_i$ which also transform under the SU(2) of $\mathbb{CP}_{\not{\,}3}$. A convenient index notation is

$$\omega_i \to \omega_{\dot{a}}, \qquad \bar{\omega}_i \to \bar{\omega}^{\dot{a}}, \tag{7.9}$$

where lower index has the plus charge of the U(1) and vice versa.

Under conjugation, all indices changes place

$$\begin{aligned}
&(\kappa^{+\,a})^\dagger = \bar{\kappa}_{+\,a} = \epsilon_{ab}\,\bar{\kappa}^{+\,b}, \quad (\kappa_{-\,a})^\dagger = \bar{\kappa}^{-\,a} = \epsilon^{ab}\,\bar{\kappa}_{-\,b}, \\
&(s^a_{\dot{b}})^\dagger = \bar{s}^{\dot{b}}_a = \epsilon^{\dot{b}\dot{a}}\epsilon_{ab}\,\bar{s}^b_{\dot{a}}, \quad (\omega_{\dot{a}})^\dagger = \bar{\omega}^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\,\omega_{\dot{b}}, \quad \epsilon_{ab}\,\epsilon^{bc} = \delta^c_a, \quad \epsilon_{\dot{a}\dot{b}}\,\epsilon^{\dot{b}\dot{c}} = \delta^{\dot{c}}_{\dot{a}},
\end{aligned} \tag{7.10}$$

where we also introduced epsilon tensors to raise and lower indices, with the convention $\epsilon_{01} = 1 = -\epsilon^{01}$. It is convenient to let the $\pm$, denoting U(1) charge of the unconjugated spinors, travel with the SU(2) index. This imply that all lower $\pm$ have negative U(1) while upper have positive.

The field content for the AdS$_4 \times \mathbb{CP}_{\not{\,}3}$ string splits up in a more complicated way than for the AdS$_5\times$S^5 string and this will complicate things rather severely as will be seen in the upcoming.

7.2.2 Light-cone Lagrangian and Hamiltonian

Having imposed the bosonic and fermionic gauges, we are in position to start extracting physical quantities from the string Lagrangian (4.98). As before, the main interest of study is the string Hamiltonian. As we have seen, in the light-cone formalism it enters in the natural way

$$\mathscr{L} = p_m\,\dot{x}^m + p_- + \text{Fermions}, \qquad m \in \{i, y, \dot{a}\}.$$

where p_- is a function of the physical fields and the auxiliary π.

[5]The spinor transforming with negative U(1) is in the conjugate representation of the SU(2) from AdS, hence the lower index.

The transverse components of π are solved for through (4.102)[6]

$$\pi_i^{(z)} = \frac{2i\, p_i^{(z)}}{4+z_i^2}, \quad \pi^{(y)} = \frac{4\, p_y}{8+y^2-\omega_{\dot a}\,\bar\omega^{\dot a}}, \tag{7.11}$$
$$\pi_{\mathrm{i}}^{(\omega)} = \frac{8\, p_{\mathrm{i}} + \omega_{\mathrm{i}}\,\bar\omega^{\dot 2}\,\pi_{\dot 2}^{(\omega)}}{8-\omega_{\mathrm{i}}\,\bar\omega^{\mathrm{i}}-\omega_{\dot a}\,\bar\omega^{\dot a}}, \quad \pi_{\dot 2}^{(\omega)} = \frac{8\, p_{\dot 2} + \omega_{\dot 2}\,\bar\omega^{\mathrm{i}}\,\pi_{\mathrm{i}}^{(\omega)}}{8-\omega_{\dot 2}\,\bar\omega^{\dot 2}-\omega_{\dot a}\,\bar\omega^{\dot a}}.$$

The expressions for $\pi_\pm$ are considerably more complicated and for these components we will only present the corresponding matrix equations[7]. To obtain π_+ we solve for p_+ in a similar way as we did above, then use this solution in the quadratic constraint (4.100) to solve for π_-,

$$\pi_+ = -\pi_-\,\frac{Str\,\Sigma_-\,G_-}{Str\,\Sigma_+\,G_-} + \frac{1}{Str\,\Sigma_+\,G_-}\Big(\shortmid_+ - Str\,\pi_t\,G_-\Big), \tag{7.12}$$
$$\pi_- = \frac{\shortmid_+ - Str\,\pi_t\,G_-}{2\,Str\,\Sigma_-\,G_-}\Big\{1 \pm \sqrt{1-\frac{(Str\,\Sigma_-\,G_-)(Str\,\Sigma_+\,G_-)\Big(Str\,\pi_t^2 + Str(\mathcal{A}_1^2)^2\Big)}{4\,(\shortmid_+ - Str\,\pi_t\,G_-)^2}}\Big\}$$
$$= \frac{(Str\,\Sigma_+\,G_-)\Big(Str\,\pi_t^2 + Str(\mathcal{A}_1^2)^2\Big)}{16\,\Big(\shortmid_+ - Str\,\pi_t\,G_-\Big)} + \ldots$$

where we introduced the short hand notation G_- for the even part of

$$\frac{i}{2}\,G_t^{-1}\,(f^{-1}(\eta)\,\Sigma_-\,f(\eta))\,G_t$$

and $\shortmid_+$ is

$$\shortmid_+ = p_+ - p_+^{WZ} = \tag{7.13}$$
$$p_+ - \kappa\frac{i}{2}\,Str\Big\{G_t^{-1}\Big(\frac{i}{2}\sqrt{1+\eta^2}\,\Sigma_-\,\eta - \frac{i}{2}\,\eta\,\Sigma_-\,\sqrt{1+\eta^2}\Big)G_t\,\Upsilon\,\mathcal{A}_1^{Odd}\,\Upsilon^{-1}\Big\},$$

where the last part is the contribution to p_+ coming from the WZ term.

[6] As can be seen, the complex components mix within each other and one might be tempted to shift the fields so this complication disappears. However, as it turns out this mixing enters only at quartic order in number of fields so for the upcoming perturbative analysis this mixing is irrelevant.

[7] However, their quadratic part is needed to determine the upcoming fermionic shift, so these parts we present in (25).

The light-cone Hamiltonian is given by

$$
\begin{aligned}
& -\mathcal{H} = p_- = \frac{\delta \mathscr{L}}{\delta \dot{x}^+} = \\
& \frac{i}{2}\, Str\, \pi\, G_t^{-1} (\Sigma_+ - \eta\, \Sigma_+\, \eta + \sqrt{1+\eta^2}\, \Sigma_+\, \sqrt{1+\eta^2})\, G_t \\
& - \kappa \frac{i}{2}\, Str \Big\{ G_t^{-1} \Big(\frac{i}{2} \sqrt{1+\eta^2}\, \Sigma_+\, \eta - \frac{i}{2}\, \eta\, \Sigma_+ \sqrt{1+\eta^2} \Big) G_t \times \\
& \Upsilon\, G_t^{-1} \Big((\frac{i}{2} \sqrt{1+\eta^2}\, \Sigma_-\, \eta - \frac{i}{2}\, \eta\, \Sigma_- \sqrt{1+\eta^2})\, x'^- + \sqrt{1+\eta^2}\, \eta' - \eta\, \partial_1 \sqrt{1+\eta^2} \Big) G_t\, \Upsilon^{-1} \Big\}.
\end{aligned}
\tag{7.14}
$$

Note that this expressions is more involved than its $AdS_5 \times S^5$ counterpart in (5.13). There we could choose a kappa gauge so that the odd part of the current were independent of the light-cone coordinate x^-. Unfortunately, this is not possible for the $AdS_4 \times \mathbb{CP}_{\not{\kappa}3}$ string, and hence the more complicated expression above.

Combining everything we have so far, we can write the string Lagrangian as

$$
\begin{aligned}
& \mathscr{L} = \\
& p_+\, \dot{x}^- + p_m\, \dot{x}^m + p_- + Str\, \pi\, G_t^{-1} \Big(-\eta\, \dot{\eta} + \sqrt{1+\eta^2}\, \partial_0 \sqrt{1+\eta^2} \Big) G_t \\
& + \frac{i}{2} \kappa\, Str\, G_t^{-1} \Big(\sqrt{1+\eta^2}\, \partial_0 \eta - \eta\, \partial_0 \sqrt{1+\eta^2} \Big)\, G_t\, \Upsilon\, \mathcal{A}_1^{Odd}\, \Upsilon^{-1}.
\end{aligned}
\tag{7.15}
$$

Together with the solutions for π and the expression for p_- in (7.14) this is the *exact* gauge fixed string Lagrangian for the $AdS_4 \times \mathbb{CP}_{\not{\kappa}3}$ superstring. It will be the starting point for a perturbative analysis in the next section. However, as was the case in the $AdS_5 \times S^5$ section also, it should be clear that the terms involving time derivatives of the fermions will have terms beyond quadratic order.

7.3 Strong coupling expansion

To be able to extract anything useful from (7.14) we have to consider some sort of perturbative expansion. The standard way to proceed is to boost, spin or deform the string in some way or another. As we did for the $AdS_5 \times S^5$ string, we will expand around a point like string configuration moving on a null geodesic. Or equivalently, a plane wave expansion ?. This BMN expansion boils down to (5.15), which we for completeness present again

$$
g \to \infty, \qquad x_m \to \frac{x_m}{\sqrt{g}}, \qquad p_m \to \frac{p_m}{\sqrt{g}}, \qquad \eta \to \frac{\eta}{\sqrt{g}}.
\tag{7.16}
$$

7.3.1 Leading order

It is a good idea to start out the perturbative analysis by fixing some of the constants we encountered so far. First of all, from now on we will fix[8]

$$p_+ = 1 \qquad \kappa = 1. \tag{7.17}$$

What we choose to do with our parameter space is of course arbitrary and the physics we want to extract is totally independent of numerical conventions. However, the choices above are very convenient in terms of notation. Having factors of κ and p_+ in the expressions makes things which are, and especially will become, complicated more involved than necessary.

It is also desirable to have the Lagrangian in such a form that the field expansions becomes as simple as possible. To achieve this we rescale the string length parameter as $\sigma \to 2\sigma$ and send[9] $\eta \to i\,\eta$. Taking this into consideration, and taking the limit (7.16) of (7.15) gives the leading order quadratic Lagrangian

$$\begin{aligned}
\frac{1}{2}\mathscr{L} &= p_i\,\dot{z}_i + p_y\,\dot{y} + \dot{w}_{\dot{a}}\,\bar{p}^{\dot{a}} + \dot{\bar{\omega}}^{\dot{a}}\,p_{\dot{a}} + i\bar{s}_a^{\dot{b}}\,\dot{s}_{\dot{b}}^a + i\bar{\kappa}_{+\,a}\,\dot{\kappa}^{+\,a} + i\bar{\kappa}^{-\,a}\,\dot{\kappa}_{-\,a} \\
&- p_i^2 - 4\bar{p}^{\dot{a}}\,p_{\dot{a}} - p_y^2 - \frac{1}{4}\big(y^2 + z_i^2 + \frac{1}{4}\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}\big) - \frac{1}{4}\big(z_i'^2 + y'^2 + \bar{\omega}'^{\dot{a}}\,\omega_{\dot{a}}'\big) \\
&- \bar{s}_a^{\dot{b}}\,s_{\dot{b}}^a - \frac{1}{2}\big(\bar{\kappa}_{+\,a}\,\kappa^{+\,a} + \bar{\kappa}^{-\,a}\,\kappa_{-\,a}\big) - i\big(\kappa_{-\,a}\,\kappa'^{+\,a} + \bar{\kappa}_{+\,a}\,\bar{\kappa}'^{-\,a}\big) \\
&- \frac{i}{2}\big(s_{\dot{b}}^a\,(s')_a^{\dot{b}} + \bar{s}_a^{\dot{b}}\,(\bar{s}')_{\dot{b}}^a\big).
\end{aligned} \tag{7.18}$$

From this we find that the fields come in heavy and light multiplets,

$$\mathbf{M} = 1; \qquad \{s_{\dot{b}}^a\,,\, z_i\,,\, y\} \qquad \mathbf{M} = \frac{1}{2}; \qquad \{\kappa^{+\,a}\,,\, \kappa_{-\,a}\,,\, \omega_{\dot{a}}\,,\, \bar{\omega}^{\dot{a}}\}.$$

This $4_{\frac{1}{2}} + 4_1$ split of the masses is a novel feature for the $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\,}3}$ string. In the last section of this thesis we will calculate loop corrections to propagators for the massive modes. There it will be argued that the heavy excitations can be viewed as composite states of light modes. For now though we view them as single excitations.

Already at the quadratic level, we get a hint of the complexity of this theory in contrast to the quadratic $\mathrm{AdS}_5{\times}\mathrm{S}^5$ theory in (5.115).

We can tidy up the notation a bit further by making the quadratic 2-d Lorentz symmetry manifest. First we introduce, $\gamma^0 = \sigma_3$ and $\gamma^1 = -i\sigma_2$, which obeys $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ with $(+,-)$ convention. We then combine the fermions into two spinors as

$$\Psi = \begin{pmatrix} \kappa^{+\,a} \\ \bar{\kappa}^{-\,a} \end{pmatrix}, \quad \bar{\Psi} = \Psi^\dagger\,\gamma^0, \qquad \chi = \begin{pmatrix} s_{\dot{b}}^a \\ \bar{s}_{\dot{b}}^a \end{pmatrix}, \quad \bar{\chi} = \chi^\dagger\,\gamma^0.$$

[8]Once again we stress that the κ here has nothing to do with the two fermions $\kappa^\pm$.
[9]This is equivalent to defining the fermionic part of the group element as $f(\eta) = \sqrt{1-\eta^2} + i\,\eta$.

Then the quadratic Lagrangian can be written as

$$\begin{aligned}\frac{1}{2}\mathscr{L} &= p_i\,\dot{z}_i + p_y\,\dot{y} + \dot{w}_{\dot{a}}\,\bar{p}^{\dot{a}} + \dot{\bar{\omega}}^{\dot{a}}\,p_{\dot{a}} \\ &- p_i^2 - 4\bar{p}^{\dot{a}}\,p_{\dot{a}} - p_y^2 - \frac{1}{4}(y^2 + z_i^2 + \frac{1}{4}\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}) - \frac{1}{4}(z_i'^2 + y'^2 + \bar{\omega}'^{\dot{a}}\,\omega'_{\dot{a}}) \\ &+ i\bar{\Psi}\,\gamma^\alpha\,\partial_\alpha\,\Psi + \frac{i}{2}\bar{\chi}\,\gamma^\alpha\,\partial_\alpha\,\chi - \frac{1}{2}\bar{\Psi}\,\Psi - \frac{1}{2}\bar{\chi}\,\chi.\end{aligned} \tag{7.19}$$

Anticipating the quantization procedure we expand the fields in Fourier coefficients as

$$\begin{aligned}
\omega_{\dot{a}} &= \frac{1}{\sqrt{2\,\pi}}\int dp\,\frac{1}{\sqrt{\omega_p}}\Big(a^{\dot{a}}\,e^{ip\sigma} + \bar{b}^{\dot{a}}\,e^{-ip\sigma}\Big), \quad p_{\dot{a}} = \frac{i}{\sqrt{2\,\pi}}\int dp\,\frac{\sqrt{\omega_p}}{4}\Big(\bar{b}^{\dot{a}}\,e^{-ip\sigma} - a^{\dot{a}}\,e^{ip\sigma}\Big),\\
y &= \frac{1}{\sqrt{2\,\pi}}\int dp\,\frac{1}{\sqrt{2\Omega_p}}\Big(y\,e^{ip\sigma} + \bar{y}\,e^{-ip\sigma}\Big), \quad p_y = \frac{1}{2}\frac{i}{\sqrt{2\,\pi}}\int dp\,\sqrt{\frac{\Omega_p}{2}}\Big(\bar{y}\,e^{-ip\sigma} - y\,e^{ip\sigma}\Big),\\
z_i &= \frac{1}{\sqrt{2\,\pi}}\int dp\,\frac{1}{\sqrt{2\Omega_p}}\Big(z_i\,e^{ip\sigma} + \bar{z}_i\,e^{-ip\sigma}\Big), \quad p_i = \frac{1}{2}\frac{i}{\sqrt{2\,\pi}}\int dp\,\sqrt{\frac{\Omega_p}{2}}\Big(\bar{z}_i\,e^{-ip\sigma} - z_i\,e^{ip\sigma}\Big),\\
s^a_{\dot{b}} &= \frac{1}{\sqrt{2\pi}}\int dp\frac{1}{\sqrt{2\Omega_p}}\Big(F_p\,\chi^a_{\dot{b}}\,e^{ip\sigma} - H_p\,\bar{\chi}^a_{\dot{b}}\,e^{-ip\sigma}\Big),\\
\kappa^{+\,a} &= \frac{1}{\sqrt{2\pi}}\int dp\frac{1}{\sqrt{2\omega_p}}\Big(f_p\,c^a\,e^{ip\sigma} - h_p\,\bar{d}^{\,a}\,e^{-ip\sigma}\Big),\\
\kappa_{-\,a} &= \frac{1}{\sqrt{2\pi}}\int dp\frac{1}{\sqrt{2\omega_p}}\Big(f_p\,d_a\,e^{ip\sigma} - h_p\,\bar{c}_a\,e^{-ip\sigma}\Big),
\end{aligned}$$

and obvious ones for conjugated fields. The frequencies and the fermionic wave functions are given by,

$$\begin{aligned}\omega_p &= \sqrt{\frac{1}{4} + p^2}, \quad f_p = \sqrt{\frac{\omega_p + \frac{1}{2}}{2}}, \quad h_p = \frac{p}{2f_p},\\ \Omega_p &= \sqrt{1+p^2}, \quad F_p = \sqrt{\frac{\Omega_p + 1}{2}}, \quad H_p = \frac{p}{2F_p},\end{aligned} \tag{7.20}$$

where the wave functions satisfy the following important identities,

$$f_p^2 + h_p^2 = \omega_p, \quad f_p^2 - h_p^2 = \frac{1}{2}, \quad F_p^2 + H_p^2 = \Omega_p, \quad F_p^2 - H_p^2 = 1.$$

If we now plug the field expansion into (7.18) and integrate over σ, we find

$$\begin{aligned}L = \int dp\,\Big(&i(\bar{b}^{\dot{b}}\,\dot{b}_{\dot{b}} + \bar{a}_{\dot{b}}\,\dot{a}^{\dot{b}} + \bar{y}\,\dot{y} + \bar{z}_i\,\dot{z}_i + \bar{\chi}^{\dot{b}}_a\,\dot{\chi}^a_{\dot{b}} + \bar{c}_a\,\dot{c}^a + \bar{d}^a\,\dot{d}_a)\\ &- \omega_p(\bar{b}^{\dot{b}}\,b_{\dot{b}} + \bar{a}_{\dot{b}}\,a^{\dot{b}} + \bar{c}_{\dot{b}}\,c^a + \bar{d}^a\,d_a) - \Omega_p(\bar{y}\,y + \bar{z}_i\,z_i + \bar{\chi}^{\dot{b}}_a\,\chi^a_{\dot{b}})\Big).\end{aligned} \tag{7.21}$$

We also need to consider the second constraint which give rise to

$$\mathcal{V} = \int dp\, p\Big(\bar{b}^{\dot{b}}\, b_{\dot{b}} + \bar{a}_{\dot{b}}\, a^{\dot{b}} + \bar{y}\, y + \bar{z}_i\, z_i + \bar{c}_{\dot{b}}\, c^a + \bar{d}^a\, d_a + \bar{\chi}^{\dot{b}}_a\, \chi^a_{\dot{b}}\Big). \tag{7.22}$$

Which is the so called level matching constraint enforcing that the sum of all mode numbers has to vanish for physical states. In the quantum theory this will be promoted to an operator whose action on a physical state should project to zero.

Promoting the oscillators to operators is now down by imposing the equal time (anti)commutators

$$\begin{aligned}
&[a(p,\tau)^{\dot{a}}, \bar{a}(p',\tau)_{\dot{b}}] = 2\pi\, \delta^{\dot{a}}_{\dot{b}}\, \delta(p-p'), \quad [b(p,\tau)_{\dot{a}}, \bar{b}(p',\tau)^{\dot{b}}] = 2\pi\, \delta^{\dot{b}}_{\dot{a}}\, \delta(p-p') \\
&[y(p,\tau), \bar{y}(p',\tau)] = 2\pi\, \delta(p-p'), \quad [z_i(p,\tau), \bar{z}_j(p',\tau)] = 2\pi\, \delta_{ij}\delta(p-p'), \\
&\{c^a(p,\tau), \bar{c}_b(p',\tau)\} = \{d_b(p,\tau), \bar{d}^a(p',\tau)\} = 2\pi\, \delta^a_b\, \delta(p-p'), \\
&\{\chi^a_{\dot{a}}(p,\tau), \bar{\chi}^{\dot{b}}_b(p',\tau)\} = 2\pi\, \delta^a_b\, \delta^{\dot{a}}_{\dot{b}}\, \delta(p-p').
\end{aligned} \tag{7.23}$$

With this we have established the quadratic Lagrangian, including field expansions and commutation relations. We would now like to proceed to the higher order contributions from (7.14). However, before extracting the sub leading terms in the light-cone Hamiltonian, we have to take care of the higher order kinetic fermions. If these were to be included then the anti commutation relations in (7.23) would receive higher order corrections. In the next section we will describe how this complication can (partially) be avoided by a appropriate shift of the fermions.

7.3.2 Canonical fermions

The focus of this section be will the piece of (7.15) that contains kinetic fermionic terms,

$$\begin{aligned}
&\mathscr{L}^{\eta}_{Kinetic} = \\
&\frac{1}{2} Str\, \pi\, G_t^{-1}\Big([\dot{\eta},\eta] + \frac{1}{4}[\eta^2, \{\dot{\eta},\eta\}]\Big) G_t \\
&-\frac{i}{2}\kappa\, Str\, G_t^{-1}\Big(\dot{\eta} - \frac{1}{2}\eta\,\dot{\eta}\,\eta\Big)\, G_t\, \Upsilon\, G_t^{-1}\big(\frac{i}{2}\,[\Sigma_-,\eta]\, x'^{-} + \eta' - \frac{1}{2}\eta\,\eta'\,\eta\big) G_t\, \Upsilon^{-1} + \mathcal{O}(\eta^6),
\end{aligned} \tag{7.24}$$

from which it is clear that the anti commutation relations in (7.23) will receive higher order contributions. In principle this is not a fundamental problem and it can be solved explicitly by a careful analysis of the Poisson structure, see for example **?**. However, from a calculational point of view, it is rather cumbersome to deal with non trivial commutation relations. For that reason we will try to avoid the problem by performing a shift of the fermionic coordinates as we did for the $AdS_5 \times S^5$ string.

By using the cyclicity of the super trace and the form of π_+, we can write[10]

$$\mathscr{L}^{\eta}_{Kinetic} = \frac{i}{4}\, Str\, \Sigma_+\, \dot{\eta}\, \eta + Str\, \dot{\eta}\, \widetilde{\Phi}(x_m, p_m, \eta), \tag{7.25}$$

where $\widetilde{\Phi}(x_m, p_m, \eta)$ is a complicated fermionic matrix, presented in (24), that can be deduced from (7.24). It starts at quadratic order in number of fields and for the analysis at hand we have to know it up to cubic order[11]

We will now show that most of the higher order terms can be removed by shifting the fermions in an appropriate way. First we introduce a, so far arbitrary, function $\Phi(x_m, p_m, \eta)$. Since we are to expand the Hamiltonian up to quartic order, we need this function to third order in number of fields. To simplify the notations we split up $\Phi(x_m, p_m, \eta)$ in number of fields and leave the bosonic dependence implicit, $\Phi(x_m, p_m, \eta) = \Phi_2(\eta) + \Phi_3(\eta)$. The idea is now to shift the fermionic matrix as

$$\eta \to \eta + \Phi(\eta). \tag{7.26}$$

Performing the shift in (7.25) and writing, $\widetilde{\Phi}(x_m, p_m, \eta) = \widetilde{\Phi}_2(\eta) + \widetilde{\Phi}_3(\eta)$, we find

$$\begin{aligned} &\mathscr{L}^{\eta}_{Kinetic} = \\ &\frac{i}{4}\, Str\, \Sigma_+\, \dot{\eta}\, \eta + Str\, \dot{\eta}\, (\widetilde{\Phi}_2(\eta) + \widetilde{\Phi}_3(\eta)) + \frac{i}{4}\, Str\, \dot{\eta}[\Phi_2(\eta) + \Phi_3(\eta), \Sigma_+] \\ &+ Str\, \dot{\eta}\, \widetilde{\Phi}_2(\eta \to \Phi_2) + Str\, \dot{\Phi}_2(\eta)\, \widetilde{\Phi}_2(\eta) + \frac{i}{4}\, Str\, \Sigma_+\, \dot{\Phi}_2(\eta)\, \Phi_2(\eta), \end{aligned} \tag{7.27}$$

where $\widetilde{\Phi}_2(\eta \to \Phi_2)$ is a cubic contribution from $\widetilde{\Phi}$ with Φ_2 as argument.

To proceed, we need to find the form of Φ. We do this by recalling that a general kappa gauge fixed fermionic element, which we again call η, can be written as a commutator, $\eta = [\Sigma_+, \chi]$ for some arbitrary, non kappa gauge fixed, fermionic matrix χ. This means that a term of the form $Str\, \dot{\eta}\, \widetilde{\Phi}$, for arbitrary fermionic $\widetilde{\Phi}$, can be written $Str\, \dot{\chi}\, [\Sigma_+, \widetilde{\Phi}]$. This imply that for Φ to remove the higher order terms, it should satisfy the matrix equation

$$[\Sigma_+, [\Phi, \Sigma_+]] + [\Sigma_+, \widetilde{\Phi}] = 0. \tag{7.28}$$

Some trial and error shows that a solution for Φ in terms of $\widetilde{\Phi}$ is

$$\Phi = \begin{pmatrix} \nVdash_{6\times 6} & 0 \\ 0 & \frac{1}{4}\nVdash_{4\times 4} \end{pmatrix} [\Sigma_+, \widetilde{\Phi}] \begin{pmatrix} \nVdash_{6\times 6} & 0 \\ 0 & \frac{1}{4}\nVdash_{4\times 4} \end{pmatrix} = \Gamma\, [\Sigma_+, \widetilde{\Phi}]\, \Gamma,$$

[10] π_+ is the only component of the auxiliary field which has a constant leading order term.

[11] The observant reader might notice that (7.24) also has a second quadratic piece $\sim Str\, \dot{\eta}\, \Upsilon\, \eta'\, \Upsilon^{-1}$. This term is, however, a total derivative and can be neglected.

which allows us to remove the $Str\,\dot\eta\,\widetilde\Phi$ terms in (7.27) by choosing,

$$\Phi = -4i\,\Gamma\,[\Sigma_+, \widetilde\Phi_2 + \widetilde\Phi_2(\eta\to\Phi_2) + \widetilde\Phi_3]\,\Gamma. \tag{7.29}$$

This leaves us with

$$\mathscr{L}^\eta_{Kin} = \frac{i}{4}\,Str\,\Sigma_+\,\dot\eta\,\eta + Str\,\dot\Phi_2\,\widetilde\Phi_2 + \frac{i}{4}\,Str\,\Sigma_+\,\dot\Phi_2\,\Phi_2, \tag{7.30}$$

which can be rewritten using (7.29) to

$$\mathscr{L}^\eta_{Kin} = \frac{i}{4}\,Str\,\Sigma_+\,\dot\eta\,\eta + \frac{1}{2}Str\,\dot\Phi_2\,\widetilde\Phi_2. \tag{7.31}$$

The last expression is unfortunately rather involved. It is of quartic order in number of fields and introduce additional time derivatives of the bosonic fields since

$$\widetilde\Phi_2 = \frac{1}{2}(\frac{i}{4}[\eta,[G^1_t,\Sigma_+]] + [\eta,\pi^1_t]) - \frac{i}{2}([G^1_t,\Upsilon]\,\eta'\,\Upsilon^{-1} + \Upsilon\,\eta'\,[G^1_t,\Upsilon^{-1}]),$$

where G^1_t and π^1_t are the pieces of G_t and π_t linear in fields. To remove the additional fermionic kinetic terms induced by the shift, one needs to isolate the $\dot\eta$ terms from (7.31) and introduce a second shift, say $\hat\Phi_3$, with the property $\frac{i}{4}Str\,\dot\eta[\hat\Phi_3,\Sigma_+] = -\frac{1}{2}Str\,\Phi_2\,\widetilde\Phi_2|_{\dot\eta}$, where the notation is meant to imply the $\dot\eta$ dependent part of $Str\,\Phi_2\,\widetilde\Phi_2$. However, this means that the $\dot\eta$ independent part contains time derivatives of the bosonic fields, so we find corrections to the transverse part of π in (7.11). Needless to say, this analysis becomes rather involved. Not only will the additional fermionic shift, $\hat\Phi_3$, complicate things further, but the additional momentum terms also give rise to complications since they will have a quadratic fermionic dependence[12].

We will tackle this problem by simply ignoring it. Or, to be more precise, we assume that the $\hat\Phi_3$ shift is performed but do not determine the form of it, nor the additional momentum terms, allowing us to maintain the canonical Poisson structure for the fermions. The reason we can do this is because $Str\,\Phi_2\,\widetilde\Phi_2$ contains two fermions and two bosons, which implies that all additional terms, both from the shift and from π_t, will end up in the mixing part of the shifted Hamiltonian, $\mathcal{H}_{BF}$. This is acceptable since this part is not needed for the upcoming analysis.

However, a nice feature of the shift is that the x'^- dependence will cancel between the shifted and the original quartic Hamiltonian[13]. Another nice consequence of the shift is that it removes all fermionic non σ derivative terms from the relevant parts of the Hamiltonian. This is important since the point particle dynamics should be fully encoded in the quadratic fluctuations.

[12] One could try to change the form of the OSP$(2,2|6)$ group element as $G = \Lambda\,G_t\,f(\eta)$ which simplifies the fermionic kinetic term with the price of fermionic dependence in the bosonic conjugate momentas from start. However, pushing through with the analysis one finds that in the end the complications are more or less the same and the fermionic shift is still very involved.

[13] This is also true for the shifted $\mathcal{H}_{BF}$ part. The additional contributions from the complicated $Str\,\Phi_2\,\widetilde\Phi_2$ does introduce any additional x^- terms.

To summarize what we have done; We introduced a fermionic shift Φ, which can be expressed in terms of $\widetilde{\Phi}$, with the property that it removes all higher order fermionic derivative terms. However, due to the presence of cubic terms in the Lagrangian, the shift adds a 'self interaction' term of the form $Str\,\Phi_2\,\widetilde{\Phi}_2$. This term is not only complicated, but it also alters the transverse part of the auxiliary field π. Instead of determining this term explicitly, we simply assume the shift is performed, which guarantees a canonical Poisson structure. This is equivalent to put $Str\,\Phi_2\,\widetilde{\Phi}_2$ to zero by hand and accept that we can not determine the mixing part, $\mathcal{H}_{BF}$, of the shifted Hamiltonian. It is a bit surprising that the fermions are of such a complicated nature. As we saw earlier, for the $\mathrm{AdS}_5\times\mathrm{S}^5$ string the corresponding shift actually simplified the resulting theory, while here it has the opposite effect. Perhaps it is related to the coset construction we use which is not as rigorous as the AdS_5 string, see **?** and **?** for a related discussion.

What we can determine though is the shifted part of the Hamiltonian containing only bosons and fermions. This we will do in the next section. In the appendix we also present the full unshifted Hamiltonian, which together with the full form of the fermionic shift allows one to determine the shifted mixing Hamiltonian.

7.3.3 Higher order Hamiltonian

Having established the relevant form of the fermionic shift we are now in position to derive the Hamiltonian (7.14) to quartic order in fields. The way to do this is a straight forward, albeit somewhat tedious, multi step process. First we use the solution for π in (7.14), impose the shift (7.26) and expand to quartic order. It should be obvious that due to the complexity of both the Hamiltonian and the shift, it is very desirable to use some sort of computer program that can handle symbolic manipulations.

Pushing through with the calculation one find that the Hamiltonian has cubic next to leading order terms. This is another novel feature compared to the $\mathrm{AdS}_5\times\mathrm{S}^5$ string which subleading terms start at quartic order.

Before we present our findings we would like to introduce yet another convenient notation,

$$Z^a_b = \sum_i z_i \sigma^a_{i,b}, \qquad Z^2 = \frac{1}{2} Tr\, Z^a_b\, Z^b_c = \sum_i z_i^2 \tag{7.32}$$
$$P^a_{z,b} = \sum_i p_i \sigma^a_{i,b}, \qquad P^2_z = \frac{1}{2} Tr\, P^a_{z,b}\, P^b_{z,c} = \sum_i p_i^2,$$

where the Pauli matrices transform as $\sigma \to g\,\sigma\,g^t$ under the AdS SU(2).

With all this, we are now in position to extract the full Hamiltonian. Starting out with the subleading cubic part, we find

$$
\begin{aligned}
&\sqrt{g}\,\mathcal{H}_3 = \qquad (7.33)\\
&(\bar{\Psi}_a\,\Psi^b)' Z_b^a + i(\bar{\Psi}\,\gamma^1\,\Psi' - \bar{\Psi}'\,\gamma^1\,\Psi)_a^b\,(Z')_b^a - 2i(\bar{\Psi}'\,\gamma^0\,\Psi - \bar{\Psi}\,\gamma^0\,\Psi')_a^b\,P^a_{z,b}\\
&+2\Big((\bar{\chi}_{a\beta}\,\gamma^1\,\Psi'^a - \bar{\chi}'_{ab}\,\gamma^1\,\Psi^a)\bar{p}^\beta + (\bar{\Psi}_a\,\gamma^1\,\chi'^{a\,\dot{b}} - \bar{\Psi}'_a\,\gamma^1\,\chi^{a\,\dot{b}})\,p_\beta\Big) + \frac{i}{4}\Big((\bar{\chi}_{a\,\dot{b}}\,\gamma^1\,\gamma^0\,\Psi^a)'\bar{\omega}^{\dot{b}}\\
&+(\bar{\Psi}_a\,\gamma^0\,\gamma^1\,\chi^{a\,\dot{b}})'\omega_{\dot{b}}\Big) + \frac{1}{2}(\bar{\chi}_{a\,\dot{b}}\,\gamma^0\,\Psi'^a - \bar{\chi}'_{a\,\dot{b}}\,\gamma^0\,\Psi^a)\bar{\omega}'^{\dot{b}} + \frac{1}{2}(\bar{\Psi}_a\,\gamma^0\,\chi'^{a\,\dot{b}} - \bar{\Psi}'_a\,\gamma^0\,\chi^{a\,\dot{b}})\omega'_{\dot{b}}\\
&+i\,y\,(\bar{p}^{\dot{b}}\,\omega_{\dot{b}} - p_{\dot{b}}\,\bar{\omega}^{\dot{b}}).
\end{aligned}
$$

A nice feature of the coordinate system we use is that the massive singlet do not mix with any of the fermionic coordinates. Let us also remark that the fermionic shift (7.26) induces additional terms already here in the cubic Hamiltonian.

We will split up the quartic Hamiltonian according to its bosonic / fermionic field content $g\,\mathcal{H}_4 = \mathcal{H}_{BB} + \mathcal{H}_{BF} + \mathcal{H}_{FF}$. For the pure bosonic contribution, we find

$$
\begin{aligned}
&\frac{g}{2}\,\mathcal{H}_{BB} = \qquad (7.34)\\
&\frac{1}{4}Z^2\,Z'^2 - \frac{3}{4}\,p_y^2\,y^2 + \frac{1}{16}y^4 - \frac{1}{16}y^2\,y'^2 - \frac{1}{16}\bar{\omega}^{\dot{a}}\,\bar{\omega}'^{\dot{b}}\,\omega_{\dot{b}}\,\omega'_{\dot{a}} - \frac{3}{32}\bar{\omega}^{\dot{a}}\,\bar{\omega}'^{\dot{b}}\,\omega_{\dot{a}}\,\omega'_{\dot{b}}\\
&-\frac{1}{128}\bar{\omega}^{\dot{a}}\,\bar{\omega}^{\dot{b}}\,\omega_{\dot{a}}\,\omega_{\dot{b}} + \frac{1}{2}\bar{p}^{\dot{a}}\,\bar{\omega}^{\dot{b}}\,p_{\dot{a}}\,\omega_{\dot{b}} + \bar{p}^{\dot{a}}\,\bar{\omega}^{\dot{b}}\,p_{\dot{b}}\,\omega_{\dot{a}} - \frac{1}{8}\bar{\omega}'^{\dot{a}}\,\omega'_{\dot{a}}\,y^2 - \frac{3}{32}\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}\,y'^2\\
&-2\,\bar{p}^{\dot{a}}\,p_{\dot{a}}\,y^2 + \frac{1}{8}p_y^2\,\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}} - \frac{1}{2}y^2\,P_z^2 - \frac{1}{8}\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}\,P_z^2 + 2\,\bar{p}^{\dot{a}}\,p_{\dot{a}}\,Z^2 + \frac{1}{8}\,\bar{\omega}'^{\dot{a}}\,\omega'_{\dot{a}}\,Z^2\\
&-\frac{1}{32}\,\bar{\omega}^{\dot{a}}\,\omega_{\dot{a}}\,Z'^2 + \frac{1}{8}y'^2\,Z^2 + \frac{1}{2}p_y^2\,Z^2 - \frac{1}{8}y^2\,Z'^2,
\end{aligned}
$$

which, for another more complicated coordinate system, was first calculated in ?.

Next we turn to the purely fermionic part which is given by[14],

$$
\begin{aligned}
g\,\mathcal{H}_{FF} = & -i\big(\kappa_{-a}\,\bar\kappa_{+b}\,\kappa^{+a}\,\kappa'^{+b} + \kappa_{-a}\,\kappa'_{-b}\,\kappa^{+a}\,\bar\kappa^{-b}\big) - \frac{i}{2}\big(\kappa_{-a}\,\bar\kappa_{+b}\,\kappa'^{+a}\,\kappa^{+b} \\
& + \kappa_{-a}\,\kappa'_{-b}\,\bar\kappa^{-a}\,\kappa^{+b} + \kappa_{-a}\,\bar\kappa'_{+b}\,\bar\kappa^{-a}\,\bar\kappa^{-b} + \bar\kappa_{+a}\,\bar\kappa_{+b}\,\kappa^{+a}\,\bar\kappa'^{-b}\big) + \big(\kappa_{-a}\,\bar\kappa_{+b}\,\bar\kappa'^{-a}\,\kappa'^{+b} \\
& + \kappa_{-a}\,\kappa_{-b}\,\kappa'^{+a}\,\kappa'^{+b} + \bar\kappa_{+a}\,\bar\kappa_{+b}\,\bar\kappa'^{-a}\,\bar\kappa'^{-b}\big) - \frac{5}{2}\big(\kappa_{-a}\,\kappa_{-b}\,\bar\kappa'^{-a}\,\bar\kappa'^{-b} + \bar\kappa_{+a}\,\bar\kappa_{+b}\,\kappa'^{+a}\,\kappa'^{+b}\big) \\
& - 3\,\kappa_{-a}\,\bar\kappa'_{+b}\,\bar\kappa'^{-a}\,\kappa^{+b} - 4\big(\kappa_{-a}\,\bar\kappa_{+b}\,\kappa'^{+a}\,\bar\kappa'^{-b} - \kappa_{-a}\,\kappa'_{-b}\,\kappa'^{+a}\,\kappa^{+b} + \kappa_{-a}\,\bar\kappa'_{+b}\,\kappa^{+a}\,\bar\kappa'^{-b} \\
& - \kappa_{-a}\,\kappa'_{-b}\,\bar\kappa^{-a}\,\bar\kappa'^{-b} - \bar\kappa_{+a}\,\bar\kappa'_{+b}\,\kappa^{+a}\,\kappa'^{+b} + \kappa_{-a}\bar\kappa'_{+b}\,\kappa'^{+a}\,\bar\kappa^{-b} + \bar\kappa_{+a}\,\kappa'_{-b}\,\bar\kappa'^{-a}\,\kappa^{+b}\big) \\
& + 5\big(\kappa_{-a}\,\kappa'_{-b}\,\bar\kappa'^{-a}\,\bar\kappa^{-b} + \bar\kappa_{+a}\,\bar\kappa'_{+b}\,\kappa'^{+a}\,\kappa^{+b}\big) + 6\,\kappa_{-a}\,\kappa'_{-b}\,\kappa^{+a}\,\kappa'^{+b} - 2\big(\kappa_{-a}\,\bar\kappa'_{+b}\,\bar\kappa^{-a}\,\kappa'^{+b} \\
& + \bar\kappa_{+a}\,\kappa'_{-b}\,\kappa^{+a}\,\bar\kappa'^{-b}\big) - \frac{1}{2}(\bar s')^{\dot a}_{a}\,(\bar s')^{\dot b}_{b}\,s^{a}_{\dot b}\,s^{b}_{\dot a} + \frac{1}{2}s^{a}_{\dot a}\,s^{b}_{\dot b}\,(s')^{c,\dot b}\,s'^{d\,\dot a}\big(\epsilon_{a\,b}\,\epsilon_{c\,d} - \epsilon_{a\,d}\,\epsilon_{c\,b}\big) \\
& - \frac{i}{4}\big(\kappa_{-a}\,\kappa'^{+a}\,\bar s^{\dot a}_{b}\,s^{b}_{\dot a} - \kappa_{-a}\,\kappa'^{+b}\,\bar s^{\dot b}_{a}\,s^{a}_{\dot a} - \kappa'_{-c}\,\kappa^{+a}\,\bar s^{\dot a}_{a}\,s^{c}_{\dot a}\big) - \frac{1}{2}\Big(\bar\kappa'_{+c}\,\kappa^{+c}\,\bar s^{\dot a}_{a}\,(s')^{a}_{\dot a} - \kappa'_{-c}\,\bar\kappa'^{-c}\,\bar s^{\dot a}_{a}\,s^{a}_{\dot a} \\
& + \kappa'_{-c}\,\bar\kappa'^{-a}\,\bar s^{\dot a}_{a}\,s^{c}_{\dot a} + \bar\kappa'_{+c}\,\kappa'^{+a}\,\bar s^{\dot a}_{a}\,s^{c}_{\dot a} + \bar\kappa_{+a}\,\kappa^{+b}\,(\bar s')^{\dot a}_{b}\,(s')^{a}_{\dot a} - \bar\kappa'_{+c}\,\kappa^{+c}\,(\bar s')^{\dot a}_{a}\,s^{a}_{\dot a} - \kappa_{-a}\,\bar\kappa'^{-a}\,\bar s^{\dot b}_{a}\,(s')^{b}_{\dot a} \\
& - \kappa_{-a}\,\bar\kappa^{-a}\,(\bar s')^{\dot a}_{b}\,(s')^{b}_{\dot a} + \kappa_{-a}\,\bar\kappa^{-b}\,(\bar s')^{\dot a}_{b}\,(s')^{a}_{\dot a}\Big) + \kappa_{-a}\,\bar\kappa'^{-a}\,(\bar s')^{\dot a}_{b}\,s^{b}_{\dot a} - \frac{3}{2}\big(\kappa_{-a}\,\bar\kappa'^{-b}\,(\bar s')^{\dot a}_{b}\,s^{a}_{\dot a} \\
& + \bar\kappa'_{+c}\,\kappa^{+a}\,(\bar s')^{\dot a}_{a}\,s^{c}_{\dot a}\big) + \frac{1}{4}\epsilon^{b\,d}\,\epsilon_{\dot a\,\dot b}\Big(i\big(\kappa_{-b}\,\bar\kappa^{-a}\,\bar s^{\dot a}_{a}\,(\bar s')^{\dot b}_{d} - \bar\kappa_{+d}\,\kappa^{+a}\,\bar s^{\dot a}_{b}\,(\bar s')^{\dot b}_{a} - 3\,\kappa_{-a}\,\bar\kappa^{-a}\,\bar s^{\dot a}_{b}\,(\bar s')^{\dot b}_{d} \\
& + 3\,\bar\kappa_{+a}\,\kappa^{+a}\,\bar s^{\dot a}_{b}\,(\bar s')^{\dot b}_{d}\big) + 2\big(\kappa_{-b}\,\kappa'^{+a}\,\bar s^{\dot a}_{a}\,(\bar s')^{\dot b}_{d} - \bar\kappa_{+d}\,\bar\kappa'^{-a}\,\bar s^{\dot a}_{b}\,(\bar s')^{\dot b}_{a} + \kappa'_{-d}\,\kappa'^{+a}\,\bar s^{\dot a}_{a}\,\bar s^{\dot b}_{b} \\
& - \kappa_{-a}\,\kappa'^{+a}\,\bar s^{\dot a}_{b}\,(\bar s')^{\dot b}_{d} - \kappa_{-b}\,\kappa^{+a}\,(\bar s')^{\dot a}_{a}\,(\bar s')^{\dot b}_{d} + \kappa'_{-b}\,\kappa^{+a}\,\bar s^{\dot a}_{a}\,(\bar s')^{\dot b}_{d}\big) + 6\big(\bar\kappa_{+b}\,\bar\kappa'^{-a}\,\bar s^{\dot a}_{a}\,(\bar s')^{\dot b}_{d} \\
& - \bar\kappa'_{+a}\,\bar\kappa^{-a}\,\bar s^{\dot a}_{b}\,(\bar s')^{\dot b}_{d}\big) - 8\,\bar\kappa'_{+d}\,\bar\kappa^{-a}\,\bar s^{\dot a}_{a}\,(\bar s')^{\dot b}_{b}\Big) + h.c.
\end{aligned}
\tag{7.35}
$$

Even though quite complicated, both $\mathcal{H}_{BB}$ and $\mathcal{H}_{FF}$ are definitely manageable expressions. Note that the pure bosonic Hamiltonian suffers from non derivative terms while the pure fermionic do not. For the latter, these were removed through the shift (7.26). For the bosonic non derivative terms these can be removed through the use of a canonical transformation as explained in **?** and **?**. However, for the upcoming analysis, these will not have any effect on the calculations, so we choose to leave them as they stand.

As was explained in the previous section, the exact form of the fermionic shift relevant for the mixing Hamiltonian has not been determined. In the appendix we present the original Hamiltonian, prior to the fermionic shift, together with the form of $\widetilde\Phi$. The brave reader interested in the full mixing Hamiltonian can from there determine the exact form of the additional shift $\hat\Phi_3$. Having established the full shift one can, together with the corrections to the transverse part of π, determine the exact form of the shifted $\mathcal{H}_{BF}$.

We have now obtained the relevant Hamiltonian up to quartic order in number of fields. It is fully gauge fixed and posses the full SU(2|2)×U(1) symmetry of the theory. In the next two sections we will perform explicit calculations with it, starting by calculating

[14] The expression is not simplified by using the two spinor notation so we choose to present it with the $s^{a}_{\dot b}$ and $\kappa^{\pm}$ terms explicit.

the energy shift for closed bosonic and fermionic subsectors and matching these with a set of light-cone Bethe equations.

7.4 Energy shifts and light-cone Bethe equations

In light of the AdS_4/CFT_3 correspondence, energies of string excitations should correspond to anomalous dimensions of single trace operators in certain three dimensional Chern-Simons theories **?**. Based on integrability and the extensive knowledge from the original AdS_5/CFT_4 correspondence **?**, there has been a very rapid progress in understanding how to encode the spectral problem of both models in terms of Bethe equations. In **?** a all loop set of asymptotic Bethe equations were proposed for the full $OSP(2,2|6)$ model which supposedly encode the energies of all possible (free) $AdS_4 \times \mathbb{CP}_{\not{k}3}$ string configurations. In **?** and **?** it was shown that the spectrum of string excitations in a closed bosonic subsectors of the theory exactly match the predictions of the Bethe equations from **?**. In this section we will review the analysis of **?** and **?** and explicitly match energy shifts against the light-cone Bethe equations. Not only will this be an important consistency check of the derived Hamiltonian, but it will also lend support to the assumed integrability of the full supersymmetric string model. It is also worth mentioning that this is the first explicit calculation probing the higher order fermionic sector of the duality.

Note that we will be rather brief in this section since all the details are spelled out in section 5.3 and the papers **?** and **?**.

7.4.1 Strings in closed subsectors

In this section we will compute the energy shifts for a closed fermionic subsector constituted of the fields $\kappa^{\pm}$. Since we have cubic interaction terms in the Hamiltonian, the standard way to obtain the energy shifts would be through second order perturbation theory. However, this is quite an involved procedure since we have to sum over intermediate zeroth order states. A much simpler approach is to remove the cubic terms through a unitary transformation of the Hamiltonian **?**, **?**

$$\mathcal{H} \to e^{iV}\,\mathcal{H}\,e^{-iV}, \tag{7.36}$$

where the guiding principle for the construction of V is that it should obey

$$i[V,\mathcal{H}_2] = -\mathcal{H}_3, \tag{7.37}$$

and thus removes the unwanted terms.

To find an appropriate generating functional we need the oscillator components of $\mathcal{H}_3$

$$\begin{aligned}\sqrt{g}\,\mathcal{H}_3 &= \mathcal{H}^{+++} + \mathcal{H}^{++-} + h.c \qquad (7.38)\\ &= \int dk\,dn\,dl\Big(C(k,n,l)^{+++}\,\bar{X}(k)\,\bar{Y}(n)\,\bar{Z}(l) + C(k,n,l)^{++-}\,\bar{X}(k)\,\bar{Y}(n)\,Z(l)\Big) + h.c.\end{aligned}$$

where the oscillators X, Y and Z takes values in the set of $8_F + 8_B$ oscillators. However, since we want the energy shifts for $\kappa^{\pm}$ and $\omega_{\dot{a}}$ excitations, we only need the piece of $\mathcal{H}_3$ that depends quadratically on these excitations, that is, the first and last line of (7.33). Considering only this part, we can construct a function V with the property (7.37) as **?**

$$\sqrt{g}\, V = \int dk\, dn\, dl \Big\{ \tag{7.39}$$
$$\frac{-iC(k,n,l)^{+++}}{w_x(k)+w_y(n)+w_z(l)} \bar{X}(k)\, \bar{Y}(n)\, \bar{Z}(l) + \frac{-iC(k,n,l)^{++-}}{w_x(k)+w_y(n)-w_z(l)} \bar{X}(k)\, \bar{Y}(n)\, Z(l) \Big\} + h.c,$$

where $w_i(m)$ is either ω_m or Ω_m depending on the mass of $Z(l)$. It is straight forward, albeit tedious, to check that this choice of V indeed removes the cubic terms. However, from (7.36) it is clear the V commuted with the cubic part of the Hamiltonian will give rise to additional quartic terms,

$$\mathcal{H}_4^{Add} = -\frac{1}{2}\{V^2, \mathcal{H}_2\} + V\, \mathcal{H}_2\, V = \frac{i}{2}[V, \mathcal{H}_3]. \tag{7.40}$$

Even though the precise form of $\mathcal{H}_4^{Add}$ is quite complicated, evaluating its matrix elements is nevertheless significantly simpler than performing second order perturbation theory with the original Hamiltonian. Thus, after the unitary transformation, the Hamiltonian is of the form

$$\mathcal{H} = \mathcal{H}_2 + \frac{1}{g}(\mathcal{H}_4 + \mathcal{H}_4^{Add}) + \mathcal{O}(g^{-3/2}), \tag{7.41}$$

and this is the Hamiltonian we will use to calculate energy shifts in first order perturbation theory.

However, before we move on to that analysis there is one important issue we should comment on - namely, normal ordering. As was the case for the AdS$_5\times$S^5 string, the next to leading order piece, which is the cubic contribution in our case, can be assumed to be normal ordered. The subleading piece can, however, not be assumed to be ordered. This is quite clear since the resulting additional quartic terms from the unitary shift are not ordered, and even though the precise prescription to order them is clear, the ordering will result in quadratic and zeroth order terms and these terms should somehow combine with an ordering prescription of the original quartic terms. This is an analysis that we have not performed since to the order of our interest, the normal ordering ambiguities can be addressed using ζ-function regularization, see **?** and **?**[15]

The states we calculate the energy shifts from will be of the form

$$\begin{aligned} F: &\quad |m_1 \dots m_M\, n_1 \dots n_N\rangle = \bar{c}_1(m_1) \dots \bar{c}_1(m_M)\, \bar{d}^2(n_1) \dots \bar{d}^2(n_N)\, |0\rangle, \\ B: &\quad |k_1 \dots k_M\, l_1 \dots l_N\rangle = \bar{a}_{\dot{1}}(k_1) \dots \bar{a}_{\dot{1}}(k_M)\, \bar{b}^{\dot{2}}(l_1) \dots \bar{b}^{\dot{2}}(l_N)\, |0\rangle, \end{aligned} \tag{7.42}$$

[15]From the point of view of the worldsheet theory, calculating energy shifts to the order we are doing is basically a tree level calculation and the additional effects originating from the ordering terms enter at loop level.

where the sum of the mode numbers has to equal zero, $\sum_{i=1}^{M} m_i + \sum_{j=1}^{N} n_j = 0$. For simplicity we only consider states where all mode numbers are distinct.

Before we move on with an explicit calculation of the energy shifts, let us comment a bit on the normal ordering of the cubic and quartic Hamiltonian. As was the case for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string, we can take the next to leading order contribution, in our case $\mathcal{H}_3$, to be normal ordered. However, what about the quartic piece? If we were to assume it to be ordered then the sum $\mathcal{H}_3 + \mathcal{H}_4$ is naturally also ordered. However, the additional terms originating from the unitary transformation (7.36) would then not be ordered and ordering them would result in additional zeroth and quadratic order terms. When calculating the energy shifts these terms result in divergent sums which has to be regularized using, for example, Zeta function regularization, see **?** for a detailed discussion. Even though it can be done and the resulting expressions are physical, it is by all means an ugly method. The most rigorous way to proceed would be to assume an ordered cubic piece but leave the quartic Hamiltonian unordered. Then by ordering the quartic part in the most general way, namely symmetrized sums for bosonic and antisymmetrized sums for fermionic modes, would allow one to order the full Hamiltonian.

Nevertheless, for the calculation at hand we can extract the energy shifts without ordering the full Hamiltonian. The resulting divergent expressions can then be shown to vanish upon Zeta function regularization, very much as was the case in **?**.

The full quartic Hamiltonian, including the additional terms from the unitary transformation, have a general structure as

$$
\begin{aligned}
& g\,\mathcal{H}_4 = \\
& \frac{1}{(2\pi)^2}\int dk\,dn\,dl\,dm\,\delta(m+l-k-n)\Big\{F(k,n,l,m)^{11}_{11}\,\bar{c}_1(k)\,\bar{c}_1(n)\,c^1(l)\,c^1(m) \\
& + F(k,n,l,m)^{22}_{22}\,\bar{d}^2(k)\,\bar{d}^2(n)\,d_2(l)\,d_2(m) + F(k,n,l,m)^{12}_{21}\,\bar{c}_1(k)\,\bar{d}^2(n)\,d_2(l)\,c^1(m) \\
& G(k,n,l,m)^{\dot{1}\dot{1}}_{\dot{1}\dot{1}}\,\bar{a}_{\dot{1}}(k)\,\bar{a}_{\dot{1}}(n)\,a^{\dot{1}}(l)\,a^{\dot{1}}(m) + G(k,n,l,m)^{\dot{2}\dot{2}}_{\dot{2}\dot{2}}\,\bar{b}^{\dot{2}}(k)\,\bar{b}^{\dot{2}}(n)\,b_{\dot{2}}(l)\,b_{\dot{2}}(m) \\
& + G(k,n,l,m)^{\dot{1}\dot{2}}_{\dot{2}\dot{1}}\,\bar{a}_{\dot{1}}(k)\,\bar{b}^{\dot{2}}(n)\,b_{\dot{2}}(l)\,a^{\dot{1}}(m)\Big\} + \text{ Non relevant terms.}
\end{aligned}
\tag{7.43}
$$

The components $F(k,n,l,m)^{ab}_{cd}$ and $G(k,n,l,m)^{\dot{a}\dot{b}}_{\dot{c}\dot{d}}$ are quite complicated functions of the frequencies and the fermionic wave functions. Luckily, their form gets constrained considerably when projected on the states (7.42),

$$
\begin{aligned}
\Delta E_F &= \langle n_N \dots n_1\, m_M \dots m_1 |\, \mathcal{H}_4\, | m_1 \dots m_M\, n_1 \dots n_N \rangle = \qquad (7.44)\\
&\frac{1}{g}\Big\{\frac{1}{16}\sum_{i,j=1}^{M}\frac{(m_i-m_j)^2}{\omega_{m_i}\,\omega_{m_j}}+\frac{1}{16}\sum_{i,j=1}^{N}\frac{(n_i-n_j)^2}{\omega_{n_i}\,\omega_{n_j}}+\frac{1}{8}\sum_{i=1}^{M}\sum_{j=1}^{N}\frac{(m_i+n_j)^2+4\,m_i\,n_j}{\omega_{m_i}\,\omega_{n_j}}\Big\},\\
\Delta E_B &= \langle l_N \dots l_1\, k_M \dots k_1 |\, \mathcal{H}_4\, | k_1 \dots k_M\, l_1 \dots l_N \rangle =\\
&\frac{N\,M}{2\,g}-\frac{1}{4\,g}\Big\{\sum_{i=1}^{M}\sum_{j=1}^{N}\frac{(k_i-l_j)^2+2\,\omega_{k_i}\,\omega_{l_j}}{\omega_{k_i}\,\omega_{l_j}}\Big\}\\
&-\frac{1}{16\,g}\Big\{\sum_{\substack{i,j\\ i\neq j}}^{N}\frac{6\,\Omega^2_{l_i+l_j}-4(1+\omega^2_{l_i}+\omega^2_{l_j})}{\omega_{l_i}\,\omega_{l_j}}+\sum_{\substack{i,j\\ i\neq j}}^{M}\frac{6\,\Omega^2_{k_i+k_j}-4(1+\omega^2_{k_i}+\omega^2_{k_j})}{\omega_{k_i}\,\omega_{k_j}}\Big\}.
\end{aligned}
$$

Since both the $\kappa^{\pm}$ and $\omega_{\dot{a}}$ part of (7.35) and the additional quartic terms are quite complicated, it is a remarkable feature of the uniform light-cone and kappa gauge that the energy shifts takes such a simple form.

In the next section we will show that these energy shifts are exactly reproduced from the asymptotic Bethe equations of **?** and **?**.

7.4.2 Bethe equations

As we did for the $AdS_5 \times S^5$ string, it is convenient to rewrite the Bethe equations in a form which make them more suitable for a large λ expansion. As in section 5.3, we start by introducing the rapidity variables $x^{\pm}$ as[16]

$$x^{\pm}+\frac{1}{x^{\pm}}=\frac{1}{h(\lambda)}\Big(u\pm\frac{i}{2}\Big), \tag{7.45}$$

where u is given in (3.37) and express the spin chain length L in terms of M and N as

$$L=J+\frac{1}{2}(M+N), \tag{7.46}$$

where J is the total charge of the ground state (3.34). We then combine the energy E and the charge J into the light-cone pair as $P_{\pm}=\pm E+J$ which allow us to write

$$J=\frac{1}{2}(P_+ + P_-). \tag{7.47}$$

Using this to rewrite L together with the identity **?**

$$\frac{u(p_k)-u(p_j)+i}{u(p_k)-u(p_j)-i}=\frac{x^+(p_k)-x^-(p_j)}{x^-(p_k)-x^+(p_j)}\cdot\frac{1-\big(x^+(p_k)\,x^-(p_j)\big)^{-1}}{1-\big(x^-(p_k)\,x^+(p_j)\big)^{-1}},$$

[16]Note that these are not the same as for the $AdS_5 \times S^5$ equations since they now depend on the interpolating scalar function $h(\lambda)$.

we can write (3.35) as

$$\Big(\frac{x^+(p_k)}{x^-(p_k)}\Big)^{\frac{1}{2}(P_+ + M + N)} = \tag{7.48}$$
$$\Big(\frac{x^+(p_k)}{x^-(p_k)}\Big)^{-\frac{1}{2}P_-} \prod_{k\neq j}^{M} \frac{x^+(p_k)-x^-(p_j)}{x^-(p_k)-x^+(p_j)} \cdot \frac{1-\big(x^+(p_k)\,x^-(p_j)\big)^{-1}}{1-\big(x^-(p_k)\,x^+(p_j)\big)^{-1}} \prod_{j=1}^{M} S_0(p_k,p_j) \prod_{j=1}^{N} S_0(p_k,q_j),$$
$$\Big(\frac{x^+(q_k)}{x^-(q_k)}\Big)^{\frac{1}{2}(P_+ + M + N)} =$$
$$\Big(\frac{x^+(q_k)}{x^-(q_k)}\Big)^{-\frac{1}{2}P_-} \prod_{k\neq j}^{N} \frac{x^+(q_k)-x^-(q_j)}{x^-(q_k)-x^+(q_j)} \cdot \frac{1-\big(x^+(q_k)\,x^-(q_j)\big)^{-1}}{1-\big(x^-(q_k)\,x^+(q_j)\big)^{-1}} \prod_{j=1}^{N} S_0(q_k,p_j) \prod_{j=1}^{M} S_0(q_k,p_j).$$

At first glance this does not seem like a convenient reformulation of the equations at all. However, in the strong coupling expansion, the current form will turn out to be extremely convenient.

The equations are unknown functions in terms of the momenta p_k and q_k, the excitation numbers M and N and the light-cone energy P_- (we will shortly identify P_+ with the coupling) and the dressing phase S_0. For the dressing phase, we will only need the leading order part **?**, also presented in section 5.3, which can be written in terms of conserved charges as

$$S_0(p_k,p_j) = \tag{7.49}$$
$$\exp\{2i \sum_{r=0}^{\infty} \Big(\frac{h(\lambda)^2}{4}\Big)^{r+2} \big(Q_{r+2}(p_k)\, Q_{r+3}(p_j) - Q_{r+3}(p_k)\, Q_{r+2}(p_j)\big)\Big\},$$

where the charges $Q_r(p_k)$ are given by

$$Q_r(p_k) = \frac{2\,\sin(\frac{r-1}{2}\,p_k)}{r-1}\Big(\frac{\sqrt{\frac{1}{4}+4\,h(\lambda)^2\sin^2\frac{p_k}{2}}-\frac{1}{2}}{h(\lambda)^2\sin\frac{p_k}{2}}\Big)^{r-1}. \tag{7.50}$$

The light-cone energy can be expressed, using (3.39), through the dispersion relation

$$\Delta - J = \sum_{j=1}^{M}\Big(\sqrt{\frac{1}{4}+4\,h(\lambda)^2\sin^2\frac{p_j}{2}}-\frac{1}{2}\Big) + \sum_{j=1}^{N}\Big(\sqrt{\frac{1}{4}+4\,h(\lambda)^2\sin^2\frac{q_j}{2}}-\frac{1}{2}\Big). \tag{7.51}$$

The numbers M and N figuring above is the total number of oscillators, or equivalently, the number of Y_1 and $Y_3^\dagger$ or fermionic impurities in (2.36).

With the identifications $P_+ = 2g$, which follows from $p_+ = 1$, and the discussion at

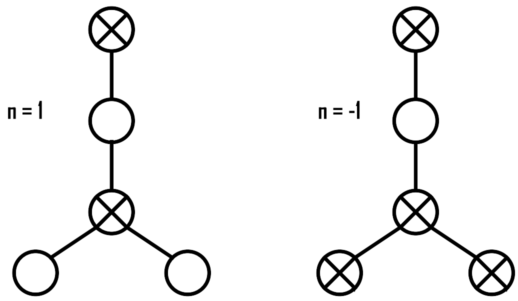

Figure 7.1: Dynkin diagrams for the two choices of gradings, $\eta = \pm 1$

the end of section 5.3, we can rewrite the equations of (7.48), to order $\mathcal{O}(g^{-3})$, as

$$\begin{aligned}
&\Big(\frac{x^+(p_k)}{x^-(p_k)}\Big)^{\frac{1}{2}\big(2g+\eta(M+N)\big)} = \\
&\Big(\frac{x^+(p_k)}{x^-(p_k)}\Big)^{-g} \prod_{k\neq j}^{M} \Big(\frac{x^+(p_k)-x^-(p_j)}{x^-(p_k)-x^+(p_j)}\Big)^{\frac{1}{2}(1+\eta)} \sqrt{\frac{1-\big(x^+(p_k)\,x^-(p_j)\big)^{-1}}{1-\big(x^+(p_j)\,x^-(p_k)\big)^{-1}}}\times \\
&\prod_{j=1}^{N} \Big(\frac{x^+(p_k)-x^-(q_j)}{x^-(p_k)-x^+(q_j)}\Big)^{\frac{1}{2}(1-\eta)} \sqrt{\frac{1-\big(x^+(q_j)\,x^-(p_k)\big)^{-1}}{1-\big(x^+(p_k)\,x^-(q_j)\big)^{-1}}} + \mathcal{O}(g^{-3}), \\
&\Big(\frac{x^+(q_k)}{x^-(q_k)}\Big)^{\frac{1}{2}\big(2g+\eta(M+N)\big)} = \\
&\Big(\frac{x^+(q_k)}{x^-(q_k)}\Big)^{-g} \prod_{k\neq j}^{M} \Big(\frac{x^+(q_k)-x^-(q_j)}{x^-(q_k)-x^+(q_j)}\Big)^{\frac{1}{2}(1+\eta)} \sqrt{\frac{1-\big(x^+(q_k)\,x^-(q_j)\big)^{-1}}{1-\big(x^+(q_j)\,x^-(q_k)\big)^{-1}}}\times \\
&\prod_{j=1}^{N} \Big(\frac{x^+(q_k)-x^-(p_j)}{x^-(q_k)-x^+(p_j)}\Big)^{\frac{1}{2}(1-\eta)} \sqrt{\frac{1-\big(x^+(p_j)\,x^-(q_k)\big)^{-1}}{1-\big(x^+(q_k)\,x^-(p_j)\big)^{-1}}} + \mathcal{O}(g^{-3}).
\end{aligned} \tag{7.52}$$

In contrast to (7.48), we generalized the equations a bit by introducing the constant $\eta = \pm 1$ which selects one of the two Dynkin diagrams in figure 7.1. The main difference between the two diagrams is the statistics of the M and N nodes, where the integers denote the number of oscillators. For $\eta = 1$ the basic spin flips in the two spin chains are the purely bosonic $\{a^1, b_{\dot 2}\}$ (corresponding to the closed $SU(2)\times SU(2)$ sector described earlier) while for $\eta = -1$ they are the fermionic $\{c^1, d_2\}$. Since we are calculating energy shifts for both bosonic and fermionic operators, we should pick the $\eta = 1$ for bosonic states and $\eta = -1$ for fermionic states, see section 5.3 for a detailed discussion.

The spectral parameters $x^{\pm}(p_k)$ can be solved for using (7.45), where u is given in (3.37) and which we state again for completeness

$$u(p_k) = \cot\frac{p_k}{2}\sqrt{\frac{1}{4} + 4\,h(\lambda)^2\sin^2\frac{p_k}{2}}. \tag{7.53}$$

As we explained earlier, the function $h(\lambda)$ is a novel feature for the AdS_4 / CFT_3 duality and is, so far, only known perturbatively **?**. It scales differently in the weak / strong coupling regimes, where in our case we only need the leading order part of the strong coupling expansion[17]

$$h(\lambda) = \sqrt{\frac{\lambda}{2}} + \mathcal{O}(\lambda^0), \tag{7.54}$$

where the 't Hooft coupling λ is related to g as

$$\lambda = \frac{g^2}{2\pi^2}. \tag{7.55}$$

The plan now is to assume a perturbative expansion for the rapidity momentas p_k and q_k that enters through $x^{\pm}(p_k)$ in the Bethe equations. Assuming an expansion as in (5.61), and with the identification $P_+ = 2g$, we have

$$p_k = \frac{p_k^0}{2g} + \frac{p_k^1}{(2g)^2}, \qquad q_k = \frac{q_k^0}{2g} + \frac{q_k^1}{(2g)^2}. \tag{7.56}$$

Using (7.45) and the explicit representation for $u(p_k)$ in (3.37), we can expand the Bethe equations (7.52) to the order of interest and solve explicitly for the components of p_k and q_k. For the leading order contribution one finds

$$p_k^0 = 4\pi\, m_k, \qquad q_k^0 = 4\,\pi\, n_k. \tag{7.57}$$

The higher order components p_k^1 and q_k^1 are a bit more involved but can straightforwardly be deduced from (7.52). Having obtained both p_k and q_k one plugs the solution into (7.51) and expands to correct order. Going through withe the calculation, and picking $\eta = 1$

[17]The reason we could ignore the ordering issues of the light-cone Hamiltonian, is because they kick in at order $\mathcal{O}(\lambda^0)$ of $h(\lambda)$, i.e. beyond the tree level approximation.

for the bosonic states and $\eta = -1$ for the fermions, one gets[18]

$$
\begin{aligned}
&\Delta E_F = \qquad (7.58)\\
&\frac{1}{4g}\sum_{k=1}^{M}\Big\{\frac{(M+N)m_k^2}{\omega_k} + \frac{8\,m_k^2}{\omega_k}\Big(\sum_{j=1}^{M}\frac{m_j(m_k-m_j)}{(1+2\omega_k)(1+2\omega_j)-4\,m_k\,m_j}\\
&-\sum_{j=1}^{N}\frac{n_j(m_k-n_j)}{(1+2\omega_k)(1+2\omega_j)-4\,m_k\,n_j} - \sum_{j=1}^{N}\frac{n_j(1+\omega_j+\omega_k)}{n_j(1+2\omega_k)-m_k(1+2\omega_j)}\Big)\Big\}\\
&+\frac{1}{4g}\sum_{k=1}^{N}\Big\{\frac{(M+N)n_k^2}{\omega_k} + \frac{8\,n_k^2}{\omega_k}\Big(\sum_{j=1}^{N}\frac{n_j(n_k-n_j)}{(1+2\omega_k)(1+2\omega_j)-4\,n_k\,n_j}\\
&-\sum_{j=1}^{M}\frac{m_j(n_k-m_j)}{(1+2\omega_k)(1+2\omega_j)-4\,n_k\,m_j} - \sum_{j=1}^{M}\frac{m_j(1+\omega_j+\omega_k)}{m_j(1+2\omega_k)-n_k(1+2\omega_j)}\Big)\Big\},
\end{aligned}
$$

while the bosonic shifts with $\eta = 1$ are given by

$$
\begin{aligned}
&\Delta E_B = \qquad (7.59)\\
&\frac{1}{4\,g}\sum_{r=1}^{M}\Big\{-\frac{(M+N)\,k_r^2}{\omega_r} + \frac{8\,k_r^2}{\omega_r}\Big(\sum_{j\neq r}^{M}\frac{k_j(1+\omega_r+\omega_j)}{k_j(1+2\,\omega_r)-k_r(1+2\,\omega_j)}\\
&+\sum_{j=1}^{M}\frac{k_j(k_r-k_j)}{(1+2\,\omega_r)(1+2\,\omega_j)-4\,k_r\,k_j} - \sum_{j=1}^{N}\frac{l_j(k_r-l_j)}{(1+2\,\omega_r)(1+2\,\omega_j)-4\,k_r\,l_j}\Big)\Big\}\\
&+\frac{1}{4\,g}\sum_{r=1}^{N}\Big\{-\frac{(M+N)\,l_r^2}{\omega_r} + \frac{8\,l_r^2}{\omega_r}\Big(\sum_{j\neq r}^{N}\frac{l_j(1+\omega_r+\omega_j)}{l_j(1+2\,\omega_r)-l_r(1+2\,\omega_j)}\\
&+\sum_{j=1}^{N}\frac{l_j(l_r-l_j)}{(1+2\,\omega_r)(1+2\,\omega_j)-4\,l_r\,l_j} - \sum_{j=1}^{M}\frac{k_j(l_r-k_j)}{(1+2\,\omega_r)(1+2\,\omega_j)-4\,l_r\,k_j}\Big)\Big\}.
\end{aligned}
$$

both these sets should be augmented with the expanded cyclicity condition (3.38),

$$\sum_{j=1}^{M} m_j + \sum_{j=1}^{N} n_j = 0. \qquad (7.60)$$

After enforcing this in the above one can show that the energy shifts calculated from the Bethe equations (7.52) precisely matches the string energies obtained from diagonalizing the string Hamiltonian in (7.44). However, it is quite tedious to show the algebraic equivalence of the two expressions and the use of a computer program able to handle symbolic manipulations is recommended.

The calculations here were the first to probe the factorization property on the string theory side, **?**. It was also the first the fermionic sectors of the theory were probed **?**.

[18] We abbreviated $\omega_{m_k} = \omega_k$ and similar for the n_k indices. Which excitation the index belong to should be clear from the context.

Figure 7.2: Self energy graphs.

Before we end this section let us also mention the result in **?**. There the authors constructed a fermionic reduction to a subsector identical to the SU(1|1) sector of the $AdS_5 \times S^5$ string **? ?**. However, this is not the sector we have studied since the form of the Bethe equations are not the same as the $SU(1|1) \subset PSU(2,2|4)$ light-cone Bethe equations in **?**. The relation between the two sectors is unclear for us and it would be nice to understand it further.

7.5 Quantum corrections to the heavy modes

The Bethe equations presented in the earlier section can be extended to the full symmetry group OSP(2,2|6) in which the Bethe roots fall into short representations of SU(2|2), that is, only $4_F + 4_B$ modes appear as fundamental excitation in the scattering matrix. At leading order these have the magnon dispersion relation, $\omega = \sqrt{\frac{1}{4} + p^2}$, so it is natural to associate these with the $4_F + 4_B$ light string modes, $\kappa^{\pm}$ and $\omega_{\dot{a}}$. However, as we have seen, critical string theory exhibits $8_F + 8_B$ oscillatory degrees of freedom, so how are we to understand the modes y, z_i and $s^a_{\dot{b}}$? From the quadratic Lagrangian it certainly seems like they are on an equal footing as the light modes, so why do they not appear as excitations in the S-matrix?

By continuing a line of research initiated by Zarembo in **?** we will try to address this question in the upcoming section. We will do this by calculating loop corrections to the propagators of the massive fields. As we will argue, the loop corrections have the effect that the pole gets shifted beyond the energy threshold for pair production of two light modes, so the heavy state dissolves into a two particle continuum.

From the analysis in the previous section, it is clear that the two type of relevant loop diagrams are a three vertex loop from (7.33) and a tadpole diagram from the full quartic Hamiltonian, see Figure 7.2. To calculate the corrections one would need to calculate the full contribution from both types of diagram. However, for the question wetter the heavy modes come as fundamental excitations or not, it is enough to focus our attention on the propagators analytic properties close to the pole. For the pure quadratic theory, at strictly infinite coupling, the massive propagators has a pole at $\bar{k}^2 = 1$. Incorporating quantum corrections, it gets shifted as

$$\Delta(k) \sim \int d^2k \frac{Z(k)}{\bar{k}^2 - 1 + \frac{1}{g}\delta m + i\epsilon},$$

where, as we will show, the relevant part of the mass corrections are of the form

$$\delta m = C(k)\sqrt{1 - \frac{1}{\bar{k}^2}}. \tag{7.61}$$

For values of $\bar{k}$ such that the difference $\bar{k}^2 - 1$ is very small, the first term in the propagator can be as important as the second one. Since the bare pole lies exactly at the branch point for pair production of two light modes, the sign of $C(k)$ may change the analytical properties of $\Delta(k)$. If the sign is positive, then the one particle pole is shifted below the threshold energy. If negative, however, the pole gets shifted beyond the threshold energy and disappears. This means that this field does not exist as a physical excitation for finite values of the coupling g.

As is well known, the behavior of a Feynman integral close to its pole is dominated by its imaginary part. Thus, the behavior of the quantum corrected pole can be extracted from the imaginary part of δm. This has the pleasant advantage that, for the calculation at hand, we can neglect the tadpole diagrams. This is easy to understand if one takes a look at the general structure of such a contribution,

$$\int d^2k \frac{G(k)}{\bar{k}^2 - m^2 + i\,\epsilon},$$

where $g(k)$ is a even polynomial in k and m is the mass of the particle in the loop. By direct inspection it is clear that there are no extra branch points associated to this integral. Of course, there are however a lot of real terms, both finite and divergent, resulting from the integral. It is however likely that supersymmetry guarantees that these terms cancel among themselves[19].

The analysis then boils down to isolating the imaginary part of the three vertex loops. Since we will only focus on the massive bosonic coordinates, the relevant part of the cubic Hamiltonian (7.33) is

$$\sqrt{g}\,\mathcal{H}_3^{loop} = (\bar{\Psi}_a\,\Psi^b)' Z^a_b + i(\bar{\Psi}\,\gamma^1\,\Psi' - \bar{\Psi}'\,\gamma^1\,\Psi)_a^{\ b}\,(Z')^a_b - 2i(\bar{\Psi}'\,\Psi - \bar{\Psi}\,\Psi')_a^{\ b}\,P^a_{z,b}, \tag{7.62}$$

from where its clear that the fields in the loops are $\omega_{\dot{a}}$ for the singlet and $\kappa^{\pm}$ for Z^a_b.

7.5.1 Massive singlet

We will start the analysis with the massive singlet y, already calculated by Zarembo in ?. The analysis basically boils down to determining the sign of the mass correction and

[19] The heavy modes are in a semi short representation of the SU(2|2) and should be BPS protected from mass renormalizations ? ?.

since we will encounter (complex) multi valued functions, some care is asked for when determining which value to take as physical. For this reason we will be rather detailed in this part of the calculation.

For the singlet we find that one loop corrected propagator equals

$$\langle\Omega|T(y(x)\,y(y))|\Omega\rangle = \frac{i}{(2\pi)^2}\int d^2k \frac{e^{-i\bar{k}\cdot(\bar{x}-\bar{y})}}{\bar{k}^2-1+i\epsilon}\Big(1-\frac{1}{\bar{k}^2-1-i\epsilon}\pi_{00}\Big), \tag{7.63}$$

where the polarization tensor is given by

$$-\pi_{00} = \frac{i}{2(2\pi)^2}\int d^2p \frac{(2p_0-k_0)^2}{(\bar{p}^2-\frac{1}{4}+i\epsilon)((\bar{p}-\bar{k})^2-\frac{1}{4}+i\epsilon)} \tag{7.64}$$

Using the standard Feynman parametrization with $\bar{q} = \bar{p} - \bar{k}\,z$, a direct computation gives

$$\begin{aligned} &-\pi_{00} = \\ &\frac{1}{4\pi}\Big\{\frac{1}{\eta}-\gamma-\log(\pi)-\int_0^1 dz\Big(\log\big(\frac{1}{4}-\bar{k}^2(1-z)z+i\epsilon\big)+\frac{(1-2z)^2k_0^2}{2(\frac{1}{4}-\bar{k}^2(1-z)z+i\epsilon)}\Big)\Big\}, \end{aligned} \tag{7.65}$$

where we used dimensional regularization to isolate the divergence. For a purely real argument the logarithm develops a imaginary $\pm i\pi$ part when $\bar{k}^2 > 1$, and to isolate it, we integrate z over the interval $\frac{1}{2}(1\pm\sqrt{1-\frac{1}{\bar{k}^2}})$. With the ϵ prescription included, we find that it gives rise to a small positive imaginary contribution, so it is the $i\pi$ part of Im (log) that we should use. Thus, for $\bar{k}^2 > 1$, its imaginary contribution is

$$\mathrm{Im}\Big[\frac{1}{4\pi}\int \log(\frac{1}{4}-\bar{k}^2(1-z)z)\Big] = \frac{1}{4}\sqrt{1-\frac{1}{\bar{k}^2}}. \tag{7.66}$$

If we introduce the short hand notation $\alpha = \frac{1}{2}\sqrt{1-\frac{1}{\bar{k}^2}-\frac{4i\epsilon}{\bar{k}^2}}$ and shift $z \to y+\frac{1}{2}$, the last term in (7.65) can be written as

$$\frac{k_0^2}{4\pi\,\bar{k}^2}\int_0^{\frac{1}{2}} dy(4+\frac{2\alpha}{y-\alpha}-\frac{2\alpha}{y+\alpha}). \tag{7.67}$$

The imaginary part of this integral comes from the middle term, where the ϵ prescription gives a negative imaginary contribution. To calculate the imaginary part of (7.67) we introduce $y-\alpha = \epsilon_0\,e^{i\theta}$, which gives

$$-\frac{k_0^2}{4\,\bar{k}^2}\sqrt{1-\frac{1}{\bar{k}^2}} = -\frac{k_0^2}{4}\sqrt{1-\frac{1}{\bar{k}^2}}+\mathcal{O}(1-\frac{1}{\bar{k}^2})^{\frac{3}{2}}, \tag{7.68}$$

where we assumed that $\bar{k}^2$ is close to the two particle threshold.

Combining the two results shows that

$$\operatorname{Im} \pi_{00} = \frac{1}{4}(1 - k_0^2)\sqrt{1 - \frac{1}{\bar{k}^2}} = -\frac{1}{4}k_1^2\sqrt{1 - \frac{1}{\bar{k}^2}} + \mathcal{O}(1 - \frac{1}{\bar{k}^2})^{\frac{3}{2}}, \tag{7.69}$$

which is negative definite close to the pole. This is almost what Zarembo calculated in **?**. The difference lies in the form of the square root, which in **?** was, $\sqrt{1 - \bar{k}^2}$, while we have $\sqrt{1 - \frac{1}{\bar{k}^2}}$. This is related to the expansion scheme and has no physical consequence. What is important is the presence of a positive definite function with the correct overall sign in front.

7.5.2 Massive AdS coordinates

Having established what happens to the singlet when loop corrections are taken into account we turn next to the remaining massive coordinates. The corrected propagator we want to calculate is

$$\langle \Omega | T\big(Z_l^k(x)\, Z_n^m(y)\big) | \Omega \rangle. \tag{7.70}$$

For this calculation, it is convenient to write the relevant part of the cubic Hamiltonian, (7.33), as

$$\sqrt{g}\,\mathcal{H}_3 = i\big(\bar{\Psi}\,\gamma^1\,\Psi' - \bar{\Psi}'\gamma^1\,\Psi + i\,\bar{\Psi}\cdot\Psi\big)_a^b (Z')_b^a - 2i\big(\bar{\Psi}'\,\Psi - \bar{\Psi}\,\Psi'\big)_a^b (P_z)_b^a. \tag{7.71}$$

Due to the fermions in the loop, we will encounter quadratic divergences along the way. However, as was the case for the singlet, these will not contribute to the imaginary part.

Due to the more complicated cubic Hamiltonian, the calculation will be more involved. However, pushing through with the calculation and using the Feynman parametrization as before, gives that the relevant terms are of the form

$$\delta m = \int d^2q \frac{F_0(\bar{k}) + F_2(\bar{k}, q_0^2, q_1^2) + F_4(\bar{k}, q_0^2\, q_1^2, q_1^4)}{(\bar{q}^2 - \bar{k}^2(1-z)z - \frac{1}{4} - i\,\epsilon)^2}, \tag{7.72}$$

where the subscript denote the power of q_i in the nominator.

To determine the form of the functions F_i, we repeat the same procedure as for the singlet computation. Unfortunately they are rather involved so we will not present them explicitly, but a straight forward, albeit somewhat tedious, calculation shows that

$$\delta m_0 = -2\,k_1^2\sqrt{1 - \frac{1}{\bar{k}^2}}, \qquad \delta m_2 = \frac{1}{3}(k_0^2 - k_0^4 + 4k_1^2 + k_1^4)\sqrt{1 - \frac{1}{\bar{k}^2}}, \tag{7.73}$$
$$\delta m_4 = \frac{1}{3}(\bar{k}^2 - 1)(k_0^2 + k_1^2)\sqrt{1 - \frac{1}{\bar{k}^2}},$$

which added together gives

$$\delta m = -k_1^2 \sqrt{1 - \frac{1}{\bar{k}^2}} (2\delta_n^k \, \delta_l^m - \delta_l^k \, \delta_n^m), \tag{7.74}$$

which is strictly negative[20] and exact for $\bar{k}^2 > 1$.

With this we conclude that all the massive bosons dissolve in a two particle continuum.

7.5.3 Massive fermions and comments

Even though we have not performed the calculation in detail, it is plausible that the massive $s_{\dot{a}}^b$ fields exhibit the same property as the massive bosons. By direct inspection of the cubic Hamiltonian it is clear that the fields in the loop will be the two light $\omega_{\dot{a}}$ and $\kappa^{\pm}$. Unfortunately, due to the rather entangled mixing between the $s_{\dot{a}}^b$ and $\kappa^{\pm}$ fields, the imaginary part of the propagator is rather involved. Nevertheless, it is still of the form

$$C(k) \sqrt{1 - \frac{1}{\bar{k}^2}} + \mathcal{O}(1 - \frac{1}{\bar{k}^2})^{\frac{3}{2}},$$

with a complicated $C(k)$ which we have not determined. Instead of pursuing this line of research, a much better way to approach the problem would be to calculate the worldsheet scattering matrix and from there study the behavior of the massive fields. Unfortunately, since it is only through loop corrections that the physical role of the massive fields emerge, the calculation of the scattering matrix would be complicated. In fact, not even for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case is the one loop BMN scattering matrix fully known. This gives a rather grim outlook for the possibility of deriving the exact one loop behavior of the $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{k}3}$ BMN string.

7.6 Summary and closing comments

In this section we have presented a detailed discussion about the type IIA superstring in $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{k}3}$. By starting directly from the $\mathfrak{osp}(2,2|6)$ superalgebra we constructed the string Lagrangian through its graded components. The string Lagrangian, covariant under SU(2)×SU(2)×U(1), was the starting point for a perturbative analysis in a strong coupling limit. We almost immediately ran into problem due to the presence of higher order kinetic terms for the fermions. These had the sad effect that they complicated the general structure of the theory to such an extent that we only presented parts of the canonical Hamiltonian. Nevertheless, we proceeded with a calculation of energy shifts for bosonic and fermionic string configurations built out of a arbitrary number of $\{a^{\dot{1}}, b_{\dot{2}}\}$ and $\{c^1, d_2\}$ oscillators. These shifts we successfully matched with the prediction coming from a conjectured set of light-cone Bethe equations.

We then moved on to an investigation of the role of the massive bosonic modes. By

[20]Or, to be precise, it is strictly negative when we restrict to the z_i propagation, $\langle T(z_i \, z_j) \rangle$.

calculating loop corrections to the propagators of the massive fields we saw that the massive modes dissolved into a two particle continuum.

We also provided an extensive appendix where the original Hamiltonian, including the kinetic terms of the fermions were spelled out in detail.

All in all we have presented a rather thorough study of the $\mathrm{AdS}_4 \times \mathbb{CP}_{\not{\vDash}3}$ superstring. Naturally a lot remains to be done, where perhaps the most stressing, at least from the point of view of our analysis, is to establish the one loop scattering matrix for the heavy modes. Even though we provided arguments for that the heavy modes dissolve in a two particle continuum, it would be desirable to see it explicitly in terms of Feynman diagrams. Unfortunately, due to the complexity of the theory, it does not seem very plausible that one can achieve this through the use of the BMN string. Perhaps a better way to approach the problem would be through the so called near flat space limit **?**, **?**.

Another interesting line of research would be to consider higher order corrections to the interpolating function $h(\lambda)$ that occurs in the magnon dispersion relation. It has been extensively studied in **? ? ?**, but its higher order structure remains unknown.

8 Closing comments, summary and future research

We are finally reaching the end of this thesis in theoretical physics. Hopefully we have managed to convey a general picture of how string theory at strong coupling behaves.

As the reader might remember we embarked upon the journey with introductory chapters reviewing the gauge theories occurring in the two AdS / CFT correspondences. We paid careful attention to the existence of integrable structures and explained how to encode the spectrum of conformal dimensions into compact sets of Bethe equations. We then turned to describe general aspects of light-cone string theory where we began with a thorough review of the bosonic aspects of the theory. Even though significantly simpler, the bosonic strings still shared many features with the full supersymmetric theory. After having established some familiarity with the formalism we introduced the full supersymmetric theory where the starting point were the symmetry algebras of each model. We then constructed the respective group elements and from there built a flat current from which we could obtain the full string Lagrangian.

The second part of the thesis were in general devoted to strong coupling analysis of the three string models, with an important focus on the light-cone Bethe equations. A major emphasis has, except for the $\mathrm{AdS}_3\times\mathrm{S}^3$ string, been to calculate string energies in order to compare these with the light-cone Bethe equations. For all the cases compared, we found a remarkable agreement.

The main output of this thesis is two fold. First, we have provided a rigorous study of strongly coupled light-cone string theory. Even though the review article **?** touched upon the subject, that review mostly focused on the foundations of the coset construction of the $\mathrm{AdS}_5\times\mathrm{S}^5$ string. In this thesis we more or less took that review as a starting point to perform a large coupling expansion beyond leading order in perturbation theory. Second, we have provided quite a considerable amount of evidence for the validity of the asymptotic Bethe equations of **?** and **?**.

Even though integrability is believed to be manifest in the AdS / CFT dualities, it is nevertheless very important to put it on a solid footing. In the literature a huge host of independent tests and checks have been performed and it has been a research field populated by a large number of scientists. The authors contribution is naturally just one small piece of the puzzle, but nevertheless, it lends argument for the existence of integrability, even in highly non trivial sectors of the theories. This is important since, as of now, the only hope to solve both sides of the AdS / CFT correspondences analytically, is through the use of integrability. It is because of integrability that we can extrapolate the values of the calculated observables and actually compare them on both sides of the duality.

8.1 Outlook

What to do next? If we start with the $\mathrm{AdS}_5\times\mathrm{S}^5$ string a natural extension of the work presented in this thesis would be to calculate finite size corrections to energies of string states. As we remember, the strong coupling limit was equivalent to a decompactification of the string worldsheet. This is a crucial fact when defining asymptotic states for the scattering theory since without these, the spectral problem can not decomposed into a set of Bethe equations **?**. Thus, to write down the spectrum for any values of the coupling, or J, would be very instructive indeed. For the string theory, this has been an active research field lately which have culminated into a mirror model which supposedly describe the energy of any operator at finite J. Since it is a rather complicated model we have chosen not to presented it in this thesis, but for some recent results see **????**.

In section 5.4 we studied the near flat space limit (NFS) of the $\mathrm{AdS}_5\times\mathrm{S}^5$ string. It would be interesting to see wetter the reduced NFS model is invariant under the full centrally extended $\mathrm{SU}(2|2)^2$ group or only invariant under some truncated part. However, since the model is obviously invariant under the bosonic $\mathrm{SU}(2)^2$, it is very likely that it possess the full symmetry.

Another line of research would be to acquire a more fundamental understanding of the $\mathrm{AdS}_5\times\mathrm{S}^5$ sigma model. For example, could one invent a gauge in which the two dimensional Lorentz symmetry is manifest even at higher order in perturbation theory? As we remember, the light-cone gauge breaks the worldsheet Lorentz symmetry beyond quadratic order. Perhaps the approach developed in **????** is an appropriate alternative. There the authors rewrite the sigma model as a gauged WZN-model whose two dimensional Lorentz symmetry is manifest.

It would also be very interesting to study higher loop effects from the string theory side. For example, deriving the full worldsheet scattering matrix of the near-BMN string to, for example, one loop. However, as it turns out, the theory suffers from infinities and at the moment it is not clear how to remove these[1].

For the $\mathrm{AdS}_3\times\mathrm{S}^3$ string its rather clear what needs to be done. First, one should finish the analysis we initiated. That is, one should show in detail that the non critical string is centrally extended in the same way as its ten dimensional $\mathrm{AdS}_5\times\mathrm{S}^5$ cousin. However, this is in one sense more or less trivial, since everything worked out exactly as for the $\mathrm{PSU}(2,2|4)$ string. Secondly, it would be much more interesting to study possible off shell extensions for the full ten dimensional $\mathrm{AdS}_3\times\mathrm{S}^3\times T^4$ string. A natural starting point for this analysis would be the paper **?** where the authors constructed the theory as a coset model with an exceptional superalgebra as G.

The most pressing issue for the $\mathrm{AdS}_4\times\mathbb{CP}_{\not{\mathbb{F}}3}$ string is to determine the higher order contributions to the interpolating function $h(\lambda)$. This function is only known to the first few orders in perturbation theory and it would indeed be interesting to see how it extrapolates between strong and weak coupling at higher loop order.

The analysis we provided for the $\mathrm{AdS}_4\times\mathbb{CP}_{\not{\mathbb{F}}3}$ string was rather hand wavy, at least from a fundamental point of view. The Hamiltonian we derived, albeit classically sound,

[1] We thank T. McLoughlin for a discussion regarding this point.

suffered from normal ordering ambiguities in the quantum theory. This issue we did not address properly and in order to present a more rigorous analysis, this issue should definitely be addressed. For some recent results concerning this, please see **?**.

The study of integrable structures in AdS / CFT correspondences is still a very active research field and there is much to be done. It is the authors belief that the field is still in its infancy and yet a lot of remarkable discoveries are to be found.

The $\mathrm{AdS}_5 \times \mathrm{S}^5$ string

1 Overview of the string results

To confront the proposed light-cone Bethe equations with the quantum string result extensive computer algebra computations have been performed to diagonalize the worldsheet Hamiltonian perturbatively. For every considered subsector, i.e. $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$, $\mathfrak{su}(1|1)$, $\mathfrak{su}(1|2)$, $\mathfrak{su}(1,1|2)$ and $\mathfrak{su}(2|3)$, we state the effective Hamiltonian and present analytic results for its eigenvalues up to three impurities, whenever available. In some cases we had to retreat to a numerical comparison with the Bethe equations, details of these investigations are given in section 2.

As one sees in table 5.1 the total number of impurities (or string excitations) is given by K_4. We also allow for confluent mode numbers, where the index $k = 1, .., K_4'$ counts the excitations with distinct modes, each with a multiplicity of ν_k, using the notation of section 5.3.1. In uniform light-cone gauge the Hamiltonian eigenvalue $-P_-$ is then given by

$$P_- = -\sum_{k=1}^{K_4} \omega_k + \delta P_- = -\sum_{k=1}^{K_4'} \nu_k\, \omega_k + \delta P_- \tag{1}$$

In order to classify the Hamiltonian eigenvalues we will make use of the $U(1)$ charges $\{S_+, S_-, J_+, J_-\}$ introduced in ?. They are light-cone combinations of the two spins S_i of AdS_5 and two angular momenta J_i on S^5, viz. $S_\pm = S_1 \pm S_2$ and $J_\pm = J_1 \pm J_2$. The charges of the string oscillators are spelled out in table 1.

1.1 The $\mathfrak{su}(2)$ sector

This sector consists of states, which are composed only of $\alpha^+_{1,n}$ creation operators. The Hamiltonian (6.10) simplifies dramatically to the effective form

$$\mathcal{H}_4^{(\mathfrak{su}(2))} = \widetilde{\lambda} \sum_{\substack{m_1+m_2\\+m_3+m_4}=0} \frac{m_2 m_4}{\sqrt{\omega_{m_1}\omega_{m_2}\omega_{m_3}\omega_{m_4}}}\, \alpha^+_{1,m_1}\alpha^+_{1,m_2}\alpha^-_{1,-m_3}\alpha^-_{1,-m_4} \quad . \tag{2}$$

This sector is of rank one and the energy shifts $-\delta P_-$ for arbitrary modes $m_1, ..., m_{K_4}$ can be evaluated to

$$\delta P_-^{(\mathfrak{su}(2))} = \frac{\widetilde{\lambda}}{2P_+} \sum_{\substack{i,j=1\\ i\neq j}}^{K_4} \frac{(m_i+m_j)^2}{\omega_{m_i}\omega_{m_j}} - \frac{\widetilde{\lambda}}{P_+} \sum_{k=1}^{K_4'} \frac{m_k^2}{\omega_{m_k}^2} \nu_k\,(\nu_k - 1) \tag{3}$$

	S_+	S_-	J_+	J_-
$Y_1,\, P_1^y,\, \alpha^+_{1,m},\, \alpha^-_{4,m}$	0	0	1	1
$Y_2,\, P_2^y,\, \alpha^+_{2,m},\, \alpha^-_{3,m}$	0	0	1	-1
$Y_3,\, P_3^y,\, \alpha^+_{3,m},\, \alpha^-_{2,m}$	0	0	-1	1
$Y_4,\, P_4^y,\, \alpha^+_{4,m},\, \alpha^-_{1,m}$	0	0	-1	-1

	S_+	S_-	J_+	J_-
$Z_1,\, P_1^z,\, \beta^+_{1,m},\, \beta^-_{4,m}$	1	1	0	0
$Z_2,\, P_2^z,\, \beta^+_{2,m},\, \beta^-_{3,m}$	1	-1	0	0
$Z_3,\, P_3^z,\, \beta^+_{3,m},\, \beta^-_{2,m}$	-1	1	0	0
$Z_4,\, P_4^z,\, \beta^+_{4,m},\, \beta^-_{1,m}$	-1	-1	0	0

	S_+	S_-	J_+	J_-
$\theta_1,\, \theta_4^\dagger,\, \theta^+_{1,m},\, \theta^-_{4,m}$	0	1	1	0
$\theta_2,\, \theta_3^\dagger,\, \theta^+_{2,m},\, \theta^-_{3,m}$	0	-1	1	0
$\theta_3,\, \theta_2^\dagger,\, \theta^+_{3,m},\, \theta^-_{2,m}$	0	1	-1	0
$\theta_4,\, \theta_1^\dagger,\, \theta^+_{4,m},\, \theta^-_{1,m}$	0	-1	-1	0

	S_+	S_-	J_+	J_-
$\eta_1,\, \eta_4^\dagger,\, \eta^+_{1,m},\, \eta^-_{4,m}$	1	0	0	1
$\eta_2,\, \eta_3^\dagger,\, \eta^+_{2,m},\, \eta^-_{3,m}$	1	0	0	-1
$\eta_3,\, \eta_2^\dagger,\, \eta^+_{3,m},\, \eta^-_{2,m}$	-1	0	0	1
$\eta_4,\, \eta_1^\dagger,\, \eta^+_{4,m},\, \eta^-_{1,m}$	-1	0	0	-1

Table 1: Charges of the annihilation and creation operators of the $AdS_5 \times S^5$ string in uniform light-cone gauge.

By rewriting this P_- shift in terms of the global energy E and the BMN quantities J and $\lambda' = \lambda/J^2$ using $P_\pm = J \pm E$, and then subsequently solving for E one obtains the $\mathfrak{su}(2)$ global energy, which precisely agrees with the results in **?** and **?**

$$E = J + \sum_{k=1}^{K_4} \bar{\omega}_k - \frac{\lambda'}{4J} \sum_{k,j=1}^{K_4} \frac{m_k^2 \bar{\omega}_j^2 + m_j^2 \bar{\omega}_k^2}{\bar{\omega}_k \bar{\omega}_j} - \frac{\lambda'}{4J} \sum_{\substack{i,j=1 \\ i \neq j}}^{K_4} \frac{(m_i + m_j)^2}{\bar{\omega}_i \bar{\omega}_j} + \frac{\lambda'}{2J} \sum_{i=1}^{K_4'} \frac{m_i^2}{\bar{\omega}_i^2} \nu_k (\nu_i - 1)$$

$$\text{with} \qquad \bar{\omega}_k := \sqrt{1 + \lambda' m_k^2}\,. \tag{4}$$

1.2 The $\mathfrak{sl}(2)$ sector

The $\mathfrak{sl}(2)$ states are composed of one flavor of $\beta^+_{1,n}$ operators. Since the structure of the Hamiltonian is identical for $\alpha^\pm_{1,n}$ and $\beta^\pm_{1,n}$ up to a minus sign one immediately has

$$\mathcal{H}_4^{(\mathfrak{sl}(2))} = -\widetilde{\lambda} \sum_{\substack{m_1+m_2 \\ +m_3+m_4}=0} \frac{m_2 m_4}{\sqrt{\omega_{m_1}\omega_{m_2}\omega_{m_3}\omega_{m_4}}} \beta^+_{1,m_1}\beta^+_{1,m_2}\beta^-_{1,-m_3}\beta^-_{1,-m_4} \tag{5}$$

$$\delta P_-^{(\mathfrak{sl}(2))} = -\delta P_-^{(\mathfrak{su}(2))} \tag{6}$$

and the global energy shift follows immediately.

1.3 The $\mathfrak{su}(1|1)$ sector

States of the $\mathfrak{su}(1|1)$ sector are formed of $\theta^+_{1,n}$ creation operators. As noted in **?** the restriction of the $\mathcal{O}(1/P_+)$ string Hamiltonian (6.10) to the pure $\mathfrak{su}(1|1)$ sector vanishes

$$\mathcal{H}_4^{(\mathfrak{su}(1|1))} \equiv 0 \;, \qquad \delta P_-^{(\mathfrak{su}(1|1))} = 0 \;. \tag{7}$$

1.4 The $\mathfrak{su}(1|2)$ sector

We now turn to the first larger rank sector $\mathfrak{su}(1|2)$ being spanned by the creation operators $\theta^+_{1,n}$ and $\alpha^+_{1,n}$ of one flavor. The effective Hamiltonian is given by

$$\mathcal{H}_4^{(\mathfrak{su}(1|2))} = \mathcal{H}_4^{(\mathfrak{su}(2))} + \widetilde{\lambda} \sum_{\substack{m_1+m_2\\+m_3+m_4}=0} \frac{X(m_1,m_2,m_3,m_4)}{\sqrt{\omega_{m_3}\omega_{m_4}}}\ \theta^+_{1,m_1}\theta^-_{1,-m_2}\alpha^+_{1,m_3}\alpha^-_{1,-m_4} \quad . \tag{8}$$

where $X(m,n,k,l)$ is defined as

$$\begin{aligned} X(m,n,k,l) := \Big[\Big(mn - \frac{(m-n)(k-l)}{4}\Big)(f_n f_m + g_n g_m) \\ - \frac{\kappa}{4\sqrt{\widetilde{\lambda}}}(k+l)(\omega_k+\omega_l)(f_n g_m + f_m g_n)\Big] , \end{aligned} \tag{9}$$

where $\kappa = \pm 1$.

Two impurities

For two impurity $\mathfrak{su}(1|2)$ states carrying the modes $m_1 = -m_2$ the Hamiltonian $\mathcal{H}_4$ forms a 4×4 matrix with eigenvalues $-\delta P_-$ where

$$\delta P_- = \Big\{ \pm 2 \frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1},\ 0,\ 0 \Big\} \ . \tag{10}$$

Three impurities with distinct modes

Considering the three impurity case with distinct mode numbers m_1, m_2, m_3 the Hamiltonian is represented by an 8×8 matrix which decomposes into 4 non mixing submatrices, where two fall into the rank one sectors $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$. The remaining pieces are two 3×3 matrices.

Since string states only mix if they carry the same charges, we can classify the submatrices and their eigenvalues by the charge of the corresponding states. One finds:
$\{S_+, S_-, J_+, J_-\} = \{0,2,3,1\}_{\theta_1^+\theta_1^+\alpha_1^+|0\rangle}$:

$$\delta P_- = \Big\{ \pm \frac{\widetilde{\lambda}}{P_+}\sum_{j=1}^{3}\frac{m_j^2}{\omega_j}, \quad \frac{\widetilde{\lambda}}{P_+\omega_1\omega_2\omega_3}\sum_{j=1}^{3} m_j^2\,\omega_j \Big\} \tag{11}$$

$\{S_+, S_-, J_+, J_-\} = \{0, 1, 3, 2\}_{\theta_1^+ \alpha_1^+ \alpha_1^+ |0\rangle}$:

$$\delta P_- = \Big\{0, \quad \frac{\widetilde{\lambda}}{P_+} \frac{m_1^2 \omega_{m_1} + m_2^2 \omega_{m_2} + m_3^2 \omega_{m_3} \pm \Xi_{m_1,m_2,m_3}}{\omega_{m_1}\omega_{m_2}\omega_{m_3}}\Big\} \tag{12}$$

$$\text{with} \quad \Xi_{a,b,c} := \sqrt{4(\omega_a^2 \chi_{b,c}^2 + \omega_b^2 \chi_{a,c}^2 + \omega_c^2 \chi_{a,b}^2) + (\xi_{a;b,c} - \xi_{b;a,c} + \xi_{c;a,b})^2 - 4\xi_{a;b,c}\xi_{c;a,b}}$$

$$\xi_{a;b,c} := -\, a(b\omega_b + c\omega_c - a\omega_a)$$

$$\chi_{a,b} := -\, ab \frac{\widetilde{\lambda} ab - (1+\omega_a)(1+\omega_b)}{\sqrt{(1+\omega_a)(1+\omega_b)}} \,.$$

Three impurities with confluent modes

In the case of confluent modes $\{m_1, m_2, m_3\} = \{m, m, -2m\}$ the submatrix with charges $\{0, 2, 3, 1\}$ collapses to a scalar whereas the submatrix of charge $\{0, 1, 3, 2\}$ reduces to 2×2 matrix with energy shifts

$$\{S_+, S_-, J_+, J_-\} = \{0, 2, 3, 1\}_{\theta_1^+ \theta_1^+ \alpha_1^+ |0\rangle} : \qquad \delta P_- = \frac{\widetilde{\lambda}}{P_+} \frac{2m^2}{\omega_m} \Big(\frac{1}{\omega_m} + \frac{1}{\omega_{2m}}\Big) \tag{13}$$

$$\{S_+, S_-, J_+, J_-\} = \{0, 1, 3, 2\}_{\theta_1^+ \alpha_1^+ \alpha_1^+ |0\rangle} : \tag{14}$$

$$\delta P_- = 2 \frac{\widetilde{\lambda} q^2}{P_+ \omega_q^2 \omega_{2q}} \Big(\omega_q + \omega_{2q} \pm \omega_q \sqrt{3 + 2\omega_{2q}^2 + 4\omega_q \omega_{2q}}\Big)$$

1.5 The $\mathfrak{su}(1,1|2)$ sector

States of the $\mathfrak{su}(1,1|2)$ sector are spanned by the set $\{\theta_{1,n}^+, \eta_{1,n}^+, \beta_{1,n}^+, \alpha_{1,n}^+\}$ of creation operators. In this sector the effective Hamiltonian takes the form

$$\begin{aligned}
\mathcal{H}_4^{(\mathfrak{su}(1,1|2))} = \; & \widetilde{\lambda} \sum_{\substack{k+l \\ +n+m}=0} \frac{kl}{\sqrt{\omega_m \omega_n \omega_k \omega_l}} (\alpha_{1,m}^+ \alpha_{1,-n}^- - \beta_{1,m}^+ \beta_{1,-n}^-)(\alpha_{1,k}^+ \alpha_{1,-l}^- + \beta_{1,k}^+ \beta_{1,-l}^-) \\
& + \widetilde{\lambda} \sum_{\substack{k+l \\ +n+m}=0} 2\,\mathrm{i}\, \frac{f_m f_n - g_m g_n}{\sqrt{\omega_k \omega_l}} (\theta_{1,m}^+ \eta_{1,n}^+ \beta_{1,-k}^- \alpha_{1,-l}^- + \theta_{1,-m}^- \eta_{1,-n}^- \beta_{1,k}^+ \alpha_{1,l}^+) \\
& + \widetilde{\lambda} \sum_{\substack{k+l \\ +n+m}=0} \frac{X(m,n,k,l)}{\sqrt{\omega_k \omega_l}} (\theta_{1,m}^+ \theta_{1,-n}^- + \eta_{1,m}^+ \eta_{1,-n}^-)(\alpha_{1,k}^+ \alpha_{1,-l}^- - \beta_{1,k}^+ \beta_{1,-l}^-) \,,
\end{aligned} \tag{15}$$

where $X(m, n, k, l)$ is given in (9).

Two impurities

The Hamiltonian matrix decomposes into several non mixing submatrices. The $\mathfrak{su}(1,1|2)$ sector contains all previous discussed sectors, whose eigenvalues we do not state again.

dimension $d = 1$

$\{S_+, S_-, J_+, J_-\}$	State pattern	Property	δP_-
$\{0,0,3,3\}$	$\alpha_1^+\alpha_1^+\alpha_1^+\|0\rangle$	$\mathfrak{su}(2)$ state	(3)
$\{3,3,0,0\}$	$\beta_1^+\beta_1^+\beta_1^+\|0\rangle$	$\mathfrak{sl}(2)$ state	(6)

dimension $d = 3$

$\{S_+, S_-, J_+, J_-\}$	State pattern	Property	δP_-
$\{0,2,3,1\}$	$\theta_1^+\theta_1^+\alpha_1^+\|0\rangle$	$\mathfrak{su}(1\|2)$ state	$\delta P_-^{\{0,2,3,1\}}$ see (11)
$\{2,0,1,3\}$	$\eta_1^+\eta_1^+\alpha_1^+\|0\rangle$	property of (15) implies	$\delta P_-^{\{2,1,0,3\}} = +\delta P_-^{\{0,2,3,1\}}$
$\{1,3,2,0\}$	$\theta_1^+\theta_1^+\beta_1^+\|0\rangle$	property of (15) implies	$\delta P_-^{\{1,3,2,0\}} = -\delta P_-^{\{0,2,3,1\}}$
$\{3,1,0,2\}$	$\eta_1^+\eta_1^+\beta_1^+\|0\rangle$	property of (15) implies	$\delta P_-^{\{3,1,0,2\}} = -\delta P_-^{\{0,2,3,1\}}$
$\{0,1,3,2\}$	$\theta_1^+\alpha_1^+\alpha_1^+\|0\rangle$	$\mathfrak{su}(1\|2)$ state	$\delta P_-^{\{0,1,3,2\}}$ see (12)
$\{1,0,2,3\}$	$\eta_1^+\alpha_1^+\alpha_1^+\|0\rangle$	property of (15) implies	$\delta P_-^{\{1,0,2,3\}} = +\delta P_-^{\{0,1,3,2\}}$
$\{2,3,1,0\}$	$\theta_1^+\beta_1^+\beta_1^+\|0\rangle$	property of (15) implies	$\delta P_-^{\{2,3,1,0\}} = -\delta P_-^{\{0,1,3,2\}}$
$\{3,2,0,1\}$	$\eta_1^+\beta_1^+\beta_1^+\|0\rangle$	property of (15) implies	$\delta P_-^{\{3,2,0,1\}} = -\delta P_-^{\{0,1,3,2\}}$

Table 2: Analytically accessible three impurity, distinct $\mathfrak{su}(1,1|2)$ energy shifts.

For the two impurity case with mode numbers $m_1 = -m_2$ one obtains the new eigenvalues:

$$\{1,1,1,1\}_{\theta_1^+\eta_1^+|0\rangle,\ \beta_1^+\alpha_1^+|0\rangle}: \qquad \delta P_- = \Big\{ \pm 4\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1},\ 0,\ 0\Big\} \tag{16}$$

$$\begin{array}{ll}\{1,2,1,0\}_{\theta_1^+\beta_1^+|0\rangle}, & \{0,1,2,1\}_{\theta_1^+\alpha_1^+|0\rangle} \\ \{2,1,0,1\}_{\eta_1^+\beta_1^+|0\rangle}, & \{1,0,1,2\}_{\eta_1^+\alpha_1^+|0\rangle}\end{array} \qquad \delta P_- = \pm 2\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1} \tag{17}$$

Three impurities with confluent modes

For higher impurities the situation becomes much more involved. Already the three impurity $\mathfrak{su}(1,1|2)$ Hamiltonian for non-confluent modes becomes a 64×64 matrix with submatrices of rank 9. We will classify the $\mathfrak{su}(1,1|2)$ submatrices with respect to their charges and dimension d. Because $\mathfrak{su}(1,1|2)$ contains previously discussed sectors, we can deduce most of the eigenvalues by using properties of the Hamiltonian $\mathcal{H}_4^{(\mathfrak{su}(1,1|2))}$. Our findings are collected in the table 2.

The structure of the 9×9 submatrices is a bit more involved. Under the oscillator exchange $\theta_{1,m} \leftrightarrow \eta_{1,m}$ and $\alpha_{1,m} \leftrightarrow \beta_{1,m}$ the effective Hamiltonian $\mathcal{H}_4^{(\mathfrak{su}(1,1|2))}$ changes its sign. This exchange translates a state with charge $\{1,1,2,2\}$ into one with $\{2,2,1,1\}$ or a $\{1,2,2,1\}$ charged state into one with $\{2,1,1,2\}$ and vice versa with mutual energy shifts of opposite signs. See table 3 for results.

dimension $d = 9$

$\{S_+, S_-, J_+, J_-\}$	State pattern		δP_-
$\{1,1,2,2\}$	$\beta_1^+\alpha_1^+\alpha_1^+\vert 0\rangle$,	$\theta_1^+\eta_1^+\alpha_1^+\vert 0\rangle$	rank 9 matrix, numerical eigenvalues see table 4
$\{2,2,1,1\}$	$\beta_1^+\beta_1^+\alpha_1^+\vert 0\rangle$,	$\theta_1^+\eta_1^+\beta_1^+\vert 0\rangle$	$\delta P_-^{\{2,2,1,1\}} = -\delta P_-^{\{1,1,2,2\}}$
$\{1,2,2,1\}$	$\theta_1^+\theta_1^+\eta_1^+\vert 0\rangle$,	$\theta_1^+\beta_1^+\alpha_1^+\vert 0\rangle$	rank 6 matrix, numerical eigenvalues see table 4
$\{2,1,1,2\}$	$\theta_1^+\eta_1^+\eta_1^+\vert 0\rangle$,	$\eta_1^+\beta_1^+\alpha_1^+\vert 0\rangle$	$\delta P_-^{\{2,1,1,2\}} = -\delta P_-^{\{1,2,2,1\}}$

Table 3: Remaining three impurity, distinct $\mathfrak{su}(1,1|2)$ shifts, which were compared numerically.

1.6 The $\mathfrak{su}(2|3)$ sector

Finally the $\mathfrak{su}(2|3)$ sector is spanned by the operators $\theta^+_{1,n}, \theta^+_{2,n}, \alpha^+_{1,n}, \alpha^+_{2,n}$. The effective form of $\mathcal{H}_4$ in this closed subsector reads

$$
\begin{aligned}
\mathcal{H}_4^{(\mathfrak{su}(2|3))} = &\\
&\widetilde{\lambda} \sum_{\substack{k+l\\+n+m}=0} \frac{kl}{\sqrt{\omega_m\omega_n\omega_k\omega_l}} (\alpha^+_{1,m}\alpha^-_{1,-n} + \alpha^+_{2,m}\alpha^-_{2,-n})(\alpha^+_{1,k}\alpha^-_{1,-l} + \alpha^+_{2,k}\alpha^-_{2,-l})\\
&+\widetilde{\lambda} \sum_{\substack{k+l\\+n+m}=0} \frac{X(m,n,k,l)}{\sqrt{\omega_k\omega_l}} (\theta^+_{1,m}\theta^-_{1,-n} + \theta^+_{2,m}\theta^-_{2,-n})(\alpha^+_{1,k}\alpha^-_{1,-l} + \alpha^+_{2,k}\alpha^-_{2,-l})\\
&-\frac{\widetilde{\lambda}}{2}\mathrm{i} \sum_{\substack{k+l\\+n+m}=0} \frac{1}{\sqrt{\omega_k\omega_l}} (\theta^+_{2,m}\theta^+_{1,n}\alpha^-_{2,-k}\alpha^-_{1,-l} + \theta^-_{2,-m}\theta^-_{1,-n}\alpha^+_{2,k}\alpha^+_{1,l})\\
&\qquad \times \Big[(m-n)(k-l)(f_n g_m - f_n g_m) + \tfrac{\kappa}{\sqrt{\widetilde{\lambda}}}(k+l)(\omega_k-\omega_l)(f_n f_m - g_m g_n)\Big]\\
&+\widetilde{\lambda} \sum_{\substack{k+l\\+n+m}=0} \begin{pmatrix} (f_m g_n + f_n g_m)(f_k g_l + f_l g_k)(mn+kl)\\ +(f_n g_k + f_k g_n)(f_m g_l + f_l g_m)(nk+ml)\\ -(f_n f_l - g_n g_l)(f_m f_k + g_m g_k)(nl+mk) \end{pmatrix} \theta^+_{2,m}\theta^-_{2,-n}\theta^+_{1,k}\theta^-_{1,-l}\,.
\end{aligned}
\tag{18}
$$

Two impurities

For two impurities with mode numbers $m_2 = -m_1$ we find the energy shifts

$$\{0,0,2,0\}_{\theta_2^+\theta_1^+|0\rangle,\ \alpha_2^+\alpha_1^+|0\rangle}: \qquad \delta P_- = \Big\{\pm 4\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1},\ 0,\ 0\Big\} \tag{19}$$

$$\begin{matrix}\{0,1,2,1\}_{\theta_1^+\alpha_1^+|0\rangle}, & \{0,1,2,-1\}_{\theta_1^+\alpha_2^+|0\rangle}\\ \{0,-1,2,1\}_{\theta_2^+\alpha_1^+|0\rangle}, & \{0,-1,2,-1\}_{\theta_2^+\alpha_2^+|0\rangle}\end{matrix} \qquad \delta P_- = \pm 2\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1} \tag{20}$$

$\mathfrak{su}(2|3)$ sector[2]

$\{S_+, S_-, J_+, J_-\}$	eigenvalues $-\delta P_-$			
{0,0 ,3,±3}	-0.0106324			
{0,±2,3,±1}	± 0.0108634	-0.0106324		
{0,±1,3,±2}	-0.0214958	0.000230962	0	
{0,±1,3,0}	0.0217267	3×-0.0214958	2×0.000230962	3×0
{0,0,3,±1}	-0.0323591	0.0110943	$2 \times \pm 0.0108634$	3×-0.0106324

$\mathfrak{su}(1,1|2)$ sector

$\{S_+, S_-, J_+, J_-\}$	eigenvalues $-\delta P_-$				
{1,1,2,2}	-0.0323591	0.0110943	$2 \times \pm 0.0108634$	2×-0.0106324	0.0106324
{1,2,2,1}, {2,1,1,2}	± 0.0217267	± 0.0214958	± 0.000230962	3×0	
{2,2,1,1}	0.0323591	-0.0110943	$2 \times \pm 0.0108634$	2×0.0106324	-0.0106324

Table 4: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for three impurity states with distinct mode numbers $m_1 = 2, m_2 = 1, m_3 = -3$. The number in front of some eigenvalues denotes their multiplicity if unequal to one.

$\mathfrak{su}(2|3)$ sector

$\{S_+, S_-, J_+, J_-\}$	eigenvalues $-\delta P_-$		
{0,±1,3,0}	2×-0.0454059	2×0.0142814	
{0,0,3,±1}	-0.0752496	0.044125	3×-0.0155623
{0,±2,3,±1}, {0,0,3,±3}	-0.0155623		
{0,±1,3,±2}	-0.0454059	0.0142814	

$\mathfrak{su}(1,1|2)$ sector

$\{S_+, S_-, J_+, J_-\}$	eigenvalues $-\delta P_-$			
{1,1,2,2}	-0.0752496	0.044125	0.0155623	2×-0.0155623
{1,2,2,1},{2,1,1,2}	± 0.0454059	± 0.0142814		
{2,2,1,1}	0.0752496	-0.044125	2×0.0155623	-0.0155623

Table 5: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for three impurity states with confluent mode numbers $m_1 = m_2 = 3, m_3 = -6$. The number in front of some eigenvalues denotes their multiplicity if unequal to one.

2 Numerical results

Here we collect the numerical results, for this we dial explicit mode numbers and values for the coupling constant λ'. The considered cases constitute certain three impurity excitations in the $\mathfrak{su}(1,1|2)$ subsector with distinct and confluent mode numbers, as well as all three impurity excitations (distinct and confluent) for the $\mathfrak{su}(2|3)$ subsector. In the tables below we state explicit results for the values $\widetilde{\lambda} = 0.1$ and $P_+ = 100$ and mode numbers $(m_1, m_2, m_3) = \{(2,1,-3),(3,3,-6)\}$. All numerical energy shifts were matched precisely with the result obtained from the Bethe equations.

[2] The ± signs at some charges are just a short form of writing several charge combinations all with the same eigenvalues. They are not related to the signatures of the eigenvalues in any sense.

3 Mixing term of the original Hamiltonian

In this appendix we present the full, non shifted, quartic Hamiltonian, which combined with the fermionic kinetic term in (7.24) encodes the full dynamics of the quartic theory.

We start out by presenting the original cubic Hamiltonian which is similar but not identical to the shifted one,

$$
\begin{aligned}
&\sqrt{g}\,\mathcal{H}_3^{ns} = \qquad\qquad (21)\\
&(\bar{\Psi}'\cdot\Psi)^a_{\ b}\,Z^b_a + \big(\bar{\Psi}\,\gamma^1\,\Psi'\big)^a_{\ b}\,(Z')^b_{\ a} + i\,y\,\omega_{\dot{a}}\,\bar{p}^{\dot{a}} + 3i\,(\kappa_{-\,a}\,\bar{s}^{a\,\dot{a}} - \bar{\kappa}_{+\,a}\,s^{a\,\dot{a}})\,p_{\dot{a}}\\
&+\big(\frac{3}{8}(\bar{\kappa}_{+\,a}\,s^{a\,\dot{a}} + \kappa_{-\,a}\,\bar{s}^{a\,\dot{a}}) + i(\kappa'_{-\,a}\,s^{a\,\dot{a}} - \bar{\kappa}'_{+\,a}\,\bar{s}^{a\,\dot{a}}) + \frac{i}{2}(\bar{\kappa}_{+\,a}\,(\bar{s}')^{a\,\dot{a}} - \kappa_{-\,a}\,(s')^{a\,\dot{a}})\big)\,\omega_{\dot{a}}\\
&+\frac{1}{2}(\kappa_{-\,a}\,(\bar{s}')^{a\,\dot{a}} - \kappa'_{-\,a}\,\bar{s}^{a\,\dot{a}} - \bar{\kappa}'_{+\,a}\,s^{a\,\dot{a}} + \bar{\kappa}_{+\,a}\,(s')^{a\,\dot{a}})\,\omega'_{\dot{a}} + h.c,
\end{aligned}
$$

where the ns superscript denotes that this is the non shifted Hamiltonian.

Next we turn to the quartic interactions, where we as before split up the Hamiltonian according to its field content. The pure bosonic part will naturally be identical to (7.34) so we will not present it again. For the pure fermionic part we find

$$
\begin{aligned}
g\,\mathcal{H}^{ns}_{FF} = {}& \kappa_{-a}\,\bar\kappa_{+b}\,\kappa^{+b}\,\bar\kappa^{-a} - \kappa_{-a}\,\bar\kappa_{+b}\,\kappa'^{+a}\,\bar\kappa'^{-b} - \kappa_{-a}\,\kappa'_{-b}\,\kappa^{+b}\,\kappa'^{+a} \qquad (22)\\
& - \kappa_{-a}\,\kappa'_{-b}\,\bar\kappa^{-b}\,\bar\kappa'^{-a} - \kappa_{-a}\,\bar\kappa'_{+b}\,\kappa^{+b}\,\bar\kappa'^{-a} - \kappa_{-a}\,\bar\kappa'_{+b}\,\bar\kappa^{-b}\,\kappa'^{+a} - \bar\kappa_{+a}\,\kappa'_{-b}\,\kappa^{+b}\,\bar\kappa'^{-a}\\
& - \bar\kappa_{+a}\,\bar\kappa'_{+b}\,\kappa^{+b}\,\kappa'^{+a} + \frac{1}{2}\big(\kappa_{-a}\,\kappa_{-b}\,\bar\kappa^{-a}\,\bar\kappa^{-b} - \kappa_{-a}\,\kappa_{-b}\,\bar\kappa'^{-a}\,\bar\kappa'^{-b} + \bar\kappa_{+a}\,\bar\kappa_{+b}\,\kappa^{+a}\,\kappa^{+b}\\
& - \bar\kappa_{+a}\,\bar\kappa_{+b}\,\kappa'^{+a}\,\kappa'^{+b}\big) - \frac{3}{2}\big(\kappa_{-a}\,\kappa_{-b}\,\kappa'^{+a}\,\kappa'^{+b} + \bar\kappa_{+a}\,\bar\kappa_{+b}\,\bar\kappa'^{-a}\,\bar\kappa'^{-b}\big) + 2\big(\kappa_{-a}\,\kappa'_{-b}\,\kappa^{+a}\,\kappa'^{+b}\\
& - \kappa_{-a}\,\bar\kappa_{+b}\,\kappa^{+a}\,\bar\kappa^{-b} + \kappa_{-a}\,\bar\kappa'_{+b}\,\bar\kappa^{-a}\,\kappa'^{+b} + \bar\kappa_{+a}\,\kappa'_{-b}\,\kappa^{+a}\,\bar\kappa'^{-b}\big) + 3\,\kappa_{-a}\,\bar\kappa_{+b}\,\kappa'^{+b}\,\bar\kappa'^{-a}\\
& + i\Big(\kappa_{-a}\,\bar\kappa_{+b}\,\bar\kappa^{-b}\,\bar\kappa'^{-a} + \kappa_{-a}\,\bar\kappa_{+b}\,\kappa^{+a}\,\kappa'^{+b} + \frac{1}{2}\big(\kappa_{-a}\,\bar\kappa_{+b}\,\kappa^{+b}\,\kappa'^{+a} + \kappa_{-a}\,\bar\kappa_{+b}\,\bar\kappa^{-a}\,\bar\kappa'^{-b}\big)\\
& - 3i\big(\kappa_{-a}\,\kappa_{-b}\,\kappa^{+a}\,\bar\kappa'^{-b} + \kappa_{-a}\,\bar\kappa'_{+b}\,\kappa^{+a}\,\kappa^{+b}\big) + i\frac{7}{2}\big(\bar\kappa_{+a}\,\bar\kappa_{+b}\,\kappa^{+a}\,\bar\kappa'^{-b} - \kappa_{-a}\,\kappa_{-b}\,\bar\kappa^{-a}\,\kappa'^{+b}\big)\Big)\\
& + \frac{i}{2}\epsilon^{\dot a\,\dot b}\,\epsilon_{a\,b}\big(s^a_{\dot a}\,s^c_{\dot b}\,s^b_{\dot d}\,(\bar s')^{\dot d}_c + \frac{1}{2}\big(s^c_{\dot a}\,s^a_{\dot b}\,(s')^b_{\dot d}\,\bar s^{\dot d}_c - s^a_{\dot d}\,s^b_{\dot a}\,(s')^c_{\dot b}\,\bar s^{\dot d}_c\big)\big) + \epsilon^{\dot a\,\dot b}\,\epsilon_{b\,c}\big(-i\,s^a_{\dot a}\,s^b_{\dot b}\big(\kappa_{-a}\,\bar\kappa'^{-c}\\
& + \bar\kappa'_{+a}\,\kappa^{+c} - \frac{i}{2}\,\bar\kappa'_{+a}\,\bar\kappa'^{-c}\big) + \frac{1}{2}s^b_{\dot a}\,(s')^a_{\dot b}\big(\bar\kappa_{+a}\,\bar\kappa'^{-c} + \bar\kappa'_{+a}\,\bar\kappa^{-c} - i\,\kappa_{-a}\,\bar\kappa^{-c} - i\,\bar\kappa_{+a}\,\kappa^{+c}\big)\\
& + \frac{1}{2}s^b_{\dot a}\,(s')^c_{\dot b}\big(i\,\bar\kappa_{+a}\,\kappa^{+a} - \bar\kappa'_{+a}\,\bar\kappa^{-a}\big) - \frac{1}{2}(s')^a_{\dot a}\,(s')^b_{\dot b}\,\bar\kappa_{+a}\,\bar\kappa^{-c}\big) + s^a_{\dot a}\,\bar s^{\dot a}_a\big(i\,\bar\kappa_{+b}\,\bar\kappa'^{-b} - i\,\kappa'_{-b}\,\kappa^{+b}\\
& + \bar\kappa_{+b}\,\kappa^{+b} - \frac{1}{2}\kappa'_{-b}\,\bar\kappa'^{-b} - \frac{5}{4}\kappa_{-b}\,\bar\kappa^{-b}\big)\\
& + s^a_{\dot a}\,\bar s^{\dot a}_b\big(i\,\kappa_{-a}\,\kappa'^{+b} - i\,\bar\kappa_{+a}\,\bar\kappa'^{-b} + \frac{1}{2}\kappa'_{-a}\,\bar\kappa'^{-b} + \frac{1}{2}\bar\kappa'_{+a}\,\kappa'^{+b}\\
& + \frac{1}{4}\kappa_{-a}\,\bar\kappa^{-b} + \frac{1}{4}\bar\kappa_{+a}\,\kappa^{+b}\big) + (s')^a_{\dot a}\,\bar s^{\dot a}_a\big(\kappa'_{-b}\,\bar\kappa^{-b} + \frac{i}{2}\bar\kappa_{+b}\,\bar\kappa^{-b} - \frac{1}{2}\kappa_{-b}\,\bar\kappa'^{-b}\\
& - \frac{1}{2}\bar\kappa_{+b}\,\kappa'^{+b} + \frac{1}{2}\bar\kappa'_{+b}\,\kappa^{+b}\big)\\
& - (s')^a_{\dot a}\,\bar s^{\dot a}_b\big(i\,\bar\kappa_{+a}\,\bar\kappa^{-b} + \frac{1}{2}\bar\kappa_{+a}\,\kappa'^{+b} + \frac{1}{2}\kappa'_{-a}\,\bar\kappa^{-b}\big) - \frac{1}{2}(s')^a_{\dot a}\,(\bar s')^{\dot a}_a\,\kappa_{-b}\,\bar\kappa^{-b}\\
& + \frac{1}{2}(s')^a_{\dot a}\,(\bar s')^{\dot a}_b\big(\kappa_{-a}\,\bar\kappa^{-b} + \bar\kappa_{+a}\,\kappa^{+b}\big) + h.c.
\end{aligned}
$$

The original mixing Hamiltonian is rather involved and is given by

$$
\begin{aligned}
&- g\,\mathcal{H}^{ns}_{BF} = \qquad (23)\\
&\frac{i}{2}y^2\, s^a_{\dot a}\,(s')^{\dot a}_a - y\, s^a_{\dot a}\,(\bar s')^{\dot a}_b\, Z^b_a - \frac{i}{4}y^2\,\bar\Psi\,\gamma^1\,\Psi' - p_y^2(2\,\bar\Psi\cdot\Psi + i\,\bar\Psi'\,\gamma^1\,\Psi) - \frac{1}{2}y'^2(\bar\Psi\cdot\Psi + \frac{i}{2}\bar\Psi'\,\gamma^1\,\Psi)\\
&- i\frac{3}{4}\,p_y\,\omega_{\dot a}\,\kappa_{-\,a}\,\bar s^{a\,\dot a} + 3(\frac{i}{2}\,y\,p_{\dot a}\,\kappa_{-\,a}\,\bar s^{a\,\dot a} - \frac{1}{16}\,y\,\omega_{\dot a}\,\kappa_{-\,a}\,\bar s^{a\,\dot a}) + \frac{i}{4}\,y\,\omega_{\dot a}\,\bar\kappa_{+\,a}\,(\bar s')^{a\,\dot a} - \frac{i}{2}\,y\,\omega_{\dot a}\,\bar\kappa'_{+\,a}\,\bar s^{a\,\dot a}\\
&+ \frac{i}{4}\,y\,\omega'_{\dot a}\,\kappa_{-\,a}\,(\bar s')^{a\,\dot a} - \frac{i}{4}\,y\,\omega'_{\dot a}\,\kappa'_{-\,a}\,\bar s^{a\,\dot a} + \frac{i}{4}\,y'\,\omega_{\dot a}\,\kappa'_{-,a}\,\bar s^{a\,\dot a} - \frac{i}{4}\,y'\,\omega_{\dot a}\,\kappa_{-\,a}\,(\bar s')^{a\,\dot a} - i\,\frac{3}{4}\,p_y\,\omega_{\dot a}\,\bar\kappa_{+\,a}\,s^{a\,\dot a}\\
&+ i\,\frac{3}{2}\,y\,p_{\dot a}\,\bar\kappa_{+\,a}\,s^{a\,\dot a} + i\,\frac{3}{16}\,y\,\omega_{\dot a}\,\bar\kappa_{+\,a}s^{a\,\dot a} - \frac{i}{2}\,y\,\omega_{\dot a}\,\kappa'_{-\,a}\,s^{a\,\dot a} + \frac{i}{4}\,y\,\omega_{\dot a}\,\kappa_{-\,a}\,(s')^{a\,\dot a} + \frac{1}{4}\,y\,\omega'_{\dot a}\,\bar\kappa'_{+\,a}\,s^{a\,\dot a}\\
&- \frac{1}{4}\,y\,\omega'_{\dot a}\,\bar\kappa_{+\,a}\,(s')^{a\,\dot a} - \frac{1}{4}\,y'\,\omega_{\dot a}\,\bar\kappa'_{+\,a}\,s^{a\,\dot a} + \frac{1}{4}\,y'\,\omega_{\dot a}\,\bar\kappa_{+\,a}\,(s')^{a\,\dot a} + \omega_{\dot a}(\bar\kappa'_{+\,a}\,s^{b\,\dot a} + \frac{1}{2}\bar\kappa_{+\,a}\,(s')^{b\,\dot a})\,Z^a_b\\
&- 2\,i\,\bar\Psi_a\,\gamma^0\,\Psi^b\,(P_z\cdot Z - \frac{1}{2}Tr(P_z\cdot Z)\not{\mathbb{1}})^a_b - \bar\Psi_a\cdot(\Psi')^b\,(Z\cdot Z' - \frac{1}{2}Tr(Z\cdot Z')\not{\mathbb{1}})^a_b\\
&- \frac{1}{2}(i\,\bar\Psi'\,\gamma^1\,\Psi + 2\,\bar\Psi\cdot\Psi)\,Tr(Pz\cdot Pz) - \frac{i}{8}\,\bar\Psi'\,\gamma^1\,\Psi\,Tr(Z\cdot Z) - \frac{i}{4}\,s^a_{\dot a}\,(s')^{\dot a}_a\,Tr(Z\cdot Z)\\
&- \frac{1}{8}(i\,\bar\Psi'\,\gamma^1\,\Psi + 2\,\bar\Psi\cdot\Psi)\,Tr(Z'\cdot Z') + \frac{1}{2}(s')^a_{\dot a}\,\bar s^{\dot a}_b(Z\cdot Z' - \frac{1}{2}Tr(Z\cdot Z')\not{\mathbb{1}})^b_a\\
&+ 2i\,s^a_{\dot a}\,\bar s^{\dot a}_b(P_z\cdot Z - \frac{1}{2}Tr(P_z\cdot Z)\not{\mathbb{1}})^b_a - (\kappa'_{-\,a}\,\bar s^{\dot a}_b + \frac{1}{2}\kappa_{-\,a}\,(\bar s')^{\dot a}_b\,)\omega_{\dot a}\,Z^{b\,a} + 2i\,x'^{-}\,\bar\Psi\,\gamma^0\,\Psi'\\
&- 4(i\,\bar\Psi'\,\gamma^1\,\Psi + 2\,\bar\Psi\cdot\Psi)\,p_{\dot a}\,\bar p^{\dot a} + 2i\,\bar\Psi\,\gamma^0\,\Psi\,p_{\dot a}\,\bar\omega^{\dot a} - \frac{1}{4}(i\,\bar\Psi'\,\gamma^1\,\Psi + 2\,\bar\Psi\cdot\Psi)\,\omega'_{\dot a}\,(\bar\omega')^{\dot a}\\
&- i\,\frac{9}{16}\,\bar\Psi\,\gamma^1\,\Psi'\,\omega_{\dot a}\,\bar\omega^{\dot a} + \frac{1}{2}(\bar\Psi'\cdot\Psi - \bar\Psi\cdot\,\Psi')\omega'_{\dot a}\,\bar\omega^{\dot a} + \frac{1}{4}(s^a_{\dot a}\,(\bar s')^{\dot b}_a - (s')^a_{\dot a}\,\bar s^{\dot b}_a)\,\omega_{\dot b}\,(\bar\omega')^{\dot a}\\
&+ \frac{1}{8}(s^a_{\dot a}\,(\bar s')^{\dot a}_a - s^a_{\dot a}\,\bar s^{\dot a}_a)\,\omega'_{\dot b}\,\bar\omega^{\dot b} + 2i\,s^a_{\dot a}\,s^{\dot b}_a\,\omega_{\dot b}\,\bar p^{\dot a} - i\,s^a_{\dot a}\,\bar s^{\dot a}_a\,\omega_{\dot b}\,\bar p^{\dot b} + \frac{i}{4}\,s^a_{\dot a}\,(s')^{\dot a}_a\,\omega_{\dot b}\,\bar\omega^{\dot b} + h.c.
\end{aligned}
$$

Note the slight asymmetry between the $\kappa^{\pm}$ fields. This is due to the fact that we have not considered the kinetic terms of the fermions, with witch one should augment the non shifted Hamiltonian.

4 Fermionic shift

The fermionic shift has to be implemented on the quadratic and cubic Hamiltonian in (7.19) and (21). In order to attain this one need the explicit form of the fermionic shift. Starting from (7.24), one can write

$$
\begin{aligned}
\frac{1}{g}\mathscr{L}^{\eta}_{Kin} &= \frac{1}{2}Str\,\dot\eta\Big\{[\eta, G_t\,\pi\,G_t^{-1}] + \frac{1}{4}([G_t\,\pi\,G_t^{-1}, \eta^3] + \eta\,[G_t\,\pi\,G_t^{-1}, \eta]\,\eta) \qquad (24)\\
&- i\,\kappa\,G_t\,\Upsilon\,G_t^{-1}\Big(\frac{i}{2}[\Sigma_-, \eta]x'^{-} + \eta' - \frac{1}{2}\eta\,\eta'\,\eta\Big)G_t\,\Upsilon^{-1}\,G_t^{-1}\\
&+ \frac{i}{2}\,\kappa\,\eta\,G_t\,\Upsilon\,G_t^{-1}\Big(\frac{i}{2}[\Sigma_-, \eta]x'^{-} + \eta' - \frac{1}{2}\eta\,\eta'\,\eta\Big)G_t\,\Upsilon^{-1}\,G_t^{-1}\,\eta\Big\} + \mathcal{O}(\eta^6).
\end{aligned}
$$

Where the leading order term is the quadratic kinetic term and the higher order terms are just the function $\widetilde{\Phi}(\eta)$ introduced earlier, which together with its self interaction terms constitute the fermionic shift.

Since we do a perturbative analysis up to quartic order, the presence of quadratic fermionic terms in the above expressions imply that we need $\pi_\pm$ to quadratic order only[3], from (7.12) we find

$$
\begin{aligned}
g\,\pi_- &= \frac{i}{4}\Big(p_y^2 + \frac{1}{2}Tr(P_z\cdot P_z) + 4\,p_{\dot a}\,\bar p^{\dot a} + \omega'_{\dot a}\,(\bar\omega')^{\dot a} + y'^2 + \frac{1}{2}Tr(Z'\cdot Z')\Big), \\
g\,\pi_+ &= \frac{i}{4} + \frac{i}{16}\Big(y^2 - \frac{1}{2}Tr(Z\cdot Z) + \frac{1}{4}\omega_{\dot a}\,\bar\omega^{\dot a}\Big) + \frac{1}{4}\Big(\bar\Psi'\,\gamma^1\,\Psi - \bar\Psi\,\gamma^1\,\Psi' - \frac{i}{2}\bar\Psi\cdot\Psi\Big).
\end{aligned}
\tag{25}
$$

Combining the solutions for $\pi_\pm$ and the transverse components of π in (7.11) one can solve for the fermionic shift (7.29) explicitly. As should be clear, the explicit form in components is quite complicated. Nevertheless, it is a straightforward task to obtain the shift for each coordinates by inverting the expressions (7.8).

To obtain the full shift that also removes the $Str\,\Phi_2\,\widetilde{\Phi}_2$ term, one need to isolate the $\dot\eta$ part and add this contribution to (7.29). The terms from $Str\,\Phi_2\,\widetilde{\Phi}_2$ without a $\dot\eta$ dependence will introduce corrections to π_t which one also need to determine explicitly. Having done all this, one can implement the full shift in the original Hamiltonian, together with the corrections to π, and determine the full mixing part of the shifted Hamiltonian. Needless to say, all this will be a rather involved procedure and is beyond the scope of this thesis.

[3]This is only true for the fermionic kinetic term. In the full Lagrangian π_- is needed to quartic order.

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Selbst"andigkeitserkl"arung

Hiermit erkläre ich, Per Sundin, dass ich diese Arbeit selbstständig verfasst und dabei auf keine anderen Hilfsmittel als jene im Text angegebenen zurueckgegriffen habe

Per Sundin

Berlin, den 02.03.2010 Per Sundin