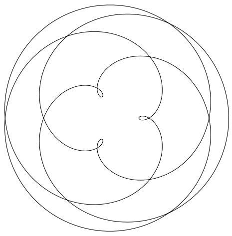

Quantization of Coset $SL(2, \mathbb{R})$ WZNW Models

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Zusammenfassung

Diese Arbeit gibt einen Überblick über die $SL(2,\mathbb{R})$-WZNW-Theorie und ihre Coset-Modelle. Ein neuer Apekt der Arbeit ist die Analyse des elliptische Sektors, der gebundend Zustände beschreibt. Es werden Ungleichzeitige Poissonklammern im elliptischen Sektor für die $SL(2,\mathbb{R})$-WZNW-Theorie und das $SL(2,\mathbb{R})/U(1)$-Modell berechnet. Die Ergebnisse verallgemeinern frühere Resultate für den hyperbolischen Sektor and deuten auf eine monodromieunabhängige Poissonklammer-Struktur hin. Der Vertexoperator des $SL(2,\mathbb{R})/U(1)$-Modells wird im hyperbolischen Sektor konstruiert mit Hilfe der Parametrisierung durch freie Felder und des Moyal-Formalismus. Der kausal Kommutator zweier Verteyoperatoren erhält die lokale Struktur des ungleichzeitigen Poissonklammer, mit einer konsistenten Quantendeformation. Die Reflektionsamplitude wird aus der Struktur des Vertexoperators in einlaufenden und auslaufenden Feldern hergeleitet. Das diskrete Spektrum des elliptischen Sektors wird durch die Nullstellen der Reflektionsamplitude auf der imaginären Achse des einlaufenden Impulses bestimmt.

Abstract

The thesis reviews the $SL(2,\mathbb{R})$ WZNW theory and its coset models. A new point of the review is the analysis of the elliptic sector, which describes bound states. Non-equal time Poisson brackets in the elliptic sector are calculated for the $SL(2,\mathbb{R})$ WZNW theory and the $SL(2,\mathbb{R})/U(1)$ black hole model. These calculations generalize the earlier obtaind results for the hyperbolic monodromy and indicate that the causal Poisson bracket structure is monodromy independent. The vertex operator of the $SL(2,\mathbb{R})/U(1)$ model is constructed in the hyperbolic sector using a free-field parameterization and the Moyal formalism. The causal commutator of the vertex operators preserves the local form of the non-equal time Poisson brackets with a consistent quantum deformation. The reflection amplitude is derived from the structure of the vertex operator in terms of incoming and outgoing fields. The discrete spectrum of the elliptic sector is found by the zeros of the reflection amplitude on the imaginary axis of the analytically continued incoming momentum.

Hilfsmittel

Diese Diplomarbeit wurde mit LaTeX 2_ε gesetzt.

Selbstständigkeitserklärkung

Hiermit erkläre ich, die vorliegende Diplomarbeit selbständig sowie ohne unerlaubte fremde Hilfe verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

Mit der Auslage meiner Diplomarbeit in den Bibliotheken der Humboldt-Universität zu Berlin bin ich einverstanden.

Berlin, den 30. April 2008

Nikolai Beck

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This diploma thesis is available at
`http://qft.physik.hu-berlin.de`.

Contents

I

Introduction

The standard model describes the dynamics of elementary particles under the influence of the fundamental forces, except gravity. The main tool of the standard model is quantum field theory [1]. It makes predictions for scattering processes, which have been verified experimentally to a high precision. However, the derivation of these predictions relies on a perturbative treatment of the interaction terms, which does not work for strong coupling regimes of the theory. The perturbative approach also fails for bound states, which is an important missing point, since their accurate treatment is essential for the understanding of the structure of matter. Furthermore, the inclusion of gravity into the standard model renders it unrenormalizable. Therefore, in spite of the great success with experimental data, the standard model is not treated as a fundamental theory, and one has to look for an alternative one.

String theory [2] seems to be one of the most promising candidates in this respect. Here, the elementary particles are no longer represented by point like objects but rather by one dimensional strings, thus removing some of the divergences encountered in field theory.

A non-perturbative treatment of physical phenomena usually requires existence of integrable structures. The investigation of such structure for $AdS_5 \times S_5$ string theory is one of the most actively discussed topics in theoretical physics today [3, 4]. Wess-Zumino-Novikov-Witten (WZNW) theory and its cosets play an important role for understanding of non-perturbative integrable structures both in quantum field and string theory.

WZNW theory [5, 6, 7] is a fascinating 2-dimensional interacting conformal field theory, which is completely integrable due to the additional Kac-Moody symmetry. The target space of WZNW theory is a semi-simple Lie group and the general solution is given as a product of chiral and antichiral fields. The invariance of the general solution under the left and right multiplication is just the Kac-Moody symmetry, which is a non-abelian analog of the Weyl symmetry for free-field theory.

The Poisson bracket algebra of the chiral fields is a basic result for any WZNW theory [8, 9, 10]. Piecing together the chiral and anti-chiral brackets one finds surprisingly simple causal (and local) non-equal time Poisson brackets for the $SL(2,\mathbb{R})$ WZNW field [11].

The chiral fields of the periodic $SL(2,\mathbb{R})$ WZNW theory are only quasi-periodic and they split into three different monodromy classes, called hyperbolic, parabolic and elliptic. The time dependent behavior of the WZNW-field essentially depends on the class of monodromy. The above mentioned chiral Poisson brackets are monodromy dependent as well. The calculation of the causal brackets in [11] was done for the hyperbolic monodromy only. One can expect that the causal Poisson bracket structure is monodromy independent, but to check this statement one needs a similar calculation in the other sectors.

The coset models, obtained from WZNW theory by a gauging procedure [12, 13], form an important class of integrable theories. An outstanding example is the

$SL(2,\mathbb{R})$ WZNW theory. Its cosets are Liouville theory [12, 14] and various black hole models [15, 16], with interesting target space geometries [17]. The different cosets are derived by gauging the $SL(2,\mathbb{R})$ WZNW model with respect to one parameter subgroups. In the Hamiltonian formulation the gauging procedure corresponds to constraints imposed on the Kac-Moody currents. There are three different types of subgroups related to the signature of the Killing form of the $sl(2,\mathbb{R})$ algebra. Liouville theory arises from the nilpotent (light-like) gauging, which leads to first class constraints, whereas the constraints for the black hole models are of the second class. Therefore, the target space of the black hole models is two dimensional. The different cosets are mutually related, and Liouville theory is not only the simplest and well investigated, but it is also fundamental for understanding the other cosets.

Liouville theory has wide applications in different areas of physics and mathematics such as non-critical strings [18], 2d gravity [19, 20], quantum groups [21], or the dynamics of branes [22, 23, 24]. Its complete integrability directly follows from the conformal symmetry. Among the remarkable results obtained in Liouville theory one has to mention the calculation of the Virasoro central charge [25], exchange algebra [21], the construction of the vertex operators [26, 27, 28], 3-point correlation function [29, 30, 31], correlation functions for boundary theory [22, 23, 32], a local form of the causal commutator [33, 34] and others. The coset interpretation gave a new insight to various aspects of Liouville theory. The thesis reviews only some of them, which are relevant for generalizations to other cosets. (for a general review of Liouville theory see [28].)

Reduction of the hyperbolic sector leads to regular Liouville fields [35], while the elliptic and parabolic sector corresponds to singular configurations. The hyperbolic sector has a smooth behavior at the time asymptotics, which allows to define the in and out free-fields. One can express the Liouville field through the in- (or out-)field and establish an explicit relation between the asymptotic fields. The free-field parameterization allows canonical quantization of Liouville theory in the Fock space related to the in-states. The operator ordering ambiguity for the Liouville field exponentials (vertex operators) can be fixed by the symmetries of the theory. Then, the relation between the asymptotic fields defines the S-matrix.

The elliptic sector comes into the game only for boundary Liouville theory [36, 21]. It describes bound states, which are analytically related to the scattering states of the hyperbolic sector. The parameter for the analytical continuation is the in-coming momentum, which in scattering sector is positive and in the elliptic sector becomes purely imaginary. Then the discrete spectrum of the elliptic sector can be obtained by the zeros of the reflection amplitude on the negative imaginary axis [32, 37, 38].

For the $SL(2,\mathbb{R})/U(1)$ model, unlike for Liouville theory, the $2d$ conformal symmetry is not sufficient for the complete integrability. Nevertheless, the general solution of the model, found in [39] in terms of chiral and antichiral fields, can be written in a form quite similar to Liouville theory. Namely, as a canonical map from free-fields to the interacting $SL(2,\mathbb{R})/U(1)$ field [40, 41]. The parameterizing free-field is now complex, and the $SL(2,\mathbb{R})/U(1)$ field can be seen as a complex version of the Liouville exponential. Then, one can apply a similar quantization scheme as in Liouville theory, based on the free-field parameterization.

The first quantum deformations, related to the parafermionic algebra of the coset currents, were calculated in [42]. The construction of the vertex operator for the $SL(2,\mathbb{R})/U(1)$-field was done in [43] and the reflection amplitude was calculated there as well (see also [44]).

The free-field parameterization of the $SL(2,\mathbb{R})/U(1)$ model used in [42, 43, 44] corresponds to the hyperbolic monodromy. The case of the elliptic and parabolic sectors is not discussed in the literature, though they describe regular field configurations. The parabolic monodromy can be treated as an intermediate between the hyperbolic and elliptic sector, and the corresponding field configurations can be reached by a limiting procedure from the hyperbolic or elliptic solutions. A special interest is to the elliptic monodromy since, similarly to the boundary Liouville theory, this sector can describe bound states.

It is the aim of this work to extend the analysis of the $SL(2,\mathbb{R})$ WZNW theory and its cosets from the hyperbolic to the elliptic sector and give a joint treatment of the monodromy classes. At the same time we intend to fill on some gaps in the constructions, providing a consisten description of the field.

Outline

In chapter 2 we introduce the language of symplectic geometry and show how it can be used to quantize canonically constrained classical (field) theories. Then we discuss symplectic and Poisson bracket structures on the space of motions and consider $2d$ massless free-field theory as an example to demonstrate the corresponding calculations. Free-field theory is also used to introduce infinite dimensional symmetry groups related to 2d conformal transformations and translations of the space of solutions.

Chapter 3 discusses the structure of the $SL(2,\mathbb{R})$ group, which is the target space of the $SL(2,\mathbb{R})$ WZNW theory. Furthemore, particle dynamics on this manifold and its cosets is studied. Here we introduce the left-right symmetries on the group manifold and the corresponding gaugings, which are generalized to WZNW theory in a field theoretical treatment later.

In chapter 4 we turn to the $SL(2,\mathbb{R})$ WZNW theory. We provide the general solution and describe its symmetries. We introduce the chiral symplectic form. By its inversion we define the basic chiral Poisson brackets. Then, we calculate the causal Poisson brackets for the full $SL(2,\mathbb{R})$ WZNW-field and, finally, describe gaugings of the Kac-Moody symmetries.

Chapter 5 is devoted to classical Liouville theory as it arises by Hamiltonian reduction from the $SL(2,\mathbb{R})$ WZNW model. We give a parameterization of the general solution in terms of free-fields and establish the Poisson bracket structure on the space of solutions. Then, the symmetries of Liouville theory are discussed and the origin of the improved term in the energy-momentum tensor is investigated.

In chapter 6 we study the $SL(2,\mathbb{R})/U(1)$ model. The general solution is derived by reduction of the space of solutions, both in the hyperbolic and elliptic sector. We relate the two sectors by an analytical continuation and show that the elliptic sector corresponds to bound states, while the hyperbolic one describes scattering processes. Causal Poisson brackets for both sectors are calculated with the help of Dirac brackets.

Chapter 7 reviews some aspects of quantum Liouville theory. Here, we construct the vertex operator in the Moyal formalism, based on the free-field parameterization and the symmetries of the theory. At the end of the chapter the reflection amplitude is calculated.

In chapter 8 the quantization of the hyperbolic sector of the $SL(2,\mathbb{R})/U(1)$ model is carried out in the Moyal formalism, similarly to Liouville theory. We construct the vertex operator and calculate the causal commutator in the fundamental domain. Furthermore, the scattering amplitude is calculated and by zeros of its analytical continuation the discrete spectrum in the bound sector is found.

The last chapter summarizes the results and gives an outlook.

The three appendices contain technical details.

II

Quantizing Classical Systems

In this chapter we introduce the language of symplectic geometry, which is a helpful tool to analyze constrained dynamical systems and find their canonical coordinates. These coordinates can then be used to quantize the reduced system canonically. We discuss symplectic and Poisson bracket structures on the space of motions. As an example we consider $2d$ massless free-field theory to demonstrate the corresponding calculations. At the end of the chapter we discuss the conformal and Weyl symmetries of free-field theory. These infinite dimensional symmetries play an important role in our further constructions.

2.1 Symplectic Geometry

We start our considerations with a Lagrange function $L(q, \dot{q})$ on the tangent bundle TM of an n dimensional configuration manifold M. The equations of motion are

$$\frac{\partial L}{\partial q^i} - \partial_t \frac{\partial L}{\partial \dot{q}^i} = 0\,. \tag{2.1}$$

If the Lagrangian is regular, i.e. $\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$, one can pass to the Hamiltonian formulation through a Legendre transformation. The Lagrangian is then replaced by the Hamilton function

$$H(p, q) = \sum_{i=1}^{n} p_i \dot{q}^i - L(q, \dot{q})\,, \quad \text{where } p_i \equiv \frac{\partial L}{\partial \dot{q}^i}\,, \tag{2.2}$$

which is now a function on the phase space T^*M parameterized by p and q, and the equations of motion are given by

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}\,. \tag{2.3}$$

An elegant way of expressing these equations of motion is by use of the Poisson brackets

$$\{f, g\} \equiv \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q^i} \right)\,, \tag{2.4}$$

with which (2.3) can be written as

$$\dot{q}^i = \{H, q^i\}, \qquad \dot{p}_i = \{H, p_i\}\,. \tag{2.5}$$

The formalism up to this point was developed with respect to specific phase space coordinates, the canonical coordinates. There exists, however, a way of deriving the same equations in coordinate independent notation. A point on T^*M

consists of a point q on M and a covector $p \in T_q^*M$. One can now construct a canonical 1-form on T^*M, given by

$$\theta = \sum_{i=1}^{n} p_i \, dq^i \, . \tag{2.6}$$

The outer derivative of this 1-form θ is called the canonical symplectic form $\omega \equiv d\theta$. In arbitrary local coordinates it can be written as an antisymmetric tensor, $\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu$, and in the canonical coordinate system from above it becomes

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \, . \tag{2.7}$$

One can use this 2-form to assign every function a vector field on T^*M by demanding that for a function f and its vector field X_f the equation

$$df + X_f \rfloor \omega = 0 \tag{2.8}$$

holds. The vector field is then

$$X_f^i = \omega^{ij} \partial_j f \, , \tag{2.9}$$

where ω with upper indices denotes the inverse matrix $\omega^{ij} \equiv (\omega^{-1})_{ij}$. One can now make an alternative definition of the Poisson bracket

$$\{f(x), g(x)\} \equiv \omega(X_f, X_g) = \omega^{ij} \partial_j f(x) \partial_i g(x) \, , \tag{2.10}$$

which for canonical coordinates p, q coincides with the previous definition (2.4). Note that the Poisson bracket is a derivation,

$$\{f(x), g(x)\} = X_f(g(x)). \tag{2.11}$$

This property can be used to extract the Poisson bracket of two functions directly from (2.8) without constructing the vector fields explicitly. The Poisson bracket is antisymmetric and obeys the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \, , \tag{2.12}$$

which is due to the fact that the symplectic form is closed, $d\omega = 0$.

The coordinate independent formulation of Hamiltonian mechanics allows a generalization of mechanics to a phase space manifold: Consider a manifold M equipped with a non-degenerate, closed 2-form ω, i.e. a symplectic manifold. Because of the non-degeneracy one can invert the 2-form and thus define a relation between functions and vector fields by (2.8) and a Poisson bracket as in (2.10). Furthermore, this Poisson bracket obeys the Jacobi identity (2.12), since ω is closed. Given a Hamiltonian H we can then determine the time evolution of an observable f through

$$\frac{df}{dt} = \{H, f\} + \partial_t f \, . \tag{2.13}$$

By Darboux's theorem one can always find a transformation to canonical variables, such that locally the symplectic form takes on the form of (2.7) and the equations of motion are given by (2.3). However, it is, in general, not possible to find a global transformation to such coordinates.

2.2 Hamiltonian Reduction

The reason we discuss symplectic geometry here is the fact that it provides a convenient way to describe a constrained system. Constrained systems arise when the Lagrangian is not regular and therefore the velocities cannot be written as a function of independent momenta. This is actually the case for many physically relevant problems like electrodynamics. In the following we will present the approach to this problem proposed by Faddeev and Jackiw [45, 46].

The first point to note is that any Lagrangian $\tilde{L}(q, \dot{q})$ can be written in linear form as

$$L(p, q, v) = p_i(\dot{q}^i - v^i) + \tilde{L}(q, v)\,. \tag{2.14}$$

Variation with respect to p_i implies $v^i = \dot{q}^i$, and substituting this into the new Lagrangian L yields the original Lagrangian $\tilde{L}$. On the other hand the variation with respect to v^i implies

$$p_i = \frac{\partial L(q, v)}{\partial v^i}\,, \tag{2.15}$$

and with the usual definition of the Hamiltonian $H(p, q, v) \equiv p_i v^i - L(q, v)$ the Lagrangian can be written as

$$L(p, q, v) = p_i \dot{q}^i - H(p, q, v)\,. \tag{2.16}$$

If the Lagrangian is regular we can express the velocities v in terms of the momenta p and thus arrive at the usual Hamiltonian formulation. If, however, the Lagrangian is irregular then there arise constraints. These cases will be discussed in the following.

Let us consider an arbitrary linear Lagrangian on an N dimensional phase space with coordinates ξ^i

$$L(\xi) = a_i(\xi)\dot{\xi}^i - H(\xi) \tag{2.17}$$

The equations of motion are then

$$f_{ij}(\xi)\dot{\xi}^j = \frac{\partial H(\xi)}{\partial \xi^i} \tag{2.18}$$

where f is the fieldstrength to the vector potential $a_i(\xi)d\xi^i$:

$$f_{ij}(\xi) \equiv \partial_i a_j(\xi) - \partial_j a_i(\xi)\,. \tag{2.19}$$

One now has to distinguish two cases:

Suppose first the matrix $f_{ij}(\xi)$ is invertible, i.e. $f^{ki} f_{ij} = \delta^k_j$, which is only possible if the phase space is even dimensional, hence $N = 2n$. Then the equations of motions simply become

$$\dot{\xi}^j = f^{ji}(\xi)\frac{\partial H(\xi)}{\partial \xi^i}\,. \tag{2.20}$$

Here, there are no constraints and the equations of motion can also be written in the Poisson formalism as

$$\dot{\xi}^j = \{H(\xi), \xi^j\}\,, \tag{2.21}$$

if we define the basic Poisson bracket as

$$\{\xi^i, \xi^j\} = -f^{ij}(\xi)\,. \tag{2.22}$$

This Poisson bracket obeys the Jacobi identity due to the Bianchi identity for f_{ij}. For two functions g, h the Poisson bracket is then

$$\{g(\xi), h(\xi)\} = f^{ij}\partial_j g(\xi)\partial_i h(\xi) \tag{2.23}$$

and comparison with (2.10) shows that f^{ij} corresponds to ω^{ij}. Therefore the 1-form $a_i(\xi)d\xi^i$ in (2.17) can be identified with the 1-form θ. This can be used to extract the symplectic form of a Lagrangian without constructing the canonical momenta explicitly. Afterwards one can then apply Darboux's theorem to find local canonical coordinates.

Let us now consider the case that $f_{ij}(\xi)$ is not invertible and therefore there exist m zero modes, $f_{ij}z^j_\alpha = 0$ with $\alpha = 1, \ldots, m$. From (2.20) we then get the m constraints

$$0 = z^i_\alpha \frac{\partial H(\xi)}{\partial \xi^i}\,. \tag{2.24}$$

By Darboux's theorem one can locally find coordinates such that the zero modes are separated from the other $N - m = 2n$ variables and these remaining variables are canonical ones. The Lagrangian thus becomes

$$L = \frac{1}{2}\omega_{ij}\xi^i\dot{\xi}^j - H(\xi, z)\,, \tag{2.25}$$

where i and j now range from 0 to $2n$, and the m constraints are

$$\frac{\partial}{\partial z_\alpha} H(\xi, z) = 0\,. \tag{2.26}$$

If $H(\xi, z)$ is non-linear in some z_α then this results in an algebraic equation for z_α. The solution of this in terms of the other variables z and ξ can then be inserted back into H to find the reduced Hamiltonian. The linearly occurring z_α's however give constraints on the other variables z and ξ. Some of these can be solved for another z_β, which can thus be eliminated from the Hamiltonian. The remaining linearly occurring z's can be interpreted as Lagrange multipliers λ

$$L = \frac{1}{2}\omega_{ij}\xi^i\dot{\xi}^j - H(\xi) - \lambda_k\Phi^k(\xi)\,. \tag{2.27}$$

To incorporate the constraints $\Phi^k = 0$ we construct a parameterization of the constrained surface. Inserting this parameterization into the Lagrangian of course results in a new Hamiltonian and a new 2-form, which can possibly again contain zero modes. One then has repeat the procedure stated above.

The method can be summarized in the the following steps:

- Identify the zero modes of the 2-form f_{ij}.
- Find new coordinates that separate zero modes z_α from other variables.
- Solve the new constraints $\frac{\partial H}{\partial z_\alpha} = 0$ for as many z_α as possible.
- Construct a parameterization that satisfies the remaining constraints and substitute it back into L.

These steps have to be repeated until one ends up with a non-singular 2-form. For the infinite dimensional case encountered in field theory it is not certain, that this will happen after a finite number of steps, but for many cases it does. Specifically for purely first class constraints, which have vanishing Poisson brackets with each other and with the Hamiltonian, or for second class constraints, where the matrix $\{\Phi_i, \Phi_j\}$ is invertible, the procedure terminates after one step. Of course, it is possible that one cannot find a parameterization of the constrained surface. Or it may not be possible to find global canonical coordinates.

2.3 The Symplectic Form on the Space of Motions

Let us consider a Hamiltonian system with the canonical symplectic form (2.7) and assume that the dynamical equations (2.5) are completely integrable in the form

$$q^i = q^i(t, x^\mu)\,, \quad p_i = p_i(t, x^\mu)\,, \quad \mu = 1, \dots, 2n\,, \tag{2.28}$$

The parameters x^μ usually are dynamical integrals of the system. Equation (2.28) defines the space of motions parameterized by x^μ. Since the time evolution is a canonical transformation the symplectic form (2.7) is time independent. Therefore, inserting (2.28) into (2.7) we find

$$\tilde{\omega} = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu\,, \quad \text{where} \quad \omega_{\mu\nu}(x) = \sum_{i=1}^{n} \left(\frac{\partial p_i}{\partial x^\mu}\frac{\partial q^i}{\partial x^\nu} - \frac{\partial p_i}{\partial x^\nu}\frac{\partial q^i}{\partial x^\mu} \right)\,. \tag{2.29}$$

In this way we get a symplectic form on the space of solutions.

Due to Darboux's theorem one can pass from x^μ to some canonical coordinates $x^\mu \to (\tilde{p}_i, \tilde{q}^i)$. Equation (2.28) then becomes

$$q^i = q^i(t, \tilde{q}, \tilde{p})\,, \quad p_i = p_i(t, \tilde{q}, \tilde{p})\,, \tag{2.30}$$

and one can apply canonical quantization in the new coordinates $(\tilde{p}, \tilde{q})$. The quantum version of (2.30) has to provide the Heisenberg operators $\hat{p}_i(t), \hat{q}^i(t)$. But, in general, if the transformation $\tilde{p}, \tilde{q} \to p, q$ is non-linear we have an ordering problem and therefore an ambiguity in the definition of these Heisenberg operators. In order to fix this ambiguity one can use equations of motion or symmetry properties of the corresponding operators.

If one imposes constraints on the space of motions it will reduce the symplectic form (2.29) in way discussed in the previous section.

2.4 Free-Field Theory

2.4.1 Free-Field Theory on a Strip

As an instructive example, and also because we will refer back to the formulas throughout this work, we now carry out the quantization of a free scalar field on a Minkowskian strip (τ, σ) given on $\mathbb{R} \times [0, \pi]$. The Lagrangian of a free field is

$$\mathcal{L} = \frac{1}{2}\left(\dot{\Phi}^2(\tau, \sigma) - \Phi'^2(\tau, \sigma)\right)\,, \tag{2.31}$$

where the dot denotes differentiation with respect to τ and the prime denotes differentiation with respect to σ. Here we will consider the boundary conditions $\Phi'(\tau,0) = \Phi'(\tau,\pi) = 0$. In the light-cone (chiral) coordinates

$$\begin{aligned} x &= \tau + \sigma\,, \qquad & \partial_x &= \frac{1}{2}(\partial_\tau + \partial_\sigma)\,, \\ \bar{x} &= \tau - \sigma\,, \qquad & \partial_{\bar{x}} &= \frac{1}{2}(\partial_\tau - \partial_\sigma)\,, \end{aligned} \tag{2.32}$$

the equation of motion takes the form

$$\partial_x \partial_{\bar{x}} \Phi(x,\bar{x}) = 0\,. \tag{2.33}$$

The general solution of this equation is given by the sum of an arbitrary chiral and an antichiral function. The boundary condition at $\sigma = 0$ implies that the two functions can differ only by a constant, which without loss of generality can be chosen to be zero. Thus, the solution is

$$\Phi(x,\bar{x}) = \phi(x) + \phi(\bar{x})\,. \tag{2.34}$$

From the boundary condition at $\sigma = \pi$ follows that the function ϕ' must be periodic, i.e. $\phi'(x+2\pi) = \phi'(x)$, and it can therefore be written as a discrete Fourier series. Integration then leads to

$$\phi(x) = q + \frac{p}{4\pi}x + \frac{i}{\sqrt{4\pi}} \sum_{n\neq 0} \frac{a_n}{n} e^{-inx}\,, \tag{2.35}$$

where $a_n^* = a_{-n}$, so that ϕ is real. With this we have constructed a parameterization of the space of motions of the system.

We will now determine the canonical 2-form on the space of motions. The canonical symplectic form and the Hamiltonian of the system (2.31) are

$$\omega = \int_0^\pi d\sigma\, d\Pi(\tau,\sigma) \wedge d\Phi(\tau,\sigma)\,, \tag{2.36}$$

$$H = \frac{1}{2}\int_0^\pi d\sigma \left(\dot{\Phi}^2(\tau,\sigma) + \Phi'(\tau,\sigma)\right)\,, \tag{2.37}$$

where the canonical momentum $\Pi(\tau,\sigma)$ of the theory is given by

$$\Pi(\tau,\sigma) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \dot{\Phi}(\tau,\sigma)\,. \tag{2.38}$$

With the help of (2.34) we then obtain the induced symplectic from on the space of motions

$$\tilde{\omega} = \frac{1}{2} dp \wedge d\phi(0) + \int_0^{2\pi} dx\, d\phi'(x) \wedge d\phi(x)\,, \tag{2.39}$$

where p is the momentum zero mode, which is a monodromy parameter for $\phi(x)$, i.e.

$$\phi(\sigma + 2\pi) = \frac{p}{2} + \phi(\sigma)\,. \tag{2.40}$$

In a further step one can insert the Fourier decomposition (2.35), which results in

$$\tilde{\omega} = dp \wedge dq + \frac{1}{2}\sum_{n\neq 0} \frac{i}{n} da_{-n} \wedge da_n\,, \tag{2.41}$$

$$H = \frac{1}{8\pi}p^2 + \sum_{n>0} |a_n|^2\,. \tag{2.42}$$

It is obvious that p, q and a_n are canonical coordinates, and indeed (2.8) yields the Poisson brackets

$$\{p, q\} = 1, \qquad \{a_m, a_n\} = in\delta_{n+m}, \qquad \{p, a_n\} = \{q, a_n\} = 0\,. \tag{2.43}$$

We can now also calculate the Poisson brackets of the field itself

$$\{\phi'(x), \phi(y)\} = \frac{1}{2}\delta(x-y), \qquad \{\phi(x), \phi(y)\} = \frac{1}{4}\epsilon(x-y)\,. \tag{2.44}$$

where δ is the periodic δ-distribution and ϵ is the stair-step function: $\epsilon(x) = 1$ for $0 < x < 2\pi$ and $\epsilon(x + k2\pi) = 2k + \epsilon(x)$ for $k \in \mathbb{Z}$.

The quantization of this system is straightforward as the solution is linear in the canonical coordinates. In order to have a ground state with zero energy the operator $\hat{H}$ is chosen to be the normal ordered operator

$$\hat{H} = \frac{1}{8\pi}\hat{p}^2 + \sum_{n>0} \hat{a}_n^\dagger \hat{a}_n\,. \tag{2.45}$$

With these operators it is possible to determine the time evolution of all observables and their expectation values.

2.4.2 Free-Field Theory on a Cylinder

We will later study the $SL(2,\mathbb{R})$ WZNW model on a Minkowskian cylinder. It is therefore useful to analyze the free-field also on the cylinder. The Lagrangian and the equations of motion in light-cone coordinates are here also given by (2.31) and (2.33). The general solution is therefore the sum of a chiral and an antichiral function

$$\Phi(x, \bar{x}) = \phi(x) + \bar{\phi}(\bar{x})\,. \tag{2.46}$$

The periodicity condition $\Phi(t, \sigma + 2\pi) = \Phi(\tau, \sigma)$ implies that the derivatives of these functions are periodic and by integration of their Fourier decomposition we find

$$\phi(x) = \frac{q}{2} + \frac{p}{4\pi}x + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{a_n}{n}e^{-inx}\,, \quad \bar{\phi}(\bar{x}) = \frac{q}{2} + \frac{p}{4\pi}\bar{x} + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{\bar{a}_n}{n}e^{-in\bar{x}}\,. \tag{2.47}$$

Note that in contrast to the strip here the chiral and antichiral function have absolutely independent non-zero modes $a_n, \bar{a}_n$. The Hamiltonian of this system in terms of the Fourier modes is

$$H = \frac{1}{2}\int_0^{2\pi} d\sigma\, \left(\dot{\Phi}^2(\tau,\sigma) + \Phi'^2(\tau,\sigma)\right) = \frac{p^2}{4\pi} + \sum_{n\neq 0}\left(|a_n|^2 + |\bar{a}_n|^2\right)\,. \tag{2.48}$$

The symplectic form on the space of motions can be obtained by inserting the momentum $\Pi(x, \bar{x}) = \dot{\Phi}(x, \bar{x})$ and the solution (2.46) into the canonical symplectic form, which yields

$$\begin{aligned}\tilde{\omega} &= \int_0^{2\pi} d\sigma\, d\Pi(\tau,\sigma) \wedge d\Phi(\tau,\sigma) \\ &= \int_0^{2\pi} d\sigma\, \left(d\phi'(x) + d\bar{\phi}'(\bar{x})\right) \wedge \left(d\phi(x) + d\bar{\phi}(\bar{x})\right)\,.\end{aligned} \tag{2.49}$$

After integration by parts using the monodromy behavior of the functions, we find

$$\begin{aligned}\tilde{\omega} =& \frac{1}{2} dp \wedge d\phi(0) + \int_0^{2\pi} dx\, d\phi'(x) \wedge d\phi(x) \\ &+ \frac{1}{2} dp \wedge d\bar{\phi}(0) + \int_0^{2\pi} d\bar{x}\, d\bar{\phi}'(\bar{x}) \wedge d\bar{\phi}(\bar{x})\,.\end{aligned} \tag{2.50}$$

Here we have used the τ independence of the first and second line by itself to change the integration variables while keeping the same limits. Inserting the Fourier series one obtains

$$\tilde{\omega} = dp \wedge dq + \frac{1}{2} \sum_{n\neq 0} \frac{i}{n} da_{-n} \wedge da_n + \frac{1}{2} \sum_{n\neq 0} \frac{i}{n} d\bar{a}_{-n} \wedge d\bar{a}_n\,. \tag{2.51}$$

Application of (2.8) yields as the only non-vanishing Poisson brackets

$$\{p, q\} = 1\,, \qquad \{a_m, a_n\} = im\delta_{n+m}\,, \qquad \{\bar{a}_m, \bar{a}_n\} = im\delta_{n+m}\,. \tag{2.52}$$

From these relations one then finds the basic Poisson brackets

$$\{\phi'(x), \phi(y)\} = \{\bar{\phi}'(x), \bar{\phi}(y)\} = \frac{1}{2}\delta(x-y) - \frac{1}{8\pi}\,, \quad \{\phi(x), \bar{\phi}(y)\} = \frac{1}{8\pi}(x-y)\,. \tag{2.53}$$

We have thus inverted the symplectic form for the free-field theory on the cylinder. It would, however, be easier for calculations if one could manage to have commuting chiral and antichiral parts. This can be achieved with the so-called Veneziano-Fubini trick [47]: We expand the phase space of zero modes and introduce a new symplectic form $\omega = \omega_L + \omega_R$ with

$$\begin{aligned}\omega_L &= \frac{1}{2} dp_L \wedge d\phi(0) + \int_0^{2\pi} dx\, d\phi'(x) \wedge d\phi(x)\,, \\ \omega_R &= \frac{1}{2} dp_R \wedge d\bar{\phi}(0) + \int_0^{2\pi} d\bar{x}\, d\bar{\phi}'(\bar{x}) \wedge d\bar{\phi}(\bar{x})\end{aligned} \tag{2.54}$$

and a new Hamiltonian $H = H_L + H_R$ with

$$H_L = \frac{p_L^2}{8\pi} + \sum_{n\neq 0} |a_n|^2\,, \qquad H_L = \frac{p_R^2}{8\pi} + \sum_{n\neq 0} |\bar{a}_n|^2\,. \tag{2.55}$$

The Poisson brackets are then (2.43) for each field. Brackets between chiral and antichiral objects vanish. The important point is now that on the physical subspace $p_L = p_R$ the Poisson brackets of functions that only depend on the sum $q = q_L + q_R$, but not their difference, are identical to their Poisson brackets in the original space because

$$\{p_L, (q_L + q_R)\} = \{p_R, (q_L + q_R)\} = \{p, q\} = 1\,. \tag{2.56}$$

Note that the Hamiltonian (2.55) is gauge invariant, and on the reduced space it is equal to the original Hamiltonian (2.48). The dynamics of the new system are therefore equivalent to those of the original one. The problem has thus been reduced to two separate free-fields. This can be understood in a more general context as a reduction by a first class constraint (see Appendix A).

2.4.3 Symmetries

The free-field theory has a conformal symmetry, since the transformation

$$\phi(x) \to \phi(\xi(x)), \qquad \bar{\phi}(\bar{x}) \to \bar{\phi}(\bar{\xi}(\bar{x})) \tag{2.57}$$

with $\xi'(x) > 0$ and $\xi(x+2\pi) = \xi(x)$ leaves the action constructed from (2.31) and the space of motions (2.46) invariant. By Noether's theorem one obtains the corresponding conserved quantity, the energy-momentum tensor, which has vanishing diagonal components and the off-diagonal elements

$$T(x) \equiv T^{\bar{x}}{}_{x}(x) = \phi'^2(x), \qquad \overline{T}(\bar{x}) \equiv T^{x}{}_{\bar{x}}(\bar{x}) = \bar{\phi}'^2(\bar{x}). \tag{2.58}$$

This tensor in turn generates conformal transformations, since

$$\{T(x), \phi(y)\} = \phi'(x)\delta(x-y) \tag{2.59}$$

is the infinitesimal version of the transformation (2.57), $\phi(x) \to \phi(x) + \epsilon(x)\phi'(x)$ with

$$\xi(x) = e^{\epsilon(x)\partial_x} x = x + \epsilon(x) + \mathcal{O}(\epsilon^2). \tag{2.60}$$

Furthermore the translation on the space of motions, $\phi(x) \to \phi(x) + h(x)$, is also a symmetry (in the sense of a symplectomorphism), which is generated by $\phi'(x)$ since

$$e^{\int_0^{2\pi} dz 2h(z)\{\phi'(z),\,\cdot\,\}}\phi(x) = \phi(x) + h(x). \tag{2.61}$$

These generators of the group of symmetry transformations form a Lie algebra

$$\{\phi'(x), \phi'(y)\} = -\frac{1}{2}\delta'(x-y), \tag{2.62a}$$

$$\{T(x), \phi'(y)\} = \phi''(y)\delta(x-y) - \phi'(y)\delta'(x-y), \tag{2.62b}$$

$$\{T(x), T(y)\} = T'(y)\delta(x-y) - 2T(y)\delta'(x-y). \tag{2.62c}$$

For the quantum objects one can show that this algebra is modified by a central extension

$$[\hat{T}(x), \hat{T}(y)] = \hat{T}'(y)\delta(x-y) - 2\hat{T}(y)\delta'(x-y) + \frac{\hbar}{24\pi}(\delta'(x-y) + \delta'''(x-y)), \tag{2.63}$$

with the normal ordered operator $\hat{T}(x) \equiv\, :\hat{\phi}'^2(x):$. The central extension of the symmetry algebra on the quantum level is a general effect, which has to be taken into account for a consistent quantization.

III

Target Space Structure

The $SL(2,\mathbb{R})$ WZNW theory describes a field that takes values in the Lie group $SL(2,\mathbb{R})$. It is therefore useful to discuss the geometric structure of this target space. Furthermore, we study the dynamics of a free particle on the $SL(2,\mathbb{R})$ group manifold and its cosets. These mechanical systems play the role of toy models for our field theoretical constructions later.

3.1 The $SL(2,\mathbb{R})$ Group

The $SL(2,\mathbb{R})$ group is the set of all real 2×2 matrices with unit determinant, where the group multiplication is the standard matrix multiplication. An element $g \in SL(2,\mathbb{R})$ and its inverse g^{-1} can be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{3.1}$$

with the restriction

$$ad - bc = 1\,. \tag{3.2}$$

The group can be understood as the level set in the four dimensional vector space of real 2×2 matrices given by $ad - bc = 1$. Because the gradient of this function is never zero on the set, it is a regular surface and can be described as a manifold. Condition (3.2) also guarantees that the multiplication is a diffeomorphism. Thus the group is a Lie group.

The unit 2×2 matrix I together with the traceless matrices

$$T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.3}$$

forms a basis of $\mathbb{R}^{2\times 2}$, and an element $g \in SL(2,\mathbb{R})$ can be expanded in this basis

$$g = cI + u^n T_n \tag{3.4}$$

with real coefficients c and u^n. In matrix form this reads

$$g = \begin{pmatrix} c + u^2 & u^1 - u^0 \\ u^1 + u^0 & c - u^2 \end{pmatrix}, \tag{3.5}$$

and condition (3.2) becomes

$$c^2 + \left(u^0\right)^2 - \left(u^1\right)^2 - \left(u^2\right)^2 = 1\,. \tag{3.6}$$

If we consider separately the planes (u^1, u^2) and (c, u^0), then choosing a point on the (u^1, u^2) plane fixes the radius on the (c, u^0) plane by $c^2 + (u^0)^2 = 1 + (u^1)^2 + (u^2)^2$. The only freedom remains in the polar angle and hence the $SL(2,\mathbb{R})$ is homeomorphic to $\mathbb{R}^2 \times S^1$. With this idea in mind we can also choose a different parameterization

$$\begin{aligned} c &= R\cos(\alpha)\,, & u_0 &= R\sin(\alpha)\,, \\ u_1 &= r\cos(\beta)\,, & u_2 &= r\sin(\beta)\,, \end{aligned} \tag{3.7}$$

with $r \in \mathbb{R}_+$, $R = \sqrt{r^2+1}$ and $\alpha, \beta \in [0, 2\pi]$.

3.2 The $sl(2,\mathbb{R})$ Algebra

The left invariant vector fields on a Lie group together with the Lie bracket form its Lie algebra. It is isomorphic to the tangent space at the unit element, which is called the space of generators. For matrix groups the algebra multiplication is the matrix commutator.

If one chooses u^n in (3.4) as local coordinates on $SL(2,\mathbb{R})$, then c is given by

$$c = \pm\sqrt{1-(u^0)^2+(u^1)^2+(u^2)^2}\,, \tag{3.8}$$

and the tangent vector space at the unit element $g = I$, which corresponds to the coordinates $(u^0, u^1, u^2) = (0,0,0)$, is spanned by the partial derivatives with respect to these coordinates, and by (3.5) $\frac{\partial g}{\partial u^n} = T_n$. Thus, the generator space is the set of all traceless 2×2 matrices, which is also clear from the relation

$$\det(e^A) = e^{tr(A)}\,. \tag{3.9}$$

The T_n satisfy the relation

$$T_n T_m = -\eta_{nm} I + \epsilon^l{}_{nm} T_l\,, \tag{3.10}$$

where $\eta_{nm} = \mathrm{diag}(+,-,-)$ is the metric tensor of the three dimensional Minkowski space and ϵ_{ijk} is the Levi-Civita tensor. Indices are raised and lowered by η. Using relation (3.10) it is easy to calculate the commutator

$$[T_n, T_m] = 2\epsilon^l{}_{nm} T_l\,, \tag{3.11}$$

from which we read of the structure constants of the $sl(2,\mathbb{R})$ algebra $f^i{}_{jk} = 2\epsilon^i{}_{jk}$.

3.3 Metric Structure

The adjoint representation of a Lie algebra $\mathcal{G}$ is defined by

$$\mathrm{ad}_A(B) \equiv [A, B]\,, \tag{3.12}$$

for $A, B \in \mathcal{G}$. The Jacobi identity of the Lie bracket ensures that

$$\mathrm{ad}_A(\mathrm{ad}_B) - \mathrm{ad}_B(\mathrm{ad}_A) = \mathrm{ad}_{[A,B]} \tag{3.13}$$

and the map $A \mapsto \mathrm{ad}_A$ is indeed a representation. In a fixed basis the operators can be written as matrices, which are related to structure constants via

$$(ad_i)^n_m = f^n{}_{im}\,. \tag{3.14}$$

The Killing form of two vectors is defined by

$$\mathcal{K}(T_i, T_j) = \mathrm{tr}(\mathrm{ad}_i\, \mathrm{ad}_j)\,, \tag{3.15}$$

and the Lie algebra is called semi-simple if the Killing form is non-degenerate. In case of the $sl(2,\mathbb{R})$ the structure constants are given by (3.11) and we have $\mathcal{K}(T_i, T_j) = 4\epsilon^n{}_{ik}\epsilon^k{}_{jn}$. Using the identity $\epsilon_{ij}{}^k\epsilon_{kmn} = \eta_{im}\eta_{jn} - \eta_{in}\eta_{jm}$ this can be evaluated to

$$\mathcal{K}(T_i, T_j) = -8\eta_{ij}\ . \tag{3.16}$$

Due to (3.10) we have $\mathrm{tr}(T_iT_j) = -2\eta_{ij}$, which is proportional to the Killing form. For the sake of simplicity we define a normalized trace

$$\langle\cdot\rangle \equiv -\frac{1}{2}\mathrm{tr}(\cdot)\,, \tag{3.17}$$

which realizes the scalar product in the $sl(2,\mathbb{R})$, i.e. $\langle T_iT_j\rangle = \eta_{ij}$. We can now identify T_0 as a timelike vector, T_1 and T_2 as spacelike vectors, and the nilpotent linear combinations

$$T_\pm \equiv T_0 \pm T_1 \tag{3.18}$$

as lightlike vectors.

Since the square of any element of the $sl(2,\mathbb{R})$ algebra is proportional to the unit matrix the exponent of an arbitrary generator $A \in sl(2,\mathbb{R})$ can be written as

$$e^A = \cosh(\sqrt{-\langle AA\rangle})I + \sinh(-\sqrt{\langle AA\rangle})\frac{A}{\sqrt{-\langle AA\rangle}} \qquad \text{for } \langle AA\rangle < 0 \tag{3.19}$$

$$e^A = \cos(\sqrt{\langle AA\rangle})I + \sin(\sqrt{\langle AA\rangle})\frac{A}{\sqrt{\langle AA\rangle}} \qquad \text{for } \langle AA\rangle > 0 \tag{3.20}$$

$$e^A = I + A \qquad \text{for } \langle AA\rangle = 0\,. \tag{3.21}$$

Thus, the $SL(2,\mathbb{R})$ group has a compact subgroup generated by timelike vectors, while the spacelike vectors generate non-compact subgroups. Explicitly, for T_2, T_0 and T_+ we find

$$e^{\theta T_2} = \begin{pmatrix} e^\theta & 0 \\ 0 & e^{-\theta}\end{pmatrix}\,, \qquad e^{\theta T_0} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\,, \qquad e^{\theta T_+} = \begin{pmatrix} 1 & 0 \\ 2\theta & 1\end{pmatrix}\,. \tag{3.22}$$

Because of the invariance of the trace with respect to cyclic permutation the pullback of the scalar product (3.17) under the multiplication from the left is identical to that under the multiplication from the right. We can therefore unambiguously define a metric G on $SL(2,\mathbb{R})$ through

$$G(X,Y) \equiv \langle g^{-1}X(g)g^{-1}Y(g)\rangle\,. \tag{3.23}$$

In coordinates (3.4) this metric becomes

$$G = \left(\eta_{ij} + \frac{1}{c^2}u_iu_j\right) du^i \otimes du^j\,, \tag{3.24}$$

which is identical to the metric of $\mathbb{H}_3$ of radius $R = 1$ in the standard coordinates which arises from the embedding in $\mathbb{R}^{(2,2)}$. Thus, the coordinates u^n realize an isomorphism between $SL(2,\mathbb{R})$ and $\mathbb{H}_3$.

3.4 Lorentz Transformations

For every group element a in a Lie group G one can define a group automorphism

$$\Psi_a(b) \equiv aba^{-1} \tag{3.25}$$

which maps the unit element I onto itself. The differential of this automorphism at the unit element is a map from the generator algebra to itself, which is denoted by Ad_a,

$$\mathrm{Ad}_a : \mathcal{G} \to \mathcal{G} \,, \qquad \mathrm{Ad}_a(A) \equiv D\Psi_a(A) \,. \tag{3.26}$$

These operators on the Lie algebra realize a representation of the group, called the adjoint representation.

For matrix groups like the $SL(2,\mathbb{R})$ the map Ψ_g is already linear and the adjoint representation is given by

$$Ad_g(A) = gAg^{-1} \,. \tag{3.27}$$

Due to the invariance of the trace (3.17) with respect to cyclic permutations the scalar product is invariant under this map

$$\langle \mathrm{Ad}_g(A)\mathrm{Ad}_g(B)\rangle = \langle AB\rangle \,, \tag{3.28}$$

and we can identify the map Ad_g with the Lorentz transformations in the $sl(2,\mathbb{R})$. The components of a vector $A \in sl(2,\mathbb{R})$, given by $A^n = \langle T^n A\rangle$, transform as

$$A^i \to \Lambda^n{}_m A^m \,, \qquad \text{where } \Lambda^n{}_m = \langle T^n g T_m g^{-1}\rangle \,. \tag{3.29}$$

The matrix elements Λ_{ij} can also be expressed in coordinates (3.5), and using the identities

$$\langle T_l T_m T_n\rangle = \epsilon_{lmn} \,, \tag{3.30}$$

$$\langle T_l T_m T_n T_k\rangle = \eta_{ln}\eta_{mk} - \eta_{lk}\eta_{mn} - \eta_{lm}\eta_{nk} \,, \tag{3.31}$$

one finds the expression

$$\Lambda_{ij}(g) = (1 - 2u^k u_k)\eta_{ij} - 2\epsilon_{ijk} c u^k + 2u_i u_j \,. \tag{3.32}$$

3.5 Particle Dynamics

Now we study the motion of a free particle on the $SL(2,\mathbb{R})$ space. The particle model is useful to get a basic understanding of dynamics on the group manifold.

3.5.1 Lagrangian Formulation

The standard Lagrangian of a free particle with mass $m = 1$ is given by

$$L = -\frac{1}{2} G_{\mu\nu}\dot{q}^\mu(t)\dot{q}^\nu(t) = -\frac{1}{2}\langle g^{-1}(t)\dot{g}(t)g^{-1}(t)\dot{g}(t)\rangle \,. \tag{3.33}$$

The equation of motion for this system is

$$\ddot{g}(t) = \dot{g}(t)g^{-1}(t)\dot{g}(t) \tag{3.34}$$

and one can immediately find its general solution

$$g(t) = g_0 e^{(t-t_0)R} \qquad \text{or} \qquad g(t) = e^{(t-t_0)L} g_0 \,, \tag{3.35}$$

with initial position $g_0 = g(t_0)$ and initial 'velocity' $R = g^{-1}(t_0)\dot{g}(t_0)$ and $L = \dot{g}(t_0)g^{-1}(t_0)$.

3.5.2 Hamiltonian Formulation

We now want to apply the first order formalism, that was described in section 2.2, to obtain the Poisson structure of the system. First, note that the linear Lagrangian

$$L = -\langle R g^{-1} \dot{g} \rangle + \frac{1}{2} \langle RR \rangle \tag{3.36}$$

is equivalent to the free particle Lagrangian (3.33), since the equations of motion for R imply

$$R = g^{-1} \dot{g} \,, \tag{3.37}$$

insertion of which reproduces the original Lagrangian. Here one can identify the first term with the 1-form θ and the second term with the Hamiltonian

$$\theta = -\langle R g^{-1} dg \rangle \,, \tag{3.38a}$$

$$H = -\frac{1}{2} \langle RR \rangle \,. \tag{3.38b}$$

The space of solutions given by (3.35) and (3.37) is parameterized by g_0 and R. The induced 2-form $\tilde{\omega} = d\theta$ on the space of motions is easily found to be

$$\tilde{\omega} = -\langle dR \wedge g_0^{-1} dg_0 \rangle + \langle R g_0^{-1} dg_0 \wedge g_0^{-1} dg_0 \rangle \,. \tag{3.39}$$

Using (2.8) we can extract the Poisson brackets of the system and find

$$\{R_n, g_0\} = -g_0 T_n \,, \qquad \{R_m, R_n\} = -2\epsilon_{mn}{}^k R_k \,, \qquad \{g_0, g_0\} = 0 \,. \tag{3.40}$$

For the function $L = \dot{g} g^{-1} = g_0 R g_0^{-1}$ one finds similar Poisson brackets

$$\{L_m, L_n\} = -2\epsilon_{mn}{}^k L_k \,, \qquad \{L_n, g_0\} = -T_n g_0 \,. \tag{3.41}$$

The Poisson bracket between R and L vanishes

$$\{R_m, L_n\} = 0 \,. \tag{3.42}$$

3.5.3 Gauged Systems

The Lagrangian (3.33) and the space of motions (3.35) are invariant under multiplication of $g(t)$ from the left or right with arbitrary constant group elements $a, b \in SL(2, \mathbb{R})$. These transformations are generated by L and R, respectively. One can now make a subgroup of this global symmetry local by the standard gauging procedure, i.e. by replacing the time derivative with the covariant derivative $\dot{g}(t) \to \dot{g}(t) - A(t)\delta g$, where δg is the infinitesimal transformation of $g(t)$ under the symmetry transformation. For the multiplication from the right with a group element generated by T_n we have

$$g(t) \to g(t) e^{\epsilon T_n} = g(t) + \epsilon g(t) T_n + \mathcal{O}(\epsilon^2) \,. \tag{3.43}$$

The gauged Lagrangian is therefore

$$L = -\frac{1}{2} \langle g^{-1}(\dot{g} - A g T_n) g^{-1}(\dot{g} - A g T_n) \rangle \,. \tag{3.44}$$

For a time- or spacelike vector T_n the algebraic equation of motion for A can be solved

$$A(t) = \frac{1}{\eta_{nn}} \langle T_n g^{-1} \dot{g} \rangle \,. \tag{3.45}$$

Substituting this solution back into the Lagrangian we find the effective Lagrangian

$$L = -\frac{1}{2} \left[\langle g^{-1} \dot{g} g^{-1} \dot{g} \rangle - \frac{1}{\eta_{nn}} \langle T_n g^{-1} \dot{g} \rangle^2 \right] . \tag{3.46}$$

In terms of the gauge invariant variables $x^i \equiv \frac{1}{2} \langle T^i g T_n g^{-1} \rangle$ the Lagrangian can be written as

$$L = -\frac{1}{2\eta_{nn}} \dot{x}_i \dot{x}^i \,, \tag{3.47}$$

which can be checked using the identities (3.30), (3.31) and

$$(T_n)_{ab} (T^n)_{cd} = \delta_{ab} \delta_{bc} - 2\delta_{ad} \delta_{bc} \,. \tag{3.48}$$

Since the coordinates must also satisfy the constraint

$$x_i x^i = \frac{\eta_{nn}}{4} \tag{3.49}$$

the solutions of the equations of motion are geodesics on this pseudo-sphere. The action as well as the constraint are invariant under Lorentz transformations, and from the Noether theorem we find the corresponding conserved quantities

$$L_n = \epsilon_n{}^{ij} x_i \dot{x}_j \,. \tag{3.50}$$

The vector L is orthogonal to the position x and the velocity $\dot{x}$, and thus the trajectories are given by the intersection of the hyperboloid (3.49) and a plane through the origin perpendicular to L.

The solutions can also be obtained by Hamiltonian reduction of the space of motions of the ungauged system. For that purpose we look at the gauged Lagrangian (3.44) in the first order formalism

$$L = -\langle R g^{-1} \dot{g} \rangle + \frac{1}{2} \langle RR \rangle + A \langle R T_n \rangle \rangle \,. \tag{3.51}$$

This is equivalent to the original Lagrangian with an additional Lagrange multiplier, that imposes the constraint

$$R_n = 0 \tag{3.52}$$

on the solution (3.35). The solution in terms of gauge invariant variables is therefore given by

$$x^i = \langle T^i g_0 e^{(R^1 T_1 + R^2 T_2)(t-t_0)} T_n e^{-(R^1 T_1 + R^2 T_2)(t-t_0)} g_0^{-1} \rangle \,. \tag{3.53}$$

As one can check the gauge invariant quantity $L = \langle T_n g_0 R g_0^{-1} \rangle$ is also orthogonal to these coordinates

$$L_n x^n = 0 \,. \tag{3.54}$$

The solutions found by Hamiltonian reduction therefore also give the geodesics on the hyperboloid. Using (3.20) we find the explicit solution for $g_0 = I$ and $n = 0$

$$x^0 = \frac{1}{2} (1 + 2 \sinh^2(|R| t)) \,, \tag{3.55a}$$

$$x^1 = \frac{R_2}{|R|} \sinh(|R| t) \cosh(|R| t) \,, \tag{3.55b}$$

$$x^2 = -\frac{R_1}{|R|} \sinh(|R| t) \cosh(|R| t) \,. \tag{3.55c}$$

with $|R| = \sqrt{R_1^2 + R_2^2}$. Solutions with arbitrary g_0 are related to this one by the Lorentz transformations (3.32).

3.5.4 Further Gaugings

Here we will study specific gaugings, which will become relevant for the WZNW theory later. These are the axial gauging with respect to the transformation

$$g(t) \to e^{\epsilon T_n} g(t) e^{\epsilon T_n} = g(t) + \epsilon(T_n g(t) + g(t) T_n) + \mathcal{O}(\epsilon^2)\,, \tag{3.56}$$

the vector gauging with respect to the transformation

$$g(t) \to e^{\epsilon T_n} g(t) e^{-\epsilon T_n} = g(t) + \epsilon(T_n g(t) - g(t) T_n) + \mathcal{O}(\epsilon^2) \tag{3.57}$$

and the nilpotent gauging with respect to two transformation generated by the nilpotent matrix T_+ from (3.18)

$$g(t) \to e^{\epsilon_1 T_+} g(t) e^{\epsilon_2 T_+} = g(t) + \epsilon_1 T_+ g(t) + \epsilon_2 g(t) T_+ + \mathcal{O}(\epsilon^2)\,. \tag{3.58}$$

Axial Gauging: The Lagrangian with the covariant derivative in place of the regular one becomes

$$L_{(n)}^{\text{ax}} = -\frac{1}{2}\langle g^{-1}(\dot g - A(T_n g + g T_n)) g^{-1}(\dot g - A(T_n g + g T_n))\rangle\,, \tag{3.59}$$

where we have suppressed the time dependence of the variables. Note that the equation of motion for $A(t)$ is purely algebraic. Substituting its solution

$$A(t) = \frac{\langle T_n g^{-1}\dot g\rangle + \langle T_n \dot g g^{-1}\rangle}{2(\eta_{nn} + \Lambda_{nn})} \tag{3.60}$$

back into the Lagrangian (3.59) leads to the gauged expression

$$L_{(n)}^{\text{ax}} = -\frac{1}{2}\left[\langle g^{-1}\dot g g^{-1}\dot g\rangle - \frac{(\langle T_n g^{-1}\dot g\rangle + \langle T_n \dot g g^{-1}\rangle)^2}{2(\eta_{nn} + \Lambda_{nn})}\right]\,. \tag{3.61}$$

In order to write this Lagrangian only in terms of the gauge invariant variables we go to coordinates (3.4) and thus arrive at

$$L_{(n)}^{\text{ax}} = -\frac{1}{2}\left[\dot u^k \dot u_k + \dot c^2 - \frac{(c\dot u_n - u_n \dot c)^2}{\eta_{nn} c^2 + u_n^2}\right]\,. \tag{3.62}$$

For the gauging with respect to the compact subgroup generated by T_0 we set $n = 0$. The final gauged Lagrangian is then

$$L_{(0)}^{\text{ax}} = \frac{1}{2}\frac{\dot u_1^2 + \dot u_2^2 + (u_1\dot u_2 - u_2\dot u_1)^2}{1 + u_1^2 + u_2^2}\,. \tag{3.63}$$

For $n = 2$, the gauging with respect to the con-compact subgroup, we get

$$L_{(2)}^{\text{ax}} = \frac{1}{2}\frac{\dot u_0^2 - \dot u_1^2 + (u_0\dot u_1 - u_1\dot u_0)^2}{u_0^2 - u_1^2 - 1}\,. \tag{3.64}$$

Vector Gauging: Similarly to the axial case the Lagrangian for the vector gauging

$$L^{\text{vec}}_{(n)} = -\frac{1}{2}\langle g^{-1}(\dot{g} - A(T_n g - gT_n))g^{-1}(\dot{g} - A(T_n g - gT_n))\rangle \tag{3.65}$$

leads to an algebraic equation for $A(t)$

$$A(t) = \frac{\langle T_n g^{-1}\dot{g}\rangle - \langle T_n \dot{g} g^{-1}\rangle}{2(\eta_{nn} - \Lambda_{nn})} \,. \tag{3.66}$$

Substituting this solution back into (3.65) gives the reduced Lagrangian, which in terms of coordinates (3.4) becomes

$$L^{\text{vec}}_{(n)} = -\frac{1}{2}\left[\dot{u}^k \dot{u}_k + \frac{1}{c^2}u^i \dot{u}_i u^j \dot{u}_j - \frac{(\epsilon_{ijn}u^i \dot{u}^j)^2}{\eta_{nn}u^k u_k - u_n^2}\right] . \tag{3.67}$$

In the compact case $n = 0$ this leads to the final expression

$$L^{\text{vec}}_{(0)} = \frac{1}{2}\,\frac{\dot{u}_0^2 + \dot{c}^2 - (u_1\dot{u}_2 - u_2\dot{u}_1)^2}{u_0^2 + c^2 - 1} \,. \tag{3.68}$$

For the non-compact case $n = 2$ the Lagrangian is the same as for the axial gauge, except in terms of c and u_2.

Nilpotent Gauging: Here we have to introduce two independent gauge fields for the two separate transformations. The Lagrangian is then

$$L_{(+)} = -\frac{1}{2}\langle g^{-1}(\dot{g} - AT_+ g - BgT_+)g^{-1}(\dot{g} - AT_+ g - BgT_+)\rangle \,. \tag{3.69}$$

The equations of motion for A and B are easily solved and we find

$$A = \frac{\langle T_+ g^{-1}\dot{g}\rangle}{\Lambda_{++}} \,, \qquad\qquad B = \frac{\langle T_+ \dot{g} g^{-1}\rangle}{\Lambda_{++}} \,. \tag{3.70}$$

Substituting these solutions into the Lagrangian we get the effective Lagrangian

$$L_{(+)} = -\frac{1}{2}\left[\langle g^{-1}\dot{g}g^{-1}\dot{g}\rangle - \frac{2}{\Lambda_{++}}\langle T_+\dot{g}g^{-1}\rangle\langle T_+\dot{g}g^{-1}\rangle\right] . \tag{3.71}$$

In order to eliminate all gauge dependent variables we insert the coordinates (3.4) and finally get

$$L_{(+)} = \frac{\dot{u}_+^2}{2u_+^2} \,. \tag{3.72}$$

Note that this Lagrangian is singular for $u_+(t) = 0$ and the solutions must be positive or negative for all times. We can therefore parameterize the trajectory by $u_+(t) = \pm e^{x(t)}$. The Lagrangian in terms of this $x(t)$ is then just

$$L_{(+)} = \frac{1}{2}\dot{x}^2(t) \tag{3.73}$$

which describes a free particle in one dimension.

Hamiltonian Reduction: For the axial or vector gauged Lagrangian one can write a linear Lagrangian in the same way as before

$$L^{\mathrm{ax/vec}}_{(n)} = -\langle Rg^{-1}\dot{g}\rangle + \left(\frac{1}{2}\langle RR\rangle + A\langle Rg^{-1}(T_n g \pm g T_n)\rangle\right) . \tag{3.74}$$

The gauge field $A(t)$ appears here as a Lagrange multiplier which imposes the constraint

$$\langle Rg^{-1}(T_n g \pm g T_n)\rangle = 0 . \tag{3.75}$$

On the space of motions given by the solutions (3.35) and (3.37) this constraint becomes

$$L_n \pm R_n = 0 , \tag{3.76}$$

which corresponds to a gauge choice $A(t) = 0$.

For the nilpotent gauging the linear Lagrangian is

$$L = -\langle Rg^{-1}\dot{g}\rangle + \left(\frac{1}{2}\langle RR\rangle + A\langle Rg^{-1}T_+ g\rangle + B\langle RT_+\rangle\right) . \tag{3.77}$$

Here we see two Lagrange multipliers, which impose the conditions

$$R_+ \equiv \langle T_+ R\rangle = 0 , \qquad L_+ \equiv \langle T_+ L\rangle = 0 . \tag{3.78}$$

3.5.5 Liouville Particle

We now want to study a reduction related to the nilpotent gauging. Consider the constraints

$$R_+ = -\rho , \qquad L_+ = \rho \tag{3.79}$$

on the space of motions. These are first class constraints, because by (3.40)-(3.42) and (3.38b) the Poisson brackets of R_+ and L_+ with themselves, with each other and with the Hamiltonian vanish. We then know that the Hamiltonian and the symplectic form are gauge invariant with respect to transformations generated by these constraints (see Appendix A). Therefore we impose an additional constraint, that fixes the gauge

$$R_2 = 0 , \qquad L_2 = 0 . \tag{3.80}$$

Writing the relation $g_0 R = L g_0$ in matrix form gives four equations, from which we find

$$g_0 = \begin{pmatrix} a & b \\ -\frac{1}{\rho}R_- b & -a \end{pmatrix} \tag{3.81}$$

with arbitrary parameters a, b. The unit determinant condition $det(g_0) = 1$ fixes R_-

$$R_- = \frac{\rho}{b^2}(1 + a^2) . \tag{3.82}$$

We now substitute the parameterization (3.81) into the 1-form (3.38a) and the Hamiltonian (3.38b) and obtain

$$\tilde{\theta} = \frac{1}{\rho} b\, da , \tag{3.83}$$

$$H = \frac{1}{2}\rho R_- = \frac{1}{2}\frac{\rho^2}{b^2}(1 + a^2) . \tag{3.84}$$

A reparameterization with $a = \frac{1}{\rho} P e^{-Q}$ and $b = e^{-Q}$ leads to the canonical symplectic form $\tilde{\omega} = d\tilde{\theta} = dP \wedge dQ$ and to the Hamiltonian

$$H = \frac{1}{2}\left(P^2 + \rho^2 e^{2Q}\right) , \tag{3.85}$$

which is that of a particle in the Liouville potential. To find the general solution we substitute the parameterization (3.81) into the general solution of the unconstrained system (3.35). Using (3.19) we obtain for the only gauge invariant component

$$e^{-Q(t)} \equiv g_{12}(t) = b \cosh\left(\sqrt{2H}\rho\, t\right) + a \frac{\rho}{\sqrt{2H}} \sinh\left(\sqrt{2H}\rho\, t\right) . \tag{3.86}$$

Here, b can be expressed in terms of a and the Hamiltonian (3.85)

$$b = \frac{\rho}{\sqrt{2H}} \sqrt{1 + a^2} . \tag{3.87}$$

With the definitions $p \equiv \sqrt{2H}$ and $e^q \equiv \sqrt{1 + a^2} + a$ the solution can then be written as

$$e^{-Q(t)} = \frac{\rho}{2p}\left[e^{q+pt} + e^{-(q+pt)}\right] . \tag{3.88}$$

The Hamiltonian in the new variables is $H = \frac{1}{2} p^2$, and one can check that these are still canonical variables, i.e. $\omega = dp \wedge dq$. We have thus found the general solution of a particle in the Liouville potential in terms of a free particle. A similar strategy will be used for the field theoretical model in chapter 5.

IV

The WZNW Theory

In this chapter we turn to WZNW theory, using the formalism and the technique established in the previous chapters. We provide both Lagrangian and Hamiltonian descriptions of the $SL(2,\mathbb{R})$ WZNW theory, where we outline the Kac-Moody and conformal symmetries of the general solution and give their Poisson bracket realization. The basic Poisson bracket relations are obtained by the inversion of the chiral symplectic form, which is monodromy dependent. Then we calculate non-equal time Poisson brackets for the full WZNW-field and, finally, describe gaugings of the Kac-Moody symmetries.

A new result for this chapter is the calculation of the non-equal time Poisson brackets in the elliptic sector of the $SL(2,\mathbb{R})$ WZNW theory. This calculation generalizes the earlier obtained result of [11] for the hyperbolic monodromy and indicates that the non-equal time Poisson bracket structure of the full WZNW-field is monodromy independent.

4.1 The WZNW Action

The WZNW theory [5, 6, 7] is described by the action

$$S_{WZNW}[g] = -\frac{k}{2}\int\limits_M \sqrt{-h}\, h^{\mu\nu}\, \langle g^{-1}\partial_\mu g g^{-1}\partial_\nu g\rangle\, d^2\xi + kI_{WZ}[g]\,, \tag{4.1}$$

which contains a sigma model part and the additional Wess-Zumino term

$$I_{WZ}[g] = \frac{1}{3}\int\limits_B \langle g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\rangle\,. \tag{4.2}$$

Here, the world surface M is a cylinder $(\xi^0,\xi^1)\in\mathbb{R}\times S^1$ with the Minkowski metric $h_{\mu\nu} = \mathrm{diag}(+,-)$, $\langle\cdot\rangle$ denotes a normalized trace, B is a volume with the boundary $\partial B = g(M)$, and g takes values in a semi-simple Lie group. In this work we will consider only $SL(2,\mathbb{R})$ valued fields, and $\langle\cdot\rangle$ is given by (3.17). The coordinates (ξ^0,ξ^1) we denote by (τ,σ) as for free-field theory and assume $\sigma\in[0,2\pi]$.

The first term is the Polyakov action of a string on $\mathbb{H}_3$ in the conformal gauge, to which the general solution is not known. The second term, the Wess-Zumino term, is a priori not given in the usual form of an integral over a Lagrangian. However, the 3-form $H = \frac{1}{3}\langle g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\rangle$ can be written as the outer derivative $H = dF_{(A,\pm)}$ of the 2-form [42]

$$F_{(A,\pm)} = \frac{\langle Adgg^{-1}\rangle\wedge\langle Ag^{-1}dg\rangle}{\langle AgAg^{-1}\rangle\pm\langle AA\rangle}\,, \tag{4.3}$$

where A is a fixed non-zero element of the $sl(2,\mathbb{R})$ algebra. Using then Stokes' theorem, we can rewrite the Wess-Zumino term as

$$I_{WZ}[g] = \int\limits_{g(M)} F = \int_M \tilde{F}\,, \tag{4.4}$$

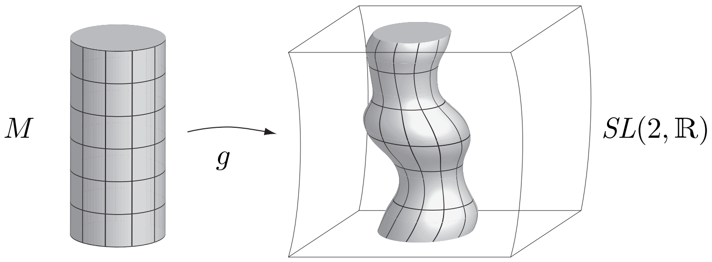

Figure 4.1: Embedding of the Minkowskian world sheet M into the 3 dimensional Lie group $SL(2,\mathbb{R})$.

where $\tilde{F}$ is the pullback of F on M. In the light cone coordinates (2.32) the metric $h_{\mu\nu}$ and its inverse are

$$h = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad h^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \tag{4.5}$$

and the Lagrangian of the system becomes

$$\mathcal{L}_{(A,\pm)} = -k\,\langle g^{-1}\partial_x g g^{-1}\partial_{\bar{x}} g\rangle - k\frac{\langle A\partial_x g g^{-1}\rangle\langle Ag^{-1}\partial_{\bar{x}} g\rangle - \langle A\partial_{\bar{x}} g g^{-1}\rangle\langle Ag^{-1}\partial_x g\rangle}{\langle AgAg^{-1}\rangle \pm \langle AA\rangle}\,. \tag{4.6}$$

The change of the sign in front of the F-term occurs because the coordinate transformation from (τ,σ) to $(x,\bar{x})$ inverts the orientation. Lagrangians with different choices of A and the sign $\pm$ differ only by a total derivative, which does not affect the equations of motion.

Using local coordinates u^n on the $SL(2,\mathbb{R})$ group manifold, the action of the system can be written as

$$S = -k\int dx d\bar{x}\ \left(G_{mn}(u) + F_{mn}(u)\right)\partial_x u^m \partial_{\bar{x}} u^n\,, \tag{4.7}$$

where $G_{mn} = \langle g^{-1}\partial_m g g^{-1}\partial_n g\rangle$ corresponds to the metric tensor (3.24), while the antisymmetric F_{mn} stands for the 2-form (4.3).

4.2 Equation of Motion, Solutions and Monodromies

In order to derive the equations of motion from the Lagrangian (4.6) first note that the part corresponding to the 2-form $\mathcal{L}_A = F_{mn}\partial_x u^m\partial_{\bar{x}} u^n$ can be expressed in terms of the outer derivative

$$\frac{\partial\mathcal{L}_A}{\partial u^n} - \partial_x\frac{\partial\mathcal{L}_A}{\partial(\partial_x u^n)} - \partial_{\bar{x}}\frac{\partial\mathcal{L}_A}{\partial(\partial_{\bar{x}} u^n)} = dF\left(Du(\partial_x), Du(\partial_{\bar{x}}), \frac{\partial}{\partial u^n}\right)\,. \tag{4.8}$$

Here Du denotes the differential of u. Then the dynamical equation obtained from (4.6) does not depend on the matrix A and it reads

$$\partial_x(g^{-1}\partial_{\bar{x}} g) = 0\,. \tag{4.9a}$$

The multiplication of this equation from left and right by g and g^{-1}, respectively, yields the equivalent equation

$$\partial_{\bar{x}}(\partial_x g g^{-1}) = 0\,. \tag{4.9b}$$

The general solution of these equations is given by the product of chiral and antichiral functions

$$g(x,\bar{x}) = g_L(x)g_R(\bar{x})\,, \qquad\qquad g_L, g_R : \mathbb{R} \to SL(2,\mathbb{R})\,. \tag{4.10}$$

However, we must also take care of the boundary conditions, which imply periodicity of the solution (4.10) in σ

$$g(\tau, \sigma + 2\pi) = g(\tau, \sigma)\,. \tag{4.11}$$

This condition requires the following behavior of the chiral and antichiral parts under the 2π shift of the arguments

$$g_L(x + 2\pi) = g_L(x)M\,, \qquad\qquad g_R(\bar{x} - 2\pi) = M^{-1}g_R(\bar{x})\,. \tag{4.12}$$

Here, $M \in SL(2,\mathbb{R})$ is a monodromy matrix, which can be written in the exponential form $M = e^B$, with some $B \in sl(2,\mathbb{R})$. The transformation

$$g_L(x) \to g_L(x)N\,, \qquad\qquad g_R(\bar{x}) \to N^{-1}g_R(\bar{x}) \tag{4.13}$$

with $N \in SL(2,\mathbb{R})$ obviously leaves the solution (4.10) invariant, but transforms the monodromy matrix by

$$M \to N^{-1}MN = e^{N^{-1}BN}\,. \tag{4.14}$$

Since $B \to N^{-1}BN$ corresponds to a Lorentz transformation of the coordinates $B_n = \langle T_n B\rangle$ (see (3.29)), the matrix B can be transformed to one of the following forms

$$B_s = \lambda T_2\,, \qquad\qquad B_t = \lambda T_0\,, \qquad\qquad B_l = T_+\,, \tag{4.15}$$

where T_+ is the nilpotent element (3.18). B_s, B_t and B_l correspond to spacelike $(0, 0, \lambda)$, timelike $(\lambda, 0, 0)$ and lightlike $(1, 1, 0)$ Minkowski vectors, respectively. Thus, there are three different classes of monodromy, which are called

- hyperbolic: $\qquad M = e^{\lambda T_2}\,,$ (4.16a)
- elliptic: $\qquad M = e^{\lambda T_0}\,, \qquad \lambda \in (0, \pi)\,,$ (4.16b)
- and parabolic: $\qquad M = e^{T_+}\,.$ (4.16c)

We mostly consider the hyperbolic and elliptic monodromies. In these cases the parameter λ becomes an important dynamical variable like the momentum zero mode p in free-field theory (see (2.40)). Note that the parabolic monodromy has no parameter λ. As we will see below, the symplectic form in that case is degenerate. For more details about monodromies see [48].

4.3 Symmetries and Conserved Currents

The action (4.7) is invariant under the $2d$ conformal transformation

$$g(x,\bar{x}) \to g(\xi(x), \bar{\xi}(\bar{x})) \tag{4.17}$$

with functions $\xi, \bar{\xi}$ as in (2.60). The corresponding conserved current constructed from the Lagrangian (4.6) by the Noether theorem, the energy-momentum tensor, has vanishing diagonal components $T^x{}_x = T^{\bar{x}}{}_{\bar{x}} = 0$, and the off-diagonal components are

$$T \equiv T^{\bar{x}}{}_x = -k\langle g^{-1}\partial_x g g^{-1}\partial_x g\rangle \,, \tag{4.18a}$$

$$\overline{T} \equiv T^x{}_{\bar{x}} = -k\langle g^{-1}\partial_{\bar{x}} g g^{-1}\partial_{\bar{x}} g\rangle \,. \tag{4.18b}$$

The local conservation law for the energy-momentum tensor then provides the chirality conditions

$$\partial_x \overline{T} = 0 \,, \qquad\qquad \partial_{\bar{x}} T = 0 \,. \tag{4.19}$$

The dynamical equations (4.9a) and (4.9b) can also be written as the chirality conditions

$$\partial_{\bar{x}} J = 0 \,, \qquad\qquad \partial_x \bar{J} = 0 \,, \tag{4.20}$$

where J and $\bar{J}$ are $sl(2,\mathbb{R})$ valued functions

$$J \equiv k\,\partial_x g g^{-1} \,, \qquad\qquad \bar{J} \equiv k\, g^{-1}\partial_{\bar{x}} g \,, \tag{4.21}$$

which are called Kac-Moody currents. Here, the coupling constant k is included in the definition for further convenience. Introducing components of the Kac-Moody currents in the basis T_n (3.3)

$$J_n(x) = \langle T_n J(x)\rangle \,, \qquad\qquad \bar{J}_n(\bar{x}) = \langle T_n \bar{J}(\bar{x})\rangle \,, \tag{4.22}$$

we get the Sugawara form of the energy-momentum tensor (4.18a), (4.18b)

$$T(x) = -\frac{1}{k} J^n(x) J_n(x) \,, \qquad\qquad \overline{T}(\bar{x}) = -\frac{1}{k} \bar{J}^n(\bar{x}) \bar{J}_n(\bar{x}) \,. \tag{4.23}$$

As it is shown below, the Kac-Moody currents are the generators of infinite dimensional symmetry transformations

$$g(x,\bar{x}) \to \tilde{g}(x,\bar{x}) = h(x) g(x,\bar{x}) \bar{h}(\bar{x}) \tag{4.24}$$

with arbitrary $SL(2,\mathbb{R})$ valued periodic functions $h(x)$ and $\bar{h}(\bar{x})$. The equation of motion (4.9a) and the space of solutions (4.10) are apparently invariant under (4.24).

4.4 Symplectic Form

In order to extract the symplectic form of the $SL(2,\mathbb{R})$ WZNW theory we pass to the Hamiltonian formulation of the system given by the Lagrangian (4.6). Using the (τ,σ) coordinates and applying the first order formalism, similarly to the particle dynamics on the group manifold, we find the action

$$S = -k \int d\tau \int_0^{2\pi} d\sigma \left(\langle R g^{-1}\dot{g}\rangle - \frac{1}{2}\left(\langle RR\rangle + \langle g^{-1}g' g^{-1}g'\rangle\right) + \tilde{F}(\partial_\sigma, \partial_\tau) \right) \,, \tag{4.25}$$

where $\dot{g} \equiv \partial_\tau g$ and $g' \equiv \partial_\sigma g$. The variation of (4.25) with respect to R provides

$$R = g^{-1}\dot{g}\,, \tag{4.26}$$

and inserting this R back into (4.25) we get the initial Lagrangian (4.6), which confirms the equivalence of (4.25) and (4.6). Thus, in the Hamiltonian formulation the phase space of the system is given as a set of functions $R(\sigma)$ and $g(\sigma)$ with values in the $sl(2,\mathbb{R})$ algebra and $SL(2,\mathbb{R})$ group, respectively. The 1-form and the Hamiltonian read

$$\theta = -k\int_0^{2\pi} d\sigma\,\left(\langle Rg^{-1}dg\rangle + \partial_\sigma\rfloor\tilde{F}\right)\,, \tag{4.27a}$$

$$H = -\frac{k}{2}\int_0^{2\pi} d\sigma\,\left(\langle RR\rangle + \langle g^{-1}g'g^{-1}g'\rangle\right)\,. \tag{4.27b}$$

With the help of the identity $d(V\rfloor\tilde{F}) = \mathcal{L}_V\tilde{F} - V\rfloor d\tilde{F}$ (see e.g. [49]), where V is a vector field and $\mathcal{L}_V$ denotes the corresponding Lie derivative, we calculate the symplectic form $\omega = d\theta$ and find

$$\omega = -k\int_0^{2\pi} d\sigma\left(\langle dR\wedge g^{-1}dg\rangle - \langle Rg^{-1}dg\wedge g^{-1}dg\rangle - \partial_\sigma\rfloor d\tilde{F} + \mathcal{L}_{\partial_\sigma}\tilde{F}\right). \tag{4.28}$$

Since $\mathcal{L}_{\partial_\sigma} = \partial_\sigma$ the last term can be integrated, and it vanishes due to the periodicity of g. Inserting now the solutions (4.10) we obtain the symplectic form $\tilde{\omega}$ on the space of motions. Further simplification based on $dF = \frac{1}{3}\langle g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\rangle$ leads to

$$\begin{aligned}\tilde{\omega}(\tau) = -k\Big[\int_0^{2\pi} &\langle\partial_x(g_L^{-1}dg_L)\wedge g_L^{-1}dg_L\rangle + \langle\partial_{\bar{x}}(dg_Rg_R^{-1})\wedge dg_Rg_R^{-1}\rangle d\sigma \\ &+ \langle dMM^{-1}\wedge g_L^{-1}dg_L\rangle\big|_{\sigma=0} + \langle dMM^{-1}\wedge dg_Rg_R^{-1}\rangle\big|_{\sigma=0}\Big]\,.\end{aligned} \tag{4.29}$$

Here the functions g_L and g_R depend on x and $\bar{x}$ respectively and M is the monodromy matrix (4.12). Since M can be reduced to one of the three monodromy classes, we get for the hyperbolic (4.16a) and elliptic (4.16b) monodromies

$$dMM^{-1} = T_n d\lambda\,, \qquad n = 2, 0\,, \tag{4.30}$$

and $dMM^{-1} = 0$ for the parabolic one (4.16c) ($n = +$).

The bulk part of the symplectic form (4.29) splits into the sum of a chiral and an antichiral part, but the boundary terms are still coupled through the monodromy parameter λ, like in free-field theory (2.50). Applying here the Veneziano-Fubini trick we introduce two parameters λ_L, λ_R instead of λ and define a new symplectic form, $\omega_{WZ} \equiv \tilde{\omega}_L + \tilde{\omega}_R$, which is the sum of a chiral and an antichiral part [8, 9, 10]

$$\tilde{\omega}_L(\tau) = -k\int_0^{2\pi} d\sigma\langle\partial_x(g_L^{-1}dg_L)\wedge g_L^{-1}dg_L\rangle - kd\lambda_L\wedge\langle T_n g_L^{-1}dg_L\rangle\big|_{\sigma=0}\,, \tag{4.31a}$$

$$\tilde{\omega}_R(\tau) = -k\int_0^{2\pi} d\sigma\langle\partial_{\bar{x}}(dg_Rg_R^{-1})\wedge dg_Rg_R^{-1}\rangle - kd\lambda_R\wedge\langle T_n dg_Rg_R^{-1}\rangle\big|_{\sigma=0}\,. \tag{4.31b}$$

Then, for gauge invariant functions (i.e. functions that have vanishing Poisson brackets with $\lambda_L - \lambda_R$ on the subspace $\lambda_L = \lambda_R$) the Poisson brackets calculated from the symplectic forms $\tilde{\omega}$ and ω_{WZ} coincide (for details see Appendix A).

A simple calculation shows that the chiral symplectic forms (4.31a) and (4.31b) are time independent. Hence, we can change the integration variable to x and $\bar{x}$ respectively, while keeping the limits 0 and 2π

$$\tilde{\omega}_L = -k\int_0^{2\pi} dx \langle \partial_x(g_L^{-1}dg_L)\wedge g_L^{-1}dg_L\rangle - kd\lambda_L \wedge \langle T_n g_L^{-1}dg_L\rangle\big|_{x=0}\,, \tag{4.32a}$$

$$\tilde{\omega}_R = -k\int_0^{2\pi} d\bar{x} \langle \partial_{\bar{x}}(dg_R g_R^{-1})\wedge dg_R g_R^{-1}\rangle - kd\lambda_R \wedge \langle T_n dg_R g_R^{-1}\rangle\big|_{\bar{x}=0}\,. \tag{4.32b}$$

The splitting into left and right parts also holds for the Hamiltonian (4.27b)

$$H = -k\int_0^{2\pi} d\sigma\ \left(\langle g_L'(x)g_L^{-1}g_L'(x)g_L^{-1}\rangle + \langle g_R'(\bar{x})g_R^{-1}g_R'(\bar{x})g_R^{-1}\rangle\right) = H_L + H_R\,, \tag{4.33}$$

which can be expressed through the energy-momentum tensor

$$H_L = \int_0^{2\pi} dx\, T(x)\,, \qquad\qquad H_R = \int_0^{2\pi} d\bar{x}\, \overline{T}(\bar{x})\,. \tag{4.34}$$

Thus in the Hamiltonian formulation the WZNW theory has a chiral structure similar to free-field theory. Only the chiral symplectic form is not canonical and one needs a certain labor to extract the Poisson brackets of the chiral fields.

4.5 Poisson Brackets

We are now going to invert the symplectic form of the system and calculate the Poisson brackets on the space of solutions. The Poisson brackets between chiral and antichiral functions vanish due to our extension. We first calculate the Poisson brackets of the chiral functions $\{g_L(x), g_L(y)\}$ using (4.32a) and (2.8). To simplify expressions we suppress the index L whenever possible.

It is helpful to first introduce the following periodic field

$$f(x) \equiv g(x)e^{-\frac{1}{2\pi}\lambda x T_n}\,, \tag{4.35}$$

in terms of which the symplectic form (4.32a) then becomes

$$\tilde{\omega} = -k\int_0^{2\pi} \langle (f^{-1}df)'\wedge f^{-1}df\rangle + \frac{\lambda}{2\pi}\langle [f^{-1}df, T_n]\wedge f^{-1}df\rangle + \frac{1}{\pi}d\lambda\wedge\langle T_n f^{-1}df\rangle dx\,. \tag{4.36}$$

4.5.1 Hyperbolic Monodromy

Let us consider the hyperbolic monodromy (4.16a) and set $n=2$ in (4.36). Application of equation (2.8) to the symplectic form (4.36) and the function λ, after partial integration yields

$$d\lambda - k\int_0^{2\pi} dx\,\left(2\langle A' f^{-1}df\rangle + \frac{\lambda}{\pi}\langle [A,T_2]f^{-1}df\rangle - \frac{1}{\pi}d\lambda\langle T_2 A\rangle\right) = 0\,, \tag{4.37}$$

where $A(x) \equiv f^{-1}(x)\{\lambda, f(x)\}$. The differentials $d\lambda$ and $df(x)$ are independent and from (4.37) we find two equations for $A(x)$

$$1 + k\int_0^{2\pi} dx\, \frac{1}{\pi}\langle T_2 A(x)\rangle = 0\,, \qquad \frac{\lambda}{2\pi}[A(x), T_2] + A'(x) = 0\,. \tag{4.38}$$

Since the Poisson bracket is a derivation we have $A \in sl(2,\mathbb{R})$, and A is represented as $A(x) = A^n(x)T_n$. The second equation of (4.38) together with the periodicity of $A(x)$ implies that $A^0 = A^1 = 0$ and $A^2 = C$, where C is some constant. The first equation fixes this constant $A^2 = \frac{1}{2k}$, and we find

$$\{\lambda, f(x)\} = \frac{1}{2k}(f(x)T_2)\,, \tag{4.39}$$

which by (4.35) yields

$$\{\lambda, g(x)\} = \frac{1}{2k}\big(g(x)T_2\big)\,. \tag{4.40}$$

In order to determine the Poisson bracket $\{f_{ab}(x), f_{cd}(y)\}$ we need to solve equation (2.8) for the function $f_{ab}(x)$. Introducing

$$A_{ab}(x,y) \equiv f^{-1}(y)\{f_{ab}(x), f(y)\} = A^n_{ab}(x,y)T_n\,, \tag{4.41}$$

we can write (2.8) as

$$df_{ab}(x) - k\int_0^{2\pi} dy\,\Big(2\partial_y A^n_{ab}\langle T_n f^{-1}df\rangle + \frac{\lambda}{\pi}A^n_{ab}\langle [T_n, T_2]f^{-1}df\rangle \\ + \frac{1}{\pi}\{f_{ab}(x),\lambda\}\langle T_2 f^{-1}df\rangle - \frac{1}{\pi}d\lambda\langle T_2 A^n_{ab}T_n\rangle\Big) = 0\,. \tag{4.42}$$

Inserting (4.40) and identifying the coefficients of the one forms $d\lambda$ and $\langle T_n f^{-1}df\rangle$, we find the equations

$$0 = \int_0^{2\pi} dy\, A^2_{ab}\,, \tag{4.43a}$$

$$0 = \partial_y A^n_{ab} - \frac{1}{2k}\delta(x-y)\big(f(y)T^n\big)_{ab} - \frac{1}{4\pi k}\delta^n{}_2\big(f(x)T_2\big)_{ab} + \frac{\lambda}{\pi}\epsilon_{k2}{}^n A^k_{ab}\,. \tag{4.43b}$$

For $n = 2$ (4.43b) is a first order differential equation, which can be easily integrated, and we get

$$A^2_{ab}(x,y) = \frac{1}{4k}\left(\epsilon(x-y) - \frac{x-y}{\pi}\right)\big(f(x)T_2\big)_{ab}\,. \tag{4.44}$$

The remaining equations can be decoupled by using the light-cone coordinates $A^{\pm}_{ab} = \frac{1}{2}(A^0_{ab} \pm A^1_{ab})$ and they take the form

$$\left(\partial_y \pm \frac{\lambda}{\pi}\right) A^{\pm}_{ab} = \frac{1}{2k}\delta(x-y)(f(y)T^{\pm})_{ab}\,. \tag{4.45}$$

The Green's function which inverts the operator $\left(\partial_z + \frac{C}{2\pi}\right)$ for $C \neq 0$ in the class of periodic functions, is given by

$$h_C(z) \equiv \frac{e^{\frac{C}{2}h(z)}}{2\sinh(\frac{C}{2})}\,, \tag{4.46}$$

where $h(z) \equiv \epsilon(z) - \frac{z}{\pi}$ is the sawtooth function. Thus, (4.45) can be written as

$$A^{\pm}_{ab}(x,y) = -\frac{1}{2k}\big(f(x)T^{\pm}\big)_{ab}\, h_{\mp 2\lambda}(x-y)\,. \tag{4.47}$$

Combining (4.44) and (4.47) we obtain the Poisson bracket

$$\begin{aligned} \{f_{ab}(x), f_{cd}(y)\} = \frac{1}{4k}\Big\{\big(f(x)T_2\big)_{ab}\big(f(y)T_2\big)_{cd}\left(\epsilon(x-y) - \frac{x-y}{\pi}\right) \\ - (f(x)T_-)_{ab}(f(y)T_+)_{cd}h_{-2\lambda}(x-y) - (f(x)T_+)_{ab}(f(y)T_-)_{cd}h_{2\lambda}(x-y)\Big\}\,. \end{aligned} \tag{4.48}$$

Using equation (4.35), the tensorial structure of (4.48) and the identity

$$e^{-xT_2}T_{\pm}e^{xT_2} = e^{\pm 2x}T_{\pm}\,, \tag{4.49}$$

we combine equations (4.40) and (4.48) in the form

$$\begin{aligned} \{g(x), \otimes g(y)\} = \frac{1}{4k}\Big\{\epsilon(x-y)(g(x)T_2)\otimes(g(y)T_2) \\ - \theta_{-2\lambda}(x-y)(g(x)T_-)\otimes(g(y)T_+) - \theta_{2\lambda}(x-y)(g(x)T_+)\otimes(g(y)T_-)\Big\}\,, \end{aligned} \tag{4.50}$$

where

$$\theta_{2\lambda}(z) \equiv \frac{e^{\lambda\epsilon(z)}}{2\sinh(\lambda)}\,. \tag{4.51}$$

Note that this is the Green's function which inverts the operator ∂_z on the class of functions $A(z)$ with the monodromy $A(z+2\pi) = a^{2\lambda}A(z)$.

The calculation for the antichiral part is similar and it yields

$$\{\lambda_R, g_R(\bar{x})\} = \frac{1}{2k}\big(T_2 g_R(\bar{x})\big)\,, \tag{4.52}$$

and

$$\begin{aligned} \{g_R(\bar{x}), \otimes g_R(\bar{y})\} = \frac{1}{4k}\Big\{\epsilon(\bar{x}-\bar{y})(T_2 g_R(\bar{x}))\otimes(T_2 g_R(\bar{y})) \\ - \theta_{2\lambda}(\bar{x}-\bar{y})(T_- g_R(\bar{x}))\otimes(T_+ g_R(\bar{y})) - \theta_{-2\lambda}(\bar{x}-\bar{y})(T_+ g_R(\bar{x}))\otimes(T_- g_R(\bar{y}))\Big\}\,. \end{aligned} \tag{4.53}$$

4.5.2 Elliptic Monodromy

We now turn to the elliptic monodromy (4.16b) and set $n = 0$ in (4.36).

The derivation of the Poisson bracket $\{\lambda, f(x)\}$ is completely analogous to the previous case and the result is

$$\{\lambda_L, g_L(x)\} = -\frac{1}{2k}\big(g_L(x)T_0\big)\,, \qquad \{\lambda_R, g_R(\bar{x})\} = -\frac{1}{2k}\big(T_0 g_R(\bar{x})\big)\,. \tag{4.54}$$

In order to obtain the Poisson bracket $\{f(x), \otimes f(y)\}$ we define three matrix valued fields fields by $A^n(x,y)\otimes f(y)T_n \equiv \{f(x), \otimes f(y)\}$. Application of (2.8) to the function $f(x)$ as above leads to the equations

$$0 = \int_0^{2\pi} dy\, A^0 \tag{4.55a}$$

$$0 = \partial_y A^n - \frac{1}{2k}\delta(x-y)f(y)T^n + \frac{1}{4\pi k}f(x)T_0\delta^n{}_0 + \frac{\lambda}{\pi}\epsilon_{k0}{}^n A^k\,. \tag{4.55b}$$

For $n = 0$ this can be integrated and similarly to the hyperbolic case we get

$$A^0(x,y) = -\frac{1}{4k}\left(\epsilon(x-y) - \frac{x-y}{\pi}\right)\left(f(x)T_0\right). \tag{4.56}$$

The remaining equations, however, decouple for the complex linear combination $\tilde{A}^{\pm} = \pm\frac{1}{2}(A^2 \mp iA^1)$ in the form

$$\left(\partial_y \pm i\frac{\lambda}{\pi}\right)\tilde{A}^{\pm} = \frac{1}{2k}(f(x)\tilde{T}^{\pm}). \tag{4.57}$$

The Green's function (4.46) then provides

$$\tilde{A}^{\pm}(x,y) = -\frac{1}{2k}\big(f(x)\tilde{T}^{\pm}\big)h_{\mp 2i\lambda}(x-y)\,, \tag{4.58}$$

where $\tilde{T}_{\pm} = iT_1 \pm T_2$. These solutions together with (4.56) lead to

$$\begin{aligned}\{f(x),\otimes f(y)\} = \frac{1}{4k}\Big\{&(f(x)T_0)\otimes(f(y)T_0)\left(\frac{x-y}{\pi} - \epsilon(x-y)\right)\\ &-(f(x)\tilde{T}_-)\otimes(f(y)\tilde{T}_+)h_{-2i\lambda}(x-y) - (f(x)\tilde{T}_+)\otimes(f(y)\tilde{T}_-)h_{2i\lambda}\Big\}.\end{aligned} \tag{4.59}$$

Now we use the identity $e^{-xT_0}\tilde{T}_{\pm}e^{xT_0} = \tilde{T}_{\pm}e^{\pm 2ix}$ and similarly to (4.50) finally obtain the Poisson bracket

$$\begin{aligned}\{g(x),\otimes g(y)\} = -\frac{1}{4k}\Big\{&\epsilon(x-y)(g(x)T_0)\otimes(g(y)T_0)\\ &+\theta_{-2i\lambda}(x-y)(g(x)\tilde{T}_-)\otimes(g(y)\tilde{T}_+) + \theta_{2i\lambda}(x-y)g(x)(\tilde{T}_+)\otimes(g(y)\tilde{T}_-)\Big\},\end{aligned} \tag{4.60}$$

where $\theta_{\pm 2i\lambda}(z)$ is the Green's function (4.51) with imaginary parameter.

The result for the antichiral part is similar

$$\begin{aligned}\{g_R(\bar{x}),\otimes g_R(\bar{y})\} = -\frac{1}{4k}\Big\{&\epsilon(\bar{x}-\bar{y})(T_0 g_R(\bar{x}))\otimes(T_0 g_R(\bar{y}))\\ &+\theta_{2i\lambda}(\bar{x}-\bar{y})(\tilde{T}_- g_R(\bar{x}))\otimes(\tilde{T}_+ g_R(\bar{y})) + \theta_{-2i\lambda}(\bar{x}-\bar{y})(\tilde{T}_+ g_R(x))\otimes(\tilde{T}_- g_R(\bar{y}))\Big\}.\end{aligned} \tag{4.61}$$

4.5.3 Parabolic Monodromy

The parabolic monodromy (4.16c) corresponds to setting $T_n = T_+$ and $\lambda = 1$ in (4.35). The 2-form (4.36) then becomes

$$\tilde{\omega} = -k\int_0^{2\pi}\langle(f^{-1}df)'\wedge f^{-1}df\rangle + \frac{1}{2\pi}\langle[f^{-1}df, T_+]\wedge f^{-1}df\rangle dx\,. \tag{4.62}$$

One could already argue that for dimensional reasons this 2-form has to be singular. However, since for the infinite dimensional case this argument is not fully convincing, we are going to show this explicitly. Applying (2.8) to $df_{ab}(x)$ results in three equations for $A_{ab} \equiv f^{-1}(y)\{f_{ab}(x), f(y)\}$

$$\partial_y A^n_{ab} - \frac{1}{2k}\delta(x-y)(f(y)T^n)_{ab} + \frac{1}{\pi}({\epsilon_{k0}}^n + {\epsilon_{k1}}^n)A^k_{ab} = 0\,. \tag{4.63}$$

For the linear combination $A^- \equiv \frac{1}{2}(A^0 - A^1)$ we find

$$\partial_y A^- = \frac{1}{2}\delta(x-y)(f(y)T^-) \tag{4.64}$$

The solution of this equation is linear in the stair-step function $\epsilon(x-y)$, which is not periodic. Therefore, the 2-form (4.62) has no inverse in the class of periodic functions.

A more general treatment of the inversion of the WZNW chiral symplectic form can be found in [10].

4.5.4 Symmetry Generators

Having established the basic Poisson brackets for $g(x)$ and λ we are now ready to derive the Poisson brackets of the conserved currents $J(x)$ and $T(x)$. With the formulas from above it is easy to show that for both monodromy classes we have

$$\{\lambda_L, J_n(x)\} = 0\,, \qquad \{\lambda_R, \bar{J}_n(\bar{x})\} = 0. \tag{4.65}$$

The Poisson bracket $\{J_n(x), g(y)\}$ can, in principle, also be derived from the basic Poisson brackets, however, it is easier to apply equation (2.8) to (4.32a) for the function $J_n(x)$. Its differential is

$$dJ_n(x) = k\langle g_L^{-1}(x) T_n g_L(x) \left(g_L^{-1}(x) dg_L(x)\right)'\rangle\,, \tag{4.66}$$

which results in

$$\{J_n(x), g_L(y)\} = -\frac{1}{2}\delta(x-y)\big(T_n g_L(y)\big)\,, \tag{4.67a}$$

$$\{\bar{J}_n(\bar{x}), g_R(\bar{y})\} = -\frac{1}{2}\delta(\bar{x}-\bar{y})\big(g_R(\bar{y})T_n\big)\,. \tag{4.67b}$$

This reveals that the currents $J(x)$, $\bar{J}(\bar{x})$ are generators of the left and right multiplications

$$g(x,\bar{x}) \to e^{-2\int_0^{2\pi} dz f^n(z)\{J_n(z),\,\cdot\,\}} g(x,\bar{x}) = e^{f^n(x)T_n} g(x,\bar{x})\,, \tag{4.68a}$$

$$g(x,\bar{x}) \to e^{-2\int_0^{2\pi} dz f^n(\bar{z})\{\bar{J}_n(\bar{z}),\,\cdot\,\}} g(x,\bar{x}) = g(x,\bar{x}) e^{f^n(\bar{x})T_n}\,, \tag{4.68b}$$

respectively.

The Poisson brackets for the Sugawara energy-momentum tensor (4.18a) follow immediately

$$\{T(x), g_L(y)\} = \delta(x-y) {g_L}'(x)\,, \tag{4.69a}$$

$$\{\overline{T}(\bar{x}), g_R(\bar{y})\} = \delta(\bar{x}-\bar{y}) {g_R}'(\bar{y})\,, \tag{4.69b}$$

and we can identify the energy momentum tensor as the generator of conformal transformations.

$$g(x,\bar{x}) \to e^{\int_0^{2\pi} dz\, \epsilon(z)\{T(z),\,\cdot\,\}} g(x,\bar{x}) = g(\xi(x),\bar{x})\,, \tag{4.70a}$$

$$g(x,\bar{x}) \to e^{\int_0^{2\pi} dz\, \bar{\epsilon}(\bar{z})\{\overline{T}(\bar{z}),\,\cdot\,\}} g(x,\bar{x}) = g(x,\bar{\xi}(\bar{x})) \tag{4.70b}$$

with $\xi(x) = x + \epsilon(x) + \mathcal{O}(\epsilon^2)$.

The calculation of the remaining Poisson brackets between the symmetry generators is straightforward and we obtain the Lie algebra

$$\{J_m(x), J_n(y)\} = \delta(x-y)\epsilon_{mn}{}^l J_l(x) + \frac{k}{2}\delta'(x-y)\eta_{mn}\,, \tag{4.71}$$

$$\{T(x), J_n(y)\} = J_n'(y)\delta(x-y) - J_n(y)\delta'(x-y)\,, \tag{4.72}$$

$$\{T(x), T(y)\} = T'(y)\delta(x-y) - 2T(y)\delta'(x-y)\,. \tag{4.73}$$

The last two equations show that $J(x)$ and $T(x)$ transform as conformal primaries with conformal weight one and two, respectively. Equation (4.71) defines the Kac-Moody algebra of the currents. It has a central term and can be treated as a non-abelian generalization of the free-field algebra (2.62a).

4.5.5 Full WZNW Field

First, note that from the Poisson bracket relations of the chiral fields with λ (see (4.40), (4.52) and (4.54)) follows that for both the hyperbolic and the elliptic monodromy we have

$$\{g_L(x)g_R(\bar{x}), (\lambda_L - \lambda_R)\} = 0\,, \tag{4.74}$$

which indicates that the full WZNW field $g(x,\bar{x}) = g_L(x)g_R(\bar{x})$ is gauge invariant in the extended phase space. This implies that the Poisson brackets of the full field $g(x,\bar{x})$ determined in the extended space are equal to the ones in the physical space $\lambda_L = \lambda_R$. The symmetry generators $J(x)$ and $T(x)$ can be written as functions of the full field and they are, therefore, gauge invariant as well, which also trivially follows from the vanishing Poisson bracket $\{J(x), \lambda\}$.

The relations (4.50), (4.53) or (4.60), (4.61) from above can thus be combined to calculate the Poisson bracket of the full WZNW-fields

$$\{g(x,\bar{x}), \otimes g(y,\bar{y})\} = (g_L(x)\otimes g_L(y))\cdot\{g_R(\bar{x}),\otimes g_R(\bar{y})\} + \{g_L(x),\otimes g_L(y)\}\cdot(g_R(\bar{x})\otimes g_R(\bar{y}))\,. \tag{4.75}$$

For the elliptic monodromy the result written in the original basis is

$$\begin{aligned}\{g(x,\bar{x}), \otimes g(y,\bar{y})\} = -\frac{1}{4k}\Bigg\{&(\epsilon+\bar{\epsilon})\big(g_L(x)T_0g_R(\bar{x})\big)\otimes\big(g_L(y)T_0g_R(\bar{y})\big)\\ &-\frac{\sin(\lambda\epsilon)+\sin(\lambda\bar{\epsilon})}{\sin\lambda}\Big[\big(g_L(x)T_1g_R(\bar{x})\big)\otimes\big(g_L(y)T_1g_R(\bar{y})\big)\\ &\qquad+\big(g_L(x)T_2g_R(\bar{x})\big)\otimes\big(g_L(y)T_2g_R(\bar{y})\big)\Big]\\ &+\frac{\cos(\lambda\epsilon)-\cos(\lambda\bar{\epsilon})}{\sinh\lambda}\Big[\big(g_L(x)T_2g_R(\bar{x})\big)\otimes\big(g_L(y)T_1g_R(\bar{y})\big)\\ &\qquad-\big(g_L(x)T_1g_R(\bar{x})\big)\otimes\big(g_L(y)T_2g_R(\bar{y})\big)\Big]\Bigg\}\,,\end{aligned} \tag{4.76}$$

where $\epsilon \equiv \epsilon(x-y)$ and $\bar{\epsilon} \equiv \epsilon(\bar{x}-\bar{y})$. In the fundamental domain $|x-y| < 2\pi$, $|\bar{x}-\bar{y}| < 2\pi$ we have $\epsilon = \pm 1$, $\bar{\epsilon} = \pm 1$ and the last term vanishes. The remaining terms can be written in a covariant way as

$$\{g(x,\bar{x}), \otimes g(y,\bar{y})\} = -\frac{1}{4k}(\epsilon+\bar{\epsilon})\big[(g_L(x)T_ng_R(\bar{x}))\otimes(g_L(y)T^ng_R(\bar{y}))\big]\,. \tag{4.77}$$

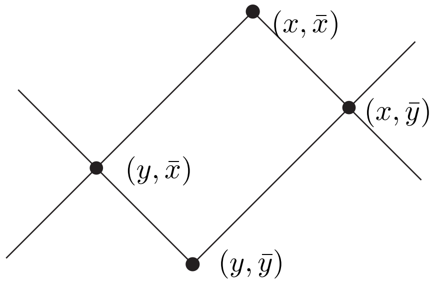

Figure 4.2: Points in space-time that contribute to the Poisson bracket of the full fields $\{g(x,\bar{x}), g(y,\bar{y})\}$ in the fundamental domain

Giving up the tensorial structure we can use the identity (3.48) and rewrite the Poisson bracket (4.77) in terms of the full fields

$$\boxed{\{g_{ab}(x,\bar{x}), g_{cd}(y,\bar{y})\} = \frac{1}{2k}\Theta\big[2g_{ad}(x,\bar{y})g_{cb}(y,\bar{x}) - g_{ab}(x,\bar{x})g_{cd}(y,\bar{y})\big]} \tag{4.78}$$

with

$$\Theta \equiv \frac{1}{2}\left(\epsilon(x-y) + \epsilon(\bar{x}-\bar{y})\right) . \tag{4.79}$$

Note that $\Theta = +1$ if $(x,\bar{x})$ is in the forward light cone of $(y,\bar{y})$, $\Theta = -1$ if it is in the backward light cone and $\Theta = 0$ if the two points are separated by a space-like distance. Hence, the non-equal time Poisson bracket (4.78) is causal. It has a local structure as well, since the Poisson bracket at two differnet space time points is expressed in terms of WZNW-fields at four points: The two original points plus two more, obtained by exchanging the light-cone coordinates (see fig. 4.2). The same result was obtained in [11] for the hyperbolic monodromy. Our calculations show that the Poisson bracket structure is monodromy independent.

4.6 Gauged WZNW Models

Let us now consider gauged $SL(2,\mathbb{R})$ models. The Lagrangian (4.6) is invariant under left and right multiplications

$$g(x,\bar{x}) \to e^{\epsilon_1 A} g(x,\bar{x}) e^{\epsilon_2 A} . \tag{4.80}$$

One can make this global symmetry local by replacing the partial derivatives in the Lagrangian with the covariant ones

$$\partial_\mu g \to D_\mu g = \partial_\mu g - A_\mu \delta g . \tag{4.81}$$

Here, we consider two cases

- axial gauging: $\epsilon_1 = \epsilon_2 = \epsilon(x,\bar{x})$ $\qquad g(x,\bar{x}) \to e^{\epsilon(x,\bar{x})A} g(x,\bar{x}) e^{\epsilon(x,\bar{x})A}$ (4.82)
  $$\delta g = Ag + gA$$
- vector gauging: $\epsilon_1 = -\epsilon_2 = \epsilon(x,\bar{x})$ $\qquad g(x,\bar{x}) \to e^{\epsilon(x,\bar{x})A} g(x,\bar{x}) e^{-\epsilon(x,\bar{x})A}$ (4.83)
  $$\delta g = Ag - gA$$

One can show that integrability of the reduced system is preserved only for these two cases [50]. As we know from Chapter 3 (see eq. (3.19)-(3.21)), the character of the subgroup $e^{\epsilon A}$ depends on whether A is time-like, space-like or light-like. One can expect that the structure of the corresponding gauged Lagrangian also depends on the metric properties of A. We show this explicitly in the next two subsections, considering the cases $A = T_0$, $A = T_2$ and $A = T_+$ as examples for timelike, space-like and light-like generators. To simplify the analysis we use the Lagrangians $\mathcal{L}_{(A,+)}$ and $\mathcal{L}_{(A,-)}$ for the axial and the vector gaugings, respectively.

4.6.1 Axial Gauging

Replacing the partial derivative according to (4.81) and (4.82) we construct the gauged Lagrangian

$$\mathcal{L}^{\rm ax}_{(n)}(g,A) \equiv \mathcal{L}_{(T_n,+)}(g, \partial_x g - A_x(T_n g + gT_n), \partial_{\bar{x}} - A_{\bar{x}}(T_n g + gT_n))\,. \qquad (4.84)$$

The equations of motion for the gauge field obtained from (4.84) are algebraic, since there is no kinetic term containing derivatives of $(A_x, A_{\bar{x}})$. From these equations we get

$$A_x = \frac{\langle T_n \partial_x g g^{-1}\rangle}{\eta_{nn} + \Lambda_{nn}}\,, \qquad A_{\bar{x}} = \frac{\langle T_n g^{-1}\partial_{\bar{x}} g\rangle}{\eta_{nn} + \Lambda_{nn}}\,, \qquad (4.85)$$

Substituting this solution back into (4.84) we obtain the gauge invariant Lagrangian expressed in terms of the g-field

$$\mathcal{L}^{\rm ax}_{(n)}(g) = -k\Big[\langle g^{-1}\partial_x g g^{-1}\partial_{\bar{x}} g\rangle - \frac{\langle T_n\partial_x g g^{-1}\rangle\langle T_n g^{-1}\partial_{\bar{x}}g\rangle + \langle T_n g^{-1}\partial_x g\rangle\langle T_n \partial_{\bar{x}} g g^{-1}\rangle}{\Lambda_{nn} + \eta_{nn}}\Big]\,. \qquad (4.86)$$

As for the particle models, the Lagrangian (4.86) depends only on the gauge invariant components of $g(x,\bar{x})$. To eliminate the gauge dependent variables we use the parameterization (3.4), which leads to

$$\mathcal{L}^{\rm ax}_{(n)}(u) = -k\Big[\partial_x u^i \partial_{\bar{x}} u_i + \frac{1}{c^2} u^i\partial_x u_i u^j \partial_{\bar{x}} u_j - \frac{1}{\eta_{nn} + (u_n)^2} \times \big(c^2 \partial_x u_n \partial_{\bar{x}} u_n + u_n\partial_x u_n u^i \partial_{\bar{x}} u_i + u_n \partial_{\bar{x}} u_n u^i \partial_x u_i + \frac{(u_n)^2}{c^2} u^i \partial_x u_i u^j \partial_{\bar{x}} u_j - \epsilon_{ijn}\epsilon_{lmn} u^i u^l \partial_x u^j \partial_{\bar{x}} u^m\big)\Big]\,. \qquad (4.87)$$

Now we consider separately the cases $n = 0$, $n = 2$ and $n = +$. The first case corresponds to the compact subgroup generated by T_0. The gauge invariant variables then are u_1 and u_2. Setting $n = 0$ in (4.87) we find that the dependence on u_0 is indeed canceled and we arrive at

$$\mathcal{L}^{\rm ax}_{(0)} = k\frac{\partial_x u_1 \partial_{\bar{x}} u_1 + \partial_x u_2 \partial_{\bar{x}} u_2}{1 + (u_1)^2 + (u_2)^2}\,. \qquad (4.88)$$

This system is known as the $SL(2,\mathbb{R})/U(1)$ black hole model [16], which we will discuss in detail in chapters 6 and 8.

The case $n = 2$ corresponds to the gauging with respect to the non-compact subgroup generated by T_2. Here, the gauge invariant components are u_0 and u_1, and (4.87) provides the Lagrangian of the 2d Minkowskian black hole model [16]

$$\mathcal{L}^{\rm ax}_{(2)} = k\frac{\partial_x u_1 \partial_{\bar{x}} u_1 - \partial_x u_0 \partial_{\bar{x}} u_0}{1 + (u_1)^2 - (u_0)^2}\,. \qquad (4.89)$$

Finally, setting $n=+$ in (4.87) leads to a Lagrangian with only one field

$$\mathcal{L}^{\mathrm{ax}}_{(+)} = \frac{\partial_x u_+ \partial_{\bar{x}} u_+}{(u_+)^2}\,. \tag{4.90}$$

This Lagrangian is singular at $u_+ = 0$ and the solutions decompose into positive and negative fields. Choosing the positive sector, with $u_+ = e^{-\varphi}$, we arrive at

$$\mathcal{L}^{\mathrm{ax}}_{(+)} = k\partial_x\varphi\partial_{\bar{x}}\varphi\,, \tag{4.91}$$

which is just the free-field Lagrangian discussed in chapter 2.

4.6.2 Vector Gauging

Similarly to the axial gauging we construct the new Lagrangian from (4.6) with covariant derivatives in place of the partial derivatives using (4.81) and (4.83)

$$\mathcal{L}^{\mathrm{vec}}_n(g,A) \equiv \mathcal{L}_{(T_n,-)}(g, \partial_x g - A_x(T_n g - gT_n), \partial_{\bar{x}} g - A_{\bar{x}}(T_n g - gT_n))\,. \tag{4.92}$$

The equations of motion for the gauge field now yield

$$A_x = \frac{\langle T_n \partial_x g g^{-1}\rangle}{\eta_{nn} - \Lambda_{nn}}\,, \qquad A_{\bar{x}} = \frac{\langle T_n g^{-1}\partial_{\bar{x}} g\rangle}{\Lambda_{nn} - \eta_{nn}}\,, \tag{4.93}$$

and its insertion back into (4.92) gives

$$\begin{aligned}\mathcal{L}^{\mathrm{vec}}_n(g) = -k\Big[&\langle g^{-1}\partial_x g g^{-1}\partial_{\bar{x}} g\rangle \\ &- \frac{1}{\Lambda_{nn} - \eta_{nn}}\left(\langle T_n\partial_x g g^{-1}\rangle\langle T_n g^{-1}\partial_{\bar{x}} g\rangle + \langle T_n g^{-1}\partial_x g\rangle\langle T_n \partial_{\bar{x}} g g^{-1}\rangle\right)\Big]\,. \end{aligned}\tag{4.94}$$

Already at this stage one can see that for the nilpotent case $n=+$ this Lagrangian is identical to (4.86) because $\eta_{++} = 0$. The reduced Lagrangian is, therefore, given by (4.91).

The Lagrangian (4.94) expressed in terms of u^n coordinates reads

$$\begin{aligned}\mathcal{L}^{\mathrm{vec}}_n(u^n) = -k\Big[&\partial_x u^i \partial_{\bar{x}} u_i + \partial_x c\partial_{\bar{x}} c - \frac{1}{u_n^2 - \eta_{nn}u^k u_k} \\ &\times\left(c^2\partial_x u_n\partial_{\bar{x}} u_n - u_n\partial_x u_n c\partial_{\bar{x}} c - u_n\partial_{\bar{x}} u_n c\partial_x c\right. \\ &\left.+ u_n^2\partial_x c\partial_{\bar{x}} c - \epsilon_{ijn}\epsilon_{lmn}u^i u^l \partial_x u^j\partial_{\bar{x}} u^m\right)\Big]\,. \end{aligned}\tag{4.95}$$

For $n=0$ the gauge invariant components are u_0, c, and we indeed find

$$\mathcal{L}^{\mathrm{vec}}_{(0)} = k\frac{\partial_x u_0\partial_{\bar{x}} u_0 + \partial_x c\partial_{\bar{x}} c}{(u_0)^2 + c^2 - 1}\,. \tag{4.96}$$

This can be interpreted as a string on the infinite trumpet [17]. Note that this Lagrangian becomes singular for $(u_0)^2 + c^2 = 1$.

Setting $n=2$ in (4.95) yields the Lagrangian

$$\mathcal{L}^{\mathrm{vec}}_{(2)} = k\frac{\partial_x u_2\partial_{\bar{x}} u_2 - \partial_x c\partial_{\bar{x}} c}{1 + (u_2)^2 - c^2}\,, \tag{4.97}$$

which is the same as (4.89), only in terms of different components of $g(x, \bar{x})$.

As we have seen in chapter 3, a gauging of the particle dynamics in the Lagrangian formulation is equivalent to imposing constraints in the Hamiltonian approach. Of course, this general statement is also valid for the gauged WZNW models. In the next chapter we consider constraints imposed on the light-like components J_+ and $\bar{J}_+$ of the Kac-Moody currents, which similar to particle model of subsection 3.5.5 can be regarded as a generalization of the nilpotent gauging. In chapter 6 we show that the $SL(2,R)/U(1)$ model is obtained by imposing constraints on the J_0 and $\bar{J}_0$ components of the Kac-Moody currents.

The reduction of the WZNW theory to its coset models was first studied in [12, 13, 14].

V

Liouville Theory

In this chapter we review the classical Liouville Theory as it arises by Hamiltonian reduction from the $SL(2,\mathbb{R})$ WZNW theory.

Liouville theory describes 2d scalar field dynamics with exponential self interaction. The Lagrangian is usually given by

$$\mathcal{L} = \frac{1}{2}\left((\partial_\tau\varphi(\tau,\sigma))^2 - (\partial_\sigma\varphi(\tau,\sigma))^2\right) - \frac{4m^2}{\gamma^2}e^{2\gamma\varphi(\tau,\sigma)}\,, \tag{5.1}$$

where m and γ are positive coupling constants. We consider the periodic case $(\tau,\sigma)\in\mathbb{R}\times S^1$. Using the light-cone coordinates (2.32), the general solution of the equation of motion

$$\partial^2_{x\bar{x}}\varphi(x,\bar{x}) + \frac{m^2}{\gamma}e^{2\gamma\varphi(x,\bar{x})} = 0 \tag{5.2}$$

can be written as

$$e^{2\gamma\varphi(x,\bar{x})} = \frac{A'(x)\bar{A}'(\bar{x})}{m^2(1+A(x)\bar{A}(\bar{x}))^2}\,. \tag{5.3}$$

This form of the general solution was found by Liouville in 1853 [51]. Later it was realized that the Liouville equation (5.2) is invariant under $2d$ conformal transformations, which is the underlying symmetry for this compact form of the general solution.

Here we reproduce (5.3) from the general solution of the $SL(2,\mathbb{R})$ WZNW theory by Hamiltonian reduction. We follow the reduction scheme described in subsection 2.2. Namely, we calculate the reduced symplectic form and find its canonical coordinates. As it turns out these coordinates are the Fourier modes of the asymptotic in-field. The general solution (5.3) is then realized as a canonical map from the in-field to the interacting Liouville field φ. We also discuss the conformal and Weyl symmetries of Liouville theory.

A new point for this chapter is the analysis of the Kac-Moody and conformal symmetries of the constrained surface. It is shown that a special linear combination of the symmetry generators leave the constrained surface invariant. The reduction of the corresponding generator on the space of gauge orbits coincides with the energy-momentum tensor of Liouville theory. This observation gives a new insight to the improved term of the energy-momentum tensor.

5.1 Hamiltonian Reduction

Let us introduce the light-like components of the Kac-Moody currents (4.22)

$$J_\pm(x) \equiv J_0(x) \pm J_1(x)\,, \tag{5.4}$$

and, like for the mechanical model (3.79), impose the constraints

$$J_+(x) + \rho = 0\,, \qquad\qquad \bar{J}_+(x) - \rho = 0\,, \tag{5.5}$$

where ρ is a non-zero constant. In order to apply the reduction scheme, one has to parameterize the constrained surface (5.5).

5.1.1 The Chiral Part

We first consider the chiral part and, as before, suppress the index L whenever possible. The Kac-Moody current (4.21) constrained by (5.5) can be written as

$$J(x) = \begin{pmatrix} -J_2(x) & \rho \\ J_-(x) & J_2(x) \end{pmatrix}, \tag{5.6}$$

and from the definition $J(x) = kg'(x)g^{-1}(x)$ we find the following parameterization of the chiral WZNW field

$$g(x) = \begin{pmatrix} \psi(x) & \chi(x) \\ \frac{1}{m}\psi'(x) + \frac{1}{\rho}J_2(x)\psi(x) & \frac{1}{m}\chi(x)' + \frac{1}{\rho}J_2(x)\chi(x) \end{pmatrix}, \tag{5.7}$$

where $m \equiv \frac{\rho}{k}$ and, since $\det(g) = 1$, the components $\psi(x)$ and $\chi(x)$ are related by

$$\psi(x)\chi'(x) - \psi'(x)\chi(x) = m\,. \tag{5.8}$$

Here, we consider only the hyperbolic monodromy (4.16a). In this case the monodromies of the chiral fields $\psi(x)$ and $\chi(x)$ are

$$\psi(x + 2\pi) = \psi(x)e^{\lambda}\,, \qquad \chi(x + 2\pi) = \chi(x)e^{-\lambda}\,. \tag{5.9}$$

In addition, we assume $\psi(x) > 0$ and $\lambda < 0$. Equation (5.8) then can be integrated uniquely in the form

$$\chi(x) = \psi(x)A(x)\,, \tag{5.10}$$

where the so-called screening charge $A(x)$ is obtained from

$$A'(x) = \frac{m}{\psi^2(x)} \tag{5.11}$$

as

$$A(x) = \int_0^{2\pi} dz\; \theta_{-2\lambda}(x - z)\frac{m}{\psi^2(z)}\,, \tag{5.12}$$

with the Green's function (4.46). Note that the conditions $\psi(x) > 0$ and $\lambda < 0$ provide regularity of the screening charge (5.12), which is apparently positive, and therefore we have $\chi(x) > 0$ as well. Later we will discuss the physical meaning of the condition $\lambda > 0$.

From (4.71), (4.72) and (4.33) follows

$$\{J_+(x), J_+(y)\}\big|_{J_+=-\rho} = 0\,, \qquad \{J_+(x), H\}\big|_{J_+=-\rho} = 0\,. \tag{5.13}$$

Thus, (5.5) is a first class constraint and it generates gauge transformations. It is important to note that $\psi(x)$ and $\chi(x)$ are gauge invariant due to (4.67a). A non gauge invariant part of the chiral field $g_L(x)$ (5.7) is given by $J_2(x)$, which due to (4.71) has the following Poisson brackets with the constraints (5.5)

$$\{J_2(x), J_+(y)\}\big|_{J_+=-\rho} = 2\rho\delta(x - y)\,. \tag{5.14}$$

The constraints (5.5) and $J_2(x) = 0$ together form second class constraints, and $J_2(x) = 0$ can be used as a gauge fixing condition. The parameterization (5.7) then simplifies to

$$g(x) = \begin{pmatrix} \psi(x) & \chi(x) \\ \frac{1}{m}\psi'(x) & \frac{1}{m}\chi'(x) \end{pmatrix} . \tag{5.15}$$

Now the calculation of the reduced symplectic form (4.32a) is a straightforward procedure, and with the help of (5.8) we arrive at

$$\tilde{\omega} = k \int_0^{2\pi} \frac{d\psi'(x)}{\psi(x)} \wedge \frac{d\psi(x)}{\psi(x)} dx + k d\lambda \wedge \frac{d\psi(0)}{\psi(0)} . \tag{5.16}$$

Of course, the same symplectic form is obtained from the parameterization (5.7) without gauge fixing $J_2(x) = 0$, but that calculation is rather lengthy.

Since $\psi(x) > 0$, we can parameterize it by

$$\psi(x) = e^{-\gamma\phi(x)} , \tag{5.17}$$

where $\gamma \equiv k^{-\frac{1}{2}}$ and $\phi(x)$ has the free-field monodromy (2.40) with $p \equiv -2\lambda\gamma^{-1}$. The symplectic form (5.16) then takes the free-field form (2.39).

From the Wronskian condition (5.8) and the parameterization (5.15) we find

$$T(x) = \frac{-1}{\gamma^2}\langle g^{-1}(x)g'(x)g^{-1}(x)g'(x)\rangle = \frac{1}{\gamma^2}\frac{\psi''(x)}{\psi(x)} = \frac{1}{\gamma^2}\frac{\chi''(x)}{\chi(x)} , \tag{5.18}$$

where $T(x)$ is the reduced energy-momentum tensor of the $SL(2,\mathbb{R})$ WZNW theory (4.18a). Therefore the reduced Hamiltonian also becomes free in terms of the ϕ-field

$$H_L = \frac{1}{\gamma^2}\int_0^{2\pi} dx \, \frac{\psi''(x)}{\psi(x)} = \int_0^{2\pi} dx \, \phi'^2(x) . \tag{5.19}$$

In this way we obtain a free-field parameterization of the reduced system, and using the mode expansion (2.35) one can pass to the canonical coordinates (p, q, a_n) with the Poisson bracket relations (2.43).

5.1.2 The Antichiral Part

The reduction of the antichiral part is similar. The constrained Kac-Moody current now is

$$\bar{J}(\bar{x}) = \begin{pmatrix} -\bar{J}_2(\bar{x}) & -\rho \\ \bar{J}_-(\bar{x}) & \bar{J}_2(\bar{x}) \end{pmatrix} , \tag{5.20}$$

and together with $\bar{J}(\bar{x}) = k g_R^{-1}(\bar{x}) g_R'(\bar{x})$ we find the parameterization

$$g_R(\bar{x}) = \begin{pmatrix} -m^{-1}\bar{\psi}'(\bar{x}) + \rho^{-1}\bar{J}_2(\bar{x})\bar{\psi}(\bar{x}) & \bar{\psi}(\bar{x}) \\ -m^{-1}\bar{\chi}'(\bar{x}) + \rho^{-1}\bar{J}_2(\bar{x})\bar{\chi}(\bar{x}) & \bar{\chi}(\bar{x}) \end{pmatrix} . \tag{5.21}$$

The gauge invariant antichiral functions $\bar{\psi}(\bar{x})$, $\bar{\chi}(\bar{x})$ are related by a similar Wronskian condition

$$\bar{\psi}(\bar{x})\bar{\chi}'(\bar{x}) - \bar{\psi}'(\bar{x})\bar{\chi}(\bar{x}) = m \tag{5.22}$$

and have the monodromies

$$\bar{\psi}(x + 2\pi) = \bar{\psi}(x)e^{\lambda_R} , \qquad \bar{\chi}(x + 2\pi) = \bar{\chi}(x)e^{-\lambda_R} . \tag{5.23}$$

The reduced symplectic form and the Hamiltonian again take the free-field form, but now in terms of the antichiral function $\bar{\phi}(\bar{x})$ defined by

$$\bar{\psi}(\bar{x}) = e^{-\gamma\bar{\phi}(\bar{x})}\,. \tag{5.24}$$

To simplify the calculation of the reduced symplectic form here one can choose the gauge fixing condition $\bar{J}_2(x) = 0$. As in the chiral case we assume $\bar{\psi}(\bar{x}) > 0$, which for $\lambda < 0$ provides $\bar{A}(\bar{x}) > 0$ and $\bar{\chi}(\bar{x}) > 0$.

5.1.3 The Liouville Field

The full $SL(2,\mathbb{R})$ WZNW theory is obtained by setting $\lambda_R = \lambda_L = \lambda$. Then, the reduced symplectic form of the system corresponds to the canonical 2-form of the periodic free-field theory (2.50). The momentum zero mode p becomes canonically conjugated to $q = q_L + q_R$, where q_L and q_R are the coordinate zero modes of $\phi(x)$ and $\bar{\phi}(\bar{x})$, respectively.

According to the parameterizations (5.7) and (5.21) the full WZNW-field $g(x,\bar{x}) = g_L(x)g_R(\bar{x})$ has only one gauge invariant component $V(x,\bar{x}) \equiv g_{12}(x,\bar{x})$, which is a physical field of the system. Due to our assumptions this field is positive and one can introduce its exponential parameterization

$$V(x,\bar{x}) = \psi(x)\bar{\psi}(\bar{x}) + \chi(x)\bar{\chi}(\bar{x}) = e^{-\gamma\varphi(x,\bar{x})}\,. \tag{5.25}$$

Using the Wronskian conditions (5.8) and (5.22), it is easy to check that the field $\varphi(x,\bar{x})$ satisfies the Liouville equation (5.2). Expressing ψ, $\bar{\psi}$, χ and $\bar{\chi}$ in terms of the screening charges via (5.10) and (5.11) and their antichiral counterparts, we recover from (5.25) the Liouville field just in the form of the general solution (5.3). The parameterizing functions $A(x)$, $\bar{A}(\bar{x})$ are monotonic and have the monodromy

$$A(x+2\pi) = A(x)e^{\gamma p}\,, \qquad\qquad \bar{A}(\bar{x}+2\pi) = \bar{A}(\bar{x})e^{\gamma p}\,, \tag{5.26}$$

with $p > 0$. One can show that this class of $A(x)$, $\bar{A}(\bar{x})$ covers all regular periodic Liouville fields [48, 28].

From (5.25) we also find a free-field parameterization of Liouville theory

$$e^{-\gamma\varphi(x,\bar{x})} = e^{-\gamma(\phi(x)+\bar{\phi}(\bar{x}))}(1 + A(x)\bar{A}(\bar{x}))\,, \tag{5.27}$$

where the product of screening charges is given by

$$A(x)\bar{A}(\bar{x}) = \frac{m^2 e^{-\gamma p}}{4\sinh^2\left(\frac{\gamma p}{2}\right)} \int_0^{2\pi} dz \int_0^{2\pi} d\bar{z}\; e^{2\gamma(\phi(z+x)+\bar{\phi}(\bar{x}+\bar{z})}\,. \tag{5.28}$$

Here, we have used the periodicity of the integrand in (5.12) and shifted the arguments $z \to z + x$ ($\bar{z} \to \bar{z} + \bar{x}$) without shifting the integration domain.

Since $p > 0$, we get the following behavior of the Liouville field exponential at the time asymptotics:

$$e^{-\gamma\varphi(x,\bar{x})} \overset{\tau\to-\infty}{\sim} e^{-\gamma(\phi(x)+\bar{\phi}(\bar{x}))}\,, \tag{5.29a}$$

$$e^{-\gamma\varphi(x,\bar{x})} \overset{\tau\to+\infty}{\sim} A(x)\bar{A}(\bar{x})e^{-\gamma(\phi(x)+\bar{\phi}(\bar{x}))}\,. \tag{5.29b}$$

Hence, the field $\Phi(\tau,\sigma) \equiv \phi(x) + \bar{\phi}(\bar{x})$ can be interpreted as the in-field of Liouville theory, while the out-field is

$$\Phi_{\text{out}}(\tau,\sigma) = \Phi(\tau,\sigma) - \frac{1}{\gamma}\log(A(x)\bar{A}(\bar{x}))\,, \tag{5.30}$$

and we get the relation

$$e^{-\gamma\varphi(x,\bar{x})} = e^{-\gamma\Phi_{\text{in}}(\tau,\sigma)} + e^{-\gamma\Phi_{\text{out}}(\tau,\sigma)}\,. \tag{5.31}$$

This structure of the Liouville field exponential can be used for the construction of the S-matrix, an issue we discuss in chapter 7.

The parameterizing in-field $\Phi(\tau,\sigma)$ can be expanded in Fourier modes

$$\Phi(\tau,\sigma) = q + \frac{p}{2\pi}\tau + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{a_n}{n}e^{-inx} + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{\bar{a}_n}{n}e^{-in\bar{x}}\,, \tag{5.32}$$

which become canonical coordinates with the Poisson bracket relations (2.52) of the periodic free-field theory. The difference to the standard case discussed in subsection 2.4.2 is that the zero mode sector of the Liouville in-field is given on the half plane $p > 0$.

For vanishing non-zero modes ($a_n = \bar{a}_n = 0$) the Liouville field is σ independent

$$e^{-\gamma\varphi}\big|_{a_n=\bar{a}_n=0} = e^{-\gamma(q+\frac{p\tau}{2\pi})} + \frac{4\pi^2m^2}{\gamma^2p^2}e^{\gamma(q+\frac{p\tau}{2\pi})}\,, \tag{5.33}$$

and it describes particle dynamics in the exponential potential considered in section 3.5.5. The zero mode p becomes the momentum of the incoming particle, which can be only positive, while the outgoing particle has a negative momentum. Thus, we see that the choice $\lambda < 0$ ($p > 0$) corresponds to the parameterization of Liouville theory in terms of the in-field. The change of sign of λ is equivalent to the time reversal, i.e. to the out-field parameterization. In general, the analysis of the mechanical model becomes quite helpful to understand the peculiarities of the Liouville field dynamics both on the classical and quantum level [52, 53].

5.2 Poisson Brackets

Having established the canonical coordinates on the space of solutions we can now calculate the Poisson bracket algebra of the chiral fields $\psi(x)$, $A(x)$, $\chi(x)$, which are the building blocks of the Liouville fields. For the exponential $\psi(x) = e^{-\gamma\phi(x)}$ one immediately finds

$$\{\psi(x),\psi(y)\} = \frac{\gamma^2}{4}\epsilon(x-y)\psi(x)\psi(y)\,. \tag{5.34}$$

The calculation of other Poisson brackets $\{\psi(x),A(y)\},\{\psi(x),\chi(y)\}$ is also straightforward, but to find their algebraic form one has to apply special identities satisfied by the stair-step function (see [11, 54]). Here, we simplify the procedure and use the Poisson bracket structure of the $SL(2,\mathbb{R})$ WZNW theory. Taking into account that $\psi(x)$ and $\chi(x)$ are gauge invariant components of the chiral WZNW-field ($\psi(x) = (g_L)_{11}(x)$, $\chi(x) = (g_L)_{12}(y)$), their Poisson bracket algebra is undeformed under the reduction. Since we are in the hyperbolic sector we apply (4.50) for the components g_{11} and g_{22}, and find (5.34) together with

$$\{\chi(x), \chi(y)\} = \frac{\gamma^2}{4}\epsilon(x-y)\chi(x)\chi(y)\,, \tag{5.35}$$

$$\{\psi(x), \chi(y)\} = \gamma^2\theta_{-\gamma p}(x-y)\chi(x)\psi(y) - \frac{\gamma^2}{4}\epsilon(x-y)\psi(x)\chi(y)\,. \tag{5.36}$$

From the definition of $A(x) = \frac{\chi(x)}{\psi(x)}$ we then get

$$\{\psi(x), A(y)\} = \gamma^2\left(\theta_{\gamma p}(x-y)\psi(x)A(x) - \frac{1}{2}\epsilon(x-y)\psi(x)A(y)\right) \tag{5.37}$$

and

$$\{A(x), A(y)\} = \gamma^2\left(\epsilon(x-y)A(x)A(y) - \theta_{-\gamma p}(x-y)A^2(x) - \theta_{\gamma p}(x-y)A^2(y)\right)\,. \tag{5.38}$$

The antichiral fields $\bar\psi(\bar x)$ and $\bar A(\bar x)$, $\bar\chi(\bar x)$ have the same Poisson bracket relations. The algebra of the building blocks of the general solution (5.25) provides the following non-equal time Poisson bracket for the Liouville field exponential $V(x,\bar x) = e^{-\gamma\varphi(x,\bar x)}$

$$\begin{aligned}\{V(x,\bar x),V(y,\bar y)\} = \gamma^2\Big[&\frac{1}{4}(\epsilon(x-y)+\epsilon(\bar x-\bar y))\Big(\psi(x)\bar\psi(\bar x)\psi(y)\bar\psi(\bar y)\\ &+\chi(x)\bar\chi(\bar x)\chi(y)\bar\chi(\bar y) - \psi(x)\bar\psi(\bar x)\chi(y)\bar\chi(\bar y) - \chi(x)\bar\chi(\bar x)\psi(y)\bar\psi(\bar y)\Big)\\ &+(\theta_{2\lambda}(x-y)+\theta_{-2\lambda}(\bar x-\bar y))\psi(y)\bar\psi(\bar x)\chi(x)\bar\chi(\bar y)\\ &+(\theta_{-2\lambda}(x-y)+\theta_{2\lambda}(\bar x-\bar y))\psi(x)\bar\psi(\bar y)\chi(y)\bar\chi(\bar x)\Big]\,.\end{aligned} \tag{5.39}$$

Then, similarly to the $SL(2,\mathbb{R})$ WZNW model, in the 'fundamental' domain, we obtain

$$\boxed{\{V(x,\bar x), V(y,\bar y)\} = \frac{\gamma^2}{2}\Theta\left[2V(x,\bar y)V(y,\bar x) - V(x,\bar x)V(y,\bar y)\right]}\,. \tag{5.40}$$

Note that these results one can also read off from (4.78), because $V(x,\bar x)$ coincides with the gauge invariant component $g_{12}(x,\bar x)$ of the WZNW-field.

5.3 Symmetries

In this section we analyze the symmetries of the reduced system. Let us consider the energy-momentum tensor $T(x)$ of the $SL(2,\mathbb{R})$ WZNW theory. By (4.72) one has

$$\{T(x), J_+(y)\}\big|_{J_+=-\rho} = \rho\,\delta'(x-y)\,, \tag{5.41}$$

and therefore $T(x)$ is not gauge invariant. In the gauge $J_2 = 0$ the energy-momentum tensor is given by

$$T(x) = \frac{1}{\gamma^2}\frac{\psi''(x)}{\psi(x)} = \phi'^2(x) - \frac{1}{\gamma}\phi''(x)\,. \tag{5.42}$$

Its Poisson bracket with a free-field exponential takes the form

$$\{T(x), e^{2\alpha\gamma\phi(y)}\} = \left(e^{2\alpha\gamma\phi(x)}\right)' \delta(x-y) - \alpha \left(e^{2\alpha\gamma\phi(y)}\right) \delta'(x-y)\,, \tag{5.43}$$

which is the infinitesimal version of a conformal transformation with conformal weight $\Delta_\alpha = \alpha$

$$e^{2\alpha\gamma\phi(x)} \to e^{\int_0^{2\pi} dz\, \epsilon(z)\{T(z),\cdot\}} e^{2\alpha\gamma\phi(x)} = \left(\xi'(x)\right)^\alpha e^{2\alpha\gamma\phi(\xi(x))}\,, \tag{5.44}$$

where $\xi(x) \approx x + \epsilon(x)$. The integrand of the screening charge (5.12) has conformal weight one and the conformal factor $\xi'(z)$ becomes the Jacobian of a transformation $z \to \xi(z)$. Since furthermore $\epsilon(x-z) = \epsilon(\xi(x) - \xi(z))$, the screening charge transforms with zero conformal weight

$$A(x) \to A(\xi(x))\,. \tag{5.45}$$

From the Poisson brackets (5.43) and (5.45) we see that the chiral fields $\psi(x)$ and $\chi(x)$ have the same conformal weight $\Delta_{-\frac{1}{2}} = -\frac{1}{2}$.

Introducing $\bar{T}(\bar{x})$ in a similar way, we find that the Liouville field transforms as

$$\varphi(x,\bar{x}) \to \varphi(\xi(x), \bar{\xi}(\bar{x})) + \frac{1}{2}\log(\xi'(x)) + \frac{1}{2}\log(\bar{\xi}'(\bar{x}))\,. \tag{5.46}$$

It is easy to check that the transformed field still satisfies the Liouville equation (5.2). Thus, the reduced energy-momentum tensor (5.42) is the generator of the conformal transformation (5.46), which is a symmetry of Liouville theory. Therefore, (5.42) is identified with the energy-momentum tensor of Liouville theory [55, 56].

The term proportional to $\phi''(x)$ in (5.42) is responsible for the creation of conformal weights and it is called the improved term. To find its origin, let us analyze the symmetries of the constrained surface $J_+ + \rho = 0$ under the action of the conformal and Kac-Moody transformation of WZNW theory. Considering a linear combination of the symmetry generators one has to find solutions to the equation

$$\int_0^{2\pi} dx\, \{\epsilon(x)T(x) + \epsilon^n(x)J_n(x), J_+(y)\}\big|_{J_+=-\rho} = 0\,. \tag{5.47}$$

Using the Poisson brackets (4.72) and (4.71) we get

$$\rho(\epsilon'(y) + \epsilon_2(y)) + (\epsilon_0(y) + \epsilon_1(y))J_2(y) + \frac{k}{2}(\epsilon_0'(y) + \epsilon_1'(y)) = 0\,. \tag{5.48}$$

In order to have the parameters gauge independent we set $\epsilon_0(x) + \epsilon_1(x) \equiv 0$ and immediately get $\epsilon_2(x) = -\epsilon'(x)$, which corresponds to the gauge invariant symmetry generator

$$\tilde{T}(x) = T(x) - J_2'(x) = \frac{1}{\gamma^2}\frac{\psi''(x)}{\psi(x)}\,. \tag{5.49}$$

It generates the following infinitesimal transformation of the WZNW-field

$$\delta g(x,\bar{x}) = \epsilon(x)\partial_x g(x,\bar{x}) - \frac{1}{2}\epsilon'(x)(T_2 g(x,\bar{x}))\,, \tag{5.50}$$

which after reduction on the constrained surface provides conformal weight of gauge invariant components of the g-field.

Another symmetry transformation of the space of solutions is a periodic translation of $\phi(x)$, generated by $\phi'(x)$

$$\{\phi'(x), e^{2\alpha\gamma\phi(y)}\} = \alpha\gamma\delta(x-y)e^{2\alpha\gamma\phi(y)}\,. \tag{5.51}$$

This corresponds to a Weyl transformation of the free-field exponential

$$e^{\gamma\phi(x)} \to e^{\int_0^{2\pi} dz\, f(x)\{\phi'(x),\,\cdot\,\}} e^{\gamma\phi(x)} = e^{\gamma\phi(x)} e^{f(x)}\,. \tag{5.52}$$

The Liouville energy-momentum tensor $T(x)$ given by (5.42) and $\phi'(x)$ form the following Lie algebra, similar to (2.62a)-(2.62c) but with central terms

$$\{\phi'(x), \phi'(y)\} = -\frac{1}{2}\delta'(x-y)\,, \tag{5.53a}$$

$$\{T(x), \phi'(y)\} = \phi''(y)\delta(x-y) - \phi'(y)\delta'(x-y) + \frac{1}{2\gamma}\delta''(x-y)\,, \tag{5.53b}$$

$$\{T(x), T(y)\} = T'(y)\delta(x-y) - 2T(y)\delta'(x-y) + \frac{1}{2\gamma^2}\delta'''(x-y)\,. \tag{5.53c}$$

A consistent quantum theory has to provide realization of this algebra in a Hilbert space.

VI

The $SL(2,\mathbb{R})/U(1)$ Model

Chapter 6 reviews the classical theory of the $SL(2,\mathbb{R})/U(1)$ model. We show that the gauged model can be treated as a constrained WZNW theory. Constraints are imposed on the space of solutions, which in the Hamiltonian formulation corresponds to the constrained $U(1)$ Kac-Moody currents. This picture enables us to extract the general solution of the model and to carry out the Hamiltonian reduction, as well.

A new result here is the calculation of the reduced symplectic form with a detailed analysis of the zero mode sector. Then we describe the space of solutions and calculate the Poisson bracket structure there. An important new point is the analysis of the elliptic sector. We show that the solutions of this sector can be interpreted as bound states, while the hyperbolic monodromy describes only scattering processes. We relate the two sectors by an analytical continuation and discuss the role of the winding number, which is an important characteristic of the solutions.

Another new result for this chapter is the calculation of the non-equal time Poisson bracket in the elliptic sector. Similarly to the $SL(2,\mathbb{R})$ WZNW model, this calculation indicates the monodromy independence of the Poisson bracket structure for the full interacting field.

6.1 Lagrangian Description

As it was shown in section 4.6 the axial gauging of the $SL(2,\mathbb{R})$ WZNW theory with respect to the compact subgroup generated by T_0 leads to the Lagrangian (see (4.88))

$$\mathcal{L} = \frac{k}{2}\,\frac{\partial_x u\partial_{\bar{x}} u^* + \partial_x u^*\partial_{\bar{x}} u}{1+|u|^2}\,. \tag{6.1}$$

where u is a complex valued field $u \equiv u_1 + iu_2$, $u^* \equiv u_1 - iu_2$. The equations of motion for this gauged system are

$$\partial^2_{x\bar{x}} u = u^*\frac{\partial_x u\partial_{\bar{x}} u}{1+|u|^2} \tag{6.2}$$

and its complex conjugate. This model is obviously conformal invariant, but in contrast to Liouville theory, the conformal symmetry is not sufficient for the complete integrability of these equations. To find the general solution we apply a reduction scheme to the space of solutions (4.10) of the $SL(2,\mathbb{R})$ WZNW theory. Before discussing this procedure we first describe a geometric interpretation of the model.

6.1.1 Embedding into Euclidean Space

The Lagrangian (6.1) provides a sigma model with the target space metric

$$ds^2 = \frac{du\,du^*}{1+|u|^2}\,. \tag{6.3}$$

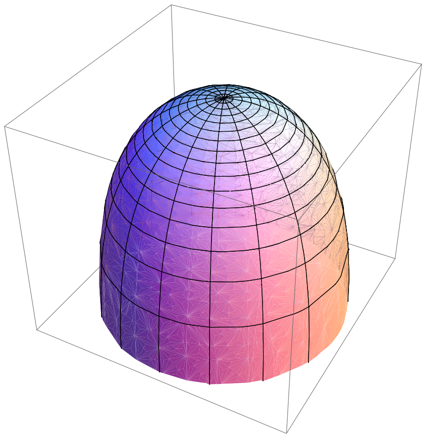

Figure 6.1: Tip of the “infinite cigar”, the target space of the $SL(2,\mathbb{R})/U(1)$ model embedded in Euclidean space

If one parameterizes u in terms of polar coordinates r and θ by

$$u = \frac{r}{\sqrt{1-r^2}}\, e^{i\theta}\,, \qquad r < 1\,, \tag{6.4}$$

then the line element (6.3) becomes

$$ds^2 = \frac{dr^2}{(1-r^2)^2} + r^2 d\theta^2\,. \tag{6.5}$$

This can be interpreted as the induced metric on a rotational surface in $\mathbb{R}^3$ given by

$$x = r\cos(\theta)\,, \qquad y = r\sin(\theta)\,, \qquad z = Z(r)\,, \tag{6.6}$$

where the function $Z(r)$ satisfies the equation

$$1 + Z'^2(r) = \frac{1}{(1-r^2)^2}\,. \tag{6.7}$$

The solution of this equation with the choice $Z'(r) < 0$ and $Z(0) = 0$ is given by

$$Z(r) = \sqrt{2-r^2} - \frac{1}{2}\log\left(\frac{\sqrt{2-r^2}+1}{\sqrt{2-r^2}-1}\right) + \frac{1}{2}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - \sqrt{2}\,. \tag{6.8}$$

The resulting surface is called the “infinite cigar” (see fig. 6.1). For $r \to 1$ (large negative z) it takes the form of a cylinder and at $r = 0$ ($z = 0$) it is pinched to one point.

This model was introduced in [16] and intensively studied in the 90’s [15, 17, 57, 36, 58].

6.1.2 Energy-Momentum Tensor

The energy-momentum tensor obtained from (6.1) has only off-diagonal components

$$T(x) \equiv T^{\bar{x}}{}_{x} = \frac{\partial_x u \partial_x u^*}{1+|u|^2}\,, \tag{6.9a}$$

$$\overline{T}(\bar{x}) \equiv T^{x}{}_{\bar{x}} = \frac{\partial_{\bar{x}} u \partial_{\bar{x}} u^*}{1+|u|^2}\,, \tag{6.9b}$$

which are chiral and antichiral, respectively, due to the equations of motion (6.2). The vanishing of the diagonal components ${T^x}_x = {T^{\bar{x}}}_{\bar{x}} = 0$ shows that the energy-momentum tensor is traceless, and confirms the conformal invariance of the theory.

6.2 General Solution

Let us recall that the initial Lagrangian (4.6) for our model is given by

$$\mathcal{L} = -k \langle g^{-1}\partial_x g g^{-1}\partial_{\bar{x}} g\rangle - k\frac{\langle T_0\partial_x g g^{-1}\rangle\langle T_0 g^{-1}\partial_{\bar{x}} g\rangle - \langle T_0\partial_{\bar{x}} g g^{-1}\rangle\langle T_0 g^{-1}\partial_x g\rangle}{1+\langle T_0 g T_0 g^{-1}\rangle}\,, \tag{6.10}$$

and the corresponding gauged Lagrangian (4.84)

$$\mathcal{L}_G = \mathcal{L}(g, \partial_x g - A_x(T_0 g + g T_0), \partial_{\bar{x}} - A_{\bar{x}}(T_0 g + g T_0)) \tag{6.11}$$

provides the solution for the gauge potential (4.85)

$$A_x = \frac{\langle T_0\partial_x g g^{-1}\rangle}{1+\langle T_0 g T_0 g^{-1}\rangle}\,, \qquad A_{\bar{x}} = \frac{\langle T_0 g^{-1}\partial_{\bar{x}} g\rangle}{1+\langle T_0 g T_0 g^{-1}\rangle}\,. \tag{6.12}$$

6.2.1 Field Strength

In order to find the general solution of the $SL(2,\mathbb{R})/U(1)$ model we first calculate the field strength from (6.12)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{6.13}$$

Note that in the polar coordinates (3.7) the Lagrangian (6.1) takes the form

$$\mathcal{L} = \frac{k}{1+r^2}\left((\partial_x r)(\partial_{\bar{x}} r) + r^2(\partial_x\beta)(\partial_{\bar{x}}\beta)\right)\,, \tag{6.14}$$

and the variation with respect to the β-field yields the equation of motion

$$\partial_x\left(\frac{r^2}{1+r^2}\partial_{\bar{x}}\beta\right) + \partial_{\bar{x}}\left(\frac{r^2}{1+r^2}\partial_x\beta\right) = 0\,. \tag{6.15}$$

The gauge field (6.12) in the polar coordinates is given by

$$A_x = \frac{1}{2}\partial_x\alpha + \frac{1}{2}\frac{r^2}{1+r^2}\partial_x\beta\,, \qquad A_{\bar{x}} = \frac{1}{2}\partial_{\bar{x}}\alpha - \frac{1}{2}\frac{r^2}{1+r^2}\partial_{\bar{x}}\beta\,, \tag{6.16}$$

and the field strength is then

$$F_{\bar{x}x} = \frac{1}{2}\partial_{\bar{x}}\left(\frac{r^2}{1+r^2}\partial_x\beta\right) + \frac{1}{2}\partial_x\left(\frac{r^2}{1+r^2}\partial_{\bar{x}}\beta\right) = 0\,, \tag{6.17}$$

which vanishes due to (6.15).

6.2.2 Solutions

First note that for a vanishing gauge field the gauged Lagrangian (6.11) is equal to the initial Lagrangian (6.10). Therefore, a solution with zero gauge field splits into the product (4.10).

Let now $(\hat{g}, A_x, A_{\bar{x}})$ be an arbitrary solution of the dynamical equations obtained from the gauged Lagrangian (6.11). Expanding the gauge field components $A_\tau = A_x + A_{\bar{x}}$ and $A_\sigma = A_x - A_{\bar{x}}$ in Fourier modes

$$A_\sigma(\tau,\sigma) = \mu(\tau) + \sum_{n\neq 0} c_n(\tau) e^{-in\sigma}\,, \quad A_\tau(\tau,\sigma) = \nu(\tau) + \sum_{n\neq 0} d_n(\tau) e^{-in\sigma}\,, \tag{6.18}$$

we find that the vanishing field strength implies

$$\partial_\tau c_n(\tau) + in d_n(\tau) = 0 \qquad \text{and} \qquad \partial_\tau \mu(\tau) = 0\,. \tag{6.19}$$

Then, a gauge transformation with the function

$$\epsilon(\tau,\sigma) \equiv -\mu\sigma + \sum_{n\neq 0} \frac{c_n(\tau)}{in} e^{-in\sigma} - \int_0^\tau \nu(t)dt \tag{6.20}$$

transforms the $\hat{g}$-field according to

$$\hat{g}(\tau,\sigma) \to \tilde{g}(\tau,\sigma) = e^{\epsilon(\tau,\sigma)T_0} \hat{g}(\tau,\sigma) e^{\epsilon(\tau,\sigma)T_0} \tag{6.21}$$

and vanishes the gauge field

$$A_\sigma \to \tilde{A}_\sigma = A_\sigma + \partial_\sigma \epsilon = 0\,, \qquad A_\tau \to \tilde{A}_\tau = A_\tau + \partial_\tau \epsilon = 0\,. \tag{6.22}$$

According to the remark above the solution is now given by the product $\tilde{g}(x,\bar{x}) = \tilde{g}_L(x)\tilde{g}_R(\bar{x})$. However, it has to be noted that for $\mu \neq 0$ the function (6.20) is not periodic in σ, which leads to a non-periodic transformed solution $\tilde{g}$

$$\tilde{g}(\tau,\sigma+2\pi) = e^{-2\pi\mu T_0} \tilde{g}(\tau,\sigma) e^{-2\pi\mu T_0}\,. \tag{6.23}$$

The gauge invariant u-field can thus be written as

$$u(x,\bar{x}) = \langle (T_1 + iT_2)\tilde{g}_L(x)\tilde{g}_R(\bar{x})\rangle \tag{6.24}$$

where due to (6.12) and the vanishing gauge potential the chiral fields are constrained by

$$\langle T_0 \partial_x \tilde{g}_L(x) \tilde{g}_L^{-1}(x)\rangle = 0\,, \qquad \langle T_0 \tilde{g}_R^{-1}(\bar{x}) \partial_{\bar{x}} \tilde{g}_R(\bar{x})\rangle = 0 \tag{6.25}$$

and must also satisfy condition (6.23).

Equation (6.23) and (6.25) can be interpreted as constraints on the chiral fields. A proper parameterization of this constrained surface and its insertion into (6.24) leads to the general solution of our model. In the next sections we will realize this scheme in detail for the hyperbolic and elliptic monodromies separately.

6.3 Hyperbolic Monodromy

In the hyperbolic sector (4.16a) the chiral and antichiral parts of the WZNW-field (6.21) have the monodromy

$$\tilde{g}_L(x+2\pi) = e^{-2\pi\mu T_0}\tilde{g}_L(x)e^{\lambda T_2}\,, \qquad \tilde{g}_R(\bar{x}+2\pi) = e^{\lambda T_2}\tilde{g}_R(\bar{x})e^{2\pi\mu T_0}\,. \tag{6.26}$$

We parameterize these fields in terms of two complex fields $\psi(x), \chi(x)$ and $\bar{\psi}(\bar{x}), \bar{\chi}(\bar{x})$ by

$$\tilde{g}_L(x) = -\sqrt{2}\begin{pmatrix}\mathrm{Im}\,\psi(x) & \mathrm{Im}\,\chi(x)\\ \mathrm{Re}\,\psi(x) & \mathrm{Re}\,\chi(x)\end{pmatrix}\,, \quad \tilde{g}_R(\bar{x}) = \sqrt{2}\begin{pmatrix}\mathrm{Re}\,\bar{\psi}(\bar{x}) & -\mathrm{Im}\,\bar{\psi}(\bar{x})\\ \mathrm{Re}\,\bar{\chi}(\bar{x}) & -\mathrm{Im}\,\bar{\chi}(\bar{x})\end{pmatrix}\,. \tag{6.27}$$

This choice is motivated by the fact that the fields now each have a closed monodromy behavior

$$\psi(x+2\pi) = \psi(x)e^{\lambda+i2\pi\mu}\,, \qquad \chi(x+2\pi) = \chi(x)e^{-\lambda+i2\pi\mu}\,, \tag{6.28}$$

and the same for the antichiral parts. On these functions we have to impose two conditions: Demanding a determinant equal to one immediately leads to

$$\psi(x)\chi^*(x) - \psi^*(x)\chi(x) = i\,. \tag{6.29}$$

The constraint on the Kac-Moody current $J_0 = 0$ can be expressed as

$$\psi'(x) = V(x)\psi^*(x)\,, \qquad \chi'(x) = V(x)\chi^*(x)\,, \tag{6.30}$$

where we have defined

$$V(x) \equiv \frac{1}{i}\langle(T_1 + iT_2)\tilde{g}'_L(x)\tilde{g}_L^{-1}(x)\rangle\,. \tag{6.31}$$

Combining these two conditions on $\psi(x)$ and $\chi(x)$ yields the Wronskian

$$\psi(x)\chi'(x) - \psi'(x)\chi(x) = i\frac{\psi'(x)}{\psi^*}\,, \tag{6.32}$$

and the ansatz $\chi(x) = \psi(x)A(x)$ results in the following equation for the screening charge $A(x)$

$$A'(x) = i\frac{\psi'(x)}{|\psi(x)|^2\psi(x)}\,. \tag{6.33}$$

Taking into account the monodromy behavior which A inherits from $\psi(x)$ and $\chi(x)$, this can be integrated to

$$A(x) = i\int_0^{2\pi} dz\;\theta_{-2\lambda}(x-z)\frac{\psi'(z)}{|\psi(z)|^2\psi(z)} \tag{6.34}$$

with the Green's function (4.46).

For the antichiral fields $\bar{\psi}(\bar{x})$ and $\bar{\chi}(\bar{x})$ the monodromy is the same and with

$$\bar{V}(\bar{x}) \equiv i\langle(T_1 + iT_2)\tilde{g}_R^{-1}(\bar{x})\tilde{g}'_R(\bar{x})\rangle \tag{6.35}$$

they also satisfy the corresponding relations (6.29) and (6.30). By the same arguments as above we can therefore write $\bar{\chi}(\bar{x}) = \bar{\psi}(\bar{x})\bar{A}(\bar{x})$ with

$$\bar{A}(\bar{x}) = i\int_0^{2\pi} d\bar{z}\,\theta_{-2\lambda}(\bar{x}-\bar{z})\frac{\bar{\psi}'(\bar{z})}{|\bar{\psi}(\bar{z})|^2\bar{\psi}(\bar{z})}\,. \tag{6.36}$$

Inserting these results into the solution (6.24) we obtain the general solution of the $SL(2,\mathbb{R})/U(1)$ model in the hyperbolic sector

$$\begin{aligned} u(x,\bar{x}) &= \psi(x)\bar{\psi}(\bar{x}) + \chi(x)\bar{\chi}(\bar{x}) \\ &= \psi(x)\bar{\psi}(\bar{x})\left(1 + A(x)\bar{A}(\bar{x})\right) \end{aligned} \tag{6.37}$$

parameterized by two arbitrary chiral and antichiral fields $\psi(x)$ and $\bar{\psi}(\bar{x})$ with monodromies defined be (6.28).

6.3.1 Hamiltonian Reduction

Although we have already described the space of solutions of the $SL(2,\mathbb{R})/U(1)$ model in the hyperbolic sector we still do not know the Poisson bracket structure there. In order to find this structure we perform the Hamiltonian reduction.

The solution (6.21) is not in the class of periodic functions, but one can remedy this non-periodicity and define a new field

$$g(x,\bar{x}) \equiv e^{\mu x T_0}\tilde{g}(x,\bar{x})e^{-\mu\bar{x}T_0}\,. \tag{6.38}$$

This g still has the structure of chiral times antichiral function and is also periodic. It is therefore in the class of WZNW-fields. The zero component of its Kac-Moody currents however is fixed by the original gauge fields zero mode

$$J_0(x) = k\mu\,, \qquad\qquad \bar{J}_0(\bar{x}) = -k\mu\,. \tag{6.39}$$

The solutions of the $SL(2,\mathbb{R})/U(1)$-model are thus parameterized by the subclass of WZNW-fields g with $J_0 = -\bar{J}_0 = \text{const.}$, and the final solution in terms of gauge invariant variables is given by

$$u(x,\bar{x}) = \langle (T_1 + iT_2)e^{-\mu x T_0}g(x,\bar{x})e^{\mu\bar{x}T_0}\rangle\,. \tag{6.40}$$

The chiral part of the periodic WZNW field is related to the fields $\psi(x)$ and $\chi(x)$ that parameterize the non-periodic solution by

$$g_L(x) = e^{\mu x T_0}\tilde{g}_L(x) = -\sqrt{2}\begin{pmatrix} \mathrm{Im}(\psi(x)e^{-i\mu x}) & \mathrm{Im}(\chi(x)e^{-i\mu x}) \\ \mathrm{Re}(\psi(x)e^{-i\mu x}) & \mathrm{Re}(\chi(x)e^{-i\mu x}) \end{pmatrix}\,. \tag{6.41}$$

We now insert this parameterization into the symplectic form (4.32a) and find the symplectic form on the subspace of solutions. Using the relations (6.29) and (6.30) and the monodromy behavior we can eliminate $\chi(x)$ without using its explicit solution in terms of $\psi(x)$. The symplectic form then reads

$$\begin{aligned} \omega = \int_0^{2\pi} dx\frac{1}{2k}&\left[\left(\frac{d\psi(x)}{\psi(x)}\right)'\wedge\frac{d\psi^*(x)}{\psi^*(x)} + \left(\frac{d\psi^*(x)}{\psi^*(x)}\right)'\wedge\frac{d\psi(x)}{\psi(x)}\right] \\ &+ \frac{1}{2k}d\lambda\wedge\left(\frac{d\psi(0)}{\psi(0)} + \frac{d\psi^*(0)}{\psi^*(x)}\right) + \frac{i\pi}{k}d\mu\wedge\left(\frac{d\psi^*(0)}{\psi*(0)} - \frac{d\psi(0)}{\psi(0)}\right)\,. \end{aligned} \tag{6.42}$$

Due to the unit determinant condition (6.29) we have $\psi(x) \neq 0$ and $\psi(x)$ can be parameterized by

$$\psi(x) = e^{\gamma\phi_1(x)+i\gamma\phi_2(x)} \tag{6.43}$$

where $\phi_1(x)$ and $\phi_2(x)$ are two real fields and $\gamma \equiv k^{-\frac{1}{2}}$. In order to reproduce the monodromy (6.28) of $\psi(x)$ the fields satisfy

$$\phi_1(x+2\pi) = \phi_1(x) + \frac{p_1}{2}\,, \qquad \phi_2(x+2\pi) = \phi_2(x) + \frac{p_2}{2} + 2\pi\frac{n}{\gamma}\,, \tag{6.44}$$

with $p_1 \equiv 2\lambda/\gamma$, $p_2 \equiv 4\pi\mu/\gamma$ and an arbitrary integer $n \in \mathbb{Z}$. The symplectic form then becomes

$$\omega = \int_0^{2\pi} \left(d\phi_1' \wedge d\phi_1 + d\phi_2' \wedge d\phi_2\right) dx + \frac{1}{2}dp_1 \wedge d\phi_1(0) + \frac{1}{2}dp_2 \wedge d\phi_2(0)\,. \tag{6.45}$$

This can be identified with two free-field symplectic forms (2.39), and therefore $\phi_1(x)$ and $\phi_2(x)$ satisfy the canonical relations

$$\{\phi_i(x), \phi_j(y)\} = \delta_{ij}\frac{1}{4}\epsilon(x-y)\,. \tag{6.46}$$

On the antichiral side an analogous calculation can be done and for the parameterization

$$\bar\psi(\bar x) = e^{\gamma(\bar\phi_1(\bar x)+i\bar\phi_2(\bar x))}\,, \tag{6.47}$$

where $\bar\phi_1$ and $\bar\phi_2$ have the monodromy

$$\bar\phi_1(\bar x+2\pi) = \bar\phi_1(\bar x) + \frac{\bar p_1}{2}\,, \qquad \bar\phi_2(\bar x+2\pi) = \bar\phi_2(\bar x) + \frac{p_2}{2} + \frac{2\pi}{\gamma}\bar n \tag{6.48}$$

with $\bar p_1 \equiv 2\lambda_R/\gamma$ and another arbitrary integer $\bar n \in \mathbb{Z}$ we also find the standard free-field symplectic form.

The full solution on the physical space $p_j = \bar p_j$ with the redefinition $q_j + \bar q_j \to q_j$ can now be written in free-field parameterization

$$u(x,\bar x) = e^{\gamma(\phi_1(x)+\bar\phi_1(\bar x))+i\gamma(\phi_2(x)+\bar\phi_2(\bar x))}\left(1 + A(x)\bar A(\bar x)\right) \tag{6.49}$$

with

$$\phi_1(x) = \frac{q_1}{2} + \frac{p_1}{4\pi}x + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{a_n}{n}e^{-inx}\,, \tag{6.50}$$

$$\phi_2(x) = \frac{q_2}{2} + \frac{p_2}{4\pi}x + \frac{n}{\gamma}x + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{b_n}{n}e^{-inx}\,, \tag{6.51}$$

$$A(x) = \frac{-i}{2\sinh(\frac{\gamma p_1}{2})}\int_0^{2\pi} dz\; e^{\frac{\gamma p_1}{2}}\,e^{-2\gamma\phi_1(x+z)}\gamma(\phi_1'(x+z) + i\phi_2'(x+z)) \tag{6.52}$$

and similarly for the antichiral part. The reduced energy-momentum tensor can be calculated from (6.31) and (6.35)

$$T(x) = \frac{1}{\gamma^2}|V|^2 = \phi_1'^2(x) + \phi_2'^2(x)\,, \quad \overline{T}(\bar x) = \frac{1}{\gamma^2}|\bar V|^2 = \bar\phi_1'^2(\bar x) + \bar\phi_2'^2(\bar x)\,, \tag{6.53}$$

and by integration we obtain the free-field Hamiltonian

$$H = \frac{p_1^2}{4\pi} + \frac{p_2^2}{4\pi} + \sum_{n>0} \left(|a_n|^2 + |\bar{a}_n|^2 + |b_n|^2 + |\bar{b}_n|^2 \right) . \tag{6.54}$$

Since we have chosen $p_1 < 0$ the asymptotic behavior of the field is

$$u(x,\bar{x}) \overset{\tau\to-\infty}{\sim} e^{\gamma\Phi_{\text{in}}(x,\bar{x})} = e^{\gamma(\phi_1(x)+\bar{\phi}_1(\bar{x})+i(\phi_2(x)+\bar{\phi}_2(\bar{x})))} , \tag{6.55a}$$

$$u(x,\bar{x}) \overset{\tau\to+\infty}{\sim} e^{\gamma\Phi_{\text{out}}(x,\bar{x})} = A(x)\bar{A}(\bar{x}) e^{\gamma(\phi_1(x)+\bar{\phi}_1(\bar{x})+i(\phi_2(x)+\bar{\phi}_2(\bar{x})))} . \tag{6.55b}$$

As in Liouville theory one can interpret

$$\Phi_1(x,\bar{x}) + i\Phi_2(x,\bar{x}) \equiv \phi_1(x) + \bar{\phi}_1(\bar{x}) + i(\phi_2(x) + \bar{\phi}_2(\bar{x})) \tag{6.56}$$

as the asymptotic in-field, which is mapped to the interacting field u.

Note that with a canonical transformation $p_2 + 2\pi\frac{n+\bar{n}}{\gamma} \to p_2$ and with $\nu \equiv n - \bar{n}$ the phase factor of the in-field can be written as

$$e^{i\gamma\Phi_2(x,\bar{x})} = e^{i\nu\sigma} e^{i\gamma(q_2 + \frac{p_2}{2\pi}\tau + \text{oscillators})} . \tag{6.57}$$

Here ν appears as a winding number of the string. Even though the target space is simply connected the asymptotic in-field lives on the cylinder and ν describes the winding of this in-field.

We now consider the vacuum solutions setting the oscillator modes equal to zero. The fields are then

$$\Phi_1(\tau,\sigma) = q_1 + \frac{p}{2\pi}\tau , \qquad \Phi_2(\tau,\sigma) = q_2 + \frac{p_2}{2\pi}\tau + \frac{\nu}{\gamma}\sigma . \tag{6.58}$$

For this simple configuration the screening charges can be integrated and we obtain the vacuum solution

$$\begin{aligned} u(\tau,\sigma) =& e^{i\gamma(q_2+\frac{p_2}{2\pi}\tau+\frac{\nu}{\gamma}\sigma)} \Big[e^{\gamma(q_1+\frac{p_1}{2\pi}\tau)} \\ & - e^{-\gamma(q_1+\frac{p_1}{2\pi}\tau)} \frac{1}{4p_1^2} \left(p_1 + i\left(p_2 + 2\pi\frac{\nu}{\gamma}\right)\right)\left(p_1 + i\left(p_2 - 2\pi\frac{\nu}{\gamma}\right)\right)\Big] . \end{aligned} \tag{6.59}$$

For $\nu = 0$ the u-field becomes σ independent and the corresponding mechanical model is similar to (3.63).

6.3.2 Poisson Brackets

We now calculate the Poisson brackets between the chiral fields $\psi(x)$ and $\chi(x)$. Interestingly, the Poisson bracket of $\psi(x)$ with itself vanishes due to the complex structure

$$\{\psi(x), \psi(y)\} = \{e^{\gamma\phi_1(x)+i\gamma\phi_2(x)}, e^{\gamma\phi_1(y)+i\gamma\phi_2(y)}\} = 0 . \tag{6.60}$$

The remaining Poisson brackets can, in principle, also be calculated using the basic relations of ϕ_1 and ϕ_2 (6.46). However, to simplify the calculations here we use the method of Dirac brackets (see Appendix A).

As we have shown, the $SL(2,\mathbb{R})/U(1)$ model can be treated as a reduced WZNW theory, constrained to the surface (6.39), where μ is a constant dynamical parameter. The constraints can also be written as

$$J_0'(x) = 0 \qquad \bar{J}_0'(\bar{x}) = 0 \tag{6.61a}$$

and

$$\int_0^{2\pi} dx\, J_0(x) + \int_0^{2\pi} d\bar{x}\, \bar{J}_0(\bar{x}) = 0\,. \tag{6.61b}$$

In terms of the Fourier modes $J_0(x) = \sum_k j_k e^{ikx}$ these constraints become

$$j_k = 0\,, \qquad \bar{j}_k = 0 \qquad \text{for } k \neq 0\,, \tag{6.62a}$$

and

$$j_0 + \bar{j}_0 = 0\,. \tag{6.62b}$$

From (4.71) we have

$$\{j_m, j_n\} = i\frac{m}{4\pi\gamma^2}\delta_{m+n}\,. \tag{6.63}$$

and similarly for the antichiral constraints. Therefore (6.62a) form second class constraints, while (6.62b) is of the first class. To calculate the Poisson brackets in the reduced space from the ones in the full space we have to use the Dirac brackets (A.16). For this we need to invert $\{j_m, j_n\}$ with $m, n \neq 0$, which is easily found to be

$$(\{j, j\})^{-1}_{lm} = i\frac{4\pi\gamma^2}{l}\delta_{l+m}\,. \tag{6.64}$$

According to (6.41) the fields ψ and χ are given by

$$\psi(x) = \frac{1}{\sqrt{2}}\langle (T_1 - T_0 + i(T_2 + I))g_L(x)\rangle e^{i\gamma^2 j_0 x}\,, \tag{6.65}$$

$$\chi(x) = \frac{1}{\sqrt{2}}\langle (I - T_2 + i(T_1 + T_0))g_L(x)\rangle e^{i\gamma^2 j_0 x}\,. \tag{6.66}$$

From (4.67a) one can now extract the Poisson brackets of $\psi(x)$ and $\chi(x)$ and construct the Dirac brackets. This procedure and the results are given in Appendix B. Finally, one arrives at the Dirac bracket of the full field

$$\begin{aligned}\{u(x,\bar{x}), u(y,\bar{y})\}_D = \gamma^2 \Big[&(\theta_{2\lambda}(x-y) + \theta_{-2\lambda}(\bar{x}-\bar{y}))\chi(x)\bar{\psi}(\bar{x})\psi(y)\bar{\chi}(\bar{y}) \\ &+ (\theta_{-2\lambda}(x-y) + \theta_{2\lambda}(\bar{x}-\bar{y}))\psi(x)\bar{\chi}(\bar{x})\chi(y)\bar{\psi}(\bar{y}) \\ &- \Theta\left(\psi(x)\bar{\psi}(\bar{x})\chi(y)\bar{\chi}(\bar{y}) + \chi(x)\bar{\chi}(\bar{x})\psi(y)\bar{\psi}(\bar{y})\right)\Big]\,.\end{aligned} \tag{6.67}$$

In the fundamental domain we find

$$\boxed{\{u(x,\bar{x}), u(y,\bar{y})\}_D = \gamma^2\Theta\left[u(x,\bar{y})u(y,\bar{x}) - u(x,\bar{x})u(y,\bar{y})\right]}\,, \tag{6.68}$$

which also has a causal structure and is given in terms of the full fields at the intersection points of their light cones.

The Dirac bracket of u with its complex conjugate can be found to be

$$\begin{aligned}\{u^*(x,\bar{x}), u(y,\bar{y})\}_D = \gamma^2 \Big[&\Theta\left(\psi^*(x)\bar{\psi}^*(\bar{x})\psi(y)\bar{\psi}(\bar{y}) + \chi^*(x)\bar{\chi}^*(\bar{x})\chi(y)\bar{\chi}(\bar{y})\right) \\ &+ (\theta_{2\lambda}(x-y) + \theta_{-2\lambda}(\bar{x}-\bar{y}))\,\chi^*(x)\bar{\psi}^*(\bar{x})\psi(y)\bar{\chi}(\bar{y}) \\ &+ (\theta_{-2\lambda}(x-y) + \theta_{2\lambda}(\bar{x}-\bar{y}))\,\psi^*(x)\bar{\chi}^*(\bar{x})\chi(y)\bar{\psi}(\bar{y})\Big]\,.\end{aligned} \tag{6.69}$$

Note that even in the fundamental domain this cannot be written as an expression of u and u^*. However, with the object

$$\mathrm{x}(x,\bar{x}) \equiv \langle (I + iT_0) g(x,\bar{x}) \rangle = -i\big(\psi(x)\bar{\psi}^*(\bar{x}) + \chi(x)\bar{\chi}^*(\bar{x})\big)\,, \tag{6.70}$$

which is related to the vector gauged model, the fundamental Poisson bracket can be expressed as

$$\boxed{\{u^*(x,\bar{x}), u(y,\bar{y})\}_D = \gamma^2 \Theta\, \mathrm{x}^*(x,\bar{y})\mathrm{x}(y,\bar{x})}\,. \tag{6.71}$$

The results derived in this section coincide with the Poisson relations obtained from the $SL(2,\mathbb{R})/U(1)$ model on the full line in [11].

6.3.3 Symmetries

The reduced energy-momentum tensor of the $SL(2,\mathbb{R})/U(1)$ model is

$$T(x) = \phi_1'^2(x) + \phi_2'^2(x)\,. \tag{6.72}$$

It generates conformal transformations of the system. Its action on a free-field exponential

$$\{T(x), e^{2\alpha\gamma(\phi_1(y)+i\phi_2(y))}\} = \Big(e^{2\alpha\gamma(\phi_1(y)+i\phi_2(y))}\Big)'\,\delta(x-y) \tag{6.73}$$

shows that $\psi(x)$ is a conformal scalar. The integrand of the screening charge however transforms as

$$\begin{aligned}\{T(x), (\phi_1'(y) + i\phi_2'(y))e^{-2\phi_1(y)}\} = {} & \Big(\big(\phi_1'(x) + i\phi_2'(x)\big)\, e^{-2\phi_1(x)}\Big)'\,\delta(x-y) \\ & - \Big(\big(\phi_1'(y) + i\phi_2'(y)\big)\, e^{-2\phi_1(y)}\Big)\,\delta'(x-y)\,,\end{aligned} \tag{6.74}$$

which indicates that it transforms with conformal weight one. Therefore the screening charge itself will transform as a conformal scalar as well. Thus the object $u(x,\bar{x})$ transforms with conformal weight zero

$$u(x,\bar{x}) \to u(\xi(x),\bar{\xi}(\bar{x}))\,, \tag{6.75}$$

which is also a solution of the equation of motion (6.2).

As in Liouville theory we have another symmetry generated by $\phi_1'(x)$ and $\phi_2'(x)$

$$\{\phi_1'(x), e^{\gamma(\phi_1(y)+i\phi_2(y))}\} = \frac{\gamma}{2}\delta(x-y)e^{\gamma(\phi_1(y)+i\phi_2(y))}\,, \tag{6.76}$$

$$\{\phi_2'(x), e^{\gamma(\phi_1(y)+i\phi_2(y))}\} = i\frac{\gamma}{2}\delta(x-y)e^{\gamma(\phi_1(y)+i\phi_2(y))}\,, \tag{6.77}$$

which corresponds to the translation on the space of solutions

$$\phi_1(x) \to \phi_1(x) + h_1(x)\,, \tag{6.78a}$$

$$\phi_2(x) \to \phi_2(x) + h_2(x)\,. \tag{6.78b}$$

The Lie algebra of the symmetry group is

$$\{\phi_j'(x), \phi_k'(y)\} = -\frac{1}{2}\delta_{jk}\delta'(x-y)\,, \tag{6.79}$$

$$\{T(x), \phi_j'(y)\} = \phi_j''(x)\delta(x-y) - \phi_j'(y)\delta'(x-y)\,, \tag{6.80}$$

$$\{T(x), T(y)\} = T'(x)\delta(x-y) - 2T(y)\delta'(x-y)\,. \tag{6.81}$$

6.4 Elliptic Monodromy

6.4.1 Solutions

So far we have only analyzed the hyperbolic sector of the theory. However, for the $SL(2,\mathbb{R})/U(1)$ theory there also exists an elliptic sector where the monodromies of the chiral and antichiral parts of the solution are

$$\tilde{g}_L(x+2\pi) = e^{-2\pi\mu T_0}\tilde{g}_L(x)e^{\lambda T_0}\,, \qquad \tilde{g}_R(\bar{x}+2\pi) = e^{\lambda T_0}\tilde{g}_R(\bar{x})e^{2\pi\mu T_0}\,. \tag{6.82}$$

A convenient parameterization of the chiral part in terms of two complex functions $\psi(x)$ and $\chi(x)$ now can be written as

$$\tilde{g}_L = \begin{pmatrix} \operatorname{Im}\psi + \operatorname{Re}\chi & \operatorname{Re}\psi + \operatorname{Im}\chi \\ \operatorname{Re}\psi - \operatorname{Im}\chi & \operatorname{Re}\chi - \operatorname{Im}\psi \end{pmatrix}\,. \tag{6.83}$$

By (6.82) the chiral functions $\psi(x)$, $\chi(x)$ have the monodromy

$$\psi(x+2\pi) = \psi(x)e^{i(\lambda+2\pi\mu)}\,, \qquad \chi(x+2\pi) = \chi(x)e^{i(-\lambda+2\pi\mu)}\,, \tag{6.84}$$

and the condition $\det(\tilde{g}_L) = 1$ implies

$$|\chi(x)|^2 - |\psi(x)|^2 = 1\,. \tag{6.85}$$

Similar to the case of the hyperbolic monodromy the constraint $J_0 = 0$ can be written as

$$\psi'(x) = -iV(x)\chi^*(x)\,, \qquad \chi'(x) = -iV(x)\psi^*(x)\,. \tag{6.86}$$

Combining these two conditions we find

$$\chi^*(x)\chi'(x) - \psi^*(x)\psi'(x) = 0\,. \tag{6.87}$$

To pass to independent variables we introduce polar coordinates

$$\psi(x) = r(x)e^{i\alpha(x)}\,, \qquad \chi(x) = R(x)e^{i\beta(x)}\,. \tag{6.88}$$

Due to (6.84) the radial functions $r(x)$ and $R(x)$ are periodic, and the monodromy of the angle variables reads

$$\alpha(x+2\pi) = \alpha(x) + 2\pi(\mu+n) + \lambda\,, \quad \beta(x+2\pi) = \beta(x) + 2\pi(\mu+m) - \lambda\,, \tag{6.89}$$

with $n, m \in \mathbb{Z}$. The conditions (6.85) and (6.87) are then

$$R^2(x) = r^2(x) + 1\,, \qquad r^2(x)\alpha'(x) - R^2(x)\beta'(x) = 0\,. \tag{6.90}$$

The solutions of these equations in terms of the angle variables $\alpha(x)$ and $\beta(x)$ with $\frac{\alpha'(x)}{\beta'(x)} > 1$ are

$$\psi(x) = \sqrt{\frac{\beta'(x)}{\alpha'(x)-\beta'(x)}}\,e^{i\alpha(x)}\,, \qquad \chi(x) = \sqrt{\frac{\alpha'(x)}{\alpha'(x)-\beta'(x)}}\,e^{i\beta(x)}\,. \tag{6.91}$$

The induced symplectic form on this space of motions is a rather complicated expression and we suspect that it is not possible to find global canonical coordinates here.

The antichiral part can be parameterized in a similar way

$$\tilde{g}_R(\bar{x}) = -\begin{pmatrix} \mathrm{Re}\,\bar{\psi}(\bar{x}) + \mathrm{Im}\,\bar{\chi}(\bar{x}) & \mathrm{Re}\,\bar{\chi}(\bar{x}) - \mathrm{Im}\,\bar{\psi}(\bar{x}) \\ \mathrm{Re}\,\bar{\chi}(\bar{x}) + \mathrm{Im}\,\bar{\psi}(\bar{x}) & \mathrm{Re}\,\bar{\psi}(\bar{x}) - \mathrm{Im}\,\bar{\chi}(\bar{x}) \end{pmatrix}. \tag{6.92}$$

The fields $\bar{\psi}(\bar{x})$ and $\bar{\chi}(\bar{x})$ then have a monodromy

$$\bar{\psi}(\bar{x}+2\pi) = \bar{\psi}(x)e^{i(2\pi\mu+\lambda)}, \qquad \bar{\chi}(\bar{x}+2\pi) = \bar{\chi}(x)e^{i(2\pi\mu-\lambda)}. \tag{6.93}$$

The conditions for the unit determinant and the zero component of the Kac-Moody current here become

$$|\bar{\psi}(\bar{x})|^2 - |\bar{\chi}(\bar{x})|^2 = 1 \tag{6.94}$$

and

$$\bar{\psi}'(\bar{x}) = i\bar{V}(\bar{x})\bar{\chi}^*(\bar{x}), \qquad \bar{\chi}'(\bar{x}) = i\bar{V}(\bar{x})\bar{\psi}^*(\bar{x}). \tag{6.95}$$

We then find a parameterization similar to (6.91) in terms of the angle variables

$$\bar{\psi}(\bar{x}) = \sqrt{\frac{\bar{\beta}'(\bar{x})}{\bar{\beta}'(\bar{x})-\bar{\alpha}'(\bar{x})}}e^{i\bar{\alpha}(\bar{x})}, \qquad \bar{\chi}(\bar{x}) = \sqrt{\frac{\bar{\alpha}'(\bar{x})}{\bar{\beta}'(\bar{x})-\bar{\alpha}'(\bar{x})}}e^{i\bar{\beta}(\bar{x})} \tag{6.96}$$

with $\frac{\bar{\beta}'(\bar{x})}{\bar{\alpha}'(\bar{x})} > 1$ and monodromies

$$\bar{\alpha}(\bar{x}+2\pi) = \bar{\alpha}(\bar{x}) + 2\pi(\mu+\bar{n}) + \lambda, \quad \bar{\beta}(\bar{x}+2\pi) = \bar{\beta}(\bar{x}) + 2\pi(\mu+\bar{m}) - \lambda. \tag{6.97}$$

The full solution (6.24) is now given by

$$u(x,\bar{x}) = \psi(x)\bar{\psi}(\bar{x}) + \chi(x)\bar{\chi}(\bar{x}). \tag{6.98}$$

Note that the radial functions are periodic and therefore we see that the elliptic sector describes bound states, where the string does not go to infinity but remains in a bounded part of the cigar.

In order to make contact with the hyperbolic vacuum solution we choose linear functions

$$\begin{aligned} \alpha(x) &= \frac{\tilde{q}_2}{2} + \pi + \left(\mu + n + \frac{\lambda}{2\pi}\right)x, & \beta(x) &= \frac{\tilde{q}_2}{2} + \left(\mu + n - \frac{\lambda}{2\pi}\right)x, \\ \bar{\alpha}(x) &= \frac{\tilde{q}_2}{2} + \left(\mu + \bar{n} + \frac{\lambda}{2\pi}\right)\bar{x}, & \bar{\beta}(\bar{x}) &= \frac{\tilde{q}_2}{2} + \left(\mu + \bar{n} - \frac{\lambda}{2\pi}\right)\bar{x}. \end{aligned} \tag{6.99}$$

With the definitions $\nu \equiv n - \bar{n}$, $\rho \equiv 2\lambda\gamma^{-1}$ and $\gamma p_2 \equiv 4\pi\mu + 2\pi(n+\bar{n})$ the solution is then

$$\begin{aligned} u(x,\bar{x}) = e^{i\gamma(\tilde{q}_2 + \frac{p_2}{2\pi}\tau)}e^{i\nu\sigma}\frac{1}{2\rho}\Bigg[&\sqrt{\left(p_2 + 2\pi\frac{\nu}{\gamma} - \rho\right)\left(2\pi\frac{\nu}{\gamma} - p_2 + \rho\right)}e^{i\gamma\frac{\rho}{2\pi}\tau} \\ &- \sqrt{\left(p_2 + 2\pi\frac{\nu}{\gamma} + \rho\right)\left(2\pi\frac{\nu}{\gamma} - p_2 - \rho\right)}e^{-i\gamma\frac{\rho}{2\pi}\tau}\Bigg] \end{aligned} \tag{6.100}$$

with the condition

$$|2\pi\nu| > \gamma(|p_2| + |\rho|)\,. \tag{6.101}$$

The hyperbolic vacuum solution (6.59) can be symmetrized with a specific choice of q_1 and q_2

$$q_1 = \frac{1}{2\gamma}\log\left|\left(2\pi\frac{\nu}{\gamma} + p_2 + ip_1\right)\left(2\pi\frac{\nu}{\gamma} - p_2 - ip_1\right)\right| - \frac{1}{\gamma}\log|2p_1| \tag{6.102a}$$

$$iq_2 = i\tilde{q}_2 + i\frac{3}{2}\pi + \frac{1}{4\gamma}\log\left(\frac{(2\pi\frac{\nu}{\gamma} + p_2 + ip_1)(2\pi\frac{\nu}{\gamma} - p_2 - ip_1)}{(2\pi\frac{\nu}{\gamma} + p_2 - ip_1)(2\pi\frac{\nu}{\gamma} - p_2 + ip_1)}\right)\,, \tag{6.102b}$$

so that it takes the form

$$u(\tau,\sigma) = e^{i\gamma(\tilde{q}_2 + \frac{p_2}{2\pi}\tau)}e^{i\nu\sigma}\frac{i}{2p_1}\left[\sqrt{\left(2\pi\frac{\nu}{\gamma} + p_2 + ip_1\right)\left(2\pi\frac{\nu}{\gamma} - p_2 - ip_1\right)}e^{\gamma\frac{p_1}{2\pi}\tau} - \sqrt{\left(2\pi\frac{\nu}{\gamma} + p_2 - ip_1\right)\left(2\pi\frac{\nu}{\gamma} - p_2 + ip_1\right)}e^{-\gamma\frac{p_1}{2\pi}\tau}\right]\,. \tag{6.103}$$

It is obvious that the specific elliptic solution can be reached from this hyperbolic solution by an analytic continuation $p_1 \to i\rho$.

6.4.2 Poisson Brackets

The symplectic form in the elliptic sector is a complicated expression and difficult to invert. However, we can obtain the Poisson brackets of the parameterizing fields by the method of Dirac brackets (A.16). This technique was already used for the hyperbolic sector in section 6.3.2.

The constraints are here also given by (6.62a) and (6.62b) and the chiral periodic WZNW-field g_L is related to the nonperiodicc solution $\tilde{g}_L$ by

$$g_L(x) = e^{\mu x T_0}\tilde{g}_L(x)\,. \tag{6.104}$$

Inserting now the parameterizationn (6.83) we find the expressions for $\psi(x)$ and $\chi(x)$ in terms of the WZNW-field

$$\psi(x) = -\langle (T_1 + iT_2)g_L(x)\rangle e^{i\gamma^2 j_0 x}\,, \tag{6.105a}$$

$$\chi(x) = -\langle (I + iT_0)g_L(x)\rangle e^{i\gamma^2 j_0 x}\,. \tag{6.105b}$$

From (4.67a) we then find their Poisson brackets with the Fourier components of $J_0(x)$

$$\{\psi(x), j_k\} = \frac{-i}{4\pi}e^{-ikx}\psi(x)\,, \qquad \{\chi(x), j_k\} = \frac{-i}{4\pi}e^{-ikx}\chi(x)\,. \tag{6.106}$$

The Poisson brackets of $\psi(x)$ and $\chi(x)$ in the unconstrained space can now be obtained from (4.60)

$$\{\psi(x),\psi(y)\} = \frac{\gamma^2}{4}\psi(x)\psi(y)\left(\epsilon(x-y)-\frac{1}{\pi}(x-y)\right), \tag{6.107a}$$

$$\{\chi(x),\chi(y)\} = \frac{\gamma^2}{4}\chi(x)\chi(y)\left(\epsilon(x-y)-\frac{1}{\pi}(x-y)\right), \tag{6.107b}$$

$$\{\psi(x),\chi(y)\} = \gamma^2\Big[-\frac{1}{4}(\epsilon(x-y)+\frac{1}{\pi}(x-y))\psi(x)\chi(y)+\theta_{2i\lambda}(x-y)\chi(x)\psi(y)\Big]. \tag{6.107c}$$

Similarly the Poisson brackets of the complex conjugate fields are

$$\{\psi^*(x),\psi(y)\} = \gamma^2\Big[\theta_{-2i\lambda}(x-y)\chi^*(x)\chi(y)-\frac{1}{4}\psi^*(x)\psi(y)\Big(\epsilon(x-y)-\frac{1}{\pi}(x-y)\Big)\Big], \tag{6.108a}$$

$$\{\chi^*(x),\chi(y)\} = \gamma^2\Big[\theta_{2i\lambda}(x-y)\psi^*(x)\psi(y)-\frac{1}{4}\chi^*(x)\chi(y)\Big(\epsilon(x-y)-\frac{1}{\pi}(x-y)\Big)\Big], \tag{6.108b}$$

$$\{\psi^*(x),\chi(y)\} = \frac{\gamma^2}{4}\psi^*(x)\chi(y)(\epsilon(x-y)+\frac{1}{\pi}(x-y)). \tag{6.108c}$$

Since the constraints are second class constraints we have to calculate the Dirac brackets (A.16). Using that the inverse of the matrix $\{j_m, j_n\}$ is given by (6.64) we find

$$\{\psi(x),\psi(y)\}_D = 0, \qquad \{\chi(x),\chi(y)\}_D = 0, \tag{6.109a}$$

$$\{\psi(x),\chi(y)\}_D = \gamma^2\left[-\frac{1}{2}\psi(x)\chi(y)+\theta_{2i\lambda}(x-y)\chi(x)\psi(y)\right]. \tag{6.109b}$$

and for the complex conjugate fields

$$\{\psi^*(x),\psi(y)\}_D = \gamma^2\theta_{-2i\lambda}(x-y)\chi^*(x)\chi(y), \tag{6.110a}$$

$$\{\chi^*(x),\chi(y)\}_D = \gamma^2\theta_{2i\lambda}(x-y)\psi^*(x)\psi(y), \tag{6.110b}$$

$$\{\psi^*(x),\chi(y)\}_D = \frac{\gamma^2}{2}\psi^*(x)\chi(y)\epsilon(x-y). \tag{6.110c}$$

Analogous results can be obtained for the antichiral objects. Combining the above relations one immediately gets the Poisson bracket of the solution $u(x,\bar{x})$ with itself

$$\begin{aligned}\{u(x,\bar{x}),u(y,\bar{y})\}_D =\Big[&(\theta_{2i\lambda}(x-y)+\theta_{-2i\lambda}(\bar{x}-\bar{y}))\psi(y)\bar{\psi}(\bar{x})\chi(x)\bar{\chi}(\bar{y})\\ &+(\theta_{-2i\lambda}(x-y)+\theta_{2i\lambda}(\bar{x}-\bar{y}))\psi(x)\bar{\psi}(\bar{y})\chi(y)\bar{\chi}(\bar{x})\\ &-\Theta(\psi(x)\bar{\psi}(\bar{x})\chi(y)\bar{\chi}(\bar{y})+\psi(y)\bar{\psi}(\bar{y})\chi(x)\bar{\chi}(\bar{x}))\Big].\end{aligned} \tag{6.111}$$

Note that this result is the analytical continuation of the corresponding expression in the hyperbolic sector (6.67) by $\lambda \to i\lambda$. In the fundamental domain we therefore also find

$$\boxed{\{u(x,\bar{x}),u(y,\bar{y})\}_D = \gamma^2\Theta\left[u(x,\bar{y})u(y,\bar{x})-u(x,\bar{x})u(y,\bar{y})\right]}. \tag{6.112}$$

For the complex conjugate field $u^*(x,\bar{x})$ the Poisson bracket relation takes a different form

$$\begin{aligned}\{u^*(x,\bar{x}),u(y,\bar{y})\}_D =&\gamma^2\Big[\Theta(\psi^*(x)\bar{\psi}^*(\bar{x})\chi(y)\bar{\chi}(\bar{y})+\chi^*(X)\bar{\chi}^*(\bar{x})\psi(y)\bar{\psi}(\bar{y}))\\ &+(\theta 2i\lambda(x-y)+\theta_{-2i\lambda}(\bar{x}-\bar{y}))\psi^*(x)\bar{\chi}^*(\bar{x})\psi(y)\bar{\chi}(\bar{y})\\ &+(\theta{-}2i\lambda(x-y)+\theta_{2i\lambda}(\bar{x}-\bar{y}))\chi^*(x)\bar{\psi}^*(\bar{x})\chi(y)\bar{\psi}(\bar{y})\Big]\,.\end{aligned} \tag{6.113}$$

However, in the fundamental domain we obtain the same result in terms of components of $\tilde{g}(x,\bar{x})$ as in the hyperbolic sector (6.70)

$$\boxed{\{u^*(x,\bar{x}),u(y,\bar{y})\}_D=\gamma^2\Theta\mathrm{x}^*(x,\bar{y})\mathrm{x}(y,\bar{x})} \tag{6.114}$$

with

$$\mathrm{x}(x,\bar{x})\equiv\langle(I+iT_0)\tilde{g}(x,\bar{x})\rangle=\psi(x)\bar{\chi}^*(\bar{x})+\chi(x)\bar{\psi}^*(\bar{x})\,. \tag{6.115}$$

VII

Quantization of Liouville Theory

In this chapter we review some aspects of quantum Liouville theory following mainly the approach proposed in [33]. Our aim is to construct the Heisenberg operator $\hat{V}(x,\bar{x})$ for the Liouville field exponential (5.25) and then to calculate the reflection amplitude of the theory, using the structure of the V-field in terms of incoming and outgoing free-fields (5.31).

We apply canonical quantization based on the standard commutation relations

$$[\hat{q},\hat{p}] = i\hbar\,, \qquad [\hat{a}_n,\hat{a}_m] = [\hat{\bar{a}}_n,\hat{\bar{a}}_m] = \hbar n\delta_{n+m}\,, \tag{7.1}$$

where $(\hat{p},\hat{q},\hat{a}_n,\hat{\bar{a}}_n)$ are the operators for the Fourier modes of the in-field (5.32) of Liouville theory.

The Hilbert space is spanned by the creation operators acting on a p-dependent vacuum, with $p>0$

$$|\{a\},p\rangle = \prod_{n>0}(\hat{a}_n^\dagger)^{k_n}|0,p\rangle\,. \tag{7.2}$$

Even though this scheme establishes the basic operators and the Hilbert space they act on, one still cannot write the quantum version of the Liouville field exponential (5.25) due to the operator ordering problem there. The operators corresponding to the Liouville field exponentials are called vertex operators. The vertex operator for (5.25) can be written as

$$\hat{V}(x,\bar{x}) = \hat{E}(x,\bar{x}) + \hat{F}(x,\bar{x})\,, \tag{7.3}$$

where $\hat{E}$ and $\hat{F}$ correspond to the exponentials of the in- and out-fields, respectively. Our strategy for the construction of these operators is based on the assumption that the classical symmetry of the theory is realized on the quantum level. More precisely, we require that

1. The operators for the symmetry generators $\hat{\phi}'(x), \hat{T}(x)$ satisfy the commutation relations of the classical Lie algebra (5.53a)-(5.53c), up to a deformation of the central terms;

2. The symmetry transformations of the physical operators are the same as for their classical counterparts;

3. The vertex operator is local, i.e.

$$[\hat{V}(\tau,\sigma_1),\hat{V}(\tau,\sigma_2)] = 0\,. \tag{7.4}$$

We follow the Moyal formalism, which appears a convenient tool for the realization of this program [59, 60, 61, 62, 63].

7.1 Moyal Formalism

The idea of the Moyal formalism is to establish a correspondence between operators on the Hilbert space of a system and functions on the phase space, called operator symbols. This correspondence has to map the algebraic structure of the space of operators to the space of functions. In particular, a product of two operators is mapped to the $*$-product (star product) of their symbols, and then, the commutators of operators create Moyal brackets on the phase space, which can be treated as a quantum deformation of the Poisson bracket structure.

7.1.1 Holomorphic Representation

As an instructive example let us shortly review the harmonic oscillator. The Hamiltonian for this system in standard variables is

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2\,, \tag{7.5}$$

and the ground state $|0\rangle$ in the q and p representation is a Gaussian about the origin

$$\psi_0(q) = Ce^{-\frac{\omega}{2\hbar}q^2}\,, \qquad \tilde{\psi}_0(p) = \tilde{C}e^{-\frac{1}{2\hbar\omega}p^2}\,. \tag{7.6}$$

It is obvious that the expectation values of $\hat{p}$ and $\hat{q}$ are zero, but for $\hat{p}^2$ and $\hat{q}^2$ we find

$$\langle 0|\hat{p}^2|0\rangle = \frac{\hbar\omega}{2}\,, \qquad \langle 0|\hat{q}^2|0\rangle = \frac{\hbar}{2\omega}\,. \tag{7.7}$$

Thus, the product of the deviations is $\Delta p\Delta q = \hbar/2$ and due to Heisenberg's uncertainty relation the ground state represents the best possible localization of a particle in both position and momentum space. Let us now introduce the operator

$$\hat{U}(p,q) \equiv e^{\frac{i}{\hbar}(p\hat{q}-q\hat{p})}\,. \tag{7.8}$$

It acts as a translation in position and momentum space since

$$\hat{U}^\dagger(p,q)\,\hat{p}\,\hat{U}(p,q) = \hat{p}+p\,, \qquad \hat{U}^\dagger(p,q)\,\hat{q}\,\hat{U}(p,q) = \hat{q}+q\,. \tag{7.9}$$

However, opposed to classical translations on phase space the group is non-abelian

$$\hat{U}(p_1,q_1)\hat{U}(p_2,q_2) = e^{\frac{i}{2\hbar}(p_1q_2-q_1p_2)}\hat{U}(p_1,+p_2,q_1+q_2) \tag{7.10}$$

The action of the translation operator on the harmonic oscillator ground state provides the coherent states [64]

$$|p,q\rangle \equiv \hat{U}(p,q)|0\rangle\,. \tag{7.11}$$

In terms of the annihilation and creation operators

$$\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{p}-i\omega\hat{q})\,, \qquad \hat{a}^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{p}+i\omega\hat{q})\,, \tag{7.12}$$

which have the commutator $[\hat{a}, \hat{a}^\dagger] = \hbar\omega$, these states can be expressed as

$$|\alpha\rangle = e^{\frac{1}{\hbar\omega}(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}|0\rangle = e^{-\frac{1}{2\hbar\omega}|\alpha|^2} e^{\frac{1}{\hbar\omega}\alpha\hat{a}^\dagger}|0\rangle\,, \tag{7.13}$$

with $\alpha = \frac{1}{\sqrt{2}}(p - i\omega q)$. Here we have used the Baker-Campbell-Hausdorff (BCH) formula and the vacuum definition $\hat{a}|0\rangle = 0$. The coherent states can thus also be characterized by a complex number α. From the definition follows that the coherent states are eigenstates of the annihilation operator $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,, \qquad\qquad \langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*\,. \tag{7.14}$$

The scalar product of two coherent states is given by

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2\hbar\omega}(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta)}\,. \tag{7.15}$$

The coherent states form a complete basis of the Hilbert space, i.e.

$$\mathbb{1} = \int d^2\alpha\; |\alpha\rangle\langle\alpha|\,, \tag{7.16}$$

where $d^2\alpha = (2\pi\hbar)^{-1} dp\, dq$. This can be checked by projection onto the number eigenstates using $\langle n|\alpha\rangle = (n!)^{-\frac{1}{2}} e^{-\frac{1}{2\hbar\omega}|\alpha|^2}(\alpha/\hbar\omega)^n$ and the property of the gamma function $\Gamma(n) = (n-1)!$. In fact the coherent states form an over complete set as any coherent state can be expressed in terms of the others.

One can now define a map from operators to symbols as the projection from left and right with the coherent states

$$\check{A}(\alpha^*, \alpha) \equiv \langle\alpha|\hat{A}|\alpha\rangle\,. \tag{7.17}$$

This symbol, known as the normal or Berezin symbol [61], is holomorphic in α and antiholomorphic in α^*. The inverse map corresponds to the normal ordering

$$\hat{A} =: \check{A}(\hat{a}^\dagger, \hat{a}) :\,. \tag{7.18}$$

7.1.2 Star-Product

The analog of the non-commutative operator product is now introduced as the $*$-product which is defined through

$$\check{A} * \check{B} \equiv \langle\alpha|\hat{A}\hat{B}|\alpha\rangle\,. \tag{7.19}$$

Inserting the completeness relation (7.16) one finds the explicit formula of this star product as the integral

$$\check{A}(\alpha^*, \alpha) * \check{B}(\alpha^*, \alpha) = \int d^2\beta\, e^{-\frac{|b|^2}{\hbar\omega}}\, \check{A}(\alpha^*, \alpha + \beta)\check{B}(\alpha^* + \beta^*, \alpha)\,. \tag{7.20}$$

If one expands the symbols into a Taylor series the integral can be carried out and one ends up with a power series in $\hbar$

$$\check{A}(\alpha^*, \alpha) * \check{B}(\alpha^*, \alpha) = \sum_{n=0}^{\infty} \frac{(\hbar\omega)^2}{n!} \left(\partial_\alpha^n \check{A}(\alpha^*, \alpha)\right)\left(\partial_{\alpha^*}^n \check{B}(\alpha^*, \alpha)\right)\,. \tag{7.21}$$

This formalism can be applied to all non-zero modes of free-field theory by replacing $\hat{a}$ with the mode operators $\hat{a}_n$ and substituting the corresponding frequencies $\omega_n = n$. However, for the zero mode operators $\hat{p}, \hat{q}$ one cannot use the holomorphic representation since here $\omega_0 = 0$. We therefore introduce another map between operators and symbols by

$$e^{2\alpha q} A(p) \longleftrightarrow e^{\alpha \hat{q}} A(\hat{p}) e^{\alpha \hat{q}} \,. \tag{7.22}$$

This is essentially the Weyl symbol, corresponding to symmetrized operators, but since in Liouville theory we only encounter operators of the form e^q it is sufficient to define this relation for the exponentials. The star-product of such two symbols is

$$e^{2\alpha q} A(p) * e^{2\beta q} B(p) = e^{2(\alpha+\beta) q} A(p - i\hbar\beta) B(p + i\hbar\alpha) \,. \tag{7.23}$$

Expanding this in terms of $\hbar$ as well we get

$$\begin{aligned} e^{2\alpha q} A(p) * e^{2\beta q} B(p) = e^{2(\alpha+\beta) q} \Bigg[& A(p)B(p) + i\hbar \left(A(p) \frac{\partial B}{\partial p} \alpha - \frac{\partial A}{\partial p} \beta \right) \\ & + \frac{(i\hbar)^2}{2!} \left(A(p) \frac{\partial^2 B}{\partial p^2} \alpha^2 + \frac{\partial^2 A}{\partial p^2} B \beta^2 - 2 \frac{\partial A}{\partial p} \frac{\partial B}{\partial p} \alpha\beta \right) + \mathcal{O}(\hbar^3) \Bigg] . \end{aligned} \tag{7.24}$$

Putting together the calculations from above we can write the full star-product of two symbols

$$\begin{aligned} \check{A} * \check{B} = & \check{A}\check{B} + \hbar \left[\sum_{\substack{n>0 \\ c=a,\bar{a}}} n \frac{\partial \check{A}}{\partial c_n} \frac{\partial \check{B}}{\partial c_n^*} + \frac{i}{2} \left(\frac{\partial \check{A}}{\partial q} \frac{\partial \check{B}}{\partial p} - \frac{\partial \check{B}}{\partial q} \frac{\partial \check{A}}{\partial p} \right) \right] \\ & + \frac{\hbar^2}{2} \Bigg[\sum_{\substack{n,m>0 \\ c,d=a,\bar{a}}} nm \frac{\partial^2 \check{A}}{\partial c_n \partial d_m} \frac{\partial^2 \check{B}}{\partial c_n^* \partial d_m^*} \\ & \qquad + i \sum_{\substack{n>0 \\ c=a,\bar{a}}} n \left(\frac{\partial^2 \check{A}}{\partial q \partial c_n} \frac{\partial^2 \check{B}}{\partial p \partial c_n^*} - \frac{\partial^2 \check{A}}{\partial p \partial c_n} \frac{\partial^2 \check{B}}{\partial q \partial c_n^*} \right) \\ & \qquad - \frac{1}{4} \left(\frac{\partial^2 \check{A}}{\partial q^2} \frac{\partial^2 \check{B}}{\partial p^2} + \frac{\partial^2 \check{A}}{\partial p^2} \frac{\partial^2 \check{B}}{\partial q^2} - 2 \frac{\partial^2 \check{A}}{\partial q \partial p} \frac{\partial^2 \check{B}}{\partial q \partial p} \right) \Bigg] + \mathcal{O}(\hbar^3). \end{aligned} \tag{7.25}$$

7.1.3 Moyal Bracket

One can now compute the symbol of the operator commutator by the star-product commutator, the so-called Moyal bracket

$$\{\check{A}.\check{B}\}_* \equiv \frac{i}{\hbar} \left(\check{A} * \check{B} - \check{B} * \check{A} \right) \,. \tag{7.26}$$

For a single non-zero mode we can substitute the expansion of the star-product in $\hbar$ into this expression, which shows that the Moyal bracket is a deformation of the classical Poisson bracket

$$\{\check{A}, \check{B}\}_* = \{\check{A}, \check{B}\}_{P.B.} + i\hbar \frac{n^2}{2!} \left(\frac{\partial^2 \check{A}}{\partial a_n^2} \frac{\partial^2 \check{B}}{(\partial a_n^*)^2} - \frac{\partial^2 \check{B}}{\partial a_n^2} \frac{\partial^2 \check{A}}{(\partial a_n^*)^2} \right) + \mathcal{O}(\hbar^2). \tag{7.27}$$

Finally the full Moyal bracket of two symbols can be found by inserting the star-product (7.25) into the definition (7.26), which results in

$$\{\check{A},\check{B}\}_* = \{\check{A},\check{B}\}_{P.B.} + \hbar X_1(\check{A},\check{B}) + \mathcal{O}(\hbar^2) \tag{7.28}$$

with

$$\begin{aligned} X_1(\check{A},\check{B}) = \frac{i}{2} \sum_{\substack{n,m>0 \\ c,d=a,\bar{a}}} nm \left[\frac{\partial^2 \check{A}}{\partial c_n \partial d_m} \frac{\partial^2 \check{B}}{\partial c_n^* \partial d_m^*} - \frac{\partial^2 \check{B}}{\partial c_n \partial d_m} \frac{\partial^2 \check{A}}{\partial c_n^* \partial d_m^*} \right] \\ + \frac{1}{2} \sum_{\substack{n>0 \\ c=a,\bar{a}}} n \left[\frac{\partial^2 \check{A}}{\partial p \partial c_n} \frac{\partial^2 \check{B}}{\partial q \partial c_n^*} - \frac{\partial^2 \check{A}}{\partial q \partial c_n} \frac{\partial^2 \check{B}}{\partial p \partial c_n^*} - \frac{\partial^2 \check{B}}{\partial p \partial c_n} \frac{\partial^2 \check{A}}{\partial q \partial c_n^*} + \frac{\partial^2 \check{B}}{\partial q \partial c_n} \frac{\partial^2 \check{A}}{\partial p \partial c_n^*} \right] . \end{aligned} \tag{7.29}$$

Note that X_1 consists of second derivatives only and higher order terms in $\hbar$ consist of higher derivatives also. Therefore the Moyal brackets of a symbol that is linear coincide with its Poisson brackets. Furthermore, if one symbol is at most quadratic then expression (7.28) will terminate after X_1.

7.2 Construction of Operators

7.2.1 Symmetry Generators and Hamiltonian

We are now going to construct the quantum versions of the symmetry generators $\phi'(x) \equiv \partial_x \Phi(x,\bar{x})$, $T(x)$ and their antichiral counterparts. The operator $\phi'(x)$ is linear in p, q and a_n and therefore there is no ordering ambiguity. The symbol of this operator is then simply the classical function itself

$$\check{\phi}'(x) = \phi'(x) \, . \tag{7.30}$$

The same argument also holds for the antichiral symbol $\check{\bar{\phi}}'(\bar{x}) = \bar{\phi}'(\bar{x})$ and since these symbols are linear they trivially satisfy the same algebra as classically

$$\{\check{\phi}'(x), f(y)\}_* = \{\phi'(x), f(y)\}_{P.B.} \, . \tag{7.31}$$

For the symbol of the operator $\hat{T}(x)$ the demand that it satisfy the classical relation (5.53b) up to a central extension

$$\{\check{T}(x), \phi'(y)\}_* = \phi''(y)\delta(x-y) - \phi'(y)\delta'(x-y) + \frac{\eta}{2\gamma}\delta''(x-y) \tag{7.32}$$

gives a variational equation for $\check{T}(x)$ which can be integrated to

$$\check{T}(x) = \phi'^2(x) - \frac{\eta}{\gamma}\phi''(x) + C(p,x) \, . \tag{7.33}$$

Note that this $\check{T}$ is quadratic and linear in ϕ and does not depend on q. Hence, its Moyal brackets are equal to the Poisson brackets plus a linear term in $\hbar$. Using the identity

$$\sum_{n=1}^{k-1} n(k-n) = \frac{k^3-k}{6} \tag{7.34}$$

the Moyal bracket of $\check{T}$ with itself can be found to be

$$\begin{aligned}\{\check{T}(x),\check{T}(y)\}_* =&\check{T}'(y)\delta(x-y) - 2\check{T}(y)\delta'(x-y) + \left(\frac{1}{2\gamma^2}\eta^2 + \frac{\hbar}{24\pi}\right)\delta'''(x-y)\\ &+ \frac{\hbar}{24\pi}\delta'(x-y) - C'(p,y)\delta(x-y) + 2C(p,y)\delta'(x-y)\,.\end{aligned} \tag{7.35}$$

Since we only allow central extensions a comparison with the classical relation (5.53c) shows that $C(p,x)\equiv 0$. However, η remains undetermined for now. The antichiral symbol $\check{\bar{T}}$ can be obtained in the same way.

In analogy to the free-field theory we choose the quantum version of the Hamiltonian (2.48) to be the normal ordered operator. The symbol then coincides with the classical function

$$\check{H} = H = \frac{p^2}{4\pi} + \sum_{n>0} a_n a_n^* + \sum_{n>0} \bar{a}_n \bar{a}_n^*\,. \tag{7.36}$$

With this choice $\phi'(x)$ trivially satisfies the Heisenberg equation as its Moyal bracket is given by its classical Poisson bracket. Also the relation

$$\check{H} = \int_0^{2\pi} d\sigma\,\left(\check{T}(\sigma) + \check{\bar{T}}(\sigma)\right) \tag{7.37}$$

is preserved because the deformation of $\check{T}$ is in the coefficient of a total derivative of a periodic function. Since furthermore the central extension of the Moyal bracket of $\check{T}$ is also a total derivative of a periodic function $\check{T}$ satisfies the Heisenberg equation

$$\{\check{H},\check{T}(x)\}_* = \partial_\tau \check{T}(x)\,. \tag{7.38}$$

7.2.2 Free-Field Exponential

We now turn to the construction of the symbols for the free-field exponentials

$$\check{E}_\alpha(x,\bar{x}) \longleftrightarrow e^{2\alpha\gamma\Phi(x,\bar{x})}\,. \tag{7.39}$$

The requirement here is that the symmetry generators act in a similar fashion as the classical ones. Commutation with $\phi'(x)$ as in (5.51) gives the condition

$$\{\phi'(x),\check{E}_\alpha(y,\bar{y})\} = \alpha\gamma\check{E}_\alpha(y,\bar{y}) \tag{7.40}$$

plus a similar condition for the antichiral symbol $\bar{\phi}'$. These are easily solved by $\check{E}_\alpha(x) = C_\alpha(p,x,\bar{x})e^{2\alpha\gamma\phi(x,\bar{x})}$ with an arbitrary coefficient $C_\alpha(p,x,\bar{x})$. Commutation with the energy-momentum tensor yields

$$\begin{aligned}\{\check{T}(x),\check{E}_\alpha(y,\bar{y})\}_* =&\partial_x\left(\check{E}_\alpha(y,\bar{y})\right)\delta(x-y) - \left(\alpha\eta - \alpha^2 b^2\right)\delta'(x-y)\check{E}_\alpha(y,\bar{y})\\ &- \partial_x C_\alpha(p,x,\bar{x})\check{E}_\alpha(x,\bar{x})\delta(x-y)\,.\end{aligned} \tag{7.41}$$

with

$$b^2 \equiv \frac{\hbar\gamma^2}{2\pi}\,. \tag{7.42}$$

Comparing this Moyal bracket (and the corresponding one with $\check{\bar{T}}$) to (5.43) we see that we have to take $C_\alpha(p,x,\bar{x})$ to be constant in x and $\bar{x}$. The Heisenberg equation is then also satisfied. Furthermore we note that the conformal weight is now deformed to

$$\Delta_\alpha = \alpha\eta - \alpha^2 b^2 \,. \tag{7.43}$$

For the symbol of the in-field, $\alpha = -1/2$, we can make a redefinition of q to absorb the p dependent factor. We have then fixed the symbol of the in-field

$$\check{E}(x,\bar{x}) = e^{-\gamma\Phi(x,\bar{x})} \,, \tag{7.44}$$

which according to the remarks about Moyal formalism is the symbol of the normal ordered operator

$$: e^{-\gamma\hat{\Phi}(x,\bar{x})} := e^{-\gamma(\hat{q}+\frac{\hat{p}}{2\pi}\tau)} e^{-\gamma(\hat{\phi}^+(x)+\hat{\bar{\phi}}^+(\bar{x}))} e^{-\gamma(\hat{\phi}^-(x)+\hat{\bar{\phi}}^-(\bar{x}))} \,. \tag{7.45}$$

with

$$\hat{\phi}^-(x) \equiv \frac{i}{\sqrt{4\pi}} \sum_{n>0} \frac{\hat{a}_n}{n} e^{-inx} \,, \qquad \hat{\phi}^+(x) \equiv \frac{-i}{\sqrt{4\pi}} \sum_{n>0} \frac{\hat{a}_n^\dagger}{n} e^{inx} \,, \tag{7.46}$$

and similar definitions of the antichiral operators.

7.2.3 Short Distance Singularity

We now consider the product of two free-field exponentials without their p dependent coefficients at equal times but separated by a distance ϵ with $|\epsilon| < 2\pi$. In the operator language this can be calculated with the BCH formula

$$\begin{aligned} &: e^{\alpha\gamma\hat{\Phi}(x,\bar{x})} :: e^{\beta\gamma\hat{\Phi}(x+\epsilon,\bar{x}-\epsilon)} : \\ &\qquad =: e^{\gamma\left(\alpha\hat{\Phi}(x,\bar{x})+\beta\hat{\Phi}(x+\epsilon,\bar{x}-\epsilon)\right)} : e^{\alpha\beta\gamma^2\left([\hat{\phi}^-(x),\hat{\phi}^+(x+\epsilon)]+[\hat{\bar{\phi}}^-(\bar{x}),\hat{\bar{\phi}}^+(\bar{x}-\epsilon)]\right)} \end{aligned} \tag{7.47}$$

In this expression the two commutators

$$[\hat{\phi}^-(x), \hat{\phi}^+(x+\epsilon)] = \frac{\hbar}{4\pi} \sum_{n>0} \frac{1}{n} e^{in\epsilon} \,, \quad [\hat{\bar{\phi}}^-(\bar{x}), \hat{\bar{\phi}}^+(\bar{x}-\epsilon)] = -\frac{\hbar}{4\pi} \sum_{n<0} \frac{1}{n} e^{in\epsilon} \tag{7.48}$$

are divergent, but one can regularize them by replacing ϵ with $\epsilon + i\delta$ where $\delta > 0$ and taking the limit $\delta \to 0$. Then we can insert the identity

$$\sum_{n>0} \frac{1}{n} z^n = -\log(1-z) \qquad\qquad |z| < 1 \,. \tag{7.49}$$

with $z = e^{in\epsilon}$. The sum of the commutators then becomes

$$[\hat{\phi}^-(x), \hat{\phi}^+(x+\epsilon)] + [\hat{\bar{\phi}}^-(\bar{x}), \hat{\bar{\phi}}^+(\bar{x}-\epsilon)] = -\frac{\hbar}{2\pi} \log\left|2\sin\frac{\epsilon}{2}\right| \tag{7.50}$$

and the product of two free-field exponentials is found to be

$$: e^{\alpha\gamma\hat{\Phi}(x,\bar{x})} :: e^{\beta\gamma\hat{\Phi}(x+\epsilon,\bar{x}-\epsilon)} :=: e^{\gamma\left(\alpha\hat{\Phi}(x,\bar{x})+\beta\hat{\Phi}(x+\epsilon,\bar{x}-\epsilon)\right)} : \left|2\sin\frac{\epsilon}{2}\right|^{-\alpha\beta\, b^2} \,. \tag{7.51}$$

The product of two free-field exponentials with coefficients of equal sign at equal space and time is therefore singular.

7.2.4 Screening Charge

Classically the out-field exponential can be written as an integral over a bilocal field

$$\chi(x)\chi(\bar{x}) = \int_0^{2\pi}\int_0^{2\pi} dz\, d\bar{z}\, B(x,\bar{x},x+z,\bar{x}+\bar{z}) \tag{7.52}$$

with

$$B(x,\bar{x},y,\bar{y}) = \frac{m^2}{4\sinh^2\left(\frac{\gamma p}{2}\right)} e^{-\gamma p} e^{-\gamma\Phi(x,\bar{x})} e^{2\gamma\Phi(y,\bar{y})}\,. \tag{7.53}$$

This field has the classical Poisson bracket with ϕ'

$$\{\phi'(x), B(y,\bar{y},z,\bar{z})\} = \frac{\gamma}{2} B(y,\bar{y},z,\bar{z})\delta(x-y) + \gamma B(y,\bar{y},z,\bar{z})\delta(x-z)\,. \tag{7.54}$$

We now assume that the quantum version of the out-field exponential $\check{F}$ can also be written as the integral over a bilocal field. Demanding that the symbol have the same Poisson bracket with $\phi'(x)$ as the classical field leads to the symbol

$$\check{B}(y,\bar{y},z,\bar{z}) = C(p,y,\bar{y},z,\bar{z}) e^{-\gamma\Phi(y,\bar{y})} e^{2\gamma\Phi(z,\bar{z})}\,. \tag{7.55}$$

The classical Poisson bracket of $T(x)$ with B can easily be found from (5.43)

$$\begin{aligned}\{T(x), B(y,\bar{y},z,\bar{z})\} =& \partial_y B\delta(x-y) + \frac{1}{2}B\delta'(x-y)\\ &+ \partial_z B\delta(x-z) - B\delta'(x-z)\,.\end{aligned} \tag{7.56}$$

The Moyal bracket with $\check{T}(x)$ can be calculated in a similar way as before and we find

$$\begin{aligned}\{T(x), \check{B}(y,\bar{y},z,\bar{z})\}_* =& \partial_y B(y,\bar{y},z,\bar{z})\delta(x-y) + \partial_z B(y,\bar{y},z,\bar{z})\delta(x-z)\\ &+ \frac{1}{2}\left(\eta + \frac{1}{2}b^2\right) B(y,\bar{y},z,\bar{z})\delta'(x-y) - \left(\eta - b^2\right) B(y,\bar{y},z,\bar{z})\delta'(x-z)\\ &+ e^{-\gamma\Phi(y,\bar{y})} e^{2\gamma\Phi(z,\bar{z})}\left(\frac{1}{2}b^2 C\cot\left(\frac{1}{2}(y-z)\right) - \partial_y C\right)\delta(x-y)\\ &- e^{-\gamma\Phi(y,\bar{y})} e^{2\gamma\Phi(z,\bar{z})}\left(\frac{1}{2}b^2 C\cot\left(\frac{1}{2}(y-z)\right) + \partial_z C\right)\delta(x-z)\,.\end{aligned} \tag{7.57}$$

Here we have used

$$\begin{aligned}2i\sum_{k>1} e^{-ik(x-y)}\sum_{m=1}^{k-1} e^{-im(y-z)} + i\sum_{k>0}\left(e^{-ik(x-y)} + e^{-ik(x-z)}\right) + \text{c.c.} =&\\ 2\pi\cot\left(\frac{1}{2}(y-z)\right)\left(\delta(x-y) - \delta(x-z)\right)&\,.\end{aligned} \tag{7.58}$$

In order for the last two anomalous lines in (7.57) to disappear C has to be of the form $C = u_p\, c(y-z,\bar{y}-\bar{z})$ where c satisfies the equation

$$\partial_y c(y-z,\bar{y}-\bar{z}) = \frac{b^2}{2}\cot\left(\frac{y-z}{2}\right) \tag{7.59}$$

The solution of this and the corresponding condition for the antichiral part is up to an integration constant

$$C(p, y-z, \bar{y}-\bar{z}) = u_p \left(4\sin^2\left(\frac{y-z}{2}\right)\right)^{\frac{b^2}{2}} \left(4\sin^2\left(\frac{\bar{y}-\bar{z}}{2}\right)\right)^{\frac{b^2}{2}} . \tag{7.60}$$

Integration of the Moyal bracket then results in the following relation of $\check{T}$ and $\check{F}$

$$\begin{aligned}\{\check{T}(x),\check{F}(y,\bar{y})\}_* &= \partial_y \check{F}(y,\bar{y})\delta(x-y) + \frac{1}{2}\left(\eta + \frac{b^2}{2}\right)\check{F}(y,\bar{y})\delta'(x-y)\\ &- \left(\eta - 1 - b^2\right)\int_0^{2\pi} d\bar{z}\, \partial_x B(y,\bar{y},x,\bar{y}+\bar{z})\\ &+ \left(\eta - 1 - b^2\right)\int_0^{2\pi} d\bar{z}\, B(y,\bar{y},y,\bar{y}+\bar{z})(e^{\gamma p}-1)\delta(x-y) .\end{aligned} \tag{7.61}$$

Requiring again that the anomalous terms vanish leads us to the condition

$$\eta = 1 + b^2 . \tag{7.62}$$

From this follows that both the in- and out-field transform with a conformal weight

$$\Delta_{-\frac{1}{2}} = -\frac{1}{2} - \frac{3}{4}b^2 \tag{7.63}$$

and therefore the full field $\check{V}$ is also a primary. Inserting (7.62) into (7.35) we fix the central charge of the Virasoro algebra [25].

In order to write the operator for the symbol $\check{F}$ note that the function C (7.60) contains the short distance factor from (7.51). The operator can therefore be written as the product of two normal ordered exponential free fields

$$\hat{F}(x,\bar{x}) = u_p \hat{E}(x,\bar{x})\hat{A}(x,\bar{x}) \tag{7.64}$$

with

$$\hat{A}(x,\bar{x}) = \int_0^{2\pi}\int_0^{2\pi} dz\, d\bar{z}\ : e^{2\gamma\Phi(x+z,\bar{x}+\bar{z})} : e^{-\gamma(p-i\hbar\gamma)} \tag{7.65}$$

here we have included a part of the p-dependent factor in the definition of $\hat{A}$, which makes for simpler exchange relations.

We have now fixed the operator $\hat{F}$ except for the function u_p. This function can be determined from the locality condition (7.4), for which we need the exchange relations of $\hat{E}$ and $\hat{F}$.

7.2.5 Exchange Algebra

In this section we establish the exchange algebra of the objects $\hat{E}$ and $\hat{F}$, that are the building blocks of the vertex operator $\hat{V}$. The exponential free-fields obey a simple exchange relation for exchange of only the chiral and antichiral dependence

$$\hat{E}(x,\bar{x})\hat{E}(y,\bar{y}) = \hat{E}(y,\bar{x})\hat{E}(x,\bar{y})e^{-\frac{i}{2}\pi b^2\epsilon(x-y)} \tag{7.66}$$

$$= \hat{E}(x,\bar{y})\hat{E}(y,\bar{x})e^{-\frac{i}{2}\pi b^2\epsilon(\bar{x}-\bar{y})} , \tag{7.67}$$

This can be calculated using the BCH formula. Combination immediately yields the full exchange relation

$$\hat{E}(x,\bar{x})\hat{E}(y,\bar{y}) = \hat{E}(y,\bar{y})\hat{E}(x,\bar{x})e^{-\frac{i}{2}\pi b^2(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y}))}\,. \tag{7.68}$$

A bit more complicated is the derivation of the exchange relation of $\hat{E}$ and $\hat{A}$. Exchanging the integrand of $\hat{A}$ with $\hat{E}$ leads to

$$\hat{E}(x,\bar{x})\hat{A}(y,\bar{y}) = \int_0^{2\pi}\int_0^{2\pi} dz\,d\bar{z}\,\hat{E}_1(x+z,\bar{x}+\bar{z})e^{-(\gamma\hat{p}-4\pi i b^2)} \times \hat{E}(x,\bar{x})e^{i\pi b^2(\epsilon(x-y-z)+\epsilon(\bar{x}-\bar{y}-\bar{z}))}\,. \tag{7.69}$$

For $x=y$ and $\bar{x}=\bar{y}$ we find $\epsilon(x-y-z)+\epsilon(\bar{x}-\bar{y}-\bar{z})=-2$ since $z\in(0,2\pi)$, and therefore

$$\hat{E}(x,\bar{x})\hat{A}(x,\bar{x}) = \hat{A}(x,\bar{x})\hat{E}(x,\bar{x})\,. \tag{7.70}$$

In the general case when $x\neq y$ we can use the identity

$$e^{i\pi b^2(\epsilon(x-y-z)+\epsilon(\bar{x}-\bar{y}\bar{z}))} = \frac{1}{\sinh(\pi P)}\Big[\sinh\left(\pi P - i\pi b^2\right)e^{i\pi b^2(\epsilon(x-y)-\epsilon(z))} + i\sin\left(\pi b^2\right)e^{\pi P(\epsilon(x-y-z)-\epsilon(x-y)+\epsilon(z))}e^{i\pi b^2(\epsilon(x-y)-\epsilon(z))}\Big] \tag{7.71}$$

with $P\equiv\frac{\gamma p}{2\pi}$, which is derived from

$$\sinh(a)e^{\pm b}+\sinh(b)e^{\pm a} = \sinh(a+b) \tag{7.72}$$

and the property of the stair-step function

$$\epsilon(x-y-z)-\epsilon(x-y)+\epsilon(z) = \pm 1\,. \tag{7.73}$$

Insertion of this identity into the exchange relation leads to four terms under the integral. One can now use the periodicity in z or $\bar{z}$ of some of these terms to shift the domain of integration, which results in

$$\begin{aligned}\hat{E}(x,\bar{x})\hat{A}(y,\bar{y}) =& \frac{\sinh^2(\pi P)}{\sinh^2(\pi(P+ib^2))}\Big[e^{i\pi b^2(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y}))}\hat{A}(y,\bar{y})\hat{E}(x,\bar{x})\\ &- 2i\sin\left(\pi b^2\right)e^{i\pi b^2\epsilon(\bar{x}-\bar{y})}\theta_{-2\pi P}(x-y)\hat{A}(x,\bar{y})\hat{E}(x,\bar{x})\\ &- 2i\sin\left(\pi b^2\right)e^{i\pi b^2\epsilon(x-y)}\theta_{-2\pi P}(\bar{x}-\bar{y})\hat{A}(y,\bar{x})\hat{E}(x,\bar{x})\\ &- 4\sin^2\left(\pi b^2\right)\theta_{-2\pi P}(x-y)\theta_{-2\pi P}(\bar{x}-\bar{y})\hat{A}(x,\bar{x})\hat{E}(x,\bar{x})\Big].\end{aligned} \tag{7.74}$$

With the exchange relation for partial exchange of $\hat{E}$ with itself (7.66) it is easy to find the exchange relation between the free-field exponential and $\hat{F}$

$$\begin{aligned}\hat{E}(x,\bar{x})\hat{F}(y,\bar{y}) =& \frac{u_{P-ib^2}}{u_P}\frac{\sinh^2(\pi P)}{\sinh^2(\pi P+i\pi b^2)}\Big[e^{\frac{i}{2}\pi b^2(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y}))}\hat{F}(y,\bar{y})\hat{E}(x,\bar{x})\\ &- 2i\sin\left(\pi b^2\right)e^{\frac{i}{2}\pi b^2\epsilon(\bar{x}-\bar{y})}\theta_{-2\pi P}(x-y)\hat{F}(x,\bar{y})\hat{E}(y,\bar{x})\\ &- 2i\sin\left(\pi b^2\right)e^{\frac{i}{2}\pi b^2\epsilon(x-y)}\theta_{-2\pi P}(\bar{x}-\bar{y})\hat{F}(y,\bar{x})\hat{E}(x,\bar{y})\\ &- 4\sin^2\left(\pi b^2\right)\theta_{-2\pi P}(x-y)\theta_{-2\pi P}(\bar{x}-\bar{y})\hat{F}(x,\bar{x})\hat{E}(y,\bar{y})\Big]\,.\end{aligned} \tag{7.75}$$

We assume here that the operator $\hat{F}$ is related to $\hat{E}$ by a unitary transformation. Therefore $\hat{F}$ must have the same exchange relation as (7.68), namely

$$\hat{F}(x,\bar{x})\hat{F}(y,\bar{y}) = \hat{F}(y,\bar{y})\hat{F}(x,\bar{x})e^{-\frac{i}{2}\pi b^2(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y}))}\,. \tag{7.76}$$

7.2.6 Locality Condition

With the exchange algebra of the building blocks of the vertex operator (7.3) we can now look at the commutator of $\hat{V}$ with itself. Locality then implies that the equal time commutator for vertex operators at different points in space must vanish (7.4). For simplicity we will set time to zero. The demand is then equivalent to the condition that the expression

$$\begin{aligned}\hat{V}(x,-x)\hat{V}(y,-y) =& \hat{E}(x,-x)\hat{E}(y,-y) + \hat{F}(x,-x)\hat{F}(y,-y) \\ &+ f_1(P)\Bigg[f_2(P)\Big(e^{-\left(\pi P + \frac{i}{2}\pi b^2\right)\epsilon(x-y)}\hat{F}(x,-y)\hat{E}(y,-x) \\ &+ e^{\left(\pi P + \frac{i}{2}\pi b^2\right)\epsilon(x-y)}\hat{F}(y,-x)\hat{E}(x,-y)\Big) \\ &+ \hat{F}(y,-y)\hat{E}(x,-x) + f_3(P)\hat{F}(x,-x)\hat{E}(y,-y)\Bigg] . \end{aligned} \tag{7.77}$$

with

$$\begin{aligned} f_1(P) &= \frac{u_{P-ib^2}}{u_P}\frac{\sinh^2(\pi P)}{\sinh^2(\pi P + i\pi b^2)}, \qquad f_2(P) = \frac{i\sin\left(\pi b^2\right)}{\sinh^2(\pi P)}, \\ f_3(P) &= \frac{u_P}{u_{P-ib^2}}\frac{\sinh^2\left(\pi P + i\pi b^2\right)}{\sinh^2(\pi P)} + \frac{\sinh^2(i\pi b^2)}{\sinh^2(\pi P)} \end{aligned} \tag{7.78}$$

is symmetric in x and y. The first three lines are symmetric as one can check using (7.68) and (7.76). Symmetry of the last line however gives a condition on u_P in form of a difference equation

$$\left(\sinh^2(\pi P) - \sinh^2(i\pi b^2)\right) u_{P-ib^2} = \sinh^2(\pi P + i\pi b^2)\, u_P . \tag{7.79}$$

This is solved by

$$u_P = m_b^2 \frac{1}{2\sinh(\pi P)} \frac{1}{2\sinh(\pi P + i\pi b^2)}, \tag{7.80}$$

where m_b is a deformed constant with the classical limit $m_b \to m$. The out-field operator is thus

$$\hat{F}(x,\bar{x}) = \frac{m_b^2}{2\sinh(\pi P)}\hat{E}(x,\bar{x})\hat{S}(x,\bar{x})\frac{1}{2\sinh(\pi P)} . \tag{7.81}$$

After insertion of the explicit operators in terms of the free-field ϕ one can use the commutation rules to group the zero modes and normal order the oscillatory parts of the two operators. The regularized operator is then

$$\begin{aligned}\hat{F}(x,\bar{x}) &= e^{\gamma\hat{q}+\hat{P}\tau}\frac{m_b}{2\sinh(\pi(P-ib^2))}\frac{m_b}{2\sinh(\pi P)}e^{-2\pi(P-ib^2)} \\ &\times \int_0^{2\pi} dz\, e^{(P-ib^2)z}(1-e^{iz})^{b^2} e^{\gamma(2\hat{\phi}^+(x+z)-\hat{\phi}^+(x))} e^{\gamma(2\hat{\phi}^-(x+z)-\hat{\phi}^-(x))} \\ &\times \int_0^{2\pi} d\bar{z}\, e^{(P-ib^2)\bar{z}}(1-e^{i\bar{z}})^{b^2} e^{\gamma(2\hat{\bar{\phi}}^+(\bar{x}+\bar{z})-\hat{\bar{\phi}}^+(\bar{x}))} e^{\gamma(2\hat{\bar{\phi}}^-(\bar{x}+\bar{z})-\hat{\bar{\phi}}^-(\bar{x}))} . \end{aligned} \tag{7.82}$$

7.3 Reflection Amplitude

Here we assume that the operators for the in- and out-field are related by a unitary transformation with the S-matrix

$$\hat{E}(x,\bar{x})\hat{\mathcal{S}} = \hat{\mathcal{S}}\hat{F}(x,\bar{x})\,. \tag{7.83}$$

Furthermore we assume that this scattering matrix can be written as the product of the parity operator $\hat{\mathcal{P}}$, which inverts the momentum $\hat{p}$ and the position $\hat{q}$, a p-dependent part $\hat{S}$ that only acts on the non-zero modes, and finally the reflection amplitude $\hat{R}$ that acts on the vacuum

$$\hat{\mathcal{S}} = \hat{\mathcal{P}}\,\hat{R}\,\hat{S}\,. \tag{7.84}$$

Projecting equation (7.83) onto two vacuum states

$$\langle 0,P|\hat{E}(x,\bar{x})\hat{\mathcal{S}}|0,P'\rangle = \langle 0,P|\hat{\mathcal{S}}\hat{F}(x,\bar{x})|0,P'\rangle\,. \tag{7.85}$$

vanishes the normal ordered oscillatory parts of the operators and we get a relation for the reflection amplitude

$$R(P) = R(P-ib^2)D(P) \tag{7.86}$$

with

$$D(P) = \frac{e^{-2(\pi P - i\pi b^2)}}{2\sinh(\pi P - i\pi b^2)}\frac{m_b^2}{2\sinh(\pi P)}\left(\int_0^{2\pi} dz\, e^{\frac{1}{\pi}(\pi P - i\pi b^2)z}\left(1-e^{iz}\right)^{b^2}\right)^2 \tag{7.87}$$

Here one can use the integral [65]

$$\int_0^{2\pi} dy\, e^{\rho y}\left(1-e^{\pm iy}\right)^{\alpha} = 2\pi\,\frac{\Gamma(1+\alpha)}{\Gamma(1\pm i\rho)\Gamma(1+\alpha\mp i\rho)}\,e^{\pi\rho} \tag{7.88}$$

to express the integral through gamma functions. Using the formulas

$$\frac{\pi}{\sinh(\pi z)} = \frac{1}{z}\Gamma(1-iz)\Gamma(1+iz) \tag{7.89}$$

and $\Gamma(1+z) = z\Gamma(z)$ for the gamma-function one can further simplify this to

$$D(P) = m_b^2\frac{(b^2)^2}{P(P-ib^2)}\Gamma^2\left(b^2\right)\frac{\Gamma\left(iP\right)\Gamma\left(-iP-b^2\right)}{\Gamma\left(-iP\right)\Gamma\left(iP+b^2\right)} \tag{7.90}$$

The ansatz

$$R(P) = \tilde{R}(P)\frac{\Gamma(iP)}{\Gamma(-iP)} \tag{7.91}$$

reduces the formula to

$$\tilde{R}(P) = m_b^2\frac{(b^2)^2}{P(P-ib^2)}\Gamma^2\left(b^2\right)\tilde{R}(P-ib^2)\,. \tag{7.92}$$

One can easily check that this equation is solved by

$$\tilde{R}(P) = -\left(m_b^2\Gamma^2\left(b^2\right)\right)^{-\frac{iP}{b^2}}\frac{\Gamma(iP/b^2)}{\Gamma(-iP/b^2)}\,. \tag{7.93}$$

The final result for the reflection amplitude with b expressed in terms of the original coupling constant γ is therefore

$$R(p) = -\left(m_b\,\Gamma\left(\frac{\hbar\gamma^2}{2\pi}\right)\right)^{-i\frac{2p}{\hbar\gamma}}\ \frac{\Gamma\left(i\frac{p}{\hbar\gamma}\right)\Gamma\left(i\frac{\gamma p}{2\pi}\right)}{\Gamma\left(-i\frac{p}{\hbar\gamma}\right)\Gamma\left(-i\frac{\gamma p}{2\pi}\right)}\,. \tag{7.94}$$

This reflection amplitude was first obtained in [31] by the analysis of the 3- and 2-point functions of Liouville theory [29, 31]. Note that for complete agreement with this result we have to set

$$m_b^2 = \frac{\sin(\frac{\pi}{2}b^2)}{\frac{\pi}{2}b^2}m^2\,. \tag{7.95}$$

This 'renormalization' of the Liouville mass was first derived in[27, 66] in the operator approach based on a free-field parameterization similar to the one used here.

VIII
Quantization of the $SL(2,\mathbb{R})/U(1)$ Model

In this chapter we consider quantization of the $SL(2,\mathbb{R})/U(1)$ model following the same line as for Liouville theory. In the last two sections we present two new results. Namely, in section 8.3 we calculate the non-equal time commutator of the interacting u-field. The commutator preserves causal and local structure of the corresponding classical Poisson bracket with a consistent quantum deformation. Finally in section 8.4 we calculate the discrete spectrum of the bound states by an analytical continuation of the reflection amplitude from the hyperbolic to the elliptic sector.

In contrast to Liouville theory here we have two free-fields

$$\Phi_1(x,\bar{x}) \equiv q_1 + \frac{p_1}{4\pi}(x+\bar{x}) + \phi_1^+(x) + \phi_1^-(x) + \bar{\phi}_1^+(\bar{x}) + \bar{\phi}_1^-(\bar{x})\,, \tag{8.1}$$

$$\Phi_2(x,\bar{x}) \equiv q_2 + \frac{p_2}{4\pi}(x+\bar{x}) + \frac{\nu}{2\gamma}(x-\bar{x}) + \phi_2^+(x) + \phi_2^-(x) + \bar{\phi}_2^+(\bar{x}) + \bar{\phi}_2^-(\bar{x})\,, \tag{8.2}$$

with

$$\phi_1^-(x) = \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{a_n}{n}e^{-inx}\,, \qquad \phi_1^+(x) = \frac{-i}{\sqrt{4\pi}}\sum_{n>0}\frac{a_n^*}{n}e^{inx}\,, \tag{8.3a}$$

$$\phi_2^-(x) = \frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{b_n}{n}e^{-inx}\,, \qquad \phi_2^+(x) = \frac{-i}{\sqrt{4\pi}}\sum_{n>0}\frac{b_n^*}{n}e^{inx}\,. \tag{8.3b}$$

and the analogous antichiral part. Since the derivatives with respect to x and $\bar{x}$ are still chiral functions we will keep the notation

$$\phi_j'(x) = \partial_x\Phi_j(x,\bar{x})\,, \qquad \bar{\phi}_j'(x) = \partial_{\bar{x}}\Phi_j(x,\bar{x})\,, \qquad j=1,2\,. \tag{8.4}$$

The operators corresponding to the canonical coordinates satisfy the standard commutation relations

$$[\hat{q}_j,\hat{p}_j] = \delta_{jk}i\hbar\,, \qquad [\hat{a}_n,\hat{a}_m] = [\hat{\bar{a}}_n,\hat{\bar{a}}_m] = [\hat{b}_n,\hat{b}_m] = [\hat{\bar{b}}_n,\hat{\bar{b}}_m] = \hbar n\delta_{n+m}\,, \tag{8.5}$$

and the Hilbert space is spanned by the creation operators acting on the momentum dependent vacuum

$$|\{a,b\},p_1,p_2\rangle = \prod_{n>0}(\hat{a}_n^\dagger)^{n_k}(\hat{b}_n^\dagger)^{n_k}|0,p_1,p_2\rangle\,. \tag{8.6}$$

Here, p_1 is negative and p_2 is quantized, due to the periodicity in q_2. The discrete spectrum we specify later. The vertex operator is assumed to be the sum of incoming and outgoing operators

$$\hat{u}(x,\bar{x}) = \hat{E}(x,\bar{x}) + \hat{F}(x,\bar{x})\,, \tag{8.7}$$

where $\hat{F}$ is given as the integral over a bilocal operator $\hat{B}$. In order to fix these operators we again follow the Moyal formalism.

8.1 Moyal Formalism

For both Φ_1 and Φ_2 we use the symbol calculus of the previous chapter. Therefore, the Moyal bracket of two symbols is given by

$$\{\check{A},\check{B}\}_* = \{\check{A},\check{B}\}_{P.B.} + \hbar X_1(\check{A},\check{B}) + \mathcal{O}(\hbar^2) \tag{8.8}$$

with

$$\begin{aligned} X_1(\check{A},\check{B}) = \frac{i}{2} \sum_{\substack{n,m>0 \\ c,d=a,\bar{a},b,\bar{b}}} nm \left[\frac{\partial^2 \check{A}}{\partial c_n \partial d_m} \frac{\partial^2 \check{B}}{\partial c_n^* \partial d_m^*} - \frac{\partial^2 \check{B}}{\partial c_n \partial d_m} \frac{\partial^2 \check{A}}{\partial c_n^* \partial d_m^*} \right] \\ + \frac{1}{2} \sum_{\substack{n>0 \\ c=a,\bar{a},b,\bar{b} \\ i=1,2}} n \left[\frac{\partial^2 \check{A}}{\partial p_i \partial c_n} \frac{\partial^2 \check{B}}{\partial q_i \partial c_n^*} - \frac{\partial^2 \check{A}}{\partial q_i \partial c_n} \frac{\partial^2 \check{B}}{\partial p_i \partial c_n^*} - \frac{\partial^2 \check{B}}{\partial p_i \partial c_n} \frac{\partial^2 \check{A}}{\partial q_i \partial c_n^*} + \frac{\partial^2 \check{B}}{\partial q_i \partial c_n} \frac{\partial^2 \check{A}}{\partial p_i \partial c_n^*} \right]. \end{aligned} \tag{8.9}$$

Note that all symbols contain e^{iq_2} only in discrete powers due to the quantization of p_2.

8.2 Construction of Operators

8.2.1 Symmetry Generators and Hamiltonian

The chiral functions $\phi_1'(x)$ and $\phi_2'(x)$ are linear in the canonical coordinates and have no ordering ambiguity. Their symbols are equal to the classical functions

$$\check{\phi}_1'(x) = \phi_1'(x) \qquad \check{\phi}_2'(x) = \phi_2'(x) \tag{8.10}$$

and similarly for the antichiral part. Their Moyal brackets obviously coincide with their Poisson brackets.

As in the previous chapter the symbol of the Hamiltonian is also undeformed

$$\check{H} = H = \frac{p_1^2}{4\pi} + \frac{p_2^2}{4\pi} + \sum_{n>0} a_n a_n^* + \sum_{n>0} \bar{a}_n \bar{a}_n^* + \sum_{n>0} b_n b_n^* + \sum_{n>0} \bar{b}_n \bar{b}_n^* \,. \tag{8.11}$$

In order to determine the symbol of the energy-momentum tensor $\check{T}(x)$ we use the algebra (6.80) with a deformed central term.

$$\{\check{T}(x), \phi_j'(y)\} = \phi_j''(y)\delta(x-y) - \phi_j'(y)\delta'(x-y) + \frac{\eta_j}{2\gamma}\delta''(x-y)\,. \tag{8.12}$$

Integration of this variational equation for $\check{T}(x)$ yields

$$\check{T}(x) = \phi_1'^2(x) + \phi_2'^2(x) - \frac{\eta_1}{\gamma}\phi_1''(x) - \frac{\eta_2}{\gamma}\phi_2''(x) + C(p_1,p_2;x)\,, \tag{8.13}$$

with an integration 'constant' $C(p_1,p_2;x)$. Then, similarly to (7.35) we find

$$\begin{aligned} \{\check{T}(x),\check{T}(y)\}_* =& \check{T}'(y)\delta(x-y) - 2T(y)\delta'(x-y) + \left(\frac{\eta_1^2}{2\gamma^2} + \frac{\eta_2^2}{2\gamma^2} + \frac{\hbar}{12\pi}\right)\delta'''(x-y) \\ &+ \frac{\hbar}{12\pi}\delta'(x-y) - C'(p_1,p_2;x)\delta(x-y) + 2C(p_1,p_2;x)\delta'(x-y)\,, \end{aligned} \tag{8.14}$$

and its comparison with (6.81) leads to $C(p_1, p_2; x) \equiv 0$. Here the coefficients η_1 and η_2 remain undetermined for now.

The antichiral symbol

$$\check{\bar{T}}(\bar{x}) = \bar{\phi}_1'^2(\bar{x}) + \bar{\phi}_2'^2(\bar{x}) - \frac{\eta_1}{\gamma}\bar{\phi}_1''(\bar{x}) - \frac{\eta_2}{\gamma}\bar{\phi}_2''(\bar{x}) \tag{8.15}$$

is obtained in a similar way and we take the same deformation parameters η_1, η_2 to preserve the chiral symmetry.

Then we find the relation

$$\check{H} = \int_0^{2\pi} dx\, \check{T}(x) + \int_0^{2\pi} d\bar{x}\, \check{\bar{T}}(\bar{x}), \tag{8.16}$$

since the improved terms disappear after integration.

8.2.2 Free-Field Exponential

Having almost fixed the symbols and thereby the operators of the symmetry generators we now turn to the construction of the symbols corresponding to the classical exponential free-field. Since Φ_1 and Φ_2 commute we can do this separately for each field. We introduce the notation

$$E(x, \bar{x}) \longleftrightarrow e^{\gamma(\Phi_1(x,\bar{x}) + i\Phi_2(x,\bar{x}))} \tag{8.17}$$

for the symbol of the in-field exponential. Demanding that the commutation with $\phi_1'(x)$ and $\phi_2'(x)$ corresponds to (6.76) and (6.77), we consider the two equations

$$\{\phi_1'(x), E(y, \bar{y})\} = \gamma\delta(x-y)E(y, \bar{y}) \tag{8.18a}$$

$$\{\phi_2'(x), E(y, \bar{y})\} = i\gamma\lambda\, \delta(x-y)E(y, \bar{y}) \tag{8.18b}$$

Here λ is a deformation parameter. We allow a deformation of $\phi_2'(x)$ part only. In principle, we could allow an additional deformation of the ϕ_1' part as well, but it could be absorbed by a redefinition of γ. The solution of (8.18a), (8.18b) and the corresponding antichiral versions is

$$E = C(p_1, p_2, x, \bar{x})e^{\gamma(\Phi_1(x,\bar{x}) + i\lambda\Phi_2(x,\bar{x}))}. \tag{8.19}$$

In principle the Moyal formalism does not require the existence of a winding number ν of the field $\Phi_2(x, \bar{x})$. However, we assume that the topological properties of the in-field are the same as for the classical solution. In order to have periodicity we must deform the winding number in (8.2) according to

$$\frac{\nu}{\gamma} \to \frac{\nu}{\gamma\lambda} \tag{8.20}$$

and the spectrum of $\hat{p}_2$ is also deformed

$$\hat{p}_2\, |\{a, b\}, p_1, n\rangle = \hbar\gamma\lambda n\, |\{a, b\}, p_1, n\rangle \qquad n \in \mathbb{Z}. \tag{8.21}$$

The commutation relation of $\hat{E}$ with $\check{T}$ can be calculated to be

$$\begin{aligned}\{\check{T}(x), E(y, \bar{y})\}_* =& \partial_y\left(E(y, \bar{y})\right)\delta(x-y) - \Delta E(y, \bar{y})\delta'(x-y) \\ &- \partial_y C(p_1, p_2, y, \bar{y})\, e^{\gamma(\Phi_1(y,\bar{y}) + i\lambda\Phi_2(x,\bar{x}))}\delta(x-y),\end{aligned} \tag{8.22}$$

with a conformal weight

$$\Delta = \frac{\eta_1}{2} + i\lambda\frac{\eta_2}{2} - (1-\lambda^2)\frac{b^2}{4}\,. \tag{8.23}$$

Here $C(p_1, p_2; x, \bar{x})$ has to be constant in x and $\bar{x}$, so that the commutation relation corresponds to (6.73).

One can now make a redefinition of q_1 and q_2 to absorb the p-dependent coefficient of the in-field exponential. Its symbol is then

$$\check{E}(x,\bar{x}) \equiv e^{\gamma(\Phi_1(x,\bar{x})+i\lambda\Phi_2(x,\bar{x}))}\,, \tag{8.24}$$

and the operator is just the normal ordered operator

$$\hat{E}(x,\bar{x}) =: e^{\gamma(\hat{\Phi}_1(x,\bar{x})+i\lambda\hat{\Phi}_2(x,\bar{x}))} :\,. \tag{8.25}$$

As in Liouville theory multiplication of two free-field exponentials generates a short-distance factor. Since Φ_1 and Φ_2 commute the results from section (7.2.3) can easily be generalized to the $SL(2,\mathbb{R})/U(1)$ model.

8.2.3 Screening Charge

As in the Liouville theory the classical out-field is given as an integral over a bilocal field

$$\chi(x)\bar{\chi}(\bar{x}) = \int_0^{2\pi}\int_0^{2\pi} dz\, d\bar{z}\, B(x,\bar{x},x+z,\bar{x}+\bar{z}) \tag{8.26}$$

with

$$\begin{aligned} B(x,\bar{x},y,\bar{y}) =& \frac{-\gamma^2 e^{\gamma p_1}}{4\sinh^2\left(\frac{\gamma p_1}{2}\right)} e^{\gamma(\Phi_1(x,\bar{x})+i\Phi_2(x,\bar{x})-2\Phi_1(y,\bar{y}))} \\ &\times \big(\phi_1'(y)+i\phi_2'(y)\big)\big(\bar{\phi}_1'(\bar{y})+i\bar{\phi}_2'(\bar{y})\big). \end{aligned} \tag{8.27}$$

In principle, the quantum version of this can be constructed from the demand that the action of the symmetry generators is the same as classically. However the calculation is more involved because one has to introduce additional objects to get a closed algebra. Requiring cancellation of anomalous terms in these Moyal relations forces us to set

$$\eta_1 = -\,b^2\,, \qquad\qquad \eta_2 = 0\,. \tag{8.28}$$

Furthermore we find as the symbol of the bilocal field

$$\begin{aligned} \check{B}(y,\bar{y},z,\bar{z}) =& f(p_1,p_2)C(y-z)C(\bar{y}-\bar{z})e^{\gamma(\Phi_1(x,\bar{x})+i\lambda\Phi_2(x,\bar{x}))}e^{-2\gamma\Phi_1(y,\bar{y})} \\ &\times \big(\lambda\Phi_1'(z)+i\Phi_2'(z)\big)\big(\lambda\bar{\Phi}_1'(\bar{z})+i\bar{\Phi}_2'(\bar{z})\big)\,, \end{aligned} \tag{8.29}$$

with a still undetermined function $f(p_1,p_2)$ and the short distance factor

$$C(y-z) = \left(4\sin^2\left(\frac{y-z}{2}\right)\right)^{\frac{b^2}{2}}\,. \tag{8.30}$$

The integral of this gives the out-field symbol

$$\check{F}(x,\bar{x}) = \int_0^{2\pi}\int_0^{2\pi} dz\, d\bar{z}\, \check{B}(x,\bar{x},x+z,\bar{x}+\bar{z})\,, \tag{8.31}$$

which transforms as conformal primary with conformal weight

$$\Delta_{\check{F}} = (3-\lambda^2)\frac{b^2}{4}\,. \tag{8.32}$$

Together with the values for η_1 and η_2 we find that the in-field transforms with the same conformal weight. Therefore the full field $\check{u}$ is also a conformal primary with the same conformal weight. For details of this calculation we refer to Appendix C.

The operator of the bilocal field $\check{B}$ is the normal ordered operator version. Note that because it contains the short distance factor of two exponential free-fields, and because the commutators with ϕ_1' and ϕ_2' arising in normal ordering cancel each other, it can also be written as

$$\begin{aligned}\hat{B}(y,\bar{y},z,\bar{z}) = f(\hat{p}_1,\hat{p}_2) :& e^{\gamma(\Phi_1(x,\bar{x})+i\lambda\Phi_2(x,\bar{x}))} : \\ & \times : e^{-2\gamma\Phi_1(y,\bar{y})}\big(\lambda\phi_1'(z)+i\phi_2'(z)\big)\big(\lambda\bar{\phi}_1'(\bar{z})+i\bar{\phi}_2'(\bar{z})\big) : \end{aligned} \tag{8.33}$$

with a redefinition of $f(p_1,p_2)$. The operator $\hat{F}$ of the out-field can thus be written as a product of the in-field and a screening charge

$$\hat{F}(x,\bar{x}) = f(\hat{p}_1,\hat{p}_2)\hat{E}(x,\bar{x})\hat{A}(x,\bar{x}) \tag{8.34}$$

with

$$\begin{aligned}\hat{A}(x,\bar{x}) = \int_0^{2\pi}\int_0^{2\pi} dz\, d\bar{z}\ :& e^{-2\gamma\Phi_1(x+z,\bar{x}+\bar{z})}\big(\lambda\phi_1'(x+z)+i\phi_2'(x+z)\big) \\ & \times \big(\lambda\bar{\phi}_1'(\bar{x}+\bar{z})+i\bar{\phi}_2'(\bar{x}+\bar{z})\big) : e^{\gamma\hat{p}_1+i\hbar\gamma^2}\,. \end{aligned} \tag{8.35}$$

As has been pointed out in [43] the terms containing $e^{-2\gamma\phi_1(x,\bar{x})}\phi_1'(x)$ are total derivatives and can be integrated. Furthermore due to the short distance singularity the product of these free-field exponentials with the in-field vanish. One can thus write the effective screening charge as

$$\hat{A}(x,\bar{x}) = -\int_0^{2\pi}\int_0^{2\pi} dz\, d\bar{z}\ : e^{-2\gamma\Phi_1(x+z,\bar{x}+\bar{z})}\phi_2'(x+z)\bar{\phi}_2'(\bar{x}+\bar{z}) : e^{\gamma p_1+i\hbar\gamma^2}\,, \tag{8.36}$$

which seemingly has a wrong classical limit. However, upon normal ordering the product of the in-field and this screening charge the ϕ_1' terms reappear by partial integration.

8.2.4 Exchange Algebra

Since $\hat{\Phi}_1$ and $\hat{\Phi}_2$ commute the exchange relations of the free-field exponentials can be carried over from (7.66) and (7.68) and we find

$$\begin{aligned}\hat{E}(x,\bar{x})\hat{E}(y,\bar{y}) &= \hat{E}(y,\bar{x})\hat{E}(x,\bar{y})\, e^{-\frac{i}{2}\pi b^2(1-\lambda^2)\epsilon(x-y)} \\ &= \hat{E}(x,\bar{y})\hat{E}(y,\bar{x})\, e^{-\frac{i}{2}\pi b^2(1-\lambda^2)\epsilon(\bar{x}-\bar{y})} \\ &= \hat{E}(y,\bar{y})\hat{E}(x,\bar{x})\, e^{-\frac{i}{2}\pi b^2(1-\lambda^2)(\epsilon(\bar{x}-\bar{y})+\epsilon(x-y))}\,. \end{aligned} \tag{8.37}$$

The Exchange algebra of $\hat{F}$ with the screening charge requires more labor. Commuting $\hat{E}$ with the screening charge leads to the integral

$$\begin{aligned}\hat{E}(x,\bar{x})\hat{A}(x,\bar{x}) = \int_0^{2\pi}\int_0^{2\pi} dz\, d\bar{z}\ :& e^{-2\gamma\phi_1(y+z,\bar{y}+\bar{z})}(\lambda\phi_1'(y+z)+i\phi_2'(y+z)) \\ (\lambda\bar{\phi}_1'(\bar{y}+\bar{z})+i\bar{\phi}_2'(\bar{y}+\bar{z})) :& e^{2\pi(P+2ib^2)}\hat{E}(x,\bar{x})\, e^{i\pi b^2(\epsilon(x-y-z)+\epsilon(\bar{x}-\bar{y}-\bar{z}))}\,. \end{aligned} \tag{8.38}$$

with $P \equiv \frac{\gamma p_1}{2\pi}$. Using the identity (7.71) this can be split into different terms. Some of these are periodic in z or $\bar{z}$ and with a shift of the integration variable the expression becomes

$$\begin{aligned}\hat{E}(x,\bar{x})\hat{A}(y,\bar{y}) = \frac{\sinh^2(\pi P)}{\sinh^2(\pi(P-ib^2))}\Big[&e^{i\pi b^2(\epsilon(x-y)+\epsilon(\bar{x}-\bar{y}))}\hat{A}(y,\bar{y})\hat{E}(x,\bar{x})\\ &-2i\sin\left(\pi b^2\right)e^{i\pi b^2\epsilon(x-y)}\theta_{2\pi P}(\bar{x}-\bar{y})\hat{A}(y,\bar{x})\hat{E}(x,\bar{x})\\ &-2i\sin\left(\pi b^2\right)e^{i\pi b^2\epsilon(\bar{x}-\bar{y})}\theta_{2\pi P}(x-y)\hat{A}(x,\bar{y})\hat{E}(x,\bar{x})\\ &-4\sin^2\left(\pi b^2\right)\theta_{2\pi P}(x-y)\theta_{2\pi P}(\bar{x}-\bar{y})\hat{A}(x,\bar{x})\hat{E}(x,\bar{x})\Big].\end{aligned} \tag{8.39}$$

Here we have a similar algebra as in Liouville theory.

Using (8.37) it is now easy to find the exchange relation between $\hat{E}$ and $\hat{F}$

$$\begin{aligned}\hat{E}(x,\bar{x})\hat{F}(y,\bar{y}) =& \frac{f(P+ib^2,\hat{p}_2)}{f(P,\hat{p}_2)}\frac{\sinh^2(\pi P)}{\sinh^2(\pi(P-ib^2))}\Big[e^{\frac{i}{2}\pi b^2(1+\lambda^2)\Theta}\hat{F}(y,\bar{y})\hat{E}(x,\bar{x})\\ &-2i\sin\left(\pi b^2\right)e^{\frac{i}{2}\pi b^2(1+\lambda^2)\epsilon(x-y)}\theta_{2\pi P}(\bar{x}-\bar{y})\hat{F}(y,\bar{x})\hat{E}(x,\bar{y})\\ &-2i\sin\left(\pi b^2\right)e^{\frac{i}{2}\pi b^2(1+\lambda^2)\epsilon(\bar{x}-\bar{y})}\theta_{2\pi P}(x-y)\hat{F}(x,\bar{y})\hat{E}(y,\bar{x})\\ &-4\sin^2\left(\pi b^2\right)\theta_{2\pi P}(x-y)\theta_{2\pi P}(\bar{x}-\bar{y})\hat{F}(x,\bar{x})\hat{E}(y,\bar{y})\Big].\end{aligned} \tag{8.40}$$

Since the operator $\hat{F}$, corresponding to the out-field, is related to the operator $\hat{E}$, that is related to the in-field, by a unitary transformation it has the same exchange algebra with itself

$$\hat{F}(x,\bar{x})\hat{F}(y,\bar{y}) = \hat{F}(y,\bar{y})\hat{F}(x,\bar{x})e^{-\frac{i}{2}\pi b^2(1-\lambda^2)(\epsilon(\bar{x}-\bar{y})+\epsilon(x-y))}. \tag{8.41}$$

8.2.5 Locality Condition

We have now determined all parts of the vertex operator $\hat{u}$ except for the p dependent factor of $\hat{F}$. In order to fix this factor we impose the locality condition, i.e. we demand that the equal time commutator of two $\hat{u}$ fields at different positions is zero. Taking time equal to zero this is equivalent to the condition that the product

$$\begin{aligned}\hat{u}(x,-x)\hat{u}(y,-y) =& \hat{E}(x,-x)\hat{E}(y,-y)+\hat{F}(x,-x)\hat{F}(y,-y)\\ &+h_1(p)\Bigg[h_2(P)\Big(e^{\left(\pi P-\frac{i}{2}\pi b^2(1+\lambda^2)\right)\epsilon(x-y)}\hat{F}(x,-y)\hat{E}(y,-x)\\ &\qquad+e^{-\left(\pi P-\frac{i}{2}\pi b^2(1+\lambda^2)\right)\epsilon(x-y)}\hat{F}(y,-x)\hat{E}(x,-y)\Big)\\ &\qquad+\hat{F}(y,-y)\hat{E}(x,-x)+h_3(P)\hat{F}(x,-x)\hat{E}(y,-y)\Bigg].\end{aligned} \tag{8.42}$$

with

$$\begin{aligned}h_1(P) &= \frac{f(P+ib^2,p_2)}{f(p_1,p_2)}\frac{\sinh^2(\pi P)}{\sinh^2(\pi(P-ib^2))}, \qquad h_2(P) = \frac{i\sin\left(\pi b^2\right)}{\sinh^2(\pi P)},\\ h_3(P) &= \frac{f(P,p_2)}{f(P+ib^2,p_2)}\frac{\sinh^2\left(\pi(P-ib^2)\right)}{\sinh^2(\pi P)}+\frac{\sinh^2(i\pi b^2)}{\sinh^2(\pi P)}\end{aligned} \tag{8.43}$$

is symmetric under exchange of x and y. The first line is symmetric by (8.37) and (8.41) and the second and third line are manifestly symmetric. The last line however gives a condition on $f(p_1, p_2)$

$$\left(\sinh^2(\pi P) - \sinh^2\left(i\pi b^2\right)\right) f(P + ib^2, p_2) = \sinh^2\left(\pi(P - ib^2)\right) f(P, p_2) . \quad (8.44)$$

This equation is fulfilled by

$$f(P, p_2) = -\rho^2 \frac{\gamma^2}{4\sinh(\pi P)\sinh(\pi(P - ib^2))} , \quad (8.45)$$

where ρ is a constant. The locality of the commutator of $\hat{u}(x, \bar{x})$ and its complex conjugate yields an equivalent condition.

The operator of the out-field is thus

$$\hat{F}(x, \bar{x}) = -\frac{\rho^2\gamma^2}{2\sinh(\pi P)} \hat{E}(x, \bar{x})\hat{S}(x, \bar{x}) \frac{1}{2\sinh(\pi P)} . \quad (8.46)$$

Here the deformations are hidden in the ordering.

8.3 Non-Equal Time Commutator

Having fixed the vertex operator completely we can now investigate its commutation relations. Motivated by the classical structure of (6.68) we make the ansatz

$$\hat{u}(x, \bar{x})\hat{u}(y, \bar{y}) = D_1 \hat{u}(y, \bar{y})\hat{u}(x, \bar{x}) + D_2 \left(\hat{u}(x, \bar{y})\hat{u}(y, \bar{x}) + \hat{u}(y, \bar{x})\hat{u}(x, \bar{y})\right) \quad (8.47)$$

with undetermined functions D_1 and D_2. This equation has to be satisfied separately for the structures $\hat{E} \cdot \hat{E}$, $\hat{F} \cdot \hat{F}$ and $\hat{E} \cdot \hat{F}$. With the exchange relation (8.37) we can now bring all terms quadratic in either $\hat{E}$ or $\hat{F}$ to the form $\hat{E}(x, \bar{x})\hat{E}(y, \bar{y})$ or $\hat{F}(x, \bar{x})\hat{F}(y, \bar{y})$. Since $\hat{E}$ and $\hat{F}$ both have the same exchange relations we read off one condition

$$1 = D_1 e^{i\pi b^2(1-\lambda^2)(\epsilon+\bar{\epsilon})} + D_2 \left(e^{\frac{i}{2}\pi b^2(1-\lambda^2)\bar{\epsilon}} + e^{\frac{i}{2}\pi b^2(1-\lambda^2)\epsilon}\right) \quad (8.48)$$

with $\epsilon \equiv \epsilon(x - y)$ and $\bar{\epsilon} \equiv \epsilon(\bar{x} - \bar{y})$. With (8.40) we can order the mixed terms such that $\hat{F}$ stands to the left of $\hat{E}$. Comparing separately the coefficients of the four terms $\hat{F}(x, \bar{x})\hat{E}(y, \bar{y})$, $\hat{F}(y, \bar{y})\hat{E}(x, \bar{x})$, $\hat{F}(x, \bar{y})\hat{E}(y, \bar{x})$ and $\hat{F}(y, \bar{x})\hat{E}(x, \bar{y})$ we find additional equations

$$\begin{aligned} \sinh^2(\pi P) - \sinh^2(i\pi b^2)(1 + e^{2\pi P\Theta}) &= D_1 \sinh^2(\pi P) e^{-i\pi b^2(1+\lambda^2)\Theta} \\ &- D_2 \sinh(\pi P)\sinh(i\pi b^2)\left(e^{-\frac{i}{2}\pi b^2(1+\lambda^2)\epsilon} e^{\pi P\bar{\epsilon}} + e^{-\frac{i}{2}\pi b^2(1+\lambda^2)\bar{\epsilon}} e^{\pi P\epsilon}\right) \end{aligned} \quad (8.49)$$

$$\begin{aligned} \sinh^2(\pi P) e^{i\pi b^2(1+\lambda^2)\Theta} &= D_1 \left(\sinh^2(\pi P) + \sinh^2(i\pi b^2)(e^{2\pi P\Theta} - 1)\right) \\ &- D_2 \sinh(\pi P)\sinh(i\pi b^2)\left(e^{\frac{i}{2}\pi b^2(1+\lambda^2)\epsilon} e^{-\pi P\bar{\epsilon}} + e^{\frac{i}{2}\pi b^2(1+\lambda^2)\bar{\epsilon}} e^{-\pi P\epsilon}\right) \end{aligned} \quad (8.50)$$

$$\begin{aligned} e^{\frac{i}{2}\pi b^2(1+\lambda^2)\bar{\epsilon}} e^{\pi P\epsilon} &= D_1 e^{-\frac{i}{2}\pi b^2(1+\lambda^2)\epsilon} e^{-\pi P\bar{\epsilon}} \\ &- D_2 \left(\frac{\sinh(\pi P)}{\sinh(i\pi b^2)}\left(1 + e^{-i\pi b^2(1+\lambda^2)(\epsilon-\bar{\epsilon})}\right) + \frac{\sinh(i\pi b^2)}{\sinh(\pi P)}\left(e^{\pi P(\epsilon-\bar{\epsilon})} - 1\right)\right) . \end{aligned} \quad (8.51)$$

In the fundamental domain where $\epsilon, \bar{\epsilon} = \pm 1$ one can check that these equations are solved by

$$D_1 = e^{i\pi b^2(1+\lambda^2)\Theta} \qquad D_2 = -\, i\Theta \sin\left(\pi b^2\right) e^{\frac{i}{2}\pi b^2(1+\lambda^2)\Theta} \,. \tag{8.52}$$

We can therefore write the commutator in the fundamental domain as

$$\boxed{\begin{aligned}[\hat{u}(x,\bar{x}), \hat{u}(y,\bar{y})] \;&= e^{\frac{i}{2}\pi b^2(1+\lambda^2)\Theta}\Big[2i \sin\left(\frac{1}{2}\pi b^2(1+\lambda^2)\Theta\right) \hat{u}(y,\bar{y})\hat{u}(x,\bar{x}) \\ &\quad - i\Theta \sin\left(\pi b^2\right) \left(\hat{u}(x,\bar{y})\hat{u}(y,\bar{x}) + \hat{u}(y,\bar{x})\hat{u}(x,\bar{y})\right)\Big] .\end{aligned}} \tag{8.53}$$

Expanding this result in powers of $\hbar$ we find

$$[\hat{u}(x,\bar{x}), \hat{u}(y,\bar{y})] \approx -i\hbar\gamma^2\Theta \left[\frac{\hat{u}(x,\bar{y})\hat{u}(y,\bar{x}) + \hat{u}(y,\bar{x})\hat{u}(x,\bar{y})}{2} - \hat{u}(y,\bar{y})\hat{u}(x,\bar{x})\right] \tag{8.54}$$

which agrees with the classical Poisson bracket (6.68).

8.4 Reflection Amplitude

As it was mentioned before we assume that the operators $\hat{E}$ and $\hat{F}$ are related by a unitary transformation by the scattering matrix $\mathcal{S}$ such that

$$\hat{E}(x,\bar{x})\hat{\mathcal{S}} = \hat{\mathcal{S}}\hat{F}(x,\bar{x}) \,. \tag{8.55}$$

We furthermore assume that the scattering matrix is the product of three operators

$$\hat{\mathcal{S}} = \hat{\mathcal{P}}_1 \hat{R} \hat{S} \,, \tag{8.56}$$

where $\hat{\mathcal{P}}_1$ is the parity operator for the zero modes p_1, q_1, $\hat{S}$ acts only on the non-zero modes, and $\hat{R}$ is the reflection amplitude of the vacuum sector

$$\hat{R}|0,P,n\rangle = R(P,n)|0,P,n\rangle \,. \tag{8.57}$$

Projection of equation (8.55) onto vacuum states leads to a difference equation for $R(P,n)$

$$R(P,n) = R(P+ib^2, n+1)D(P,n) \tag{8.58}$$

with

$$\begin{aligned} D(P,n) = \;& \frac{e^{2\pi(P+ib^2)}}{2\sinh\left(\pi(P+ib^2)\right)} \frac{-\rho^2}{2\sinh\left(\pi P\right)} \left(\int_0^{2\pi} dz\, e^{-(P+ib^2)z}\left(1-e^{iz}\right)^{b^2}\right)^2 \\ & \times \left(\frac{\lambda}{2}(P+ib^2) + \frac{i}{4\pi}\left(\hbar\gamma^2\lambda n + \frac{\nu}{\lambda}\right)\right)\left(\frac{\lambda}{2}(P+ib^2) + \frac{i}{4\pi}\left(\hbar\gamma^2\lambda n - \frac{\nu}{\lambda}\right)\right) .\end{aligned} \tag{8.59}$$

This corresponds to the difference for the reflection amplitude in Liouville theory (7.87) with an additional factor. We therefore make the following ansatz using the Liouville reflection amplitude

$$R(p_1,n) = -\tilde{R}(p_1,n)\left(\rho\,\Gamma\left(b^2\right)\right)^{i2P/b^2} \frac{\Gamma\left(-iP/b^2\right)\Gamma\left(-iP\right)}{\Gamma\left(iP/b^2\right)\Gamma\left(iP\right)} \,. \tag{8.60}$$

Substituting this into the original difference equation (8.58) leads to the new relation $\tilde{R}(P,n) = \tilde{D}(P,n)\tilde{R}(P+ib^2, n+1)$ with

$$\tilde{D}(P,n) = -\left(\frac{\lambda}{2}(P+ib^2) + \frac{i}{4\pi}\left(\hbar\gamma^2\lambda n + \frac{\nu}{\lambda}\right)\right)\left(\frac{\lambda}{2}(P+ib^2) + \frac{i}{4\pi}\left(\hbar\gamma^2\lambda n - \frac{\nu}{\lambda}\right)\right). \tag{8.61}$$

This equation is solved by the two functions

$$\tilde{R}(P,n) = \left(\lambda b^2\right)^{i\frac{2P}{b^2}} \frac{\Gamma\left(i\frac{P}{2b^2} \pm \frac{n}{2} + \frac{\nu}{4\pi b^2\lambda^2} + \frac{1}{2}\right)\Gamma\left(i\frac{P}{2b^2} \pm \frac{n}{2} - \frac{\nu}{4\pi b^2\lambda^2} + \frac{1}{2}\right)}{\Gamma\left(-i\frac{P}{2b^2} \pm \frac{n}{2} + \frac{\nu}{4\pi b^2\lambda^2} + \frac{1}{2}\right)\Gamma\left(-i\frac{P}{2b^2} \pm \frac{n}{2} - \frac{\nu}{4\pi b^2\lambda^2} + \frac{1}{2}\right)}. \tag{8.62}$$

Since the reflection amplitude has to respect the symmetry $p_2 \to -p_2$ of the system we choose the solution

$$\tilde{R}(P,n) = \left(\lambda b^2\right)^{i\frac{2P}{b^2}} \frac{\Gamma\left(i\frac{P}{2b^2} + \frac{|n|}{2} + \frac{\nu}{4\pi b^2\lambda^2} + \frac{1}{2}\right)\Gamma\left(i\frac{P}{2b^2} + \frac{|n|}{2} - \frac{\nu}{4\pi b^2\lambda^2} + \frac{1}{2}\right)}{\Gamma\left(-i\frac{P}{2b^2} + \frac{|n|}{2} + \frac{\nu}{4\pi b^2\lambda^2} + \frac{1}{2}\right)\Gamma\left(-i\frac{P}{2b^2} + \frac{|n|}{2} - \frac{\nu}{4\pi b^2\lambda^2} + \frac{1}{2}\right)}. \tag{8.63}$$

The final result for the reflection amplitude in terms of the original variables is thus

$$\begin{aligned} R(p_1,n) = &-\left(\rho\lambda\frac{\hbar\gamma^2}{2\pi}\Gamma\left(\frac{\hbar\gamma^2}{2\pi}\right)\right)^{i\frac{2p_1}{\hbar\gamma}} \\ &\times \frac{\Gamma\left(-i\frac{p_1}{\hbar\gamma}\right)\Gamma\left(-i\frac{\gamma p_1}{2\pi}\right)\Gamma\left(i\frac{p_1}{2\hbar\gamma} + \frac{|n|}{2} + \frac{\nu}{2\hbar\lambda^2\gamma^2} + \frac{1}{2}\right)\Gamma\left(i\frac{p_1}{2\hbar\gamma} + \frac{|n|}{2} - \frac{\nu}{2\hbar\lambda^2\gamma^2} + \frac{1}{2}\right)}{\Gamma\left(i\frac{p_1}{\hbar\gamma}\right)\Gamma\left(i\frac{\gamma p_1}{2\pi}\right)\Gamma\left(-i\frac{p_1}{2\hbar\gamma} + \frac{|n|}{2} + \frac{\nu}{2\hbar\lambda^2\gamma^2} + \frac{1}{2}\right)\Gamma\left(-i\frac{p_1}{2\hbar\gamma} + \frac{|n|}{2} - \frac{\nu}{2\hbar\lambda^2\gamma^2} + \frac{1}{2}\right)}. \end{aligned} \tag{8.64}$$

The deformation parameter λ can be fixed by demanding closed exchange relations of parafermions [42], which gives

$$\lambda = (1+2b^2)^{-\frac{1}{2}}. \tag{8.65}$$

This result for the reflection amplitude is a generalization of the expressions found in [43, 44].

As it was shown in chapter 6 a part of the bound states is reached by an analytical continuation from the hyperbolic sector $p_1 \to i\rho$. A comparison with the scattering of a quantum mechanical particle described by a wave function

$$\psi(x) = e^{ipx} + R(p)e^{-ipx} \tag{8.66}$$

shows that in order to have a normalizable state for an analytical continuation $p_1 \to i\rho$ with $\rho > 0$ the reflection amplitude must be zero, $R(i\rho) = 0$. Since the gamma function has poles at negative integers we find that bound states exist for positive ρ with

$$\boxed{\rho = -\hbar\gamma(2k + |n| + 1) + \frac{|\nu|}{\gamma\lambda^2}, \qquad k \in \mathbb{N}}. \tag{8.67}$$

This can be regarded as the quantum version of the classical condition (6.101).

As it was mentioned before the periodic Liouville theory has no continuation to the elliptic sector and therefore the bound states are missing there. But Liouville theory on a strip admits the elliptic sector, which was studied in the 80's and 90's by Gervais and his collaborators [67, 67]. Later the investigation was continued both in the Euclidean [22, 32, 23] and Minkowskian [54, 38] cases. The corresponding discrete spectrum analyzed in [32, 37, 38] is quite similar to our spectrum (8.67).

IX
Conclusion

Summary and Discussion

We have analyzed the periodic $SL(2,\mathbb{R})$ WZNW theory: its general solution, symmetries, the chiral symplectic structure and monodromies. By inversion of the chiral symplectic form we have derived the non-equal time Poisson brackets in the elliptic sector. In the fundamental domain this result coincides with the earlier obtained result of [11] for the hyperbolic monodromy, indicating the monodromy independence of the causal Poisson bracket structure for the full WZNW-field.

Then, we have investigated the $SL(2,\mathbb{R})/U(1)$ model. By reduction of the space of solutions of the $SL(2,\mathbb{R})$ WZNW theory we have derived the general solution for the hyperbolic and also for the elliptic sector. While the hyperbolic sector describes scattering processes, similarly to Liouville theory, the elliptic sector corresponds to bound states. The symplectic form of the hyperbolic sector has been written in the canonical coordinates related to the Fourier modes of the incoming free-field. By an analytical continuation of the incoming zero-mode momentum to the imaginary axis we reached the solutions of the elliptic sector, thereby establishing an analytical relation between the two sectors. Using the Dirac bracket method we have reduced the causal Poisson bracket structure of the $SL(2,\mathbb{R})$ WZNW theory to its coset model.

Using the free-field parameterization we have then quantized the hyperbolic sector of the $SL(2,\mathbb{R})/U(1)$ model. The vertex operator has been fixed in the Moyal formalism, which had already proved useful in Louville theory. We have calculated the causal commutator for the vertex operator and obtained its compact form, which preserves causal and localstructure of the corresponding classical Poisson bracket with a consistent deformation. Furthermore, the reflection amplitude has been constructed from the structure of the vertex operator in terms of incoming and outgoing fields. The analytical continuation of the reflection amplitude to the imaginary axis of the incoming momentum has zeros, like mechanical models with a bound states or the boundary Liouville theory. Therefore, these zeros have been identified with the discrete spectrum of the elliptic sector.

Outlook

The description of the elliptic sector of the $SL(2,\mathbb{R})/U(1)$ model is still incomplete as we lack the necessary symmetries to cover the full space of solutions. It is therefore highly desirable to find an analog of the translation symmetry that is present in the hyperbolic sector. This additional symmetry would then allow a semiclassical treatment of the bound sector.

Furthermore it was found that the non-equal time Poisson bracket of the hyperbolic $SL(2,\mathbb{R})/U(1)$ field with its complex conjugate is given in terms of the field related to the vector gauged $SL(2,\mathbb{R})/U(1)$ model. We presume that similarly a relation between the two models exists on the quantum level. To further analyze this it will be necessary to repeat the quantization procedure carried out for the Euclidean black hole model in this work for the vector gauged model as well.

Appendix A

Poisson Brackets in Constrained Systems

Let us consider a $2n$ dimensional manifold M with a symplectic form ω and impose m constraints $\phi_i = 0$ $(i = 1, \ldots, m)$, where ϕ_i are smooth functions on M, such that $d\phi_i$ are linearly independent on the constrained surface

$$\Sigma \equiv \{x \in M | \phi_i(x) = 0\} . \tag{A.1}$$

Hence, Σ is a $2n - m$ dimensional regular surface in M.

The aim is to analyze under which conditions one can construct Poisson brackets on Σ (or on some subspace of Σ) from the induced 2-form

$$\tilde{\omega} = \omega\big|_\Sigma , \tag{A.2}$$

and how the new Poisson brackets are related to the Poisson brackets on the full space M. We consider two cases:

- first class constraints $\quad \{\phi_i, \phi_j\}\big|_\Sigma = 0\,, \ \ \forall\, i, j \in \{1, \ldots, m\}$ (A.3)
- second class constraints $\quad \det\left(\{\phi_i, \phi_j\}\right)_{1 \leq i,j \leq m}\big|_\Sigma \neq 0$ (A.4)

First Class Constraints

Here we have $X_{\phi_i}(\phi_j) = 0$ on Σ and therefore $X_i \equiv X_{\phi_i}|_\Sigma \in T\Sigma$. To simplify the analysis we introduce local coordinates x^μ on M, such that x^μ for $\mu = 1, \ldots, 2n-m$ are coordinates on Σ and $x^{2n-m+i} = \phi_i$, $i = 1, \ldots, m$. We furthermore choose the coordinates on Σ such that the following consitions are fulfilled:

$$\begin{aligned} \partial_k\big|_\Sigma &\in T\Sigma && \text{for } k \in \{1, \ldots, 2n - 2m\}\,, \\ \partial_j\big|_\Sigma &= X_{j-2n+2m} && \text{for } j \in \{2n - 2m + 1, \ldots, 2n - m\}\,, \\ \partial_l \phi_i(x)\big|_\Sigma &= \delta_{k(2n-m+l)}\,, && \text{for } l \in \{2n - m + 1, \ldots, 2n\},\, i \in \{1, \ldots, m\}\,. \end{aligned} \tag{A.5}$$

In these coordinates we thus have $d\phi_i = dx^{2n-m+i}$ and $X^\mu_l = \delta^\mu_{2n-2m+l}$ and applying equation (2.8) to ϕ_i yields

$$dx^{2n-m+i} = \omega_{\mu(2n-2m+i)} dx^\mu\,, \qquad i = 1, \ldots, m\,. \tag{A.6}$$

From this equation we read off $\omega_{\mu(2n-2m+i)} = \delta_{\mu(2n-m+i)}$, and we can write ω in the block form

$$\omega(x)\big|_\Sigma = \begin{pmatrix} A(x) & 0 & B(x) \\ 0 & 0 & -\mathbb{1} \\ -B^T(x) & \mathbb{1} & C(x) \end{pmatrix} . \tag{A.7}$$

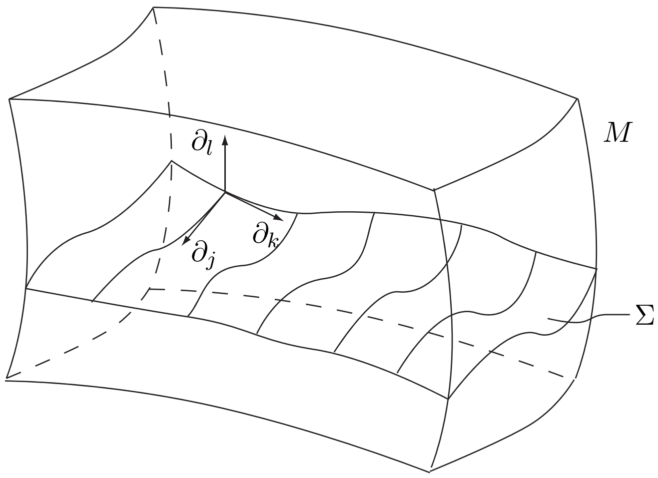

Figure A.1: Hypersurface Σ with coordinates

Similarly from equation (2.9) we find $\delta_{\mu(2n-2m+l)} = \omega^{\mu(2n-m+l)}$ for $l = 1, \ldots, m$ and we can write the inverse of the symplectic form as

$$\omega^{-1}(x)\big|_{\Sigma} = \begin{pmatrix} D(x) & E(x) & 0 \\ -E^T(x) & F(x) & \mathbb{1} \\ 0 & -\mathbb{1} & 0 \end{pmatrix} . \tag{A.8}$$

It is obvious that the induced two form on Σ is singular. We can, however, consider the quotient space of Σ with respect to the equivalence classes generated by the flow of X_i:

$$\tilde{M} \equiv \Sigma/G , \qquad \text{with } [x]_G \equiv \{e^{\alpha^i\{\phi_i,\cdot\}}x \,|\, \alpha^i \in \mathbb{R}\} . \tag{A.9}$$

Coordinates on $\tilde{M}$ in terms of representatives are given by x^i with $i = 1, \ldots, 2n-2m$ and the induced two form in these coordinates, $\tilde{\omega}(x) \equiv A(x)$ is invertible as $\tilde{\omega}^{-1} = D$. Furthermore it is independent of the choice of representatives as can be seen from the following: The Poisson bracket of x^i, x^j for $i, j = 1, \ldots, 2n-2m$ is given by $\{x^i, x^j\} = \tilde{\omega}^{ij}$ and therefore the derivative of the induced two form is

$$X_k(\tilde{\omega}^{ij}(x)) = \{\phi_k, \{x^i, x^j\}\} . \tag{A.10}$$

Due to the Jacobi identity (2.12) and the fact that $\{\phi_k, x^i\} = \partial_{k+2n-2m}x^i = 0$ this vanishes. Thus the pullback of ω onto $\tilde{M}$ is well defined, non-degenerate and closed, since ω is closed. We have thus shown that $(\tilde{M}, \tilde{\omega})$ is a symplectic manifold.

A function f that satisfies

$$\{f, \phi_i\}\big|_{\Sigma} = 0 \tag{A.11}$$

is called gauge invariant. By definition a gauge invariant function can be uniquely reduced to $\tilde{M}$. It is now easy to verify that for two gauge invariant functions the Poisson brackets calculated in (M, ω) evaluated at Σ is identical to the Poisson bracket in $(\tilde{M}, \tilde{\omega})$.

$$\{f, g\}\big|_{\Sigma} = \{f|_{\Sigma}, g|_{\Sigma}\}_{\sim} . \tag{A.12}$$

Second class constraints

We now consider $m = 2k$ constraints $\phi_i = 0, i = 1, \ldots, 2k$ and assume $\det(\{\phi_i, \phi_j\}) \neq 0$. In this case one can choose local coordinates with the properties

$$\begin{aligned} \partial_i \phi_l \big|_\Sigma &= 0 \quad \text{for } i = 1, \ldots, 2(n-k)\,, \quad l = 1, \ldots, 2k\,, \\ \partial_j \phi_l \big|_\Sigma &= \delta_{ij} \quad \text{for } j = 2(n-k)+1 \ldots, 2n\,, \quad l = 1, \ldots, 2k\,. \end{aligned} \tag{A.13}$$

In these coordinates the symplectic form and its inverse have the following block form

$$\omega(x) = \begin{pmatrix} \tilde{\omega}(x) & A(x) \\ -A^T(x) & B(x) \end{pmatrix} \qquad \omega^{-1}(x) = \begin{pmatrix} C(x) & D(x) \\ -D^T(x) & -E(x) \end{pmatrix}\,. \tag{A.14}$$

The matrix $(\{\phi_i, \phi_j\})$ is then given by

$$\{\phi_i, \phi_j\} = \omega^{\mu\nu} \partial_\nu \phi_i \partial_\mu \phi_j = \omega^{ji} = E_{ij}\,. \tag{A.15}$$

In the following we will show that the Dirac bracket of two functions f, g, defined by

$$\{f, g\}_D \equiv \{f, g\} - \{f, \phi_k\}(\{\phi_i, \phi_j\})^{-1}_{kl}\{\phi_l, g\}\,, \tag{A.16}$$

is equal to the Poisson bracket obtained by inverting the reduced symplectic form $\tilde{\omega}$.

In the coordinate system chosen above the Dirac bracket becomes

$$\begin{aligned} \{f, g\}_D &= \omega^{\mu\nu} \partial_\nu f \partial_\mu g - \omega^{\mu\nu} \partial_\nu f \partial_\mu \phi_k E^{-1}_{kl} \omega^{\rho\sigma} \partial_\sigma \phi_l \partial_\rho g \\ &= \omega^{\mu\nu} \partial_\nu f \partial_\mu g + \partial_\rho g \omega^{\rho l} E^{-1}_{lk} \omega^{k\nu} \partial_\nu f \end{aligned} \tag{A.17}$$

where k and l range from $2(n-k)+1$ to $2n$, and we have used $\partial_i \phi_j = \delta_{ij}$ together with the antisymmetry of E. Writing this in the matrix form we get

$$\begin{aligned} \{f, g\}_D &= (\partial g)^T \left(\begin{pmatrix} C & D \\ -D^T & -E \end{pmatrix} + \begin{pmatrix} D \\ -E \end{pmatrix} (E^{-1}) \begin{pmatrix} -D^T & -E \end{pmatrix} \right) (\partial f) \\ &= (\partial g)^T \begin{pmatrix} C - DE^{-1}D^T & 0 \\ 0 & 0 \end{pmatrix} (\partial f) \\ &= (C - DE^{-1}D^T)_{kl} \partial_l f \partial_k g \end{aligned} \tag{A.18}$$

The first row of the matrix equation $\omega\omega^{-1} = \mathbb{1}$ now tells us that $\tilde{\omega} D E^{-1} = A$ and furthermore $\tilde{\omega} C - AD^T = \mathbb{1}$. Combining these two relations then leads to $\tilde{\omega}(C - DE^{-1}D^T) = \mathbb{1}$, which shows that $\tilde{\omega}$ is invertible and

$$\tilde{\omega}^{-1} = C - DE^{-1}D^T\,. \tag{A.19}$$

As a result we can write

$$\{f, g\}_D = \tilde{\omega}^{kl} \partial_l f \partial_k g\,. \tag{A.20}$$

Appendix B
$SL(2,\mathbb{R})/U(1)$ Poisson Brackets in the Hyperbolic Sector

In this section we construct the Dirac brackets (A.16) of the chiral fields in the hyperbolic sector of the $SL(2,\mathbb{R})/U(1)$ model.

From (4.67a) one can extract the Poisson brackets of the chiral fields (6.65) and (6.66) with the Fourier modes j_k of the Kac-Moody current J_0

$$\{\psi(x), j_k\} = -\frac{i}{4\pi}e^{-ikx}\psi(x) \qquad \{\chi(x), j_k\} = -\frac{i}{4\pi}e^{-ikx}\chi(x)\,, \tag{B.1}$$

and because J_0 is real we can use $j_k^* = j_{-k}$ to find

$$\{\psi^*(x), j_k\} = \frac{i}{4\pi}e^{-ikx}\psi(x) \qquad \{\chi^*(x), j_k\} = \frac{i}{4\pi}e^{-ikx}\chi(x)\,. \tag{B.2}$$

Using these relations and the equation (4.50) we can now calculate the Poisson brackets in the unconstrained space

$$\{\psi(x), \psi(y)\} = \frac{\gamma^2}{4}\left(\epsilon(x-y) - \frac{1}{\pi}(x-y)\right)\psi(x)\psi(y)\,, \tag{B.3a}$$

$$\{\chi(x), \chi(y)\} = \frac{\gamma^2}{4}\left(\epsilon(x-y) - \frac{1}{\pi}(x-y)\right)\chi(x)\chi(y)\,, \tag{B.3b}$$

$$\{\psi(x), \chi(y)\} = \frac{\gamma^2}{4}\left((-\frac{1}{\pi}(x-y) - \epsilon(x-y))\psi(x)\chi(y) + 4\theta_{2\lambda}(x-y)\chi(x)\psi(y)\right)\,. \tag{B.3c}$$

For the complex conjugate fields one can derive in the same manner the relations

$$\{\psi^*(x), \psi(y)\} = \frac{\gamma^2}{4}\left(\epsilon(x-y) + \frac{1}{\pi}(x-y)\right)\psi^*(x)\psi(y)\,, \tag{B.4a}$$

$$\{\chi^*(x), \chi(y)\} = \frac{\gamma^2}{4}\left(\epsilon(x-y) + \frac{1}{\pi}(x-y)\right)\chi^*(x)\chi(y)\,, \tag{B.4b}$$

$$\{\psi^*(x), \chi(y)\} = \frac{\gamma^2}{4}\left((\frac{1}{\pi}(x-y) - \epsilon(x-y))\psi^*(x)\chi(y) + 4\theta_{2\lambda}(x-y)\chi^*(x)\psi(y)\right)\,. \tag{B.4c}$$

Finally we insert these relations and (6.64) into the Dirac bracket (A.16).

For $\psi(x)$ and $\psi(y)$ we obtain

$$\{\psi(x), \psi(y)\}_D = \{\psi(x), \psi(y)\} - \sum_{l,m\neq 0}\{\psi(x), j_l\}i\frac{4\pi\gamma^2}{l}\delta_{l+m}\{j_m, \psi(y)\} = 0 \tag{B.5}$$

which reproduces (6.60) calculated in free-field parameterization. The remaining Dirac brackets are

$$\{\chi(x), \chi(y)\}_D = 0\,, \tag{B.6a}$$

$$\{\psi(x), \chi(y)\}_D = \gamma^2\left(\theta_{2\lambda}(x-y)\chi(x)\psi(y) - \frac{1}{2}\epsilon(x-y)\psi(x)\chi(y)\right)\,. \tag{B.6b}$$

For the complex conjugate fields we find similarly

$$\{\psi^*(x), \psi(y)\}_D = \frac{\gamma^2}{2}\epsilon(x-y)\psi^*(x)\psi(y) \tag{B.7}$$

$$\{\chi^*(x), \chi(y)\}_D = \frac{\gamma^2}{2}\epsilon(x-y)\chi^*(x)\chi(y) \tag{B.8}$$

$$\{\psi^*(x), \chi(y)\}_D = \gamma^2\theta_{2\lambda}(x-y)\chi^*(x)\psi(y)\,. \tag{B.9}$$

As in Liouville theory we can complete this algebra by the Poisson bracket relations with the screening chargesusing that $A(x) = \frac{\chi(x)}{\psi(x)}$

$$\{\psi(x), A(y)\}_D = \gamma^2\left(\theta_{2\lambda}(x-y)A(x)\psi(x) - \frac{1}{2}\epsilon(x-y)\psi(x)A(y)\right)\,, \tag{B.10a}$$

$$\{A(x), A(y)\}_D = \gamma^2\left(\epsilon(x-y)A(x)A(y) - \theta_{2\lambda}(x-y)A^2(x) - \theta_{-2\lambda}(x-y)A^2(y)\right)\,, \tag{B.10b}$$

$$\{\psi^*(x), A(y)\}_D = \gamma^2\left(\theta_{2\lambda}(x-y)A^*(x)\psi^*(x) - \frac{1}{2}\epsilon(x-y)\psi^*(x)A(y)\right)\,, \tag{B.10c}$$

$$\begin{aligned}\{A^*(x), A(y)\}_D = \gamma^2\Big(&\epsilon(x-y)A^*(x)A(y) \\ &- \theta_{2\lambda}(x-y)A^{*2}(x) - \theta_{-2\lambda}(x-y)A^2(y)\Big).\end{aligned} \tag{B.10d}$$

The Dirac bracket algebra for the antichiral part is the same.

Combining now the chiral and the antichiral calculations one obtains the non-equal time Poisson bracket structure for the interacting field in the hyperbolic monodromy (6.67) and (6.70).

Appendix C

Construction of the Out-Field Operator

In this section we show in detail the construction of the symbol that corresponds to the classical bilocal field

$$\begin{aligned} B(x,\bar{x},y,\bar{y}) =& \frac{-\gamma^2 e^{\gamma p_1}}{4\sinh^2\left(\frac{\gamma p_1}{2}\right)} e^{\gamma(\Phi_1(x,\bar{x})+i\Phi_2(x,\bar{x})-2\Phi_1(y,\bar{y}))} \\ & \times \left(\phi_1'(y)+i\phi_2'(y)\right)\left(\bar{\phi}_1'(\bar{y})+i\bar{\phi}_2'(\bar{y})\right). \end{aligned} \tag{C.1}$$

As one can check using the basic Poisson brackets this field does not form a closed algebra with $\phi_i'(x)$. To close the algebra we will have to introduce additional fields

$$s(x,\bar{x},y,\bar{y}) \equiv e^{\gamma(\Phi_1(x,\bar{x})+i\Phi_2(x,\bar{x}))} e^{-2\gamma\Phi_1(y,\bar{y})}(\phi_1'(y)+i\phi_2'(y)) \tag{C.2a}$$

$$\bar{s}(x,\bar{x},y,\bar{y}) \equiv e^{\gamma(\Phi_1(x,\bar{x})+i\Phi_2(x,\bar{x}))} e^{-2\gamma\Phi_1(y,\bar{y})}(\bar{\phi}_1'(\bar{y})+i\bar{\phi}_2'(\bar{y})) \tag{C.2b}$$

$$f(y,\bar{y},z,\bar{z}) \equiv e^{\gamma(\Phi_1(y,\bar{y})+i\Phi_2(y,\bar{y}))} e^{-2\gamma\Phi_1(z,\bar{z})}\,. \tag{C.2c}$$

We will start the construction of symbols with the symbol of f since its relation with $\phi_i'(x)$ are closed. The classical Poisson brackets of f impose the conditions

$$\{\phi_1'(x),\check{f}(y,\bar{y},z,\bar{z})\}_* = \frac{\gamma}{2}\check{f}(y,\bar{y},z,\bar{z})\delta(x-y) - \gamma f(y,\bar{y},z,\bar{z})\delta(x-z) \tag{C.3a}$$

$$\{\phi_2'(x),\check{f}(y,\bar{y},z,\bar{z})\}_* = id\frac{\gamma}{2}\check{f}(y,\bar{y},z,\bar{z})\delta(x-y)\,. \tag{C.3b}$$

The solution of these equations is simply

$$\check{f}(x,\bar{x},y,\bar{y}) = C(p_1,p_2,x,\bar{x},y,\bar{y}) e^{\gamma(\Phi_1(x,\bar{x})+i\lambda\Phi_2(x,\bar{x}))} e^{-2\gamma\Phi_1(y,\bar{y})}\,. \tag{C.4}$$

Here we have chosen the same deformation of the coupling constant in front of ϕ_2 as for the in-field $\check{E}$ (cf. (8.19)). Other deformations would be inconsistent with our assumptions for the conformal properties and the scattering matrix. The Moyal bracket of this $\check{f}$ with $\check{T}$ is

$$\begin{aligned} \{\check{T}(x),&\check{f}(y,\bar{y},z,\bar{z})\}_* = \partial_y\check{f}\delta(x-y) + \partial_z\check{f}\delta(x-z) \\ &- \frac{1}{2}\left(\eta_1 + i\lambda\eta_2 - (1-\lambda^2)\frac{b^2}{2}\right)\check{f}\delta'(x-y) + \left(\eta_1+b^2\right)\check{f}\delta'(x-z) \\ &+ \check{f}\left(\frac{b^2}{2}\cot\left(\frac{1}{2}(y-z)\right) - (\partial_y C)\frac{1}{C}\right)\delta(x-y) \\ &- \check{f}\left(\frac{b^2}{2}\cot\left(\frac{1}{2}(y-z)\right) + (\partial_z C)\frac{1}{C}\right)\delta(x-z)\,. \end{aligned} \tag{C.5}$$

Comparing this with the classical Poisson bracket

$$\{T(x),f(y,\bar{y},z,\bar{z})\} = \partial_y f(y,\bar{y},z,\bar{z})\delta(x-y) + \partial_z f((y,\bar{y},z,\bar{z})\delta(x-z) \tag{C.6}$$

we allow a deformation of the conformal weight but require that the last two terms to vanish. This leads to the condition $C = f(p_1, p_2)\, c(y-z, \bar{y}-\bar{z})$ with

$$\partial_y c(y-z, \bar{y}-\bar{z}) = c(y-z, \bar{y}-\bar{z}) \frac{b^2}{2} \cot\left(\frac{1}{2}(y-z)\right) . \tag{C.7}$$

The solution of this and the equivalent antichiral condition from commutation with $\check{\bar{T}}(\bar{x})$ is the short distance factor

$$C = f_p \left(4\sin^2\left(\frac{y-z}{2}\right)\right)^{\frac{b^2}{2}} \left(4\sin^2\left(\frac{\bar{y}-\bar{z}}{2}\right)\right)^{\frac{b^2}{2}} . \tag{C.8}$$

To construct the symbols $\check{s}$ and $\check{\bar{s}}$ we have to satisfy the conditions

$$\begin{aligned} \{\phi_1'(x), \check{s}(y,\bar{y},z,\bar{z})\}_* =& \frac{\gamma}{2}\check{s}(y,\bar{y},z,\bar{z})\delta(x-y) - \gamma\check{s}(y,\bar{y},z,\bar{z})\delta(x-z) \\ &- \frac{1}{2}\check{f}(y,\bar{y},z,\bar{z})\delta'(x-z) \end{aligned} \tag{C.9a}$$

$$\{\phi_2'(x), \check{s}(y,\bar{y},z,\bar{z})\}_* = i\frac{\gamma}{2}\check{s}(y,\bar{y},z,\bar{z})\delta(x-y) - i\frac{1}{2}\check{f}(y,\bar{y},z,\bar{z})\delta'(x-z) \tag{C.9b}$$

$$\{\bar{\phi}_1'(x), \check{s}(y,\bar{y},z,\bar{z})\}_* = \frac{\gamma}{2}\check{s}(y,\bar{y},z,\bar{z})\delta(\bar{x}-\bar{y}) - \gamma\check{s}(y,\bar{y},z,\bar{z})\delta(\bar{x}-\bar{z}) \tag{C.9c}$$

$$\{\bar{\phi}_2'(x), \check{s}(y,\bar{y},z,\bar{z})\}_* = i\frac{\gamma}{2}\check{s}(y,\bar{y},z,\bar{z})\delta(x-y) , \tag{C.9d}$$

and similar relations for $\check{\bar{s}}$. Here no further deformations can be allowed. The solutions of the variational equations are

$$\check{s}(y,\bar{y},z,\bar{z}) = \check{f}((y,\bar{y},z,\bar{z}))\left(c_1\phi_1'(z) + ic_2\phi_2(z) + C_1(p_1,p_2,y,\bar{y},z,\bar{z})\right) \tag{C.10a}$$

$$\check{\bar{s}}(y,\bar{y},z,\bar{z}) = \check{f}((y,\bar{y},z,\bar{z}))\left(c_1\bar{\phi}_1'(\bar{z}) + ic_2\bar{\phi}_2(\bar{z}) + C_2(p_1,p_2,y,\bar{y},z,\bar{z})\right) . \tag{C.10b}$$

Finally the classical Poisson brackets of B give conditions for the symbol $\check{B}$

$$\begin{aligned} \{\phi_1'(x), \check{B}(y,\bar{y},z,\bar{z})\}_* =& \frac{\gamma}{2}\check{B}(y,\bar{y},z,\bar{z})\delta(x-y) - \gamma\check{B}(y,\bar{y},z,\bar{z})\delta(x-z) \\ &- \frac{1}{2}\delta'(x-z)\check{\bar{s}}(y,\bar{y},z,\bar{z}) \end{aligned} \tag{C.11}$$

$$\{\phi_2'(x), \check{B}(y,\bar{y},z,\bar{z})\}_* = i\frac{\gamma}{2}\check{B}(y,\bar{y},z,\bar{z})\delta(x-y) - i\frac{1}{2}\delta'(x-z)\check{\bar{s}}(y,\bar{y},z,\bar{z}) . \tag{C.12}$$

We can therefore write the symbol of the out-field as

$$\begin{aligned} \check{B}(y,\bar{y},z,\bar{z}) = \check{f}(y,\bar{y},z,\bar{z})&\left(c_1\phi_1'(z) + i\phi_2'(z) + C_1(p_1,p_2,y,\bar{y},z,\bar{z})\right) \\ &\left(c_1\bar{\phi}_1'(\bar{z}) + i\bar{\phi}_2'(\bar{z}) + C_2(p_1,p_2,y,\bar{y},z,\bar{z})\right) . \end{aligned} \tag{C.13}$$

As before we now compare the classical Poisson bracket with $T(x)$

$$\begin{aligned} \{T(x), B(y,\bar{y},z,\bar{z})\} =& \partial_y B(y,\bar{y},z,\bar{z})\delta(x-y) + \partial_z B(y,\bar{y},z,\bar{z})\delta(x-z) \\ &- B(y,\bar{y},z,\bar{z})\delta'(x-z) \end{aligned} \tag{C.14}$$

with the Moyal bracket with $\check{T}$

$$
\begin{aligned}
\{\check{T}(x), \check{B}(y,\bar{y},z,\bar{z})\}_* &= \partial_y B\delta(x-y) + \partial_z B\delta(x-z) \\
&- \frac{1}{2}\left(\eta_1 + i\lambda\eta_2 - (1-\lambda^2)\frac{b^2}{2}\right)\check{B}\delta'(x-y) - \left(1-\eta_1-b^2\right)\check{B}\delta'(x-z) \\
&+ \left(\frac{\eta_1}{2\gamma} + i\lambda\frac{\eta_2}{2\gamma} + \frac{b^2}{2\gamma}\right)\check{\check{s}}\delta''(x-z) \\
&- \left(\partial_y C_1\check{\check{s}} + \partial_y C_2\check{s}\right)\delta(x-y) - \left(\partial_z C_1\check{\check{s}} + \partial_z C_2\check{s}\right)\delta(x-z) + C_1\check{\check{s}}\delta'(x-z) \\
&- (c_1-\lambda)\frac{b^2}{4\gamma}\check{\check{s}}\cot\left(\frac{1}{2}(x-z)\right)\left(\delta(x-y)-\delta(x-z)\right) \\
&- i(c_1-\lambda)\check{\check{s}}\frac{b^2}{8\pi\gamma}\sum_{k>0}\left(e^{ik(x-y)} - e^{-ik(x-y)}\right) .
\end{aligned}
\tag{C.15}
$$

In order for the anomalous terms to vanish we find that we have to set $c_1 = \lambda$, $C_1 = 0 = C_2$ and

$$
\eta_2 = \frac{i}{\lambda}\left(\eta_1 + b^2\right) . \tag{C.16}
$$

Integration of the bilocal field yields the out-field

$$
\check{F}(y,\bar{y}) = \int_0^{2\pi}\int_0^{2\pi} dz\, d\bar{z}\, \check{B}(y,\bar{y},y+z,\bar{y}+\bar{z}) . \tag{C.17}
$$

We therefore find the Moyal bracket of $\check{T}(x)$ with the out-field to be

$$
\begin{aligned}
\{\check{T}(x), \check{F}(y,\bar{y})\}_* &= \partial_y \check{F}(y,\bar{y})\delta(x-y) + (3-\lambda^2)\frac{b^2}{4}F(y,\bar{y})\delta'(x-y) \\
+ \left(\eta_1 + b^2\right)&\left[\int_0^{2\pi} d\bar{z}\, \partial_0 x B(y,\bar{y},x,\bar{y}+\bar{z}) - B(y,\bar{y},y,\bar{y}+\bar{z})(e^{-\gamma p_1}-1)\delta(x-y)\right] .
\end{aligned}
\tag{C.18}
$$

Here we demand that the out-field transform as a conformal primary and find the condition

$$
\eta_1 = -b^2 . \tag{C.19}
$$

This also implies that $\eta_2 = 0$.

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