Non-local Symmetries of Wilson Loops

MASTERARBEIT

zur Erlangung des akademischen Grades
Master of Science
(M. Sc.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät I
Institut für Physik
Humboldt-Universität zu Berlin

von
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geboren am 02.05.1988 in Dernbach

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eingereicht am: 30. November 2013

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1. Introduction

Within the huge class of quantum field theories, gauge theories have proved to be most significant for describing elementary particles and their interactions. Particularly Yang-Mills theories, which have been introduced nearly 60 years ago [1], form an integral part of the standard model of elementary particle physics, being one of the most successful theories ever. However, despite their predictive success we have to acknowledge that quantum Yang-Mills theories in general and especially those with non-abelian gauge group like quantum chromodynamics (QCD) still present major challenges within the field of mathematical physics. Even after half a century of active research we have, for instance, neither gained a profound understanding of the mathematical foundations of quantum Yang-Mills theories, nor of the dynamics of the strongly coupled low energy regime, where perturbation theory is not applicable and phenomena like confinement occur.

Indeed, quantum Yang-Mills theories seem to be a tough nut to crack. A natural approach to to the subject is to study idealized Yang-Mills models, which are simpler in some respect. One of these models is $\mathcal{N} = 4$ super Yang-Mills theory ($\mathcal{N} = 4$ SYM) [2,3] with gauge group $\mathrm{SU}(N)$. In contrast to ordinary Yang-Mills theories, this model has an additional symmetry, called supersymmetry, which relates bosons to fermions and vice versa. It has in fact the highest possible degree of supersymmetry a gauge theory can have and it is therefore often referred to as the maximal supersymmetric, interacting gauge theory in four dimensions. Although its Lagrangian looks rather complicated, $\mathcal{N} = 4$ SYM theory is much simpler than a generic four-dimensional gauge theory, due to a few remarkable properties, which are all tightly linked to the supersymmetry of the model. The presence of supersymmetric partner fields, for instance, causes all ultraviolet divergences to cancel to all orders in perturbation theory. This implies that the β-function of the theory, which describes the energy dependence of the coupling constant, vanishes exactly [4–7]. Thus, $\mathcal{N} = 4$ SYM theory is not only superconformally invariant at the classical level, but also at the quantum level, making this theory much more tractable than any other interacting four-dimensional gauge theory. Although the model itself as well as the above mentioned properties were already discovered in the late 70s and early 80s, the most intriguing findings have come to light during the last 15 years. Much research activity was triggered by Juan Maldacena, who conjectured in 1997 that $\mathcal{N} = 4$ SYM theory has a dual description in terms of a type IIB superstring theory on a ten-dimensional curved background, which is $AdS_5 \times S^5$ [8]. One of the striking features of this correspondence is that it maps the strongly coupled sector of the one theory to the weakly coupled sector of the other. Accordingly, it relates the perturbatively inaccessible strong coupling regime of the gauge theory to the weakly coupled low energy regime of the string theory, which is computationally under control using string perturbation theory. While this feature makes it one the one hand really hard to prove or even verify that the conjectured correspondence holds completely true, it is on the other hand a great advantage since we have a new powerful tool to analyze the usually inaccessible strong coupling regions on both sides of the duality. Just to mention one point in passing: the AdS/CFT correspondence is not less interesting with respect to the role of gravity within the framework of quantum field theory, as it relates a theory that naturally contains gravity, i.e. string theory, to a gauge theory with no gravity at all.

After the duality was conjectured, $\mathcal{N} = 4$ SYM theory gained a renewed interest among scientists. In the course of this development it gradually became clear that the $\mathcal{N} = 4$ SYM model has an extremely rich structure with many hidden secrets. One of the most remarkable of them being the integrability or, simply speaking, the exact solvability of the planar [1] model. The first integrable structures in $\mathcal{N} = 4$ SYM theory were detected in the context of the spectral problem, where great progress was achieved by reformulating the problem of finding the scaling dimensions of local gauge invariant operators in terms of eigenvalue problems of integrable (dynamic) super spin chains, see [9] for an overview. The subsequent investigations of the spectra, which were performed by using Bethe ansätze and their generalizations, allowed for the extraction of anomalous dimensions far beyond the limits of perturbation theory. The ultimate reason allowing for this vast reduction of complexity of the spectral problem is a hidden infinite-dimensional symmetry, which is not respected by the action of the model, but only shows up at the level of observables in the planar limit. In fact, the existence of an infinite number of conserved charges commuting with the spin chain Hamiltonian is inseparably connected to the success of Bethe ansatz techniques. From an algebraic point of view these hidden conserved charges enhance the finite-dimensional superconformal algebra to an infinite-dimensional quantum algebra of Yangian type. Another sector where unexpected simplicity was found and integrable structures were shown to exist is that of scattering amplitudes. It was discovered that tree-level superamplitudes are not only invariant under superconformal transformations, but also enjoy an additional symmetry called dual superconformal symmetry [10]. It was further shown that these two symmetry algebras combine to a Yangian algebra [11]. Beside scattering amplitudes and local gauge invariant operators, there exists another important class of observables in Yang-Mills theories, namely the Wilson loops. It is probably fair to say that so far not much is known about integrable structures in this sector. Of course, due to the existence of an intimate relation between polygonal light-like (super) Wilson loops and scattering amplitudes [12–15], the question whether Wilson loops possess integrable structures, such as hidden symmetries, has been partially addressed [16]. However, since the ultraviolet divergences, arising due to the presence of cusps in the contour, typically spoil the symmetry, integrable structures are very hard to find within this null polygonal domain. For this reason, this thesis will investigate another type of loop operator: the Maldacena-Wilson loop as originally proposed by Juan Maldacena in [17]. In contrast to the ordinary Wilson loop, this operator does not only couple to the gauge field of the theory but also to the six adjoint scalars.

[1] The planar model is obtained by taking the number of colors N to infinity, while the product $\lambda = g^2N$ is held fixed. The word planar refers to the fact that in this limit only those diagrams survive that can be drawn on a plane without any crossings.

While Wilson loops generally have divergent expectation values, even if the contour is smooth, this does not apply to smooth Maldacena-Wilson loops. Given that smooth Maldacena-Wilson loops are finite gauge invariant observables in $\mathcal{N} = 4$ SYM theory, it is natural to ask [2] whether they possess any integrable structures, such as hidden Yangian symmetries. In what follows we will address this question in perturbation theory. More specifically, we will derive a concrete expression for the Yangian level-one momentum generator and apply it to the one-loop expectation value of a smooth Maldacena-Wilson loop. The result will show that Maldacena-Wilson loops are not invariant under the non-local transformation in question. Yet, we will find that the level-one generator annihilates the one-loop expectation value of the appropriately supersymmetrized Maldacena-Wilson loop operator.

In the best of all possible worlds a fully uncovered Yangian symmetry could be exploited to determine the planar expectation value of a smooth (supersymmetrically completed) Maldacena-Wilson loop operator to arbitrary loop order. This would bring us closer to an exact solution of the (planar) $\mathcal{N} = 4$ SYM model, which is widely believed to have an important impact on our present understanding of gauge theories.

[2]This idea was pointed out by Prof. Jan Plefka and Dr. Nadav Drukker.

Overview

This thesis is divided up into five chapters as outlined below.

Chapter 1 is the current chapter and forms the introduction to this thesis. It provides the motivation for this work and contextualizes our research project.

Chapter 2 concentrates on $\mathcal{N} = 4$ SYM theory and its symmetries. We will start with a discussion of spinors and Clifford algebras in various dimensions and continue by deriving $\mathcal{N} = 4$ SYM theory by dimensional reduction of $\mathcal{N} = 1$ SYM theory in ten dimensions. We will then focus on the Lagrangian symmetries of the theory and subsequently address the algebraic foundations of integrability, i.e. we discuss the definition of the Yangian of a semisimple Lie (super)algebra. The remaining part of this chapter offers a review of the emergence of the Yangian in the context of tree-level superamplitudes. This will be helpful later on, when we investigate Yangian symmetries of smooth Maldacena-Wilson loops.

Chapter 3 deals with smooth Maldacena-Wilson loops and their symmetries. Here, we will first review the definition of the ordinary Wilson loop, then introduce its (locally) supersymmetric cousin called the Maldacena-Wilson loop and proceed by briefly looking at the relation between Wilson loops and scattering amplitudes in $\mathcal{N} = 4$ SYM theory. Subsequently, we elaborate on the notion of conformal symmetry and explicitly show that smooth Maldacena-Wilson loops are conformally invariant at one-loop order. Finally, we will turn to the question of Yangian symmetries. For this, we will consider the conformal algebra as the level-zero algebra, construct the level-one momentum generator and investigate whether it annihilates the one-loop expectation value.

In **chapter 4** we will, based on the insights gained from the analysis carried out in the previous chapter, extend the level-zero algebra to the superconformal algebra and subsequently derive the full Yangian level-one momentum generator. We will argue that the appropriate loop operator to consider is the supersymmetrically completed Maldacena-Wilson loop and partly construct it, using supersymmetry as a guiding principle. What remains then is to apply the full level-one momentum generator to the one-loop expectation value of the supersymmetrically completed Maldacena-Wilson loop. This will provide evidence that the supersymmetrized Maldacena-Wilson loop indeed possesses hidden Yangian symmetries.

In **chapter 5** we will present our conclusions, make contact to the result found on string side of the AdS/CFT duality and point out possible future research directions.

The results presented in this thesis concerning the Yangian symmetries of smooth supersymmetrized Maldacena-Wilson loops have been published in the paper: D.Müller, H. Münkler, J. Plefka, J. Pollok and K. Zarembo, *"Yangian symmetry of smooth Wilson Loops in $\mathcal{N} = 4$ super Yang-Mills Theory"*, JHEP 1311, 081 (2013).

2. $\mathcal{N} = 4$ Super Yang-Mills Theory

In this chapter we will discuss various aspects of $\mathcal{N} = 4$ SYM theory. We will start with a brief review on spinors and Clifford algebras in various dimensions and subsequently derive $\mathcal{N} = 4$ SYM theory by dimensional reduction of the $\mathcal{N} = 1$ SYM model in ten dimensions. We will then focus on the global symmetries of the classical and the quantum theory and introduce the superconformal algebra. Having discussed this, we will review the concept of the universal enveloping algebra of a Lie algebra and subsequently introduce one of the central object of this thesis, the Yangian. The remainder of the chapter then deals with scattering amplitudes in $\mathcal{N} = 4$ SYM theory. Firstly, we will briefly set out the basic formalism and then review superconformal symmetry, dual superconformal symmetry and the emergence of the Yangian.

2.1. Preliminaries

In this section we will briefly review some well known facts about spinors and Clifford algebras in four, six and ten dimensions. The motivation to do so is twofold: first, to set out the basics for the following section and second, to provide a perfect framework for setting our conventions and stating the identities which will be used in later calculations. The presentation is based on that of Belitsky et al. [18], and the one in [19].

2.1.1. Spinors and Vectors in 3+1 Dimensions

Unless stated otherwise, the spacetime we consider throughout this thesis is the usual Minkowski space $\mathbb{R}^{1,3}$. Formally, Minkowski space is a four-dimensional real vector space equipped with a metric $\eta_{\mu\nu}$, which in our conventions reads $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. The associated spinor space is denoted by $\Delta_{1,3}$ and isomorphic to $\mathbb{C}^4$. The elements of this space are four-component vectors called Dirac spinors and we will choose the following convention

$$\Psi = \begin{pmatrix} \lambda_\alpha \\ \tilde{\lambda}^{\dot{\alpha}} \end{pmatrix} \qquad \text{with} \quad \alpha, \dot{\alpha} \in \{1, 2\}\,. \tag{2.1}$$

In our basis, a Dirac spinor decomposes into a pair of Weyl spinors. A left-handed Weyl spinor is denoted by λ_α and transforms in the fundamental representation of $\mathrm{SL}(2, \mathbb{C})$, which is the double covering group of the proper, orthochronous Lorentz group $\mathrm{SO}^+(1, 3)$. In contrast to that, a right-handed Weyl spinor $\tilde{\lambda}_{\dot{\alpha}}$ transforms in the conjugate representation of $\mathrm{SL}(2, \mathbb{C})$. Weyl indices can be raised and lowered using the totally antisymmetric tensor in two dimension.

We choose the following convention

$$\lambda^\alpha = \varepsilon^{\alpha\beta}\,\lambda_\beta \qquad \lambda_\alpha = \lambda^\beta\,\varepsilon_{\beta\alpha} \qquad \tilde{\lambda}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\,\tilde{\lambda}^{\dot\beta} \qquad \tilde{\lambda}^{\dot\alpha} = \tilde{\lambda}_{\dot\beta}\,\varepsilon^{\dot\beta\dot\alpha}\,, \tag{2.2}$$

with

$$\varepsilon^{12} = \varepsilon_{12} = 1 \qquad\qquad \varepsilon^{\dot1\dot2} = \varepsilon_{\dot1\dot2} = -1\,. \tag{2.3}$$

We note that in this convention we have

$$\varepsilon^{\alpha\beta}\,\varepsilon_{\gamma\beta} = \delta^\alpha_\gamma \qquad\qquad \varepsilon^{\dot\alpha\dot\beta}\,\varepsilon_{\dot\gamma\dot\beta} = \delta^{\dot\alpha}_{\dot\gamma}\,. \tag{2.4}$$

Before we come to the Clifford algebra, let us introduce the four-dimensional sigma matrices

$$\sigma^{\mu\dot\alpha\beta} = (\mathbb{1}, \vec\sigma) \qquad\qquad \bar\sigma^\mu_{\alpha\dot\beta} = (\mathbb{1}, -\vec\sigma)\,, \tag{2.5}$$

where $\mathbb{1}$ stands for the identity matrix and $\vec\sigma$ denotes the three-vector of Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.6}$$

If the spinor indices of the sigma matrices are suppressed, the index position of σ and $\bar\sigma$ is given by (2.5). We further note that these two matrices can be identified as follows

$$\sigma^{\mu\dot\alpha\beta} = \varepsilon^{\beta\gamma}\,\bar\sigma^\mu_{\gamma\dot\delta}\,\varepsilon^{\dot\delta\dot\alpha} = \bar\sigma^{\mu\beta\dot\alpha} \qquad\qquad \bar\sigma^\mu_{\alpha\dot\beta} = \varepsilon_{\dot\beta\dot\gamma}\,\sigma^{\mu\dot\gamma\delta}\,\varepsilon_{\delta\alpha} = \sigma^\mu_{\dot\beta\alpha}\,. \tag{2.7}$$

Let us also mention identities for products of sigma matrices with contracted spacetime indices as we will need them later on

$$\bar\sigma^\mu_{\alpha\dot\beta}\,\bar\sigma_{\mu\gamma\dot\delta} = -2\,\varepsilon_{\alpha\gamma}\,\varepsilon_{\dot\beta\dot\delta} \qquad\qquad \sigma^\mu_{\dot\alpha\beta}\,\sigma_{\mu\dot\gamma\delta} = -2\,\varepsilon_{\beta\delta}\,\varepsilon_{\dot\alpha\dot\gamma}\,. \tag{2.8}$$

These identities can easily be proved by a direct calculation, see [19]. We also need the following trace identities

$$\begin{aligned}
\mathrm{Tr}\,(\bar\sigma^\mu\,\sigma^\nu) &= 2\,\eta^{\mu\nu} \\
\mathrm{Tr}\,(\bar\sigma^\mu\,\sigma^\nu\,\bar\sigma^\rho\,\sigma^\kappa) &= 2\,(\eta^{\mu\nu}\,\eta^{\rho\kappa} + \eta^{\nu\rho}\,\eta^{\mu\kappa} - \eta^{\mu\rho}\,\eta^{\nu\kappa} - i\,\varepsilon^{\mu\nu\rho\kappa}) \\
\mathrm{Tr}\,(\sigma^\mu\,\bar\sigma^\nu\,\sigma^\rho\,\bar\sigma^\kappa) &= 2\,(\eta^{\mu\nu}\,\eta^{\rho\kappa} + \eta^{\nu\rho}\,\eta^{\mu\kappa} - \eta^{\mu\rho}\,\eta^{\nu\kappa} + i\,\varepsilon^{\mu\nu\rho\kappa})\,.
\end{aligned} \tag{2.9}$$

You can find a proof of the trace identities in appendix A.1. Using the so-defined four-dimensional sigma matrices we can write down the following representation of the Dirac algebra

$$\gamma^\mu = \begin{pmatrix} 0 & \bar\sigma^\mu_{\alpha\dot\beta} \\ \sigma^{\mu\dot\alpha\beta} & 0 \end{pmatrix} \qquad\qquad \{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\,. \tag{2.10}$$

This representation is often referred to as the Weyl or chiral representation. The matrix γ^5 is as usual defined by

$$\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.11}$$

The spinor space $\Delta_{1,3}$ can be shown to be the direct sum of the two eigenspaces Δ^+ and Δ^- of γ^5 with eigenvalues $+1$ and -1 respectively. A spinor is said to satisfy a Weyl condition, if it obeys

$$\gamma^5\,\Psi = c\,\Psi\,, \tag{2.12}$$

with c fixed to $+1$ or -1. Projection operators onto the eigenspaces are given by

$$P_L = \frac{1}{2}\left(1+\gamma^5\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad P_R = \frac{1}{2}\left(1-\gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.13}$$

Hence, the elements of Δ^+ are the left-handed Weyl spinors while the elements of Δ^- are the right-handed Weyl spinors. Now, let us briefly recall the definition of a Majorana spinor. A fermion is called a Majorana fermion if the associated spinor satisfies the following condition

$$\Psi^T\,C_4 = \Psi^\dagger\,\gamma^0 = \bar{\Psi}\,, \tag{2.14}$$

where C_4 is the charge conjugation matrix and is given by

$$C_4 = i\,\gamma^2\,\gamma^0 = \begin{pmatrix} -\varepsilon^{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \tag{2.15}$$

The Majorana condition (2.14) is essentially a reality condition written in a Lorentz invariant way. It is worth mentioning that the ability to impose a Majorana or a Weyl condition on a Dirac spinor depends on the spacetime structure, i.e. the signature of the metric and the dimension. Furthermore, while the Majorana and the Weyl condition cannot be imposed simultaneously in four dimensions it is possible in Minkowski spacetimes of dimension two and ten.

In the beginning of this section we mentioned the group homomorphism between $\mathrm{SO}^+(1,3)$ and $\mathrm{SL}(2,\mathbb{C})$. The question we want to address now is how we can assign a bi-spinor to a four-vector which then transforms under the corresponding $\mathrm{SL}(2,\mathbb{C})$ representations of the Lorentz group. Using the four-dimensional sigma matrices (2.5), we define

$$x^{\alpha\dot{\alpha}} := \bar{\sigma}^{\mu\alpha\dot{\alpha}}\,x_\mu = \sigma^{\mu\dot{\alpha}\alpha}\,x_\mu =: x^{\dot{\alpha}\alpha}\,, \tag{2.16}$$

where x^μ is a spacetime vector and the equality sign in the middle holds true due to the identification (2.7). If we plug (2.5) into (2.16), we find

$$\tilde{x} := \sigma^\mu\,x_\mu = \begin{pmatrix} x_0 + x_3 & x_1 - i\,x_2 \\ x_1 + i\,x_2 & x_0 - x_3 \end{pmatrix} \qquad\qquad \det\left(\tilde{x}\right) = x^\mu\,x_\mu\,. \tag{2.17}$$

Now, if $N \in \mathrm{SL}(2,\mathbb{C})$, the map $\tilde{x} \to N\tilde{x}\,N^\dagger$ preserves the hermicity of $\tilde{x}$ and, since $\det\left(N\right) = 1$, also the determinant, i.e. the Minkowski norm of the vector x^μ. Using the inverse relation to (2.16), one can directly write down the explicit form of the so-defined group homomorphism between $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO}^+(1,3)$

$$\Lambda^\mu{}_\nu(N) = \frac{1}{2}\,\mathrm{Tr}\left(\bar{\sigma}^\mu\,N\,\sigma_\nu\,N^\dagger\right). \tag{2.18}$$

Note that since (2.16) is our exclusive rule for assigning a bi-spinor to a vector, it also applies to the partial derivative ∂_μ. This obviously implies

$$\partial^{\alpha\dot{\alpha}}\, x_{\dot{\beta}\beta} = \bar{\sigma}^{\mu\alpha\dot{\alpha}}\, \sigma^{\nu}_{\dot{\beta}\beta}\, \partial_\mu\, x_\nu = \varepsilon^{\alpha\gamma}\, \sigma^{\mu}_{\dot{\gamma}\gamma}\, \varepsilon^{\dot{\gamma}\dot{\alpha}}\, \sigma_{\mu\dot{\beta}\beta} = 2\, \delta^{\alpha}_{\beta}\, \delta^{\dot{\alpha}}_{\dot{\beta}}\,, \tag{2.19}$$

where we used (2.7) and (2.8). In the following course of this thesis we will encounter different superspaces, which do not only have ordinary spacetime dimension with coordinates x^μ, but also anticommuting dimensions with coordinates θ^A_α and $\bar{\theta}_{A\dot{\alpha}}$. These coordinates are Grassmann-valued and transform as Weyl spinors under Lorentz transformations. The meaning of the capital Latin index will become clear later on. Derivatives with respect to these fermionic coordinates are defined by

$$\frac{\partial \theta^B_\beta}{\partial \theta^A_\alpha} = \delta^B_A\, \delta^\alpha_\beta \qquad\qquad \frac{\partial \bar{\theta}_{B\dot{\beta}}}{\partial \bar{\theta}_{A\dot{\alpha}}} = \delta^A_B\, \delta^{\dot{\alpha}}_{\dot{\beta}}\,. \tag{2.20}$$

At this point it is important to note that the above given definitions, together with the rules (2.2), imply that

$$\frac{\partial \theta^{B\beta}}{\partial \theta^{A\alpha}} = \varepsilon^{\beta\gamma}\, \varepsilon_{\delta\alpha}\, \frac{\partial \theta^B_\gamma}{\partial \theta^A_\delta} = \varepsilon^{\beta\gamma}\, \varepsilon_{\delta\alpha}\, \delta^B_A\, \delta^\delta_\gamma = \delta^B_A\, \varepsilon^{\beta\gamma}\, \varepsilon_{\gamma\alpha} = -\delta^B_A\, \delta^\beta_\alpha\,, \tag{2.21}$$

and a similar relation for $\bar{\theta}^{\dot{\alpha}}_A$. These somewhat counterintuitive derivative rules are sometimes cured by defining

$$\frac{\partial}{\partial \theta^{A\alpha}} := -\varepsilon_{\gamma\alpha}\, \frac{\partial}{\partial \theta^A_\gamma} \qquad\qquad \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}_A} := -\varepsilon_{\dot{\alpha}\dot{\gamma}}\, \frac{\partial}{\partial \bar{\theta}_{A\dot{\gamma}}}\,, \tag{2.22}$$

but we will not employ this convention. Instead, we will raise and lower indices without exception according to the rules (2.2). Let us proceed by defining

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} := \frac{i}{2}\Big(\bar{\sigma}^{\mu}_{\alpha\dot{\gamma}}\, \sigma^{\nu\dot{\gamma}\beta} - \bar{\sigma}^{\nu}_{\alpha\dot{\gamma}}\, \sigma^{\mu\dot{\gamma}\beta}\Big) \qquad \bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} := \frac{i}{2}\Big(\sigma^{\mu\dot{\alpha}\gamma}\, \bar{\sigma}^{\nu}_{\gamma\dot{\beta}} - \sigma^{\nu\dot{\alpha}\gamma}\, \bar{\sigma}^{\mu}_{\gamma\dot{\beta}}\Big)\,. \tag{2.23}$$

These two expressions allow us to assign two bi-spinors to an antisymmetric two-tensor $F^{\mu\nu}$.

$$F^{\alpha\beta} := F_{\mu\nu}\, \sigma^{\mu\nu\alpha\beta} \qquad\qquad F^{\dot{\alpha}\dot{\beta}} := F_{\mu\nu}\, \bar{\sigma}^{\mu\nu\dot{\alpha}\dot{\beta}} \tag{2.24}$$

In appendix A.1 we prove that these bi-spinors are related to $F^{\alpha\dot{\alpha}\beta\dot{\beta}} := F_{\mu\nu}\, \bar{\sigma}^{\mu\alpha\dot{\alpha}}\, \bar{\sigma}^{\nu\beta\dot{\beta}}$ by the following identity

$$F^{\alpha\dot{\alpha}\beta\dot{\beta}} = \frac{i}{2}\, \varepsilon^{\dot{\alpha}\dot{\beta}}\, F^{\alpha\beta} + \frac{i}{2}\, \varepsilon^{\alpha\beta}\, F^{\dot{\alpha}\dot{\beta}}\,. \tag{2.25}$$

Let us close this section by stating some more loosely related identities and definitions which we need later on. Since there is only one antisymmetric two-tensor in two dimensions, we have the following decompositions

$$\begin{aligned} \Lambda_{\alpha\beta} &= \Lambda_{(\alpha\beta)} - \frac{1}{2}\, \varepsilon_{\alpha\beta}\, \Lambda^{\gamma}{}_{\gamma} \\ \Lambda_{\dot{\alpha}\dot{\beta}} &= \Lambda_{(\dot{\alpha}\dot{\beta})} - \frac{1}{2}\, \varepsilon_{\dot{\alpha}\dot{\beta}}\, \Lambda_{\dot{\gamma}}{}^{\dot{\gamma}}\,, \end{aligned} \tag{2.26}$$

where $\Lambda_{(\alpha\beta)} = 1/2\,(\Lambda_{\alpha\beta} + \Lambda_{\beta\alpha})$. In order to prove that the antisymmetric part comes with the right coefficient, we contract the first of the two equations with $\varepsilon^{\alpha\beta}$.

$$\begin{aligned}\Lambda_{\alpha\beta}\,\varepsilon^{\alpha\beta} &= -\frac{1}{2}\,\varepsilon_{\alpha\beta}\,\varepsilon^{\alpha\beta}\,\Lambda^{\gamma}{}_{\gamma}\\ \Lambda^{\gamma}{}_{\gamma} &= \Lambda^{\gamma}{}_{\gamma}\end{aligned} \tag{2.27}$$

Since the last line is a true statement, the first of the two equations (2.26) is proved. For the second one the calculation works exactly the same. Another spinor identity that will be extensively employed later on is the following Fierz identity

$$\tilde{\xi}^{\dot\alpha}\,\xi^{\beta} = \frac{1}{2}\,\sigma^{\mu\dot\alpha\beta}\left(\tilde{\xi}^{\dot\gamma}\,\sigma_{\mu\dot\gamma\delta}\,\xi^{\delta}\right), \tag{2.28}$$

which can easily be shown to hold true by using (2.8).

2.1.2. Spinors and Vectors in 6 Dimensions

Consider the vector space $\mathbb{R}^6$ with the metric $\eta^{ij} = \mathrm{diag}(-1,-1,-1,-1,-1,-1)$. The associated spinor representation space is denoted by $\Delta_{0,6}$ and isomorphic to $\mathbb{C}^8$. The spacetime dimension is even, thus there is a natural way to build the analogon of γ^5. As in the four-dimensional case, this implies that the spinor space decomposes into a direct sum of the two eigenspaces which motivates the following notation for a Dirac spinor

$$\Psi = \begin{pmatrix}\chi^A\\ \chi_A\end{pmatrix} \qquad \text{with} \quad A \in \{1,2,3,4\}\,. \tag{2.29}$$

We start by defining the sigma matrices because we want to express the gamma matrices in terms of those.

$$\begin{aligned}(\Sigma^{1AB},\dots,\Sigma^{6AB}) &= (\eta_{1AB},\eta_{2AB},\eta_{3AB},i\,\bar\eta_{1AB},i\,\bar\eta_{2AB},i\,\bar\eta_{3AB})\\ (\bar\Sigma^{1}_{AB},\dots,\bar\Sigma^{6}_{AB}) &= (\eta_{1AB},\eta_{2AB},\eta_{3AB},-i\,\bar\eta_{1AB},-i\,\bar\eta_{2AB},-i\,\bar\eta_{3AB})\end{aligned} \tag{2.30}$$

The so-defined sigma matrices are expressed in terms of the 't Hooft symbols which explicitly read

$$\eta_{iAB} = \varepsilon_{iAB4} + \delta_{iA}\delta_{4B} - \delta_{iB}\delta_{4A} \qquad \bar\eta_{iAB} = \varepsilon_{iAB4} - \delta_{iA}\delta_{4B} + \delta_{iB}\delta_{4A}\,. \tag{2.31}$$

The sigma matrices can be shown to satisfy the following relations

$$\Sigma^{iAB}\,\bar\Sigma^{j}_{BC} + \Sigma^{jAB}\,\bar\Sigma^{i}_{BC} = 2\,\eta^{ij}\,\delta^{A}{}_{C} \qquad \bar\Sigma^{i}_{AB}\,\Sigma^{jBC} + \bar\Sigma^{j}_{AB}\,\Sigma^{iBC} = 2\,\eta^{ij}\,\delta_{A}{}^{C}\,, \tag{2.32}$$

which imply that a representation of the Clifford algebra is given by

$$\hat\gamma^i = \begin{pmatrix}0 & \Sigma^{iAB}\\ \bar\Sigma^{i}_{AB} & 0\end{pmatrix} \qquad \{\hat\gamma^i,\hat\gamma^j\} = 2\,\eta^{ij}\,. \tag{2.33}$$

The chiral and the charge conjugation matrix can then be defined as follows

$$\hat\gamma^7 = i\,\hat\gamma^1\,\hat\gamma^2\,\hat\gamma^3\,\hat\gamma^4\,\hat\gamma^5\,\hat\gamma^6 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad C_6 = \hat\gamma^1\,\hat\gamma^2\,\hat\gamma^3 = \begin{pmatrix}0 & \delta_{A}{}^{B}\\ \delta^{A}{}_{B} & 0\end{pmatrix}. \tag{2.34}$$

Let us proceed by stating some more identities obeyed by the sigma matrices (2.30)

$$\begin{aligned}
\bar{\Sigma}^i_{AB} &= \frac{1}{2}\,\varepsilon_{ABCD}\,\Sigma^{iCD} & \Sigma^{iAB} &= \frac{1}{2}\,\varepsilon^{ABCD}\,\bar{\Sigma}^i_{CD} \\
\bar{\Sigma}^i_{AB}\,\bar{\Sigma}^i_{CD} &= 2\,\varepsilon_{ABCD} & \Sigma^{iAB}\,\bar{\Sigma}^j_{AB} &= 4\,\delta^{ij}\,, \qquad (2.35)
\end{aligned}$$

where ε_{ABCD} $\big(\varepsilon_{1234}=\varepsilon^{1234}=1\big)$ is the totally antisymmetric four-tensor. These identities can easily be derived by using the basic properties of the t' Hooft symbols, see [20]. A product of two epsilon tensors with one or two indices contracted can be expressed as follows

$$\begin{aligned}
\varepsilon_{DABC}\,\varepsilon^{DKLM} &= \delta^{KLM}_{ABC} + \delta^{MKL}_{ABC} + \delta^{LMK}_{ABC} - \delta^{LKM}_{ABC} - \delta^{MLK}_{ABC} - \delta^{KML}_{ABC} \\
\varepsilon_{ABGK}\,\varepsilon^{CDGK} &= 2\Big(\delta^{CD}_{AB} - \delta^{DC}_{AB}\Big)\,, \qquad (2.36)
\end{aligned}$$

where $\delta^{A..D}_{E..H} := \delta^A_E..\delta^E_H$. As in the four-dimensional case (2.16) we can use the sigma matrices (2.30) to assign antisymmetric (4×4)-matrices to a vector $\phi^i \in \mathbb{R}^6$.

$$\phi^{AB} := \frac{1}{\sqrt{2}}\,\Sigma^{iAB}\,\phi^i \qquad\qquad \bar{\phi}_{AB} := \frac{1}{\sqrt{2}}\,\bar{\Sigma}^i_{AB}\,\phi^i \qquad (2.37)$$

Using the identities (2.35), it can easily be shown that ϕ^{AB} and $\bar{\phi}_{AB}$ are related to each other as follows

$$\phi^{AB} = \frac{1}{2}\,\varepsilon^{ABCD}\,\bar{\phi}_{CD} \qquad\qquad \bar{\phi}_{AB} = \frac{1}{2}\,\varepsilon_{ABCD}\,\phi^{CD}\,. \qquad (2.38)$$

For the scalar product of two vectors we find the following trace expression

$$X^{AB}\,\bar{Y}_{AB} = \frac{1}{2}\,\Sigma^{iAB}\,\bar{\Sigma}^j_{AB}\,X^i\,Y^j = 2\,X^i\,Y^i = -2\,X^i\,Y_i\,. \qquad (2.39)$$

In preparation of the subsequent discussion on Maldacena-Wilson loops, let us specialize to the case that the $\mathbb{R}^6$ vector, from now on referred to by n^i, squares to minus one, i.e. $n^i\,n_i = -1$. A trivial consequence of (2.39) is

$$n^{AB}\,\bar{n}_{AB} = 2\,, \qquad (2.40)$$

with n^{AB} being the matrix assigned to n^i. Furthermore, we note

$$\begin{aligned}
n^{AB}\,\bar{n}_{BC} &= \frac{1}{2}\,\Sigma^{iAB}\,\bar{\Sigma}^j_{BC}\,n^i\,n^j \\
&= \frac{1}{2}\,\Big(\Sigma^{iAB}\,\bar{\Sigma}^j_{BC} + \Sigma^{jAB}\,\bar{\Sigma}^i_{BC}\Big)\,n^i\,n^j \\
&= -\frac{1}{2}\,\delta^A{}_C\,, \qquad (2.41)
\end{aligned}$$

where we have used (2.32).

2.1.3. Spinors in 9+1 Dimensions

Having recapitulated some basic knowledge about spinors in four-dimensional Minkowski space as well as in six-dimensional Euclidean space, we now move on to investigating spinors in ten-dimensional Minkowski space. In the subsequent discussion we will need this knowledge in order to perform the dimensional reduction of $\mathcal{N}=1$ SYM theory in ten dimensions to $\mathcal{N}=4$ SYM theory in four dimensions. Let us consider the vector space $\mathbb{R}^{1,9}$ with the metric $g^{NM}=\mathrm{diag}(1,-1,\dots,-1)$. The spinor representation space is denoted by $\Delta_{1,9}$ and isomorphic to $\mathbb{C}^{32}$. We choose the following notation for a Dirac spinor $\Psi\in\Delta_{1,9}$

$$\Psi=\begin{pmatrix}\chi^A\\ \chi_A\end{pmatrix}\qquad\text{with}\quad A\in\{1,2,3,4\}\,. \tag{2.42}$$

where all χ^A and χ_A are now four component spinors with a four-dimensional Dirac like substructure, i.e.

$$\chi^A=\begin{pmatrix}\psi^A_\alpha\\ \tilde{\psi}^{A\dot\alpha}\end{pmatrix}\qquad\qquad \chi_A=\begin{pmatrix}\psi_{A\alpha}\\ \tilde{\psi}^{\dot\alpha}_A\end{pmatrix}\,. \tag{2.43}$$

This notation will become clear when we construct a representation of the appropriate Clifford algebra, which acts on $\Delta_{1,9}$. We note that the spinor (2.42) can also be written as tensor product

$$\Psi=\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}\psi^A_\alpha\\ \tilde{\psi}^{A\dot\alpha}\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}\psi_{A\alpha}\\ \tilde{\psi}^{\dot\alpha}_A\end{pmatrix}\,, \tag{2.44}$$

where the ordering of the components in the $\mathbb{C}^{16}$-vector is such that if one multiplies out the tensor product and adds up the two vectors of (2.44), the resulting $\mathbb{C}^{32}$-vector coincides with (2.42). Using the Clifford algebra representations in four (2.10) and six dimensions (2.33), we can easily construct a representation that acts on $\Delta_{1,9}$

$$\Gamma^M=\begin{cases}\mathbb{1}_8\otimes\gamma^\mu & M=\mu\in\{0,1,2,3\}\\ \hat{\gamma}^i\otimes\gamma^5 & M=i+3\in\{4,5,6,7,8,9\}\,.\end{cases} \tag{2.45}$$

A simple calculation shows that these matrices indeed satisfy the Clifford algebra relation, i.e. $\{\Gamma^M,\Gamma^N\}=2\,g^{MN}$. The charge conjugation matrix C_{10} and the chiral matrix Γ^{11} can be defined in a similar way

$$C_{10}=C_6\otimes C_4\qquad\qquad \Gamma^{11}=\hat{\gamma}^7\otimes\gamma^5\,. \tag{2.46}$$

As we will later on explicitly calculate $\bar{\Psi}$, let us introduce a further matrix which we will call $\tilde{\Gamma}^0$

$$\tilde{\Gamma}^0=\begin{pmatrix}\delta_A{}^B & 0\\ 0 & \delta^A{}_B\end{pmatrix}\otimes\begin{pmatrix}0 & \delta^{\dot\alpha}{}_{\dot\beta}\\ \delta_\alpha{}^\beta & 0\end{pmatrix}\,. \tag{2.47}$$

The Dirac adjoint spinor is then defined as follows

$$\bar{\Psi}=\Psi^\dagger\,\tilde{\Gamma}^0\,. \tag{2.48}$$

Numerically $\tilde{\Gamma}^0$ obviously equals Γ^0 as it should. However, since Γ^0 has the wrong index structure to build the quantity $\bar{\Psi}$, we are forced to redefine this matrix. Now, having introduced the appropriate Gamma matrices, it becomes clear that the notation (2.42) reflects a certain choice of basis, for which the tensor product in (2.45) can be calculated explicitly using the Kronecker product for matrices. However, for computational purposes it is more convenient to work with the tensorially decomposed spinor (2.44). In the context of dimensional reduction the spinor will be a Majorana-Weyl spinor. Accordingly, let us focus on how these two conditions reduce the degrees of freedom of a Dirac spinor. In analogy to the lower-dimensional case (2.12) the Weyl condition reads

$$\Gamma^{11}\,\Psi = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\psi^A_\alpha\\ -\tilde{\psi}^{A\dot{\alpha}}\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}-\psi_{A\alpha}\\ \tilde{\psi}^{\dot{\alpha}}_A\end{pmatrix} \stackrel{!}{=} \Psi\,. \tag{2.49}$$

This condition obviously removes 16 complex degrees of freedom, leaving us with

$$\Psi = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\psi^A_\alpha\\ 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\ \tilde{\psi}^{\dot{\alpha}}_A\end{pmatrix}\,. \tag{2.50}$$

The next step is to impose the Majorana condition on this Weyl spinor

$$\Psi^T\, C_{10} = \bar{\Psi}\,. \tag{2.51}$$

In order to see the implications of this condition in terms of the components, let us start by computing $\bar{\Psi}$.

$$\begin{aligned}\Psi^\dagger\,\tilde{\Gamma}^0 &= \left(\begin{pmatrix}1 & 0\end{pmatrix} \otimes \begin{pmatrix}(\psi^B_\beta)^* & 0\end{pmatrix} + \begin{pmatrix}0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & (\tilde{\psi}^{\dot{\beta}}_B)^*\end{pmatrix}\right) \begin{pmatrix}\delta_B{}^A & 0\\ 0 & \delta^B{}_A\end{pmatrix} \otimes \begin{pmatrix}0 & \delta^{\dot{\beta}}{}_{\dot{\alpha}}\\ \delta_\beta{}^\alpha & 0\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & (\psi^A_\alpha)^*\end{pmatrix} + \begin{pmatrix}0 & 1\end{pmatrix} \otimes \begin{pmatrix}(\tilde{\psi}^{\dot{\alpha}}_A)^* & 0\end{pmatrix}\end{aligned} \tag{2.52}$$

For the left side of equation (2.51) we find

$$\begin{aligned}\Psi^T\, C_{10} &= \left(\begin{pmatrix}1 & 0\end{pmatrix} \otimes \begin{pmatrix}\psi^B_\beta & 0\end{pmatrix} + \begin{pmatrix}0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & \tilde{\psi}^{\dot{\beta}}_B\end{pmatrix}\right) \begin{pmatrix}0 & \delta_B{}^A\\ \delta^B{}_A & 0\end{pmatrix} \otimes \begin{pmatrix}-\varepsilon^{\beta\alpha} & 0\\ 0 & -\varepsilon_{\dot{\beta}\dot{\alpha}}\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\end{pmatrix} \otimes \begin{pmatrix}\psi^{A\alpha} & 0\end{pmatrix} + \begin{pmatrix}1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & \tilde{\psi}_{A\dot{\alpha}}\end{pmatrix}.\end{aligned} \tag{2.53}$$

Thus, on the level of components, the Majorana condition (2.51) implies

$$(\psi^A_\alpha)^* = \tilde{\psi}_{A\dot{\alpha}} \qquad\qquad (\tilde{\psi}^{\dot{\alpha}}_A)^* = \psi^{A\alpha}\,. \tag{2.54}$$

Since we did not introduce a new symbol for the Majorana-Weyl spinor, we in principal always need to announce which spinor is meant when we write Ψ. However, from now on, Ψ will always denote a Majorana-Weyl spinor. In anticipation of the subsequent discussion on dimensional reduction, let us close this section by computing the component expression for $\bar{\Xi}\,\Gamma^M\Psi$, where $\bar{\Xi}$ and Ψ are both Majorana-Weyl spinors. For $M \in \{0,1,2,3\}$ we find

$$\begin{aligned}\bar{\Xi}\,\Gamma^\mu\,\Psi &= \bar{\Xi}\begin{pmatrix}\delta^A{}_B & 0\\ 0 & \delta_A{}^B\end{pmatrix} \otimes \begin{pmatrix}0 & \bar{\sigma}^\mu_{\alpha\dot{\beta}}\\ \sigma^{\mu\dot{\alpha}\beta} & 0\end{pmatrix}\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\psi^B_\beta\\ 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\ \tilde{\psi}^{\dot{\beta}}_B\end{pmatrix}\right)\\ &= \bar{\Xi}\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\ \sigma^{\mu\dot{\alpha}\beta}\,\psi^A_\beta\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}\bar{\sigma}^\mu_{\alpha\dot{\beta}}\,\tilde{\psi}^{\dot{\beta}}_A\\ 0\end{pmatrix}\right)\\ &= \tilde{\xi}_{A\dot{\alpha}}\,\sigma^{\mu\dot{\alpha}\beta}\,\psi^A_\beta + \xi^{A\alpha}\,\bar{\sigma}^\mu_{\alpha\dot{\beta}}\,\tilde{\psi}^{\dot{\beta}}_A\,,\end{aligned} \tag{2.55}$$

where $\tilde{\xi}_{A\dot{\alpha}}$ and $\xi^{A\alpha}$ denote the components of the spinor $\bar{\Xi}$. If $M \in \{4,5,6,7,8,9\}$, the structure of the gamma matrices is slightly different. We therefore get

$$\begin{aligned}\bar{\Xi}\,\Gamma^{i+3}\,\Psi &= \bar{\Xi}\begin{pmatrix}0 & \Sigma^{iAB}\\ \bar{\Sigma}^i_{AB} & 0\end{pmatrix}\otimes\begin{pmatrix}\delta_\alpha{}^\beta & 0\\ 0 & -\delta^{\dot{\alpha}}{}_{\dot{\beta}}\end{pmatrix}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}\psi^B_\beta\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\ \tilde{\psi}^{\dot{\beta}}_B\end{pmatrix}\right)\\ &= \bar{\Xi}\left(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\ -\Sigma^{iAB}\,\tilde{\psi}^{\dot{\alpha}}_B\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}\bar{\Sigma}^i_{AB}\,\psi^B_\alpha\\0\end{pmatrix}\right)\\ &= -\tilde{\xi}_{A\dot{\alpha}}\,\Sigma^{iAB}\,\tilde{\psi}^{\dot{\alpha}}_B+\xi^{A\alpha}\,\bar{\Sigma}^i_{AB}\,\psi^B_\alpha\,.\end{aligned}\tag{2.56}$$

2.2. Dimensional Reduction of $\mathcal{N}=1$ SYM Theory in 10d

A nice way to obtain $\mathcal{N}=4$ Super Yang-Mills theory in four dimensions is to derive it by dimensional reduction of $\mathcal{N}=1$ Super-Yang Mills theory in ten dimensions [2]. In this section we will discuss this procedure in detail, again based on the presentation in Belitky et al. [20] and [19]. Let us start by introducing $\mathcal{N}=1$ SYM in ten-dimensional Minkowski space $\mathbb{R}^{1,9}$, equipped with metric $g^{NM}=\mathrm{diag}(1,-1,\dots,-1)$. The fundamental fields of the theory are the gauge fields A^a_M and the fermionic fields described by Majorana-Weyl Spinors Ψ^a, see section 2.1.3. The Majorana and the Weyl condition reduce the fermionic on-shell degrees of freedom to eight so that there is an exact balance between bosonic and fermionic on-shell degrees of freedom. This is required for a linear realization of supersymmetry without auxiliary fields. It is convenient to introduce matrix-valued fields, which are defined as follows

$$A_M = A^a_M\,T^a \qquad\qquad \Psi = \Psi^a\,T^a\,.\tag{2.57}$$

The matrices T^a are the generators of $\mathrm{SU}(N)$ in the fundamental representation and normalized according to

$$\mathrm{Tr}\big(T^a\,T^b\big)=\frac{\delta^{ab}}{2}\,.\tag{2.58}$$

Under a gauge transformation these fields transform as follows

$$A_M \to U(z)\left(A_M + i\,\partial_M\right)U^\dagger(z) \qquad\qquad \Psi \to U(z)\,\Psi\,U^\dagger(z)\,,\tag{2.59}$$

where $U(z)=\exp(i\,\theta^a(z)\,T^a)$. We note that the fermionic field transforms in the adjoint representation which is necessary because supersymmetry transforms fermions into bosons and vice versa. Having introduced the fields, let us move on and write down the action of the $\mathcal{N}=1$ SYM model.

$$S_{\mathcal{N}=1}=\frac{1}{g^2_{10}}\int \mathrm{d}^{10}z\;\mathrm{Tr}\left(-\frac{1}{2}\,F_{MN}\,F^{MN}+i\,\bar{\Psi}\,\Gamma_M\,D^M\,\Psi\right)\tag{2.60}$$

The field strength F_{MN} and the covariant derivative D_M are given by

$$F_{MN}=\partial_M\,A_N-\partial_N\,A_M-i\big[A_M,A_N\big] \qquad\qquad D_M\,(\cdot)=\partial_M\,(\cdot)-i\big[A_M,(\cdot)\big]\,.\tag{2.61}$$

The action (2.60) can be shown to be invariant under the following supersymmetry transformations

$$\delta\Psi = \tfrac{i}{2}\, F_{MN}\, \Gamma^{MN}\, \Xi \qquad\qquad \delta A_M = -i\, \bar{\Xi}\, \Gamma_M \Psi\,, \tag{2.62}$$

where $\Gamma_{MN} := \frac{i}{2}\,(\Gamma_M\,\Gamma_N - \Gamma_N\,\Gamma_M)$. For a proof of this statement see [19]. Before we start with the dimensional reduction of the theory, let us briefly analyze the mass dimensions of the fields and the coupling constant. One easily finds

$$[g_{10}] = -3 \qquad\qquad [A_M] = 1 \qquad\qquad [\Psi] = [\bar{\Psi}] = \frac{3}{2}\,. \tag{2.63}$$

All fields in the action (2.60) have been rescaled by the dimensionful coupling constant g_{10} so that the mass dimensions of the fields match with those found in four-dimensional Minkowski space. For the reduction we now assume that six of the ten dimensions are compactified in such a way that the spacetime structure is $\mathbb{R}^{1,3} \times T^6$. The six-dimensional torus T^6 will be treated as an internal space with volume V_6. We split the ten-dimensional coordinates according to

$$z^M = \begin{pmatrix} x^\mu & y^i \end{pmatrix} \qquad\qquad \mu = 0,1,2,3 \qquad\qquad i = 4,\ldots,9\,, \tag{2.64}$$

with $y^4,\ldots,y^6$ being the internal coordinates on the torus T^6. For the gauge fields A_M^a we write

$$A_M^a = \begin{pmatrix} A_\mu^a & \phi_i^a \end{pmatrix}, \tag{2.65}$$

where the indices take the same values as in (2.64). The first four components will become the four-dimensional gauge fields A_μ^a, whereas the latter components, referred to by ϕ_i^a, will give rise to the scalars of $\mathcal{N} = 4$ SYM. Using this notation, we will now split the ten-dimensional Lagrangian of (2.60) into its four- and six-dimensional part. Since the torus is treated as an internal space, we employ that the fields do only depend on the first four coordinates, i.e.

$$\partial_i\, A_M^a(x) = 0 \qquad\qquad \partial_i\, \Psi^a(x) = 0\,. \tag{2.66}$$

Let us start by focusing on the bosonic part of (2.60). We find

$$\begin{aligned} F_{MN}\, F^{MN} &= F_{\mu\nu}\, F^{\mu\nu} + F_{ij}\, F^{ij} + 2\, F_{\mu i}\, F^{\mu i} \\ &= F_{\mu\nu}\, F^{\mu\nu} - \left[\phi_i, \phi_j\right]\left[\phi^i, \phi^j\right] + 2\,(D_\mu\, \phi_i)(D^\mu\, \phi^i) \\ &= F_{\mu\nu}\, F^{\mu\nu} - \tfrac{1}{4}\left[\phi^{AB}, \phi^{CD}\right]\left[\bar{\phi}_{AB}, \bar{\phi}_{CD}\right] - \left(D_\mu\, \phi^{AB}\right)\left(D^\mu\, \bar{\phi}_{AB}\right), \end{aligned} \tag{2.67}$$

where (2.39) was used to get from the second to the third line. We have already computed the component expression for $\bar{\Psi}\, \Gamma^M\, \Psi$ ((2.55) and (2.56)), so the reduction of the fermionic term is fairly easy. Treating $M = \mu$ first, we obtain

$$\begin{aligned} \mathrm{Tr}\left(\bar{\Psi}\, \Gamma^\mu\, D_\mu\, \Psi\right) &= \mathrm{Tr}\left(\tilde{\psi}_{A\dot{\alpha}}\, \sigma^{\mu\dot{\alpha}\beta}\, D_\mu\, \psi_\beta^A + \psi^{A\alpha}\, \bar{\sigma}^\mu_{\alpha\dot{\beta}}\, D_\mu\, \tilde{\psi}_A^{\dot{\beta}}\right) \\ &= \mathrm{Tr}\left(\tilde{\psi}_{A\dot{\alpha}}\, \sigma^{\mu\dot{\alpha}\beta}\, D_\mu\, \psi_\beta^A + \psi_\beta^A\, \sigma^{\mu\dot{\alpha}\beta}\, \partial_\mu\, \tilde{\psi}_{A\dot{\alpha}} - i\, \psi_\beta^{aA}\, \sigma^{\mu\dot{\alpha}\beta}\, A_\mu^b\, \tilde{\psi}^c_{A\dot{\alpha}}\, T^a\left[T^b, T^c\right]\right) \\ &= \mathrm{Tr}\left(\tilde{\psi}_{A\dot{\alpha}}\, \sigma^{\mu\dot{\alpha}\beta}\, D_\mu\, \psi_\beta^A - \left(\partial_\mu\, \tilde{\psi}_{A\dot{\alpha}}\right) \sigma^{\mu\dot{\alpha}\beta}\, \psi_\beta^A - i\, \tilde{\psi}^c_{A\dot{\alpha}}\, \sigma^{\mu\dot{\alpha}\beta}\, A_\mu^b\, \psi_\beta^{aA}\left[T^b, T^a\right] T^c\right) \\ &= 2\ \mathrm{Tr}\left(\tilde{\psi}_{A\dot{\alpha}}\, \sigma^{\mu\dot{\alpha}\beta}\, D_\mu\, \psi_\beta^A\right) + \text{total div}\,. \end{aligned} \tag{2.68}$$

In the second line we raised the Weyl indices of the sigma matrix according to the rule (2.7). The third line is obtained by noting that the spinors are anticommuting Grassmann-valued objects and by using the cyclicity of the trace. Integration by parts then yields the final result. Let us finally work out the terms where $M=i$.

$$\begin{aligned}\bar{\Psi}\,\Gamma^i\,D_i\,\Psi &= i\,\tilde{\psi}_{A\dot{\alpha}}\,\Sigma^{iAB}\big[\phi_i,\tilde{\psi}^{\dot{\alpha}}_B\big]-i\,\psi^{A\alpha}\,\bar{\Sigma}^i_{AB}\big[\phi_i,\psi^B_\alpha\big]\\ &= -i\,\sqrt{2}\,\tilde{\psi}_{A\dot{\alpha}}\big[\phi^{AB},\tilde{\psi}^{\dot{\alpha}}_B\big]+i\,\sqrt{2}\,\psi^{A\alpha}\big[\bar{\phi}_{AB},\psi^B_\alpha\big]\,,\end{aligned}\tag{2.69}$$

where we have employed (2.37). We will need the supersymmetry transformations of the fields of $\mathcal{N}=4$ SYM in section 4.2. Therefore let us derive them as well, starting from the ten-dimensional supersymmetry transformations (2.62). For the gauge field and the scalars one instantly finds

$$\begin{aligned}\delta A^\mu &= -i\,\xi^{A\alpha}\,\bar{\sigma}^\mu_{\alpha\dot{\beta}}\,\tilde{\psi}^{\dot{\beta}}_A - i\,\tilde{\xi}_{A\dot{\alpha}}\,\sigma^{\mu\dot{\alpha}\beta}\,\psi^A_\beta\\ \delta\phi^i &= -i\,\xi^{A\alpha}\,\bar{\Sigma}^i_{AB}\,\psi^B_\alpha + i\,\tilde{\xi}_{A\dot{\alpha}}\,\Sigma^{iAB}\,\tilde{\psi}^{\dot{\alpha}}_B\,,\end{aligned}\tag{2.70}$$

where (2.55) and (2.56) have been used again. Contracting the second equation with $1/\sqrt{2}\,\Sigma^{iAB}$ and using identities (2.35) and (2.36) yields

$$\begin{aligned}\delta\phi^{AB} &= -\tfrac{i}{\sqrt{2}}\,\xi^{C\alpha}\,\Sigma^{iAB}\,\bar{\Sigma}^i_{CD}\,\psi^D_\alpha + \tfrac{i}{\sqrt{2}}\,\tilde{\xi}_{C\dot{\alpha}}\,\Sigma^{iAB}\,\Sigma^{iCD}\,\tilde{\psi}^{\dot{\alpha}}_D\\ &= -\tfrac{i}{\sqrt{2}}\,\xi^{C\alpha}\,2\left(\delta^A_C\,\delta^B_D-\delta^A_D\,\delta^B_C\right)\psi^D_\alpha + \tfrac{i}{\sqrt{2}}\,\tilde{\xi}_{C\dot{\alpha}}\,2\,\varepsilon^{ABCD}\,\tilde{\psi}^{\dot{\alpha}}_D\\ &= -i\,\sqrt{2}\left(\xi^{A\alpha}\,\psi^B_\alpha-\xi^{B\alpha}\,\psi^A_\alpha-\varepsilon^{ABCD}\,\tilde{\xi}_{C\dot{\alpha}}\,\tilde{\psi}^{\dot{\alpha}}_D\right).\end{aligned}\tag{2.71}$$

To derive the supersymmetry transformations of the fermions we proceed as follows

$$\tfrac{i}{2}\,F_{MN}\,\Gamma^{MN}\,\Xi = \tfrac{i}{2}\,F_{\mu\nu}\,\Gamma^{\mu\nu}\,\Xi + \tfrac{i}{2}\,F_{ij}\,\Gamma^{ij}\,\Xi + i\,F_{\mu i}\,\Gamma^{\mu i}\,\Xi\,.\tag{2.72}$$

This calculation is a bit longer than the one before which is why we will compute all three terms individually. Let us start with the first one. Using the definition of $\Gamma^{\mu\nu}$ and (2.23), we get

$$\begin{aligned}\tfrac{i}{2}\,F_{\mu\nu}\,\Gamma^{\mu\nu}\,\Xi &= \tfrac{i}{2}\,F_{\mu\nu}\left(\mathbb{1}_8\otimes\tfrac{i}{2}\big[\gamma^\mu,\gamma^\nu\big]\right)\Xi = \tfrac{i}{2}\,F_{\mu\nu}\left(\mathbb{1}_8\otimes\begin{pmatrix}\sigma^{\mu\nu}{}_\alpha{}^\beta & 0\\ 0 & \bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}}\end{pmatrix}\right)\Xi\\ &= \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}\tfrac{i}{2}\,F_{\mu\nu}\,\sigma^{\mu\nu}{}_\alpha{}^\beta\,\xi^A_\beta\\ 0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\ \tfrac{i}{2}\,F_{\mu\nu}\,\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}}\,\tilde{\xi}^{\dot{\beta}}_A\end{pmatrix}.\end{aligned}\tag{2.73}$$

For the second term of (2.72), we find

$$\begin{aligned}\tfrac{i}{2}\,F_{ij}\,\Gamma^{ij}\,\Xi &= \tfrac{i}{2}\,F_{ij}\left(\tfrac{i}{2}\big[\hat{\gamma}^i,\hat{\gamma}^j\big]\otimes\mathbb{1}_4\right)\Xi\\ &= \tfrac{i}{4}\big[\phi_i,\phi_j\big]\left(\begin{pmatrix}\Sigma^{iAB}\,\bar{\Sigma}^j_{BC}-\Sigma^{jAB}\,\bar{\Sigma}^i_{BC} & 0\\ 0 & \bar{\Sigma}^i_{AB}\,\Sigma^{jBC}-\bar{\Sigma}^j_{AB}\,\Sigma^{iBC}\end{pmatrix}\otimes\mathbb{1}_4\right)\Xi\\ &= \left(\begin{pmatrix}i\,\big[\phi^{AB},\bar{\phi}_{BC}\big] & 0\\ 0 & i\,\big[\bar{\phi}_{AB},\phi^{BC}\big]\end{pmatrix}\otimes\mathbb{1}_4\right)\Xi\\ &= \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}i\,\big[\phi^{AB},\bar{\phi}_{BC}\big]\,\xi^C_\alpha\\ 0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\ i\,\big[\bar{\phi}_{AB},\phi^{BC}\big]\,\tilde{\xi}^{\dot{\alpha}}_C\end{pmatrix}.\end{aligned}\tag{2.74}$$

The second line has been obtained by noting that $F_{ij}=-i[\phi_i,\phi_j]$ due to the dimensional reduction. What remains is the computation of the last term of (2.72). Since $F_{\mu i}=D_\mu\,\phi_i$ we get

$$\begin{aligned}
i\,F_{\mu i}\,\Gamma^{\mu i}\,\Xi &= i\,(D_\mu\,\phi_i)\,\Big(i\,\hat{\gamma}^i\otimes\gamma^\mu\,\gamma^5\Big)\,\Xi \\
&= -\,(D_\mu\,\phi_i)\left(\begin{pmatrix}0 & \Sigma^{iAB}\\ \bar{\Sigma}^i_{AB} & 0\end{pmatrix}\otimes\begin{pmatrix}0 & -\bar{\sigma}^\mu{}_{\alpha\dot{\beta}}\\ \sigma^{\mu\dot{\alpha}\beta} & 0\end{pmatrix}\right)\Xi \\
&= \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}-\sqrt{2}\,(D_\mu\,\phi^{AB})\,\bar{\sigma}^\mu{}_{\alpha\dot{\beta}}\,\tilde{\xi}^{\dot{\beta}}_B\\ 0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\ \sqrt{2}\,(D_\mu\,\bar{\phi}_{AB})\,\sigma^{\mu\dot{\alpha}\beta}\xi^B_\beta\end{pmatrix}\,,
\end{aligned}\tag{2.75}$$

where we have raised the index of ϕ_i with the help of the metric η^{ij}. Adding up all three terms then yields the supersymmetry transformations of the fermions.

$$\begin{aligned}
\delta\psi^A_\alpha &= \tfrac{i}{2}\,F_{\mu\nu}\,\sigma^{\mu\nu}{}_\alpha{}^\beta\,\xi^A_\beta+i\,[\phi^{AB},\bar{\phi}_{BC}]\,\xi^C_\alpha-\sqrt{2}\,(D_\mu\,\phi^{AB})\,\bar{\sigma}^\mu{}_{\alpha\dot{\beta}}\,\tilde{\xi}^{\dot{\beta}}_B \\
\delta\tilde{\psi}^{\dot{\alpha}}_A &= \tfrac{i}{2}\,F_{\mu\nu}\,\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}}\,\tilde{\xi}^{\dot{\beta}}_A+i\,[\bar{\phi}_{AB},\phi^{BC}]\,\tilde{\xi}^{\dot{\alpha}}_C+\sqrt{2}\,(D_\mu\,\bar{\phi}_{AB})\,\sigma^{\mu\dot{\alpha}\beta}\xi^B_\beta
\end{aligned}\tag{2.76}$$

2.3. The Fields, the Action and the Propagators

Let us first summarize the results of the last section. Through dimensional reduction of $\mathcal{N}=1$ SYM theory, $\mathcal{N}=4$ SYM theory in four-dimensional Minkowski space has been obtained. The field content of the theory consists of a gauge field A_μ, four complex Weyl fermions ψ^A_α and six real scalars ϕ^{AB}. All fields transform in the adjoint representation of the gauge group, which in this case is $\mathrm{SU}(N)$. In total we have

$$\begin{array}{lll}
\text{gluon} & A^a_\mu & \mu=0,\dots,3, a=1,\dots,N^2-1 \\
\text{6 scalars} & \phi^{aAB} & A,B=0,\dots,4, a=1,\dots,N^2-1 \\
\text{4 Weyl fermions} & \psi^{aA}_\alpha,\tilde{\psi}^{a\dot{\alpha}}_A & A=0,\dots,4, a=1,\dots,N^2-1, \alpha,\dot{\alpha}=1,2\,,
\end{array}\tag{2.77}$$

accompanied by the complex conjugation properties

$$\left(\phi^{aAB}\right)^*=\bar{\phi}^a_{AB}\qquad\qquad \left(\psi^{aA}_\alpha\right)^*=\tilde{\psi}^a_{A\dot{\alpha}}\,.\tag{2.78}$$

Adding up (2.67), (2.68) and (2.69) with the appropriate prefactors, we find that the action of the $\mathcal{N}=4$ SYM model is given by

$$\begin{aligned}
S_{\mathcal{N}=4}=\frac{1}{g^2}\int\mathrm{d}^4x\;\mathrm{Tr}\Big(&-\frac{1}{2}\,F_{\mu\nu}\,F^{\mu\nu}+\frac{1}{2}\,(D_\mu\,\phi^{AB})(D^\mu\,\bar{\phi}_{AB})+\frac{1}{8}\,[\phi^{AB},\phi^{CD}]\,[\bar{\phi}_{AB},\bar{\phi}_{CD}] \\
&+2\,i\,\tilde{\psi}_{A\dot{\alpha}}\,\sigma^{\mu\dot{\alpha}\beta}\,D_\mu\,\psi^A_\beta+\sqrt{2}\,\tilde{\psi}_{A\dot{\alpha}}\,[\phi^{AB},\tilde{\psi}^{\dot{\alpha}}_B]-\sqrt{2}\,\psi^{A\alpha}\,[\bar{\phi}_{AB},\psi^B_\alpha]\Big)\,,
\end{aligned}\tag{2.79}$$

where

$$F_{\mu\nu}=\partial_\mu\,A_\nu-\partial_\nu\,A_\mu-i\big[A_\mu,A_\nu\big]\qquad\qquad D_\mu\,(\cdot)=\partial_\mu\,(\cdot)-i\big[A_\mu,(\cdot)\big]\,.\tag{2.80}$$

The volume integral over the internal space has been absorbed into a redefinition of the coupling constant

$$g^2:=\frac{g^2_{10}}{V_6}\,,\tag{2.81}$$

which is now dimensionless due to the fact that V_6 has mass dimension -6. Beside the action, we also derived the supersymmetry transformations of $\mathcal{N} = 4$ SYM theory. From the canonical point of view, the field transformations are related to the 16 supercharges in the following way

$$\delta W = -i \left[\xi^A_\alpha \, \mathcal{Q}^\alpha_A + \tilde{\xi}_{A\dot{\alpha}} \, \bar{\mathcal{Q}}^{A\dot{\alpha}}, W\right] \qquad\qquad W \in \left\{A^\mu, \phi^{AB}, \psi^A_\alpha, \tilde{\psi}^{\dot{\alpha}}_A\right\} . \tag{2.82}$$

Here, the transformation parameters ξ^A_α and $\tilde{\xi}_{A\dot{\alpha}}$ are the components of a constant ten-dimensional Majorana-Weyl spinor and $\mathcal{Q}^\alpha_A$ and $\bar{\mathcal{Q}}^{A\dot{\alpha}}$ are the sixteen supercharges, which are defined as the spatial integral of the zeroth component of the supersymmetry current. For later computational convenience it is useful to split up the parameters of the transformation and to define formal operators, which generate supersymmetry transformations when applied to a field. Thus, we define

$$\delta W = \xi^A_\alpha \, \mathfrak{q}^\alpha_A(W) + \tilde{\xi}_{A\dot{\alpha}} \, \bar{\mathfrak{q}}^{A\dot{\alpha}}(W) , \tag{2.83}$$

where W again stands for an arbitrary field and $\mathfrak{q}^\alpha_A$ and $\bar{\mathfrak{q}}^{A\dot{\alpha}}$ denote the formal generators of supersymmetry variations of the fields. Using this definition and the results (2.70), (2.71) and (2.76), it is now fairly easy to show that these supersymmetry generators act on fields as follows

$$\begin{aligned}
\mathfrak{q}^\alpha_A(A^{\beta\dot{\beta}}) &= 2\, i\, \varepsilon^{\alpha\beta} \, \tilde{\psi}^{\dot{\beta}}_A & \bar{\mathfrak{q}}^{A\dot{\alpha}}(A^{\beta\dot{\beta}}) &= -2\, i\, \varepsilon^{\dot{\alpha}\dot{\beta}} \, \psi^{A\beta} \\
\mathfrak{q}^\alpha_A(\bar{\phi}_{BC}) &= \sqrt{2}\, i\, \varepsilon_{ABCD} \, \psi^{D\alpha} & \bar{\mathfrak{q}}^{A\dot{\alpha}}(\bar{\phi}_{BC}) &= -\sqrt{2}\, i\, (\tilde{\psi}^{\dot{\alpha}}_B \, \delta^A_C - \tilde{\psi}^{\dot{\alpha}}_C \, \delta^A_B) \\
\mathfrak{q}^\alpha_A(\psi^{B\beta}) &= \tfrac{i}{2} \, F^{\alpha\beta} \, \delta^B_A + i\, \varepsilon^{\beta\alpha} \left[\bar{\phi}_{AC}, \phi^{BC}\right] & \bar{\mathfrak{q}}^{A\dot{\alpha}}(\psi^{B\beta}) &= -\sqrt{2}\, D^{\beta\dot{\alpha}} \, \phi^{AB} \\
\mathfrak{q}^\alpha_A(\tilde{\psi}^{\dot{\beta}}_B) &= -\sqrt{2}\, D^{\dot{\beta}\alpha} \, \bar{\phi}_{AB} & \bar{\mathfrak{q}}^{A\dot{\alpha}}(\tilde{\psi}^{\dot{\beta}}_B) &= -\tfrac{i}{2} \, F^{\dot{\alpha}\dot{\beta}} \, \delta^A_B + i\, \varepsilon^{\dot{\alpha}\dot{\beta}} \left[\phi^{AC}, \bar{\phi}_{BC}\right] .
\end{aligned}$$

Given the action (2.79), we can now derive the Feynman rules of this theory. As this is a standard textbook exercise, we will not go into too much detail. In particular, we shall only derive the Feynman rules for the propagators in position space, since only those will be needed in the subsequent discussion. Let us start by writing down the component expression for the interesting part of the Lagrangian. Using (2.58), we find

$$\begin{aligned}
\mathcal{L}_p &= -\frac{1}{4g^2} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \frac{1}{4g^2} \left(\partial_\mu \phi^{aAB}\right) \left(\partial^\mu \bar{\phi}^a_{AB}\right) + \frac{i}{g^2} \, \tilde{\psi}^a_{A\dot{\alpha}} \, \sigma^{\mu\dot{\alpha}\alpha} \, \partial_\mu \, \psi^{aA}_\alpha \\
&= \frac{1}{2g^2} \, A^{a\mu} \left(\eta_{\mu\nu} \, \partial_\rho \, \partial^\rho - \partial_\nu \, \partial_\mu\right) A^{a\nu} - \frac{1}{2g^2} \, \phi^{ai} \left(\partial_\mu \, \partial^\mu\right) \phi^{ai} + \frac{i}{g^2} \, \tilde{\psi}^a_{A\dot{\alpha}} \, \sigma^{\mu\dot{\alpha}\alpha} \, \partial_\mu \, \psi^{aA}_\alpha ,
\end{aligned} \tag{2.84}$$

where the second line has been obtained by employing integration by parts and the trace identity (2.39). It is well-known that the operator in (2.84) that acts on the gluon fields is not invertible. However, choosing Feynman gauge allows us to neglect the second part of the operator and to work with the part proportional to the metric instead. For details concerning this gauge fixing prescription see [21]. From the last line of (2.84) we can directly read off the propagators. We get

$$\begin{aligned}
\left\langle \psi^{aA}_\alpha(x_1) \, \tilde{\psi}^b_{\dot{\alpha}B}(x_2) \right\rangle &= -\, g^2 \, \delta^{ab} \, \delta^A_B \, (\partial_{x_1})_{\alpha\dot{\alpha}} \, G(x_1 - x_2) \\
\left\langle A^a_\mu(x_1) \, A^b_\nu(x_2) \right\rangle &= -i \, g^2 \, \eta_{\mu\nu} \, \delta^{ab} \, G(x_1 - x_2) \\
\left\langle \phi^{ai}(x_1) \, \phi^{bj}(x_2) \right\rangle &= -i \, g^2 \, \eta^{ij} \, \delta^{ab} \, G(x_1 - x_2) ,
\end{aligned} \tag{2.85}$$

where

$$G(x_1-x_2)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\frac{e^{-ik\cdot(x_1-x_2)}}{k^2+i\varepsilon}\,. \tag{2.86}$$

To obtain the position space representation of these propagators we have to carry out the Fourier integral. This can be done by performing a Wick rotation, which leads to a well-defined Euclidean integral and by using Schwinger parametrization to rewrite the denominator. Details on this calculation can be found in [22]. The result reads

$$G(x_1-x_2)=\frac{i}{4\pi^2}\,\frac{1}{(x_1-x_2)^2}\,. \tag{2.87}$$

Plugging this expression into (2.85) yields

$$\begin{aligned}\big\langle\psi_\alpha^{aA}(x_1)\,\tilde{\psi}^b_{\dot\alpha B}(x_2)\big\rangle&=\frac{ig^2}{2\pi^2}\,\frac{\delta^A_B\,\delta^{ab}\,(x_1-x_2)_{\alpha\dot\alpha}}{(x_1-x_2)^4}\\ \big\langle A^a_\mu(x_1)\,A^b_\nu(x_2)\big\rangle&=\frac{g^2}{4\pi^2}\,\frac{\eta_{\mu\nu}\,\delta^{ab}}{(x_1-x_2)^2}\\ \big\langle\phi^{ai}(x_1)\,\phi^{bj}(x_2)\big\rangle&=\frac{g^2}{4\pi^2}\,\frac{\eta^{ij}\,\delta^{ab}}{(x_1-x_2)^2}\,.\end{aligned} \tag{2.88}$$

The last thing we want to derive is the Feynman propagator of the scalar fields in case that the fields are characterized by antisymmetric 4×4 matrices, see (2.37).

$$\begin{aligned}\big\langle\bar\phi^a_{AB}(x_1)\,\bar\phi^b_{CD}(x_2)\big\rangle&=\frac{1}{2}\,\bar\Sigma^i_{AB}\,\bar\Sigma^j_{CD}\,\big\langle\phi^{ai}(x_1)\,\phi^{bj}(x_2)\big\rangle\\ &=-\frac{g^2}{8\pi^2}\,\bar\Sigma^i_{AB}\,\bar\Sigma^i_{CD}\,\frac{\delta^{ab}}{(x_1-x_2)^2}\\ &=-\frac{g^2}{4\pi^2}\,\frac{\varepsilon_{ABCD}\,\delta^{ab}}{(x_1-x_2)^2}\end{aligned} \tag{2.89}$$

2.4. Symmetries

One of the most remarkable properties of $\mathcal{N}=4$ SYM theory is the high degree of symmetry this model possesses compared to other quantum field theories. In this section we will first discuss the symmetries of the classical action, followed by a short statement on whether or not these symmetries survive the quantization procedure. The remaining part of the section is then dedicated to the algebraic structures of the hidden symmetries, which are strongly related to the integrability of the theory and which show up on the level of gauge invariant observables in a certain limit.

2.4.1. Symmetries of the Classical Action

As already mentioned in the abstract of this section, we will start by discussing the global symmetries of the action (2.79). The most obvious classical symmetries are the Poincaré symmetry and the invariance under scale transformations. While the first one is manifest due to the Lagrangian being a Lorentz scalar, the latter one can easily be seen by analyzing the mass dimensions of the quantities appearing in the action. One finds

$$[g]=0\qquad[A^\mu]=[\phi^{AB}]=[D^\mu]=[\partial^\mu]=1\qquad[\Psi]=[\bar\Psi]=\frac{3}{2}\,, \tag{2.90}$$

from which we conclude that the summands in the Lagrangian scale uniformly. Hence, the theory is scale invariant at the classical level. In the case of $\mathcal{N} = 4$ SYM theory the Poincaré symmetry and the scale invariance extend to a full conformal symmetry (the unfamiliar reader might wish to consult section 3.4 for an introduction to conformal symmetry). Therefore, the conformal group of $\mathbb{R}^{1,3}$, i.e. $\mathrm{SO}(2,4)$, has to be a subgroup of the full symmetry group of the theory. Beside the conformal symmetry, the theory also has a global $\mathrm{SU}(4)$ invariance, which is called R-symmetry. Under a R-symmetry transformation the supercharges get rotated according to

$$Q^{\alpha}_{A} \to U_{A}{}^{B}\, Q^{\alpha}_{B} \qquad\qquad \bar{Q}^{A\dot{\alpha}} \to U^{A}{}_{B}\, \bar{Q}^{B\dot{\alpha}}\,, \tag{2.91}$$

where U is a global $\mathrm{SU}(4)$-matrix. On the level of the Lagrangian the R-symmetry is realized as a flavor symmetry of the fields, which is manifest due to all capital Latin letters being contracted. As we are dealing with a supersymmetric gauge theory, we know that the two bosonic subalgebras $\mathfrak{so}(2,4)$ and $\mathfrak{su}(4)$ must be part of a larger symmetry algebra which also involves the 16 supercharges, whose action on fields we have already derived in section 2.3. In fact, the supersymmetry generators Q^{α}_{A} and $\bar{Q}^{A\dot{\alpha}}$ extend the Poincaré algebra, which is a subalgebra of the conformal algebra, to a super Poincaré algebra. The relations which define this superalgebra are the usual Poincaré commutators supplemented by the anticommutator

$$\left\{Q^{\alpha}_{A}, \bar{Q}^{B\dot{\alpha}}\right\} = 2\,i\,\delta^{B}_{A}\, P^{\alpha\dot{\alpha}}\,, \tag{2.92}$$

and commutators which state that the supercharges transform as left-/right-handed spinors under Lorentz transformations. All the other (anti-)commutators vanish. However, since the commutator between the generator of special conformal transformations and the supercharges

$$\left[K_{\alpha\dot{\alpha}}, Q^{\beta}_{A}\right] = 2\,\delta^{\beta}_{\alpha}\, \bar{S}_{A\dot{\alpha}} \qquad\qquad \left[K_{\alpha\dot{\alpha}}, \bar{Q}^{A\dot{\beta}}\right] = 2\,\delta^{\dot{\beta}}_{\dot{\alpha}}\, S^{A}_{\alpha}\,, \tag{2.93}$$

yields an element which we have not introduced so far, even this is not the whole story. For the closure of the algebra we need to introduce the generators S^{A}_{α} and $\bar{S}_{A\dot{\alpha}}$ which are often called conformal supercharges. From an algebraic point of view, S^{A}_{α} and $\bar{S}_{A\dot{\alpha}}$ stand to $K_{\alpha\dot{\alpha}}$ in the way that Q^{α}_{A} and $\bar{Q}^{A\dot{\alpha}}$ stand to $P_{\alpha\dot{\alpha}}$, i.e.

$$\left\{S^{A}_{\alpha}, \bar{S}_{B\dot{\alpha}}\right\} = -2\,i\,\delta^{A}_{B}\, K_{\alpha\dot{\alpha}}\,. \tag{2.94}$$

The full symmetry algebra of the theory is now given by the R-symmetry algebra $\mathfrak{su}(4)$ as well as the conformal algebra $\mathfrak{so}(2,4)$ which form together with the generators Q^{α}_{A}, $\bar{Q}^{A\dot{\alpha}}$,S^{A}_{α} and $\bar{S}_{A\dot{\alpha}}$ the Lie superalgebra $\mathfrak{psu}(2,2|4)$. Thus, $\mathcal{N} = 4$ SYM theory is at the classical level "much more symmetric" than other four-dimensional quantum field theories. Due to the fact that additional supercharges would require to introduce fields with spin higher than one, $\mathcal{N} = 4$ SYM theory is often described as the most symmetric interacting gauge theory in four dimensions.

2.4.2. The β-Function

So far, we have discussed the classical symmetries of the action (2.79). There are other quantum field theories beside $\mathcal{N} = 4$ SYM theory that are scale invariant at the classical level, for example massless QCD. However, in the majority of cases, this symmetry is broken at the quantum level. A quantity, which indicates whether the conformal symmetry is quantum mechanically broken or not is the renormalization group β-function. In a theory that remains scale invariant at the quantum level the β-function has to vanish. For a gauge theory with n_f Weyl fermions and n_s complex scalars the one-loop β-function [23] is given by

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{6}T(\text{adj}) - \frac{1}{3}\sum_a T(r_a) - \frac{1}{6}\sum_n T(r_n)\right) , \tag{2.95}$$

where the first sum runs over the fermions, the second over complex scalars and $T(r_{a,n})$ denotes the index of the representation. In the case of $\mathcal{N}=4$ SYM theory, we have four Weyl fermions and three complex scalars which all transform in the adjoint representation of $\mathrm{SU}(N)$. Hence, the one-loop β-function vanishes. Furthermore, since it is believed that the β-function vanishes to all orders in perturbation theory [4, 5, 7], the theory remains exactly scale invariant. The β-function thus indicates that the symmetry group $\mathrm{PSU}(2,2|4)$ is unbroken by quantum corrections, which makes $\mathcal{N}=4$ SYM theory a very remarkable quantum field theory.

2.4.3. Integrability and Yangian Symmetries

Although $\mathcal{N}=4$ SYM theory was discovered in the late 70s [2, 3], the most fascinating results have been found in the last 15 years. In particular, much evidence has been accumulated indicating that in the planar limit, in which the number of colors N is taken to infinity, while the 't Hooft coupling

$$\lambda := g^2 N \tag{2.96}$$

is held fixed, $\mathcal{N}=4$ SYM theory becomes integrable. Generally, integrability can be viewed as an infinite-dimensional (hidden) symmetry, which imposes powerful constraints onto all physical observables so that they are, at least in principle, completely determined. In $\mathcal{N}=4$ SYM theory this infinite-dimensional symmetry algebra arises as an extension of the finite-dimensional symmetry algebra $\mathfrak{psu}(2,2|4)$ and is due to the necessity of the planar limit not respected by the Lagrangian. In fact, it can only be observed at the level of observables with a non-trivial dependence on the 't Hooft coupling and is even there far away from being manifest. From a mathematical point of view the extended algebra forms an infinite-dimensional Hopf algebra (see [24] for a proper definition) of Yangian type. Since Yangian symmetries are at the heart of this thesis, we will now give a basic introduction of the algebraic framework. For reasons of simplicity we will focus on the case where the underlying Lie algebra is semisimple and non-graded, but everything can also be extended to Lie superalgebras. Our presentation is partly influenced by that in [25].

Let $\mathfrak{g}$ be a semisimple, finite-dimensional Lie algebra with structure constants f_{ab}^c and generators $J_a^{(0)}$, $a = 1, \ldots, \dim(\mathfrak{g})$.

$$\left[J_a^{(0)}, J_b^{(0)}\right] = f_{ab}^c\, J_c^{(0)} , \tag{2.97}$$

where $[\ ,\ ]$ denotes the Lie bracket, which satisfies the Jacobi identity

$$\left[J_a^{(0)},\left[J_b^{(0)},J_c^{(0)}\right]\right]+\text{cyclic}=0\,. \tag{2.98}$$

The upper index (0) has been introduced for later convenience. Note that in the above definition, $[\ ,\ ]$ really denotes the abstract Lie bracket, which simply assigns to two given elements another element of the Lie algebra. Only on representation spaces the Lie bracket coincides with the usual commutator, i.e. $[x,y]=x\,y-y\,x$. Now, given such a Lie algebra $\mathfrak{g}$ there exists an invariant non-degenerate symmetric bilinear form

$$K(x,y)=\mathrm{Tr}\big(\mathrm{ad}(x)\ \mathrm{ad}(y)\big) \qquad\qquad x,y\in\mathfrak{g}\,, \tag{2.99}$$

known as the Killing form, which defines a (pseudo) inner product on the Lie algebra. In the above definition $\mathrm{ad}(x)$ denotes a Lie algebra element in the adjoint representation. Symmetry and invariance (or associativity) means that for all $x,y,z\in\mathfrak{g}$

$$K(x,y)=K(y,x) \qquad\qquad K([x,y],z)=K\big(x,[y,z]\big)\,. \tag{2.100}$$

Evaluated on basis elements, the Killing form reads

$$K_{ab}=K\big(J_a^{(0)},J_b^{(0)}\big)=\mathrm{Tr}\big(\mathrm{ad}(J_a^{(0)})\ \mathrm{ad}(J_b^{(0)})\big)=f_{ad}^{c}\,f_{bc}^{d}\,, \tag{2.101}$$

Since K_{ab} is non-degenerate, its inverse exists and will be denoted by K^{ab}. Hence, we can use the Killing form as a metric tensor to raise and lower group indices. For instance, we have

$$f_{ab}^{c}=K^{cd}\,f_{abd} \qquad\qquad f_a^{bc}=K^{bd}\,f_{ad}^{c}\,. \tag{2.102}$$

Before we come to the definition of the Yangian, let us briefly introduce the notion of the universal enveloping algebra and comment a little bit on its Hopf-algebraic structure. The universal enveloping algebra is a concept to embed a Lie algebra into a much bigger associative algebra with unit element. An important advantage of this extended algebra is that one can take associative products of Lie algebra elements. In general, such products are, as mentioned before, only defined in a representation $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$. However, some relations that involve associative products are of course valid in any representation, for example

$$\rho(x)\,\rho(y)=\rho(z)+\rho(y)\,\rho(x) \qquad\qquad \text{if} \qquad [x,y]=z\,. \tag{2.103}$$

Let us take this as a first motivation to study associative products of Lie algebra elements on more general grounds, using the universal enveloping algebra. This being a rather abstract concept, the most intuitive approach from the perspective of a physicist is to construct the universal enveloping algebra $U(\mathfrak{g})$ explicitly, including an appropriate basis. Let us start by introducing the tensor algebra of $\mathfrak{g}$

$$T(\mathfrak{g}):=\bigoplus_{n=0}^{\infty}\mathfrak{g}^{\otimes n}=\mathbb{K}\oplus g\oplus(g\otimes g)\oplus\dots\,, \tag{2.104}$$

where $\mathbb{K}$ is the underlying field. This tensor algebra is isomorphic to the free algebra spanned by all linear combinations

$$P(J_a^{(0)})=\sum_{k=0}a^{i_1\dots i_k}\,J_{i_1}^{(0)}\dots J_{i_k}^{(0)}\,, \tag{2.105}$$

of formal products of the generators $J_a^{(0)}$ with coefficients $a^{i_1\cdots i_k} \in \mathbb{K}$. Hence, the elements of the free algebra are the ordered polynomials of Lie algebra elements. Based on this, the universal enveloping algebra $U(\mathfrak{g})$ can be defined as the tensor/free algebra of $\mathfrak{g}$ where, simply speaking, one has additionally identified the abstract Lie bracket with commutators, i.e. for all $x, y \in \mathfrak{g}$

$$[x, y] \equiv x\,y - y\,x\,. \tag{2.106}$$

Here, the product is the tensor product or simply the formal concatenation. More specifically, the universal enveloping algebra is obtained by taking the tensor/free algebra of $\mathfrak{g}$ and dividing it into equivalence classes. Two elements belong two the same equivalence class, if they are equal modulo the commutation relations. For example, let M and N be two monomials in $T(\mathfrak{g})$. If $[x, y] = z$, the following two elements are equivalent

$$M\,(x\,y - y\,x)\,N \equiv M\,z\,N\,. \tag{2.107}$$

Using mathematical language, the process of dividing the tensor algebra into equivalence classes can be formulated as follows

$$U(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} \Big/ \Big(x\,y - y\,x - [x, y]\Big)\,. \tag{2.108}$$

The universal enveloping algebra is therefore the quotient space obtained by taking the tensor algebra of $\mathfrak{g}$ and dividing out the two-sided ideal generated by all elements of the form $x\,y - y\,x - [x, y]$. Now, the Poincaré-Birkhoff-Witt theorem states that given a set of generators $J_a^{(0)}$, which form a basis of the underlying Lie algebra, the set of lexicographically ordered monomials

$$J_{i_1}^{(0)}\,J_{i_2}^{(0)} \ldots J_{i_k}^{(0)} \qquad \big(i_1 \le i_2 \le \ldots \le i_k;\; k \in \mathbb{N}\big)\,, \tag{2.109}$$

provide a basis of $U(\mathfrak{g})$ [26]. In this way, we have obtained a concrete realization of the universal enveloping algebra. An interesting property of this algebra is that it can easily be promoted to an Hopf algebra by defining the following (trivial) coproduct [1]

$$\Delta(J_a^{(0)}) = J_a^{(0)} \otimes \mathbb{1} + \mathbb{1} \otimes J_a^{(0)}\,, \tag{2.110}$$

The so-defined coproduct, which maps an element of the algebra into the tensor product $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ satisfies the following conditions

(1) Coassociativity

$$(\Delta \otimes \mathbb{1})\,\Delta(x) = (\mathbb{1} \otimes \Delta)\,\Delta(x)\,, \tag{2.111}$$

(2) Δ is an algebra homomorphism

$$\Delta(x\,y) = \Delta(x)\,\Delta(y)\,. \tag{2.112}$$

[1] In principal one also needs to define a counit and an antipode to obtain a complete Hopf algebra. But since these maps are irrelevant here, we will not specify them explicitly.

Using the second property, it can readily be shown that the coproduct (2.110) is also compatible the Lie algebra structure, i.e.

$$\begin{aligned}\Delta(x\,y-y\,x) &= \Delta(x)\,\Delta(y)-\Delta(y)\,\Delta(x)\\ &= (x\,y-y\,x)\otimes\mathbb{1}+\mathbb{1}\otimes(x\,y-y\,x)\\ &= \Delta([x,y])\end{aligned}\tag{2.113}$$

From a physical point of view, the coproduct specifies how the symmetry algebra acts on a multi-particle state. Taking this perspective, the property of being coassociativ ensures that the action of a Lie algebra element on a multi-particle state is unique. This is most transparent at the level of a three-particle state. Furthermore, since Δ is an algebra homomorphism, the coproduct respects the Lie algebra structure, which guarantees that the multi-particle states carry representations of the symmetry algebra. Having briefly recalled some basic facts about the universal enveloping algebra and their Hopf algebra structure, we can now define the Yangian.

The Yangian algebra $Y(\mathfrak{g})$, as introduced by Drinfeld [27, 28], is the enveloping algebra generated by the level-zero generators $J_a^{(0)}$ and a second set of generators $J_a^{(1)}$, in the adjoint representation of $\mathfrak{g}$ so that

$$\left[J_a^{(0)},J_b^{(0)}\right]=f_{ab}^{c}\,J_c^{(0)}\qquad\qquad \left[J_a^{(0)},J_b^{(1)}\right]=f_{ab}^{c}\,J_c^{(1)}\,.\tag{2.114}$$

The coproduct $\Delta: Y(\mathfrak{g})\to Y(\mathfrak{g})\otimes Y(\mathfrak{g})$ is defined by

$$\begin{aligned}\Delta(J_a^{(0)}) &= J_a^{(0)}\otimes\mathbb{1}+\mathbb{1}\otimes J_a^{(0)}\\ \Delta(J_a^{(1)}) &= J_a^{(1)}\otimes\mathbb{1}+\mathbb{1}\otimes J_a^{(1)}+\frac{\alpha}{2}\,f_a^{bc}\,J_c^{(0)}\otimes J_b^{(0)}\,,\end{aligned}\tag{2.115}$$

where the free parameter $\alpha\in\mathbb{C}$. By requiring that $\Delta(J_a^{(1)})$ be a homomorphism (see [29]) one finds the following two additional constraints

$$\left[J_{(a}^{(1)},\left[J_b^{(1)},J_{c)}^{(0)}\right]\right]=\frac{\alpha^2}{4}\,c_{abc}^{deg}\left\{J_d^{(0)},J_e^{(0)},J_g^{(0)}\right\}\tag{2.116}$$

$$\begin{aligned}&\left[\left[J_a^{(1)},J_b^{(1)}\right],\left[J_c^{(0)},J_d^{(1)}\right]\right]+\left[\left[J_c^{(1)},J_d^{(1)}\right],\left[J_a^{(0)},J_b^{(1)}\right]\right]\\ &\quad=\frac{\alpha^2}{4}\left(c_{abk}^{gef}\,f_{cd}^{k}+c_{cdk}^{gef}\,f_{ab}^{k}\right)\left\{J_g^{(0)},J_e^{(0)},J_f^{(1)}\right\},\end{aligned}\tag{2.117}$$

where (abc) denotes the sum of all cyclic permutations and

$$c_{abc}^{deg}=f_{ai}^{d}\,f_{bj}^{e}\,f_{ck}^{g}\,f^{ijk}\qquad\qquad \{x_1,x_2,x_3\}=\frac{1}{6}\sum_{i\neq j\neq k}x_i\,x_j\,x_k\,.\tag{2.118}$$

These relations, which take values in the enveloping algebra of $J_a^{(0)},J_b^{(1)}$, are often referred to as Serre relations. Since (2.116) implies (2.117) for $\mathfrak{g}\neq\mathfrak{su}(2)$, the latter relation is often not stated explicitly. From now on we will only take into account the first relation (2.116) because we are interested in Lie algebras $\mathfrak{g}\neq\mathfrak{su}(2)$. A natural way to think about the Yangian $Y(\mathfrak{g})$ is as a graded algebra spanned by an infinite set of level generators $J_a^{(0)},J_b^{(1)},J_c^{(2)},\ldots$, with $J_a^{(0)},J_b^{(1)}$ simply being the first two sets at grades zero and one respectively. Given these two sets, the higher grade generators can be obtained by computing commutators of lower grade generators, for example

$$\left[J_a^{(1)}, J_b^{(1)}\right] = f_{ab}^c\, J_c^{(2)} + X_{ab}\,, \tag{2.119}$$

where X_{ab} is an extra term, which is necessary in order to fulfill the Serre relation. To see this, we first note that (2.116) can also be written as

$$\left[J_{(a}^{(1)}, \left[J_b^{(1)}, J_{c)}^{(0)}\right]\right] = f_{(bc}^d \left[J_{a)}^{(1)}, J_d^{(1)}\right] = \mathrm{rhs}(2.116)\,. \tag{2.120}$$

If we now plug in (2.119), we find

$$f_{(bc}^d \Big(f_{a)d}^e\, J_e^{(2)} + X_{a)d}\Big) = f_{(bc}^d\, X_{a)d} = \frac{\alpha^2}{4}\, c_{abc}^{deg} \left\{J_d^{(0)}, J_e^{(0)}, J_g^{(0)}\right\}. \tag{2.121}$$

Since the term including the level-two generator vanishes (due to the Jacobi-identity), we obviously need X_{ab} in order to ensure that (2.116) is satisfied. For this reason, it is convenient to think of the Serre relation as a constraint on the commutators of higher level generators. We will close this section by mentioning an explicit formula for the level-one generators. Let us assume that we are dealing with a specific representation of the level-zero generators $j_a^{(0)}$ over some vector space $\mathbb{V}$. By applying the coproduct (2.110) n-times one obtains a representation that acts on the tensor product of vector spaces $\mathbb{V}_n$ by

$$J_a^{(0)} = \sum_i j_{ia}^{(0)}\,. \tag{2.122}$$

Based on such a representation, one can write down the following formula for the additional level-one generators [30]

$$J_a^{(1)} = f_a^{cb} \sum_{i<j} j_{ib}^{(0)}\, j_{jc}^{(0)} + \sum_k c_k\, j_{ka}^{(0)}\,. \tag{2.123}$$

The bi-local part is clearly related to the non-trivial coproduct (2.115). For many algebras and representations the formula (2.123) yields valid level-one generators in the sense that the axioms (2.114) and (2.116) are satisfied. Indeed, (2.123) is the form in which the level-one generators appear in the context of spin chains and scattering amplitudes in $\mathcal{N}=4$ SYM theory.

2.5. A Glimpse on Scattering Amplitudes

The aim of this section is to complete the discussion of the Yangian algebra by taking a look at scattering amplitudes as a particular example for an important class of physical observables that possess Yangian symmetries. Moreover, it provides the necessary background information needed to formulate the relation between scattering amplitudes and Wilson loops. We will start our discussion of the symmetry structures of tree-level scattering amplitudes by briefly reviewing the general formalism used in this business. In case that the reader has not already been exposed to this topic, it might be useful to consult [31] for a more complete introduction.

2.5.1. General Formalism

Scattering amplitudes in $\mathcal{N}=4$ SYM theory with gauge group $\mathrm{SU}(N)$ are most conveniently written in a special color basis, whose basis elements are given by single traces over products of generators. The technique, which allows one to bring a generic scattering amplitude to this form is called color decomposition. Let us see how it works by considering a generic color factor as it could arises in a tree-level Feynman graph. First, we recall that all fields of $\mathcal{N}=4$ SYM theory are in the adjoint of the gauge group. Hence, the color factors are given by products of structure constants. By employing the identity

$$f^{abc} = -2\,i\ \mathrm{Tr}\big(T^a\,[T^b, T^c]\big)\,, \tag{2.124}$$

we obtain a product of traces over generators in the fundamental representation. In the next step one uses the completeness relation for $\mathrm{SU}(N)$ generators

$$T^a_{ij}\,T^a_{kl} = \frac{1}{2}\,\delta_{il}\,\delta_{jk} - \frac{1}{2N}\,\delta_{ij}\,\delta_{kl}\,, \tag{2.125}$$

in order to successively rewrite products of two traces, where additionally two adjoint indices are contracted, as single traces. For example, we can write

$$\big(T^a_{jm}\,T^b_{mi}\,T^c_{ij}\big)\big(T^c_{kl}\,T^d_{ln}\,T^e_{nk}\big) = \frac{1}{2}\,\big(T^a_{jm}\,T^b_{mi}\,T^d_{in}\,T^e_{nj}\big) + \,\dots\,, \tag{2.126}$$

where the dots represent the term arising from the piece of (2.125) that carries an additional factor of $1/2N$. Those contributions are subleading, if the number of colors N is large, so one can neglect them in the planar limit. Using these two tricks, one can bring each tree-level scattering amplitude to a form, where the color degrees of freedom are spanned by single traces over products of generators. At this point, it should be mentioned that at loop-level multiple-trace structures will appear as well. But these are also subleading, so it remains true that in the planar limit a generic n-point scattering amplitude can be decomposed as follows

$$\mathcal{A}_n\big(\{p_i, h_i, a_i\}\big) = \delta^{(4)}\big(\sum_{i=1}^{n} p_i\big) \sum_{\sigma\in S_n/Z_n} 2^{n/2}\,g^{n-2}\,\mathrm{Tr}\big(T^{a_{\sigma(1)}}\dots T^{a_{\sigma(n)}}\big)\,A_n\big(\sigma(1^{h_1},\dots,n^{h_n})\big)\,, \tag{2.127}$$

where each scattering particle (gluon, gluino or scalar) carries on-shell momentum p_i ($p_i^2=0$) and helicity $h_i \in \{-1,-1/2,0,+1/2,+1\}$, which we have written more compactly as $\{p_i,h_i\} := i^{h_i}$. All particles are further treated as outgoing and the permutation sum is over $S_n/Z_n \cong S_{n-1}$, i.e. the set of all non-cyclic permutations of n elements. The coefficients $A_n\big(\sigma(1^{h_1},\dots,n^{h_n})\big)$ are called partial or color-ordered amplitudes. They only depend on the momenta and helicities of the involved particles and admit a perturbative expansion in the parameter $a = \lambda/8\pi^2$. In the following we will only focus on these partial amplitudes.

Color-ordered amplitudes are in general functions of light-like four-momenta and external polarization vectors/spinors. However, the spinor helicity formalism provides a much more efficient description of these degrees of freedom. Let us introduce this formalism by focusing on the momentum of a scattering particle. First, we note that the on-shell condition $p_i^2 = 0$ is automatically solved if we write $p_i^{\alpha\dot\alpha}$ as the product of two commuting Weyl spinors, i.e.

$$p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \,\tilde\lambda_i^{\dot\alpha} \qquad\qquad \tilde\lambda_i^{\dot\alpha} = \pm(\lambda_i^\alpha)^* \,. \tag{2.128}$$

The sign in the conjugation relation is determined by the sign of the energy component of the associated four-momentum. Note that in this section (and only in this section) we will use other conventions for the sigma matrices as well as for raising and lowering Weyl indices. More specifically, we will employ the conventions used in most of the papers on the subject. They read

$$\lambda^\alpha = \tilde\varepsilon^{\alpha\beta}\,\lambda_\beta \qquad \lambda_\alpha = \tilde\varepsilon_{\alpha\beta}\,\lambda^\beta \qquad \tilde\lambda_{\dot\alpha} = \tilde\varepsilon_{\dot\alpha\dot\beta}\,\tilde\lambda^{\dot\beta} \qquad \tilde\lambda^{\dot\alpha} = \tilde\varepsilon^{\dot\alpha\dot\beta}\,\tilde\lambda_{\dot\beta}\,, \tag{2.129}$$

with

$$\tilde\varepsilon_{12} = \tilde\varepsilon_{\dot1\dot2} = -\tilde\varepsilon^{12} = -\tilde\varepsilon^{\dot1\dot2} = 1 \qquad \tilde\varepsilon_{\alpha\beta}\,\tilde\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma \qquad \tilde\varepsilon_{\dot\alpha\dot\beta}\,\tilde\varepsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}^{\dot\gamma}\,. \tag{2.130}$$

The sigma matrices are defined as follows

$$\sigma^\mu_{\alpha\dot\alpha} = (\mathbb{1},\vec\sigma) \qquad\qquad \bar\sigma^{\mu\dot\alpha\alpha} = \tilde\varepsilon^{\dot\alpha\dot\beta}\,\tilde\varepsilon^{\alpha\beta}\,\sigma^\mu_{\beta\dot\beta}\,. \tag{2.131}$$

It is convenient to introduce the following shorthand notation for the $\mathrm{SL}(2,\mathbb{C})$ invariant bilinear forms

$$\langle\lambda_i\,\lambda_j\rangle = \langle ij\rangle := \lambda_i^\alpha\,\lambda_{j\alpha} \qquad\qquad [\lambda_i\,\lambda_j] = [ij] := \tilde\lambda_{i\dot\alpha}\,\tilde\lambda_j^{\dot\alpha}\,. \tag{2.132}$$

Given these products, it becomes obvious that the mass-shell condition is solved by the ansatz (2.128). We further note that for a given four-momentum p^μ the spinors λ_i^α and $\tilde\lambda_i^{\dot\alpha}$ are only determined up to a complex phase. This $\mathrm{U}(1)$ phase can be identified with the particle helicity at point i. It is convenient to assign the helicities $-1/2$ and $+1/2$ to λ_i^α and $\tilde\lambda_i^{\dot\alpha}$ respectively. The local helicity generator thus reads

$$h_i = \frac{1}{2}\left[-\lambda_i^\alpha\,\frac{\partial}{\partial\lambda_i^\alpha} + \tilde\lambda_i^{\dot\alpha}\,\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\right]. \tag{2.133}$$

Using the so-defined helicity spinors, the gluon polarization vectors can be expressed as

$$\epsilon_{i,+}^{\alpha\dot\alpha} = -\sqrt2\,\frac{\tilde\lambda_i^{\dot\alpha}\,\mu_i^\alpha}{\langle\lambda_i\,\mu_i\rangle} \qquad\qquad \epsilon_{i,-}^{\alpha\dot\alpha} = \sqrt2\,\frac{\lambda_i^\alpha\,\tilde\mu_i^{\dot\alpha}}{[\lambda_i\,\mu_i]}\,, \tag{2.134}$$

where μ_i^α and $\tilde\mu_i^{\dot\alpha}$ are arbitrary reference spinors which reflect the freedom to perform local gauge transformations. The fermionic polarization spinors are related to the helicity spinors as follows

$$u_{i,+}(p_i) = v_{i,-}(p_i) = \begin{pmatrix}\lambda_{i\alpha}\\ 0\end{pmatrix} \qquad\qquad u_{i,-}(p_i) = v_{i,+}(p_i) = \begin{pmatrix}0\\ \tilde\lambda_i^{\dot\alpha}\end{pmatrix}, \tag{2.135}$$

where $u_{i,\pm}(p_i)$ and $v_{i,\pm}(p_i)$ are the degenerated particle and anti-particle solutions of the massless Dirac equation. The underlying representation of the Dirac algebra is that mentioned in section 2.1.1, except that γ^1, γ^2 and γ^3 carry an additional factor of -1.

Having introduced the notion of color-ordered amplitudes as well as the spinor helicity formalism, we could start talking about scattering amplitudes in $\mathcal{N}=4$ SYM theory. However, before doing this, let us just briefly recall another concept: that of an on-shell superwavefunction. The idea behind this is to assemble all on-shell states into a single superwavefunction, which then provides a uniform and very efficient description of all the different asymptotic scattering states (gluons, gluinos, scalars).The on-shell superwavefunction is conveniently defined as

$$\begin{aligned}\Phi(p,\eta)=&G_+(p)+\eta^A\,\Gamma_A(p)+\frac{1}{2}\,\eta^A\,\eta^B\,S_{AB}(p)+\frac{1}{3!}\,\eta^A\,\eta^B\,\eta^C\,\varepsilon_{ABCD}\,\bar{\Gamma}^D(p)\\&+\frac{1}{4!}\,\eta^A\,\eta^B\,\eta^C\,\eta^D\,\varepsilon_{ABCD}\,G_-(p)\,,\end{aligned}\tag{2.136}$$

where $G_+(p)$ represents a gluon with helicity $+1$, $\Gamma_A(p)$ describes the four fermionic states with helicity $+1/2$, $S_{AB}(p)$ labels the six scalars and the remaining terms describe the gluino/gluon states carrying negative helicities. The Grassmann-valued variables η^A transform in the fundamental representation of the R-symmetry group $\mathrm{SU}(4)$ and it is convenient to assign helicity $+1/2$ to them so that all terms in (2.136) have helicity $+1$. Given these definitions, it is natural to consider color-ordered scattering amplitudes of on-shell superwavefunctions

$$\mathbb{A}_n(\{\lambda_i,\tilde{\lambda}_i,\eta_i\})=\mathbb{A}(\Phi_1,\ldots,\Phi_n)\,,\tag{2.137}$$

where an external leg is now characterized by a point in supermomentum space $\{\lambda_i,\tilde{\lambda}_i,\eta_i\}$. On grounds of the $\mathrm{SU}(4)$ R-symmetry it is clear that the superamplitude can only be a polynomial in powers of η^4. Moreover, due to the Grassmann property of the η-variables, it can at most have degree $4n$. The coefficients of this polynomial are the different component amplitudes involving gluons, gluinos and scalars as external particles. For example, the n-point MHV gluon amplitude with negative helicity gluons sitting at positions i and j will be given by the coefficient of $(\eta_i)^4\ (\eta_j)^4$, where $(\eta_i)^4 := 1/4!\,\varepsilon_{ABCD}\,\eta_i^A\,\eta_i^B\,\eta_i^C\,\eta_i^D$. In fact, the MHV class is the first non-vanishing class of amplitudes, i.e. the η-expansion of the superamplitude (2.137) starts at order η^8. To see this, one has to take into account supersymmetry which, at the level of an n-point superamplitude, is realized by the generators

$$q^{A\alpha}=\sum_{i=1}^{n}\lambda_i^\alpha\,\eta_i^A\qquad\qquad \bar{q}_A^{\dot{\alpha}}=\sum_{i=1}^{n}\tilde{\lambda}_i^{\dot{\alpha}}\,\frac{\partial}{\partial\eta_i^A}\,,\tag{2.138}$$

as can easily be seen by computing their anticommutator

$$\left\{q^{A\alpha},\bar{q}_B^{\dot{\alpha}}\right\}=\delta_B^A\,\sum_{i=1}^{n}\lambda_i^\alpha\,\tilde{\lambda}_i^{\dot{\alpha}}=\delta_B^A\,p^{\alpha\dot{\alpha}}\,.\tag{2.139}$$

In order to make contact to the former section about the Yangian, it is worth mentioning that this n-leg representation can formally be obtained by applying the coproduct (2.110) n-times to the single-leg representation. However, the notation used above, where the identities are suppressed is more common in this context. Now, turning back to the discussion of the minimal degree of the η-polynomial, we first note that the generator $q^{A\alpha}$ acts just multiplicative in our on-shell superspace. For this reasons, the invariance of the superamplitude under this transformation can only be realized by a Grassmann delta function

$$\delta^{(8)}(q) = \delta^{(8)}\Big(\sum_{i=1}^{n} \lambda_i^\alpha\, \eta_i^A\Big) = \prod_{\alpha=1}^{2}\prod_{A=1}^{4}\Big(\sum_{i=1}^{n} \lambda_i^\alpha\, \eta_i^A\Big) \sim \mathcal{O}(\eta^8)\,. \tag{2.140}$$

But then the former statement is immediate. Based on the above discussion, we conclude that a n-point superamplitude has the following general form

$$\mathbb{A}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\}) = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 12\rangle\,\langle 23\rangle\,\ldots\,\langle n1\rangle}\,\mathbb{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})\,, \tag{2.141}$$

where the factor in the denominator is of course pure convention. The function $\mathbb{P}_n(\{\lambda_i, \tilde{\lambda}_i, \eta_i\})$ is a polynomial with terms of degree $(\eta^4)^m$ corresponding to N^{m}MHV amplitudes. While at tree-level this function is finite and, more remarkably, completely known [32], both statements do not hold true at loop-level.

2.5.2. Symmetries of Tree-Level Superamplitudes

At the end of the last subsection we already mentioned that Drummond and Henn [32] derived an explicit formula for all tree-level amplitudes in $\mathcal{N}=4$ SYM theory by solving the supersymmetric recursion relations. From the perspective of symmetries, one would expect that the existence of such a relatively simple result is related to a powerful symmetry, which highly constrains the form of tree-level scattering amplitudes. Indeed, in [11] it was shown that tree-level superamplitudes are invariant under the Yangian algebra $Y(\mathfrak{psu}(2,2|4))$. In what follows, we shall briefly discuss the superconformal and the dual superconformal invariance of tree-level superamplitudes and review how these two algebras combine to the Yangian.

To begin with, we review the superconformal invariance of tree-level superamplitudes. A representation of the superconformal algebra $(\mathfrak{p})\mathfrak{su}(2,2|4)$ that acts on the tensor product of on-shell superspaces is given by

$$j_a = \sum_i j_{ia} \qquad j_{ia} \in \big\{p_i^{\dot\alpha\alpha}, q_i^{A\alpha}, \bar{q}_{iA}^{\dot\alpha}, k_{i\alpha\dot\alpha}, s_{i\alpha A}, \bar{s}_{i\dot\alpha}^{A}, m_{i\alpha\beta}, \bar{m}_{i\dot\alpha\dot\beta}, d_i, c_i, r_{i\;B}^{A}\big\}\,, \tag{2.142}$$

with

$$\begin{gathered}
p_i^{\dot\alpha\alpha} = \tilde\lambda_i^{\dot\alpha}\,\lambda_i^\alpha \qquad q_i^{A\alpha} = \lambda_i^\alpha\,\eta_i^A \qquad \bar{q}_{iA}^{\dot\alpha} = \tilde\lambda_i^{\dot\alpha}\,\partial_{iA} \qquad m_{i\alpha\beta} = \lambda_{i(\alpha}\,\partial_{i\beta)} \\
k_{i\alpha\dot\alpha} = \partial_{i\alpha}\,\partial_{i\dot\alpha} \qquad s_{iA\alpha} = \partial_{i\alpha}\,\partial_{iA} \qquad \bar{s}_{i\dot\alpha}^{A} = \eta_i^A\,\partial_{i\dot\alpha} \qquad \bar{m}_{i\dot\alpha\dot\beta} = \tilde\lambda_{i(\dot\alpha}\,\partial_{i\dot\beta)} \\
d_i = \tfrac{1}{2}\,\lambda_i^\alpha\,\partial_{i\alpha} + \tfrac{1}{2}\,\tilde\lambda_i^{\dot\alpha}\,\partial_{i\dot\alpha} + 1 \qquad r_{i\;B}^{A} = -\eta_i^A\,\partial_{iB} + \tfrac{1}{4}\,\delta_B^A\,\eta_i^C\,\partial_{i\,C} \\
c_i = 1 + \tfrac{1}{2}\,\lambda_i^\alpha\,\partial_{i\alpha} - \tfrac{1}{2}\,\tilde\lambda_i^{\dot\alpha}\,\partial_{i\dot\alpha} - \tfrac{1}{2}\,\eta_i^A\,\partial_{iA}\,,
\end{gathered} \tag{2.143}$$

where we abbreviated $\partial_{i\alpha} = \partial/\partial\lambda_i^\alpha$, $\partial_{i\dot\alpha} = \partial/\partial\tilde\lambda_i^{\dot\alpha}$ and $\partial_{iA} = \partial/\partial\eta_i^A$. By computing commutators and anticommutators

$$
\begin{aligned}
&\left\{q^{A\alpha}, \bar q^{\dot\alpha}_B\right\} = \delta^A_B\, p^{\alpha\dot\alpha} \qquad \left\{s_{A\alpha}, \bar s^B_{\dot\alpha}\right\} = \delta^B_A\, k_{\alpha\dot\alpha} \qquad \left[p^{\alpha\dot\alpha}, s_{A\beta}\right] = -\delta^\alpha_\beta\, \bar q^{\dot\alpha}_A\\
&\left[k_{\alpha\dot\alpha}, q^{A\beta}\right] = \delta^\beta_\alpha\, \bar s^A_{\dot\alpha} \qquad \left[k_{\alpha\dot\alpha}, \bar q^{\dot\beta}_A\right] = \delta^{\dot\beta}_{\dot\alpha}\, s_{A\alpha} \qquad \left[p^{\alpha\dot\alpha}, \bar s^A_{\dot\beta}\right] = -\delta^{\dot\alpha}_{\dot\beta}\, q^{\alpha A}\\
&\left\{q^{A\alpha}, s_{B\beta}\right\} = m^\alpha{}_\beta\, \delta^A_B + \delta^\alpha_\beta\, r^A{}_B + \tfrac{1}{2}\,\delta^\alpha_\beta\, \delta^A_B \left(d + c\right)\\
&\left\{\bar q^{\dot\alpha}_A, \bar s^B_{\dot\beta}\right\} = \bar m^{\dot\alpha}{}_{\dot\beta}\, \delta^B_A - \delta^{\dot\alpha}_{\dot\beta}\, r^B{}_A + \tfrac{1}{2}\,\delta^{\dot\alpha}_{\dot\beta}\, \delta^B_A \left(d - c\right)\\
&\left[k_{a\dot\alpha}, p^{\beta\dot\beta}\right] = \delta^\beta_\alpha\, \delta^{\dot\beta}_{\dot\alpha}\, d + m_\alpha{}^\beta\, \delta^{\dot\beta}_{\dot\alpha} + \bar m_{\dot\alpha}{}^{\dot\beta}\, \delta^\beta_\alpha\,,
\end{aligned}
\tag{2.144}
$$

we find that the so-defined generators make up the algebra $\mathfrak{su}(2,2|4)$ with central charge $c = \sum_i 1 - h_i$. Since our on-shell superwavefunctions have helicity $+1$, the superamplitudes satisfy

$$
h_i\, \mathbb{A}(\Phi_1, \dots, \Phi_n) = \left(-\tfrac{1}{2}\,\lambda_i^\alpha\, \partial_{i\alpha} + \tfrac{1}{2}\,\tilde\lambda_i^{\dot\alpha}\, \partial_{i\dot\alpha} + \tfrac{1}{2}\,\eta_i^A\, \partial_{iA}\right) \mathbb{A}(\Phi_1, \dots, \Phi_n) = \mathbb{A}(\Phi_1, \dots, \Phi_n)\,. \tag{2.145}
$$

Imposing this helicity condition makes the central charge vanish, so that the algebra, which acts on the space of superamplitudes, is really $\mathfrak{psu}(2,2|4)$. Given the representation (2.143), the statement that a n-point tree-level superamplitude is superconformal invariant translates to

$$
j_a\, \mathbb{A}_n^{\text{tree}} = 0\,. \tag{2.146}
$$

In fact, this statement is not completely exact, since it holds only true up to contact terms [33]. But as these terms only appear for some particular momentum configurations, for example, if two adjacent momenta become collinear, one can neglect them at tree-level. Beside the expected superconformal symmetry, superamplitudes possess an additional symmetry called dual superconformal symmetry [10]. To see this, one conveniently introduces the dual variables x_i and θ_i which parametrize a chiral superspace.

$$
x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha} = \lambda_i^\alpha\, \tilde\lambda_i^{\dot\alpha} = p_i^{\alpha\dot\alpha} \qquad\qquad \theta_i^{A\alpha} - \theta_{i+1}^{A\alpha} = \lambda_i^\alpha\, \eta_i^A = q_i^{A\alpha}\,. \tag{2.147}
$$

Using these relations, one can eliminate the variables $\{\tilde\lambda_i, \eta_i\}$ in the superamplitude (2.141) in favor of $\{x_i, \theta_i\}$.

$$
\mathbb{A}_n(\{\lambda_i, x_i, \theta_i\}) = \frac{\delta^{(4)}(x_1 - x_{n+1})\, \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12\rangle\, \langle 23\rangle\, \dots\, \langle n1\rangle}\, \mathbb{P}_n(\{x_i, \theta_i\}) \tag{2.148}
$$

We can think of the relations (2.147) as defining a surface in the full superspace $\{\lambda_i, \tilde\lambda_i, x_i, \theta_i, \eta_i\}$. The amplitudes can then be interpreted as functions on this surface. In order to discuss the dual superconformal symmetry of amplitudes infinitesimally one needs to deduce the action of the generators of dual superconformal transformations in the full superspace $\{x_i, \theta_i, \lambda_i, \tilde\lambda_i, \eta_i\}$. This can be done as follows. Starting with the canonical representation of $\mathfrak{psu}(2,2|4)$ on the chiral superspace $\{x_i, \theta_i\}$ (see (2.149) and neglect terms which contain λ_i, $\tilde\lambda_i$ or η_i), one can find the representation on the full superspace by extending the canonical generators in such a way that they commute with the constraints (2.147) modulo constraints. This prescription leads to the following representation of the $\mathfrak{su}(2,2|4)$ algebra

$$
\begin{aligned}
P_{i\alpha\dot\alpha} &= \partial_{i\alpha\dot\alpha} \qquad\qquad Q_{i\alpha A} = \partial_{i\alpha A} \qquad\qquad \bar{Q}^A_{i\dot\alpha} = \theta_i^{\alpha A}\,\partial_{i\alpha\dot\alpha} + \eta_i^A\,\partial_{i\dot\alpha}\\
M_{i\alpha\beta} &= x_{i(\alpha}{}^{\dot\alpha}\,\partial_{i\beta)\dot\alpha} + \theta^A_{i(\alpha}\,\partial_{i\beta)A} + \lambda_{i(\alpha}\,\partial_{i\beta)} \qquad \bar{M}_{i\dot\alpha\dot\beta} = x_{i(\dot\alpha}{}^{\alpha}\,\partial_{i\dot\beta)\alpha} + \tilde\lambda_{i(\dot\alpha}\,\partial_{i\dot\beta)}\\
R_i^A{}_B &= \theta_i^{\alpha A}\,\partial_{i\alpha B} + \eta_i^A\,\partial_{iB} - \tfrac{1}{4}\,\delta_B^A\,\theta_i^{\alpha C}\,\partial_{i\alpha C} - \tfrac{1}{4}\,\delta_B^A\,\eta_i^C\,\partial_{iC}\\
D_i &= -x_i^{\alpha\dot\alpha}\,\partial_{i\alpha\dot\alpha} - \tfrac{1}{2}\,\theta_i^{\alpha A}\,\partial_{i\alpha A} - \tfrac{1}{2}\,\lambda_i^\alpha\,\partial_{i\alpha} - \tfrac{1}{2}\,\tilde\lambda_i^{\dot\alpha}\,\partial_{i\dot\alpha}\\
C_i &= -\tfrac{1}{2}\,\lambda_i^\alpha\,\partial_{i\alpha} + \tfrac{1}{2}\,\tilde\lambda_i^{\dot\alpha}\,\partial_{i\dot\alpha} + \tfrac{1}{2}\,\eta_i^A\,\partial_{iA} \qquad \bar{S}_{i\dot\alpha A} = x_{i\dot\alpha}{}^{\beta}\,\partial_{i\beta A} + \tilde\lambda_{i\dot\alpha}\,\partial_{iA}\\
S^A_{i\alpha} &= -\theta^B_{i\alpha}\,\theta_i^{\beta A}\,\partial_{i\beta B} + x_{i\alpha}{}^{\dot\beta}\,\theta_i^{\beta A}\,\partial_{i\beta\dot\beta} + \lambda_{i\alpha}\,\theta_i^{\gamma A}\,\partial_{i\gamma} + x_{i+1\alpha}{}^{\dot\beta}\,\eta_i^A\,\partial_{i\dot\beta} - \theta^B_{i+1\alpha}\,\eta_i^A\,\partial_{iB}\\
K_{i\alpha\dot\alpha} &= x_{i\alpha}{}^{\dot\beta}\,x_{i\dot\alpha}{}^{\beta}\,\partial_{i\beta\dot\beta} + x_{i\dot\alpha}{}^{\beta}\,\theta^B_{i\alpha}\,\partial_{i\beta B} + x_{i\dot\alpha}{}^{\beta}\,\lambda_{i\alpha}\,\partial_{i\beta} + x_{i+1\alpha}{}^{\dot\beta}\,\tilde\lambda_{i\dot\alpha}\,\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\,\theta^B_{i+1\alpha}\,\partial_{iB}\,,
\end{aligned}
\tag{2.149}
$$

where we abbreviated $\partial_{i\alpha\dot\alpha} = \partial/\partial x_i^{\alpha\dot\alpha}$ and $\partial_{i\alpha A} = \partial/\partial\theta_i^{\alpha A}$. An n-leg representation or, to be more specific, a representation that acts on the tensor product of superspaces is obtained by summing the generators (2.149) over the number of legs. We will denote these generators collectively by J_a. While most of the J_a's annihilate the amplitude, this does not hold true for $K^{\alpha\dot\alpha}$, $S^{\alpha A}$, D and C. Instead, it can be shown that superamplitudes transform covariantly under these transformations, i.e.

$$
K^{\alpha\dot\alpha}\,\mathbb{A}_n^{\text{tree}} = -\sum_i x_i^{\alpha\dot\alpha}\,\mathbb{A}_n^{\text{tree}} \qquad S^{\alpha A}\,\mathbb{A}_n^{\text{tree}} = -\sum_i \theta_i^{\alpha A}\,\mathbb{A}_n^{\text{tree}} \qquad (D,C)\,\mathbb{A}_n^{\text{tree}} = n\,\mathbb{A}_n^{\text{tree}}\,. \tag{2.150}
$$

By slightly redefining these four generators

$$
K'^{\alpha\dot\alpha} = K^{\alpha\dot\alpha} + \sum_i x_i^{\alpha\dot\alpha} \qquad S'^{\alpha A} = S^{\alpha A} + \sum_i \theta_i^{\alpha A} \qquad (D',C') = (D,C) - n\,, \tag{2.151}
$$

one obtains a set of generators J'_a, that satisfy the algebra relations of $\mathfrak{psu}(2,2|4)$ and annihilate all tree-level amplitudes

$$
J'_a\,\mathbb{A}_n^{\text{tree}} = 0 \qquad\qquad J'_a \in \big\{P_{\alpha\dot\alpha}, Q_{\alpha A}, \bar{Q}^A_{\dot\alpha}, K'^{\alpha\dot\alpha}, S'^A_\alpha, \bar{S}^{\dot\alpha}_A, M_{\alpha\beta}, \bar{M}_{\dot\alpha\dot\beta}, D', R^A{}_B\big\}\,. \tag{2.152}
$$

Hence, tree-level superamplitudes are invariant under dual superconformal transformations. Having reviewed the superconformal and the dual superconformal symmetry of tree-level superamplitudes, let us now turn to the discussion of how these two algebras combine to the Yangian of $\mathfrak{psu}(2,2|4)$. In order to treat both algebras on the same footing, it is useful to restrict the dual superconformal generators such that they act only on the on-shell superspace coordinatized by $\{\lambda_i, \tilde\lambda_i, \eta_i\}$. Doing this, one finds that the generators $P_{\alpha\dot\alpha}$ and $Q_{\alpha A}$ become trivial, while the generators $\{\bar{Q}^A_{\dot\alpha}, M_{\alpha\beta}, \bar{M}_{\dot\alpha\dot\beta}, D', R^A{}_B, \bar{S}^{\dot\alpha}_A\}$ coincide up to signs with the superconformal ones as defined above (2.144). The only non-trivial generators which do not belong to the Lie superalgebra spanned by the j_a, after having restricted them to on-shell superspace, are S'^A_α and $K'^{\alpha\dot\alpha}$. A natural question that arises is which algebraic structure is generated by the j_a and S'^A_α, $K'^{\alpha\dot\alpha}$. The answer was given by Drummond, Henn and Plefka in [11] who showed that the resulting algebra is the Yangian $Y(\mathfrak{psu}(2,2|4))$. Let us discuss this a bit more in depth. The level-zero is given by the Lie superalgebra itself, which can be represented on the tensor product of on-shell superspaces as follows

$$
j_a^{(0)} = \sum_i j_{ia}^{(0)} \qquad\qquad j_{ia}^{(0)} \in \big\{(2.143)\big\}\,. \tag{2.153}
$$

We have already mentioned in section 2.4.3 that given such a representation, the additional level-one generators take the following form

$$j_a^{(1)} = f_a^{cb} \sum_{i<j} j_{ib}^{(0)} \, j_{jc}^{(0)} + \sum_k c_k \, j_{ka}^{(0)} \,. \tag{2.154}$$

Now, the key point in order to prove that the j_a and S'^A_α, $K'^{\alpha\dot\alpha}$ generate the Yangian of $\mathfrak{psu}(2,2|4)$ is to show that the latter two generators can be brought to the above mentioned standard form (2.154). Indeed, it was demonstrated in [11] that by adding terms, which annihilate the amplitudes on their own, S'^A_α and $K'^{\alpha\dot\alpha}$ can be manipulated such that

$$\begin{aligned} S'^A_\alpha \quad &\rightarrow \quad q^{(1)A}{}_\alpha = \sum_{i>j} \Big[m_{i\alpha}^{\gamma} \, q_{j\gamma}^{A} - \tfrac{1}{2} \, (d_i + c_i) \, q_{j\alpha}^{A} + p_{i\alpha}^{\dot\beta} \, \bar{s}_{j\dot\beta}^{A} + q_{i\alpha}^{B} \, r_{j}{}^{A}{}_{B} - (i \leftrightarrow j) \Big] \\ K'_{\alpha\dot\alpha} \quad &\rightarrow \quad p_{\alpha\dot\alpha}^{(1)} = \sum_{i>j} \Big[\Big(m_{i\alpha}^{\gamma} \, \delta_{\dot\alpha}^{\dot\gamma} + \bar{m}_{i\dot\alpha}^{\dot\gamma} \, \delta_\alpha^\gamma - d_i \, \delta_{\dot\alpha}^{\dot\gamma} \, \delta_\alpha^\gamma \Big) \, p_{j\gamma\dot\gamma} + \bar{q}_{i\dot\alpha C} \, q_{j\alpha}^{C} - (i \leftrightarrow j) \Big] \,. \end{aligned} \tag{2.155}$$

Thus, the level-one generators are completely bi-local expressions, i.e. all c_k are zero. Furthermore, it has been argued in [11] that the Serre relations are indeed satisfied, so that the resulting algebra is really $Y(\mathfrak{psu}(2,2|4))$. Summarizing, we have reviewed how the superconformal and the dual superconformal symmetry algebras combine to a much larger Yangian algebra that is respected by tree-level superamplitudes in the sense that

$$y \, \mathbb{A}_n^{\text{tree}} = 0 \qquad\qquad y \in Y(\mathfrak{psu}(2,2|4)) \,. \tag{2.156}$$

3. Maldacena-Wilson Loops in $\mathcal{N} = 4$ SYM Theory

Having presented the necessary background information on $\mathcal{N} = 4$ SYM theory, we will now turn to the central object of this thesis, the Maldacena-Wilson loop. We begin with a short review of the definition of Wilson loops in non-abelian gauge theories and continue by commenting briefly on the connection between a rectangular Wilson loop and the potential between two static charges. In what follows we will then introduce the Maldacena-Wilson loop operator, discuss some of its properties and address the relation between scattering amplitudes and Wilson loops in $\mathcal{N} = 4$ SYM theory. In preparation for the discussion of hidden non-local symmetries we will present a technical introduction to conformal symmetry, construct a representation of the conformal generators that acts on the space of curves and explicitly show that the one-loop expectation value of a smooth Maldacena-Wilson loop is annihilated by these generators. Finally, we consider the conformal algebra as the level-zero algebra of the Yangian, construct the level-one momentum generator and investigate whether it annihilates the one-loop expectation value.

3.1. The Wilson Loop Operator in Yang-Mills Theories

Wilson loops are the most general gauge invariants in Yang-Mills theories [21] and therefore play an important role in studying the general structure of gauge theories. Furthermore, they are of fundamental importance in lattice gauge theories, where they can be used to study the non-perturbative phenomenon of confinement, for example. In this section we will first construct the Wilson loop operator and continue by sketching how the static quark-antiquark potential can be derived by considering a special loop operator. This introductory part is mainly based on chapter 82 of [34] and [35].

Let us start by defining the infinitesimal Wilson link as follows

$$W_{\text{link}}(x+\varepsilon, x) := \exp\Big(i\,\varepsilon^\mu A_\mu(x)\Big)\,, \tag{3.1}$$

where $A_\mu(x)$ is a matrix-valued gauge field defined by

$$A_\mu(x) = A_\mu^a(x)\,T^a\,, \tag{3.2}$$

and T^a are $\mathrm{SU}(N)$ generators in the fundamental representation. Since the Wilson link is defined for two infinitesimally separated spacetime points, we can also expand the exponential and consider terms up to first order in ε

$$W_{\text{link}}(x+\varepsilon, x) = 1 + i\,\varepsilon^\mu A_\mu(x) + \mathcal{O}(\varepsilon^2)\,. \tag{3.3}$$

The next step is to investigate how the Wilson link transforms under a gauge transformation. If we use that the gauge field $A_\mu(x)$ transforms as

$$A_\mu(x) \quad \to \quad U(x)\Big(A_\mu(x) + i\,\partial_\mu\Big)U^\dagger(x)\,, \tag{3.4}$$

we find

$$W_{\text{link}}(x+\varepsilon,x) \quad \to \quad 1 + i\,\varepsilon^\mu\,U(x)\,A_\mu(x)\,U^\dagger(x) - \varepsilon^\mu\,U(x)\,\partial_\mu\,U^\dagger(x) + \mathcal{O}(\varepsilon^2)\,. \tag{3.5}$$

Using

$$U(x)\,U^\dagger(x) = 1 \qquad \xrightarrow{\partial_\mu} \qquad -U(x)\,\partial_\mu\,U^\dagger(x) = (\partial_\mu\,U(x))\,U^\dagger(x)\,, \tag{3.6}$$

equation (3.5) can be written as

$$W_{\text{link}}(x+\varepsilon,x) \quad \to \quad ((1+\varepsilon^\mu\,\partial_\mu)\,U(x))\,U^\dagger(x) + i\,\varepsilon^\mu\,U(x)\,A_\mu(x)\,U^\dagger(x) + \mathcal{O}(\varepsilon^2)\,. \tag{3.7}$$

At the cost of $\mathcal{O}(\varepsilon^2)$-terms this can be cast into the following form

$$W_{\text{link}}(x+\varepsilon,x) \quad \to \quad ((1+\varepsilon^\nu\,\partial_\nu)U(x))\Big(1 + i\,\varepsilon^\mu\,A_\mu(x)\Big)U^\dagger(x) + \mathcal{O}(\varepsilon^2)\,. \tag{3.8}$$

Form this expression we conclude that under a gauge transformation the Wilson link transforms as

$$W_{\text{link}}(x+\varepsilon,x) \quad \to \quad U(x+\varepsilon)\,W_{\text{link}}(x+\varepsilon,x)\,U^\dagger(x)\,. \tag{3.9}$$

We will now define the finite Wilson line as the product of infinitesimal Wilson links. Let ε_j be a sequence of infinitesimal displacement vectors that approximate the line Γ_{yx} starting at point x and ending at point y.

$$W_{\text{line}}(\Gamma_{yx}) := \lim_{n\to\infty} W_{\text{link}}(y, y-\varepsilon_n)\,\ldots\,W_{\text{link}}(x+\varepsilon_1+\varepsilon_2, x+\varepsilon_1)\,W_{\text{link}}(x+\varepsilon_1,x) \tag{3.10}$$

In a continuous limit the Wilson line operator is then formally given by

$$W_{\text{line}}(\Gamma_{yx}) = \mathcal{P}\,\exp\left(i\int_{\Gamma_{yx}} \mathrm{d}s\;A_\mu(x)\,\dot{x}^\mu\right), \tag{3.11}$$

where $x^\mu(s)$ parametrizes the line Γ_{yx} and $\mathcal{P}$ denotes path-ordering which is defined as follows

$$\mathcal{P}\left\{A_\mu(x(s_1))\,A_\nu(x(s_2))\right\} = \begin{cases} A_\nu(x(s_2))\,A_\mu(x(s_1)) & \text{for} \quad s_1 < s_2 \\ A_\mu(x(s_1))\,A_\nu(x(s_2)) & \text{for} \quad s_1 > s_2\,. \end{cases} \tag{3.12}$$

Under a gauge transformation the Wilson line transforms as

$$W_{\text{line}}(\Gamma_{yx}) \quad \to \quad U(y)\,W_{\text{line}}(\Gamma_{yx})\,U^\dagger(x) \tag{3.13}$$

which can easily be shown by using the definition (3.10), equation (3.9) and the fact that $U(x)$ is unitary, i.e. $U^\dagger(x) = U^{-1}(x)$. Note that under a gauge transformation the objects

$$W_{\text{line}}(\Gamma_{yx})\,\Psi(x) \quad \overset{g.t.}{\sim} \quad \Psi(y) \tag{3.14}$$

transform similarly, where $\Psi(x)$ is a field in the fundamental representation. The Wilson line operator therefore plays the role of a parallel transporter.

We are now in the position to construct the gauge invariant Wilson loop operator. Based on the definition of the Wilson line operator (3.11) we define the Wilson loop operator as follows

$$W_{\text{loop}}(C) := \frac{1}{N}\,\mathrm{Tr}\,W_{\text{line}}(C) = \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\exp\left(i\oint_C \mathrm{d}s\,A_\mu(x)\,\dot{x}^\mu\right), \tag{3.15}$$

where C is now a closed, oriented path in spacetime. Since the trace is cyclic, the so-defined loop operator is gauge invariant. The factor of $1/N$ ensures that the zeroth order term in a perturbative expansion is one.

3.1.1. The Rectangular Wilson Loop

Within the huge class of Wilson loops the rectangular Wilson loop as depicted in 3.1 is the most prominent one.

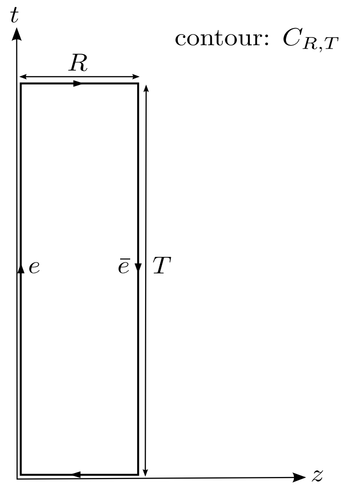

Figure 3.1.: A rectangular Wilson loop of size $R\times T$ $(T \gg R)$ which lies in the $t-z-$plane.

Its physical significance lies in the fact that the expectation value of this loop operator allows one to extract the potential between two static charges, for example a quark and an antiquark. In what follows we will sketch how the relation between the Wilson loop and the potential of two charges arises. Since we want to consider static charges, we can think of them as being infinitely heavy, so that no dynamics will be present. It is therefore sufficient to study the problem in a pure gauge theory. To keep things simple, we will choose the gauge group to be $\mathrm{U}(1)$. It is convenient in this context to work in a Wick rotated framework where the spacetime is Euclidean and the metric tensor is $\delta_{\mu\nu}$. The action then reads

$$S_E = \int \mathrm{d}^4x\,\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \qquad\qquad F_{\mu\nu} = \partial_\mu\,A_\nu - \partial_\nu\,A_\mu\,. \tag{3.16}$$

It is well-known that for large Euclidean time T, the path integral is proportional to the exponential of the ground state energy times T, i.e.

$$Z[0] = \int \mathcal{D}A\; e^{-S_E[A]} \;\overset{T\to\infty}{\sim}\; e^{-E_0\,T}\,. \tag{3.17}$$

If we want to consider a field configuration which screens a pair of two oppositely charged point particles, existing for a time T and separated by a distance R, we can do this by simply adding the following source term to the path integral

$$Z[J] = \int \mathcal{D}A \; e^{-S_E[A] + \int \mathrm{d}^4x \, J_\mu \, A^\mu} , \tag{3.18}$$

with

$$J_\mu = i\, e\, \delta^{(3)}(\vec{x} - \vec{0})\, \delta^0_\mu - i\, e\, \delta^{(3)}(\vec{x} - \vec{R})\, \delta^0_\mu \, . \tag{3.19}$$

The difference between the ground state energy of J-dependent Hamiltonian and that of the unmodified Hamiltonian, i.e. the potential $V(R)$, is then simply given by

$$V(R) = - \lim_{T\to\infty} \frac{1}{T} \ln\left(\frac{Z[J]}{Z[0]}\right) . \tag{3.20}$$

Let us now focus on the quotient of the above expression. We note that it can equally well be written as

$$\frac{Z[J]}{Z[0]} = \frac{1}{Z[0]} \int \mathcal{D}A \; \exp\left(-S_E[A] + i\, e \int \mathrm{d}t \; A^\mu(x^l)\, \dot{x}^l_\mu + i\, e \int \mathrm{d}t \; A^\mu(x^r)\, \dot{x}^r_\mu\right) , \tag{3.21}$$

where

$$x^l_\mu = (t, 0, 0, 0) \qquad\qquad x^r_\mu = (-t, 0, 0, R) \, . \tag{3.22}$$

If T is very large, we can close the contour without changing the potential $V(R)$. The quotient therefore equals the expectation value of the rectangular Wilson loop as depicted in 3.1.

$$\frac{Z[J]}{Z[0]} = \langle W_{\text{loop}}(C_{R,T})\rangle \qquad \text{with} \qquad W_{\text{loop}}(C_{R,T}) = \exp\left(i\, e \oint_{C_{R,T}} \mathrm{d}t \; A_\mu(x)\, \dot{x}^\mu\right) \tag{3.23}$$

The potential can then be found by computing

$$V(R) = - \lim_{T\to\infty} \frac{1}{T} \ln\Big(\langle W_{\text{loop}}(C_{R,T})\rangle\Big) \, . \tag{3.24}$$

Despite the fact that we chose the gauge group to be $\mathrm{U}(1)$, the last statement also holds true for other gauge groups [36]. Since perturbation theory can only be applied at weak coupling, these types of calculations are often carried out in lattice gauge theory. There, the expectation value of Wilson loops can be computed at weak coupling as well as at strong coupling, making it an appropriate framework for studying the non-perturbative phenomenon of confinement. In order to decide, whether a gauge theory is confining or not for a certain value of the coupling constant, it is convenient to study the behavior of the exponent of the rectangular Wilson loop while the loop size is scaled up to infinity [36].

$$\begin{aligned} &\lim_{R,T\to\infty} \langle W(C_{R,T})\rangle \sim e^{-\kappa\, P} && \text{Coulomb phase} \\ &\lim_{R,T\to\infty} \langle W(C_{R,T})\rangle \sim e^{-\sigma\, A} && \text{confinement phase}\, , \end{aligned} \tag{3.25}$$

where $P = 2(R+T)$ is the perimeter and $A = RT$ is the area of the rectangle. In the confining phase the exponent of the Wilson loop scales with the area of the rectangle. The potential at large distances is therefore given by

$$V(R) \approx \sigma\, R \, , \tag{3.26}$$

which shows that in this case it takes an infinite amount of energy to separate the two charges by an infinite distance.

3.2. The Maldacena-Wilson Loop Operator

In the last section the ordinary Wilson loop operator, as it is usually defined in non-abelian gauge theories, was introduced. However, there is a more natural object in $\mathcal{N}=4$ SYM theory called the Maldacena-Wilson loop. The operator was originally proposed by Juan Maldacena [17] and is given by

$$W(C) = \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp\left(i \oint_C \mathrm{d}s \, \left(A_\mu(x)\, \dot{x}^\mu + \Phi_i(x)\, n^i \, |\dot{x}| \right) \right), \tag{3.27}$$

where $x^\mu(s) : [a,b] \to \mathbb{R}^{1,3}$ parametrizes the integration contour C and n^i is a constant unit six-vector $(n^i \, n_i = -1)$ which specifies a point on S^5. Note that throughout this thesis the modulus $|\dot{x}|$ is defined as

$$|\dot{x}| = \begin{cases} \|\dot{x}\| & \text{if } \dot{x}^2 \geq 0 \\ i\,\|\dot{x}\| & \text{if } \dot{x}^2 < 0 \end{cases} \qquad \text{with} \quad \|\dot{x}\| = \sqrt{|\dot{x}^2|}\,, \tag{3.28}$$

and is therefore imaginary in the case that the tangent vector along the curve is space-like. In contrast to the ordinary Wilson loop the Maldacena-Wilson loop not only couples to the gauge field of the theory but also to the six adjoint scalars. The origin of these additional couplings can be understood by considering an ordinary Wilson loop in ten-dimensional $\mathcal{N}=1$ SYM theory and performing a dimensional reduction down to four spacetime dimensions. Using the notation of section 2.2, we find

$$\begin{aligned} W(C) &= \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp\left(i \oint_C \mathrm{d}s \, A_M(\mathrm{z})\, \dot{\mathrm{z}}^M \right) \\ &= \frac{1}{N} \operatorname{Tr} \mathcal{P} \exp\left(i \oint_C \mathrm{d}s \, \left(A_\mu(x)\, \dot{x}^\mu + \Phi_i(x)\, \dot{y}^i \right) \right), \end{aligned} \tag{3.29}$$

where $x^\mu(s)$ parametrizes the actual loop in four-dimensional Minkowski space and $y_i(s)$ determines the loop in the six extra dimensions which have been compactified. A loop operator of the form (3.29) belongs to the class of Maldacena-Wilson loops if the contour satisfies the following additional constraint

$$\dot{x}_\mu \, \dot{x}^\mu + \dot{y}_i \, \dot{y}^i = 0\,. \tag{3.30}$$

From the ten-dimensional point of view, (3.30) is of course just a light-likeness condition. However, it should be mentioned that this light-likeness condition is crucial, since it is related to nearly all the nice properties that Maldacena-Wilson loop operators have. One way to ensure that (3.30) is satisfied is to choose $\dot{y}_i$ as follows

$$\dot{y}_i = n_i \, |\dot{x}| \qquad \text{with} \quad n_i = \text{const.}. \tag{3.31}$$

In principal the unit six-vector n_i could of course depend on s, but as is done in most papers on the topic, it is assumed to be constant throughout this work. At this point we note that our definition of the modulus (3.28) is just a trick to satisfy the light-likeness condition (3.30), even if the tangent vector $\dot{x}^\mu(s)$ along the loop is space-like. Having motivated the concrete form of the Maldacena-Wilson loop operator, let us now review some of its basic properties. An important one is that in contrast to the ordinary Wilson loop the Maldacena-Wilson loop (3.27) is locally supersymmetric. To see this, we compute how the operator transforms under an infinitesimal supersymmetry transformation (2.62)

$$\delta_\varepsilon W(C) = \Big[(-1) \oint_C \mathrm{d}s\, \bar{\Psi}(x)\left(\Gamma_\mu \dot{x}^\mu + \Gamma_i n^i |\dot{x}|\right)\xi\Big] \,, \tag{3.32}$$

where ξ is the parameter of the transformation and a constant ten-dimensional Majorana-Weyl spinor. The square brackets in the above expression denote the path-ordered expectation value in the presence of a Maldacena-Wilson loop operator, i.e.

$$\Big[\mathcal{O}(x)\Big] := \frac{1}{N} \operatorname{Tr} \mathcal{P} \left\{ \exp\left(i \oint_C \mathrm{d}s\, \left(A_\mu(x)\,\dot{x}^\mu + \Phi_i(x)\, n^i |\dot{x}|\right)\right) \mathcal{O}(x) \right\} . \tag{3.33}$$

The number of conserved supercharges is determined by the number of linearly independent solutions to the following equation

$$A\,\xi = 0 \qquad\qquad \text{with} \quad A := \left(\Gamma_\mu \dot{x}^\mu + \Gamma_i n^i |\dot{x}|\right) . \tag{3.34}$$

In appendix A.3 we show that, at least for time-like curves $x^\mu(s)$ or, stated differently, for curves which satisfy the light-likeness condition (3.30) without employing our trick, equation (3.34) has eight linearly independent Majorana-Weyl solutions for any given s. This fact is often phrased as: "The Maldacena-Wilson loop is locally 1/2 BPS". Local supersymmetry however, is not a symmetry of the action of $\mathcal{N}=4$ SYM theory. Global supersymmetry on the other side requires the eigenvectors to be independent of s. In the case that the underlying four-dimensional space is Euclidean, it was argued in [37] that the eigenvalue equation (3.34) has no global solutions unless the contour C is a straight line. Even though the argument presented there does not completely go through in Minkowski space, it seems very likely that straight lines are in fact the only geometries for which (3.34) has eight linearly independent, constant solutions. At this point it should however be mentioned that in the above discussion we only took into account the 16 Poincaré supercharges. If one incorporates the 16 conformal supercharges, there is at least one more geometry which is globally 1/2 BPS and that is the circle. The situation also changes, if one allows for an s-dependence in the S^5 vector n_i. In this case there exist various loop operators which globally preserve some fraction of the supersymmetry, see [38] for example. On the perturbative level, the supersymmetry of the loop operators seems to be related to a certain simplicity of their expectation value. For example, the expectation value of a Maldacena-Wilson loop operator that depends on a straight line of infinite length does not receive any radiative corrections and is therefore given by

$$\langle W(\text{straight line})\rangle = 1\,. \tag{3.35}$$

Another important property that makes the Maldacena-Wilson loop operator (3.27) a very interesting object is that in the strong coupling limit its expectation value can be calculated by employing the dual string theory description of $\mathcal{N}=4$ SYM theory. More specifically, the expectation value can be computed by minimizing the area of the string world sheet that extends into the bulk of the AdS space and ends on the contour C on the conformal boundary of AdS. In this thesis however, we will only deal with the gauge theory side.

3.2.1. The MWL in Perturbation Theory

In the weak coupling limit the expectation value of the Maldacena-Wilson loop operator (3.27) can be computed by using perturbation theory. Expanding the exponential leads to

$$\begin{aligned}\langle W(C)\rangle = 1 - \frac{\mathrm{Tr}(T^aT^b)}{2N}\int \mathrm{d}s_1\,\mathrm{d}s_2\,\Big(&\langle A^a_\mu(x_1)\,A^b_\nu(x_2)\rangle\,\dot{x}_1^\mu\,\dot{x}_2^\nu \\ &+ |\dot{x}_1||\dot{x}_2|\,n^i\,n^j\,\langle\Phi^a_i(x_1)\,\Phi^b_j(x_2)\rangle\Big) + \ldots\,,\end{aligned} \tag{3.36}$$

where x_1^μ is short for $x^\mu(s_1)$. We note that path-ordering can be neglected at one-loop order, due to the cyclicity of the trace. If we plug in the expressions for the scalar and the gluon propagator (2.88) and use that the $\mathrm{SU}(N)$ generators are normalized according to (2.58), the expectation value (3.36) simplifies to

$$\langle W(C)\rangle = 1 - \frac{\lambda}{16\pi^2}\int \mathrm{d}s_1\,\mathrm{d}s_2\,\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{(x_1-x_2)^2} + \mathcal{O}(\lambda^2)\,, \tag{3.37}$$

where we have additionally used that

$$\frac{g^2\delta^{aa}}{16\pi^2 N} = \frac{g^2(N^2-1)}{16\pi^2 N} \xrightarrow{\text{large } N} \frac{g^2 N}{16\pi^2}\,. \tag{3.38}$$

For later convenience we introduce the following shorthand notation

$$\langle W(C)\rangle_{(1)} = -\frac{\lambda}{16\pi^2}\int \mathrm{d}s_1\,\mathrm{d}s_2\,I_{12} \qquad\qquad I_{12} = \frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{x_{12}^2}\,, \tag{3.39}$$

where we abbreviated $x_{12}^\mu = x_1^\mu - x_2^\mu$. The expectation value of the Maldacena-Wilson loop operator (3.37) is obviously a functional, which maps a curve C into a number. Since we want to investigate the symmetries of this functional, it is crucial to know whether the expectation value is finite or not. If it were not finite, one would have to introduce a regulator, which could potentially break the symmetry. In the case that the spacetime is Euclidean and the contour C is a smooth, non-intersecting loop, the Maldacena-Wilson loop is believed to be finite to all orders in perturbation theory [39]. However, if we consider curves in Minkowski space things become more subtle, as the inner product is no longer positive definite. This causes in particular that the tangent vector along the loop can get light-like and, more importantly, that the denominator of (3.37) not only gets zero if x_1 and x_2 coincide, but also if the two points are light-like separated. To avoid the necessity to deal with these issues as well as to have the possibility to parametrize the loop contour by its arc length, we will from now on restrict to smooth, non-intersecting, closed, space-like ($\dot{x}^2 < 0,\ \forall s$) curves, which furthermore fulfill the property that every two points along the curve are space-like separated. Despite the fact that for this class of curves everything should in principal work out as in the Euclidean case, let us explicitly demonstrate that the one-loop expectation value $\langle W(C)\rangle_{(1)}$ is finite. To do so, we will assume that the curve under consideration is parametrized by its arc length (i.e. $|\dot{x}| = i$), which in our case can be done without a loss of generality. In this parametrization the first order correction reads

$$\langle W(C)\rangle_{(1)} = -\frac{\lambda}{16\pi^2}\int_0^L \mathrm{d}s_1\,\mathrm{d}s_2\,\frac{\dot{x}_1\cdot\dot{x}_2 + 1}{x_{12}^2}\,. \tag{3.40}$$

We note that since the denominator cannot become light-like, it only approaches zero if x_1 and x_2 coincide. In order to check that the integrand stays finite in this case, we will choose s_2 close to s_1, i.e. $s_2 = s_1 + \varepsilon$ and expand the integrand in ε

$$\begin{aligned} I_{s_1,s_1+\varepsilon} =& \frac{\dot{x}_1 \cdot \left(\dot{x}_1 + \varepsilon\, \ddot{x}_1 + \frac{\varepsilon^2}{2}\, x_1^{(3)} + \mathcal{O}(\varepsilon^3)\right) + 1}{\left(x_1 - x_1 - \varepsilon\, \dot{x}_1 - \frac{\varepsilon^2}{2}\, \ddot{x}_1 - \mathcal{O}(\varepsilon^3)\right)^2} = \frac{\frac{\varepsilon^2}{2}\, \dot{x}_1 \cdot x_1^{(3)} + \mathcal{O}(\varepsilon^3)}{-\varepsilon^2 + \mathcal{O}(\varepsilon^4)} \\ =& -\frac{1}{\varepsilon^2}\left(\frac{\varepsilon^2}{2}\, \dot{x}_1 \cdot x_1^{(3)} + \mathcal{O}(\varepsilon^3)\right), \end{aligned} \tag{3.41}$$

where we employed that in arc length parametrization

$$\dot{x}^2 = -1 \qquad \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}s}} \qquad \dot{x} \cdot \ddot{x} = 0\,. \tag{3.42}$$

Since the integrand is finite for all possible x_1 and x_2, the whole double integral stays finite. We also note that the $1/\varepsilon^2$ pole of the gauge part in (3.41) is exactly canceled by the pole due to the scalar propagator, indicating that the Maldacena-Wilson loop has better UV properties than the ordinary Wilson loop. At this point our motivation for the modulus definition (3.28) becomes clear once more. If we had defined the modulus without the i, the expectation value would not have been finite for the class of curves that we want to consider.

3.3. Wilson Loops and Scattering Amplitudes

In the introductory chapter on $\mathcal{N}=4$ SYM theory we dedicated a whole section to scattering amplitudes and their symmetries. Since this master thesis is about Maldacena-Wilson loops one could of course question why we did that. One reason is because Wilson loops and scattering amplitudes are deeply related in $\mathcal{N}=4$ SYM theory. Indeed, it has been discovered that certain classes of amplitudes exhibit a dual description in terms of special (supersymmetrized) Wilson loops [12–15]. In what follows we will briefly discuss one of the best studied cases, which is the duality between planar MHV amplitudes and polygonal light-like Wilson loops. Our presentation is influenced by a review article of J. Drummond [40].

To begin, let us recall that in the language of superamplitudes, MHV amplitudes are the expansion coefficients of the degree eight terms, if the superamplitude is expanded in powers of η_i. In general, the MHV part of the superamplitude can be written as a product of the tree-level amplitude and a loop correction function

$$\mathbb{A}_n^{\mathrm{MHV}} = \frac{\delta^{(4)}(p)\, \delta^{(8)}(q)}{\langle 12\rangle \langle 23\rangle \dots \langle n1\rangle}\, M_n = \mathbb{A}_{n;\mathrm{tree}}^{\mathrm{MHV}}\, M_n\,, \tag{3.43}$$

where M_n is a series in $a = \lambda/8\pi^2$ that starts out with one. The concrete form of this function depends on the regularization prescription used to regulate the infrared divergences one encounters at loop-level. In this business it is convenient to employ a variant of dimensional regularization called dimensional reduction [41] with $D = 4 - 2\varepsilon_{\mathrm{ir}}$ and $\varepsilon_{\mathrm{ir}} < 0$, which makes the loop correction function depend on the regulator $\varepsilon_{\mathrm{ir}}$ and some associated scale μ. Since the divergences appear in an exponentiated form, it is convenient to consider the logarithm of the loop correction function. In general, this logarithm takes the following form

$$\log M_n = -\frac{1}{4}\sum_{l=1}^{\infty} a^l \left[\frac{\Gamma^{(l)}_{\text{cusp}}}{(l\varepsilon_{\text{ir}})^2} + \frac{\Gamma^{(l)}_{\text{col}}}{l\varepsilon_{\text{ir}}}\right] \sum_{i=1}^{n}\left(\frac{\mu^2_{\text{ir}}}{-s_{i,i+1}}\right)^{l\varepsilon_{\text{ir}}} + F_n^{\text{MHV}}(p_1,\dots,p_n;a) + \mathcal{O}(\varepsilon_{\text{ir}})\,, \qquad (3.44)$$

where F_n^{MHV} denotes the finite part, $\mu^2_{ir} = 4\pi e^{-\gamma_E}\mu^2$, $s_{i,i+1} = (p_i + p_{i+1})^2$ and

$$\Gamma_{\text{cusp}}(a) = \sum_{l=1}^{\infty} a^l\,\Gamma^{(l)}_{\text{cusp}} = 2\,a - 2\,\zeta_2\,a^2 + \mathcal{O}(a^3) \qquad \Gamma_{\text{col}}(a) = \sum_{l=1}^{\infty} a^l\,\Gamma^{(l)}_{\text{col}} = -\zeta_3\,a^2 + \mathcal{O}(a^3)\,. \quad (3.45)$$

The quantity Γ_{cusp} is the so-called cusp anomalous dimension [42,43] and describes the leading ultraviolet divergences of Wilson loops evaluated of contours with light-like cusps. Its appearance in the infrared divergent part of scattering amplitudes is not accidental, but points to the deep-lying relation between Wilson loops and scattering amplitudes. In fact, it was realized in [44–46] that the infrared divergences of planar scattering amplitudes are intimately related to the cusp ultraviolet divergences of specific polygonal light-like Wilson loops. However, while this relation holds true in any gauge theory, the distinguishing feature of $\mathcal{N}=4$ SYM theory is that there the connection goes much beyond the structure of divergences. To make this statement exact, let us first recall the definition of the dual coordinates (2.147) introduced in the context of dual superconformal symmetry.

$$x_i^\mu - x_{i+1}^\mu = p_i^\mu \qquad \longrightarrow \qquad (x_i - x_{i+1})^2 = 0 \qquad (3.46)$$

If one interprets the dual coordinates as the coordinates of some configuration space, the formula on the left-hand side assigns a sequence of light-like segments to the momenta p_i. Due to momentum conservation, these light-like segments form a n-sided polygon whose contour we will denote by C_n. A natural object which can be associated with such a closed contour is the (Maldacena-)Wilson loop

$$\langle W(C_n)\rangle = \frac{1}{N}\langle 0|\,\mathrm{Tr}\,\mathcal{P}\,\exp\left(i\oint_{C_n}\mathrm{d}s\,A_\mu(x)\,\dot{x}^\mu\right)|0\rangle\,, \qquad (3.47)$$

where we have already dropped the part involving $|\dot{x}|$, since it vanishes for light-like contours. While smooth Maldacena-Wilson loops have finite expectation values, the above expression is divergent, due to the presence of cusps and light-like edges. As in the case of scattering amplitudes it is convenient to regularize these divergences using the dimensional reduction scheme, but now with $D = 4 - 2\varepsilon_{\text{uv}}$ and $\varepsilon_{\text{uv}} > 0$. Employing this prescription, the logarithm of the expectation value takes the following form

$$\log\langle W(C_n)\rangle = -\frac{1}{4}\sum_{l=1}^{\infty} a^l \left[\frac{\Gamma^{(l)}_{\text{cusp}}}{(l\varepsilon_{\text{uv}})^2} + \frac{\Gamma^{(l)}}{l\varepsilon_{\text{uv}}}\right]\sum_{i=1}^{n}\left(-\mu^2_{\text{uv}}\,x^2_{i,i+2}\right)^{l\varepsilon_{\text{uv}}} + F_n^{\text{WL}}(x_1,\dots,x_n;a) + \mathcal{O}(\varepsilon_{\text{uv}})\,, \qquad (3.48)$$

where $x^\mu_{i,i+2} = x_i^\mu - x_{i+2}^\mu$ and

$$\Gamma(a) = \sum_{l=1}^{\infty} a^l\,\Gamma^{(l)} = -7\,\zeta_3\,a^2 + \mathcal{O}(a^3)\,. \qquad (3.49)$$

If we compare this expression with the logarithm of the loop correction function (3.44), we immediately note that the structure of the singularities is the same. In fact, by using (3.46) and performing an appropriate change of regularization parameters (see [47]), the divergent terms in (3.48) and (3.44) can be matched exactly. However, this is not the full story. In fact, there is much evidence that in $\mathcal{N} = 4$ SYM theory the finite parts are equal as well

$$F_n^{\mathrm{WL}}(x_1, \ldots, x_n; a) = F_n^{\mathrm{MHV}}(p_1, \ldots, p_n; a) + \mathrm{const}\,, \tag{3.50}$$

upon identification of kinematical variables (3.46). This is a very non-trivial statement that, moreover, seems to be of non-perturbative nature, since it has also holds true at strong coupling [12].

From the viewpoint of the duality the dual superconformal symmetry of scattering amplitudes seems to be very natural, since it can be identified with the ordinary conformal symmetry of the associated Wilson loop. Due to the fact that a Wilson loop evaluated over a polygonal light-like contour C_n is divergent, one expects the conformal symmetry to be broken by the regularization procedure. This indeed turns out to be the case, but the symmetry breaking is well understood and can be described by anomalous Ward identities, see [48]. Inspired by the duality, it is natural to ask whether the Yangian symmetries of scattering amplitudes carry over to the Wilson loop sector. To investigate this question, one could of course study the symmetries of light-like Wilson loops. However, since divergences typically cause a great deal of complication, it seems to be a good idea to widen the class of loop operators a little bit and to study the problem for smooth Maldacena-Wilson loops. This has the obvious advantage that one does not need to care about potential symmetry breakings due to regularization. In what follows, we will exactly address this question, i.e. we will investigate the potential Yangian symmetries of smooth Maldacena-Wilson loops in planar $\mathcal{N} = 4$ SYM theory.

3.4. Conformal Symmetry

Before we investigate the symmetries of the expectation value of a smooth Maldacena-Wilson loop operator, let us briefly review some basic facts about conformal transformations. The presentation is based on chapter four of [49], where more details can be found. The spacetime we want to consider is the usual four-dimensional Minkowski space equipped with the metric $\eta_{\mu\nu}$. Conformal transformations are by definition transformations $x^\mu \to x'^\mu$ which leave the metric tensor invariant up to a local scaling factor

$$\frac{\partial x'^\rho}{\partial x^\mu}\,\frac{\partial x'^\sigma}{\partial x^\nu}\,\eta_{\rho\sigma} = \Lambda(x)\,\eta_{\mu\nu}\,, \tag{3.51}$$

where $\Lambda(x)$ is a positive function. Conformal transformations therefore preserve angles, but not necessarily lengths. Let us now determine what the most general infinitesimal transformation, compatible with (3.51), looks like. Making the ansatz

$$x'^\mu = x^\mu + \varepsilon^\mu(x)\,, \tag{3.52}$$

and plugging it into (3.51), we find

$$\Big(\delta^\rho_\mu + \partial_\mu\,\varepsilon^\rho\Big)\Big(\delta^\sigma_\nu + \partial_\nu\,\varepsilon^\sigma\Big)\,\eta_{\rho\sigma} = \Big(1 + \rho(x)\Big)\,\eta_{\mu\nu}\,, \tag{3.53}$$

where we have written $\Lambda(x) = 1 + \rho(x)$ which makes sense since the conformal factor must be one if $\varepsilon^\mu(x) = 0$. From the last equation we see that up to first order in ε the requirement that the transformation be conformal implies

$$\partial_\mu\,\varepsilon_\nu + \partial_\nu\,\varepsilon_\mu = \rho(x)\,\eta_{\mu\nu}\,. \tag{3.54}$$

By taking the trace on both sides we can solve for $\rho(x)$. If we reinsert this expression into the former equation we find

$$\partial_\mu\,\varepsilon_\nu + \partial_\nu\,\varepsilon_\mu = \frac{1}{2}\,(\partial\cdot\varepsilon)\,\eta_{\mu\nu}\,. \tag{3.55}$$

The most general solution to this equation is given by

$$\varepsilon_\mu(x) = a_\mu + \omega_{\mu\nu}\,x^\nu + \lambda\,x_\mu + 2\,(b\cdot x)\,x_\mu - b_\mu\,x^2 \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}\,. \tag{3.56}$$

More details on this can be found in chapter four of [49]. The first two terms correspond to infinitesimal translations and Lorentz transformations respectively. The Poincaré group is therefore a subgroup of the conformal group. The third term represents infinitesimal scale transformations while the last two terms form the infinitesimal version of what is called a special conformal transformation. Let us now introduce generators for the infinitesimal conformal transformations. By comparing

$$x'_\rho = \Big(1 + a^\mu\,P_\mu - \frac{1}{2}\,\omega^{\mu\nu}\,M_{\mu\nu} + \lambda\,D - b^\mu\,K_\mu\Big)\,x_\rho\,, \tag{3.57}$$

with (3.56), we find

$$\begin{aligned}
P_\mu &= \partial_\mu && \text{(translations)}\\
M_{\mu\nu} &= x_\mu\,\partial_\nu - x_\nu\,\partial_\mu && \text{(Lorentz transformations)}\\
D &= x^\mu\,\partial_\mu && \text{(dilatations)}\\
K_\mu &= x^2\,\partial_\mu - 2\,x_\mu\,x^\nu\,\partial_\nu && \text{(special conformal transformations)}\,.
\end{aligned} \tag{3.58}$$

At first, equation (3.57) might look a little bit unfamiliar due to various minus signs and the absence of i's but it reflects, of course, only a certain choice of basis in the conformal algebra. The generators (3.58) obey the following commutation relations

$$\begin{aligned}
\Big[M_{\mu\nu}, M_{\rho\sigma}\Big] &= \eta_{\mu\sigma}\,M_{\nu\rho} + \eta_{\nu\rho}\,M_{\mu\sigma} - \eta_{\mu\rho}\,M_{\nu\sigma} - \eta_{\nu\sigma}\,M_{\mu\rho}\\
\Big[M_{\mu\nu}, P_\lambda\Big] &= \eta_{\nu\lambda}\,P_\mu - \eta_{\mu\lambda}\,P_\nu & \Big[P_\mu, P_\nu\Big] &= 0\\
\Big[P_\mu, K_\nu\Big] &= 2\,M_{\mu\nu} - 2\,\eta_{\mu\nu}\,D & \Big[D, K_\mu\Big] &= K_\mu\\
\Big[M_{\mu\nu}, K_\rho\Big] &= \eta_{\nu\rho}\,K_\mu - \eta_{\mu\rho}\,K_\nu & \Big[D, P_\mu\Big] &= -P_\mu\\
\Big[K_\mu, K_\nu\Big] &= 0 & \Big[D, M_{\mu\nu}\Big] &= 0\,,
\end{aligned} \tag{3.59}$$

which in fact define the conformal algebra $\mathfrak{conf}(1,3)$. Since this Lie algebra can be shown to be isomorphic to $\mathfrak{so}(2,4)$ we will always refer to the conformal algebra as $\mathfrak{so}(2,4)$. The finite transformations corresponding to (3.58) are given by

$$\begin{aligned} x'^{\mu} &= x^{\mu} + a^{\mu} && \text{(translations)} \\ x'^{\mu} &= \Lambda^{\mu}{}_{\nu}\, x^{\nu} && \text{(Lorentz transformations)} \\ x'^{\mu} &= \alpha\, x^{\mu} && \text{(dilatations)} \\ x'^{\mu} &= \frac{x^{\mu} - b^{\mu}\, x^2}{1 - 2\,(b\cdot x) + b^2\, x^2} && \text{(special conformal transformations)}\,. \end{aligned} \tag{3.60}$$

The first three transformations are easily obtained by exponentiating the infinitesimal ones. For special conformal transformations it is less trivial to find the finite transformations, but we will not stress this here. In order to give the unfamiliar reader a more geometrical interpretation of special conformal transformations we mention that such a transformation can also be represented as an inversion

$$\big(I(x)\big)^{\mu} = \frac{x^{\mu}}{x^2}\,, \tag{3.61}$$

followed by a translation T_b with vector $-b^{\mu}$ and another inversion, i.e.

$$\big(K_b(x)\big)^{\mu} = \big((I\circ T_b\circ I)\,(x)\big)^{\mu} = \frac{\frac{x^{\mu}}{x^2} - b^{\mu}}{\left(\frac{x^{\nu}}{x^2} - b^{\nu}\right)\left(\frac{x_{\nu}}{x^2} - b_{\nu}\right)} = \frac{x^{\mu} - b^{\mu}\, x^2}{1 - 2\,(b\cdot x) + b^2\, x^2}\,. \tag{3.62}$$

Note that the inversion (3.61) is an element of the conformal group not connected to the identity. Hence, the inversion does not have an infinitesimal generator.

3.4.1. Conformal Symmetry of the MWL

Having discussed conformal transformations in general, let us now turn to the symmetries of the Maldacena-Wilson loop operator. We have already shown that the one-loop expectation value $\langle W(C)\rangle_{(1)}$ is finite for the class of curves we want to consider. Furthermore, the finiteness property is, as mentioned before, believed to hold to all orders in perturbation theory. Since conformal transformations leave the action of $\mathcal{N}=4$ SYM invariant, we clearly expect the expectation value of the Maldacena-Wilson loop operator to be invariant under conformal transformations which map the closed contour C to a contour $\tilde{C}$. Nevertheless, let us explicitly demonstrate here that the Maldacena-Wilson loop is conformally invariant at leading order in perturbation theory. In order to work infinitesimally, we first have to construct an appropriate representation of the conformal algebra $\mathfrak{so}(2,4)$. Let p_{μ}, $m_{\mu\nu}$, d, k_{μ} be the densities of the conformal generators

$$\begin{aligned} p_{\mu}(\tau) &= \frac{\delta}{\delta x^{\mu}(\tau)} & k_{\mu}(\tau) &= x^{\nu}(\tau)\, x_{\nu}(\tau)\, \frac{\delta}{\delta x^{\mu}(\tau)} - 2\, x_{\mu}(\tau)\, x^{\nu}(\tau)\, \frac{\delta}{\delta x^{\nu}(\tau)} \\ d(\tau) &= x^{\mu}(\tau)\, \frac{\delta}{\delta x^{\mu}(\tau)} & m_{\mu\nu}(\tau) &= x_{\mu}(\tau)\, \frac{\delta}{\delta x^{\nu}(\tau)} - x_{\nu}(\tau)\, \frac{\delta}{\delta x^{\mu}(\tau)}\,. \end{aligned} \tag{3.63}$$

The generators are then defined as the integral of the above given densities, i.e

$$\Big(P_{\mu}, M_{\mu\nu}, D, K_{\mu}\Big) = \int \mathrm{d}\tau\, \Big(p_{\mu}(\tau), m_{\mu\nu}(\tau), d(\tau), k_{\mu}(\tau)\Big)\,. \tag{3.64}$$

The so-defined generators satisfy the conformal algebra (3.59). Furthermore, it can be shown that these generators indeed implement conformal transformations of the integration contour C. For example, for an infinitesimal Lorentz transformation we have

$$\begin{aligned}\delta x^\mu(s) = x'^\mu(s) - x^\mu(s) &= -\frac{1}{2}\,\omega_{\rho\sigma}\,M^{\rho\sigma}\,x^\mu(s) \\ &= -\frac{1}{2}\,\omega_{\rho\sigma}\int \mathrm{d}\tau\,\Big(x^\rho(\tau)\,\eta^{\mu\sigma} - x^\sigma(\tau)\,\eta^{\mu\rho}\Big)\delta(\tau - s) \\ &= \omega^\mu{}_\rho\,x^\rho(s).\end{aligned} \tag{3.65}$$

A finite Lorentz transformation is as usual obtained by exponentiating, i.e.

$$\Lambda = \exp\left(-\frac{1}{2}\,\omega_{\mu\nu}\,M^{\mu\nu}\right). \tag{3.66}$$

Under such a transformation the expectation value of the Maldacena-Wilson loop (3.37) changes as follows

$$\langle W(\tilde{C})\rangle = \exp\left(-\frac{1}{2}\,\omega_{\mu\nu}\,M^{\mu\nu}\right)\langle W(C)\rangle\,. \tag{3.67}$$

Hence, if $M^{\mu\nu}$ generates a symmetry of the Maldacena-Wilson loop, it has to annihilate the whole expectation value. Moreover, since the expectation value is a power series in the coupling valid for all values of λ as long as they are small enough, $M^{\mu\nu}$ has to annihilate each order individually. We therefore conclude that any symmetry generator necessarily annihilates the one-loop expectation value $\langle W(C)\rangle_{(1)}$. If we search for a further non-local symmetry generator later on, this will be our criterion. But before we move on to non-local symmetries let us demonstrate that the Maldacena-Wilson loop is indeed conformally invariant at one-loop order. Of course, in the case of conformal transformations, there is no need to work infinitesimally as the corresponding finite transformations are known. But since this section is intended as an introduction to the more involved calculation in section 3.5.2 we will adopt the infinitesimal point of view.

Translational Symmetry

From a global viewpoint the one-loop expectation value (3.39) is manifest translationally invariant, since it only depends on derivatives and differences. To prove this symmetry infinitesimally, we have to show that the following expression vanishes

$$P^\mu\,\langle W(C)\rangle_{(1)} = -\frac{\lambda}{16\pi^2}\int \mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}\tau\,\frac{\delta I_{12}}{\delta x_\mu(\tau)} \qquad\qquad I_{12} = \frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{x_{12}^2}\,. \tag{3.68}$$

With the basic functional derivatives given by

$$\begin{aligned}\frac{\delta x^\nu(s)}{\delta x_\mu(\tau)} &= \eta^{\mu\nu}\,\delta(s-\tau) \\ \frac{\delta \dot{x}^\nu(s)}{\delta x_\mu(\tau)} &= \eta^{\mu\nu}\,\partial_s\delta(s-\tau) \\ \frac{\delta|\dot{x}(s)|}{\delta x_\mu(\tau)} &= \frac{\dot{x}^\mu(s)}{|\dot{x}(s)|}\,\partial_s\delta(s-\tau),\end{aligned} \tag{3.69}$$

we can immediately calculate the first functional derivative of the integrand I_{12}.

$$\frac{\delta I_{12}}{\delta x_\mu(\tau)} = \frac{1}{x_{12}^2}\left[\left(\dot{x}_2^\mu - \frac{|\dot{x}_2|}{|\dot{x}_1|}\,\dot{x}_1^\mu\right)\partial_{s_1}\delta(s_1-\tau) + \left(\dot{x}_1^\mu - \frac{|\dot{x}_1|}{|\dot{x}_2|}\,\dot{x}_2^\mu\right)\partial_{s_2}\delta(s_2-\tau)\right] \\ -\frac{2}{x_{12}^4}\left[\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\,x_{12}^\mu\Big(\delta(s_1-\tau)-\delta(s_2-\tau)\Big)\right] \tag{3.70}$$

Since

$$\int \mathrm{d}\tau\;\partial_s\delta(s-\tau) = \partial_s\int \mathrm{d}\tau\;\delta(s-\tau) = \partial_s\,1 = 0 \\ \int \mathrm{d}\tau\,\Big(\delta(s_1-\tau)-\delta(s_2-\tau)\Big) = 0, \tag{3.71}$$

we readily conclude that the generator P^μ annihilates $\langle W(C)\rangle_{(1)}$.

Lorentz Symmetry

The integrand I_{12} is obviously a Lorentz scalar, meaning that it transforms under the trivial representation. For the infinitesimal discussion we again apply the corresponding generator to I_{12}. Using (3.70), we find

$$M_{\mu\nu}\,I_{12} = \int \mathrm{d}\tau\,\Bigg\{\Bigg(\frac{1}{x_{12}^2}\left[\left(\dot{x}_{2\nu} - \frac{|\dot{x}_2|}{|\dot{x}_1|}\,\dot{x}_{1\nu}\right)\partial_{s_1}\delta(s_1-\tau)\,x_\mu(\tau) + \left(\dot{x}_{1\nu} - \frac{|\dot{x}_1|}{|\dot{x}_2|}\,\dot{x}_{2\nu}\right)\partial_{s_2}\delta(s_2-\tau)\,x_\mu(\tau)\right] \\ -\frac{2}{x_{12}^4}\left[\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\,x_{12\nu}\,x_\mu(\tau)\,\Big(\delta(s_1-\tau)-\delta(s_2-\tau)\Big)\right]\Bigg) - \Big(\mu\leftrightarrow\nu\Big)\Bigg\}.$$

After we have integrated out the delta functions, the result reads

$$M_{\mu\nu}\,I_{12} = \Bigg(\frac{1}{x_{12}^2}\left[\left(\dot{x}_{2\nu} - \frac{|\dot{x}_2|}{|\dot{x}_1|}\,\dot{x}_{1\nu}\right)\dot{x}_{1\mu} + \left(\dot{x}_{1\nu} - \frac{|\dot{x}_1|}{|\dot{x}_2|}\,\dot{x}_{2\nu}\right)\dot{x}_{2\mu}\right] \\ -\frac{2}{x_{12}^4}\left[\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)x_{12\nu}\,x_{12\mu}\right]\Bigg) - \Bigg(\mu\leftrightarrow\nu\Bigg) = 0\,, \tag{3.72}$$

which vanishes, due to the symmetry under exchange of $(\mu\leftrightarrow\nu)$.

Dilatation Symmetry

Despite the fact that the scale invariance of the integrand I_{12} is obvious we will also prove it infinitesimally. Using the previous result, we note that the only thing we need to do is to leave out the term where $(\mu\leftrightarrow\nu)$ and to contract indices in the remaining part.

The result reads

$$D\,I_{12} = +\frac{2}{x_{12}^2}\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big) - \frac{2}{x_{12}^2}\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big) = 0\,. \tag{3.73}$$

Hence, the one-loop expectation value is scale invariant.

Special Conformal Symmetry

The calculation which shows that $\langle W(C)\rangle_{(1)}$ is also invariant under special conformal transformations is a bit more involved. For computational purposes we will split the generator of special conformal transformations into the sum of two parts.

$$K_\mu^{(1)} = \int \mathrm{d}\tau\, x^2(\tau)\,\frac{\delta}{\delta x^\mu(\tau)} \qquad\qquad K_\mu^{(2)} = -2\int \mathrm{d}\tau\, x_\mu(\tau)\, x^\nu(\tau)\,\frac{\delta}{\delta x^\nu(\tau)} \tag{3.74}$$

Let us start by applying the first part $K_\mu^{(1)}$ to I_{12}.

$$\begin{aligned} K_\mu^{(1)}\, I_{12} = {}& \frac{2}{x_{12}^2}\left[\left(\dot{x}_{2\mu} - \frac{|\dot{x}_2|}{|\dot{x}_1|}\,\dot{x}_{1\mu}\right) x_1\cdot\dot{x}_1 + \left(\dot{x}_{1\mu} - \frac{|\dot{x}_1|}{|\dot{x}_2|}\,\dot{x}_{2\mu}\right) x_2\cdot\dot{x}_2\right] \\ & -\frac{2}{x_{12}^4}\left[\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\, x_{12\mu}\Big(x_1^2 - x_2^2\Big)\right] \end{aligned} \tag{3.75}$$

For the action of the second part $K_\mu^{(2)}$ we find

$$\begin{aligned} K_\mu^{(2)}\, I_{12} = {}& -\frac{2}{x_{12}^2}\left[\left(\dot{x}_{2\nu} - \frac{|\dot{x}_2|}{|\dot{x}_1|}\,\dot{x}_{1\nu}\right)\Big(\dot{x}_{1\mu}\, x_1^\nu + x_{1\mu}\,\dot{x}_1^\nu\Big) + \left(\dot{x}_{1\nu} - \frac{|\dot{x}_1|}{|\dot{x}_2|}\,\dot{x}_{2\nu}\right)\Big(\dot{x}_{2\mu}\, x_2^\nu + x_{2\mu}\,\dot{x}_2^\nu\Big)\right] \\ & +\frac{4}{x_{12}^4}\left[\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\, x_{12\nu}\Big(x_{1\mu}\, x_1^\nu - x_{2\mu}\, x_2^\nu\Big)\right] \\ = {}& -\frac{2}{x_{12}^2}\Bigg[x_1\cdot\dot{x}_2\,\dot{x}_{1\mu} - \frac{|\dot{x}_2|}{|\dot{x}_1|}\, x_1\cdot\dot{x}_1\,\dot{x}_{1\mu} + x_2\cdot\dot{x}_1\,\dot{x}_{2\mu} \\ & \qquad - \frac{|\dot{x}_1|}{|\dot{x}_2|}\, x_2\cdot\dot{x}_2\,\dot{x}_{2\mu} + \Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\Big(x_{1\mu} + x_{2\mu}\Big)\Bigg] \\ & +\frac{4}{x_{12}^4}\left[\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\Big(x_{1\mu}\Big(x_1^2 - x_1\cdot x_2\Big) - x_{2\mu}\Big(x_1\cdot x_2 - x_2^2\Big)\Big)\right]. \end{aligned} \tag{3.76}$$

If we add up these two parts, some terms immediately cancel out and we are left with

$$\begin{aligned} \Big(K_\mu^{(1)} + K_\mu^{(2)}\Big)\, I_{12} = {}& \frac{2}{x_{12}^2}\Bigg[x_1\cdot\dot{x}_1\,\dot{x}_{2\mu} + x_2\cdot\dot{x}_2\,\dot{x}_{1\mu} - x_1\cdot\dot{x}_2\,\dot{x}_{1\mu} - x_2\cdot\dot{x}_1\,\dot{x}_{2\mu} \\ & \qquad - \Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\Big(x_{1\mu} + x_{2\mu}\Big)\Bigg] \\ & +\frac{2}{x_{12}^4}\left[\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\Big(x_{1\mu}\, x_{12}^2 + x_{2\mu}\, x_{12}^2\Big)\right]. \end{aligned} \tag{3.77}$$

Since the term in the second line cancels the one in the third line, we find

$$K_\mu\, I_{12} = \frac{2}{x_{12}^2}\Big[\dot{x}_1\cdot x_{12}\,\dot{x}_{2\mu} - \dot{x}_2\cdot x_{12}\,\dot{x}_{1\mu}\Big]. \tag{3.78}$$

This can however be rewritten as follows

$$K_\mu \, I_{12} = \left(\dot{x}_{1\mu} \, \partial_{s_2} + \dot{x}_{2\mu} \, \partial_{s_1}\right) \ln\left(-x_{12}^2\right). \tag{3.79}$$

We note that in contrast to the other conformal generators, K_μ does not annihilate the integrand I_{12}. Instead, it leads to a total derivative, which, together with the Wilson loop integrals, allows us to conclude that

$$K_\mu \, \langle W(C) \rangle_{(1)} = 0 \,. \tag{3.80}$$

This completes the proof that the expectation value of the Maldacena-Wilson loop operator is conformally invariant at one-loop order.

3.5. Towards a Yangian Symmetry

3.5.1. The Bosonic Level-One Generator

In the last section we successfully verified the conformal invariance of smooth Maldacena-Wilson loops. This symmetry is, as we have argued before, absolutely expected. Now, we want to investigate the question whether the symmetry algebra of smooth Maldacena-Wilson loops is actually a much bigger algebra of Yangian type that contains the conformal algebra as a subalgebra. To do so, we will take the following approach: first, we shall explicitly derive the non-local part of the level-one momentum generator using a continuous version of the bi-local formula (2.123). In the second step we will then apply it to the one-loop expectation value of the Maldacena-Wilson loop operator. The result will then show whether and how the generator has to be modified by a local term, in order to obtain a real symmetry generator. The non-local structure we expect to find is of course the same as the one which was derived in the context of scattering amplitudes. More explicitly, it was shown in [11] that tree-level superamplitudes are annihilated by the following level-one momentum generator

$$p^{(1)}_{\alpha\dot{\alpha}} = \sum_{i>j} \Big[\Big(m_{i\alpha}^{\gamma} \, \delta_{\dot{\alpha}}^{\dot{\gamma}} + \bar{m}_{i\dot{\alpha}}^{\dot{\gamma}} \, \delta_{\alpha}^{\gamma} - d_i \, \delta_{\dot{\alpha}}^{\dot{\gamma}} \, \delta_{\alpha}^{\gamma} \Big) \, p_{j\gamma\dot{\gamma}} + \bar{q}_{i\dot{\alpha}C} \, q_{j\alpha}^{C} - (i \leftrightarrow j) \Big] \,, \tag{3.81}$$

where $m_{\alpha\beta}$, $\bar{m}_{\dot{\alpha}\dot{\beta}}$, d, $\bar{q}_{\dot{\alpha}C}$ and q_α^C are the superconformal generators in on-shell superspace, see (2.143). In principal, we could of course try to translate the above expression to our framework, but since (3.81) was derived using different conventions for spinors as well as for the underlying Lie superalgebra, we will not follow this approach. Instead, we will start from the level-zero algebra and derive the non-local part of the level-one momentum generator again using our conventions. At the end of section 2.4.3 we mentioned that in a discrete setup, like for example in the context spin chains or scattering amplitudes, the bi-local part of the level-one generators is given by

$$J_a^{(1)} = f_a^{bc} \sum_{i<j} j_{ic}^{(0)} \, j_{jb}^{(0)} \,, \tag{3.82}$$

where the $j_{ic}^{(0)}$ collectively denote the generators of the Lie (super)algebra, which act on a single site or leg. In a continuous framework the same equation reads

$$J_a^{(1)} = f_a^{bc} \int \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, \theta(\tau_2 - \tau_1) \, j_c^{(0)}(\tau_1) \, j_b^{(0)}(\tau_2) \,, \tag{3.83}$$

where the $j_c^{(0)}(\tau)$ denote the generator densities as introduced in the last section. To obtain the concrete level-one generators, we now need to specify the underlying Lie (super)algebra. The most canonical choice would be the symmetry algebra of $\mathcal{N} = 4$ SYM theory, namely $\mathfrak{psu}(2,2|4)$. However, motivated by the fact that Maldacena-Wilson loops couple only to the bosonic fields of the theory, we will first try to establish an invariance under the Yangian algebra of the bosonic subalgebra $\mathfrak{so}(2,4)$. Eventually, it will turn out that one cannot consistently restrict to the conformal subsector. But as the bosonic computation has to be carried out anyway, let us postpone the discussion of the full $\mathfrak{psu}(2,2|4)$. To get started, we obviously need the dual structure constants of $\mathfrak{so}(2,4)$, since they appear in the definition of the bosonic level-one generator (3.83). These are derived in appendix A.2 and explicitly given by

$$f_{P_\mu}^{\hat{D}\,\hat{P}^\rho} = \frac{1}{8}\,\delta_\mu^\rho = -f_{P_\mu}^{\hat{P}^\rho\,\hat{D}} \qquad\qquad f_{P_\mu}^{\hat{M}^{\sigma\rho}\,\hat{P}^\lambda} = \frac{1}{8}\left(\eta^{\sigma\lambda}\,\delta_\mu^\rho - \eta^{\rho\lambda}\,\delta_\mu^\sigma\right) = -f_{P_\mu}^{\hat{P}^\lambda\,\hat{M}^{\sigma\rho}}\,, \tag{3.84}$$

where we have listed only those non-vanishing dual structure constants which have P_μ as their lower index. Now we can use formula (3.83) and write down the following expression for the bi-local part of the bosonic level-one momentum generator

$$\begin{aligned} P^{(1)}_{\mu,\,\mathrm{nl,\,bos}} = \int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\theta(\tau_2-\tau_1)\Big(& f_{P_\mu}^{\hat{D}\,\hat{P}^\rho}\,p_\rho(\tau_1)\,d(\tau_2) + f_{P_\mu}^{\hat{P}^\rho\,\hat{D}}\,d(\tau_1)\,p_\rho(\tau_2) \\ &+ \sum_{\sigma<\rho} f_{P_\mu}^{\hat{M}^{\sigma\rho}\,\hat{P}^\lambda}\,p_\lambda(\tau_1)\,m_{\sigma\rho}(\tau_2) \\ &+ \sum_{\sigma<\rho} f_{P_\mu}^{\hat{P}^\lambda\,\hat{M}^{\sigma\rho}}\,m_{\sigma\rho}(\tau_1)\,p_\lambda(\tau_2)\Big)\,, \end{aligned} \tag{3.85}$$

where summation over repeated indices is implied unless there is an explicit sigma sign in front. The ordered sums reflect the fact that all sums are really over Lie algebra indices a, which means that every linearly independent generator should appear at most once in a sum. For computational purposes it is however useful to replace the ordered sums by usual ones and to compensate for this by the inclusion of a factor of $1/2$. Substituting (3.84) into the former expression yields

$$\begin{aligned} P^{(1)}_{\mu,\,\mathrm{nl,\,bos}} &= \frac{1}{8}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\theta(\tau_2-\tau_1)\Big(p_\mu(\tau_1)\,d(\tau_2) - d(\tau_1)\,p_\mu(\tau_2) + \tfrac{1}{2}\left(\eta^{\sigma\lambda}\,\delta_\mu^\rho - \eta^{\rho\lambda}\,\delta_\mu^\sigma\right)p_\lambda(\tau_1)\,m_{\sigma\rho}(\tau_2) \\ &\qquad\qquad - \tfrac{1}{2}\left(\eta^{\sigma\lambda}\,\delta_\mu^\rho - \eta^{\rho\lambda}\,\delta_\mu^\sigma\right)m_{\sigma\rho}(\tau_1)\,p_\lambda(\tau_2)\Big)\Big) \\ &= \frac{1}{8}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\theta(\tau_2-\tau_1)\Big(p_\mu(\tau_1)\,d(\tau_2) - d(\tau_1)\,p_\mu(\tau_2) - p^\rho(\tau_1)\,m_{\mu\rho}(\tau_2) + m_{\mu\rho}(\tau_1)\,p^\rho(\tau_2)\Big) \\ &= \frac{1}{8}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\theta(\tau_2-\tau_1)\Big(\Big(m_{\mu\lambda}(\tau_1) - d(\tau_1)\,\eta_{\mu\lambda}\Big)p^\lambda(\tau_2) - \Big(\tau_1 \leftrightarrow \tau_2\Big) \\ &\qquad\qquad + \Big[p_\mu(\tau_1), d(\tau_2)\Big] - \Big[p^\lambda(\tau_1), m_{\mu\lambda}(\tau_2)\Big]\Big)\,. \end{aligned} \tag{3.86}$$

As a consequence of the algebra relations (3.59) the two commutators can only produce terms which are proportional to $p_\mu(\tau_2)$ times a delta function. By substituting explicit expressions for commutators and performing a change of variables in the part where $(\tau_1 \leftrightarrow \tau_2)$, we obtain

$$P^{(1)}_{\mu,\,\mathrm{nl,\,bos}} = \frac{1}{8}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\Big(\theta(\tau_2-\tau_1) - \theta(\tau_1-\tau_2)\Big)\Big(\Big(m_{\mu\lambda}(\tau_1) - d(\tau_1)\,\eta_{\mu\lambda}\Big)p^\lambda(\tau_2)\Big) + \frac{1}{2}\,\theta(0)\int \mathrm{d}\tau\,p_\mu(\tau)\,.$$

Since level-one generators can always be modified by adding a constant times the respective level-zero generator without changing the algebra relations (2.114), we will drop the latter term. In order to compute the action of the above given generator on the one-loop expectation value of the Maldacena-Wilson loop, it is useful to rewrite it a little bit. To do this, we first plug in the generator densities (3.63) and then factor out all operators.

$$P^{(1)\,\mu}_{\mathrm{nl,\,bos}} = \frac{1}{8} \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \left(\delta^\mu_\rho\, \delta^\nu_\kappa - \delta^\nu_\rho\, \delta^\mu_\kappa - \eta_{\rho\kappa}\, \eta^{\mu\nu}\right) x^\rho(\tau_1) \frac{\delta}{\delta x_\kappa(\tau_1)} \frac{\delta}{\delta x^\nu(\tau_2)} \times \Big(\theta(\tau_2 - \tau_1) - \theta(\tau_1 - \tau_2)\Big) \tag{3.87}$$

At this point, we note that the product of functional derivatives potentially gives rise to singular terms like $\delta(0)$. This will precisely happen when the delta functions, originating from the functional derivatives, localize both generator integrals to the same point. Naively, one might think that this only gives rise to terms like $\theta(0)$, but since the generator acts on an expression that also contains $\dot{x}(s)$, the partial derivatives will lead to $\delta(0)$ terms. To render the arising divergences finite, we have to introduce an appropriate regulator. Since all the divergences we will encounter are caused by the fact that the generator parameters can both take the same value, it is natural to introduce a point-splitting regulator. A reparametrization invariant way to do this is to substitute

$$\Big(\theta(\tau_2 - \tau_1) - \theta(\tau_1 - \tau_2)\Big) \quad \longrightarrow \quad \Big(\theta(\tau_2 - d(\tau_2, \varepsilon) - \tau_1) - \theta(\tau_1 - d(\tau_1, \varepsilon) - \tau_2)\Big) \tag{3.88}$$

in the above generator, where the function $d(\tau, \varepsilon)$ is implicitly defined by

$$\int\limits_{\tau - d(\tau,\varepsilon)}^{\tau} \mathrm{d}s\, \|\dot{x}\| \stackrel{!}{=} \varepsilon\,. \tag{3.89}$$

This regularization prescription guarantees that the two points $x^\mu(\tau_1)$ and $x^\nu(\tau_2)$ are at least separated by a distance ε, where ε is the distance measured along the curve. We further note that when we restrict to arc length parametrization, which we will do after having applied the two functional derivatives, the function $d(\tau, \varepsilon)$ becomes trivial, i.e.

$$d(\tau, \varepsilon) = \varepsilon \qquad\qquad \text{if} \quad \dot{x}^2 = -1\,. \tag{3.90}$$

For computational purposes we rewrite (3.88) using the identity

$$\theta(x) = 1 - \theta(-x) \qquad\qquad \text{with the convention} \qquad \theta(0) = \frac{1}{2}\,. \tag{3.91}$$

The result reads

$$\Big(\theta(\tau_2 - d(\tau_2, \varepsilon) - \tau_1) + \theta(\tau_2 + d(\tau_1, \varepsilon) - \tau_1) - 1\Big)\,. \tag{3.92}$$

We note that if the ordered integral in (3.87) is replaced by an unordered integral over the full parameter space, the level-one generator factorizes into a product of two level-zero generators which definitely annihilate $\langle W(C)\rangle_{(1)}$, see section 3.4.1. Hence, we can forget about the last summand in (3.92). Taking this into account, we obtain the following expression for the non-local piece of the regularized bosonic level-one generator

$$\begin{aligned} P^{(1)\,\mu}_{\mathrm{nl,\,bos,}\,\varepsilon} =& \frac{1}{8}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\Big(\delta^\mu_\rho\,\delta^\nu_\kappa - \delta^\nu_\rho\,\delta^\mu_\kappa - \eta_{\rho\kappa}\,\eta^{\mu\nu}\Big)\,x^\rho(\tau_1)\,\frac{\delta}{\delta x_\kappa(\tau_1)}\,\frac{\delta}{\delta x^\nu(\tau_2)}\,\theta(\tau_2 - d(\tau_2,\varepsilon) - \tau_1) \\ &+ \Big(d(\tau_2,\varepsilon) \to -d(\tau_1,\varepsilon)\Big)\,. \end{aligned} \tag{3.93}$$

3.5.2. The Bosonic Computation

We will now compute the action of the generator (3.93) on the one-loop expectation value of a smooth Maldacena-Wilson loop operator, i.e.

$$P^{(1)\,\sigma}_{\mathrm{nl,\,bos,}\,\varepsilon}\,\langle W(C)\rangle_{(1)} = -\frac{\lambda}{16\pi^2}\,P^{(1)\,\sigma}_{\mathrm{nl,\,bos,}\,\varepsilon}\int \mathrm{d}s_1\,\mathrm{d}s_2\,I_{12}\,. \tag{3.94}$$

We will start by calculating the second functional derivative of the integrand I_{12} multiplied by $x^\rho(\tau_1)$

$$\begin{aligned} & x^\rho(\tau_1)\frac{\delta}{\delta x_\mu(\tau_1)}\frac{\delta}{\delta x^\nu(\tau_2)}\left(\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{x_{12}^2}\right) \\ =& \frac{1}{x_{12}^2}\Bigg[\partial_{s_1}\delta(\tau_2-s_1)\,\partial_{s_2}\delta(\tau_1-s_2)\,x^\rho(\tau_1)\,\delta^\mu_\nu + \partial_{s_1}\delta(\tau_1-s_1)\,\partial_{s_2}\delta(\tau_2-s_2)\,x^\rho(\tau_1)\,\delta^\mu_\nu \\ &\quad - \left\{\frac{\dot{x}_2^\mu\,\dot{x}_{1\nu}}{|\dot{x}_1||\dot{x}_2|}\,\partial_{s_2}\delta(\tau_1-s_2) - \left(\frac{|\dot{x}_2|}{|\dot{x}_1|^3}\,\dot{x}_1^\mu\,\dot{x}_{1\nu} - \frac{|\dot{x}_2|}{|\dot{x}_1|}\,\delta^\mu_\nu\right)\partial_{s_1}\delta(\tau_1-s_1)\right\}x^\rho(\tau_1)\,\partial_{s_1}\delta(\tau_2-s_1) \\ &\quad - \left\{\frac{\dot{x}_1^\mu\,\dot{x}_{2\nu}}{|\dot{x}_1||\dot{x}_2|}\,\partial_{s_1}\delta(\tau_1-s_1) - \left(\frac{|\dot{x}_1|}{|\dot{x}_2|^3}\,\dot{x}_2^\mu\,\dot{x}_{2\nu} - \frac{|\dot{x}_1|}{|\dot{x}_2|}\,\delta^\mu_\nu\right)\partial_{s_2}\delta(\tau_1-s_2)\right\}x^\rho(\tau_1)\,\partial_{s_2}\delta(\tau_2-s_2)\Bigg] \\ &+\frac{2}{x_{12}^4}\Bigg[\left(\frac{|\dot{x}_2|}{|\dot{x}_1|}\,\dot{x}_{1\nu} - \dot{x}_{2\nu}\right)x_{12}^\mu\,x^\rho(\tau_1)\Big(\delta(\tau_1-s_1)-\delta(\tau_1-s_2)\Big)\partial_{s_1}\delta(\tau_2-s_1) \\ &\quad + \left(\frac{|\dot{x}_1|}{|\dot{x}_2|}\,\dot{x}_{2\nu} - \dot{x}_{1\nu}\right)x_{12}^\mu\,x^\rho(\tau_1)\Big(\delta(\tau_1-s_1)-\delta(\tau_1-s_2)\Big)\partial_{s_2}\delta(\tau_2-s_2) \\ &\quad + \left(\frac{|\dot{x}_2|}{|\dot{x}_1|}\,\dot{x}_1^\mu - \dot{x}_2^\mu\right)x_{12\nu}\,x^\rho(\tau_1)\Big(\delta(\tau_2-s_1)-\delta(\tau_2-s_2)\Big)\partial_{s_1}\delta(\tau_1-s_1) \\ &\quad + \left(\frac{|\dot{x}_1|}{|\dot{x}_2|}\,\dot{x}_2^\mu - \dot{x}_1^\mu\right)x_{12\nu}\,x^\rho(\tau_1)\Big(\delta(\tau_2-s_1)-\delta(\tau_2-s_2)\Big)\partial_{s_2}\delta(\tau_1-s_2) \\ &\quad - \Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)\delta^\mu_\nu\,x^\rho(\tau_1)\Big(\delta(\tau_1-s_1)-\delta(\tau_1-s_2)\Big)\Big(\delta(\tau_2-s_1)-\delta(\tau_2-s_2)\Big)\Bigg] \\ &+\frac{8}{x_{12}^6}\Bigg[\Big(\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|\Big)x_{12}^\mu\,x_{12\nu}\,x^\rho(\tau_1)\Big(\delta(\tau_1-s_1)-\delta(\tau_1-s_2)\Big)\Big(\delta(\tau_2-s_1)-\delta(\tau_2-s_2)\Big)\Bigg]. \end{aligned}$$

After having applied the two functional derivatives, we now fix the parametrization to arc length, i.e. we set $|\dot{x}| = i$. Integrating out the two generator parameters τ_1 and τ_2 with the Heaviside function in mind and applying the partial derivatives yields

$$\begin{aligned} &\frac{1}{x_{12}^2}\Bigg[\Big(\delta^\mu_\nu\,\dot{x}_2^\rho + \dot{x}_2^\mu\,\dot{x}_{1\nu}\,\dot{x}_2^\rho\Big)\delta(s_1-\varepsilon-s_2) + \Big(\delta^\mu_\nu\,\dot{x}_1^\rho + \dot{x}_1^\mu\,\dot{x}_{2\nu}\,\dot{x}_1^\rho\Big)\delta(s_2-\varepsilon-s_1) \\ &\quad - \Big(\dot{x}_2^\mu\,\dot{x}_{1\nu}\,x_2^\rho + \delta^\mu_\nu\,x_2^\rho\Big)\partial_{s_1}\delta(s_1-\varepsilon-s_2) - \Big(\dot{x}_1^\mu\,\dot{x}_{2\nu}\,x_1^\rho + \delta^\mu_\nu\,x_1^\rho\Big)\partial_{s_2}\delta(s_2-\varepsilon-s_1) \end{aligned}$$

$$
\begin{aligned}
&\quad - \Big(\dot{x}_1^{\mu}\, \dot{x}_{1\nu} + \delta^{\mu}_{\nu}\Big)\Big(\dot{x}_1^{\rho}\, \delta(\varepsilon) + x_1^{\rho}\, \partial_\varepsilon \delta(\varepsilon)\Big) - \Big(\dot{x}_2^{\mu}\, \dot{x}_{2\nu} + \delta^{\mu}_{\nu}\Big)\Big(\dot{x}_2^{\rho}\, \delta(\varepsilon) + x_2^{\rho}\, \partial_\varepsilon \delta(\varepsilon)\Big)\Bigg] \\
&+ \frac{2}{x_{12}^4}\Bigg[\dot{x}_{12\nu}\, x_{12}^{\mu}\Big(\Big(x_1^{\rho}\, \delta(\varepsilon) - x_2^{\rho}\, \delta(s_1 - \varepsilon - s_2)\Big) - \Big(x_1^{\rho}\, \delta(s_2 - \varepsilon - s_1) - x_2^{\rho}\, \delta(\varepsilon)\Big)\Big) \\
&\quad + \dot{x}_{12}^{\mu}\, x_{12\nu}\Big(\dot{x}_1^{\rho}\, \Big(\theta(-\varepsilon) - \theta(s_2 - \varepsilon - s_1)\Big) - x_1^{\rho}\Big(\delta(\varepsilon) - \delta(s_2 - \varepsilon - s_1)\Big)\Big) \\
&\quad - \dot{x}_{12}^{\mu}\, x_{12\nu}\Big(\dot{x}_2^{\rho}\Big(\theta(s_1 - \varepsilon - s_2) - \theta(-\varepsilon)\Big) - x_2^{\rho}\Big(\delta(s_1 - \varepsilon - s_2) - \delta(\varepsilon)\Big)\Big) \\
&\quad - \Big(\dot{x}_1 \cdot \dot{x}_2 + 1\Big)\delta^{\mu}_{\nu}\Big(x_1^{\rho}\Big(\theta(-\varepsilon) - \theta(s_2 - \varepsilon - s_1)\Big) - x_2^{\rho}\Big(\theta(s_1 - \varepsilon - s_2) - \theta(-\varepsilon)\Big)\Big)\Bigg] \\
&+ \frac{8}{x_{12}^6}\Bigg[\Big(\dot{x}_1 \cdot \dot{x}_2 + 1\Big)x_{12}^{\mu}\, x_{12\nu}\Big(x_1^{\rho}\Big(\theta(-\varepsilon) - \theta(s_2 - \varepsilon - s_1)\Big) - x_2^{\rho}\Big(\theta(s_1 - \varepsilon - s_2) - \theta(-\varepsilon)\Big)\Big)\Bigg].
\end{aligned}
$$

To obtain the last result, we also used integration by parts. The arising boundary terms were of the following two types

(1)

$$
\begin{aligned}
&\int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}\tau_2\, G(s_1, s_2)\, \theta(\tau_2 - \varepsilon - s_2)\, \partial_{s_2}\delta(\tau_2 - s_2) \\
&= -\int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}\tau_2\, G(s_1, s_2)\, \theta(\tau_2 - \varepsilon - s_2)\, \partial_{\tau_2}\delta(\tau_2 - s_2) \\
&= -\int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, G(s_1, s_2)\left(\Big[\theta(\tau_2 - \varepsilon - s_2)\, \delta(\tau_2 - s_2)\Big]_0^L - \int_0^L \mathrm{d}\tau_2\, \delta(\tau_2 - \varepsilon - s_2)\, \delta(\tau_2 - s_2)\right) \\
&= -\int_0^L \mathrm{d}s_1\Big(G(s_1, L) - G(s_1, 0)\Big)\theta(-\varepsilon) + \int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, G(s_1, s_2)\, \delta(\varepsilon)
\end{aligned} \tag{3.95}
$$

(2)

$$
\begin{aligned}
&\int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}\tau_2\, F(s_1, s_2)\, \delta(\tau_2 - \varepsilon - s_2)\, \partial_{s_2}\delta(\tau_2 - s_2) \\
&= -\int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}\tau_2\, F(s_1, s_2)\, \delta(\tau_2 - \varepsilon - s_2)\, \partial_{\tau_2}\delta(\tau_2 - s_2) \\
&= -\int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, F(s_1, s_2)\left(\Big[\delta(\tau_2 - \varepsilon - s_2)\, \delta(\tau_2 - s_2)\Big]_0^L - \int_0^L \mathrm{d}\tau_2\, \delta(\tau_2 - s_2)\, \partial_{\tau_2}\delta(\tau_2 - \varepsilon - s_2)\right) \\
&= -\int_0^L \mathrm{d}s_1\Big(F(s_1, L) - F(s_1, 0)\Big)\delta(-\varepsilon) - \int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, F(s_1, s_2)\, \partial_\varepsilon \int_0^L \mathrm{d}\tau_2\, \delta(\tau_2 - \varepsilon - s_2)\, \delta(\tau_2 - s_2) \\
&= -\int_0^L \mathrm{d}s_1\Big(F(s_1, L) - F(s_1, 0)\Big)\delta(-\varepsilon) - \int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, F(s_1, s_2)\, \partial_\varepsilon\delta(\varepsilon),
\end{aligned} \tag{3.96}
$$

where $G(s_1,s_2)$ and $F(s_1,s_2)$ denote some general functions of $x^\mu(s_1), x^\mu(s_2), \dot{x}^\mu(s_1), \dot{x}^\mu(s_2)$, etc. with suppressed Lorentz indices. The boundary contributions obviously vanish due to the periodicity of the curves. The expression on the previous page can be further simplified by noting that it is symmetric under the exchange $s_1 \leftrightarrow s_2$. Performing a change of variables $s_1 \leftrightarrow s_2$ in half of the terms leaves us with

$$\begin{aligned}
&\frac{2}{x_{12}^2}\Big[\Big(\delta^\mu_\nu\,\dot{x}_2^\rho + \dot{x}_2^\mu\,\dot{x}_{1\nu}\,\dot{x}_2^\rho\Big)\delta(s_1-\varepsilon-s_2) - \Big(\dot{x}_2^\mu\,\dot{x}_{1\nu}\,x_2^\rho + \delta^\mu_\nu\,x_2^\rho\Big)\partial_{s_1}\delta(s_1-\varepsilon-s_2)\\
&\qquad - \Big(\dot{x}_2^\mu\,\dot{x}_{2\nu} + \delta^\mu_\nu\Big)\Big(\dot{x}_2^\rho\,\delta(\varepsilon) + x_2^\rho\,\partial_\varepsilon\delta(\varepsilon)\Big)\Big]\\
&+\frac{4}{x_{12}^4}\Big[\dot{x}_{12\nu}\,x_{12}^\mu\Big(x_1^\rho\,\delta(\varepsilon) - x_2^\rho\,\delta(s_1-\varepsilon-s_2)\Big)\\
&\qquad - \dot{x}_{12}^\mu\,x_{12\nu}\Big(\dot{x}_2^\rho\Big(\theta(s_1-\varepsilon-s_2)-\theta(-\varepsilon)\Big) - x_2^\rho\Big(\delta(s_1-\varepsilon-s_2)-\delta(\varepsilon)\Big)\Big)\\
&\qquad - \Big(\dot{x}_1\cdot\dot{x}_2+1\Big)\delta^\mu_\nu\,x_1^\rho\Big(\theta(-\varepsilon)-\theta(s_2-\varepsilon-s_1)\Big)\Big]\\
&+\frac{16}{x_{12}^6}\Big[\Big(\dot{x}_1\cdot\dot{x}_2+1\Big)x_{12}^\mu\,x_{12\nu}\,x_1^\rho\Big(\theta(-\varepsilon)-\theta(s_2-\varepsilon-s_1)\Big)\Big].
\end{aligned}\tag{3.97}$$

Of course, these simplifications were only possible due to the suppressed Wilson loop integrals over s_1 and s_2. The next step is to contract this expression with the tensorial part of (3.93). We start by carrying out the contractions in the terms that are proportional to $2/x_{12}^2$.

$$\begin{aligned}
&\Big(\delta^\sigma_\rho\delta^\nu_\mu - \delta^\nu_\rho\delta^\sigma_\mu - \eta_{\rho\mu}\eta^{\nu\sigma}\Big)\frac{2}{x_{12}^2}\Big[\ldots\Big]\\
&= \frac{2}{x_{12}^2}\Big[\Big(2\,\dot{x}_2^\sigma + \dot{x}_1^\sigma\Big)\delta(s_1-\varepsilon-s_2) - \Big(\dot{x}_1\cdot\dot{x}_2\,x_2^\sigma - \dot{x}_1\cdot x_2\,\dot{x}_2^\sigma - \dot{x}_2\cdot x_2\,\dot{x}_1^\sigma + 2\,x_2^\sigma\Big)\partial_{s_1}\delta(s_1-\varepsilon-s_2)\\
&\qquad - \Big(\delta^\sigma_\rho - 2\,\dot{x}_2^\sigma\,\dot{x}_{2\rho}\Big)\Big(\dot{x}_2^\rho\,\delta(\varepsilon) + x_2^\rho\,\partial_\varepsilon\,\delta(\varepsilon)\Big)\Big]
\end{aligned}\tag{3.98}$$

Collecting all terms and adding the same terms with $\varepsilon \to -\varepsilon$ (cf. (3.93)) yields

$$\begin{aligned}
(A) = \frac{2}{x_{12}^2}\Big[&\Big(2\,\dot{x}_2^\sigma + \dot{x}_1^\sigma\Big)\Big(\delta(s_1-\varepsilon-s_2)+\delta(s_1+\varepsilon-s_2)\Big) - 6\,\dot{x}_2^\sigma\,\delta(\varepsilon)\\
&- \Big(\dot{x}_1\cdot\dot{x}_2\,x_2^\sigma - \dot{x}_1\cdot x_2\,\dot{x}_2^\sigma - \dot{x}_2\cdot x_2\,\dot{x}_1^\sigma + 2\,x_2^\sigma\Big)\partial_\varepsilon\Big(\delta(s_1+\varepsilon-s_2)-\delta(s_1-\varepsilon-s_2)\Big)\Big].
\end{aligned}$$

Next we carry out the contractions in the terms that are proportional to $4/x_{12}^4$.

$$\begin{aligned}
&\Big(\delta^\sigma_\rho\delta^\nu_\mu - \delta^\nu_\rho\delta^\sigma_\mu - \eta_{\rho\mu}\eta^{\nu\sigma}\Big)\frac{4}{x_{12}^4}\Big[\ldots\Big]\\
&= \frac{4}{x_{12}^4}\Big[\dot{x}_{12}\cdot x_{12}\Big(x_1^\sigma\,\delta(\varepsilon) - x_2^\sigma\,\delta(s_1-\varepsilon-s_2)\Big) - x_{12}^\sigma\Big(\dot{x}_{12}\cdot x_1\,\delta(\varepsilon) - \dot{x}_{12}\cdot x_2\,\delta(s_1-\varepsilon-s_2)\Big)\\
&\qquad - \dot{x}_{12}^\sigma\Big(x_{12}\cdot x_1\,\delta(\varepsilon) - x_{12}\cdot x_2\,\delta(s_1-\varepsilon-s_2)\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\dot{x}_{12}\cdot x_{12}\Big(\dot{x}_2^\sigma\Big(\theta(s_1-\varepsilon-s_2)-\theta(-\varepsilon)\Big)-x_2^\sigma\Big(\delta(s_1-\varepsilon-s_2)-\delta(\varepsilon)\Big)\Big)\\
&+\dot{x}_{12}^\sigma\Big(x_{12}\cdot\dot{x}_2\Big(\theta(s_1-\varepsilon-s_2)-\theta(-\varepsilon)\Big)-x_{12}\cdot x_2\Big(\delta(s_1-\varepsilon-s_2)-\delta(\varepsilon)\Big)\Big)\\
&+x_{12}^\sigma\Big(\dot{x}_{12}\cdot\dot{x}_2\Big(\theta(s_1-\varepsilon-s_2)-\theta(-\varepsilon)\Big)-\dot{x}_{12}\cdot x_2\Big(\delta(s_1-\varepsilon-s_2)-\delta(\varepsilon)\Big)\Big)\\
&-2\Big(\dot{x}_1\cdot\dot{x}_2+1\Big)x_1^\sigma\Big(\theta(-\varepsilon)-\theta(s_2-\varepsilon-s_1)\Big)\Big]
\end{aligned}
\tag{3.99}
$$

At this point, we note that due to the suppressed Wilson Loop integrals over s_1 and s_2, which allow us to perform a change of variables, the contributions proportional to $\delta(\varepsilon)$ drop out. Thus we are left with

$$
\begin{aligned}
&\frac{4}{x_{12}^4}\Big[\Big(x_{12}^\sigma\,\dot{x}_{12}\cdot\dot{x}_2+\dot{x}_{12}^\sigma\,x_{12}\cdot\dot{x}_2-\dot{x}_{12}\cdot x_{12}\,\dot{x}_2^\sigma\Big)\Big(\theta(s_1-\varepsilon-s_2)-\theta(-\varepsilon)\Big)\\
&\qquad-2\Big(\dot{x}_1\cdot\dot{x}_2+1\Big)x_1^\sigma\Big(\theta(-\varepsilon)-\theta(s_2-\varepsilon-s_1)\Big)\Big].
\end{aligned}
\tag{3.100}
$$

If we now add the part where $\varepsilon\to-\varepsilon$ and set $\theta(\varepsilon)=1$ as well as $\theta(-\varepsilon)=0$, we find

$$
(B)=\frac{4}{x_{12}^4}\Big[\Big(\dot{x}_1^\sigma\,x_{12}\cdot\dot{x}_2-\dot{x}_2^\sigma\,x_{12}\cdot\dot{x}_1-\Big(x_1^\sigma+x_2^\sigma\Big)\Big(\dot{x}_1\cdot\dot{x}_2+1\Big)\Big)\Big(\theta(s_1-\varepsilon-s_2)-\theta(s_2-\varepsilon-s_1)\Big)\Big],
$$

where we employed the identity (3.91). Due to the antisymmetry under exchange of s_1 and s_2, the last expression integrates to zero, i.e.

$$
\int_0^L ds_1\,ds_2\,\Big(\delta_\rho^\sigma\delta_\mu^\nu-\delta_\rho^\nu\delta_\mu^\sigma-\eta_{\rho\mu}\eta^{\nu\sigma}\Big)\frac{4}{x_{12}^4}\Big[...\Big]=0\,. \tag{3.101}
$$

Finally, we carry out the contractions in the terms that are proportional to $16/x_{12}^6$.

$$
\begin{aligned}
&\Big(\delta_\rho^\sigma\delta_\mu^\nu-\delta_\rho^\nu\delta_\mu^\sigma-\eta_{\rho\mu}\eta^{\nu\sigma}\Big)\frac{16}{x_{12}^6}\Big[...\Big]\\
&=\frac{16}{x_{12}^6}\Big[\Big(\dot{x}_1\cdot\dot{x}_2+1\Big)x_{12}^2\,x_1^\sigma\Big(\theta(-\varepsilon)-\theta(s_2-\varepsilon-s_1)\Big)\\
&\qquad-2\Big(\dot{x}_1\cdot\dot{x}_2+1\Big)x_{12}^\sigma\,x_1\cdot x_{12}\Big(\theta(-\varepsilon)-\theta(s_2-\varepsilon-s_1)\Big)\Big]
\end{aligned}
\tag{3.102}
$$

Again, we add the part where $\varepsilon\to-\varepsilon$. After some short manipulations, which are similar to the ones we did before, we find

$$
(C)=-\frac{16}{x_{12}^4}\Big(\dot{x}_1\cdot\dot{x}_2+1\Big)x_1^\sigma\Big(\theta(s_1-\varepsilon-s_2)-\theta(s_2-\varepsilon-s_1)\Big). \tag{3.103}
$$

Adding up (A), (B) and (C) leads to the first result for the action of our non-local level-one generator (3.93) on the one-loop expectation value $\langle W(C)\rangle_{(1)}$.

$$
\begin{aligned}
P^{(1)\,\mu}_{\mathrm{nl,\,bos,}\,\varepsilon}\,\langle W(C)\rangle_{(1)} &= -\frac{\lambda}{128\pi^2}\int_0^L \mathrm{d}s_1\,\mathrm{d}s_2\,\Big((A)+(B)+(C)\Big) \\
&= -\frac{\lambda}{128\pi^2}\int_0^L \mathrm{d}s_1\,\mathrm{d}s_2\,\Bigg\{\frac{1}{x_{12}^2}\Bigg[6\,\dot{x}_1^\mu\Big(\delta(s_1-\varepsilon-s_2)+\delta(s_1+\varepsilon-s_2)\Big) \\
&\quad -6\Big(\dot{x}_1^\mu+\dot{x}_2^\mu\Big)\delta(\varepsilon)+\Big(2\,x_{12}^\mu+\dot{x}_2\cdot\dot{x}_1\,x_1^\mu-\dot{x}_2\cdot x_1\,\dot{x}_1^\mu-\dot{x}_1\cdot x_1\,\dot{x}_2^\mu-\dot{x}_1\cdot\dot{x}_2\,x_2^\mu \\
&\quad +\dot{x}_1\cdot x_2\,\dot{x}_2^\mu+\dot{x}_2\cdot x_2\,\dot{x}_1^\mu\Big)\,\partial_\varepsilon\Big(\delta(s_1+\varepsilon-s_2)-\delta(s_1-\varepsilon-s_2)\Big)\Bigg] \\
&\quad +\frac{16}{x_{12}^4}\Big(\dot{x}_1\cdot\dot{x}_2+1\Big)x_{12}^\mu\,\theta(s_2-\varepsilon-s_1)\Bigg\}\,,
\end{aligned}
\tag{3.104}
$$

where this concrete form of the result has been obtained by exploiting the freedom to perform changes of integration variables. The next step will be to integrate out the delta functions and expand the result in ε, which will reveal the structure of the potential divergences. Since the curve under consideration is parametrized by arc length, we note that the following relations hold true

$$
\dot{x}^2=-1 \qquad \dot{x}\cdot\ddot{x}=0 \qquad \ddot{x}^2=-\dot{x}\cdot x^{(3)}\,. \tag{3.105}
$$

To begin with, we calculate the epsilon expansion of the denominator after the delta function has been integrated out.

$$
\frac{1}{D}=\frac{1}{(x_1-x(s_1+\varepsilon))^2}=\frac{1}{\Big(\varepsilon\,\dot{x}_1+\frac{\varepsilon^2}{2}\,\ddot{x}_1+\frac{\varepsilon^3}{6}\,x_1^{(3)}+\mathcal{O}(\varepsilon^4)\Big)^2}=-\frac{1}{\varepsilon^2}\left(1-\frac{\varepsilon^2}{12}\,\ddot{x}_1^2+\mathcal{O}(\varepsilon^3)\right) \tag{3.106}
$$

Let us now expand the following integral

$$
\begin{aligned}
\partial_\varepsilon\int_0^L \mathrm{d}s_1\,\mathrm{d}s_2\,\frac{1}{x_{12}^2}\Bigg[&2\,x_{12}^\mu-\dot{x}_1\cdot x_1\,\dot{x}_2^\mu-\dot{x}_2\cdot x_1\,\dot{x}_1^\mu+\dot{x}_2\cdot\dot{x}_1\,x_1^\mu \\
&+\dot{x}_2\cdot x_2\,\dot{x}_1^\mu+\dot{x}_1\cdot x_2\,\dot{x}_2^\mu-\dot{x}_1\cdot\dot{x}_2\,x_2^\mu\Bigg]\,\delta(s_1+\varepsilon-s_2)\,.
\end{aligned}
\tag{3.107}
$$

If we perform the s_2 integration and plug in the epsilon expansion for all terms in the first line and for the undotted terms in the second line, some terms immediately drop out and we are left with

$$
\begin{aligned}
\partial_\varepsilon\int_0^L \mathrm{d}s_1\,\frac{1}{D}\Bigg[&-2\left(\varepsilon\,\dot{x}_1^\mu+\frac{\varepsilon^2}{2}\,\ddot{x}_1^\mu+\frac{\varepsilon^3}{6}\,x_1^{(3)\mu}\right)+\dot{x}(s_1+\varepsilon)\cdot\left(\varepsilon\,\dot{x}_1+\frac{\varepsilon^2}{2}\,\ddot{x}_1+\frac{\varepsilon^3}{6}\,x_1^{(3)}\right)\dot{x}_1^\mu+\mathcal{O}(\varepsilon^4) \\
&+\dot{x}_1\cdot\left(\varepsilon\,\dot{x}_1+\frac{\varepsilon^2}{2}\,\ddot{x}_1+\frac{\varepsilon^3}{6}\,x_1^{(3)}\right)\dot{x}^\mu(s_1+\varepsilon)-\dot{x}_1\cdot\dot{x}(s_1+\varepsilon)\left(\varepsilon\,\dot{x}_1^\mu+\frac{\varepsilon^2}{2}\,\ddot{x}_1^\mu+\frac{\varepsilon^3}{6}\,x_1^{(3)\mu}\right)\Bigg].
\end{aligned}
$$

After plugging in the expansion for the remaining terms and using the identities (3.105), this expression reduces to

$$
\partial_\varepsilon\int_0^L \mathrm{d}s_1\,\frac{1}{\varepsilon^2}\left(3\,\varepsilon\,\dot{x}_1^\mu+\frac{3}{2}\,\varepsilon^2\,\ddot{x}_1^\mu+\frac{2}{3}\,\varepsilon^3\,x_1^{(3)\mu}-\frac{5}{12}\,\varepsilon^3\,\ddot{x}_1^2\,\dot{x}_1^\mu+\mathcal{O}(\varepsilon^4)\right). \tag{3.108}
$$

We note that the first three terms integrate to zero. Together with the part where $\varepsilon \to -\varepsilon$ we find the following expansion

$$\begin{aligned}\partial_\varepsilon \int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, \frac{1}{x_{12}^2} \Big[2\, x_{12}^\mu - \dot{x}_1 \cdot x_1\, \dot{x}_2^\mu - \dot{x}_2 \cdot x_1\, \dot{x}_1^\mu + \dot{x}_2 \cdot \dot{x}_1\, x_1^\mu + \dot{x}_2 \cdot x_2\, \dot{x}_1^\mu + \dot{x}_1 \cdot x_2\, \dot{x}_2^\mu \\ - \dot{x}_1 \cdot \dot{x}_2\, x_2^\mu \Big] \Big(\delta(s_1 + \varepsilon - s_2) - \delta(s_1 - \varepsilon - s_2) \Big) \\ = \int_0^L \mathrm{d}s_1 \left(-\frac{5}{6}\, \ddot{x}_1^2\, \dot{x}_1^\mu + \mathcal{O}(\varepsilon) \right) . \end{aligned} \tag{3.109}$$

The expansion of the integral which contains delta functions but no derivative with respect to ε is easily done

$$\begin{aligned}\int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, \frac{6\, \dot{x}_1^\mu}{x_{12}^2} \Big(\delta(s_1 + \varepsilon - s_2) + \delta(s_1 - \varepsilon - s_2) \Big) \\ = \int_0^L \mathrm{d}s_1 \left(\ddot{x}_1^2\, \dot{x}_1^\mu + \mathcal{O}(\varepsilon) \right) . \end{aligned} \tag{3.110}$$

Inserting (3.109) and (3.110) into (3.104) yields

$$\begin{aligned} P^{(1)\,\mu}_{\mathrm{nl,\,bos,}\,\varepsilon} \langle W(C) \rangle_{(1)} = -\frac{\lambda}{128\pi^2} \Bigg\{ \frac{1}{6} \int_0^L \mathrm{d}s\, \ddot{x}^2\, \dot{x}^\mu - 6 \int_0^L \mathrm{d}s_1\, \mathrm{d}s_2\, \frac{\dot{x}_1^\mu + \dot{x}_2^\mu}{x_{12}^2}\, \delta(\varepsilon) \\ + 16 \int_0^L \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\dot{x}_1 \cdot \dot{x}_2 + 1}{x_{12}^4}\, x_{12}^\mu\, \theta(s_2 - s_1 - \varepsilon) + \mathcal{O}(\varepsilon) \Bigg\} . \end{aligned} \tag{3.111}$$

In a general parametrization $\delta(\varepsilon)$ corresponds to $\delta(d(s_1, \varepsilon))$, where $d(s_1, \varepsilon)$ is a function that depends on the curve parameter as well as on the regulating distance ε. By construction this function is positive on its whole support for any finite value of ε. Since

$$\delta(d(s_1, \varepsilon)) = 0 \qquad \text{for} \quad \varepsilon \in \mathbb{R}^+ \tag{3.112}$$

holds true as an identity for delta functions, we will drop the $\delta(\varepsilon)$-terms from now on. Given the result in arc length parametrization, it is natural to ask whether one can rewrite it in a reparametrization invariant form. While the arc length constraint can easily be lifted in the bi-local expression, it is less obvious how to rewrite the local term as a proper curve integral. The easiest way to do this is by noting that for a unit speed curve $\ddot{x}^2$ describes the square of its local scalar curvature $\kappa = |\ddot{x}|$. A reparametrization invariant expression for this quantity can be found in any book on elementary differential geometry and is given by

$$\kappa^2 = \frac{\dot{x}^2 \ddot{x}^2 - (\dot{x} \cdot \ddot{x})^2}{\dot{x}^6} . \tag{3.113}$$

With the help of this formula, (3.111) can be rewritten as follows

$$P^{(1)\,\mu}_{\text{nl, bos}, \varepsilon}\,\langle W(C)\rangle_{(1)} = -\frac{\lambda}{128\pi^2}\Bigg\{\frac{1}{6}\int \mathrm{d}s\left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x}\cdot\ddot{x})^2}{\dot{x}^6}\right)\dot{x}^\mu + \mathcal{O}(\varepsilon) \\ + 16\int \mathrm{d}s_1\,\mathrm{d}s_2\,\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{x_{12}^4}\,x_{12}^\mu\,\theta(s_2 - d(s_2,\varepsilon) - s_1)\Bigg\}\,. \tag{3.114}$$

This expression is now valid for any parametrization. Having acted on the one-loop expectation value with our non-local generator, we see that two different terms remain. Let us first focus on the unproblematic one, i.e. the local term. Its appearance is not unexpected since we only took into account the canonical non-local piece of the level-one generator. In fact, as we will argue in more detail at the end of the next chapter, we can cope with this term by defining the complete generator as follows

$$P^{(1)\,\mu}_{\text{nl, bos}} := P^{(1)\,\mu}_{\text{nl, bos}} + c\,\lambda\int \mathrm{d}s\,\left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x}\cdot\ddot{x})^2}{\dot{x}^6}\right)\dot{x}^\mu\,. \tag{3.115}$$

The constant c is chosen such that the shift term cancels the local contribution in (3.114) when the complete generator is applied to the full expectation value

$$\langle W(C)\rangle = 1 + \langle W(C)\rangle_{(1)} + \mathcal{O}(\lambda^2)\,. \tag{3.116}$$

However, the presence of a bi-local term in (3.114) suggests that it is not consistent to restrict to the bosonic subalgebra $\mathfrak{so}(2,4)$. Instead, it seems very likely that one really has to consider the full $\mathfrak{psu}(2,2|4)$ as the underlying level-zero algebra. From the discussion of Yangian symmetries of scattering amplitudes in section 2.5.2 we know that in this case the non-local part of the level-one momentum generator gets an additional contribution of the form

$$\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\Big(\theta(\tau_2-\tau_1) - \theta(\tau_1-\tau_2)\Big)\,\bar{q}^{A\dot{\alpha}}(\tau_1)\,\bar{\sigma}^\mu_{\alpha\dot{\alpha}}\,q^\alpha_A(\tau_2)\,, \tag{3.117}$$

where $\bar{q}^{A\,\dot{\alpha}}(\tau_1)$ and $q^\alpha_A(\tau_2)$ are the generator densities of supertranslations to be defined in a moment. It will turn out that the inclusion of supercharges makes it necessary to consider a generalized Maldacena-Wilson loop operator that depends on a path in superspace and couples to all the fields of $\mathcal{N}=4$ SYM theory. However, since the functional form of the bi-local term is that of a fermion propagator, this seems to be the right way.

4. Supersymmetric Completion of the Maldacena-Wilson Loop Operator

In this chapter we will first introduce a superspace representation of the symmetry algebra of $\mathcal{N} = 4$ SYM theory and subsequently discuss how the bosonic level-one momentum generator gets modified when extending the level-zero algebra to the full superconformal algebra. In what follows we will then construct a generalized loop operator, whose body part $\theta = \bar{\theta} = 0$ agrees with the usual Maldacena-Wilson loop while involving all the fields of $\mathcal{N} = 4$ SYM theory and depending on a path in a non-chiral superspace. More specifically, we will derive the top components of the soul part by demanding that the expectation value of the generalized operator be invariant under supersymmetry transformations. This construction principle makes it natural to think of this generalized loop operator as the supersymmetrically completed Maldacena-Wilson loop. Having established the concrete form of the operator, we will then compute its expectation value to one-loop order and explicitly demonstrate that it is annihilated by the generators of supersymmetry transformations. Finally, we shall apply our modified level-one momentum generator to the one-loop expectation value of this new operator. This investigation will provide evidence that the supersymmetrically completed Maldacena-Wilson loop indeed possesses Yangian symmetries.

4.1. The Superconformal Algebra

In the last chapter we tried to establish the invariance of smooth Maldacena-Wilson loops under transformations generated by the bosonic level-one momentum generator $P^{(1)\,\mu}_{\mathrm{bos}} \in Y(\mathfrak{so}(2,4))$, but we found that such an invariance does not exist. This result is not surprising, because we only took into account a subalgebra of the complete symmetry algebra of the theory when constructing the non-local part of the level-one momentum generator. A natural approach is now to extend the level-zero algebra to the full $\mathfrak{psu}(2,2|4)$ and to investigate the question concerning the Yangian symmetries once again. For that purpose, we need a representation of the full superconformal algebra in terms of differential operators. Such a representation requires some type of superspace which, in addition to the usual bosonic coordinates, contains anticommuting (fermionic) coordinates. We already know how the Lie superalgebra $\mathfrak{psu}(2,2|4)$ can be represented on a chiral superspace coordinatized by $\{x^\mu, \theta^A_\alpha\}$, see section 2.5.2. However, for reasons that will become clear later on, we now need a representation on a non-chiral superspace with at least four bosonic coordinates $\{x^\mu\}$ and 16 fermionic coordinates $\{\theta^A_\alpha, \bar{\theta}_{A\dot{\alpha}}\}$. One might think that a representation of the algebra in question on this superspace can be obtained by taking the chiral representation (2.149) and simply adding the respective $\bar{\theta}$ terms. This approach, however, does not lead to a valid representation of $\mathfrak{psu}(2,2|4)$ because some algebra relations do not close. To solve this problem, we go to a slightly larger superspace which, compared to the non-chiral superspace mentioned above, has 16 additional bosonic coordinates ${y_A}^B$ labeled by two SU(4) indices A and B. A representation of (a slightly enlarged version) of our superconformal algebra on this superspace, coordinatized by $\{x^\mu, \theta^A_\alpha, \bar\theta_{A\dot\alpha}, {y_A}^B\}$, is given by

$$
\begin{aligned}
M_{\alpha\beta} &= 2\,i\,x_{\dot\gamma(\alpha}\,{\partial_{\beta)}}^{\dot\gamma} + 4\,i\,\theta^A_{(\alpha}\,\partial_{\beta)A} & \bar M_{\dot\alpha\dot\beta} &= 2\,i\,{x^\gamma}_{(\dot\alpha}\,\partial_{\dot\beta)\gamma} - 4\,i\,\bar\theta_{A(\dot\alpha}\,\partial^A_{\dot\beta)} \\
D &= \tfrac12\,x_{\alpha\dot\alpha}\,\partial^{\alpha\dot\alpha} + \tfrac12\,\theta^B_\beta\,\partial^\beta_B + \tfrac12\,\bar\theta_{B\dot\beta}\,\partial^{B\dot\beta} & P^{\alpha\dot\alpha} &= \partial^{\alpha\dot\alpha} \\
K_{\alpha\dot\alpha} &= -x_{\alpha\dot\gamma}\,x_{\dot\alpha\gamma}\,\partial^{\gamma\dot\gamma} - 2\,x_{\dot\alpha\gamma}\,\theta^C_\alpha\,\partial^\gamma_C - 2\,x_{\alpha\dot\gamma}\,\bar\theta_{C\dot\alpha}\,\partial^{C\dot\gamma} + 4\,i\,\theta^A_\alpha\,\bar\theta_{B\dot\alpha}\,{\partial_A}^B \\
Q^\alpha_A &= -\partial^\alpha_A + {y_A}^B\,\partial^\alpha_B + i\,\bar\theta_{A\dot\alpha}\,\partial^{\alpha\dot\alpha} & \bar Q^{A\dot\alpha} &= \partial^{A\dot\alpha} + {y_B}^A\,\partial^{B\dot\alpha} - i\,\theta^A_\alpha\,\partial^{\alpha\dot\alpha} \\
S^A_\alpha &= \left(\delta^A_B + {y_B}^A\right)\left(x_{\alpha\dot\gamma}\,\partial^{B\dot\gamma} + 2\,i\,\theta^C_\alpha\,{\partial_C}^B\right) - i\,x_{\alpha\dot\gamma}\,\theta^A_\beta\,\partial^{\beta\dot\gamma} - 2\,i\,\theta^A_\beta\,\theta^C_\alpha\,\partial^\beta_C \\
\bar S_{A\dot\alpha} &= \left(-\delta^B_A + {y_A}^B\right)\left(x_{\dot\alpha\gamma}\,\partial^\gamma_B - 2\,i\,\bar\theta_{D\dot\alpha}\,{\partial_B}^D\right) + i\,x_{\dot\alpha\gamma}\,\bar\theta_{A\dot\beta}\,\partial^{\gamma\dot\beta} + 2\,i\,\bar\theta_{A\dot\beta}\,\bar\theta_{C\dot\alpha}\,\partial^{C\dot\beta} \\
{R'^A}_B &= 2\,i\left(-\delta^D_B + {y_B}^D\right)\left(\delta^A_C + {y_C}^A\right){\partial_D}^C + 2\,i\left(-\delta^C_B + {y_B}^C\right)\theta^A_\gamma\,\partial^\gamma_C \\
&\quad + 2\,i\left(\delta^A_C + {y_C}^A\right)\bar\theta_{B\dot\alpha}\,\partial^{C\dot\alpha} + 2\,\bar\theta_{B\dot\alpha}\,\theta^A_\alpha\,\partial^{\alpha\dot\alpha} \\
{R^A}_B &= {R'^A}_B - \tfrac14\,\delta^A_B\,{R'^C}_C \\
C &= \tfrac14\left(\theta^D_\alpha\,\partial^\alpha_D - \bar\theta_{C\dot\alpha}\,\partial^{C\dot\alpha} + i\,\theta^A_\alpha\,\bar\theta_{A\dot\alpha}\,\partial^{\alpha\dot\alpha} - {\partial_A}^A \right. \\
&\quad \left. + {y_A}^B\,\theta^A_\alpha\,\partial^\alpha_B + {y_A}^B\,\bar\theta_{B\dot\alpha}\,\partial^{A\dot\alpha} + {y_A}^C\,{y_C}^B\,{\partial_B}^A\right),
\end{aligned}
\tag{4.1}
$$

where we have used the following shorthand notation

$$
\partial^{\alpha\dot\alpha} := \frac{\partial}{\partial x_{\alpha\dot\alpha}} \qquad \partial^{A\dot\alpha} := \frac{\partial}{\partial\bar\theta_{A\dot\alpha}} \qquad \partial^\alpha_A := \frac{\partial}{\partial\theta^A_\alpha} \qquad {\partial_B}^A := \frac{\partial}{\partial {y_A}^B}\,. \tag{4.2}
$$

While Grassmann derivatives act as defined in section 2.1.1, the ${y_A}^B$-derivatives act canonically, i.e.

$$
\frac{\partial {y_A}^B}{\partial {y_C}^D} = \delta^C_A\,\delta^B_D\,. \tag{4.3}
$$

The prefactors in the above given generators are chosen in such a way that the conformal generators agree, when restricted to pure Minkowski space, with the ones introduced in the last chapter. The conventions underlying the last statement are those stipulated in section 2.1.1. For the sake of completeness and to have them once and for all, let us now give a full list of all non-trivial (anti)commutation relations satisfied by the generators (4.1). The conformal dilatation generator satisfies commutation relations of the form

$$
\Big[D, J\Big] = \dim(J)\,J\,, \tag{4.4}
$$

where the non-vanishing dimensions are

$$
\dim(P) = -\dim(K) = -1 \qquad \dim(Q) = \dim(\bar Q) = -\frac12 \qquad \dim(S) = \dim(\bar S) = \frac12\,. \tag{4.5}
$$

The remaining part of the conformal algebra, written in bi-spinor notation, reads

$$
\begin{aligned}
\Big[M_{\alpha\beta}, M_{\gamma\delta}\Big] &= -2\,i\left(\varepsilon_{\alpha\gamma}\,M_{\beta\delta} + \varepsilon_{\alpha\delta}\,M_{\beta\gamma} + \varepsilon_{\beta\gamma}\,M_{\alpha\delta} + \varepsilon_{\beta\delta}\,M_{\alpha\gamma}\right) \\
\Big[\bar M_{\dot\alpha\dot\beta}, \bar M_{\dot\gamma\dot\delta}\Big] &= 2\,i\left(\varepsilon_{\dot\alpha\dot\gamma}\,\bar M_{\dot\beta\dot\delta} + \varepsilon_{\dot\alpha\dot\delta}\,\bar M_{\dot\beta\dot\gamma} + \varepsilon_{\dot\beta\dot\gamma}\,\bar M_{\dot\alpha\dot\delta} + \varepsilon_{\dot\beta\dot\delta}\,\bar M_{\dot\alpha\dot\gamma}\right)
\end{aligned}
$$

$$\begin{aligned}
\Big[M_{\alpha\beta}, P_{\gamma\dot\gamma}\Big] &= -2\,i\left(\varepsilon_{\alpha\gamma}\,P_{\beta\dot\gamma} + \varepsilon_{\beta\gamma}\,P_{\alpha\dot\gamma}\right) &\qquad \Big[M_{\alpha\beta}, K_{\gamma\dot\gamma}\Big] &= -2\,i\left(\varepsilon_{\alpha\gamma}\,K_{\beta\dot\gamma} + \varepsilon_{\beta\gamma}\,K_{\alpha\dot\gamma}\right)\\
\Big[\bar M_{\dot\alpha\dot\beta}, P_{\gamma\dot\gamma}\Big] &= -2\,i\left(\varepsilon_{\dot\alpha\dot\gamma}\,P_{\gamma\dot\beta} + \varepsilon_{\dot\beta\dot\gamma}\,P_{\gamma\dot\alpha}\right) &\qquad \Big[\bar M_{\dot\alpha\dot\beta}, K_{\gamma\dot\gamma}\Big] &= -2\,i\left(\varepsilon_{\dot\alpha\dot\gamma}\,K_{\gamma\dot\beta} + \varepsilon_{\dot\beta\dot\gamma}\,K_{\gamma\dot\alpha}\right)\\
\Big[P_{\alpha\dot\alpha}, K_{\beta\dot\beta}\Big] &= i\,\varepsilon_{\alpha\beta}\,\bar M_{\dot\alpha\dot\beta} + i\,\varepsilon_{\dot\alpha\dot\beta}\,M_{\alpha\beta} + 4\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,D\,.
\end{aligned} \tag{4.6}$$

The non-zero commutators involving the generators of translations $P_{\alpha\dot\alpha}$, conformal boosts $K_{\alpha\dot\alpha}$ and their fermionic partners, i.e. the generators of supertranslations Q^α_A, $\bar Q^{A\dot\alpha}$ and superboosts S^A_α, $\bar S_{A\dot\alpha}$, are given by

$$\begin{aligned}
\Big\{Q^\alpha_A, \bar Q^{B\dot\alpha}\Big\} &= 2\,i\,\delta^B_A\,P^{\alpha\dot\alpha} &\qquad \Big\{S^A_\alpha, \bar S_{B\dot\alpha}\Big\} &= -2\,i\,\delta^A_B\,K_{\alpha\dot\alpha}\\
\Big[P^{\alpha\dot\alpha}, S^A_\beta\Big] &= 2\,\delta^\alpha_\beta\,\bar Q^{A\dot\alpha} &\qquad \Big[P^{\alpha\dot\alpha}, \bar S_{A\dot\beta}\Big] &= 2\,\delta^{\dot\alpha}_{\dot\beta}\,Q^\alpha_A\\
\Big[K_{\alpha\dot\alpha}, Q^\beta_A\Big] &= 2\,\delta^\beta_\alpha\,\bar S_{A\dot\alpha} &\qquad \Big[K_{\alpha\dot\alpha}, \bar Q^{A\dot\beta}\Big] &= 2\,\delta^{\dot\beta}_{\dot\alpha}\,S^A_\alpha\\
\Big\{Q^\alpha_A, S^B_\beta\Big\} &= \delta^B_A\,M^\alpha{}_\beta + \delta^\alpha_\beta\,R^B{}_A + 2\,i\,\delta^B_A\,\delta^\alpha_\beta\,(D+C)\\
\Big\{\bar Q^{A\dot\alpha}, \bar S_{B\dot\beta}\Big\} &= -\delta^A_B\,\bar M^{\dot\alpha}{}_{\dot\beta} - \delta^{\dot\alpha}_{\dot\beta}\,R^A{}_B + 2\,i\,\delta^A_B\,\delta^{\dot\alpha}_{\dot\beta}\,(D-C)\,.
\end{aligned} \tag{4.7}$$

The Lorentz generators $M_{\alpha\beta}$ and $\bar M_{\dot\alpha\dot\beta}$ act on the generators of supertranslations Q^α_A, $\bar Q^{A\dot\alpha}$ and superboosts S^A_α, $\bar S_{A\dot\alpha}$ as follows

$$\begin{aligned}
\Big[M_{\alpha\beta}, S^A_\gamma\Big] &= -2\,i\left(\varepsilon_{\alpha\gamma}\,S^A_\beta + \varepsilon_{\beta\gamma}\,S^A_\alpha\right) &\qquad \Big[\bar M_{\dot\alpha\dot\beta}, \bar S_{A\dot\gamma}\Big] &= -2\,i\left(\varepsilon_{\dot\alpha\dot\gamma}\,\bar S_{A\dot\beta} + \varepsilon_{\dot\beta\dot\gamma}\,\bar S_{A\dot\alpha}\right)\\
\Big[M_{\alpha\beta}, Q_{A\gamma}\Big] &= -2\,i\left(\varepsilon_{\alpha\gamma}\,Q_{A\beta} + \varepsilon_{\beta\gamma}\,Q_{A\alpha}\right) &\qquad \Big[\bar M_{\dot\alpha\dot\beta}, \bar Q^A_{\dot\gamma}\Big] &= -2\,i\left(\varepsilon_{\dot\alpha\dot\gamma}\,\bar Q^A_{\dot\beta} + \varepsilon_{\dot\beta\dot\gamma}\,\bar Q^A_{\dot\alpha}\right).
\end{aligned} \tag{4.8}$$

Finally, we list all the commutators involving generators of R-symmetry transformations $R^A{}_B$.

$$\begin{aligned}
\Big[R^A{}_B, Q^\alpha_C\Big] &= 4\,i\,\delta^A_C\,Q^\alpha_B - i\,\delta^A_B\,Q^\alpha_C &\qquad \Big[R^A{}_B, \bar Q^{C\dot\alpha}\Big] &= -4\,i\,\delta^C_B\,\bar Q^{A\dot\alpha} - i\,\delta^A_B\,\bar Q^{C\dot\alpha}\\
\Big[R^A{}_B, \bar S_{C\dot\alpha}\Big] &= 4\,i\,\delta^A_C\,\bar S_{B\dot\alpha} - i\,\delta^A_B\,\bar S_{C\dot\alpha} &\qquad \Big[R^A{}_B, S^C_\alpha\Big] &= -4\,i\,\delta^C_B\,S^A_\alpha - i\,\delta^A_B\,S^C_\alpha\\
\Big[R^A{}_B, R^C{}_D\Big] &= 4\,i\,\delta^A_D\,R^C{}_B - 4\,i\,\delta^C_B\,R^A{}_D
\end{aligned} \tag{4.9}$$

4.1.1. The Full Level-One Generator

Having obtained an appropriate representation of the level-zero algebra as well as all algebra relations including the right factors, we can now derive the non-local part of the full level-one momentum generator. To do so, we could in principal use same approach as in section 3.5.1 and compute all the relevant dual structure constants of $\mathfrak{su}(2,2|4)$. However, since the conformal subalgebra of the above mentioned $\mathfrak{su}(2,2|4)$ algebra agrees with the conformal algebra introduced in the last chapter, nothing will change in the $\mathfrak{so}(2,4)$ part of the level-one generator, except that generator densities $m^{\mu\nu}(\tau)$ and $d(\tau)$ will now also contain θ- and $\bar\theta$-extensions. From the discussion in section 3.5.1, we know that the only new term that will arise is one that is quadratic in the generator densities of supertranslations.

Thus, we can directly start with the following ansatz

$$P_{\mathrm{nl}}^{(1)\,\mu} = \frac{1}{8}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\theta(\tau_2-\tau_1)\,\Big(\Big(m^{\mu\nu}(\tau_1) - d(\tau_1)\,\eta^{\mu\nu}\Big)p_\nu(\tau_2) \\ + c\,\bar{q}^{A\dot\alpha}(\tau_1)\,\bar\sigma^{\mu}_{\alpha\dot\alpha}\,q^\alpha_A(\tau_2) - (\tau_1\leftrightarrow\tau_2)\Big)\,, \tag{4.10}$$

where $q^\alpha_A(\tau_1)$ and $\bar{q}^{A\dot\alpha}(\tau_2)$ are the generator densities of supersymmetry transformations which explicitly read

$$\begin{aligned} Q^\alpha_A &= \int \mathrm{d}\tau\, q^\alpha_A(\tau) \qquad & q^\alpha_A(\tau) &= -\frac{\delta}{\delta\theta^A_\alpha(\tau)} + i\,\bar\theta_{A\dot\alpha}(\tau)\,\frac{\delta}{\delta x_{\alpha\dot\alpha}(\tau)} + y_A{}^B(\tau)\,\frac{\delta}{\delta\theta^B_\alpha(\tau)} \\ \bar{Q}^{A\dot\alpha} &= \int \mathrm{d}\tau\,\bar{q}^{A\dot\alpha}(\tau) \qquad & \bar{q}^{A\dot\alpha}(\tau) &= +\frac{\delta}{\delta\bar\theta_{A\dot\alpha}(\tau)} - i\,\theta^A_\alpha(\tau)\,\frac{\delta}{\delta x_{\alpha\dot\alpha}(\tau)} + y_B{}^A(\tau)\,\frac{\delta}{\delta\bar\theta_{B\dot\alpha}(\tau)}\,. \end{aligned} \tag{4.11}$$

Now, before elaborating a little bit on the question what the underlying space is on which the above given generators act on, let us fix the coefficient c in (4.10), using the algebra relations (4.4)-(4.9). We will fix c by requiring that the following commutator vanishes

$$\Big[P_{\mathrm{nl}}^{(1)\,\mu}, Q^\alpha_A\Big] \stackrel{!}{=} 0\,, \tag{4.12}$$

which has to be the case according to the definition of the Yangian (2.114). To begin with, let us derive an expression for the commutator between the generators of Lorentz transformations $M^{\mu\nu}$ and those of supertranslations Q^α_A. Using the relations (4.6), we find

$$\begin{aligned} \Big[M^{\mu\nu}, Q^\alpha_A\Big] &= \tfrac{1}{4}\,\bar\sigma^\mu_{\gamma\dot\gamma}\,\bar\sigma^\nu_{\delta\dot\delta}\,\Big[M^{\gamma\dot\gamma\delta\dot\delta}, Q^\alpha_A\Big] \\ &= \tfrac{i}{8}\,\bar\sigma^\mu_{\gamma\dot\gamma}\,\bar\sigma^\nu_{\delta\dot\delta}\,\varepsilon^{\dot\gamma\dot\delta}\Big[M^{\gamma\delta}, Q^\alpha_A\Big] \\ &= \tfrac{1}{4}\Big(\bar\sigma^{\mu\alpha}{}_{\dot\gamma}\,\bar\sigma^{\nu}{}_{\gamma}{}^{\dot\gamma}\,Q^\gamma_A + \bar\sigma^\mu_{\gamma\dot\gamma}\,\bar\sigma^{\nu\alpha\dot\gamma}\,Q^\gamma_A\Big)\,. \end{aligned} \tag{4.13}$$

Substituting (4.10) into (4.12) and employing the product rule for commutators yields

$$\Big[P_{\mathrm{nl}}^{(1)\,\mu}, Q^\alpha_A\Big] = \frac{1}{8}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau\,\theta(\tau_2-\tau_1)\,\Big(\Big[m^{\mu\nu}(\tau_1), q^\alpha_A(\tau)\Big]p_\nu(\tau_2) - \Big[d(\tau_1), q^\alpha_A(\tau)\Big]p^\mu(\tau_2) \\ - c\,\bar\sigma^\mu_{\beta\dot\beta}\,\Big\{\bar{q}^{B\dot\beta}(\tau_1), q^\alpha_A(\tau)\Big\}\,q^\beta_B(\tau_2) - \big(\tau_1\leftrightarrow\tau_2\big)\Big)\,. \tag{4.14}$$

After plugging in explicit expressions for the commutators and anticommutators and integrating out the delta function, we obtain

$$\Big[\,,\Big] = \frac{1}{8}\int \mathrm{d}\tau_1\int \mathrm{d}\tau_2\,\theta(\tau_2-\tau_1)\,\Big(\tfrac{1}{4}\Big(\bar\sigma^{\mu\alpha}{}_{\dot\gamma}\,\bar\sigma^{\nu}{}_{\gamma}{}^{\dot\gamma}\,q^\gamma_A(\tau_1) + \bar\sigma^\mu_{\gamma\dot\gamma}\,\bar\sigma^{\nu\alpha\dot\gamma}\,q^\gamma_A(\tau_1)\Big)\,p_\nu(\tau_2) \\ + \tfrac{1}{2}\,q^\alpha_A(\tau_1)\,p^\mu(\tau_2) - 2\,i\,c\,\bar\sigma^\mu_{\beta\dot\beta}\,p^{\alpha\dot\beta}(\tau_1)\,q^\beta_A(\tau_2) - \big(\tau_1\leftrightarrow\tau_2\big)\Big)\,. \tag{4.15}$$

From now on we will only focus on the integrand of the above given expression. Substituting $p^\mu(\tau_2) = 1/2\,\bar\sigma^\mu_{\gamma\dot\gamma}\,p^{\gamma\dot\gamma}(\tau_2)$ in the third term, we get

$$\tfrac{1}{4}\,\bar\sigma^{\mu\alpha}{}_{\dot\gamma}\,q^\delta_A(\tau_1)\,p_\delta{}^{\dot\gamma}(\tau_2) + \tfrac{1}{4}\,\bar\sigma^\mu_{\gamma\dot\gamma}\,\Big(q^\gamma_A(\tau_1)\,p^{\alpha\dot\gamma}(\tau_2) + q^\alpha_A(\tau_1)\,p^{\gamma\dot\gamma}(\tau_2)\Big) - 2\,i\,c\,\bar\sigma^\mu_{\gamma\dot\gamma}\,p^{\alpha\dot\gamma}(\tau_1)\,q^\gamma_A(\tau_2) - \big(\tau_1\leftrightarrow\tau_2\big)\,.$$

The last expression can easily be rewritten as follows

$$\tfrac{1}{2}\Big(\bar\sigma^{\mu}_{\gamma\dot\gamma}\, q_A^{(\gamma}(\tau_1)\, p^{\alpha)\dot\gamma}(\tau_2) - \tfrac{1}{2}\,\varepsilon^{\gamma\alpha}\,\bar\sigma^{\mu}_{\gamma\dot\gamma}\, q_A^{\delta}(\tau_1)\, p_{\delta}{}^{\dot\gamma}(\tau_2)\Big) + 2\,i\,c\,\bar\sigma^{\mu}_{\gamma\dot\gamma}\, q_A^{\gamma}(\tau_1)\, p^{\alpha\dot\gamma}(\tau_2) - \big(\tau_1 \leftrightarrow \tau_2\big)\,, \tag{4.16}$$

where we have used the fact that the generators of translations commute with those of supertranslations. Using identity (2.26), we find

$$\big(\tfrac{1}{2} + 2\,i\,c\big)\,\bar\sigma^{\mu}_{\gamma\dot\gamma}\, q_A^{\gamma}(\tau_1)\, p^{\alpha\dot\gamma}(\tau_2) - \big(\tau_1 \leftrightarrow \tau_2\big)\,. \tag{4.17}$$

From this expression we conclude that $c = i/4$. The full regularized level-one momentum generator is thus given by

$$\begin{aligned} P^{(1)\,\mu}_{\mathrm{nl},\,\varepsilon} = \frac{1}{8}\int_0^L \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\bigg(\Big(m^{\mu\nu}(\tau_1) - d(\tau_1)\,\eta^{\mu\nu}\Big)p_\nu(\tau_2) + \frac{i}{4}\,\bar q^{A\dot\alpha}(\tau_1)\,\bar\sigma^{\mu}_{\alpha\dot\alpha}\, q_A^{\alpha}(\tau_2)\bigg) \\ \times\Big(\theta(\tau_2 - \tau_1 - \varepsilon) - \theta(\tau_1 - \tau_2 - \varepsilon)\Big)\,. \end{aligned} \tag{4.18}$$

We note that in contrast to the bosonic level-one momentum generator (3.87), which acts on the space of bosonic curves $x^\mu(s)$, the full level-one momentum generator (4.18) acts on the space of supercurves parametrized by

$$x_{\alpha\dot\alpha}(s) = \bar\sigma^{\mu}_{\alpha\dot\alpha}\, x_\mu(s)\,, \quad \theta^A_\alpha(s)\,, \quad \bar\theta_{A\dot\alpha}(s)\,, \quad y_A{}^B(s)\,. \tag{4.19}$$

In view of the result for the action of the bosonic level-one generator on the expectation value of the Maldacena-Wilson loop operator (3.114), we are primarily interested in that part of the full generator that will give rise to corrections to the bosonic result when applied to the expectation value of the supersymmetrically completed Maldacena-Wilson loop operator to be defined in a moment. One easily convinces oneself that the only piece of (4.18) that can lead to corrections to (3.114) is that which contains the product of the two first terms of the generator densities of supertranslations. For later computational convenience let us introduce the following notation

$$P^{(1)\,\mu}_{\mathrm{nl},\,\varepsilon} = P^{(1)\,\mu}_{\mathrm{nl,\,bos},\,\varepsilon} + P^{(1)\,\mu}_{\mathrm{nl,\,ferm},\,\varepsilon} + \mathcal{O}(y,\theta,\bar\theta)\,, \tag{4.20}$$

where $P^{(1)\,\mu}_{\mathrm{nl,\,bos},\,\varepsilon}$ is the bosonic generator previously defined and $P^{(1)\,\mu}_{\mathrm{nl,\,ferm},\,\varepsilon}$ is the piece of (4.18) that will lead to corrections to the bosonic result (3.114). Explicitly, $P^{(1)\,\mu}_{\mathrm{nl,\,ferm},\,\varepsilon}$ is given by

$$P^{(1)\,\mu}_{\mathrm{nl,\,ferm},\,\varepsilon} = -\frac{i}{32}\int_0^L \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\bigg(\frac{\delta}{\delta\bar\theta_{A\dot\alpha}(\tau_1)}\,\bar\sigma^{\mu}_{\alpha\dot\alpha}\,\frac{\delta}{\delta\theta^A_\alpha(\tau_2)}\bigg)\Big(\theta(\tau_2 - \tau_1 - \varepsilon) - \theta(\tau_1 - \tau_2 - \varepsilon)\Big)\,. \tag{4.21}$$

At this point it also becomes clear why we need to consider a non-chiral superspace. If, instead of a non-chiral superspace, we had chosen a chiral one, no $\bar\theta$- (and no y-) dependence would be present and, consequently, $P^{(1)\,\mu}_{\mathrm{nl},\,\varepsilon}$ would not contain any terms that could give rise to corrections to the bosonic result (3.114), when applied to the one-loop expectation value of the supersymmetrically completed Maldacena-Wilson loop operator.

4.2. Construction of the Supersymmetrized Maldacena-Wilson Loop

Having discussed how the superconformal algebra as well as the non-local part of the full Yangian level-one momentum generator can be represented on the space of supercurves, we now need to establish a generalized Maldacena-Wilson loop operator that depends on a superpath parametrized by

$$x_{\alpha\dot{\alpha}}(s) = \bar{\sigma}^{\mu}_{\alpha\dot{\alpha}}\, x_\mu(s)\,, \quad \theta^A_\alpha(s)\,, \quad \bar{\theta}_{A\dot{\alpha}}(s)\,, \quad {y_A}^B(s)\,. \tag{4.22}$$

However, since we are for the time being only interested in corrections to the bosonic result (3.114), we will restrict to the surface defined by ${y_A}^B = 0$ and only construct the extension in the anticommuting Grassmann variables. We note that this constraint is compatible with our non-local symmetry generator in the sense that $P^{(1)\,\mu}_{\mathrm{nl}}$ preserves the constraint surface. To derive the first few Grassmann extensions we make the following ansatz for the generalized Maldacena-Wilson loop operator

$$\mathcal{W}(C) = \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\exp\Big(i\, I\left[A,\psi,\tilde{\psi},\phi;x,\theta,\bar{\theta}\right]\Big)\,, \tag{4.23}$$

with the exponent I given by

$$I\left[A,\psi,\tilde{\psi},\phi;x,\theta,\bar{\theta}\right] = \oint_C \mathrm{d}s\,\Big(\mathcal{I}_0 + \mathcal{I}_1 + \bar{\mathcal{I}}_1 + \mathcal{I}_{2m} + \mathcal{I}_2 + \bar{\mathcal{I}}_2 + \mathcal{O}\left(\{\bar{\theta}^3\theta^{3-i}\}\right)\Big)\,, \tag{4.24}$$

where the subscript denotes the order in the fermionic coordinates. Being of order zero in θ and $\bar{\theta}$, $\mathcal{I}_0$ is of course the exponent of the usual Maldacena-Wilson loop operator (3.27). The summands $\mathcal{I}_1$ and $\bar{\mathcal{I}}_1$ depend linearly on θ and $\bar{\theta}$ respectively. The mixed term $\mathcal{I}_{2m}$ depends on the product $\theta\bar{\theta}$. Due to the structure of the fermionic part of the level-one momentum generator (4.21), it is evident that the only terms of the vacuum expectation value $\langle\mathcal{W}(C)\rangle$ that will give rise to corrections to the bosonic result (3.114) are those proportional to $\bar{\theta}\sigma\theta$. It is therefore in principle sufficient to determine $\mathcal{I}_1$, $\bar{\mathcal{I}}_1$ and $\mathcal{I}_{2m}$, since only their contractions will contribute to the desired term of $\langle\mathcal{W}(C)\rangle$. Nevertheless, for reasons of completeness we will also derive $\mathcal{I}_2$ and $\bar{\mathcal{I}}_2$. We will determine the unknown summands in (4.24) by demanding that $\langle\mathcal{W}(C)\rangle$ be invariant under supersymmetry transformations, i.e.

$$Q^\alpha_A\,\langle\mathcal{W}(C)\rangle = 0 \qquad\qquad \bar{Q}^{A\dot{\alpha}}\,\langle\mathcal{W}(C)\rangle = 0\,. \tag{4.25}$$

These two relations will hold true if the exponent I satisfies the equations

$$\mathfrak{q}^\alpha_A(I) = Q^\alpha_A(I) \qquad\qquad \bar{\mathfrak{q}}^{A\dot{\alpha}}(I) = \bar{Q}^{A\dot{\alpha}}(I)\,, \tag{4.26}$$

where $\mathfrak{q}^\alpha_A$ and $\bar{\mathfrak{q}}^{A\,\dot{\alpha}}$ act on fields (see section 2.3) in contrast to Q^α_A and $\bar{Q}^{A\dot{\alpha}}$, which act on the space of superpaths. The argument that the former two equations guarantee the invariance of the expectation value of the generalized Maldacena-Wilson loop under supersymmetry transformations goes as follows. We first note that the expectation value of the supersymmetry variations of the loop operator $\mathcal{W}(C)$ generated by $\mathfrak{q}^\alpha_A$ and $\bar{\mathfrak{q}}^{A\,\dot{\alpha}}$ vanish. This is most transparently seen by adopting the canonical point of view. Using (2.82), we can write

$$\langle\mathfrak{q}^\alpha_A\,\mathcal{W}(C)\rangle = -i\,\langle 0|\left[\mathcal{Q}^\alpha_A,\mathcal{W}(C)\right]|0\rangle = -i\,\langle 0|\,\mathcal{Q}^\alpha_A\,\mathcal{W}(C)\,|0\rangle + i\,\langle 0|\,\mathcal{W}(C)\,\mathcal{Q}^\alpha_A|0\rangle = 0\,, \tag{4.27}$$

where the zero on the right-hand side follows from the invariance of the vacuum state under supersymmetry transformations

$$\mathcal{Q}^{\alpha}_{A}\,|0\rangle = 0\,. \tag{4.28}$$

The last statement certainly holds true as we are dealing with a theory with unbroken supersymmetry. Having this in mind, it is now easy to see that the left equation of (4.26) guarantees the invariance $\langle \mathcal{W}(C)\rangle$ under supersymmetry transformations generated by the differential operator Q^{α}_{A}.

$$Q^{\alpha}_{A}\,\langle \mathcal{W}(C)\rangle = \langle Q^{\alpha}_{A}\,\mathcal{W}(C)\rangle = \Big[i\,Q^{\alpha}_{A}(I)\Big] = \Big[i\,\mathfrak{q}^{\alpha}_{A}(I)\Big] = \langle \mathfrak{q}^{\alpha}_{A}\,\mathcal{W}(C)\rangle = 0\,, \tag{4.29}$$

where the square brackets again denote the path-ordered expectation value in the presence of a loop operator, i.e.

$$\Big[i\,Q^{\alpha}_{A}(I)\Big] := \left\langle \frac{1}{N}\,\mathrm{Tr}\,\mathcal{P}\left\{\exp\big(i\,I\big)\,i\,Q^{\alpha}_{A}(I)\right\}\right\rangle\,. \tag{4.30}$$

The same argument goes through with $\mathfrak{q}^{\alpha}_{A}$,$\mathcal{Q}^{\alpha}_{A}$ and Q^{α}_{A} replaced by $\bar{\mathfrak{q}}^{\alpha}_{A}$,$\bar{\mathcal{Q}}^{\alpha}_{A}$ and $\bar{Q}^{\alpha}_{A}$.

The equations (4.26) allow us to successively construct the individual summands of (4.24). For computational purposes we introduce the following notation

$$\begin{aligned} Q^{\alpha}_{A\,(1)} &= -\int \mathrm{d}\tau\,\frac{\delta}{\delta\theta^{A}_{\alpha}(\tau)} & \qquad Q^{\alpha}_{A\,(2)} &= \int \mathrm{d}\tau\,i\,\bar{\theta}_{A\dot{\alpha}}(\tau)\,\frac{\delta}{\delta x_{\alpha\dot{\alpha}}(\tau)} \\ \bar{Q}^{A\dot{\alpha}}_{(2)} &= \int \mathrm{d}\tau\,\frac{\delta}{\delta\bar{\theta}_{A\dot{\alpha}}(\tau)} & \qquad \bar{Q}^{A\dot{\alpha}}_{(2)} &= -\int \mathrm{d}\tau\,i\,\theta^{A}_{\alpha}(\tau)\,\frac{\delta}{\delta x_{\alpha\dot{\alpha}}(\tau)}\,, \end{aligned} \tag{4.31}$$

where we have split the operators (4.11) in an intuitive way. We note that the problem of finding the individual summands can be divided up as follows

$$\mathfrak{q}^{\alpha}_{A}(I_0) = Q^{\alpha}_{A\,(1)}(I_1) \tag{4.32}$$
$$\mathfrak{q}^{\alpha}_{A}(I_1) = Q^{\alpha}_{A\,(1)}(I_2) \tag{4.33}$$
$$\mathfrak{q}^{\alpha}_{A}(\bar{I}_1) = Q^{\alpha}_{A\,(2)}(I_0) + Q^{\alpha}_{A\,(1)}(I_{2m}) \tag{4.34}$$
$$\bar{\mathfrak{q}}^{A\,\dot{\alpha}}(I_0) = \bar{Q}^{A\,\dot{\alpha}}_{(1)}(\bar{I}_1) \tag{4.35}$$
$$\bar{\mathfrak{q}}^{A\,\dot{\alpha}}(\bar{I}_1) = \bar{Q}^{A\,\dot{\alpha}}_{(1)}(\bar{I}_2) \tag{4.36}$$
$$\bar{\mathfrak{q}}^{A\,\dot{\alpha}}(I_1) = \bar{Q}^{A\,\dot{\alpha}}_{(2)}(I_0) + \bar{Q}^{A\,\dot{\alpha}}_{(1)}(I_{2m})\,, \tag{4.37}$$

where we introduced the notation

$$I_x = \int \mathrm{d}s\,\mathcal{I}_x\,. \tag{4.38}$$

Since we are only interested in the one-loop contribution to $\langle \mathcal{W}(C)\rangle$ we will neglect all terms in the $\mathcal{I}_x$'s that are not linear in the fields. Let us start by calculating how $\mathcal{I}_0$ transforms under supersymmetry transformations generated by $\mathfrak{q}^{\alpha}_{A}$ and $\bar{\mathfrak{q}}^{A\,\dot{\alpha}}$ respectively. To have a more compact notation we will mostly consider the equations (4.32)-(4.37) on the level of the integrand and only write the integral when using integration by parts.

Using the basic field transformations listed in section 2.3, one finds

$$\begin{aligned} \mathfrak{q}^{\alpha}_{A}(\mathcal{I}_0) &= i\epsilon^{\alpha\beta}\,\tilde{\psi}^{\dot\beta}_{A}\,\dot{x}_{\beta\dot\beta} - \sqrt{2}\,i\,\psi^{D\alpha}\,\bar{n}_{AD}\,|\dot{x}| \\ \bar{\mathfrak{q}}^{A\dot\alpha}(\mathcal{I}_0) &= -i\,\epsilon^{\dot\alpha\dot\beta}\,\psi^{A\beta}\,\dot{x}_{\beta\dot\beta} - \sqrt{2}\,i\,\tilde{\psi}^{\dot\alpha}_{D}\,n^{AD}\,|\dot{x}|\,. \end{aligned} \tag{4.39}$$

To keep equations short, we have suppressed the dependence on the curve parameter s. Obviously, the equations (4.32) and (4.35) are satisfied if we choose $\mathcal{I}_1$ and $\bar{\mathcal{I}}_1$ as follows

$$\begin{aligned} \mathcal{I}_1 &= i\,\theta^{B\beta}\,\tilde{\psi}^{\dot\beta}_{B}\,\dot{x}_{\beta\dot\beta} + \sqrt{2}\,i\,\theta^{C}_{\beta}\,\psi^{D\beta}\,\bar{n}_{CD}\,|\dot{x}| \\ \bar{\mathcal{I}}_1 &= -i\,\bar\theta^{\dot\beta}_{B}\,\psi^{B\beta}\,\dot{x}_{\beta\dot\beta} - \sqrt{2}\,i\,\bar\theta_{C\dot\beta}\,\tilde{\psi}^{\dot\beta}_{D}\,n^{CD}\,|\dot{x}|\,. \end{aligned} \tag{4.40}$$

As we know how $\mathfrak{q}^{\alpha}_{A}$ and $\bar{\mathfrak{q}}^{A\dot\alpha}$ act on fields, writing down how $\mathcal{I}_1$ and $\bar{\mathcal{I}}_1$ transform under supersymmetry transformations poses no difficulties.

$$\begin{aligned} \mathfrak{q}^{\alpha}_{A}(\mathcal{I}_1) &= \sqrt{2}\,i\,\theta^{B\beta}\Big(\partial^{\dot\beta\alpha}\,\bar\phi_{AB}\Big)\dot{x}_{\beta\dot\beta} + \tfrac{1}{\sqrt{2}}\,\theta^{C}_{\beta}\,F^{\alpha\beta}_{lin}\,\bar{n}_{CA}\,|\dot{x}| \\ \mathfrak{q}^{\alpha}_{A}(\bar{\mathcal{I}}_1) &= -\tfrac{1}{2}\,\bar\theta^{\dot\beta}_{A}F^{\alpha\beta}_{lin}\dot{x}_{\beta\dot\beta} - 2\,i\,\bar\theta_{C\dot\beta}\Big(\partial^{\dot\beta\alpha}\,\bar\phi_{AB}\Big)n^{CB}\,|\dot{x}| \\ \bar{\mathfrak{q}}^{A\dot\alpha}(\mathcal{I}_1) &= -\tfrac{1}{2}\,\theta^{A\beta}F^{\dot\alpha\dot\beta}_{lin}\dot{x}_{\beta\dot\beta} + 2\,i\,\theta^{C}_{\beta}\Big(\partial^{\beta\dot\alpha}\,\phi^{AB}\Big)\bar{n}_{CB}\,|\dot{x}| \\ \bar{\mathfrak{q}}^{A\dot\alpha}(\bar{\mathcal{I}}_1) &= -\sqrt{2}\,i\,\bar\theta^{\dot\beta}_{B}\Big(\partial^{\beta\dot\alpha}\,\phi^{AB}\Big)\dot{x}_{\beta\dot\beta} + \tfrac{1}{\sqrt{2}}\,\bar\theta_{C\dot\beta}\,F^{\dot\alpha\dot\beta}_{lin}\,n^{CA}\,|\dot{x}| \end{aligned} \tag{4.41}$$

In these equations $F^{\alpha\beta}_{lin}$ and $F^{\dot\alpha\dot\beta}_{lin}$ denote the parts of (2.24) which are linear in the gauge field. $\mathcal{I}_2$ and $\bar{\mathcal{I}}_2$ can now be constructed by imposing that the equations (4.33) and (4.36) hold true. The result reads

$$\begin{aligned} \mathcal{I}_2 &= -\tfrac{i}{\sqrt{2}}\,\theta^{C}_{\gamma}\,\theta^{B\beta}\Big(\partial^{\dot\beta\gamma}\,\bar\phi_{CB}\Big)\dot{x}_{\beta\dot\beta} + \tfrac{1}{2\sqrt{2}}\,\theta^{C}_{\beta}\,\theta^{D}_{\gamma}\,F^{\gamma\beta}_{lin}\,\bar{n}_{CD}\,|\dot{x}| + \sqrt{2}\,i\,\theta^{C}_{\gamma}\,\dot\theta^{B\gamma}\,\bar\phi_{CB} \\ \bar{\mathcal{I}}_2 &= -\tfrac{i}{\sqrt{2}}\,\bar\theta_{C\dot\gamma}\,\bar\theta^{\dot\beta}_{B}\Big(\partial^{\beta\dot\gamma}\,\phi^{CB}\Big)\dot{x}_{\beta\dot\beta} - \tfrac{1}{2\sqrt{2}}\,\bar\theta_{C\dot\beta}\,\bar\theta_{D\dot\gamma}\,F^{\dot\gamma\dot\beta}_{lin}\,n^{CD}\,|\dot{x}| + \sqrt{2}\,i\,\bar\theta_{C\dot\gamma}\,\dot{\bar\theta}^{\dot\gamma}_{B}\,\phi^{CB}\,. \end{aligned} \tag{4.42}$$

Since the calculations which show that the equations (4.33) and (4.36) are indeed satisfied are a little bit more involved we will give some details on at least one of them. Applying $Q^{\alpha}_{A\,(1)}$ to I_2 yields

$$\begin{aligned} Q^{\alpha}_{A\,(1)}(I_2) &= \int \mathrm{d}s\,\Big(-\sqrt{2}\,i\,\theta^{B}_{\beta}\Big(\partial^{\dot\beta(\alpha}\,\bar\phi_{AB}\Big)\dot{x}^{\beta)}{}_{\dot\beta} + \tfrac{1}{\sqrt{2}}\,\theta^{C}_{\beta}\,F^{\alpha\beta}_{lin}\,\bar{n}_{CA}|\dot{x}| - \sqrt{2}\,i\,\dot\theta^{B\alpha}\,\bar\phi_{AB}\Big) \\ &= \int \mathrm{d}s\,\Big(\sqrt{2}\,i\,\theta^{B\beta}\Big(\partial^{\dot\beta\alpha}\,\bar\phi_{AB}\Big)\dot{x}_{\beta\dot\beta} - \tfrac{i}{\sqrt{2}}\,\theta^{B\alpha}\Big(\partial^{\dot\beta\beta}\,\bar\phi_{AB}\Big)\dot{x}_{\beta\dot\beta} \\ &\qquad + \tfrac{1}{\sqrt{2}}\,\theta^{C}_{\beta}\,F^{\alpha\beta}_{lin}\,\bar{n}_{CA}|\dot{x}| - \sqrt{2}\,i\,\dot\theta^{B\alpha}\,\bar\phi_{AB}\Big) \\ &= \int \mathrm{d}s\,\Big(\sqrt{2}\,i\,\theta^{B\beta}\Big(\partial^{\dot\beta\alpha}\,\bar\phi_{AB}\Big)\dot{x}_{\beta\dot\beta} + \tfrac{1}{\sqrt{2}}\,\theta^{C}_{\beta}\,F^{\alpha\beta}_{lin}\,\bar{n}_{CA}\,|\dot{x}|\Big)\,. \end{aligned} \tag{4.43}$$

In order to get the second line we used identity (2.26). Note that the last term in the second line can be rewritten as a derivative with respect to the curve parameter s acting on $\bar\phi_{AB}$. Using integration by parts, we find that the rewritten term cancels the $\dot\theta$-term. Similarly, it can be shown that $\bar{I}_2$ satisfies equation (4.36). Let us now turn to the construction of $\mathcal{I}_{2m}$. While $\mathcal{I}_2$ and $\bar{\mathcal{I}}_2$ are not necessarily needed for our purpose, this does not apply to $\mathcal{I}_{2m}$.

In contrast to the construction of $\mathcal{I}_1$, $\bar{\mathcal{I}}_1$, $\mathcal{I}_2$ and $\bar{\mathcal{I}}_2$ we now have two equations for one expression and it is not clear whether they are compatible with each other. We start by calculating how Q^α_A acts on I_0.

$$\begin{aligned}
Q^\alpha_A(I_0) &= \int \mathrm{d}s\, \mathrm{d}\tau\, i\,\bar{\theta}_{A\dot{\alpha}}(\tau)\, \frac{\delta}{\delta x_{\alpha\dot{\alpha}}(\tau)} \left(\tfrac{1}{2}\, A^{\beta\dot{\beta}}\, \dot{x}_{\beta\dot{\beta}} - \tfrac{1}{2}\, \phi^{CD}\, \bar{n}_{CD}\, |\dot{x}| \right) \\
&= \int \mathrm{d}s\, \left(\tfrac{i}{2}\, \bar{\theta}_{A\dot{\alpha}} \left(\partial^{\alpha\dot{\alpha}}\, A^{\beta\dot{\beta}} \right) \dot{x}_{\beta\dot{\beta}} + i\, \dot{\bar{\theta}}_{A\dot{\alpha}}\, A^{\alpha\dot{\alpha}} - \tfrac{i}{2}\, \bar{\theta}_{A\dot{\alpha}} \left(\partial^{\alpha\dot{\alpha}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| - \tfrac{i}{2}\, \dot{\bar{\theta}}_{A\dot{\alpha}}\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|} \right) \\
&= \int \mathrm{d}s\, \left(\tfrac{i}{2}\, \bar{\theta}_{A\dot{\alpha}} \left(\partial^{\alpha\dot{\alpha}}\, A^{\beta\dot{\beta}} - \partial^{\beta\dot{\beta}}\, A^{\alpha\dot{\alpha}} \right) \dot{x}_{\beta\dot{\beta}} - \tfrac{i}{2}\, \bar{\theta}_{A\dot{\alpha}} \left(\partial^{\alpha\dot{\alpha}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| - \tfrac{i}{2}\, \dot{\bar{\theta}}_{A\dot{\alpha}}\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|} \right) \\
&= \int \mathrm{d}s\, \left(-\tfrac{1}{4}\, \bar{\theta}^{\dot{\beta}}_A\, F^{\alpha\beta}_{lin}\, \dot{x}_{\beta\dot{\beta}} - \tfrac{1}{4}\, \bar{\theta}_{A\dot{\alpha}}\, F^{\dot{\alpha}\dot{\beta}}_{lin}\, \dot{x}^\alpha{}_{\dot{\beta}} - \tfrac{i}{2}\, \bar{\theta}_{A\dot{\alpha}} \left(\partial^{\alpha\dot{\alpha}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| - \tfrac{i}{2}\, \dot{\bar{\theta}}_{A\dot{\alpha}}\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|} \right)
\end{aligned}$$

First, we applied the functional derivative to I_0 and integrated out the delta functions by evaluating the generator integral. In going from the second to the third line we integrated the second term by parts. The last line follows by using identity (2.25). The calculation including $\bar{Q}^{A\dot{\alpha}}(I_0)$ works completely analogously.

$$\bar{Q}^{A\dot{\alpha}}(I_0) = \int \mathrm{d}s\, \left(-\tfrac{1}{4}\, \theta^A_\alpha\, F^{\alpha\beta}_{lin}\, \dot{x}_\beta{}^{\dot{\alpha}} - \tfrac{1}{4}\, \theta^{A\beta}\, F^{\dot{\alpha}\dot{\beta}}_{lin}\, \dot{x}_{\beta\dot{\beta}} + \tfrac{i}{2}\, \theta^A_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| + \tfrac{i}{2}\, \dot{\theta}^A_\alpha\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|} \right)$$

By requiring that equation (4.34) holds true, $\mathcal{I}_{2m}$ can be determined (up to the term including $\dot{\theta}$) to be

$$\begin{aligned}
\mathcal{I}_{2m} = {}& \tfrac{1}{4}\, \theta^B_\gamma\, \bar{\theta}^{\dot{\beta}}_B\, F^{\gamma\beta}_{lin}\, \dot{x}_{\beta\dot{\beta}} + \tfrac{1}{4}\, \theta^{B\beta}\, \bar{\theta}_{B\dot{\gamma}}\, F^{\dot{\gamma}\dot{\beta}}_{lin}\, \dot{x}_{\beta\dot{\beta}} + 2\, i\, \theta^B_\gamma\, \bar{\theta}_{C\dot{\beta}} \left(\partial^{\dot{\beta}\gamma}\, \bar{\phi}_{BE} \right) n^{CE}\, |\dot{x}| \\
& - \tfrac{i}{2}\, \theta^B_\gamma\, \bar{\theta}_{B\dot{\gamma}} \left(\partial^{\gamma\dot{\gamma}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| + \tfrac{i}{2}\, \dot{\theta}^B_\beta\, \bar{\theta}_{B\dot{\beta}}\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\beta\dot{\beta}}}{|\dot{x}|} - \tfrac{i}{2}\, \theta^B_\beta\, \dot{\bar{\theta}}_{B\dot{\beta}}\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\beta\dot{\beta}}}{|\dot{x}|}\,. \qquad (4.44)
\end{aligned}$$

The application of $Q^\alpha_{A\,(1)}$ to $\mathcal{I}_{2m}$ yields

$$\begin{aligned}
Q^\alpha_{A\,(1)}(\mathcal{I}_{2m}) = {}& -\tfrac{1}{4}\, \bar{\theta}^{\dot{\beta}}_A\, F^{\alpha\beta}_{lin}\, \dot{x}_{\beta\dot{\beta}} + \tfrac{1}{4}\, \bar{\theta}_{A\dot{\alpha}}\, F^{\dot{\alpha}\dot{\beta}}_{lin}\, \dot{x}^\alpha{}_{\dot{\beta}} - 2\, i\, \bar{\theta}_{C\dot{\alpha}} \left(\partial^{\alpha\dot{\alpha}}\, \bar{\phi}_{AB} \right) n^{CB}\, |\dot{x}| \\
& + \tfrac{i}{2}\, \bar{\theta}_{A\dot{\alpha}} \left(\partial^{\alpha\dot{\alpha}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| + \tfrac{i}{2}\, \dot{\bar{\theta}}_{A\dot{\alpha}}\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|}\,. \qquad (4.45)
\end{aligned}$$

We instantly see that equation (4.34) is indeed satisfied. We will now show that (4.44) also solves equation (4.37). Therefore, we calculate

$$\begin{aligned}
\bar{Q}^{A\dot{\alpha}}_{(1)}(\mathcal{I}_{2m}) = {}& +\tfrac{1}{4}\, \theta^A_\alpha\, F^{\alpha\beta}_{lin}\, \dot{x}_\beta{}^{\dot{\alpha}} - \tfrac{1}{4}\, \theta^{A\alpha}\, F^{\dot{\alpha}\dot{\beta}}_{lin}\, \dot{x}_{\alpha\dot{\beta}} - 2\, i\, \theta^B_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \bar{\phi}_{BC} \right) n^{AC}\, |\dot{x}| \\
& + \tfrac{i}{2}\, \theta^A_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| - \tfrac{i}{2}\, \dot{\theta}^A_\alpha\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|}\,. \qquad (4.46)
\end{aligned}$$

The third term can be rewritten as follows

$$\begin{aligned}
2\, i\, \theta^B_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \bar{\phi}_{BC} \right) n^{AC}\, |\dot{x}| &= \tfrac{i}{2}\, \theta^B_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \phi^{KL} \right) \bar{n}_{MN}\, |\dot{x}|\, \varepsilon_{BCKL}\, \varepsilon^{ACMN} \\
&= i\, \theta^A_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| - 2\, i\, \theta^C_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \phi^{AB} \right) \bar{n}_{CB}\, |\dot{x}|\,, \qquad (4.47)
\end{aligned}$$

where we employed the identity (2.36) . Inserting (4.47) in (4.46) yields

$$\begin{aligned}
\bar{Q}^{A\dot{\alpha}}_{(1)}(\mathcal{I}_{2m}) = {}& +\tfrac{1}{4}\, \theta^A_\alpha\, F^{\alpha\beta}_{lin}\, \dot{x}_\beta{}^{\dot{\alpha}} - \tfrac{1}{4}\, \theta^{A\beta}\, F^{\dot{\alpha}\dot{\beta}}_{lin}\, \dot{x}_{\beta\dot{\beta}} - \tfrac{i}{2}\, \theta^A_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \phi^{CD} \right) \bar{n}_{CD}\, |\dot{x}| \\
& + 2\, i\, \theta^C_\alpha \left(\partial^{\alpha\dot{\alpha}}\, \phi^{AB} \right) \bar{n}_{CB}\, |\dot{x}| - \tfrac{i}{2}\, \dot{\theta}^A_\alpha\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|}\,. \qquad (4.48)
\end{aligned}$$

If we combine this equation with the result for $\bar{Q}^{A\dot{\alpha}}(I_0)$ we note that equation (4.37) holds true as well.

In summary, we have established explicit expressions for the first few components of the exponent I of our generalized Maldacena-Wilson loop operator.

$$\begin{aligned}
\mathcal{I}_0 &= \tfrac{1}{2}\, A^{\beta\dot\beta}\, \dot{x}_{\beta\dot\beta} - \tfrac{1}{2}\, \phi^{CD}\, \bar{n}_{CD}\, |\dot{x}| \\
\mathcal{I}_1 &= i\, \theta^{B\beta}\, \tilde{\psi}^{\dot\beta}_{B}\, \dot{x}_{\beta\dot\beta} + \sqrt{2}\, i\, \theta^{C}_{\beta}\, \psi^{D\beta}\, \bar{n}_{CD}\, |\dot{x}| \\
\bar{\mathcal{I}}_1 &= -i\, \bar\theta^{\dot\beta}_{B}\, \psi^{B\beta}\, \dot{x}_{\beta\dot\beta} - \sqrt{2}\, i\, \bar\theta_{C\dot\beta}\, \tilde{\psi}^{\dot\beta}_{D}\, n^{CD}\, |\dot{x}| \\
\mathcal{I}_2 &= -\tfrac{i}{\sqrt{2}}\, \theta^{C}_{\gamma}\, \theta^{B\beta} \Big(\partial^{\dot\beta\gamma}\, \bar\phi_{CB}\Big) \dot{x}_{\beta\dot\beta} + \tfrac{1}{2\sqrt{2}}\, \theta^{C}_{\beta}\, \theta^{D}_{\gamma}\, F^{\gamma\beta}_{lin}\, \bar{n}_{CD}\, |\dot{x}| + \sqrt{2}\, i\, \theta^{C}_{\gamma}\, \dot\theta^{B\gamma}\, \bar\phi_{CB} \\
\bar{\mathcal{I}}_2 &= -\tfrac{i}{\sqrt{2}}\, \bar\theta_{C\dot\gamma}\, \bar\theta^{\dot\beta}_{B} \Big(\partial^{\beta\dot\gamma}\, \phi^{CB}\Big) \dot{x}_{\beta\dot\beta} - \tfrac{1}{2\sqrt{2}}\, \bar\theta_{C\dot\beta}\, \bar\theta_{D\dot\gamma}\, F^{\dot\gamma\dot\beta}_{lin}\, n^{CD}\, |\dot{x}| + \sqrt{2}\, i\, \bar\theta_{C\dot\gamma}\, \dot{\bar\theta}^{\dot\gamma}_{B}\, \phi^{CB} \\
\mathcal{I}_{2m} &= \tfrac{1}{4}\, \theta^{B}_{\gamma}\, \bar\theta^{\dot\beta}_{B}\, F^{\gamma\beta}_{lin}\, \dot{x}_{\beta\dot\beta} + \tfrac{1}{4}\, \theta^{B\beta}\, \bar\theta_{B\dot\gamma}\, F^{\dot\gamma\dot\beta}_{lin}\, \dot{x}_{\beta\dot\beta} + 2\, i\, \theta^{B}_{\gamma}\, \bar\theta_{C\dot\beta} \Big(\partial^{\dot\beta\gamma}\, \bar\phi_{BE}\Big) n^{CE}\, |\dot{x}| \\
&\quad - \tfrac{i}{2}\, \theta^{B}_{\gamma}\, \bar\theta_{B\dot\gamma} \Big(\partial^{\gamma\dot\gamma}\, \phi^{CD}\Big) \bar{n}_{CD}\, |\dot{x}| + \tfrac{i}{2}\, \dot\theta^{B}_{\beta}\, \bar\theta_{B\dot\beta}\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\beta\dot\beta}}{|\dot{x}|} - \tfrac{i}{2}\, \theta^{B}_{\beta}\, \dot{\bar\theta}_{B\dot\beta}\, \phi^{CD}\, \bar{n}_{CD}\, \tfrac{\dot{x}^{\beta\dot\beta}}{|\dot{x}|}
\end{aligned} \tag{4.49}$$

Since we constructed this loop operator in such a way that its expectation value is invariant under supersymmetry transformations, it is natural to think of it as the supersymmetrically completed Maldacena-Wilson loop. In contrast to the Maldacena-Wilson loop operator, this operator now depends on a path in superspace parametrized by $\{x_{\alpha\dot\alpha}(s),\, \theta^{A}_{\alpha}(s),\, \bar\theta_{A\,\dot\alpha}(s)\}$ and furthermore involves couplings to the fermionic fields of the theory. However, as the unwanted bi-local contribution in (3.114) looks exactly like a fermion propagator, we seem to be well on the way towards finding a new non-local symmetry.

4.3. The Expectation Value

We will now compute the vacuum expectation value of the supersymmetrically completed Maldacena-Wilson loop operator. Inserting the decomposition of the exponent (4.24) into (4.23) and expanding the exponential yields

$$\begin{aligned}
\langle \mathcal{W}(C)\rangle = 1 - \frac{\mathrm{Tr}(T^a T^b)}{2N} \int \mathrm{d}s_1\, \mathrm{d}s_2\, \Big(&\overbrace{\big\langle \mathcal{I}^a_0(s_1)\, \mathcal{I}^b_0(s_2)\big\rangle}^{\text{(A)}} + 2\, \overbrace{\big\langle \mathcal{I}^a_1(s_1)\, \bar{\mathcal{I}}^b_1(s_2)\big\rangle}^{\text{(B)}} \\
&+ 2\, \underbrace{\big\langle \mathcal{I}^a_0(s_1)\, \mathcal{I}^b_{2m}(s_2)\big\rangle}_{\text{(C)}} + \mathcal{O}(\theta^2, \bar\theta^2)\Big)\,,
\end{aligned} \tag{4.50}$$

where only those contractions are explicitly displayed,which will give rise to a correction to the bosonic result after having applied the full level-one momentum generator. As before, the T^a are the $\mathrm{SU}(N)$ generators in the fundamental representation, normalized according to (2.58). In what follows we will compute (A), (B) and (C) individually.

Computation of (A)

Since I_0 is the exponent of the usual Maldacena-Wilson loop operator, its vacuum expectation value has already been computed in (3.37).

$$(\mathrm{A}) = \langle \mathcal{I}_0^a(s_1)\, \mathcal{I}_0^b(s_2)\rangle = \frac{g^2\delta^{ab}}{4\pi^2}\,\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{x_{12}^2} \tag{4.51}$$

Computation of (B)

We start by plugging in the expressions for $\mathcal{I}_1^a(s_1)$ and $\bar{\mathcal{I}}_1^b(s_2)$.

$$\begin{aligned}(\mathrm{B}) = \langle \mathcal{I}_1^a(s_1)\, \bar{\mathcal{I}}_1^b(s_2)\rangle = &- \theta_1^{A\alpha}\, \bar{\theta}_{2B}^{\dot{\beta}} \langle \tilde{\psi}_A^{\dot{\alpha}a}(x_1)\, \psi^{B\beta b}(x_2)\rangle\, \dot{x}_{1\alpha\dot{\alpha}}\, \dot{x}_{2\beta\dot{\beta}} &(4.52)\\ &- 2\, \theta_{1\alpha}^{A}\, \bar{\theta}_{2C\dot{\beta}} \langle \psi^{B\alpha a}(x_1)\, \tilde{\psi}_D^{\dot{\beta}b}(x_2)\rangle\, \bar{n}_{AB}\, n^{CD} |\dot{x}_1||\dot{x}_2|\,, &(4.53)\end{aligned}$$

with $\theta_1^{A\alpha} := \theta^{A\alpha}(s_1)$. Inserting the gluino propagator (2.88) into the former expression leads to

$$\begin{aligned}(\mathrm{B}) =&\frac{ig^2\delta^{ab}}{2\pi^2}\left(\theta_1^{A\alpha}\, \bar{\theta}_{2A}^{\dot{\beta}}\, \frac{x_{21}^{\dot{\alpha}\beta}}{x_{12}^4}\, \dot{x}_{1\alpha\dot{\alpha}}\, \dot{x}_{2\beta\dot{\beta}} - 2\, \theta_{1\alpha}^{A}\, \bar{\theta}_{2C\dot{\beta}}\, \frac{x_{12}^{\dot{\beta}\alpha}}{x_{12}^4}\, \bar{n}_{AD}\, n^{CD} |\dot{x}_1||\dot{x}_2|\right) &(4.54)\\ =&\frac{ig^2\delta^{ab}}{2\pi^2}\left(\frac{1}{2}\, (\bar{\theta}_2\sigma_\mu\theta_1)\sigma^{\mu\dot{\beta}\alpha}\bar{\sigma}^{\nu}_{\alpha\dot{\alpha}}\sigma^{\rho\dot{\alpha}\beta}\bar{\sigma}^{\sigma}_{\beta\dot{\beta}}\, \frac{x_{12\rho}}{x_{12}^4}\, \dot{x}_{1\nu}\, \dot{x}_{2\sigma} + (\bar{\theta}_2\sigma_\mu\theta_1)\, \frac{x_{12}^{\mu}}{x_{12}^4}\, |\dot{x}_1||\dot{x}_2|\right). &(4.55)\end{aligned}$$

The second line has been obtained by employing identity (2.28), (2.41) and using the definition (2.16) for how a bi-spinor is assigned to a four-vector. The trace over four sigma matrices can be rewritten in terms of metric tensors and the totally antisymmetric Levi-Civita tensor. By plugging in the trace identity, we arrive at

$$(\mathrm{B}) = -\frac{ig^2\delta^{ab}}{2\pi^2}\, (\bar{\theta}_2\sigma_\mu\theta_1)\left(\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{x_{12}^4}\, x_{12}^{\mu} - \frac{x_{12}\cdot\dot{x}_2}{x_{12}^4}\, \dot{x}_1^{\mu} - \frac{x_{12}\cdot\dot{x}_1}{x_{12}^4}\, \dot{x}_2^{\mu} - i\, \varepsilon^{\mu\nu\rho\sigma}\, \frac{x_{12\rho}}{x_{12}^4}\, \dot{x}_{1\nu}\, \dot{x}_{2\sigma}\right).$$

Note that the two terms in the middle can be written as follows

$$\frac{x_{12}\cdot\dot{x}_2}{x_{12}^4}\, \dot{x}_1^{\mu} = \partial_{s_2}\left(\frac{1}{2}\frac{\dot{x}_1^{\mu}}{x_{12}^2}\right) \qquad\qquad \frac{x_{12}\cdot\dot{x}_1}{x_{12}^4}\, \dot{x}_2^{\mu} = \partial_{s_1}\left(-\frac{1}{2}\frac{\dot{x}_2^{\mu}}{x_{12}^2}\right). \tag{4.56}$$

The integrals in (4.50) allow us to integrate these terms by parts. The final result then reads

$$\begin{aligned}-\frac{ig^2\delta^{ab}}{2\pi^2}\int \mathrm{d}s_1\, \mathrm{d}s_2 \Bigg\{ &(\bar{\theta}_2\sigma_\mu\theta_1)\left(\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{x_{12}^4}\, x_{12}^{\mu} + i\, \varepsilon^{\mu\rho\nu\sigma}\, \frac{x_{12\rho}}{x_{12}^4}\, \dot{x}_{1\nu}\, \dot{x}_{2\sigma}\right)\\ &- (\bar{\theta}_2\sigma_\mu\dot{\theta}_1)\frac{1}{2}\frac{\dot{x}_2^{\mu}}{x_{12}^2} + (\dot{\bar{\theta}}_2\sigma_\mu\theta_1)\frac{1}{2}\frac{\dot{x}_1^{\mu}}{x_{12}^2}\Bigg\}. \end{aligned}\tag{4.57}$$

Computation of (C)

The computation of (C) is the most involved one. In order to keep things simple we will first rewrite $\mathcal{I}_{2m}$ in terms of Lorentz indices, so that we can use the propagators of section 2.3. We start by plugging in the expressions for $F^{\gamma\beta}_{lin}$ and $F^{\dot\gamma\dot\beta}_{lin}$ as given in section 2.1.1. The identities (2.26) then allow us to rewrite $\mathcal{I}_{2m}$ as follows

$$\begin{aligned}\mathcal{I}_{2m} =& \tfrac{1}{4}\,\theta^B_\gamma\,\bar\theta^{\dot\beta}_B\Big(2\,i\,\partial^\gamma{}_{\dot\gamma}A^{\dot\gamma\beta} - i\,\varepsilon^{\gamma\beta}\partial_{\alpha\dot\alpha}A^{\dot\alpha\alpha}\Big)\dot x_{\beta\dot\beta} + \tfrac{1}{4}\,\theta^{B\beta}\,\bar\theta_{B\dot\gamma}\Big(2\,i\,\partial^{\dot\gamma\gamma}A_\gamma{}^{\dot\beta} - i\,\varepsilon^{\dot\gamma\dot\beta}\partial_{\alpha\dot\alpha}A^{\dot\alpha\alpha}\Big)\dot x_{\beta\dot\beta}\\ &+ 2\,i\,\theta^B_\gamma\,\bar\theta_{C\dot\beta}\Big(\partial^{\dot\beta\gamma}\,\bar\phi_{BE}\Big)n^{CE}\,|\dot x| - \tfrac{i}{2}\,\theta^B_\gamma\,\bar\theta_{B\dot\gamma}\Big(\partial^{\dot\gamma\gamma}\,\phi^{CD}\Big)\bar n_{CD}\,|\dot x|\\ &+ \tfrac{i}{2}\,\dot\theta^B_\beta\,\bar\theta_{B\dot\beta}\,\phi^{CD}\,\bar n_{CD}\,\tfrac{\dot x^{\dot\beta\beta}}{|\dot x|} - \tfrac{i}{2}\,\theta^B_\beta\,\dot{\bar\theta}_{B\dot\beta}\,\phi^{CD}\,\bar n_{CD}\,\tfrac{\dot x^{\dot\beta\beta}}{|\dot x|}\,. \end{aligned}\tag{4.58}$$

We note that the terms which include $\partial_{\alpha\dot\alpha}A^{\dot\alpha\alpha}$ cancel out each other. The next step is to rewrite all terms using Fierz identity (2.28). The result reads

$$\begin{aligned}\mathcal{I}_{2m} =& \tfrac{i}{4}\,(\bar\theta\sigma_\mu\theta)\Big(\sigma^{\mu\dot\beta\gamma}\Big(\partial_{\gamma\dot\gamma}A^{\dot\gamma\beta}\Big) - \sigma^{\mu\dot\gamma\beta}\Big(\partial_{\gamma\dot\gamma}A^{\dot\beta\gamma}\Big)\Big)\dot x_{\beta\dot\beta}\\ &- i\,(\bar\theta_C\sigma_\mu\theta^B)\,\sigma^{\mu\dot\beta\gamma}\Big(\partial_{\gamma\dot\beta}\,\bar\phi_{BE}\Big)n^{CE}\,|\dot x| + \tfrac{i}{4}\,(\bar\theta\sigma_\mu\theta)\,\sigma^{\mu\dot\gamma\gamma}\Big(\partial_{\gamma\dot\gamma}\,\phi^{CD}\Big)\bar n_{CD}\,|\dot x|\\ &- \tfrac{i}{4}\,(\bar\theta\sigma_\mu\dot\theta)\,\phi^{CD}\,\bar n_{CD}\,\sigma^{\mu\dot\beta\beta}\,\tfrac{\dot x_{\beta\dot\beta}}{|\dot x|} + \tfrac{i}{4}\,(\dot{\bar\theta}\sigma_\mu\theta)\,\phi^{CD}\,\bar n_{CD}\,\sigma^{\mu\dot\beta\beta}\,\tfrac{\dot x_{\beta\dot\beta}}{|\dot x|}\,. \end{aligned}\tag{4.59}$$

Using the definition (2.16), we see that the first line of the above expression can be written as

$$\tfrac{i}{4}\,(\bar\theta\sigma_\mu\theta)\Big(\sigma^{\mu\dot\beta\gamma}\bar\sigma^\nu_{\gamma\dot\gamma}\sigma^{\rho\dot\gamma\beta}\bar\sigma^\sigma_{\beta\dot\beta} - \sigma^{\mu\dot\gamma\beta}\bar\sigma^\nu_{\gamma\dot\gamma}\sigma^{\rho\dot\beta\gamma}\bar\sigma^\sigma_{\beta\dot\beta}\Big)(\partial_\nu A_\rho)\dot x_\sigma = -(\bar\theta\sigma_\mu\theta)\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu A_\rho)\dot x_\sigma\,,\tag{4.60}$$

where we have again employed the trace identity (2.9). In the remaining terms the sigma matrices can be used to restore Lorentz indices according to (2.16). Finally, we find the following spinor-index-free expression

$$\begin{aligned}\mathcal{I}_{2m} =& -(\bar\theta\sigma_\mu\theta)\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu A_\rho)\dot x_\sigma - 2\,i\,(\bar\theta_C\sigma_\mu\theta^B)\Big(\partial^\mu\,\bar\phi_{BE}\Big)n^{CE}\,|\dot x|\\ &+ \tfrac{i}{2}\,(\bar\theta\sigma_\mu\theta)\Big(\partial^\mu\,\phi^{CD}\Big)\bar n_{CD}\,|\dot x| - \tfrac{i}{2}\,(\bar\theta\sigma_\mu\dot\theta)\,\phi^{CD}\,\bar n_{CD}\,\tfrac{\dot x^\mu}{|\dot x|} + \tfrac{i}{2}\,(\dot{\bar\theta}\sigma_\mu\theta)\,\phi^{CD}\,\bar n_{CD}\,\tfrac{\dot x^\mu}{|\dot x|}\,. \end{aligned}\tag{4.61}$$

Now we are able to compute the vacuum expectation value (C).

$$\begin{aligned}(\mathrm{C}) = \langle\mathcal{I}^a_0(s_1)\,\mathcal{I}^b_{2m}(s_2)\rangle =& -(\bar\theta_2\sigma_\mu\theta_2)\,\varepsilon^{\mu\nu\rho\sigma}\partial_{x_2\,\nu}\,\langle A^a_\kappa(x_1)\,A^b_\rho(x_2)\rangle\,\dot x_1^\kappa\,\dot x_{2\sigma}\\ &+ i\,(\bar\theta_{2F}\sigma_\mu\theta^B_2)\partial^\mu_{x_2}\,\langle\bar\phi^a_{CD}(x_1)\,\bar\phi^b_{BE}(x_2)\rangle\,n^{CD}\,n^{FE}\,|\dot x_1||\dot x_2|\\ &- \tfrac{i}{4}\,(\bar\theta_2\sigma_\mu\theta_2)\partial^\mu_{x_2}\,\langle\bar\phi^a_{CD}(x_1)\,\bar\phi^b_{EF}(x_2)\rangle\,n^{CD}\,n^{EF}\,|\dot x_1||\dot x_2|\\ &+ \tfrac{i}{4}\,(\bar\theta_2\sigma_\mu\dot\theta_2)\,\langle\bar\phi^a_{CD}(x_1)\,\bar\phi^b_{EF}(x_2)\rangle\,n^{CD}\,n^{EF}\,\tfrac{|\dot x_1|}{|\dot x_2|}\,\dot x_2^\mu\\ &- \tfrac{i}{4}\,(\dot{\bar\theta}_2\sigma_\mu\theta_2)\,\langle\bar\phi^a_{CD}(x_1)\,\bar\phi^b_{EF}(x_2)\rangle\,n^{CD}\,n^{EF}\,\tfrac{|\dot x_1|}{|\dot x_2|}\,\dot x_2^\mu \end{aligned}\tag{4.62}$$

The propagators for the gauge field and the scalar field can be found in section 2.3. Plugging these in, we get

$$\begin{aligned}(\mathrm{C}) =& -\frac{g^2\delta^{ab}}{4\pi^2}\,(\bar\theta_2\sigma_\mu\theta_2)\,\varepsilon^{\mu\nu\rho\sigma}\partial_{x_2\,\nu}\left(\frac{1}{x_{12}^2}\right)\dot x_{1\rho}\,\dot x_{2\sigma} - \frac{ig^2\delta^{ab}}{4\pi^2}\,(\bar\theta_{2F}\sigma_\mu\theta^B_2)\partial^\mu_{x_2}\left(\frac{\varepsilon_{CDBE}}{x_{12}^2}\right)n^{CD}\,n^{FE}\,|\dot x_1||\dot x_2|\\ &- \frac{ig^2\delta^{ab}}{16\pi^2}\,(\bar\theta_2\sigma_\mu\dot\theta_2 - \dot{\bar\theta}_2\sigma_\mu\theta_2)\left(\frac{\varepsilon_{CDEF}}{x_{12}^2}\right)n^{CD}\,n^{EF}\,\frac{|\dot x_1|}{|\dot x_2|}\,\dot x_2^\mu\\ &+ \frac{ig^2\delta^{ab}}{16\pi^2}\,(\bar\theta_2\sigma_\mu\theta_2)\partial^\mu_{x_2}\left(\frac{\varepsilon_{CDEF}}{x_{12}^2}\right)n^{CD}\,n^{EF}\,|\dot x_1||\dot x_2|\,. \end{aligned}\tag{4.63}$$

Firstly, we use the totally antisymmetric tensors to rewrite n^{CD} in terms of $\bar{n}_{CD}$, see (2.38). Further simplifications can then be achieved by use of the identities (2.40) and (2.41). As a final result, we get

$$(\mathrm{C}) = -\frac{g^2\delta^{ab}}{2\pi^2}\left(\left(\bar{\theta}_2\sigma_\mu\theta_2\right)\varepsilon^{\mu\nu\rho\sigma}\frac{x_{12\nu}\,\dot{x}_{1\rho}\,\dot{x}_{2\sigma}}{x_{12}^4} + \frac{i}{2}\left(\bar{\theta}_2\sigma_\mu\dot{\theta}_2 - \dot{\bar{\theta}}_2\sigma_\mu\theta_2\right)\frac{1}{x_{12}^2}\frac{|\dot{x}_1|}{|\dot{x}_2|}\,\dot{x}_2^\mu\right). \tag{4.64}$$

The desired part of the one-loop vacuum expectation value of the supersymmetrically completed Maldacena-Wilson loop operator is now easily obtained by adding up the individual contributions (A), (B) and (C).

$$\begin{aligned}\langle\mathcal{W}(C)\rangle_{(1)} = -\frac{\lambda}{4\pi^2}\int \mathrm{d}s_1\,\mathrm{d}s_2 \Bigg\{ & \left(\frac{1}{4} - i\Big(\bar{\theta}_2\sigma_\mu\theta_1\Big)\frac{x_{12}^\mu}{x_{12}^2}\right)\left(\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1|\,|\dot{x}_2|}{x_{12}^2}\right) \\ & + \Big(\bar{\theta}_2\sigma_\mu\theta_1 - \bar{\theta}_2\sigma_\mu\theta_2\Big)\frac{\varepsilon^{\mu\nu\rho\kappa}\dot{x}_{1\,\nu}\dot{x}_{2\,\rho}x_{12\,\kappa}}{x_{12}^4} \\ & + \frac{i}{2}\left(\bar{\theta}_2\sigma_\mu\dot{\theta}_1\right)\frac{\dot{x}_2^\mu}{x_{12}^2} - \frac{i}{2}\Big(\dot{\bar{\theta}}_2\sigma_\mu\theta_1\Big)\frac{\dot{x}_1^\mu}{x_{12}^2} \\ & - \frac{i}{2}\Big(\bar{\theta}_2\sigma_\mu\dot{\theta}_2 - \dot{\bar{\theta}}_2\sigma_\mu\theta_2\Big)\frac{1}{x_{12}^2}\frac{|\dot{x}_1|}{|\dot{x}_2|}\dot{x}_2^\mu\Bigg\}\end{aligned} \tag{4.65}$$

4.4. Check of Supersymmetry

Having computed the vacuum expectation value of the supersymmetrically completed Maldacena-Wilson loop operator (4.65), we can now explicitly check the invariance of this object under supersymmetry transformations generated by Q_A^α and $\bar{Q}^{A\dot{\alpha}}$. The following calculation is of course to be interpreted as a consistency check, since the expectation value should be supersymmetric by construction. Furthermore, we can only test the Q_A^α ($\bar{Q}^{A\dot{\alpha}}$) symmetry at order $\bar{\theta}$ (θ) due to the fact that we did not calculate any other terms except the ones proportional to $\bar{\theta}\sigma\theta$. Let us start by investigating how Q_A^α acts on $\langle\mathcal{W}(C)\rangle_{(1)}$.

$$Q_A^\alpha\,\langle\mathcal{W}(C)\rangle_{(1)} = Q_{A\,(1)}^\alpha\,\langle\mathcal{W}(C)\rangle_{(1)} + Q_{A\,(2)}^\alpha\,\langle W(C)\rangle_{(1)} + \mathcal{O}(\theta\bar{\theta}^2) \tag{4.66}$$

The first term of this equation is easily computed by noting that $Q_{A\,(1)}^\alpha$ is a simple functional derivative with respect to the fermionic coordinate $\theta_\alpha^A(\tau)$ integrated over τ. The action on the basic objects is given by

$$\begin{aligned} Q_{A\,(1)}^\alpha\big(\bar{\theta}_2\sigma_\nu\theta_x\big) &= \int \mathrm{d}\tau\;\sigma_\nu^{\dot{\alpha}\alpha}\,\bar{\theta}_{A\dot{\alpha}}(s_2)\,\delta(s_x-\tau) = \sigma_\nu^{\dot{\alpha}\alpha}\,\bar{\theta}_{A\dot{\alpha}}(s_2) \\ Q_{A\,(1)}^\alpha\big(\bar{\theta}_2\sigma_\nu\dot{\theta}_x\big) &= \int \mathrm{d}\tau\;\sigma_\nu^{\dot{\alpha}\alpha}\,\bar{\theta}_{A\dot{\alpha}}(s_2)\,\partial_{s_x}\delta(s_x-\tau) = 0\,. \end{aligned}$$

Using these relations, we find

$$\begin{aligned} Q_{A\,(1)}^\alpha\,\langle\mathcal{W}(C)\rangle_{(1)} = \frac{i\lambda}{4\pi^2}\,\sigma_\nu^{\dot{\alpha}\alpha}\int \mathrm{d}s_1\,\mathrm{d}s_2\Bigg\{ & \bar{\theta}_{A\dot{\alpha}}(s_2)\,\frac{x_{12}^\nu}{x_{12}^2}\left(\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1|\,|\dot{x}_2|}{x_{12}^2}\right) \\ & + \dot{\bar{\theta}}_{A\dot{\alpha}}(s_2)\,\frac{1}{2}\frac{\dot{x}_1^\nu}{x_{12}^2} - \dot{\bar{\theta}}_{A\dot{\alpha}}(s_2)\,\frac{1}{2}\frac{|\dot{x}_1|}{|\dot{x}_2|}\,\frac{\dot{x}_2^\nu}{x_{12}^2}\Bigg\}. \end{aligned} \tag{4.67}$$

The next step is to calculate the second term of (4.66). We are exclusively interested in terms being linear in $\bar{\theta}$ so it is sufficient to consider the action of $Q^{\alpha}_{A\,(2)}$ on the bosonic part $\langle W(C)\rangle_{(1)}$ of the one-loop expectation value.

$$Q^{\alpha}_{A\,(2)}\,\langle W(C)\rangle_{(1)} = -\frac{i\lambda}{16\pi^2}\,\sigma^{\nu\dot{\alpha}\alpha}\underbrace{\int \mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}\tau\,\bar{\theta}_{A\dot{\alpha}}(\tau)\frac{\delta}{\delta x^{\nu}(\tau)}\left(\frac{\dot{x}_1\cdot\dot{x}_2-|\dot{x}_1|\,|\dot{x}_2|}{x_{12}^2}\right)}_{\text{(D)}} \tag{4.68}$$

The first functional derivative with respect to $x^{\nu}(\tau)$ has already been computed in section 3.4.1. Substituting (3.70) into (D) and integrating out τ leads to

$$\begin{aligned}\text{(D)} &= \int \mathrm{d}s_1\,\mathrm{d}s_2\left\{-\frac{1}{x_{12}^2}\left[\left(\frac{|\dot{x}_2|}{|\dot{x}_1|}\dot{x}_{1\nu}-\dot{x}_{2\nu}\right)\dot{\bar{\theta}}_{A\dot{\alpha}}(s_1)+\left(\frac{|\dot{x}_1|}{|\dot{x}_2|}\dot{x}_{2\nu}-\dot{x}_{1\nu}\right)\dot{\bar{\theta}}_{A\dot{\alpha}}(s_2)\right]\right.\\ &\qquad\left.-\frac{2}{x_{12}^4}\left[\Big(\dot{x}_1\cdot\dot{x}_2-|\dot{x}_1||\dot{x}_2|\Big)x_{12\nu}\Big(\bar{\theta}_{A\dot{\alpha}}(s_1)-\bar{\theta}_{A\dot{\alpha}}(s_2)\Big)\right]\right\}\\ &= \int \mathrm{d}s_1\,\mathrm{d}s_2\left\{\frac{2}{x_{12}^2}\left(\dot{x}_{1\nu}-\frac{|\dot{x}_1|}{|\dot{x}_2|}\dot{x}_{2\nu}\right)\dot{\bar{\theta}}_{A\dot{\alpha}}(s_2)+\frac{4}{x_{12}^4}\Big(\dot{x}_1\cdot\dot{x}_2-|\dot{x}_1||\dot{x}_2|\Big)x_{12\nu}\,\bar{\theta}_{A\dot{\alpha}}(s_2)\right\},\end{aligned} \tag{4.69}$$

where the second line has been obtained by performing a change of variables in some of the terms. If we insert this back into equation (4.68), we find

$$\begin{aligned}Q^{\alpha}_{A\,(2)}\,\langle W(C)\rangle_{(1)} = -\frac{i\lambda}{4\pi^2}\,\sigma^{\dot{\alpha}\alpha}_{\nu}\int \mathrm{d}s_1\,\mathrm{d}s_2&\left\{\dot{\bar{\theta}}_{A\dot{\alpha}}(s_2)\,\frac{1}{2}\frac{\dot{x}_1^{\nu}}{x_{12}^2}-\dot{\bar{\theta}}_{A\dot{\alpha}}(s_2)\,\frac{1}{2}\frac{|\dot{x}_1|}{|\dot{x}_2|}\,\frac{\dot{x}_2^{\nu}}{x_{12}^2}\right.\\ &\left.+\bar{\theta}_{A\dot{\alpha}}(s_2)\,\frac{x_{12}^{\nu}}{x_{12}^2}\left(\frac{\dot{x}_1\cdot\dot{x}_2-|\dot{x}_1||\dot{x}_2|}{x_{12}^2}\right)\right\}.\end{aligned} \tag{4.70}$$

Combining this result with (4.67) yields

$$Q^{\alpha}_{A}\,\langle \mathcal{W}(C)\rangle_{(1)} = 0+\mathcal{O}(\theta\bar{\theta}^2)\,, \tag{4.71}$$

which indeed shows that the expectation value $\langle\mathcal{W}(C)\rangle$ is invariant under supersymmetry transformations generated by Q^{α}_{A}. Since the calculation involving $\bar{Q}^{A\dot{\alpha}}$ is completely similar to the one involving Q^{α}_{A}, we are not going to present it here.

4.5. Yangian Symmetries

Finally, we turn to the key question of this thesis: is the supersymmetrically completed Maldacena-Wilson loop invariant under the non-local symmetry generated by $P^{(1)\,\mu}_{\mathrm{nl}}$? Since we have already computed the action of the bosonic part $P^{(1)\,\mu}_{\mathrm{nl,\,bos},\,\varepsilon}$ on the expectation value of the Maldacena-Wilson loop operator $\langle W(C)\rangle$, we will now investigate how this result gets modified by fermionic corrections. We therefore calculate

$$P^{(1)\,\mu}_{\mathrm{nl,\,ferm},\,\varepsilon}\,\langle\mathcal{W}(C)\rangle\,, \tag{4.72}$$

with

$$P^{(1)\,\mu}_{\mathrm{nl,\,ferm},\,\varepsilon} = -\frac{i}{32}\int_0^L \mathrm{d}\tau_1\,\mathrm{d}\tau_2\left(\frac{\delta}{\delta\bar{\theta}_{A\dot{\alpha}}(\tau_1)}\,\bar{\sigma}^{\mu}_{\alpha\dot{\alpha}}\,\frac{\delta}{\delta\theta^{A}_{\alpha}(\tau_2)}\right)\Big(\theta(\tau_2-\tau_1-\varepsilon)-\theta(\tau_1-\tau_2-\varepsilon)\Big)\,. \tag{4.73}$$

Again, we compute the action of $P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}$ on the basic objects.

$$\begin{aligned}
\frac{\delta}{\delta\bar{\theta}_{A\dot{\alpha}}(\tau_1)}\,\bar{\sigma}^{\mu}_{\alpha\dot{\alpha}}\,\frac{\delta}{\delta\theta^A_{\alpha}(\tau_2)}\,\big(\bar{\theta}_x\sigma_\nu\theta_y\big) &= -8\,\delta^\mu_\nu\,\delta(\tau_1 - s_x)\,\delta(\tau_2 - s_y)\\
\frac{\delta}{\delta\bar{\theta}_{A\dot{\alpha}}(\tau_1)}\,\bar{\sigma}^{\mu}_{\alpha\dot{\alpha}}\,\frac{\delta}{\delta\theta^A_{\alpha}(\tau_2)}\,\big(\dot{\bar{\theta}}_x\sigma_\nu\theta_y\big) &= -8\,\delta^\mu_\nu\,\partial_{s_x}\delta(\tau_1 - s_x)\,\delta(\tau_2 - s_y)\\
\frac{\delta}{\delta\bar{\theta}_{A\dot{\alpha}}(\tau_1)}\,\bar{\sigma}^{\mu}_{\alpha\dot{\alpha}}\,\frac{\delta}{\delta\theta^A_{\alpha}(\tau_2)}\,\big(\bar{\theta}_x\sigma_\nu\dot{\theta}_y\big) &= -8\,\delta^\mu_\nu\,\delta(\tau_1 - s_x)\,\partial_{s_y}\delta(\tau_2 - s_y)
\end{aligned} \tag{4.74}$$

Starting from these relations it can easily be shown that

$$\begin{aligned}
P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\big(\bar{\theta}_2\sigma_\nu\theta_1\big) &= \tfrac{i}{4}\,\delta^\mu_\nu\Big(\theta(s_1 - s_2 - \varepsilon) - \theta(s_2 - s_1 - \varepsilon)\Big)\\
P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\big(\bar{\theta}_2\sigma_\nu\dot{\theta}_1\big) &= \tfrac{i}{4}\,\delta^\mu_\nu\Big(\delta(s_1 - \varepsilon - s_2) + \delta(s_1 + \varepsilon - s_2)\Big)\\
P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\big(\bar{\theta}_2\sigma_\nu\dot{\theta}_2 - \dot{\bar{\theta}}_2\sigma_\nu\theta_2\big) &= i\,\delta^\mu_\nu\delta(\varepsilon)\\
P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\big(\dot{\bar{\theta}}_2\sigma_\nu\theta_1\big) &= -P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\big(\bar{\theta}_2\sigma_\nu\dot{\theta}_1\big)\\
P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\big(\bar{\theta}_2\sigma_\nu\theta_2\big) &= 0\,.
\end{aligned} \tag{4.75}$$

It is now straightforward to apply $P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}$ to the expectation value (4.65). As we are dealing with our regularized level-one generator, we fix the parametrization to arc length.

$$\begin{aligned}
P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\langle\mathcal{W}(C)\rangle_{(1)} = \frac{\lambda}{16\pi^2}\int_0^L \mathrm{d}s_1\,\mathrm{d}s_2\,\Bigg\{ &-\frac{\dot{x}_1\cdot\dot{x}_2 + 1}{x_{12}^2}\,\frac{x_{12}^\mu}{x_{12}^2}\Big(\theta(s_1 - s_2 - \varepsilon) - \theta(s_2 - s_1 - \varepsilon)\Big)\\
&- i\,\frac{\varepsilon^{\mu\nu\rho\kappa}\,\dot{x}_{1\nu}\,\dot{x}_{2\rho}\,x_{12\kappa}}{x_{12}^4}\Big(\theta(s_1 - s_2 - \varepsilon) - \theta(s_2 - s_1 - \varepsilon)\Big)\\
&+ \frac{1}{2}\,\frac{\dot{x}_1^\mu + \dot{x}_2^\mu}{x_{12}^2}\Big(\delta(s_1 - \varepsilon - s_2) + \delta(s_1 + \varepsilon - s_2)\Big) - 2\,\frac{\dot{x}_2^\mu}{x_{12}^2}\,\delta(\varepsilon)\Bigg\}
\end{aligned}$$

Further simplifications can now be achieved by performing a change of variables in some of the terms. In particular, we note that the second line integrates to zero. Moreover, we will again neglect all $\delta(\varepsilon)$-terms for reasons given at the end of section 3.5.2. The result then reads

$$\begin{aligned}
P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\langle\mathcal{W}(C)\rangle_{(1)} = \frac{\lambda}{8\pi^2}\int_0^L \mathrm{d}s_1\,\mathrm{d}s_2\,\Bigg\{ &\frac{\dot{x}_1\cdot\dot{x}_2 + 1}{x_{12}^2}\,\frac{x_{12}^\mu}{x_{12}^2}\,\theta(s_2 - s_1 - \varepsilon)\\
&+ \frac{1}{2}\,\frac{\dot{x}_1^\mu}{x_{12}^2}\Big(\delta(s_1 - \varepsilon - s_2) + \delta(s_1 + \varepsilon - s_2)\Big)\Bigg\}\,.
\end{aligned} \tag{4.76}$$

The epsilon expansion of the last term has already been computed in section 3.5.2. Inserting (3.110) into the former equation with the appropriate coefficient and lifting the constraint on the parametrization leads to

$$\begin{aligned}
P^{(1)\,\mu}_{\text{nl, ferm}, \varepsilon}\,\langle\mathcal{W}(C)\rangle_{(1)} = \frac{\lambda}{8\pi^2}\Bigg\{ &\frac{1}{12}\int \mathrm{d}s\,\left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x}\cdot\ddot{x})^2}{\dot{x}^6}\right)\dot{x}^\mu + \mathcal{O}(\varepsilon)\\
&+ \int \mathrm{d}s_1\,\mathrm{d}s_2\,\left(\frac{\dot{x}_1\cdot\dot{x}_2 - |\dot{x}_1||\dot{x}_2|}{x_{12}^2}\,\frac{x_{12}^\mu}{x_{12}^2}\right)\theta(s_2 - d(s_2,\varepsilon) - s_1)\Bigg\}\,.
\end{aligned} \tag{4.77}$$

If we combine the two results (3.114) and (4.77) we see that the unwanted bi-local contribution indeed cancels out. What remains in the limit $\varepsilon \to 0$ is given by

$$\lim_{\varepsilon\to 0} P^{(1)\,\mu}_{\mathrm{nl},\,\varepsilon} \left\langle \mathcal{W}(C)\right\rangle_{(1)}\Big|_{\substack{\theta=0\\ \bar\theta=0\\ y=0}} = \frac{7}{96}\,\frac{\lambda}{8\pi^2}\int \mathrm{d}s\,\left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x}\cdot\ddot{x})^2}{\dot{x}^6}\right)\dot{x}^\mu\,, \tag{4.78}$$

which is a simple reparametrization invariant curve integral. The result that our generator annihilates the one-loop expectation value only up to a local term is of course not unexpected because we only took into account the canonical non-local piece. Inspired by how the Yangian can be represented on a tensor product of vector spaces (2.123), one could expect that the local term should be expressible as follows

$$\int \mathrm{d}\tau\, c(\tau)\, p^\mu(\tau)\ \left\langle \mathcal{W}(C)\right\rangle_{(1)} \overset{?}{=} -\frac{7}{96}\,\frac{\lambda}{8\pi^2}\int \mathrm{d}s\,\left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x}\cdot\ddot{x})^2}{\dot{x}^6}\right)\dot{x}^\mu\,, \tag{4.79}$$

where $c(\tau)$ is an arbitrary function of the curve parameter and $p^\mu(\tau)$ is the density of the level-zero momentum generator, see (3.63). However, since the left-hand side of the above equation contains three integrals but only one delta function (arising from the action of $p^\mu(\tau)$) the local term can probably not be rewritten in this fashion. Instead, it seems a reasonable assumption that the curve integral (4.78) in fact defines the local term in the sense that the complete level-one momentum generator is given by

$$P^{(1)\,\mu} := P^{(1)\,\mu}_{\mathrm{nl}} - \frac{7}{96}\,\frac{\lambda}{8\pi^2}\int \mathrm{d}s\,\left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x}\cdot\ddot{x})^2}{\dot{x}^6}\right)\dot{x}^\mu\,. \tag{4.80}$$

Indeed, the so-defined generator annihilates the expectation value $\langle\mathcal{W}(C)\rangle$ at leading order in perturbation theory and order zero in the superpath variables θ, $\bar\theta$ and y, i.e.

$$\begin{aligned} P^{(1)\,\mu}\left\langle \mathcal{W}(C)\right\rangle\Big|_{\substack{\theta=0\\ \bar\theta=0\\ y=0}} &= \left(P^{(1)\,\mu}_{\mathrm{nl}} - \frac{7}{96}\,\frac{\lambda}{8\pi^2}\int \mathrm{d}s\,\left(\frac{\ddot{x}^2}{\dot{x}^4} - \frac{(\dot{x}\cdot\ddot{x})^2}{\dot{x}^6}\right)\right)\left(1 + \left\langle \mathcal{W}(C)\right\rangle_{(1)} + \ldots\right)\Big|_{\substack{\theta=0\\ \bar\theta=0\\ y=0}} \\ &= 0 + \mathcal{O}(\lambda^2)\,. \end{aligned}$$

A natural question that arises is whether the definition (4.80) is consistent with the algebra relations (2.114). We note that the local term is translationally invariant, has the correct scaling weight and transforms as a vector under Lorentz transformations. Therefore, it is clear that three out of four conformal commutators which involve $P^{(1)\,\mu}$ work out correctly, i.e.

$$\left[P^{(0)\,\mu}, P^{(1)\,\nu}\right] = 0 \qquad \left[D^{(0)}, P^{(1)\,\mu}\right] = -P^{(1)\,\mu} \qquad \left[M^{(0)}_{\mu\nu}, P^{(1)}_{\lambda}\right] = \eta_{\nu\lambda}\,P^{(1)}_{\mu} - \eta_{\mu\lambda}\,P^{(1)}_{\nu}\,. \tag{4.81}$$

Of course, to be completely sure that the addition of the local term does not change the conformal level-one algebra relations one also needs to check the commutator between $K^{(0)\,\mu}$ and $P^{(1)\,\mu}$. However, a non-zero local term on the right-hand side of this commutator would probably only indicate that $M^{(1)}_{\mu\nu}$ and/or $D^{(1)}$ get local contributions as well. In a subsequent investigation one would then have to answer the question whether these local contributions are compatible with the remaining conformal level-one algebra relations. Now, while there is evidence that everything works out fine in the conformal sector, this does not apply to the superconformal sector. To see this, let us focus on the commutator between $Q^{(0)\,\alpha}_{A}$ and $P^{(1)\,\mu}$, which should vanish according to (2.114). Since the local term does not depend on the anticommuting fermionic coordinates, two of the three pieces of ${Q^{(0)}}^{\alpha}_{A}$, namely those which involve a Grassmann derivative, commute with our local term. However, as the structure of the third piece is the same as that of the x-part of $D^{(0)}$ with x replaced by $\bar{\theta}$, the complete commutator does obviously not vanish. But this probably only suggests that the local term also receives contributions in the anticommuting coordinates θ and $\bar{\theta}$. In sum, we have found substantial evidence that the supersymmetrized Maldacena-Wilson loop indeed possesses hidden Yangian symmetries.

5. Conclusions and Outlook

Quantum integrability has turned out to be one of the most important concepts to overcome the limitations of perturbation theory and to gain a profound understanding of quantum gauge theories. On the level of gauge invariant observables, integrability manifests itself through an infinite number of hidden non-local symmetries which, together with the generators of the global symmetry group $\mathrm{PSU}(2,2|4)$, form an infinite-dimensional quantum algebra of Yangian type.

In this thesis we focused on the class of smooth Maldacena-Wilson loops, investigating whether they possess such hidden non-local symmetries pointing to an underlying integrability. Since Maldacena-Wilson loops couple only to the bosonic fields of $\mathcal{N}=4$ SYM theory, our first attempt was to establish the invariance of their expectation values under transformations generated by elements of the Yangian algebra of $\mathfrak{so}(2,4)$. For this, we derived a functional representation of the conformal algebra that acts on the space of curves $x^\mu(s)$, then we explicitly constructed the non-local part of the level-one momentum generator and subsequently applied it to the one-loop expectation value of a smooth Maldacena-Wilson loop. The result, consisting of the sum of a single curve integral and a bi-local term, however showed that such an invariance does not exist. The functional form of the unwanted bi-local contribution, being that of a fermion propagator in position space, led us to the assumption that the expectation value of the supersymmetrically completed Maldacena-Wilson loop operator would be invariant under the non-local symmetry generated by the full Yangian level-one momentum generator $P^{(1)\,\mu} \in Y(\mathfrak{psu}(2,2|4))$. To verify this, we represented the superconformal algebra as functional derivatives acting on the space of superpaths $\{x^\mu(s), \theta_a^A(s), \bar{\theta}_{A\dot{\alpha}}(s), {y_A}^B(s)\}$ and discussed how the non-local part of the Yangian level-one momentum generator gets modified when the level-zero algebra is extended to the full superconformal algebra. Using supersymmetry as a guiding principle, we then established the supersymmetrically completed Maldacena-Wilson loop operator up to quadratic order in the anticommuting Grassmann variables and order zero in the bosonic coordinates ${y_A}^B(s)$. After computing the one-loop expectation value of the completed loop operator, we applied the non-local part of the full Yangian level-one momentum generator to it and projected onto the subspace of bosonic curves $x^\mu(s)$. This time, the unwanted bi-local contribution canceled out and we found that the non-local part of the generator annihilates the one-loop expectation value modulo a single curve integral, i.e. a local term. Finally, we concluded that the full invariance of the expectation value can be restored by defining the Yangian level-one momentum generator as the non-local piece shifted by this single curve integral.

The question whether smooth Maldacena-Wilson loops possess hidden non-local symmetries can also be investigated on the string side of the AdS/CFT duality, as has been done in our paper on the subject [50]. Compared to the perturbative discussion at weak coupling, it turned out that in the strong coupling limit it is possible to restrict to a purely bosonic discussion without fermions. More specifically, it was shown that there exist 15 conformal level-one generators which annihilate the expectation value of the Maldacena-Wilson loop in the limit $\lambda \to \infty$. The form of the level-one momentum generator at strong coupling agrees perfectly with the one we found on the gauge theory side, provided that we set to zero all θ-, $\bar{\theta}$- and y-dependence in our generator. However, the coefficient in front of the local term differs by a factor of $7/24$.

In conclusion, we have presented substantial evidence that smooth supersymmetric Maldacena-Wilson loops in $\mathcal{N} = 4$ SYM theory possess hidden Yangian symmetries. So far, our analysis on the gauge theory side has been limited to $P^{(1)\mu}$ and to the subsector defined by $\theta = \bar{\theta} = y = 0$. A natural continuation would be to push the analysis further to higher orders in the additional superspace coordinates. This would require the construction of the supersymmetrically completed Maldacena-Wilson loop operator to higher order in the fermionic coordinates and in y. However, to feel confident that everything works out fine in the y-sector, including the invariance under the level-zero generators which do not preserve the surface $y = 0$, the first y-terms should be derived anyway at some point. Another extension of the weak coupling discussion would be to go to two loop order and to perform the same type of calculations once again. This would not only be a nice consistency check, but also provide evidence for or against the existence of a non-trivial interpolation function $f(\lambda)$ in front of the local term, which could explain the difference between the coefficients. If such an interpolation function does not exist, the inclusion of fermions on the string side will most likely affect the local term. Beside these rather technical aspects, the question whether and how this hidden symmetries can be exploited to establish exact results for (supersymmetrically completed) Maldacena-Wilson loops opens an exciting field of investigation.

A. Appendix

A.1. Trace and Spinor Identities

A.1.1. Trace Identities

In this appendix we prove the following trace identities

$$\begin{aligned}
\mathrm{Tr}\,(\bar{\sigma}^{\mu}\,\sigma^{\nu}) &= 2\,\eta^{\mu\nu} \\
\mathrm{Tr}\,(\bar{\sigma}^{\mu}\,\sigma^{\nu}\,\bar{\sigma}^{\rho}\,\sigma^{\kappa}) &= 2\,(\eta^{\mu\nu}\,\eta^{\rho\kappa} + \eta^{\nu\rho}\,\eta^{\mu\kappa} - \eta^{\mu\rho}\,\eta^{\nu\kappa} - i\,\varepsilon^{\mu\nu\rho\kappa}) \\
\mathrm{Tr}\,(\sigma^{\mu}\,\bar{\sigma}^{\nu}\,\sigma^{\rho}\,\bar{\sigma}^{\kappa}) &= 2\,(\eta^{\mu\nu}\,\eta^{\rho\kappa} + \eta^{\nu\rho}\,\eta^{\mu\kappa} - \eta^{\mu\rho}\,\eta^{\nu\kappa} + i\,\varepsilon^{\mu\nu\rho\kappa})\ .
\end{aligned} \tag{A.1}$$

We will do this by repeatedly using the Clifford algebra relation (2.10) as well as the cyclicity of the trace. In terms of the sigma matrices the Clifford algebra relation reads

$$\bar{\sigma}^{\mu}_{\alpha\dot{\gamma}}\,\sigma^{\nu\dot{\gamma}\beta} + \bar{\sigma}^{\nu}_{\alpha\dot{\gamma}}\,\sigma^{\mu\dot{\gamma}\beta} = 2\,\eta^{\mu\nu}\,\delta^{\beta}_{\alpha} \qquad\qquad \sigma^{\mu\dot{\alpha}\gamma}\,\bar{\sigma}^{\nu}_{\gamma\dot{\beta}} + \sigma^{\nu\,\dot{\alpha}\gamma}\,\bar{\sigma}^{\mu}_{\gamma\dot{\beta}} = 2\,\eta^{\mu\nu}\,\delta^{\dot{\alpha}}_{\dot{\beta}}\,. \tag{A.2}$$

The first identity of (A.1) can easily be seen to hold true by taking the trace of one of the above equations and using (2.7).

$$2\ \mathrm{Tr}\,(\bar{\sigma}^{\mu}\,\sigma^{\nu}) = 4\,\eta^{\mu\nu} \tag{A.3}$$

In order to prove the two latter identities we calculate

$$\begin{aligned}
\mathrm{Tr}\,(\bar{\sigma}^{\mu}\,\sigma^{\nu}\,\bar{\sigma}^{\rho}\,\sigma^{\kappa}) &= 2\,\eta^{\mu\nu}\ \mathrm{Tr}\,(\bar{\sigma}^{\rho}\,\sigma^{\kappa}) - \mathrm{Tr}\,(\bar{\sigma}^{\nu}\,\sigma^{\mu}\,\bar{\sigma}^{\rho}\,\sigma^{\kappa}) \\
&= 4\,\eta^{\mu\nu}\,\eta^{\rho\kappa} - 2\,\eta^{\mu\rho}\ \mathrm{Tr}\,(\bar{\sigma}^{\nu}\,\sigma^{\kappa}) + \mathrm{Tr}\,(\bar{\sigma}^{\nu}\,\sigma^{\rho}\,\bar{\sigma}^{\mu}\,\sigma^{\kappa}) \\
&= 4\,\eta^{\mu\nu}\,\eta^{\rho\kappa} - 4\,\eta^{\mu\rho}\,\eta^{\nu\kappa} + 4\,\eta^{\mu\kappa}\,\eta^{\nu\rho} - \mathrm{Tr}\,(\bar{\sigma}^{\nu}\,\sigma^{\rho}\,\bar{\sigma}^{\kappa}\,\sigma^{\mu})\ ,
\end{aligned} \tag{A.4}$$

where we made repeated use of (A.2). The cyclicity of the trace allows us to rewrite this as follows

$$\mathrm{Tr}\,(\bar{\sigma}^{\mu}\,\sigma^{\nu}\,\bar{\sigma}^{\rho}\,\sigma^{\kappa}) + \mathrm{Tr}\,(\sigma^{\mu}\,\bar{\sigma}^{\nu}\,\sigma^{\rho}\,\bar{\sigma}^{\kappa}) = 4\,\eta^{\mu\nu}\,\eta^{\rho\kappa} - 4\,\eta^{\mu\rho}\,\eta^{\nu\kappa} + 4\,\eta^{\mu\kappa}\,\eta^{\nu\rho}\,. \tag{A.5}$$

To complete the proof we compute the difference between these two trace expressions. To do this, it is useful to note that the difference is completely antisymmetric under the exchange of two Lorentz indices. This can be verified by using (A.2). For example, we have

$$\begin{aligned}
\mathrm{Tr}\,(\bar{\sigma}^{\nu}\,\sigma^{\mu}\,\bar{\sigma}^{\rho}\,\sigma^{\kappa}) - \mathrm{Tr}\,(\sigma^{\nu}\,\bar{\sigma}^{\mu}\,\sigma^{\rho}\,\bar{\sigma}^{\kappa}) &= +\,2\,\eta^{\nu\mu}\ \mathrm{Tr}\,(\bar{\sigma}^{\rho}\,\sigma^{\kappa}) - \mathrm{Tr}\,(\bar{\sigma}^{\mu}\,\sigma^{\nu}\,\bar{\sigma}^{\rho}\,\sigma^{\kappa}) \\
&\quad - 2\,\eta^{\nu\mu}\ \mathrm{Tr}\,(\sigma^{\rho}\,\bar{\sigma}^{\kappa}) + \mathrm{Tr}\,(\sigma^{\mu}\,\bar{\sigma}^{\nu}\,\sigma^{\rho}\,\bar{\sigma}^{\kappa}) \\
&= -\,\mathrm{Tr}\,(\bar{\sigma}^{\mu}\,\sigma^{\nu}\,\bar{\sigma}^{\rho}\,\sigma^{\kappa}) + \mathrm{Tr}\,(\sigma^{\mu}\,\bar{\sigma}^{\nu}\,\sigma^{\rho}\,\bar{\sigma}^{\kappa})\ .
\end{aligned} \tag{A.6}$$

Since there is only one completely antisymmetric four-tensor in four dimensions, we can make the following ansatz

$$\mathrm{Tr}\,(\bar{\sigma}^\mu\,\sigma^\nu\,\bar{\sigma}^\rho\,\sigma^\kappa) - \mathrm{Tr}\,(\sigma^\mu\,\bar{\sigma}^\nu\,\sigma^\rho\,\bar{\sigma}^\kappa) = c\,\varepsilon^{\mu\nu\rho\sigma}\,. \tag{A.7}$$

The coefficient c is easily determined by evaluating the above equation for the case that $\mu = 0$, $\nu = 1$, $\rho = 2$ and $\kappa = 3$. The result reads

$$\mathrm{Tr}\,(\bar{\sigma}^\mu\,\sigma^\nu\,\bar{\sigma}^\rho\,\sigma^\kappa) - \mathrm{Tr}\,(\sigma^\mu\,\bar{\sigma}^\nu\,\sigma^\rho\,\bar{\sigma}^\kappa) = -4\,i\,\varepsilon^{\mu\nu\rho\sigma}\,. \tag{A.8}$$

The trace identities follow now by combining the two equations (A.5) and (A.8).

A.1.2. Spinor Identity

In this subsection we prove that the following identity holds true

$$F^{\alpha\dot{\alpha}\beta\dot{\beta}} = \frac{i}{2}\,\varepsilon^{\dot{\alpha}\dot{\beta}}\,F^{\alpha\beta} + \frac{i}{2}\,\varepsilon^{\alpha\beta}\,F^{\dot{\alpha}\dot{\beta}}\,, \tag{A.9}$$

where

$$F^{\alpha\dot{\alpha}\beta\dot{\beta}} := F_{\mu\nu}\,\bar{\sigma}^{\mu\alpha\dot{\alpha}}\,\bar{\sigma}^{\nu\beta\dot{\beta}}\,,$$

and

$$F^{\alpha\beta} := \frac{i}{2}\,F_{\mu\nu}\,\varepsilon^{\alpha\gamma}\left(\bar{\sigma}^\mu_{\gamma\dot{\gamma}}\,\sigma^{\nu\dot{\gamma}\beta} - \bar{\sigma}^\nu_{\gamma\dot{\gamma}}\,\sigma^{\mu\dot{\gamma}\beta}\right) \tag{A.10}$$

$$F^{\dot{\alpha}\dot{\beta}} := \frac{i}{2}\,F_{\mu\nu}\,\varepsilon^{\dot{\gamma}\dot{\beta}}\left(\sigma^{\mu\dot{\alpha}\gamma}\,\bar{\sigma}^\nu_{\gamma\dot{\gamma}} - \sigma^{\nu\dot{\alpha}\gamma}\,\bar{\sigma}^\mu_{\gamma\dot{\gamma}}\right). \tag{A.11}$$

We start by substituting the definitions (A.10) and (A.11) into the right-hand side of (A.9). This yields the following expression

$$\mathrm{rhs(A.9)} = -\frac{1}{4}\,\varepsilon^{\dot{\alpha}\dot{\beta}}\,F_{\mu\nu}\left(\bar{\sigma}^{\mu\alpha}{}_{\dot{\gamma}}\,\sigma^{\nu\dot{\gamma}\beta} - \bar{\sigma}^{\nu\alpha}{}_{\dot{\gamma}}\,\sigma^{\mu\dot{\gamma}\beta}\right) - \frac{1}{4}\,\varepsilon^{\alpha\beta}\,F_{\mu\nu}\left(\sigma^{\mu\dot{\alpha}\gamma}\,\bar{\sigma}^\nu{}_\gamma{}^{\dot{\beta}} - \sigma^{\nu\dot{\alpha}\gamma}\,\bar{\sigma}^\mu{}_\gamma{}^{\dot{\beta}}\right).$$

Starting from (2.26), one easily derives the following identities

$$\begin{aligned}\varepsilon^{\alpha\beta}\,\Lambda^\gamma{}_\gamma &= \Lambda^{\beta\alpha} - \Lambda^{\alpha\beta}\\ \varepsilon^{\dot{\alpha}\dot{\beta}}\,\Lambda_{\dot{\gamma}}{}^{\dot{\gamma}} &= \Lambda^{\dot{\beta}\dot{\alpha}} - \Lambda^{\dot{\alpha}\dot{\beta}}\,,\end{aligned} \tag{A.12}$$

which are a consequence of the fact that there is only one antisymmetric tensor in two dimensions. Using these identities, we can rewrite the above expression as follows

$$\begin{aligned}\mathrm{rhs(A.9)} = &-\frac{1}{4}\,F_{\mu\nu}\left(\bar{\sigma}^{\mu\alpha\dot{\beta}}\,\sigma^{\nu\dot{\alpha}\beta} - \bar{\sigma}^{\nu\alpha\dot{\beta}}\,\sigma^{\mu\dot{\alpha}\beta}\right) + \frac{1}{4}\,F_{\mu\nu}\left(\bar{\sigma}^{\mu\alpha\dot{\alpha}}\,\sigma^{\nu\dot{\beta}\beta} - \bar{\sigma}^{\nu\alpha\dot{\alpha}}\,\sigma^{\mu\dot{\beta}\beta}\right)\\ &-\frac{1}{4}\,F_{\mu\nu}\left(\sigma^{\mu\dot{\alpha}\beta}\,\bar{\sigma}^{\nu\alpha\dot{\beta}} - \sigma^{\nu\dot{\alpha}\beta}\,\bar{\sigma}^{\mu\alpha\dot{\beta}}\right) + \frac{1}{4}\,F_{\mu\nu}\left(\sigma^{\mu\dot{\alpha}\alpha}\,\bar{\sigma}^{\nu\beta\dot{\beta}} - \sigma^{\nu\dot{\alpha}\alpha}\,\bar{\sigma}^{\mu\beta\dot{\beta}}\right).\end{aligned} \tag{A.13}$$

The first two terms in the first line cancel against the first two terms in the second line. To complete the proof we relabel Lorentz indices in half of the remaining terms and employ the antisymmetry of the tensor $F_{\mu\nu}$. Finally, we use the identification property of sigma matrices (2.7).

$$\begin{aligned}\mathrm{rhs(A.9)} =&\frac{1}{4}\,F_{\mu\nu}\left(\bar{\sigma}^{\mu\alpha\dot{\alpha}}\,\bar{\sigma}^{\nu\beta\dot{\beta}} + \bar{\sigma}^{\mu\alpha\dot{\alpha}}\,\bar{\sigma}^{\nu\beta\dot{\beta}}\right) + \frac{1}{4}\,F_{\mu\nu}\left(\bar{\sigma}^{\mu\alpha\dot{\alpha}}\,\bar{\sigma}^{\nu\beta\dot{\beta}} + \bar{\sigma}^{\mu\alpha\dot{\alpha}}\,\bar{\sigma}^{\nu\beta\dot{\beta}}\right)\\ =&F_{\mu\nu}\,\bar{\sigma}^{\mu\alpha\dot{\alpha}}\,\bar{\sigma}^{\nu\beta\dot{\beta}}\\ =&F^{\alpha\dot{\alpha}\beta\dot{\beta}}\end{aligned} \tag{A.14}$$

A.2. The Dual Structure Constants of $\mathfrak{so}(2,4)$

In this appendix we determine the dual structure constants f_a^{bc} of the conformal Lie algebra $\mathfrak{so}(2,4)$. These dual structure constants appear in the definition of the non-local part of the Yangian generator (3.83). Thus, we need them to construct the level-one momentum generator.

If we denote the conformal generators $\{P_\mu, M_{\mu\nu}, D, K_\mu\}$ collectively by J_a, the commutation relations of the conformal algebra (3.59) can formally be written as follows

$$\Big[J_a, J_b\Big] = f_{ab}^c\, J_c\,. \tag{A.15}$$

The dual structure constants f_a^{bc} are now those, which appear in the commutation relations of the dual generators. Formally, we have

$$\Big[\hat{J}^a, \hat{J}^b\Big] = f_c^{ab}\, \hat{J}^c\,, \tag{A.16}$$

where $\hat{J}^a$ are the dual generators defined by

$$\hat{J}^a = K^{ab}\, J_b\,. \tag{A.17}$$

Here, K^{ab} is the inverse of the Killing metric (2.99) and is given by

$$K_{ab} = \Big\langle J_a, J_b\Big\rangle = \mathrm{Tr}\Big(\mathrm{ad}(J_a)\,\mathrm{ad}(J_b)\Big)\,, \tag{A.18}$$

where $\mathrm{ad}(J_a)$ stands for a generator in the adjoint representation. From the definition of the dual generators it is clear that they are orthogonal to the ordinary generators with respect to the pseudo inner product defined by the Killing metric, i.e.

$$\Big\langle \hat{J}^a, J_b\Big\rangle = K^{ac}\Big\langle J_c, J_b\Big\rangle = K^{ac}\,K_{cb} = \delta_b^a\,. \tag{A.19}$$

The last relation can also be seen as the defining relation for the dual generators. In order to perform explicit calculations, it is convenient to perform a change of basis in the Lie algebra such that the new basis coincides with the standard basis of $\mathfrak{so}(2,4)$. Explicitly, this change of basis is established by the relations

$$D = \mathbf{M}_{44'} \qquad P_\mu = \mathbf{M}_{\mu 4'} + \mathbf{M}_{\mu 4} \qquad K_\mu = \mathbf{M}_{\mu 4'} - \mathbf{M}_{\mu 4} \qquad M_{\mu\nu} = \mathbf{M}_{\mu\nu}\,, \tag{A.20}$$

where $\mathbf{M}_{MN}$ are the generators of $\mathfrak{so}(2,4)$ with indices $M, N \in \{\mu, 4, 4'\}$. The commutation relations satisfied by the $\mathbf{M}_{MN}$ read

$$\begin{aligned}\Big[\mathbf{M}_{MN}, \mathbf{M}_{KL}\Big] &= \eta_{ML}\,\mathbf{M}_{NK} + \eta_{NK}\,\mathbf{M}_{ML} - \eta_{MK}\,\mathbf{M}_{NL} - \eta_{NL}\,\mathbf{M}_{MK} \\ &= \Big(\eta_{ML}\,\delta_N^X\,\delta_K^Y + \eta_{NK}\,\delta_M^X\,\delta_L^Y - \eta_{MK}\,\delta_N^X\,\delta_L^Y - \eta_{NL}\,\delta_M^X\,\delta_K^Y\Big)\,\mathbf{M}_{XY}\,,\end{aligned} \tag{A.21}$$

where $\eta_{ML} = \mathrm{diag}(1,-1,-1,-1,-1,1)$. Let us now calculate the Killing metric on this algebra. From the above equation we can directly read off the adjoint representation of the generators. One finds

$$\left(\mathbf{M}_{MN}\right)^{XY}{}_{KL} = 2\,\Big(\eta_{ML}\,\delta_N^{[X}\,\delta_K^{Y]} + \eta_{NK}\,\delta_M^{[X}\,\delta_L^{Y]} - \eta_{MK}\,\delta_N^{[X}\,\delta_L^{Y]} - \eta_{NL}\,\delta_M^{[X}\,\delta_K^{Y]}\Big)\,, \tag{A.22}$$

where the square brackets denote antisymmetrization including a factor of 1/2. Note that the index pairs (X,Y) and (K,L) only take ordered values with $X<Y$ and $K<L$, so that the number of rows and columns really equals the number of linearly independent generators. The Killing metric is now easily obtained by taking the trace over the product of two generators in the adjoint representation.

$$\begin{aligned}\Big\langle \mathbf{M}_{MN}, \mathbf{M}_{KL}\Big\rangle &= \sum_{X<Y}\sum_{Z<W} 4\left(\mathbf{M}_{MN}\right)^{XY}{}_{ZW}\left(\mathbf{M}_{KL}\right)^{ZW}{}_{XY} \\ &= \sum_{X,Y,Z,W}\Big(\eta_{MW}\,\delta_N^{[X}\,\delta_Z^{Y]} + \eta_{NZ}\,\delta_M^{[X}\,\delta_W^{Y]} - \eta_{MZ}\,\delta_N^{[X}\,\delta_W^{Y]} - \eta_{NW}\,\delta_M^{[X}\,\delta_Z^{Y]}\Big) \\ &\qquad\quad \times\Big(\eta_{KY}\,\delta_L^{[Z}\,\delta_X^{W]} + \eta_{LX}\,\delta_K^{[Z}\,\delta_Y^{W]} - \eta_{KX}\,\delta_L^{[Z}\,\delta_Y^{W]} - \eta_{LY}\,\delta_K^{[Z}\,\delta_X^{W]}\Big)\end{aligned} \tag{A.23}$$

In going from the first to the second line we replaced the ordered sums by sums where all indices run over the full range. Simultaneously, we multiplied the whole expression by a factor of 1/4 in order to correct for the overcounting. Since all sums are now over the full index range and the latter factor in the product is already antisymmetric in X and Y, we can drop the antisymmetrization brackets in the first term. Using an analogue argument for Z and W we can drop the square brackets in the second term as well. We then get the following expression for the Killing metric

$$\begin{aligned}\Big\langle \mathbf{M}_{MN}, \mathbf{M}_{KL}\Big\rangle &= \sum_{X,Y,Z,W}\Big(\eta_{MW}\,\delta_N^{X}\,\delta_Z^{Y} + \eta_{NZ}\,\delta_M^{X}\,\delta_W^{Y} - \eta_{MZ}\,\delta_N^{X}\,\delta_W^{Y} - \eta_{NW}\,\delta_M^{X}\,\delta_Z^{Y}\Big) \\ &\qquad\quad \times\Big(\eta_{KY}\,\delta_L^{Z}\,\delta_X^{W} + \eta_{LX}\,\delta_K^{Z}\,\delta_Y^{W} - \eta_{KX}\,\delta_L^{Z}\,\delta_Y^{W} - \eta_{LY}\,\delta_K^{Z}\,\delta_X^{W}\Big) \\ &= 16\,\eta_{L[M}\,\eta_{N]K}\,.\end{aligned} \tag{A.24}$$

Using this metric, it is easy to convince oneself that the following generators

$$\hat{D} = \frac{1}{8}\,D \qquad \hat{P}^\mu = -\frac{1}{16}\,\eta^{\mu\nu}\,K_\nu \qquad \hat{K}^\mu = -\frac{1}{16}\,\eta^{\mu\nu}\,P_\nu \qquad \hat{M}^{\mu\nu} = -\frac{1}{8}\,\eta^{\mu\lambda}\eta^{\nu\rho}\,M_{\lambda\rho}\,, \tag{A.25}$$

satisfy the defining relation (A.17) and are therefore the so-called dual generators we are looking for. As such, they can be shown to satisfy the following commutation relations

$$\begin{aligned}\Big[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}\Big] &= \frac{1}{8}\Big(\eta^{\mu\rho}\,\hat{M}^{\nu\sigma} + \eta^{\nu\sigma}\,\hat{M}^{\mu\rho} - \eta^{\mu\sigma}\,\hat{M}^{\nu\rho} - \eta^{\nu\rho}\,\hat{M}^{\mu\sigma}\Big) \\ \Big[\hat{M}^{\mu\nu}, \hat{P}^{\lambda}\Big] &= \frac{1}{8}\Big(\eta^{\mu\lambda}\,\hat{P}^{\nu} - \eta^{\nu\lambda}\,\hat{P}^{\mu}\Big) \qquad\qquad \Big[\hat{P}^{\mu}, \hat{P}^{\nu}\Big] = 0 \\ \Big[\hat{P}^{\mu}, \hat{K}^{\nu}\Big] &= \frac{1}{16}\Big(\eta^{\mu\nu}\hat{D} - \hat{M}^{\mu\nu}\Big) \qquad\qquad \Big[\hat{D}, \hat{K}^{\mu}\Big] = -\frac{1}{8}\,\hat{K}^{\mu} \\ \Big[\hat{M}^{\mu\nu}, \hat{K}^{\rho}\Big] &= \frac{1}{8}\Big(\eta^{\mu\rho}\,\hat{K}^{\nu} - \eta^{\nu\rho}\,\hat{K}^{\mu}\Big) \qquad\qquad \Big[\hat{D}, \hat{P}^{\mu}\Big] = \frac{1}{8}\,\hat{P}^{\mu} \\ \Big[\hat{K}^{\mu}, \hat{K}^{\nu}\Big] &= 0 \qquad\qquad \Big[\hat{D}, \hat{M}^{\mu\nu}\Big] = 0\,.\end{aligned} \tag{A.26}$$

The dual structure constants of interest are those, where there appears a P^μ on the right-hand side of the algebra relations. Explicitly, one reads off the following dual structure constants

$$f_{P_\mu}^{\hat{D}\,\hat{P}^\rho} = \frac{1}{8}\,\delta_\mu^\rho = -f_{P_\mu}^{\hat{P}^\rho\,\hat{D}} \qquad\qquad f_{P_\rho}^{\hat{M}^{\mu\nu}\,\hat{P}^\lambda} = \frac{1}{8}\Big(\eta^{\mu\lambda}\,\delta_\rho^\nu - \eta^{\nu\lambda}\,\delta_\rho^\mu\Big) = -f_{P_\rho}^{\hat{P}^\lambda\,\hat{M}^{\mu\nu}}\,. \tag{A.27}$$

A.3. Local Supersymmetry of the MWL

In this appendix we prove that the Maldacena-Wilson loop operator (3.27) is, at least for time-like curves $x^\mu(s)$, locally 1/2 BPS. To do this, we need to show that the equation

$$A\,\xi = 0 \qquad\qquad A := \left(\Gamma_\mu\,\dot{x}^\mu + \Gamma_i\,n^i\,|\dot{x}|\right)\,, \tag{A.28}$$

has eight linearly independent Majorana-Weyl solutions for any given s. Using the Clifford algebra relation $\{\Gamma_M, \Gamma_N\} = 2\,g_{MN}$ we readily verify that the above given matrix squares to zero.

$$A^2 = \tfrac{1}{2}\Big(\big\{\Gamma_\mu, \Gamma_\nu\big\}\,\dot{x}^\mu\,\dot{x}^\nu + \big\{\Gamma_i, \Gamma_j\big\}\,n^i\,n^j\,\dot{x}^2\Big) = 0 \tag{A.29}$$

Given this property, it is straightforward to show that zero is the only eigenvalue of A and, moreover, by considering the Jordan normal form of the matrix one easily convinces oneself that A can at most have rank 16. Thus, there exist at least 16 linearly independent eigenvectors for any given s. Now, the question that we have to answer is whether one can construct eight linearly independent Majorana-Weyl spinors from these eigenvectors. To do this, we take the following approach. First, we will explicitly solve the equation (A.28) for the case that

$$\dot{z}^M_{sp} = \begin{pmatrix}1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T \qquad\qquad \text{with} \quad \dot{z}^M = \left(\dot{x}^\mu \;\; n^i|\dot{x}|\right), \tag{A.30}$$

where we have employed the ten-dimensional notation of section 2.2. We will find that in this case there indeed exist eight independent Majorana-Weyl spinors which get mapped to zero by the corresponding A. Subsequently, we shall explain how one can use these solutions to construct the eight Majorana-Weyl solutions to a general A corresponding to an arbitrary light-like $\dot{z}^M$. So, let us start by solving the equation

$$(\Gamma_0 + \Gamma_4)\;\xi = 0\,. \tag{A.31}$$

We note that since this linear combination of gamma matrices squares to zero, every non-zero column of the above matrix is in fact a solution to (A.31). For this reason, we do not need to perform any explicit calculations. We just project out the eigenvectors which do not satisfy the Weyl condition (2.49) by applying the appropriate projection operator from the left. Using this prescription and the explicit form of the gamma matrices presented in section 2.1.3, we find the following eight eigenvectors

$$\begin{aligned}
&\xi_{1W} = \begin{pmatrix}1_1\\ -1_{31}\end{pmatrix} \quad \xi_{2W} = \begin{pmatrix}1_2\\ -1_{32}\end{pmatrix} \quad \xi_{3W} = \begin{pmatrix}1_5\\ -1_{27}\end{pmatrix} \quad \xi_{4W} = \begin{pmatrix}1_6\\ -1_{28}\end{pmatrix}\\
&\xi_{5W} = \begin{pmatrix}1_9\\ 1_{23}\end{pmatrix} \quad \xi_{6W} = \begin{pmatrix}1_{10}\\ 1_{24}\end{pmatrix} \quad \xi_{7W} = \begin{pmatrix}1_{13}\\ 1_{19}\end{pmatrix} \quad \xi_{8W} = \begin{pmatrix}1_{14}\\ 1_{20}\end{pmatrix}\,,
\end{aligned} \tag{A.32}$$

where the subscript denotes the position of the respective entry in the 32-component vector and all other entries are zero. Note that the above given solutions ξ_{iW} satisfy the Weyl condition (2.49), but not the Majorana condition (2.54). It is however easy to convince oneself that the following linear combinations satisfy both conditions

$$\xi_1 = \xi_{1W} - \xi_{8W} \qquad \xi_3 = \xi_{2W} + \xi_{7W} \qquad \xi_5 = \xi_{3W} - \xi_{6W} \qquad \xi_7 = \xi_{4W} + \xi_{5W}$$

$$\xi_2 = i(\xi_{1W} + \xi_{8W}) \quad \xi_4 = i(\xi_{2W} - \xi_{7W}) \quad \xi_6 = i(\xi_{3W} + \xi_{6W}) \quad \xi_8 = i(\xi_{4W} - \xi_{5W})\,. \tag{A.33}$$

Thus, we have obtained eight linearly independent Majorana-Weyl solutions to the equation

$$\Gamma_M\, \dot{z}^M_{sp}\, \xi = 0\,, \tag{A.34}$$

which is equation (A.31) written in ten-dimensional space. Having solved (A.28) for the special $\dot{z}^M_{sp}$ mentioned above, we can now construct the solutions for a general light-like $\dot{z}^M$ by performing a Lorentz transformation. Indeed, it is well-known that there always exist a Lorentz transformation such that

$$\dot{z}^M_{sp} = \Lambda^M{}_N\, \dot{z}^N\,, \tag{A.35}$$

where $\dot{z}^N$ is an arbitrary ten-dimensional light-like vector. Now, let $S(\Lambda)$ denote the associated transformation in the spinor space. The claim then is that

$$S(\Lambda)^{-1}\,\xi\,, \tag{A.36}$$

solves equation (A.28) for the corresponding light-like $\dot{z}^M$, see (A.35). To prove this, we start from the expression (A.34), insert an identity and multiply the whole equation by $S(\Lambda)^{-1}$.

$$S(\Lambda)^{-1}\,\Gamma_M\,\dot{z}^M_{sp}\,S(\Lambda)\,S(\Lambda)^{-1}\,\xi = 0 \tag{A.37}$$

Using the identity

$$S(\Lambda)^{-1}\,\Gamma_M\,S(\Lambda) = \left(\Lambda^{-1}\right)^N{}_M\,\Gamma_N\,, \tag{A.38}$$

we find

$$\Gamma_N\,\left(\Lambda^{-1}\right)^N{}_M\,\dot{z}^M_{sp}\,S(\Lambda)^{-1}\,\xi = 0\,, \tag{A.39}$$

which translates to

$$\Gamma_N\,\dot{z}^N\,S(\Lambda)^{-1}\,\xi = 0\,. \tag{A.40}$$

Hence, the statement is proved. Since the Majorana and the Weyl condition are Lorentz invariant, we have shown that for time-like contours $x^\mu(s)$ or, stated differently, for light-like curves $z^M(s)$, there always exist eight linearly independent Majorana-Weyl solutions to equation (A.28) for any given s. Having proved this, the question arises what happens if the contour $x^\mu(s)$ is space-like. The answer is that in this case there exist eight linearly independent Weyl solutions for any point along the loop but one cannot build a single Majorana solution out of this set. Accordingly, there are still eight linear combinations of $\mathfrak{q}^\alpha_A$ and $\bar{\mathfrak{q}}^{A\dot{\alpha}}$ which locally annihilate the Maldacena-Wilson loop operator, but the parameters of the transformation do not satisfy the Majorana condition (2.54). However, since $\mathfrak{q}^\alpha_A$ and $\bar{\mathfrak{q}}^{A\dot{\alpha}}$ are independent symmetries of the theory it might be a valid position to say that in this case the loop operator is still locally 1/2 BPS while the ten-dimensional embedding is lost.

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Acknowledgments

First of all, I wish to express my sincere gratitude to my supervisor Prof. Jan Plefka for giving me the opportunity to work on this challenging subject and for continuously offering his time to facilitate the progress of my work. Over the last months we had many fruitful and enlightening discussions and I am aware that without his profound encouragement and his willingness to share his ideas and insights with me this thesis would not have been possible.
I also would like to thank Prof. Matthias Staudacher, who kindly offered his support as the second examiner of this thesis.
Last but not least, I greatly benefited from the stimulating and productive collaboration with Prof. Konstantin Zarembo, Jonas Pollok and especially Hagen Münkler, whose sharp analytical understanding and thorough knowledge I admire and whose friendship I value.

Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Berlin, den 30.11.2013