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Supersymmetric Wilson Loops in the AdS/CFT Correspondence

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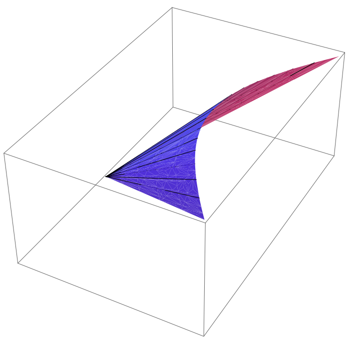

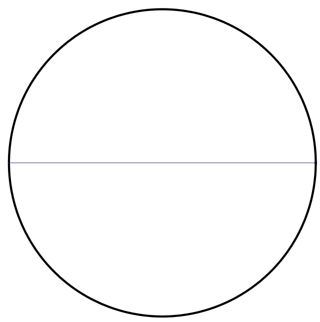

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Zusammenfassung

Diese Arbeit untersucht Wilson-Schleifen im Rahmen der AdS/CFT Korrespondenz. Auf der Eichtheorieseite der Dualität werden supersymmetrische Wilson-Schleifen auf den Mannigfaltigkeiten $\mathbb{R}^4$, S^3, $\mathbb{R}^{1,3}$ und $\mathbb{H}_3$ untersucht. Insbesondere wird erforscht, wieviele Supersymmetrien generelle, sowie einige spezielle Wilson-Schleifen erhalten. Für einige der Wilson-Schleifen kann der Erwartungswert in der Eichtheorie explizit berechnet werden. Auf der Stringtheorieseite der Korrespondenz werden die dualen Wilson-Schleifen untersucht und ihr Erwartungswert mit der Eichtheorie verglichen.

Schlüsselwörter

Supersymmetrie, Wilson-Schleifen, Stringtheorie, AdS/CFT

Abstract

This thesis investigates Wilson loops in the context of the AdS/CFT correspondence. On the gauge theory side Wilson loops on the mannifolds $\mathbb{R}^4$, S^3, $\mathbb{R}^{1,3}$ and $\mathbb{H}_3$ are examined. In particular the focus is put on how many supersymmetries are preserved by general as well as special Wilson loops. For some of the loops we can explicitly calculate the expectation value in the gauge theory. On the string theory side of the correspondence we calculate the dual Wilson loops and compare their expectation value to the gauge theory.

Keywords

Supersymmetry, Wilson Loops, String Theory, AdS/CFT

Hilfsmittel

Diese Diplomarbeit wurde mit $\LaTeX\,2_\varepsilon$ gesetzt. Die in dieser Arbeit enthaltenen Grafiken und Rechnungen wurden unter Einbeziehung von Mathematica 6 erstellt.

Selbstständigkeitserklärkung

Hiermit erkläre ich, die vorliegende Diplomarbeit selbständig sowie ohne unerlaubte fremde Hilfe verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

Mit der Auslage meiner Diplomarbeit in den Bibliotheken der Humboldt-Universität zu Berlin bin ich einverstanden.

Berlin, 22. April 2008

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This diploma thesis is available at:
http://qft.physik.hu-berlin.de.

Contents

I
Introduction

Since Einstein, physicists dream of unifying different fundamental theories into a single theory, a so called theory of everything (TOE). But on the other hand it was also Einstein, who laid the foundation for two theories, namely the theory of general relativity and quantum field theory, which up two now still cannot be unified. The theory of general relativity is a classical description of gravity, valid at large distances, whereas quantum field theory charaterizes the behaviour of atoms, molecules and subatomic particles at small distances. Although both theories are confirmed by experiment with enormous precision, it is still unknown how to unify them.

Today we know four fundamental interactions in nature. Apart from the gravitational and the electromagnetic force, there are the weak and the strong interaction. Except for gravity, all other forces can succesfully be described by renormalizable quantum field theories; this led to the famous *standard model* of elementary particles. When trying to apply the concepts of quantum field theory to gravity one faces a big obstacle, namely that there is no consistent way of quantizing gravity due to its nonrenormalizability. One of the aims of string theory is to be a theory of quantum gravity. Although string theory is still far away from beeing well understood, it is nevertheless one of the few consistent candidates. Historically, string theory was presented to be a description of the strong interaction, which guides the behaviour between nucleons. The bosonic string, which was first considered in the end of the 1960's suffered from many unphysical properties, such as containing a tachyon, being only consistent in 26 spacetime dimension and the existence of massless spin 2 particles [1]. These are of course misplaced in a theory describing short distance physics. Additionally, there was the rise of another theory, called *quantum chromodynamics (QCD)*, which could accurately describe the experimental data. Until today, QCD can be regarded as a successfull theory, since it explains the phenomena of confinement and asymptotic freedom, but because it is a strongly coupled theory it is hard to perform concrete calculations.

Another important idea that unifies different concepts of fundamental physics is *supersymmetry*. Supersymmetry provides a link between particles that are responsible for mediating forces (bosons) and particles that constitute the matter (fermions) on which the forces act. The idea of supersymmetry has had a wide impact on theoretical physics. Though it is still an open question if supersymmetry is realized in nature, which will hopefully be answered by the LHC in near future. Apart from the question if supersymmetry is a physical concept, it can be seen as a tool, which can simplify calculations in field theories drastically.

If one unifies the bosonic string and the concept of supersymmetry, leading to superstring theory, one gets rid of the tachyon in the spectrum of the superstring

and the number of dimensions needed for consistency is reduced to ten. The existence of massless spin 2 particles can then be reinterpreted as a description of gravity.
At present, superstring theory is also heading towards another direction. Already in the 1970's 't Hooft speculated that in a certain limit, non-abelian gauge theories might also be described by a string theory [2]; but for a concrete formulation of this idea people had to wait until 1997, the year in which Maldacena formulated the famous *AdS/CFT correspondence* [3]. This correspondence links two totally different theories with each other. On the one hand, one has a maximally supersymmetric non-abelian gauge theory in four dimensions and on the other hand one deals with a ten dimensional string theory on the manifold $AdS_5 \times S^5$. The correspondence states that both theories can be used to describe the same physics. This is highly non-trivial, since it is a duality linking a weakly and and a strongly coupled theory.
One of the most interesting observables in the context of the AdS/CFT correspondence is the *Wilson loop*. It is a gauge invariant non-local operator, whose importance in non-abelian gauge theories has been known for a long time. One year after the proposal of the AdS/CFT correspondence again Maldacena suggested how the Wilson loop operator has to be interpreted in the context of the correspondence: On the string theory side it is decribed by a minimal surface which ends on the contour of the Wilson loop [4], whereas in the gauge theory one has to extend the usual Wilson loop operator in such a way that it is invariant under supersymmetry variation. Calculating the expectation value of the circular Wilson loop on both sides of the correspondence has been one of the first successfull tests of the duality. Up to now a lot of effort has been put in finding constructions that can be applied to general Wilson loops. The first construction of this kind was given by Zarembo [5], where the loops are constrained to flat space $\mathbb{R}^4$. A construction for loops that are bound to S^3 was put forward by Drukker et. al. [6], [7], [8].
In this thesis, we will focus on Wilson loops in Minkowskian signature. Recently, it was proposed how to calculate gluon scattering amplitudes at strong coupling using lightlike Wilson loops; this is the reason why a lot of effort was made to investigate Wilson loops in Minkowskian signature. Some of the results presented in this thesis can be applied to improve the current research.

Outline

This thesis is organized as follows:
After reviewing as much of the AdS/CFT correspondence as will be needed throughout the thesis, we study supersymmetric Wilson loops on the gauge theory side of the correspondence. In the gauge theory, we concentrate on calculating the amount of supersymmetry that is preserved by various Wilson loops and evaluate their expectation value.
In the second chapter, we review Wilson loops in flat space.
In the third chapter, we focus on Wilson loops which are restricted to the manifolds S^2 and S^3.
Chapter four deals with Wilson loops constrained to hyperbolic three space $\mathbb{H}_3$. We point out the relation between loops on the sphere and loops in hyperbolic space.
The fifth chapter investigates Wilson loops in Minkowskian signature that are not restricted to $\mathbb{H}_3$.
The last chapter is concerned with the string theory duals of the various Wilson loops we centered in the gauge theory. We examine the minimal surfaces that describe these Wilson loops in string theory, evaluate their classical supergravity action and compare it to the results obtained from the gauge theory.
Finally, we emphasize the relation between the content of this thesis and the current research on gluon scattering amplitudes and give an outlook about questions that remain open.

II
Wilson Loops in AdS/CFT

In this chapter we review the aspects of the AdS/CFT correspondence that will be needed throughout this thesis. After a short summary of *AdS* spaces and their geometry we take a short look at $\mathcal{N}=4$ SYM. Finally we discuss the basic ideas of the AdS/CFT correspondence and show how the Wilson loop operator fits into the correspondence. For a review of the AdS/CFT correspondence the reader might take a look at [9]; for a very detailed introduction we recommend [10]; introductions in the form of lectures are provided by [11, 12].

2.1 Basics of AdS Spaces

Five dimensional Anti-de Sitter space, abbreviated as AdS_5, is usually defined by considering the hyperboloid with curvature radius L

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -L^2\,. \tag{2.1}$$

in flat space $\mathbb{R}^{2,4}$ with the metric

$$ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2 + dY_2^2 + dY_3^2 + dY_4^2\,. \tag{2.2}$$

This space has the isometry group $SO(2,4)$, which will be important in formulating the AdS/CFT correspondence.

2.1.1 Global Coordinates

The standard parametrization of (2.1) is given by

$$\begin{aligned} Y_{-1} &= L\cosh\rho\,\cos\tau, \qquad Y_0 = L\cosh\rho\,\sin\tau \\ Y_i &= L\sinh\rho\,\Omega_i, \qquad i=1,\ldots,4 \end{aligned} \tag{2.3}$$

with Ω_i denoting polar coordinates satisfying $\Sigma_{i=1}^4\Omega_i^2=1$. Given this parametrization the metric on AdS_5 takes the following form

$$ds^2 = L^2(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega_3^2)\,. \tag{2.4}$$

If we choose the restrictions $0\le\rho$ and $0\le\tau\le 2\pi$, we cover the whole hyperboloid. This is the reason why the coordinates (τ,ρ,Ω_3) are usually called *global coordinates*. It is easy to see that near $\rho\simeq 0$ the hyperboloid has the topology of $S^1\times\mathbb{R}^4$, since the metric behaves as $ds^2\simeq L^2(-d\tau^2+d\rho^2+\rho^2 d\Omega_3^2)$.

2.1.2 Poincaré Patch

Additionally to the global coordinate system, there is another set of coordinates that will be used frequently, the so called *Poincaré patch*. It is related to the embedding coordinates by

$$\begin{aligned} Y_{-1} &= \frac{1}{2u}\left(1+u^2(L^2+\vec{x}^2-x_0^2)\right), \qquad Y_0 = L\,u\,x_0 \\ Y_i &= L\,u\,x_i, \qquad i=1,2,3 \\ Y_4 &= \frac{1}{2u}\left(1-u^2(L^2-\vec{x}^2+x_0^2)\right). \end{aligned} \tag{2.5}$$

These coordinates only cover half of the hyperboloid (2.1) and we obtain the following line element

$$ds^2 = L^2\left(\frac{du^2}{u^2}+u^2(-dx_0^2+d\vec{x}^2)\right). \tag{2.6}$$

Substituting $u=1/y$ we get an equivalent form of the metric

$$ds^2 = \frac{L^2}{y^2}\left(dy^2-dx_0^2+d\vec{x}^2\right). \tag{2.7}$$

At the conformal boundary $y=0$ the metric has the topology of four dimensional Minkowski space, which is another crucial fact for formulating the AdS/CFT correspondence.

2.1.3 Euclidian AdS

Perfoming a Wick rotation in $\tau \to \tau_E = -i\tau$ is reflected in the original coordinates (2.1) as $Y_{-1} \to Y_E = -iY_{-1}$ resulting in

$$-Y_0^2+Y_E^2+Y_1^2+Y_2^2+Y_3^2+Y_4^2 = -L^2. \tag{2.8}$$

After the Wick rotation the metric (2.2) changes to

$$ds^2 = -dY_0^2+dY_E^2+dY_1^2+dY_2^2+dY_3^2+dY_4^2. \tag{2.9}$$

Expressed in global and Poincaré coordinates we get the following line element

$$\begin{aligned} ds_E^2 &= L^2\left(d\rho^2+\cosh^2\rho\,d\tau_E^2+\sinh^2\rho\,d\Omega_3^2\right) \\ &= \frac{L^2}{y^2}\left(dy^2+dx_1^2+\cdots+dx_5^2\right). \end{aligned} \tag{2.10}$$

The coordinate system just introduced is usually called *Euclidian AdS*.

2.1.4 Geometry of S^5

The five dimensional unit sphere can be embedded in $\mathbb{R}^6$ and parametrized via polar coordinates. In this thesis, we will only consider an S^2 subspace of S^5. For completeness we give the metric on S^2 parametrized in terms of polar coordinates

$$ds^2 = L^2(d\vartheta^2+\sin^2\vartheta d\varphi^2). \tag{2.11}$$

2.2 Aspects of $\mathcal{N}=4$ SYM

$\mathcal{N}=4$ SYM is a remarkable theory. It is the *maximal supersymmetric gauge theory* in four dimensions with four spinor supercharges $\mathcal{N}=4$ and gauge group $SU(N)$. Its field content consists of one gauge field A_μ, four Majorana spinors Ψ_i and six scalars Φ_I, all in the adjoint representation of the gauge group. Furthermore, one has to mention that $\mathcal{N}=4$ SYM is a conformal theory, therefore it is scale invariant and has vanishing beta function. The theory has two parameters, the Yang-Mills coupling constant g_{YM} and the number of colors N. Its Lagrangian is completely determined by supersymmetry [9]

$$\mathcal{L} = \mathrm{Tr}\Bigg(-\frac{1}{2g}F^{\mu\nu}F_{\mu\nu} + \frac{\theta_I}{8\pi^2}F^{\mu\nu}\tilde{F}_{\mu\nu} - (D_\mu\Phi_I)^2 - i\bar{\Psi}\,\not{D}\Psi \\ + g\rho^I\Psi[\Phi_I,\Psi] + g\rho^I\bar{\Psi}[\Phi_I,\bar{\Psi}] + \frac{g^2}{2}[\Phi_I,\Phi_J]^2\Bigg). \quad (2.12)$$

Here, the matrices ρ^I belong to the Clifford algebra of the R-symmetry group $SO(6)$. The Lagrangian is invariant under the supersymmetry transformations studied at the end of this section.

2.2.1 Symmetries of $\mathcal{N}=4$ SYM

In this subsection, we summarize the global symmetries of $\mathcal{N}=4$ SYM. The theory as a whole is invariant under the supergroup $SU(2,2|4)$ [13], nevertheless let us take a look at the different subsectors:

- *Conformal Symmetry*,
  given by the group $SO(2,4)\sim SU(2,2)$,
  generated by translations P_μ, Lorentz transformations $M_{\mu\nu}$,
  dilatations D and special conformal transformations K_μ;

- *R-symmetry*, rotating the six scalars, generated by R^A,
  given by the group $SO(6)\sim SU(4)$, $A=1,\ldots,15$;

- *Poincaré supersymmetries*,
  generated by the supercharges $\mathcal{Q}^a{}_\alpha$ and their adjoint $\bar{\mathcal{Q}}_{\dot{\alpha}a}$, $a=1,\ldots,4$

- *Conformal supersymmetries*,
  generated by the conformal supercharges $S_{\alpha a}$ and their adjoint $\bar{S}^a{}_{\dot{\alpha}}$

Before turning to the supersymmetry transformations let us write down the non-trivial commutation relations of the superconformal algebra of $\mathcal{N}=4$ SYM. Apart from the algebra of the generators of the conformal group

$$[M_{\mu\nu},M_{\alpha\beta}] = -i[\eta_{\mu\beta}M_{\nu\alpha}+\eta_{\nu\alpha}M_{\mu\beta}-\eta_{\mu\alpha}M_{\nu\beta}-\eta_{\nu\beta}M_{\mu\alpha}] \quad (2.13)$$

$$[M_{\mu\nu},P_\alpha] = -i[\eta_{\nu\alpha}P_\mu-\eta_{\nu\alpha}P_\nu], \qquad [M_{\mu\nu},K_\alpha] = -i[\eta_{\nu\alpha}K_\mu-\eta_{\mu\alpha}K_\nu]$$

$$[D,K_\mu]=iK_\mu, \qquad [D,P_\mu]=-iP_\mu, \qquad [P_\mu,K_\nu]=-2i[\eta_{\mu\nu}D-2M_{\mu\nu}]$$

we also have the commutator/anticommutators involving the superconformal algebra (in order not to blow up the notation we omit spinor and group indices)

$$[D,\mathcal{Q}]=-\frac{i}{2}\mathcal{Q},\qquad [D,S]=\frac{i}{2}S,\qquad [K_\mu,\bar{\mathcal{Q}}]=\gamma_\mu S,\qquad [P_\mu,S]=\gamma_\mu\bar{\mathcal{Q}} \tag{2.14}$$

$$\{\mathcal{Q},\bar{\mathcal{Q}}\}=2\gamma^\mu P_\mu,\qquad \{S,\bar{S}\}=2\gamma^\mu K_\mu,\qquad \{\mathcal{Q},S\}=\frac{1}{2}\gamma^{\mu\nu}M_{\mu\nu}+D+R\,.$$

In the following we will quite frequently make use of the supersymmetry transformations of the bosonic fields A_μ and Φ^I (for completeness we also write down the variation of the fermionic fields)

$$\begin{aligned}\delta_\epsilon A_\mu &\rightarrow \bar{\psi}\gamma_u\epsilon(x)\\ \delta_\epsilon\Phi_I &\rightarrow \bar{\psi}\rho^I\gamma^5\epsilon(x)\\ \delta_\epsilon\Psi &\rightarrow (F^{\mu\nu}\gamma_{\mu\nu}+\rho^{IJ}[\Phi_I,\Phi_J])\epsilon(x)\\ \delta_\epsilon\bar{\Psi} &\rightarrow (\gamma_\mu D^\mu\Phi_I\rho^I\gamma^5)\epsilon(x)\,.\end{aligned} \tag{2.15}$$

The parameter of the transformation is given by ϵ. In the following we will choose ϵ to be a conformal Killing spinor composed of two constant spinors ϵ_0 and ϵ_1 as

$$\epsilon(x)=\epsilon_0+x^\mu\gamma_\mu\epsilon_1\,. \tag{2.16}$$

When it comes to actual calculations we will specify the properties of the gamma matrices appearing in (2.15). We will always assume that the spacetime gamma matrices γ_μ commute with the ρ^I matrices.
The vacuum of $\mathcal{N}=4$ SYM has 32 supersymmetries, that split up in 16 Poincaré supersymmetries generated by the spinor ϵ_0 and 16 conformal supersymmetries generated by the spinor ϵ_1. If one brings an object into the vacuum, it usually destroys all or at least part of the 32 supersymmetries. There are only special observables that can preserve some of the original supersymmetries. These objects are called *BPS operators*. They annihilate part of the supercharges and the supersymmetry representation suffers multiplet shortening. The amount of supersymmetry preserved by a BPS operator is determined by the number of independent components of the spinors ϵ_0 and ϵ_1. Finally, we want to mention that BPS operators are protected from receiving quantum corrections, meaning that in a unitary representation the dimension Δ of these operators is not renormalized [9].

2.2.2 Aspects of conformal Geometry

A conformal transformation preserves the metric up to a scale factor $g_{\mu\nu}\rightarrow\Omega(x)^2 g_{\mu\nu}$. Apart from Poincaré transformations the conformal group of Minkowski space consists of dilatations

$$x^\mu\rightarrow\lambda\,x^\mu,\qquad \lambda\in\mathbb{R} \tag{2.17}$$

and special conformal transformations

$$x^\mu\rightarrow\frac{x^\mu+a^\mu x^2}{1+2x\cdot a+a^2x^2}\,. \tag{2.18}$$

One of the special conformal transformations that we will use in the following is the stereographic projection between S^2 and $\mathbb{R}_2$ as well as $\mathbb{H}_2$ and the unit disc.

2.3 The AdS/CFT Correspondence

The *AdS/CFT correspondence* or *Maldacena conjecture* states the duality of two totally different theories [3]:

- *Type IIB superstring theory* on the $AdS_5 \times S^5$ background, where both AdS_5 and S^5 have the same curvature radius L

- $\mathcal{N} = 4$ *SYM theory* in 4 dimensions with gauge group $SU(N)$

In the correspondence the gauge theory lives on the conformal boundary of AdS_5, which is 4 dimensional Minkowski space as we have already mentioned. This is the reason why the correspondence is often called *holographic*, since everything happening on the boundary completely determines what is happening in the bulk. The parameters on the gauge theory side are the number of colors N and the Yang Mills coupling constant g_{YM}. On the string theory side one has the string coupling constant g_S and the curvature radius L of the AdS_5 and S^5 spaces.

The correspondence matches the parmeters of both theories in the following way

$$\boxed{4\pi g_S = g_{YM}^2, \qquad L^4 = 4\pi g_S N \alpha'^2\,.} \tag{2.19}$$

Although a general proof of the conjecture does not exist, there are a many strong indications for its exactness. Perhaps the strongest argument comes from taking a look at the symmetry group of both theories. The conformal group in 4 dimensions is given by $SO(2,4)$, which is the isometry group of AdS_5. The R-symmetry group of $\mathcal{N} = 4$ SYM is given by $SO(6)$, which is the rigid symmetry group of S^5. Together with the fermionic degrees of freedom both theories are invariant under the supergroup $SU(2,2|4)$.

In the form defined above, the conjecture is to hold for all values of N and $4\pi g_S = g_{YM}^2$. Actual computations in this strongest form of the conjecture are hard to manage, therefore one is interested in taking simplifying limits.

2.3.1 The 't Hooft Limit

In the 't Hooft limit one is interested in $N \to \infty$ while keeping the 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed. On the field theory side, this limit is well-defined and corresponds to a topological expansion of the field theory's Feynman diagrams. On the *AdS* side, the 't Hooft limit describes a *weakly coupled string theory*, since the string coupling constant can be expressed as $g_S \propto \lambda/N$ with λ kept fixed.

2.3.2 The large λ Limit

Once the 't Hooft limit is taken, the only remaining parameter is λ. In the Yang-Mills theory one is usually interested in the perturbative regime $\lambda \ll 1$, while on the string theory side it is more natural to look at $\lambda \gg 1$. This is quite remarkable, since now we have a correspondence between a weakly and a strongly coupled theory! In the large λ limit one can make an expansion in $\lambda^{-1/2}$ in the gauge theory; this corresponds to *classical Type IIB supergravity* on $AdS_5 \times S^5$ with the action

$$\mathcal{S}_{Sugra} = \frac{1}{16\pi G^{(10)}} \int d^{10}x \sqrt{-g} e^{-2\phi_D} (R + 4\partial^\mu \phi_D \partial_\mu \phi_D + \dots) . \tag{2.20}$$

Here $G^{(10)}$ is the ten-dimensional Newton constant, ϕ_D is the Dilaton field and the dots indicate contributions from other fields.

2.3.3 Correlation Functions in the AdS/CFT Correspondence

We have already seen that a strong indication for the duality between $\mathcal{N} = 4$ SYM living on the conformal boundary of AdS_5 and type IIB superstring theory on $AdS_5 \times S^5$ comes from both theories having the same global symmetries. But matching global symmetries is of course just the first step; we need a precise prescription for each operator $\mathcal{O}(\vec{x})$ in $\mathcal{N} = 4$ SYM to be identified with a field $\phi(\vec{x})$ in the bulk of AdS_5. This prescription can then be used to compute correlation functions on both sides of the correspondence. The specification for matching correlation functions in the AdS/CFT correspondence was given in [14, 15] and is usually called Witten prescription. It matches the *generating functional* on the field theory side with the *string partition function*

$$\left\langle e^{\int d^4x \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \right\rangle_{CFT} = \mathcal{Z}_{String}[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})] . \tag{2.21}$$

In the last equality, $\mathcal{O}(\vec{x})$ is a gauge invariant local operator and $\phi_0(\vec{x})$ specifies the boundary values of the field $\phi(\vec{x})$. We note that the *Witten prescription* matches the central objects of both theories with each other. Nevertheless calculating the string partition function in general is hard to manage. Luckily in the large N and large λ limit (2.21) simplifies as

$$\mathcal{Z}_{String} \approx e^{-S_{Sugra}} . \tag{2.22}$$

Since Wilson loops are non-local operators, the prescription (2.21) has to be modified in their case.

2.4 Wilson Loops

In this section we shortly motivate the definition of the Wilson loop. The basic properties of the Wilson loop operator can be found in standard quantum field theory books, for example [16]. Originating in lattice gauge theory Wilson loops became famous since the expectation value of a rectangular loop with infinite temporal sides T and spatial length L describes the static quark-antiquark potential, where L is the distance between quark and antiquark. Additionally Wilson loops can be used to characterize confinement in QCD [17].

2.4.1 The Wilson loop operator in gauge theories

Historically, Wilson loops were considered to construct a parallel transport in gauge theories. If we consider a vector field ϕ and want to perform a parallel transport along some curve C connecting two spacetime points y and z

$$U(y,z)\phi(z) = \phi(y) \tag{2.23}$$

and additionally require that $U(y,z)$ has the right transformation law under an abelian gauge transformation (Σ denotes an element of the gauge group)

$$U(y,z) \to \Sigma(y)U(y,z)\Sigma(z)^{-1} \tag{2.24}$$

then we are lead to the following object

$$U(y,z) = \exp\left(i\int_C A_\mu dx^\mu\right) . \tag{2.25}$$

The quantity $U(y,z)$ is usually called the *Wilson line*. When we consider a closed path, $U(y,y)$ is called the *Wilson loop*. Now we want to generalize the Wilson loop to a non-abelian theory. In this case subtleties arise since we have to deal with the exponential of non-commuting matrices. It turns out that the replacement $A_\mu \to A_\mu^a T^a$ is not the correct generalization of (2.25), since the matrices T^a do not commute at different points. This ordering ambiguity is solved by introducing a path ordering, which is denoted by $\mathcal{P}$. Let t be the parameter of the path P running from 0 at $x = y$ to 1 at $x = z$. We can then define the Wilson loop operator as a power series of the exponential with the matrices T^a ordered in the way that higher values of the path parameter t stand to the left in every term. The Wilson loop generalized to non-abelian gauge theories then becomes

$$U(y,z) = \mathcal{P}\exp\left(i\oint_C dx^\mu A_\mu^a T^a\right) . \tag{2.26}$$

Taking a closed path (2.26) would no longer be gauge invariant, that is why we have to introduce the trace into the definition of the Wilson loop for non-abelian gauge theories (usually in the fundamental representation). Now we can finally write down the object of interest

$$W = \frac{1}{N}\mathrm{Tr}\ \mathcal{P}\exp\left(i\oint_C dx^\mu A_\mu^a T^a\right) . \tag{2.27}$$

The factor $1/N$ is introduced for convenience. To conclude this section, let us mention some crucial properties of the Wilson loop operator. First of all, it is a non-local operator since it depends on the path C. Secondly, it is constructed as a gauge invariant object. All gauge invariant functions of A_μ can be derived from Wilson loops by choosing appropriate paths [16], since they form a complete basis of gauge invariant operators for the pure Yang-Mills theory. Especially due to the last property Wilson loops are central objects in non-abelian gauge theories. As already mentioned, the most prominent application of the

Wilson loop in physics is the calculation of the static quark-antiquark potential [17]. For this purpose one considers a rectangular loop with temporal length T and spatial length R with $T \gg R$. We can then extract the static quark antiquark-potential from the Wilson loop operator via

$$\langle W \rangle \propto \exp\left(-V_{q\bar{q}}T\right) = \exp\left(-\sigma T R\right) = \exp\left(-\sigma A_{min}\right) . \tag{2.28}$$

The last equality is the famous area law. We realize that $V_{q\bar{q}}$ gives rise to a linear potential leading to a constant force independently of the distance between quark and antiquark.

2.4.2 The Wilson Loop Operator in AdS/CFT

The Wilson loop operator in the AdS/CFT correspondence was first presented in [4] and [18]. A review article about Wilson loops in the context of AdS/CFT is given by [19]. In contrast to the normal Wilson loop considered in gauge theories like QCD, the Wilson loop operator in the AdS/CFT correspondence has to be modified. First of all, we present the Wilson loop operator in $\mathcal{N} = 4$ SYM and then investigate its dual object in the string theory. As we have already seen, strings in the AdS/CFT correspondence propagate in $AdS_5 \times S^5$; the conformal boundary of five dimensional AdS is the setup of the gauge theory. This leads to the natural proposal that the string in AdS should end on the contour of the Wilson loop. In addition, we have to take care of the fact that strings also extend to S^5. Remember that the S^5-part in the string theory corresponds to the R-symmetry group of $\mathcal{N} = 4$ SYM. Let θ^I be coordinates on S^5, on account of this one extends the usual Wilson loop operator by taking the θ^I as coupling to the six scalars in the gauge theory. This motivates the definition of the generalized supersymmetric Wilson loop [4]

$$W = \frac{1}{N}\mathrm{Tr}\, \mathcal{P} \exp \oint dt \left(iA_\mu \dot{x}^\mu(t) + |\dot{x}|\Theta^I(t)\Phi^I\right) . \tag{2.29}$$

Since the generalized Wilson loop operator was originally proposed by Maldacena, (2.29) is sometimes called *Maldacena-Wilson loop*. Since we are dealing with a supersymmetric string theory, we also have to make sure that (2.29) is invariant under supersymmetry variation. To compute the expectation value of the Wilson loop operator in string theory one has to generalize the prescription (2.21), since the Wilson loop is a non local operator. The mapping between the non-local Wilson loop operator in the gauge theory and the string partition function $\mathcal{Z}(C)$ has also been proposed in [4] and was further generalized in [18]

$$\langle W(C) \rangle = \mathcal{Z}(C) = e^{-S(C)} . \tag{2.30}$$

In the rest of this thesis, we will be interested in computing the expectation value of the Wilson loop in the supergravity approximation. In this limit the string worldsheet is described by a *minimal surface*. The area of this surface can be obtained by extremizing the *Nambu-Goto action*

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-\det_{\alpha,\beta} G_{MN}(\partial_\alpha X^M \partial_\beta X^N)} . \tag{2.31}$$

In the definition of the Nambu-Goto action, G_{MN} is the ten-dimensional background metric and the X^M are the string coordinates in ten dimensional space time; the set $\{\tau, \sigma\}$ parametrizes the string worldsheet. There is another action that will be used frequently, the so called *Polyakov action*, which is much easier to handle than the Nambu-Goto action. The transition from the Nambu-Goto action to the Polyakov action requires to introduce an auxilliary field h^{ab}. To show that the Nambu-Goto action is equivalent to the Polyakow action on the classical level gives rise to a constraint, the so called *Virasoro constraint* [1]. The Polyakov action reads

$$S_{Pol} = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-h} h^{ab} G_{MN} (\partial_\alpha X^M \partial_\beta X^N) \,. \tag{2.32}$$

The appeareance of the auxilliary metric h^{ab} and its determinant h is new in contrast to the Nambu-Goto action as already mentioned. The Polyakov action has a huge amount of symmetry [1], it is invariant under diffeomorphisms and Weyl transformations. One can now cleverly use these symmetries to bring the Polyakow action into a simple form, namely if we choose the *conformal gauge*, which means that we take the worldsheet metric to be Euclidian, leading to

$$S_{Pol} = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left(\dot{X}^2 + X'^2 \right) \,. \tag{2.33}$$

The Polyakov action in conformal gauge is accompanied by the Virasoro constraint

$$\dot{X}^2 = X'^2 \,. \tag{2.34}$$

For more details about the Nambu-Goto and the Polyakov action the reader might take a look at standard textbooks about string theory, for example [1, 20]. Just for completeness, we want to mention that it is also possible to describe Wilson loops in *AdS* by D-Branes [21, 22, 23, 24].

2.4.3 Regulating the Action

After having found a minimal surface by solving the classical equations of motion and therefore minimizing the action, we will face an obstacle, namely that in general such a surface will be infinite. There are two possibilities for this to happen, first from being close to the conformal boundary of *AdS* and secondly due to the fact that the Wilson loop is non-compact like in the case of an infinite line. We will face both of these situations when calculating Wilson loops in *AdS*. The infinity occurring from being close to the boundary can be regulated easily: Since there are many actions whose equation of motion are solved by minimal surfaces (they differ by total derivatives or equivantly by boundary terms), we have to find the minimal surface with the correct boundary conditions. It was shown in [25] how this can be achieved: One has to consider the Legendre transformation of the original Lagrangian with respect to the coordinate y orthogonal to the boundary. We will see later that through the regularization procedure the action becomes negative. In the original reference [25] it was also shown that not every minimal surface can be regulated in that way: Only Wilson loops satisfying the constraint that the magnitude of the coupling to the gauge field is the same as the magnitude of the coupling to the scalars can be treated this way. In practice this means that Θ^I in (2.29) has to be a unit vector $\vec{\Theta} \cdot \vec{\Theta} = 1$.

2.5 Cusp anomalous Dimensions

Up to now, we quietly assumed that the curve belonging to the Wilson loop is smooth; nevertheless Wilson loops with a cusp have many interesing properties. In QCD they are describing the trajectory of a heavy quark which suddenly changes its velocity at the location of the cusp. The ultraviolet divergence appearing in the calculation of the cusped Wilson loop can then be interpreted as bremsstrahlung of soft gluons emitted by the quark while it is changing its velocity. In addition the finite part also has an important physical interpretation in QCD, leading to the scattering amplitudes of gluons expressed in terms of Mandelstam variables.

The *cusp anomalous dimension* γ_{cusp} depends on the angle at the cusp θ in Euclidian space or the change of rapidity in Minkowskian space. For large values of θ it is given by the expression

$$\gamma_{Cusp} = \frac{\theta}{2} f(g_{YM}^2, N) \,. \tag{2.35}$$

The function $f(g_{YM}^2, N)$ can be calculated perturbatively, in the planar limit it is only a function of the 't Hooft coupling λ.

At present there is big interest in the cusp anomalous dimensions in $\mathcal{N} = 4$ SYM and its relation to gluon scattering amplitudes. Up to now gluon scattering amplitudes in the gauge theory have been computed at five loops [26, 27, 28, 29]. Since the proposal of Alday and Maldacena [30], [31] relating gluon scattering amplitudes and lightlike Wilson loops, it is by now possible to do calculations in the weakly and the strongly coupled regime. For the actual status of the research we recommend the review articles [32] and [33].

III

Wilson Loops in flat Space

In this chapter, we first of all review the circular Wilson loop coupling to one scalar since it is is very well understood. Additionally, we use this object to present techniques that will be needed in the following. Afterwards, we present the Zarembo construction [5], which can be applied to a general curve on $\mathbb{R}^4$.

3.1 Circular Loop coupling to one Scalar

The circular Wilson loop coupling to one scalar is the most prominent example of a supersymmetric Wilson loop in the AdS/CFT correspondence. After showing that this operator is supersymmetric, we will investigate the perturbation series of this object. A circle with unit radius ($x^\mu = (\cos t, \sin t, 0, 0)$) coupling to one scalar gives the following Wilson loop operator

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \oint dt \left(i A_\mu \dot{x}^\mu + \Phi^1 \right) . \tag{3.1}$$

In order to check whether this operator is supersymmetric, we use the supersymmetry transformations of the bosonic fields (2.15). Vanishing of the supersymmetry variation for the circular loop requires

$$\begin{aligned} \delta W \simeq & \quad (i\dot{x}^\mu \gamma_\mu + \rho^1 \gamma^5)(\epsilon_0 + x^\mu \gamma_\mu \epsilon_1) \qquad (3.2) \\ = & \quad (-i \sin t \gamma_1 + i \cos t \gamma_2 + \rho^1 \gamma^5)(\epsilon_0 + \cos t \gamma_1 \epsilon_1 + \sin t \gamma_2 \epsilon_1) = 0 \, . \end{aligned}$$

The transformation parameter ϵ is a conformal Killing spinor, γ_μ are flat space gamma matrices and ρ^1 belongs to the Clifford algebra of $SO(6)$. Now we have to separate the terms appearing in the supersymmetry variation according to their functional dependence on the loop parameter t which should all vanish independently

$$\begin{aligned} \sin t : & \qquad i\gamma_1 \epsilon_0 = \rho^1 \gamma^5 \gamma_2 \epsilon_1 \\ \cos t : & \qquad i\gamma_2 \epsilon_0 = -\rho^1 \gamma^5 \gamma_1 \epsilon_1 \\ 1 : & \qquad i\gamma_{12} \epsilon_1 = \rho^1 \gamma^5 \epsilon_0 \, . \end{aligned} \tag{3.3}$$

It is easy to realize that only one of the three equations is independent

$$i\gamma_{12} \epsilon_1 = \rho^1 \gamma^5 \epsilon_0 \, . \tag{3.4}$$

The constraint relates every component of ϵ_0 to a component of ϵ_1, hence we are left with 16 independent components. Therefore the circular Wilson loop coupling to one scalar preserves half of the original supersymmetries and is a 1/2 BPS operator.

3.1.1 Perturbative Calculation/Matrix Models

There has been a big effort in computing the expectation value for the circular Wilson loop coupling to one scalar, or more general cases like the correlation function of two circular loops [34, 35]. What is quite remarkable about the circular loop coupling to one scalar, is the fact that one can sum up the whole pertubation series. When computing the expectation value of the Wilson loop operator in the gauge theory, one expands the exponential in a power series and performs the usual Wick contractions. Up to lowest order this looks like (with the propagators in Feynman gauge)

$$\begin{aligned} \langle W \rangle = &\quad 1 + \frac{1}{2N} \mathrm{Tr}\,(T^a T^b) \int dt_1 dt_2 \langle (i\dot{x}^\mu A^a_\mu(x) + \phi^a_1(x))(i\dot{y}^\mu A^b_\mu(y) + \phi^b_1(y)) \rangle \\ = &\quad 1 + \frac{g^2_{YM} N}{(4\pi)^2} \int dt_1 dt_2 \frac{-\dot{x}\cdot\dot{y} + 1}{(x-y)^2} \,. \end{aligned} \tag{3.5}$$

In the case of the circular loop we have

$$(x-y)^2 = 2(1 - \dot{x}\cdot\dot{y}) \tag{3.6}$$

resulting in

$$\frac{g^2_{YM} N}{(4\pi)^2} \int_0^{2\pi} dt_1 dt_2 \frac{1 - \dot{x}\cdot\dot{y}}{(x-y)^2} = \frac{\lambda}{8} \tag{3.7}$$

in lowest non-trivial order of pertubation theory. It would of course be interesting to further investigate the perturbative expansion. There are two types of Feynman diagrams that contribute to the perturbation series: On the one hand we have the so called ladder diagramms, which do not contain any interaction vertices, and on the other hand we of course have the interacting graphs. In the case of the circular Wilson loop, interacting graphs cancel each other at order λ^2 [36], leading to the conjecture that only ladder diagrams contribute to the perturbative series. This assumption is justified, since the sum of ladders in the gauge theory correctly reproduces the strong coupling behaviour obtained by an *AdS* calculation. From (3.7) we realize that the combined scalar vector propagator for the circle is dimensionsless leading to the idea of a zero dimensional field theory. Since there is no space time dependence left, it is possible to resum the whole pertubation series via a *matrix model* (it has recently been proven that the matrix model is Gaussian [37])

$$\langle W \rangle = \left\langle \frac{1}{N} \mathrm{Tr}\, \exp M \right\rangle = \frac{1}{Z} \int \mathcal{D}M \frac{1}{N} \mathrm{Tr}\, \exp(M) \exp\left(-\frac{2N}{\lambda} \mathrm{Tr}\, M^2 \right) \,. \tag{3.8}$$

This matrix model can be solved analytically (although the calculation is very tedious) and the result can be expressed as an expansion in powers of $1/N$ [38]

$$\left\langle \frac{1}{N} \mathrm{Tr}\, \exp M \right\rangle = \frac{2}{\sqrt{\lambda}} I_1 \sqrt{\lambda} + \frac{\lambda}{48 N^2} I_2 \sqrt{\lambda} + O\left(\frac{1}{N}\right) \,. \tag{3.9}$$

Here, the I_n denote modified Bessel functions. Coming back to the circular loop the leading order result is given by

$$\langle W \rangle = \frac{2}{\sqrt{\lambda}} I_1 \sqrt{\lambda} \,. \tag{3.10}$$

For large λ equation (3.10) behaves as

$$\langle W \rangle \sim \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}} \,. \tag{3.11}$$

Later we will see that this result is in agreement with the string theory prediction of AdS/CFT. We want to mention that this was one of the first "experimental" tests of the Maldacena conjecture.

3.2 The Zarembo Construction

The first general method of extending the Wilson loop operator so that it becomes supersymmetric was given by Zarembo [5]. In this construction the curves associated to the Wilson loop live in flat space $\mathbb{R}^4$. Before turning to the actual construction, let us consider a Wilson loop as defined in (2.29), with Θ^I a unit six vector, satisfying $\vec{\Theta} \cdot \vec{\Theta} = 1$

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp \oint dt \left(i A_\mu \dot{x}^\mu(t) + |\dot{x}| \Theta^I \Phi^I \right) \,. \tag{3.12}$$

If we consider the supersymmetry variation of this operator some supersymmetries will be preserved if we find a solution to

$$(i\gamma_\mu \dot{x}^\mu + |\dot{x}| \rho^I \gamma^5 \Theta^I) \epsilon(x) = 0 \,. \tag{3.13}$$

The expression $(i\gamma_\mu \dot{x}^\mu + |\dot{x}| \rho^I \gamma^5 \Theta^I)$ squares to zero, therefore (3.13) has eight independent solutions at every point t of the loop. For a general curve these solutions will of course depend on t, consequently the ansatz only leads to *local supersymmetry*. If we want to achieve *global supersymmetry* we have to make sure that there is a finite number of constraints acting on the spinor $\epsilon(x)$. To reduce the number of constraints to a finite number we make the following ansatz for the coupling to the scalars

$$\Theta^I = M^I{}_\mu \frac{\dot{x}^\mu}{|\dot{x}|} \,. \tag{3.14}$$

The matrix $M^i{}_\mu$ is a rectangular 4×6 matrix. We do not have to give an explicit form of it, since $\mathcal{N} = 4$ SYM has the global symmetry of $SO(4) \times SO(6)$. In a geometric sense, the ansatz maps the position on S^5 to the tagent vector $\dot{x}_\mu$ in spacetime. Using the ansatz (3.14) we can now evaluate the supersymmetry variation

$$i\dot{x}^\mu (\gamma_\mu - i M^I{}_\mu \rho^I \gamma^5) \epsilon(x) = 0 \,. \tag{3.15}$$

Some supersymmetry will be preserved if we find a solution to

$$(\gamma_\mu - i M^I{}_\mu \rho^I \gamma^5) \epsilon(x) = 0 \,. \tag{3.16}$$

Expanding the terms in the supersymmetry variation we find that a general curve in $\mathbb{R}^4$ preserves one Poincaré supersymmetry [5]. This is why this family of Wilson loops is often called $\mathcal{Q}$-invariant. This construction guarantees that a curve in $\mathbb{R}^1$ is 1/2 BPS. Inisde $\mathbb{R}^2$ it will be 1/4 BPS, in $\mathbb{R}^3$ it is 1/8 BPS and a generic curve in $\mathbb{R}^4$ is 1/16 BPS. We will later study the supersymmetry variation of this class of Wilson loops more detailed.

3.2.1 Circle

Let us take a look at a circle with unit radius and evaluate the supersymmetry variation in the Zarembo construction

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\oint dt\left(-iA_0\sin t + iA_1\cos t - \sin t\Phi^1 + \cos t\Phi^2\right)\,. \qquad (3.17)$$

To be supersymmetric four equations have to hold

$$(i\gamma_j + \rho^j\gamma^5)\epsilon_0 = 0, \qquad (i\gamma_j + \rho^j\gamma^5)\epsilon_1 = 0, \qquad j = 1,2\,. \qquad (3.18)$$

Since there a two constraints acting on both of the spinors, both ϵ_0 and ϵ_1 have four independent components left. Therefore the circle in the Zarembo construction is 1/4 BPS. We note that the constraints do not mix Poincaré and superconformal supersymmetries.

3.2.2 Expectation Value

One can explicitly check that the VEV of this class of Wilson loops is trivial. In the original work of Zarembo this has been checked at one loop for a general curve. The basic ingredient in this calculation is the equality of the scalar and the vector propagator in Feynman gauge. Furthermore, it is known that the one loop corrected scalar and vector propagator are still equal up to total derivatives [36]. For the circle and a line this result was also confirmed from an *AdS* calculation in the original work. In that reference it was conjectured that planar Wilson loops, which preserve 1/4 of the original supersymmetries do not receive quantum corrections, which was proven in [39] and [40]. The absence of quantum corrections for the case of a general Wilson loop in the Zarembo construction was finally proven in [41] on the string theory side of the correspondence.

IV

Wilson Loops on S^3

In this chapter we review Wilson loops which are restricted to S^3. First of all, we take a look at a general curve and then present some examples, which preserve more supersymmetries. This family of Wilson loops was introduced in [7], [6] and [8].

4.1 General Curve

The group manifold S^3 is defined by the property

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1\,. \tag{4.1}$$

The coupling to the scalars of $\mathcal{N} = 4$ SYM is constructed with the help of one forms

$$\begin{aligned}
\sigma_1^{R,L} &= 2\left[\pm(x^2dx^3 - x^3dx^2) + (x^4dx^1 - x^1dx^4)\right] \\
\sigma_2^{R,L} &= 2\left[\pm(x^3dx^1 - x^1dx^3) + (x^4dx^2 - x^2dx^4)\right] \\
\sigma_3^{R,L} &= 2\left[\pm(x^1dx^2 - x^2dx^1) + (x^4dx^3 - x^3dx^4)\right]\,.
\end{aligned} \tag{4.2}$$

Here, σ_i^R are the right one-forms and σ_i^L are the left one-forms. These are dual to left (right) invariant vector fields generating right (left) group actions. Without loss of generality we take the right invariant one forms to define a coupling to three of the six scalars of $\mathcal{N} = 4$ SYM. The one forms satisfy (as a consequence of $x^2 = 1$)

$$\sigma_i^R\sigma_i^R = 4dx^\mu dx_\mu\,. \tag{4.3}$$

Written in form notation, the ansatz for the supersymmetric Wilson loop operator including the coupling to the scalars then looks like

$$W = \frac{1}{N} TrP\exp\oint\left(iA + \frac{1}{2}\sigma_i^R {M^i}_I\Phi^I\right)\,. \tag{4.4}$$

Here γ_μ and ρ^I are the gamma matrices belonging to the Clifford algebras of $SO(4)$ and $SO(6)$, and $\epsilon(x)$ is a conformal Killing spinor. ${M^i}_I$ is a 3×6 matrix which mediates the coupling to the scalars; this is necessary since we only want to couple to three of the six scalars of $\mathcal{N} = 4$ SYM. For explicit calculations we can take ${M^1}_1 = {M^2}_2 = {M^3}_3 = 1$ and all other entries to be zero. The supersymmetry variation of the Wilson loop operator is then proportional to

$$\delta W \propto \left(idx^\mu\gamma_\mu + \frac{1}{2}\sigma_i^R {M^i}_I\rho^I\gamma^5\right)\epsilon(x) \tag{4.5}$$

and can be rewritten due to (4.1)

$$\delta W \propto idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_1 + \frac{1}{2}\sigma_i^R M^i{}_I \rho^I \gamma^5 \epsilon_0 - x^\alpha \gamma_\alpha \left(idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_0 + \frac{1}{2}\sigma_i^R M^i{}_I \rho^I \gamma^5 \epsilon_1 \right) . \tag{4.6}$$

The first two and analogously the last two terms can be decomposed into chiralities via $\epsilon^\pm = \frac{1}{2}(1 \pm \gamma^5)$. The resulting expression can then be simplified with the help of the identity

$$idx^\mu x^\nu \gamma_{\mu\nu}\epsilon^\mp = \pm\frac{1}{2}\tau^i \sigma_i^{R,L}\epsilon^\mp \tag{4.7}$$

so that the first two terms of (4.6) take the form

$$\begin{aligned} idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_1 + \frac{1}{2}\sigma_i^R M^i{}_I \rho^I \gamma^5 \epsilon_0 = \\ \frac{1}{2}\left(\sigma_i^R(\tau^i\epsilon_1^- - M^i{}_I\rho^I\epsilon_0^-) - (\sigma_i^L\tau^i\epsilon_1^+ - \sigma_i^R M^i{}_I\rho^I\epsilon_0^+)\right) \end{aligned} \tag{4.8}$$

and a similar expression holds for the last two terms.

We conclude that for an arbitrary curve on S_3 without linear relations between the $\sigma_i^{R,L}$ and non-trivial coordinates x^μ, the only solution to the supersymmetry variation of the Wilson loop is given by

$$\begin{aligned} \tau^i\epsilon_1^- &= M^i{}_I\rho^I\epsilon_0^- , \qquad i = 1,2,3 \\ \epsilon_1^+ &= \epsilon_0^+ = 0 . \end{aligned} \tag{4.9}$$

In order to solve this set of three equations

$$\tau_k\epsilon_1^- = \rho^k\epsilon_0^- , \qquad k = 1,2,3 \tag{4.10}$$

we eliminate ϵ_0^- and use the Lie-Algebra of the Pauli matrices to get

$$i\tau_1\epsilon_1^- = -\rho^{23}\epsilon_1^- , \qquad i\tau_2\epsilon_1^- = -\rho^{31}\epsilon_1^- , \qquad i\tau_3\epsilon_1^- = -\rho^{12}\epsilon_1^- . \tag{4.11}$$

This is a set of three constraints, but only two of them are independent. ϵ_0^- has eight reals components and the other spinor ϵ_1^- is completely determined by it. Since there are two constraints acting on ϵ_0^- a general curve on S^3 preserves 1/16 of the original supersymmetries. To determine the spinor ϵ_0^- we use the same techniques we will later study in detail on $\mathbb{H}_3$ leading to the result

$$\epsilon_0^- = -\epsilon_1^- . \tag{4.12}$$

Ultimately, let us write down the two supercharges preserved by a general curve on S^3

$$\bar{\mathcal{Q}}^a = \varepsilon^{\dot\alpha\dot a}\left(\bar{Q}^a_{\dot\alpha\dot a} - \bar{S}^a_{\dot\alpha\dot a}\right) , \tag{4.13}$$

before we turn to some special curves.

4.2 Special Loops on S^3

In this section we put forward some loops that preserve more supersymmetries than a general curve. We will later see that most of these curves have at least one dual object on $\mathbb{H}_3$.

4.2.1 The submanifold S^2

The submanifold S^2 is determined by the condition $x^4 = 0$. The right and left invariant one forms are no longer independent, but satisfy

$$\sigma_i^R = -\sigma_i^L = \epsilon_{ijk} x^j dx^k \,. \tag{4.14}$$

Since now there is a linear relation between left and right invariant one forms, equation (4.8) has the additional solutions

$$\tau^i \epsilon_1^+ = -M^i{}_I \rho^I \epsilon_0^+ \,. \tag{4.15}$$

The general solution can then be written in the compact form

$$i\gamma_{ij}\epsilon_1 = \epsilon_{ijk} M^i{}_I \rho^I \gamma^5 \epsilon_0, \qquad i = 1,2,3 \,. \tag{4.16}$$

Note that the constraints are no longer chiral. After all we realize that a general curve on S^2 is 1/8 BPS. Since restricting the curve from S^3 to S^2 doubles the supersymmetries, there are now four conserved supercharges

$$\mathcal{Q}^a = \varepsilon^{\alpha r}\left(Q^a_{\alpha r} + S^a_{\alpha r}\right), \qquad \bar{\mathcal{Q}}^a = \varepsilon^{\dot\alpha \dot r}\left(\bar{Q}^a_{\dot\alpha \dot r} - \bar{S}^a_{\dot\alpha \dot r}\right) \,. \tag{4.17}$$

If we consider a general smooth curve on S^2, we can find an interesting symmetry: Given the above construction the Wilson loop operator will couple to the gauge field via $\dot{\vec{x}}$, whereas the coupling to the scalars is given by (4.14) or written as a vector $\vec{x} \times \dot{\vec{x}}$. Now we can also consider $\vec{x} \times \dot{\vec{x}}$ as a curve on S^2 and use it to define a gauge coupling. If we furthermore assume that the curve is nowhere a geodesic (meaning that $\ddot{\vec{x}} \neq 0$) we get the new couplings to the scalars from (4.14) again

$$(\vec{x} \times \dot{\vec{x}}) \times (\vec{x} \times \ddot{\vec{x}}) = -\vec{x}(\dot{\vec{x}} \cdot (\vec{x} \times \ddot{\vec{x}})) \propto \vec{x} \,. \tag{4.18}$$

Since we assumed that the curve is nowhere a geodesic the proportionality constant in front of $\vec{x}$ will not be zero. We realize that for a smooth curve on S^2 there exists a dual curve with interchanged gauge and scalar couplings.

4.2.2 Great circle

Consider a circle in the (x_1, x_2) plane parametrized by

$$x^\mu = (\cos t, \sin t, 0, 0) \,. \tag{4.19}$$

Such a loop will couple to a single scalar ϕ^3

$$\vec{\sigma}^R = 2(0,0,1)dt \,. \tag{4.20}$$

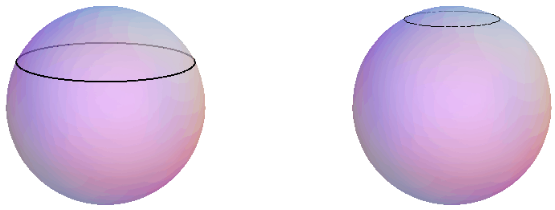
The supersymmetry variation leads to a single constraint

$$i\gamma_{12}\epsilon_1 = \rho^3 \gamma^5 \epsilon_0 \,. \tag{4.21}$$

It is pleasent to realize that the S^3 SUSY construction reproduces the long known 1/2 BPS circular Wilson loop coupling to one scalar that we have already studied in the second chapter. The circle preserves 16 supercharges (with $A = 1, \ldots, 4$)

$$\mathcal{Q}_A = i\gamma_{12} Q_A + (\rho^3 S)_A, \qquad \bar{\mathcal{Q}}^A = i\gamma_{12}\bar{Q}^A - (\rho^3 \bar{S})^A \,. \tag{4.22}$$

The vacuum expectation value of the circular loop coupling to one scalar is captured by a Gaussian matrix model as we have seen in the second chapter.



(c) A latitude on S^2

(d) The scalar couplings are also described by a latitude

Figure 4.1: A Latitude on S^2 and its scalar couplings

4.2.3 Latitude

A non-maximal circle on S^2, which is a latitude, can be parametrized by

$$x^\mu = (\sin\theta_0 \cos t, \sin\theta_0 \sin t, \cos\theta_0, 0)\,. \tag{4.23}$$

This example was first considered in [42] as an interpolation between the usual circle coupling to one scalar and the 1/4 BPS circle coupling to two scalars presented in [5]. The special example of the latitude finally led to the general supersymmetry construction on S^3 presented at the beginning of this chapter. There has been further interest in this particular Wilson loop: The correlation function between the 1/4 BPS Wilson loop and a chiral primary operator has been computed in [43] and a description in string theory in terms of D3 Branes was given in [44]. By a conformal transformation it is possible to map the Wilson loop studied in [42] to the latitude. The latitude couples to three scalars

$$\vec{\sigma}^R = 2\sin\theta_0(-\cos\theta_0\cos t, -\cos\theta_0 \sin t, \sin\theta_0)dt\,. \tag{4.24}$$

The supersymmetry variation leads to two independent equations

$$\begin{aligned} \cos\theta_0(\gamma_{12} + \rho^{12})\epsilon_1 &= 0, \\ \rho^3\gamma^5\epsilon_0 &= \left[i\gamma_{12} + \gamma_3\rho^2\gamma^5\cos\theta_0(\gamma_{23}+\rho^{23})\right]\epsilon_1\,. \end{aligned} \tag{4.25}$$

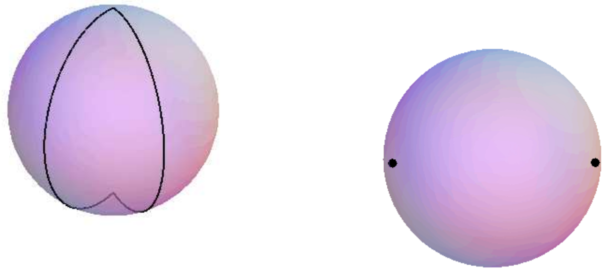
Note that for $\cos\theta_0 = 0$ the first equation vanishes and the second one reduces to the constraint for the large circle. For $\cos\theta_0 \neq 0$ there are two independent constraints, therefore this system is 1/4 BPS. As in the case of the great circle the propagator is a constant

$$\langle W \rangle_{Latitude} = 1 + \frac{g_{YM}^2 N}{8}\sin\theta_0^2 + O(g^4)\,. \tag{4.26}$$

In the original reference [42], it was shown that the pertubation series can be summed up by the same matrix model as in the case of the great circle with a replacement of the coupling constant $\lambda \to \sin^2\theta_0\lambda$. Extrapolated to the strong coupled regimes the expectation value is given by

$$\langle W \rangle_{Latitude} \sim e^{\sqrt{\lambda}\sin\theta_0}\,. \tag{4.27}$$

We will later see that this is in agreement with the string theory calculation.



(a) Two longitudes on S^2

(b) Two longitudes couple to two points on the equator of S^2

Figure 4.2: Two longitudes on S^2

4.2.4 Two Longitudes

Two longitudes intersecting at the north and southpole of a twodimensional sphere can be parametrized by

$$\begin{aligned} x^\mu &= (\sin t, 0, \cos t, 0), && 0 \le t \le \pi \\ x^\mu &= (-\cos\delta \sin t, -\sin\delta \sin t, \cos t, 0), && \pi \le t \le 2\pi\,. \end{aligned} \tag{4.28}$$

This Wilson loop has two cusps, one at each pole of the sphere. The Wilson loop will couple to Φ^2 along the first arc and to $-\cos\delta\,\Phi^2 + \sin\delta\,\Phi^1$ along the second arc. Being supersymmetric each arc produces a single constraint

$$\begin{aligned} \rho^2\gamma^5\epsilon_0 &= i\gamma_{31}\epsilon_1, \\ (\cos\delta\rho^2\gamma^5 - \sin\delta\rho^1\gamma^5)\epsilon_0 &= i(\cos\delta\gamma_{31} - \sin\delta\gamma_{23})\epsilon_1\,. \end{aligned} \tag{4.29}$$

As long as $\sin\delta \ne 0$ we can combine both equations to give the two constraints

$$\rho^2\gamma^5\epsilon_0 = i\gamma_{31}\epsilon_1, \qquad \rho^1\gamma^5\epsilon_0 = i\gamma_{23}\epsilon_1\,. \tag{4.30}$$

Consequently, this system is also 1/4 BPS. Calculating the expectation value for the two longitudes in $\mathcal{N}=4$ SYM up to lowest non-trivial order turns out to be harder than in the case of the circle and the latitude

$$\langle W\rangle = 1 + \frac{g_{YM}^2 N}{(4\pi)^2}\int dt_1 dt_2 \frac{(\cos\delta\cos t_1\cos t_2 - \sin t_1 \sin t_2) - \cos\delta}{2(1+\cos\delta\sin t_1\sin t_2 - \cos t_1 \cos t_2)}\,. \tag{4.31}$$

Nevertheless, it is easier to attack this problem by an *AdS* calculation, which will be presented in the following.

4.2.5 Hopf Fibers

The S^3 can be parametrized in terms of three Euler angles

$$\begin{aligned} x^1 &= -\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}, & x^2 &= \sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}, \\ x^3 &= \cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}, & x^4 &= \cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}\,. \end{aligned} \tag{4.32}$$

The range of the Euler angles is given by $0 \leq \theta \leq \pi,\ 0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. There is a bigger structure hidden in this parametrization, to get aware of that let us take a look at the metric in terms of the Euler angles

$$ds^2 = \frac{1}{4}\left(d\theta^2 + \sin\theta^2 d\phi^2 + (d\psi + \cos\theta d\phi)^2\right) . \tag{4.33}$$

This form of the metric shows that we have written the S^3 as an S^1 fiber over S^2. The S^1 fiber is parametrized by ψ, whereas the base S^2 of the fibration is decsribed by (θ, ϕ). Now we want to consider a Wilson loop along an arbitrary fiber. It will sit at constant (θ_0, ϕ_0) whereas ψ varies along the curve. Inserting the parametrization of S^3 in terms of Euler angles in the definition of the one forms (4.2) we get

$$\begin{aligned} \sigma_1^R &= -\sin\psi\, d\theta + \cos\psi\, \sin\theta\, d\phi \\ \sigma_2^R &= \cos\psi\, d\theta + \sin\psi\, \sin\theta\, d\phi \\ \sigma_3^R &= d\psi + \cos\theta\, d\phi\,. \end{aligned} \tag{4.34}$$

For constant (θ_0, ϕ_0) we realize that such a loop will couple to a single scalar Φ^3. This is exactly what we expected since a single fiber is a great circle. If we study the supersymmetry variation it leads to the following constraints on the chiralities

$$\rho^3\epsilon_0^- = \tau_3\epsilon_1^-, \qquad \rho^3\epsilon_0^+ = \sigma_i^L\tau^i\epsilon_1^+ . \tag{4.35}$$

With the help of the left invariant one forms

$$\begin{aligned} \sigma_1^L &= \sin\phi\, d\theta - \cos\phi\, \sin\theta\, d\psi \\ \sigma_2^L &= \cos\phi\, d\theta + \sin\phi\, \sin\theta\, d\psi \\ \sigma_3^L &= d\phi + \cos\theta\, d\psi \end{aligned} \tag{4.36}$$

we can explicitly write down

$$\sigma_i^L(\theta_0, \phi_0)\tau^i = \cos\theta_0\tau^3 - \sin\theta_0(\cos\phi_0\tau^1 - \sin\phi_0\tau^2)\,. \tag{4.37}$$

Now we want to study a system of several fibers. We notice that the first equation of (4.35) does not depend on (θ_0, ϕ_0). This is why a system of arbitrary many fibers preserves the same anti-chiral combinations of $\bar{\mathcal{Q}}$ and $\bar{\mathcal{S}}$ as a single fiber. In order to see if the two fibers can share any of the chiral symmetries we take a look at the two constraints

$$\rho^3\epsilon_0^+ = \sigma_i^L(\theta_0, \phi_0)\tau^i\epsilon_1^+, \qquad \rho^3\epsilon_0^+ = \sigma_i^L(\theta_1, \phi_1)\tau^i\epsilon_1^+ . \tag{4.38}$$

Now we substract the two equations and take a look at the determinant of the resulting matrix

$$\begin{aligned} &\det([\sigma_i^L(\theta_0, \phi_0) - \sigma_i^L(\theta_1, \phi_1)]\tau^i) = \\ &\qquad 2(-1 + \cos\theta_0\cos\theta_1 + \sin\theta_0\sin\theta_1\cos(\phi_0 - \phi_1))d\psi\,. \end{aligned} \tag{4.39}$$

The only possibility for this expression to vanish is to have $\theta_0 = \theta_1$ and $\phi_0 = \phi_1$, which means that a combined system of two or more Hopf fibers does not preserve any of the chiral supercharges. But we already saw that the 8 anti-chiral supersymmetries are conserved

$$-i\gamma_{12}\epsilon_1^- = \rho^3\epsilon_0^- \,. \tag{4.40}$$

This is why a system of two or more Hopf fibers is 1/4 BPS. When taking a look at the supercharges conserved by such a system, we basically find the same combination of $\bar{\mathcal{Q}}$ and $\bar{\mathcal{S}}$ as in the case of the circular loop, but this time we have to stick to the anti-chiral supersymmetries

$$\bar{\mathcal{Q}}^A = i\gamma_{12}\bar{Q}^A - (\rho^3\bar{S})^A, \qquad A = 1, \dots, 4\,. \tag{4.41}$$

If we study the propagator between two arbitrary points along two of the circles, then we find it to be constant again which indicates that the perturbative expansion of the Hopf fibers is also given by a Gaussian matrix model.

4.2.6 Infinitesimal Loops

It we consider a Wilson loop close to a point on S^3, for example near $x^4 = 1$, the curvature of the manifold will no longer recognizeable. In this limit, we regain Zarembo's construction [5], since the scalar couplings (4.2) become exact differentials

$$\sigma_i^{R,L} \sim 2dx_i, \qquad i = 1, 2, 3\,. \tag{4.42}$$

The Wilson loop operator then reduces to

$$W = \frac{1}{N} TrP \exp \oint dt \left(iA_i\dot{x}^i + \dot{x}^i {M^i}_I \Phi_I\right), \qquad i = 1, 2, 3 \tag{4.43}$$

which we already know from the second chapter.

4.3 Ansätze to generalize SUSY on S^2

In this section we present some ansätze how the different SUSY constructions on S^2 discussed before can or cannot be combined. The coupling to the scalars will be denoted by the vector $\vec{\Theta}$. Since we now want to eventually couple to more than three scalars, the matrix ${M^i}_I$ will be an $i \times 6$ matrix, where i is the number of scalars used in the ansatz. The general ansatz for the Wilson loop operator then takes the form

$$W = \frac{1}{N} TrP \exp \oint dt \left(iA_\mu\dot{x}^\mu + \Theta^i {M^i}_I \Phi^I\right)\,. \tag{4.44}$$

4.3.1 Zarembo's Construction on $\mathbb{R}^3$

Before moving to S^2 let us take a short look at a general curve on $\mathbb{R}^3$. The construction of Zarembo then tells us that we have to take the following scalar couplings

$$\vec{\Theta} = (\dot{x}^1, \dot{x}^2, \dot{x}^3)\,. \tag{4.45}$$

The supersymmetry variation leads to three independent equations

$$\begin{aligned}(i\gamma_k + \rho^k\gamma^5)\epsilon_0 &= 0, \qquad k = 1,2,3, \\ \epsilon_1 &= 0\,.\end{aligned} \tag{4.46}$$

Using this construction a general curve on $\mathbb{R}^3$ preserves 1/16 of the original supersymmetries, but it destroys all superconformal symmetries.

4.3.2 Zarembo's Construction on S^2

S^2 can be regarded as a submanifold of $\mathbb{R}^3$, hence we can apply Zarembo's prescription. Since we restrict our curve to lie on S^2 we are provided with additional linear relations between the different terms appearing in the supersymmetry variation. The ansatz for the Wilson loop operator will be the same as in (4.45), but this time the supersymmetry variation leads to six equations

$$\begin{aligned}(i\gamma_m + \rho^m\gamma^5)\epsilon_0 = 0, \qquad m = 1,2,3 \\ (i\gamma_n + \rho^n\gamma^5)\epsilon_1 = 0, \qquad n = 1,2,3\,.\end{aligned} \tag{4.47}$$

Counting the supersymmetries we find that a general curve will be 1/8 BPS. Compared to $\mathbb{R}^3$ we realize that confining the curve to S^2 doubles the supersymmetries.

4.3.3 Combining One forms/Zarembo coupling to six Scalars

The next thing one is tempted to try is to restrict the curve to S^2 and combine the scalar couplings given by the Zarembo construction and the one forms (4.14). This can be achieved by introducing a parameter α that interpolates between the two different constructions

$$W = \frac{1}{N} TrP \exp \oint dt \left(iA_\mu \dot{x}^\mu + \sin\alpha\, \epsilon_{ijk} x^j \dot{x}^k M^i{}_I \Phi^I + \cos\alpha\, \dot{x}^l M^l{}_L \Phi^L \right)\,. \tag{4.48}$$

The parameters in this ansatz satfisfy $i = 1,2,3$ and $l = 4,5,6$. We know that we find a solution to the supersymmetry variation for $\alpha = 0$ and $\alpha = \pi/2$. Studying the supersymmetry variation for an arbitrary value of α leads to too many constraints on the spinors ϵ_0 and ϵ_1, hence the ansatz does not lead to a supersymmetric Wilson loop operator.

4.3.4 Combining One forms/Zarembo coupling to three Scalars

Although the ansatz considered before did not lead to a supersymmetric Wilson loop operator, we will basically use the same one, but this time we will restrict to couple only to three of the six scalars

$$\begin{aligned}\Theta_1 &= \sin\alpha(x^2\dot{x}^3 - x^3\dot{x}^2) + \cos\alpha\ \dot{x}^1 \\ \Theta_2 &= \sin\alpha(x^3\dot{x}^1 - x^1\dot{x}^3) + \cos\alpha\ \dot{x}^2 \\ \Theta_3 &= \sin\alpha(x^1\dot{x}^2 - x^2\dot{x}^1) + \cos\alpha\ \dot{x}^3\,.\end{aligned} \tag{4.49}$$

The ansatz for the Wilson loop operator then looks like this ($i = 1, 2, 3$)

$$W = \frac{1}{N} Tr P \exp \oint dt \left(i A_\mu \dot{x}^\mu + \sin\alpha \, \epsilon_{ijk} x^j \dot{x}^k M^i{}_I \Phi^I + \cos\alpha \, \dot{x}^i M^i{}_I \Phi^I \right) . \tag{4.50}$$

First of all, let us take a look at the norm of the scalar couplings. It is easy to see that they satisfy

$$\vec{\Theta} \cdot \vec{\Theta} = (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2 . \tag{4.51}$$

We might ask if this construction gives a new family of supersymmetric Wilson loops; unfortunately, this is not the case. If we go back to the one forms (4.2) presented at the beginning of this chapter and set x^4 to be a constant then we can derive the scalar couplings in (4.49) by an appropriate rescaling of the coordinates. This result suggests that the construction given at the beginning of this chapter might be the most general ansatz on S^3.

4.3.5 Stereographic Projection

The conformal transformation between a Wilson line and a circular loop was studied in [38]. It was found that although $\mathcal{N} = 4$ SYM is a conformal field theory, a circular Wilson loop is different from a Wilson line. The discrepancy was associated to the point at infinity which is needed to perform a conformal transformation (an inversion to be specific) between a line and a circle.
In this section we want to use the Zarembo construction, but this time we will use the stereographic projection between S^2 and $\mathbb{R}^2$ to derive a coupling to the scalars. The stereographic projection is defined via

$$(X, Y) = \left(\frac{x^1}{1 - x^3}, \frac{x^2}{1 - x^3} \right) . \tag{4.52}$$

Here (X, Y) are coordinates on the plane and (x^1, x^2, x^3) on S^2. The Zarembo prescription tells us to take a look at $(\dot{X}, \dot{Y})$

$$(\dot{X}, \dot{Y}) = \left(\frac{\dot{x}^3 x^1 - \dot{x}^1 x^3 + \dot{x}^1}{(1 - x^3)^2}, \frac{\dot{x}^3 x^2 - \dot{x}^2 x^3 + \dot{x}^2}{(1 - x^3)^2} \right) . \tag{4.53}$$

When we want to use $(\dot{X}, \dot{Y})$ as scalar couplings for the Wilson loop, we need the norm of $\vec{\Theta}$ to be equal to $\dot{x}^2$. Since the scalars of $\mathcal{N} = 4$ SYM do not transform as a tensor under a general coordinate transformation we have to introduce a conformal factor of $\Delta = (1 - x^3)$ into the scalar couplings

$$\Theta_4 = \frac{\dot{x}^3 x^1 - \dot{x}^1 x^3 + \dot{x}^1}{(1 - x^3)}, \qquad \Theta_5 = \frac{\dot{x}^3 x^2 - \dot{x}^2 x^3 + \dot{x}^2}{(1 - x^3)} . \tag{4.54}$$

Calculating the norm of the vector $\vec{\Theta}$ we find

$$\Theta_4^2 + \Theta_5^2 = (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2 . \tag{4.55}$$

At this point we are ready to evaluate the supersymmetry variation, which leads to the following set of three constraints

$$\begin{aligned} i\gamma_3\epsilon_0 &= -\rho^4\gamma^5\gamma_1\epsilon_1 \\ i\gamma_3\epsilon_0 &= -\rho^5\gamma^5\gamma_2\epsilon_1 \\ \gamma_3\epsilon_0 &= -\epsilon_1\,. \end{aligned} \tag{4.56}$$

A general curve with this construction is therefore 1/8 BPS.

4.3.6 Coupling to five Scalars on S^2

Now we want to couple to five scalars by combining the one forms with the couplings derived by the stereographic projection. The ansatz for the Wilson loop operator then takes the form (with an interpolation parameter α)

$$W = \frac{1}{N} Tr P \exp \oint dt \left(iA_\mu \dot{x}^\mu + \sin\alpha\, \epsilon_{ijk} x^j \dot{x}^k M^i{}_I \Phi^I + \cos\alpha\, \Theta_l M^l{}_L \Phi^L \right) . \tag{4.57}$$

The parameters i and l satisfy $i = 1, 2, 3$ and $l = 4, 5$. We know that for $\alpha = \pi/2$ and $\alpha = 0$ a general curve on S^2 preserves 1/8 of the original supersymmetries. Nevertheless, it is interesting to study if there is a solution in between for a generic value of α. But after a rather lengthy calculation, it turns out that this ansatz leads to inconsistent constraints on the spinors ϵ_0 and ϵ_1 and does therefore not give a supersymmetric Wilson loop.

V

Wilson Loops in $\mathbb{H}_3$

In this section we study Wilson loops that are restricted to hyperbolic space $\mathbb{H}_3$ and how they have to couple to the scalars in order to be supersymmetric. We will extend the supersymmetry construction from S^3 to $\mathbb{H}_3$ by appropriate analytic continuations in the scalar couplings. First, we consider the case of a general curve. Afterwards we stick to some special loops, which will preserve more supersymmetries than a general curve.

5.1 General Curve on $\mathbb{H}_3$

To define three dimensional hyperbolic $\mathbb{H}_3$ space we take our manifold to be flat Minkowski space $\mathbb{R}^{1,3}$ and impose the additional constraint

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -1\,. \tag{5.1}$$

In order to make the Wilson loop operator on $\mathbb{H}_3$ invariant under supersymmetry variation, we make use of the following one forms

$$\begin{aligned}\omega_1^{\pm} &= 2\left[\pm i(x^2dx^3 - x^3dx^2) + (x^0dx^1 - x^1dx^0)\right] \\ \omega_2^{\pm} &= 2\left[\pm i(x^3dx^1 - x^1dx^3) + (x^0dx^2 - x^2dx^0)\right] \\ \omega_3^{\pm} &= 2\left[\pm i(x^1dx^2 - x^2dx^1) + (x^0dx^3 - x^3dx^0)\right]\,.\end{aligned} \tag{5.2}$$

We can now either choose the ω_i^+ or the ω_i^- to define a coupling to the scalars of $\mathcal{N} = 4$ SYM; without loss of generality we will take the ω_i^+. Therefore we will couple to three of the six scalars, namely Φ^1, Φ^2, Φ^3. To check if this ansatz leads to local supersymmetry we have to evaluate $\omega_i^+\omega_i^+$

$$\omega_i^+\omega_i^+ = 4dx^\mu dx_\mu\,. \tag{5.3}$$

In the computation of the last equation we used the following relation, which can be derived by differentiating (5.1) and squaring the generated expression

$$\begin{aligned}x_0^2dx_0^2 + x_1^2dx_1^2 + x_2^2dx_2^2 + x_3^2dx_3^3 = \\ 2(x_0x_1dx_0dx_1 + x_0x_2dx_0dx_2 + x_0x_3dx_0dx_3 \\ - x_1x_2dx_1dx_2 - x_1x_3dx_1dx_3 - x_1x_2dx_1dx_2)\,.\end{aligned} \tag{5.4}$$

Written in form notation, the ansatz for the Wilson loop operator then looks like

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\oint\left(iA + \frac{1}{2}\omega_i^+ M^i{}_I\Phi^I\right)\,. \tag{5.5}$$

The properties of the matrix ${M^i}_I$ will be the same as in the previous chapter. The supersymmetry variation of the Wilson loop operator (5.5) is then proportional to

$$\delta W \propto \left(idx^\mu \gamma_\mu + \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \gamma^5 \right) \epsilon(x) \tag{5.6}$$

The matrices ρ^I are again gamma matrices belonging to the Clifford Algebra of $SO(6)$, whereas the γ_μ belong to $SO(1,3)$. Explicit representations can be found in the appendix. As usual $\epsilon(x)$ is a conformal Killing spinor. Since our curve is restricted to $\mathbb{H}_3$ (5.1) the supersymmetry variation can be rewritten as:

$$\delta W \propto idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_1 + \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \gamma^5 \epsilon_0 + x^\alpha \gamma_\alpha \left(idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_0 - \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \gamma^5 \epsilon_1 \right) \tag{5.7}$$

Quite similar as on S^3 it turns out to be necessary to go into a Weyl basis and decompose the spinors ϵ_0 and ϵ_1 into chiralities with the help of the usual projection operators $\epsilon^\pm = \frac{1}{2}\left(1 \pm \gamma^5\right)\epsilon$. Expanding the first two terms of (5.7) into their chiralities yields

$$\begin{aligned} &idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_1 + \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \gamma^5 \epsilon_0 \\ &\quad = idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_1^+ + idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_1^- + \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \epsilon_0^+ - \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \epsilon_0^- \,. \end{aligned} \tag{5.8}$$

The key relation used to further simplify the supersymmetry variation is the following linear relation between the terms being proportional to $\gamma_{\mu\nu}$ and the scalar couplings ω_i^+

$$idx^\mu x^\nu \gamma_{\mu\nu}\epsilon^\pm = \mp \frac{i}{2}\tau^i \omega_i^\pm \epsilon^\pm \,. \tag{5.9}$$

The matrices τ on the right hand side are Pauli matrices. To prove the relation we have to use the explicit representations of the gamma matrices defined in the appendix and write out all terms explicitely. Using (5.9) we can further simplify (5.8)

$$\begin{aligned} &idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_1 + \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \gamma^5 \epsilon_0 = \\ &\quad -\frac{i}{2}\omega_i^+ \tau^i \epsilon_1^+ + \frac{i}{2}\omega_i^- \tau^i \epsilon_1^- + \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \epsilon_0^+ - \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \epsilon_0^- = \\ &\qquad \frac{1}{2}\left(\omega_i^+(-i\tau^i \epsilon_1^+ + {M^i}_I \rho^I \epsilon_0^+) + (i\omega_i^- \tau^i \epsilon_1^- - \omega_i^+ {M^i}_I \rho^I \epsilon_0^-)\right) \end{aligned}$$

and we can make an analogues calculation for the last two terms of (5.7)

$$\begin{aligned} &idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_0 - \frac{1}{2}\omega_i^+ {M^i}_I \rho^I \gamma^5 \epsilon_1 = \\ &\qquad \frac{1}{2}\left(-\omega_i^+(i\tau^i \epsilon_0^+ + {M^i}_I \rho^I \epsilon_1^+) + (i\omega_i^- \tau^i \epsilon_0^- + \omega_i^+ {M^i}_I \rho^I \epsilon_1^-)\right) \,. \end{aligned}$$

We conclude that for an arbitrary curve on $\mathbb{H}_3$ the only solution to the supersymmetry variation of the Wilson loop is given by

$$\begin{aligned} i\tau^i \epsilon_1^+ &= {M^i}_I \rho^I \epsilon_0^+, \\ \epsilon_1^- &= \epsilon_0^- = 0 \,. \end{aligned} \tag{5.10}$$

In order to solve this set of three equations

$$i\tau_k \epsilon_1^+ = \rho^k \epsilon_0^+ , \qquad k = 1, 2, 3 \tag{5.11}$$

we eliminate ϵ_0^+ and use the Lie algebra of the Pauli matrices to get

$$i\tau_1 \epsilon_1^+ = -\rho^{23} \epsilon_1^+ , \qquad i\tau_2 \epsilon_1^+ = -\rho^{31} \epsilon_1^+ , \qquad i\tau_3 \epsilon_1^+ = -\rho^{12} \epsilon_1^+ . \tag{5.12}$$

These equations are consistent with each other. On the other hand only two of the equations are independent, since the commutator of two of them gives the third one, for example

$$[(i\tau_1 + \rho^{23}), (i\tau_2 + \rho^{31})]\epsilon_1^+ = 2[(-i\tau_3 + \rho^{21})]\epsilon_1^+ . \tag{5.13}$$

ϵ_1^+ has eight real components and ϵ_0^+ is fully determinded by ϵ_1^+. For a general curve there are two independent constraints acting on ϵ_1^+, hence a general curve on $\mathbb{H}_3$ preserves 1/16 of the original supersymmetries.

To solve the equations (5.12) we have to take care of the fact that by choosing three of the six scalars of $\mathcal{N} = 4$ SYM we broke the R-symmetry group $SU(4)$ down to $SU(2)_A \times SU(2)_B$. Here $SU(2)_A$ referes to rotations of Φ^1, Φ^2, Φ^3 and $SU(2)_B$ to rotations of Φ^4, Φ^5, Φ^6 respectively.

That is why we conclude that the matrices appearing in the set of equations (5.12) are the generators of $SU(2)_A$ and $SU(2)_R$. Equation (5.12) tells us that ϵ_1^+ is a singlet of the diagonal sum of $SU(2)_R$ and $SU(2)_A$, while it is a doublet of $SU(2)_B$.

We can always find a basis in which the ρ^I act as Pauli matrices on the $SU(2)_A$ indices, hence we can write (5.12) as

$$(\tau_k^R + \tau_k^A)\epsilon_1^+ = 0, \qquad k = 1, 2, 3 . \tag{5.14}$$

The $SU(4)$ index in ϵ_1^+ can be splitted into two indices $\dot{a}$ and a refering to $SU(2)_A$ and $SU(2)_B$

$$\epsilon_{1,\dot{\alpha}}^A = \epsilon_{1,\dot{\alpha}\dot{a}}^a . \tag{5.15}$$

The solution of (5.14) can now be written as

$$\epsilon_1^a = \epsilon_{01}^a - \epsilon_{10}^a = \epsilon^{\dot{\alpha}\dot{a}} \epsilon_{1,\dot{\alpha}\dot{a}}^a . \tag{5.16}$$

Finally ϵ_0^+ can be determined by (5.11) and (5.14)

$$\epsilon_0^+ = i\tau_3^R \rho^3 \epsilon_1^+ = i\tau_3^R \tau_3^A \epsilon_1^+ = -i\epsilon_1^+ . \tag{5.17}$$

We finish this section by realizing that a general curve on $\mathbb{H}_3$ preserves two supercharges like in the case of S^3.

5.2 Special Loops on $\mathbb{H}_3$

In this section, we put forward some special loops which preserve more supersymmetries than an arbitrary curve. Since in general curves on $\mathbb{H}_3$ are non-compact, we use the Poincaré disc model to visualize them. In this model the curves on $\mathbb{H}_3$ are brought to the unit disc by a stereographic projection. Hyperbolic lines are represented by arcs that end orthogonal to the boundary of the unit disc. For a concrete form of the stereographic projection the reader might take a look at the appendix.

5.2.1 The hyperbolic Plane $\mathbb{H}_2$

The submanifold $\mathbb{H}_2$ is determined by the condition $x^3 = 0$. The one forms then reduce to

$$\vec{\omega}^+ = 2\left((x^0dx^1 - x^1dx^0), (x^0dx^2 - x^2dx^0), i(x^1dx^2 - x^2dx^1)\right) . \tag{5.18}$$

In order to look for a general solution we go back to (5.7)

$$\begin{aligned} &idx^\mu x^\nu \gamma_{\mu\nu}\epsilon_1 + \frac{1}{2}\omega_i^+ M_I^i \rho^I \gamma^5 \epsilon_0 = \\ &\quad i(dx^0x^1 - dx^1x^0)\gamma_{01}\epsilon_1 + i(dx^0x^2 - dx^2x^0)\gamma_{02}\epsilon_1 + i(dx^1x^2 - dx^2x^1)\gamma_{12}\epsilon_1 \\ &+ (x^0dx^1 - x^1dx^0)\rho^1\gamma^5\epsilon_0 + (x^0dx^2 - x^2dx^0)\rho^2\gamma^5\epsilon_0 + i(x^1dx^2 - x^2dx^1)\rho^3\gamma^5\epsilon_0 \,. \end{aligned} \tag{5.19}$$

In addition, the last two terms of (5.7) also lead to the following three independent equations

$$i\gamma_{01}\epsilon_1 = \rho^1\gamma^5\epsilon_0, \qquad i\gamma_{02}\epsilon_1 = \rho^2\gamma^5\epsilon_0, \qquad \gamma_{12}\epsilon_1 = \rho^3\gamma^5\epsilon_0 \,. \tag{5.20}$$

We conclude that a general curve on $\mathbb{H}_2$ preserves 1/8 of the supersymmetries. Furthermore, we observe that on $\mathbb{H}_2$ it is not necessary to decompose the spinors into chiralities. Restricting the curve to $\mathbb{H}_2$ doubles the number of supersymmetries, consequently we get two supercharges for each chirality.
As on S^2, there is one interesting fact that can be applied to a general curve restricted to $\mathbb{H}_2$, which is nowhere a geodesic (meaning $\ddot{x}^\mu \neq 0$). For a general curve on $\mathbb{H}_2$ the coupling to the gauge field is given by $\dot{x}^\mu$ and the coupling to the scalars is defined by $\vec{\omega}^+$ (5.18). Since $\vec{\omega}^+$ can also be interpreted as a vector on $\mathbb{H}_2$ (at least if we allow an analytic continuation of the coordinates x^μ), we can now choose $\vec{\omega}^+$ as coupling to the gauge fields and will get a new prescription (denoted by $\tilde{\omega}$) for how to couple to the scalars. Surprisingly it turns out that the new scalar couplings will be proportional to x^μ

$$(\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3) \propto (x^0, x^1, x^2) \,. \tag{5.21}$$

The terms appearing in front of x^μ do include imaginary expressions. This is the same duality between gauge and scalar couplings degrees of freedom as on S^3, but we realize that on $\mathbb{H}_3$ it is only defined in the context of analytic continuations.

5.2.2 S^2 as a Submanifold of $\mathbb{H}_3$

We can rewrite the defining property of $\mathbb{H}_3$ as

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = -1 + (x^0)^2 \,. \tag{5.22}$$

If we are at a constant time $x^0 = z$, equation (5.22) is the definition of S^2 with curvature radius $R^2 = z^2 - 1$. Note that has z to obey the constraint $|z| > 1$ to get a reasonable definition of S^2. The scalar couplings then look like

$$\begin{aligned} \omega_1^+ &= 2\left[zdx^1 + i(x^2dx^3 - x^3dx^2)\right] \\ \omega_2^+ &= 2\left[zdx^2 + i(x^3dx^1 - x^1dx^3)\right] \\ \omega_3^+ &= 2\left[zdx^3 + i(x^1dx^2 - x^2dx^1)\right] . \end{aligned} \tag{5.23}$$

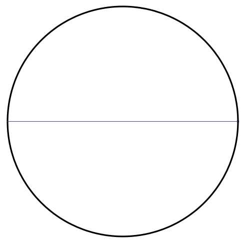


Figure 5.1: Line on the Poincaré disc

These scalar couplings can be interpreted in an interesting way: The terms proportional to z are the scalar couplings given by the construction of Zarembo in flat space [5]. The other terms are up to an factor of i the scalar couplings for a general curve on S^2 examined in the last chapter. Studying the supersymmetry variation one finds that a general curve is still 1/16 BPS, but the constraints are not chiral

$$\begin{aligned} \gamma_{ij}\epsilon_1 &= \varepsilon_{ijk}\rho^k\gamma^5\epsilon_0, \qquad i = 1,2,3 \\ \rho^1\gamma^5\epsilon_1 &= i\gamma_{10}\epsilon_0\,. \end{aligned} \tag{5.24}$$

The first set of equations are the constraints for a general curve on S^2.

5.2.3 Hyperbolic Line

The object corresponding to a great circle on S^2 is a hyperbolic line. It can be parametrized as follows

$$x^\mu = (\cosh t, \sinh t, 0, 0), \qquad -\infty \le t \le \infty\,. \tag{5.25}$$

Like the circle, the line will couple to a single scalar. Therefore the Wilson loop operator takes the familiar form

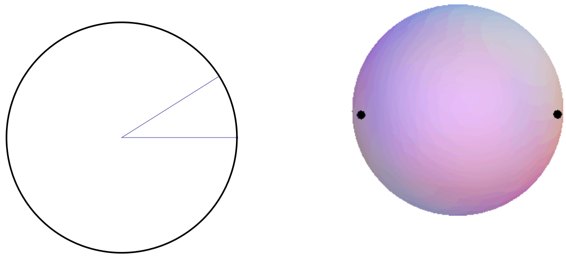
$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\oint\left(iA_\mu\dot{x}^\mu + \Phi^1\right)dt\,. \tag{5.26}$$

Although from this perspective the hyperbolic line looks quite similar to the circular loop, there is a big difference between the two objects, since the circle is compact whereas the hyperbolic line is not. It is easy to realize that supersymmetry requires a single constraint

$$i\gamma_{01}\epsilon_1 = \rho^1\gamma^5\epsilon_0\,. \tag{5.27}$$

Consequently, the hyperbolic line is a 1/2 BPS operator. Due to (5.27) we can immediately write down the 16 conserved supercharges (with $A = 1,\ldots,4$)

$$\mathcal{Q}_A = i\gamma_{01}Q_A + (\rho^1 S)_A, \qquad \bar{\mathcal{Q}}^A = i\gamma_{01}\bar{Q}^A - (\rho^1\bar{S})^A\,. \tag{5.28}$$



(a) Two rays with a cusp on the Poincaré disc

(b) Scalar Couplings on S^2

Figure 5.2: Two rays with a cusp and their scalar couplings

Now let us take a look at the expectation value of the hyperbolic line at order g_{YM}^2

$$\begin{aligned}\langle W \rangle_{Line} &= 1 + \frac{g_{YM}^2 N}{(4\pi)^2} \int dt_1 dt_2 \frac{-\cosh(t_1 - t_2) + 1}{-2 + 2\cosh(t_1 - t_2)} \\ &= 1 + \frac{g_{YM}^2 N}{(4\pi)^2} \int dt_1 dt_2 \left(-\frac{1}{2}\right) + O(g^4) .\end{aligned} \tag{5.29}$$

We realize that, as in the case of the circular loop, the integrand is a constant. Nevertheless, if we want to investigate the pertubation series we have to regulate the integrals, which will be done at the end of this chapter.

5.2.4 Two Rays with a Cusp

The next Wilson loop operator we want to study is a system of two rays with a cusp in the origin. These are the dual objects to the two longitudes on S^2. To parametrize this system, we take a ray in the (1,2) plane (x^μ), rotate it by an angle δ and add another ray in the (0,2) plane (y^μ)

$$\begin{aligned} x^\mu &= (\cosh t, \sinh t \cos\delta, \sinh t \sin\delta, 0), \qquad 0 \le t \le \infty \\ y^\mu &= (\cosh t, 0, \sinh t, 0) \end{aligned} \tag{5.30}$$

Along the first ray the Wilson loop couples to two scalars via

$$\vec{\omega}^+ = 2(\cos\delta, \sin\delta, 0)dt \tag{5.31}$$

and for the second ray it couples to Φ^2. Studying the supersymmetry variation for the rotated ray one finds the following equations, arranged by the functional dependence on t

$\sinh t$:

$$i\gamma_0\epsilon_0 = (\cos^2\delta\rho^1\gamma_1 + \sin\delta\cos\delta(\rho^1\gamma_2 + \rho^2\gamma_1) + \sin^2\delta\rho^2\gamma_2)\gamma^5\epsilon_1 \tag{5.32}$$

$\cosh t$:

$$i(\cos\delta\gamma_1 + \sin\delta\gamma_2)\epsilon_0 = (\cos\delta\rho^1\gamma_0 + \sin\delta\rho^2\gamma_0)\gamma^5\epsilon_1 \tag{5.33}$$

1 :

$$(\cos\delta\rho^1 + \sin\delta\rho^2)\gamma^5\epsilon_0 = i(\cos\delta\gamma_{01} + \sin\delta\gamma_{02})\epsilon_1 . \tag{5.34}$$

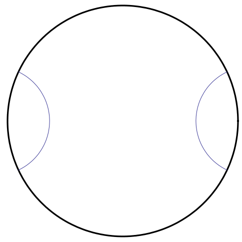


Figure 5.3: Two non intersecting lines on the Poincaré disc

It is easy to see that only one of the three constraints is independent; adding the supersymmetry variation for the second ray leads to the following two equations

$$\begin{aligned}(\cos\delta\rho^1\gamma^5+\sin\delta\rho^2\gamma^5)\epsilon_0 &= (i\cos\delta\gamma_{01}+i\sin\delta\gamma_{02})\epsilon_1 \\ \rho^2\gamma^5\epsilon_0 &= i\gamma_{02}\epsilon_1\,.\end{aligned} \tag{5.35}$$

In the case of $\sin\delta\neq 0$ there are two consistent equations,

$$\rho^1\gamma^5\epsilon_0=i\gamma_{01}\epsilon_1,\qquad \rho^2\gamma^5\epsilon_0=i\gamma_{02}\epsilon_1 \tag{5.36}$$

therefore this system is 1/4 BPS like the two longitudes on S^2. When taking a look at the expectation value of the two rays, we find the combined scalar vector propagator to be a complicated expression

$$\begin{aligned}\langle W\rangle_{Rays} &= 1+\frac{g_{YM}^2N}{(4\pi)^2}\int dt_1dt_2 \\ &\times\left(\frac{\sinh t_1\sinh t_2-\cosh t_1\cosh t_2\sin\delta+\sin\delta}{2(-1+\cosh t_1\cosh t_2-\sinh t_1\sinh t_2\sin\delta)}\right)\,.\end{aligned} \tag{5.37}$$

Despite the fact that the expectation value of the rays seems not to be accessible from the field theory side, we will study it in AdS.

5.2.5 Two non-intersecting Lines

If one considers two geodesics in $\mathbb{H}_2$ one cannot find an analogue on S^2. This fact is originating from the special properties of hyperbolic geometry. It has been kwown since the work of Lobatschewski in the 19th century that hyperbolic geometry explicitely violates the parallel axiom of Euclidian geometry. Two non-intersecting lines can be parametrized by $(j=1,2)$

$$x^\mu=(\cosh t\cosh\beta_j,\cosh t\sinh\beta_j,\sinh t,0),\qquad -\infty\leq t\leq\infty\,. \tag{5.38}$$

The scalar couplings for this particular Wilson loop are given by

$$\vec{\omega}^+=2(0,\cosh\beta_j,i\sinh\beta_j)dt,\qquad j=1,2\,. \tag{5.39}$$

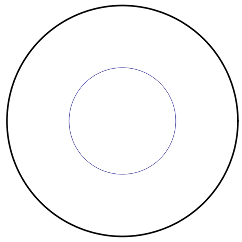


Figure 5.4: Circle in hyperbolic space

Arranged by functional dependence supersymmetry requires for the two lines ($j = 1, 2$)

$$\begin{aligned}
&\cosh t : \\
&\qquad i\gamma_2\epsilon_0 = [\cosh^2\beta_j\rho^2\gamma_0 + \cosh\beta_j\sinh\beta_j(\rho^2\gamma_1 + i\rho^3\gamma_0) \\
&\qquad\qquad + i\sinh^2\beta_j\rho^3\gamma_1]\gamma^5\epsilon_1 && (5.40)\\
&\sinh t : \\
&\qquad i(\cosh\beta_j\gamma_0 + \sinh\beta_j\gamma_1)\epsilon_0 = (\cosh\beta_j\rho^2\gamma_2 + i\sinh\beta_j\rho^3\gamma_2)\gamma^5\epsilon_1 && (5.41)\\
&1 : \\
&\qquad -(\cosh\beta_j\rho^2 + i\sinh\beta_j\rho^3)\gamma^5\epsilon_0 = i(\cosh\beta_j\gamma_{20} + \sinh\beta_j\gamma_{21})\epsilon_1\,. && (5.42)
\end{aligned}$$

It is easy to realize that only one of the three equations is independent

$$(\cosh\beta_j\rho^2\gamma^5 + i\sinh\beta_j\rho^3\gamma^5)\epsilon_0 = (i\cosh\beta_j\gamma_{02} + i\sinh\beta_j\gamma_{12})\epsilon_1\,. \qquad (5.43)$$

If we consider the system of two non-intersecting lines ($j = 1, 2$) supersymmetry requires only two consistent equations

$$\rho^2\gamma^5\epsilon_0 = i\gamma_{02}\epsilon_1, \qquad \rho^3\gamma^5\epsilon_0 = \gamma_{12}\epsilon_1\,. \qquad (5.44)$$

We conclude that this system is also 1/4 BPS.

To lowest non-trivial order the vacuum expectation value for the two non-intersecting lines is given by

$$\begin{aligned}
\langle W\rangle = 1 + \frac{g_{YM}^2 N}{(4\pi)^2}\int dt_1 dt_2 \\
\times\left(\frac{\cosh(\beta_1-\beta_2)(1+\sinh t_1\sinh t_2) - \cosh t_1\cosh t_2}{-2 + 2\cosh(\beta_1-\beta_2)\cosh t_1\cosh t_2 - 2\sinh t_1\sinh t_2}\right)\,. \qquad (5.45)
\end{aligned}$$

Even if we regularize the integral by a cutoff it is hard to handle.

5.2.6 Circle in hyperbolic Space

A circle with an arbitrary radius on $\mathbb{H}_2$ is the analogue of a latitude on S^2, the main difference between these two loops is the fact, that the radius of the circle in $\mathbb{H}_2$ is unbound, although this is not of importance in a conformal theory like $\mathcal{N} = 4$ SYM. The circle in hyperbolic space is related to the latitude on S^2, which we studied in (4.2.3) by an analytic continuation in the coordinate $\theta_0 \to i\theta_0$. Comparing the circle to the other loops on $\mathbb{H}_2$ one should be aware of the fact that the circle is compact in contrast to the various hyperbolic lines. We can parametrize the circle by (with constant θ_0)

$$x^\mu = (\cosh\theta_0, \sinh\theta_0\cos t, \sinh\theta_0\sin t, 0), \qquad 0 \le t \le 2\pi\,. \tag{5.46}$$

In this parametrization the radius of the circle is given by $\rho = \sinh\theta_0$. Like the latitude the circle couples to three scalars

$$\vec{\omega} = 2\sinh\theta_0(-\cosh\theta_0\sin t, \cosh\theta_0\cos t, i\sinh\theta_0)dt\,. \tag{5.47}$$

The supersymmetry variation results in the following constraints

$$\begin{aligned}
&\sin t:\\
&\quad (i\gamma_1 + \cosh\theta_0\rho^1\gamma^5)\epsilon_0 = (-i\cosh\theta_0\gamma_{10} - \cosh^2\theta_0\rho^1\gamma^5\gamma_0\\
&\qquad\qquad + i\sinh^2\theta_0\rho^3\gamma^5\gamma_2)\epsilon_1
\end{aligned} \tag{5.48}$$

$$\begin{aligned}
&\cos t:\\
&\quad -(i\gamma_2 + \cosh\theta_0\rho^2\gamma^5)\epsilon_0 = (i\cosh\theta_0\gamma_{20} + \cosh^2\theta_0\rho^2\gamma^5\gamma_0\\
&\qquad\qquad + i\sinh^2\theta_0\rho^3\gamma^5\gamma_1)\epsilon_1
\end{aligned} \tag{5.49}$$

$$\begin{aligned}
&1:\\
&\quad -i\rho^3\gamma^5\epsilon_0 = (i\gamma_{21} + \cosh\theta_0(\rho^2\gamma^5\gamma_2 + i\rho^3\gamma^5\gamma_0))\epsilon_1\,.
\end{aligned} \tag{5.50}$$

There is another constraint which is proportional to $\cos^2 t$ and $\sin t\cos t$, namely $(\rho^1\gamma_1 - \rho^2\gamma_2)\epsilon_1 = 0$. Global supersymmetry then requires two independent conditions

$$\begin{aligned}
(\gamma_{12} - \rho^{12})\epsilon_1 &= 0\\
\rho^3\gamma^5\epsilon_0 &= [\gamma_{12} + i\cosh\theta_0\gamma_0\rho^1\gamma^5(i\gamma_{20} + \rho^{13})]\epsilon_1\,.
\end{aligned} \tag{5.51}$$

Consequently the circle in $\mathbb{H}_2$ also is a 1/4 BPS operator. Taking a look at the expectation value we find

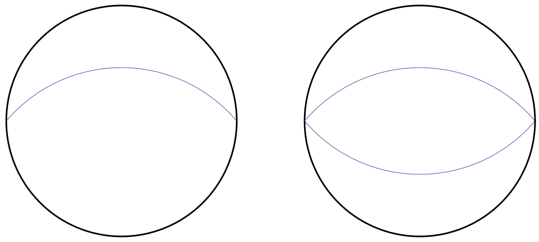
$$\begin{aligned}
\langle W\rangle_{Circle} &= 1 + \frac{g_{YM}^2 N}{(4\pi)^2}\int dt_1 dt_2\\
&\times \sinh\theta_0^2\left(\frac{-\cos(t_1-t_2) + \cosh\theta_0^2\cos(t_1-t_2) - \sinh\theta_0^2}{-2 + 2\cosh\theta_0^2 - 2\sinh\theta_0^2\cos(t_1-t_2))}\right) + O(g^4)\,.
\end{aligned} \tag{5.52}$$

The integrand turns out to be a constant again, the integrations over t become trivial

$$\langle W\rangle_{Circle} = 1 - \frac{g_{YM}^2 N}{8}\sinh\theta_0^2 + O(g^4)\,. \tag{5.53}$$

Compared to the latitude on S^2 we realize that the coupling constant is replaced as

$$\lambda \to -\sinh^2\theta_0\lambda\,. \tag{5.54}$$



(a) Moved hyperbolic line (b) Two moved hyperbolic lines

To evaluate the expectation value we sum up the ladder diagrams. By the same arguments used for the latitude on S^2 we only have to consider non-interacting graphs in the perturbative expansion. We can then use the same matrix model, but now the result is given by a Bessel function of the first kind due to the minus sign in front of the 't Hooft coupling

$$\langle W \rangle = \frac{2}{\sqrt{\lambda'}} J_1(\sqrt{\lambda'}), \qquad \lambda' = \lambda \sinh^2 \theta_0 \,. \tag{5.55}$$

For great arguments Bessel functions of the first kind asymptotically behave as

$$J_a(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{a\pi}{2} - \frac{\pi}{4}\right) . \tag{5.56}$$

We conclude that at strong coupling the vacuum expectation value for the circle in hyperbolic space behaves as

$$\langle W \rangle \sim \sqrt{\frac{2}{\pi}} \frac{1}{\lambda'^{3/4}} \cos\left(\sqrt{\lambda'} - \frac{3\pi}{4}\right) . \tag{5.57}$$

We will find the same oscillating behaviour of the expectation value from a string theory calculation.

If we reparametrize the circle by $x^\mu = (\sqrt{1+\rho_i^2}, \rho_i \cos t, \rho_i \sin t)$ and want to compute the correlation function between two circular loops with radii ρ_1, ρ_2, the integrand in (5.52) becomes more complicated

$$-\frac{\rho_1 \rho_2}{2} \frac{\cos(t_1 - t_2)(-1 + \sqrt{1+\rho_1^2}\sqrt{1+\rho_2^2}) - \rho_1\rho_2}{1 - \sqrt{1+\rho_2^2}\sqrt{1+\rho_1^2} + \rho_1\rho_2 \cos(t_1 - t_2)} \,. \tag{5.58}$$

The correlation function between two circular loops in the plane both coupling to one scalar has been contemplated in [34]. The difference between that calculation and the two circles in hyperbolic space is given by the different scalar couplings.

5.2.7 Moved hyperbolic Line

Another way of generalizing a single hyperbolic line is given by the following parametrization (with constant κ_0)

$$x^\mu = (\cosh \kappa_0 \cosh t, \cosh \kappa_0 \sinh t, \sinh \kappa_0, 0) \qquad -\infty \le t \le \infty \,. \tag{5.59}$$

It is worth mentioning that for $\kappa_0 = 0$ we recover the hyperbolic line studied in section (5.2.3). This generalized hyperbolic line can also be obtained by an analytic continuation of the latitude on S^2 studied in (4.2.3), namely by setting $\theta \to (\pi/2 + i\kappa_0)$ and $t \to it$. Therefore we realize that there are two objects on $\mathbb{H}_2$ dual to the latitude on S^2. Like the latitude the moved hyperbolic line will couple to three scalars

$$\vec{\omega}^+ = 2\cosh\kappa_0(\cosh\kappa_0, -\sinh\kappa_0\sinh t, -i\sinh\kappa_0\cosh t)dt\,. \tag{5.60}$$

The supersymmetry variation results in the following constraints

$\sinh t$:

$$\begin{aligned}(-i\gamma_0 + \sinh\kappa_0\rho^2\gamma^5)\epsilon_0 = (&i\sinh\kappa_0\gamma_{02} + \cosh^2\kappa_0\rho^1\gamma^5\gamma_1 \\ &- \sinh^2\kappa_0\rho^2\gamma^5\gamma_2)\epsilon_1\end{aligned} \tag{5.61}$$

$\cosh t$:

$$\begin{aligned}-(i\gamma_2 + \sinh\kappa_0\rho^2\gamma^5)\epsilon_0 = (&i\sinh\kappa_0\gamma_{12} + \cosh^2\kappa_0\rho^1\gamma^5\gamma_0 \\ &- i\sinh^2\kappa_0\rho^3\gamma^5\gamma_2)\epsilon_1\end{aligned} \tag{5.62}$$

1 :

$$-\rho^1\gamma^5\epsilon_0 = (i\gamma_{10} + \sinh\kappa_0(\rho^1\gamma^5\gamma_2 - i\rho^3\gamma^5\gamma_0))\epsilon_1\,. \tag{5.63}$$

In addition, there is another constraint which is proportional to $\sinh^2 t$ and $\sinh t\cosh t$, namely $(\rho^2\gamma_1 + i\rho^3\gamma_0)\epsilon_1 = 0$. Only two of the constraints are independent

$$\begin{aligned}\sinh\kappa_0(i\gamma_{01} + \rho^{23})\epsilon_1 &= 0 \\ (i\gamma_{01} + \sinh\kappa_0\gamma^5\rho^3\gamma_2(i\gamma_{02} + \rho^{13})\epsilon_1 &= \rho^1\gamma^5\epsilon_0\,.\end{aligned} \tag{5.64}$$

In the case of $\kappa_0 = 0$ the first equation disappears and the second equation reduces to the constraint for the hyperbolic line (5.27). For $\kappa_0 \neq 0$ the supersymmetry variation leads to two consistent equations. Consequently we are dealing with a 1/4 BPS operator. Calculating the propagator for the moved hyperbolic line, we find it to be a constant

$$\begin{aligned}\langle W\rangle &= 1 + \frac{g_{YM}^2 N}{(4\pi)^2}\int dt_1 dt_2 \\ &\times\left(\frac{\cosh^2\kappa_0}{2}\frac{-\cosh(t_1-t_2) + \cosh^2\kappa_0 - \sinh^2\kappa_0\cosh(t_1-t_2)}{-1+\cosh^2\kappa_0\cosh(t_1-t_2)+\sinh^2\kappa_0}\right) + O(g^4) \\ &= 1 - \frac{g_{YM}^2 N}{(4\pi)^2}\int dt_1 dt_2\frac{\cosh^2\kappa_0}{2} + O(g^4)\,.\end{aligned} \tag{5.65}$$

We can also consider a system of two lines which are 1/8 BPS. Two such lines can be parametrized by

$$x^\mu = (\cosh\kappa_l\cosh t, \cosh\kappa_l\sinh t, \sinh\kappa_l), \qquad l = 1,2\,. \tag{5.66}$$

For such a system the integrand in (5.65) will no longer be constant but proportional to

$$\frac{\cosh\kappa_1\cosh\kappa_2}{2}\frac{(\cosh\kappa_1\cosh\kappa_2 - \cosh(t_1-t_2))(1+\sinh\kappa_1\sinh\kappa_2)}{-1+\cosh(t_1-t_2)\cosh\kappa_1\cosh\kappa_2 - \sinh\kappa_1\sinh\kappa_2}\,. \tag{5.67}$$

For $\kappa_1 = \kappa_2$ the integrand reduces to (5.65) as expected.

5.2.8 No analogue of Hopf Fibers

In the last chapter we saw that on S^3 the Hopf fibers form a two parameter family of circles and are therefore very interesting objects. In order to find an analogue on $\mathbb{H}_3$ we have to solve the ODE's $\omega_1^+ = \omega_2^+ = 0$ and $\omega_3^+ = 1$ under the side condition (5.1). There exists a solution to this system of diffenrential equations, but it includes imaginary coordinates and is therefore not physically interesting. Just for completeness we want to mention that one can find a similar fibration on a manifold with two timelike directions.

5.3 Extension to the Light Cone

In order to extend the prescription studied in this chapter from $\mathbb{H}_3$ to the light cone we have to define $\mathbb{H}_3$ with curvature radius $x^2 = -R^2$ and take the limit $R \to 0$. Performing a dimensional analysis of the scalar couplings defined in (5.2), we realize that it is necessary to introduce a factor of $1/R$

$$\begin{aligned} \omega_1^\pm &= \frac{2}{R}\left[\pm i(x^2dx^3 - x^3dx^2) + (x^0dx^1 - x^1dx^0)\right] \\ \omega_2^\pm &= \frac{2}{R}\left[\pm i(x^3dx^1 - x^1dx^3) + (x^0dx^2 - x^2dx^0)\right] \\ \omega_3^\pm &= \frac{2}{R}\left[\pm i(x^1dx^2 - x^2dx^1) + (x^0dx^3 - x^3dx^0)\right] . \end{aligned} \tag{5.68}$$

For a general curve there will not exist a smooth limit $R \to 0$.

5.3.1 One Line in the Light Cone

A single line in the light cone, parametrized by

$$x^\mu = (x^0, x^0, 0, 0)\,, \tag{5.69}$$

will be supersymmetric without any coupling to the scalars. This can also be seen from (5.68), since all of the scalar couplings will already vanish before taking the limit $R \to 0$. Supersymmetry for a single line in the light cone requires

$$(\gamma_0 + \gamma_1)\epsilon_0 = 0\,. \tag{5.70}$$

Since the SUSY variation is independent of ϵ_1 none of the superconformal charges will be annihilated, in addition there is one constraint acting on ϵ_0; this is the reason why one line in the light cone is 3/4 BPS.

5.3.2 Two Lines in the Lightcone

Two lines in the light cone can be parametrized via

$$x^\mu = (x^0, x^0, 0, 0), \qquad y^\mu = (y^0, 0, y^0, 0)\,. \tag{5.71}$$

Being supersymmetric for the first and respectively the second line requires

$$(\gamma_0 + \gamma_1)\epsilon_0 = 0, \qquad (\gamma_0 + \gamma_2)\epsilon_0 = 0\,. \tag{5.72}$$

Again the supersymmetry variation is independent of ϵ_1. In order to look for shared supersymmetries we evaluate the commutator between the two constraints

$$[(\gamma_0 + \gamma_1), (\gamma_0 + \gamma_2)]\epsilon_0 = 2(\gamma_{02} + \gamma_{10} + \gamma_{12})\epsilon_0\,. \tag{5.73}$$

Since the last two terms acting on ϵ_0 vanish due to the second equation of (5.72), shared supersymmetry requires $\gamma_{02}\epsilon_0 = 0$, which is inconsistent with the second equation of (5.72), therefore ϵ_0 has to vanish. Nevertheless two lines in the light cone are 1/2 BPS, since they preserve all superconformal charges.

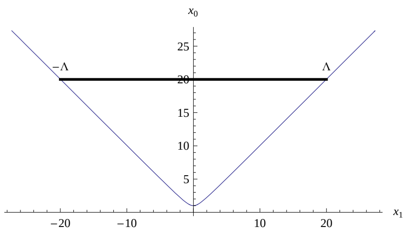


Figure 5.5: Cutoff on the hyperbolic line

5.4 The hyperbolic Line at order λ^2

In this section we study the expectation value of the hyperbolic line. We already saw that the propagator (5.29) is a constant

$$\langle W\rangle = 1 + \frac{g_{YM}^2 N}{(4\pi)^2}\int dt_1 dt_2 \left(-\frac{1}{2}\right) + O(g^4)\,. \tag{5.74}$$

To regulate equation (5.74) we integrate the coordinate x_0 twice from its minimal value 1 to a cutoff Λ and get

$$\langle W\rangle = 1 - \frac{g_{YM}^2 N}{(4\pi)^2}\frac{1}{2}(2(\Lambda-1))^2 + O(g^4)\,. \tag{5.75}$$

To further investigate the perturbative behaviour let us take a look at the next order $g_{YM}^4 N^2$. In addition to the cutoff on the integrals it turns out to be necessary to also introduce a dimensional regulator.

5.4.1 Contributions at order λ^2

The quantum corrections at order $g_{YM}^4 N^2$ for the circular Wilson loop have been calculated in [36], a cusped Wilson loop at two loops has been contemplated in [45]. The expressions of the contributing Feynman diagrams can be modified to the case of the hyperbolic line. Apart from ladder diagrams

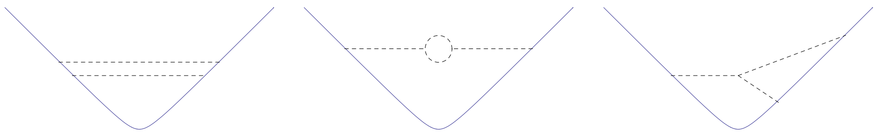
$$\Sigma_1 = \frac{g_{YM}^4 N^2}{6}\int_{t_1>t_2>t_3>t_4} dt_1 dt_2 dt_3 dt_4 \frac{(1-\dot{x}^{(1)}\cdot\dot{x}^{(2)})(1-\dot{x}^{(3)}\cdot\dot{x}^{(4)})}{[(x^{(1)}-x^{(2)})^2(x^{(3)}-x^{(4)})^2]^{\omega-1}} \tag{5.76}$$

we also have one loop corrections to the combined vector/scalar propagator

$$\Sigma_2 = -g_{YM}^4 N^2 \frac{\Gamma^2(\omega-1)}{2^7\pi^{2\omega}(2-\omega)(2\omega-3)}\int dt_1 dt_2 \frac{1-\dot{x}^{(1)}\cdot\dot{x}^{(2)}}{[(x^{(1)}-x^{(2)})^2]^{2\omega-3}} \tag{5.77}$$

and additionally a diagram with one internal vertex attching to three points on the Wilson loop

$$\Sigma_3 = -\frac{g_{YM}^4 N^2}{4}\int dt_1 dt_2 dt_3 \epsilon(t_1,t_2,t_3)(1-\dot{x}^{(1)}\cdot\dot{x}^{(3)})\dot{x}^{(2)}\cdot\frac{\partial}{\partial x^{(1)}}G(x^{(1)},x^{(2)},x^{(3)})\,. \tag{5.78}$$



(a) Σ_1 (b) Σ_2 (c) Σ_3

Figure 5.6: Diagrams at order λ^2 for the hyperbolic line

Here $\epsilon(t_1, t_2, t_3)$ performs antisymmetrization of t_1, t_2 and t_3 and the scalar three-point function is given by

$$G(x^{(1)}, x^{(2)}, x^{(3)}) = \int d^{2\omega} w \Delta(x^{(1)} - w)\Delta(x^{(2)} - w)\Delta(x^{(3)} - w) \,. \tag{5.79}$$

Introducing Feynman parameters and performing the w integration we arrive at

$$G(x^{(1)}, x^{(2)}, x^{(3)}) = \frac{\Gamma(2\omega - 3)}{2^6 \pi^{2\omega}} \times \int_0^1 \frac{d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma)}{[\alpha\beta(x^{(1)} - x^{(2)})^2 + \alpha\gamma(x^{(1)} - x^{(3)})^2 + \beta\gamma(x^{(3)} - x^{(2)})^2]^{2\omega-3}} \,. \tag{5.80}$$

The differentiation with respect to $x^{(1)}$ gives

$$\Sigma_3 = g_{YM}^4 N^2 \frac{\Gamma(2\omega - 2)}{2^7 \pi^{2\omega}} \int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma) \int dt_1 dt_2 dt_3 \times \epsilon(t_1, t_2, t_3) \frac{(1 - \dot{x}^{(1)} \cdot \dot{x}^{(3)})(\alpha(1 - \alpha)\dot{x}^{(2)} \cdot x^{(1)} - \alpha\gamma \dot{x}^{(2)} \cdot x^{(3)} - \alpha\beta \dot{x}^{(2)} \cdot x^{(2)})}{[\alpha\beta(x^{(1)} - x^{(2)})^2 + \alpha\gamma(x^{(1)} - x^{(3)})^2 + \beta\gamma(x^{(3)} - x^{(2)})^2]^{2\omega-2}} \,. \tag{5.81}$$

After discussing the Feynman graphs at order λ^2 in general, we now evaluate the contributions $\Sigma_1, \Sigma_2, \Sigma_3$ for the case of the hyperbolic line.

5.4.2 The hyperbolic Line

At first we define $t_{ij} := t_i - t_j$ and choose the standard parametrization $x^{(i)} = (\cosh t_i, \sinh t_i, 0, 0)$ leading to the following identities

$$(x^{(i)} - x^{(j)})^2 = 2(-1 + \cosh t_{ij}), \qquad x^{(i)} \cdot \dot{x}^{(j)} = \sinh t_{ij}, \qquad \dot{x}^{(i)} \cdot \dot{x}^{(j)} = \cosh t_{ij} \,. \tag{5.82}$$

First of all let us take a look at the ladder diagramm Σ_1 in $\omega = 2$ dimensions

$$\Sigma_1 = \frac{g_{YM}^4 N^2}{6} \int_{t_1 > t_2 > t_3 > t_4} dt_1 dt_2 dt_3 dt_4 \frac{1}{4} \,. \tag{5.83}$$

Using the same cutoff procedure as for (5.74) we get

$$\Sigma_1 = \frac{g_{YM}^4 N^2}{6} \frac{1}{4} (2(\Lambda - 1))^4 \,. \tag{5.84}$$

Secondly, let us specify (5.81) to the case of the hyperbolic line

$$\Sigma_3 = g_{YM}^4 N^2 \frac{\Gamma(2\omega-2)}{2^{2\omega+5}\pi^{2\omega}} \int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2}\delta(1-\alpha-\beta-\gamma)\int dt_1 dt_2 dt_3 \\ \times \epsilon(t_1,t_2,t_3)\frac{(1-\cosh t_{13})(\alpha(1-\alpha)\sinh t_{12}+\alpha\gamma\sinh t_{23})}{[\alpha\beta(1-\cosh t_{12})+\alpha\gamma(1-\cosh t_{13})+\beta\gamma(1-\cosh t_{23})]^{2\omega-2}}\,. \tag{5.85}$$

For compactness we will introduce another abbreviation

$$\Delta = \alpha\beta(1-\cosh t_{12})+\alpha\gamma(1-\cosh t_{13})+\beta\gamma(1-\cosh t_{23})\,. \tag{5.86}$$

If we consider the identity

$$\int dt_1 dt_2 dt_3 \frac{\partial}{\partial t_1}\frac{\epsilon(t_1,t_2,t_3)(1-\cosh t_{13})}{\Delta^{2\omega-3}} = 0 \tag{5.87}$$

together with

$$\frac{\partial}{\partial t_1}\epsilon(t_1,t_2,t_3) = 2\delta(t_{12}) - 2\delta(t_{13}) \tag{5.88}$$

we can derive the following relation

$$\int dt_1 dt_2 dt_3 \left[\frac{-\sinh t_{13}(\alpha\beta(1-\cosh t_{12})+\alpha\gamma(1-\cosh t_{13})+\beta\gamma(1-\cosh t_{23}))}{\Delta^{2\omega-2}}\right. \\ \left. +\ (2\omega-3)\frac{(1-\cosh t_{13})(\alpha\beta\sinh t_{12}+\alpha\gamma\sinh t_{13})}{\Delta^{2\omega-2}}\right]\epsilon(t_1,t_2,t_3) \\ = -2\int dt_1 dt_2 \frac{1}{[\gamma(1-\gamma)]^{2\omega-3}}\frac{1}{[1-\cosh t_{12}]^{2\omega-4}}\,. \tag{5.89}$$

In the last equation we used that the integrand vanishes in the case of $t_1 = t_3$. By cyclic permutations of t_1, t_2, t_3 and α, β, γ respectively and the fact that the Feynman parameters are fixed by $\alpha+\beta+\gamma = 1$, we rewrite the first term on the left hand side of (5.89) as

$$-\frac{(1-\cosh t_{13})}{\Delta^{2\omega-2}}\left(\alpha\gamma\sinh t_{32}+\alpha\gamma\sinh t_{13}+\alpha(1-\alpha)\sinh t_{21}+\alpha\beta\sinh t_{12}\right)\,. \tag{5.90}$$

With the help of the above relation, equation (5.89) can be rewritten as

$$\int dt_1 dt_2 dt_3 \epsilon(t_1,t_2,t_3)\left[\frac{(1-\cosh t_{13})(\alpha(1-\alpha)\sinh t_{12}+\alpha\gamma\sinh t_{23})}{\Delta^{2\omega-2}}\right. \\ \left. +(2\omega-4)\frac{(1-\cosh t_{13})(\alpha\beta\sinh t_{12}+\alpha\gamma\sinh t_{13})}{\Delta^{2\omega-2}}\right] \\ = -2\int dt_1 dt_2 \frac{1}{[\gamma(1-\gamma)]^{2\omega-3}}\frac{1}{[1-\cosh t_{12}]^{2\omega-4}}\,. \tag{5.91}$$

Note that the first term on the left hand side is exactly the integrand appearing in (5.85).

Conducting integration by parts we can alter the second term as (A denotes the surface term)

$$\begin{aligned}\frac{2\omega-4}{2\omega-3}\int dt_1dt_2dt_3\epsilon(t_1,t_2,t_3)(1-\cosh t_{13})\frac{\partial}{\partial t_1}\frac{1}{\Delta^{2\omega-3}}\\ =\frac{2\omega-4}{2\omega-3}\left[\int dt_1dt_2dt_3\epsilon(t_1,t_2,t_3)\frac{\sinh t_{13}}{\Delta^{2\omega-3}}\right.\\ \left.-2\int dt_1dt_2\frac{1}{[\gamma(1-\gamma)]^{2\omega-3}}\frac{1}{[1-\cosh t_{12}]^{2\omega-4}}\right]+A\,. \quad (5.92)\end{aligned}$$

Hence we arrive at

$$\begin{aligned}\int dt_1dt_2dt_3\epsilon(t_1,t_2,t_3)\frac{(1-\cosh t_{13})(\alpha(1-\alpha)\sinh t_{12}+\alpha\gamma\sinh t_{23})}{\Delta^{2\omega-2}}\\ =\frac{2\omega-4}{2\omega-3}\int dt_1dt_2dt_3\epsilon(t_1,t_2,t_3)\frac{\sinh t_{13}}{\Delta^{2\omega-3}}\\ -\frac{2}{2\omega-3}\int dt_1dt_2\frac{1}{[\gamma(1-\gamma)]^{2\omega-3}}\frac{1}{[1-\cosh t_{12}]^{2\omega-4}}+A\,. \quad (5.93)\end{aligned}$$

We will later study the first term on the right hand side, for the moment we will abbreviate it with B. The integral over the Feynman parameters gives

$$\int_0^1\frac{(\alpha\beta\gamma)^{\omega-2}}{[\gamma(1-\gamma)]^{2\omega-3}}\delta(1-\alpha-\beta-\gamma)=\frac{\Gamma^2(\omega-1)}{\Gamma(2\omega-2)(2-\omega)}\,. \quad (5.94)$$

After all the different manipulations the contribution Σ_3 takes the form

$$\Sigma_3=-g_{YM}^4N^2\frac{\Gamma^2(\omega-1)}{2^{2\omega+4}\pi^{2\omega}(2-\omega)(2\omega-3)}\int dt_1dt_2\frac{1}{(1-\cosh t_{12})^{2\omega-4}}+A+B\,. \quad (5.95)$$

In the case of the hyperbolic line the self-energy contribution Σ_2 looks like

$$\Sigma_2=g_{YM}^4N^2\frac{\Gamma^2(\omega-1)}{2^{2\omega+4}\pi^{2\omega}(2-\omega)(2\omega-3)}\int dt_1dt_2\frac{1}{(1-\cosh t_{12})^{2\omega-4}}\,. \quad (5.96)$$

Having arrived at this point we note that the self-energy diagram is canceled and we are left with the terms originating from the integration by parts

$$\Sigma_2+\Sigma_3=A+B\,. \quad (5.97)$$

We observe that the one loop calculation in the case of the hyperbolic line is more complicated than for the circular loop [36]. In the latter the contributions A and B vanish indicating that the whole perturbative expansion is given by ladder diagrams.

Coming back to the hyperbolic line let us first of all investigate the B term. It is proportional to

$$\begin{aligned}B\propto\frac{2\omega-4}{2\omega-3}\int dt_1dt_2dt_3\epsilon(t_1,t_2,t_3)\\ \times\frac{\sinh t_{13}}{\alpha\beta(1-\cosh t_{12})+\alpha\gamma(1-\cosh t_{13})+\beta\gamma(1-\cosh t_{23})^{2\omega-3}}\,. \quad (5.98)\end{aligned}$$

At this stage of the calculation we face the problem that the B term has a prefactor which vanishes in $2\omega = 4$ dimensions, whereas the integration may not be finite without setting a cutoff. Unfortunately, there does not seem to be a consistent way to link both limits. Nevertheless, if we set a cutoff and could bring the integrand in the form of

$$B \propto (2\omega - 4) f(\Lambda) \tag{5.99}$$

then the contribution from the B term vanishes in $2\omega = 4$ dimensions.

In addition to the B term we also have to take into account the surface term A, which is proportional to

$$A \propto \frac{2\omega - 4}{2\omega - 3} \int dt_2 dt_3 \\ \times \left. \frac{1 - \cosh t_{13}}{(\alpha\beta(1 - \cosh t_{12}) + \alpha\gamma(1 - \cosh t_{13}) + \beta\gamma(1 - \cosh t_{23}))^{2\omega-3}} \right|_{t_1=-\Lambda}^{t_1=\Lambda} \tag{5.100}$$

and has to be evaluated at the two cutoffs. Basically we are facing the same problem as before when evaluating the surface term. One way to proceed would be to remove the Feynman parameters and rewrite A in terms of the scalar three point function $G(x^{(1)}, x^{(2)}, x^{(3)})$. We could then assume that $(t_2, t_3) \leq t_1$ and use the asymptotic behaviour of the scalar propagators.

Although we ware unable to give a final result of the perturbative behaviour for the hyperbolic line, there are a lot of strong indications that the expectation value is not captured by summing up ladder diagramms. First of all, we expect the hyperbolic line to get contributions from infinity since it is infinite. Secondly, if we would assume for the moment that the complete perturbation series is given by ladder diagrams, the expectation value would be given by a Bessel function of the first kind like in the case of the circle in hyperbolic space. When extrapolated to the strong coupling regime the Bessel function shows an oscillating behaviour, which is in disagreement with the string theory result that will be presented later.

VI

Wilson loops in Minkowski Space

In this chapter we study Wilson loops in Minkowski space $\mathbb{R}^{1,3}$ which are not constrained to $\mathbb{H}_3$. Such a curve can be made supersymmetric by a simple analytic continuation of the Zarembo construction [5]. Such analytic continuations have already been considered in [46] for some special curves. The prescription can then also be applied to curves on the light cone.

6.1 General Curve

We investigate a Wilson loop operator in flat Minkowski space with signature $(-,+,+,+)$

$$W = \frac{1}{N}\mathrm{Tr}\ \mathcal{P}\exp\oint \left(iA_\mu \dot{x}^\mu(t) + \Theta_i M^i{}_I \Phi^I\right) dt\,. \tag{6.1}$$

The properties of the matrix $M^i{}_I$ are the same as in (3.2), but this time the coupling to the scalars is given by

$$\vec{\Theta} = (i\dot{x}^0, \dot{x}^1, \dot{x}^2, \dot{x}^3)\,. \tag{6.2}$$

When studying the supersymmetry variation acting on a conformal Killing spinor, we get the following constraints

$$\begin{aligned} (\gamma_0 + \rho^1\gamma^5)\epsilon_0 &= 0, \qquad (i\gamma_j + \rho^{(j+1)}\gamma^5)\epsilon_0 = 0, \qquad j = 1,2,3 \\ \epsilon_1 &= 0\,. \end{aligned} \tag{6.3}$$

Since there are four constraints acting on ϵ_0 a general curve preserves one Poincaré supersymmetry while it destroys all superconformal symmetries.

6.1.1 Restricting the Curve to $\mathbb{H}_3$

If we restrain the curve to the light cone or to hyperbolic three space $\mathbb{H}_3$, supersymmetry will be further enhanced since we are provided with additional linear relations in the supersymmetry variation of (6.1). For a general curve the constraints read

$$\begin{aligned} (\gamma_0 + \rho^1\gamma^5)\epsilon_\alpha = 0, &\qquad (i\gamma_1 + \rho^2\gamma^5)\epsilon_\alpha = 0 \\ (i\gamma_2 + \rho^3\gamma^5)\epsilon_\alpha = 0, &\qquad (i\gamma_3 + \rho^4\gamma^5)\epsilon_\alpha = 0, \qquad \alpha = 0,1\,. \end{aligned} \tag{6.4}$$

We realize that restraining the curve doubles the supersymmetries. A general curve then preserves one Poincaré supersymmetry and one superconformal symmetry and is therefore a 1/16 BPS operator.

6.1.2 Hyperbolic Line

Let us briefly take a look at the hyperbolic line ($x^\mu = (\cosh t, \sinh t, 0, 0)$) and use the prescription given by (6.2). Studying the supersymmetry variation results in the constraints

$$(\gamma_0 + \rho^1\gamma^5)\epsilon_\alpha = 0, \qquad (i\gamma_1 + \rho^2\gamma^5)\epsilon_\alpha = 0, \qquad \alpha = 0, 1\,. \tag{6.5}$$

Like the circle in flat space with the Zarembo construction the above hyperbolic line will be 1/4 BPS, since there are two constraints for ϵ_0 and ϵ_1.

6.1.3 Expectation Value

It seems very likely that the expectation value of this class of Wilson loops is trivial by the same arguments used in the case of Zarembo's loops [5] in flat space. To give a definite statement of course requires a more careful analysis.

6.2 Some special Curves

In this section we investigate some special curves in Minkowski space. The scalar couplings used here are not derived by a general construction.

6.2.1 Circle at a constant Time

Let us a briefly contemplate a circle in the lightcone at a constant time z. It is given by

$$x^\mu = (z, z\cos t, z\sin t)\,. \tag{6.6}$$

Coupling to one scalar via z supersymmetry requires one constraint

$$i\gamma_1\epsilon_0 = z(i\gamma_{01} + \rho^1\gamma^5\gamma_2)\epsilon_1\,. \tag{6.7}$$

Consequently the circle in the lightcone is the usual 1/2 BPS operator, whose propagator is constant.

6.2.2 Helix

Another example of a supersymmetric Wilson loop on the lightcone is a helix, given by

$$x^\mu = (e^{\xi t}, e^{\xi t}\cos t, e^{\xi t}\sin t, 0), \qquad -\infty \le t \le \infty\,. \tag{6.8}$$

Since (6.8) only covers the future part of the lightcone, we define the past lightcone helix via

$$x^\mu = -(e^{\xi t}, e^{\xi t}\cos t, e^{\xi t}\sin t, 0), \qquad -\infty \le t \le \infty\,. \tag{6.9}$$

To make this Wilson loop supersymmetric we have to couple to one scalar via $e^{\xi t}$ for the future helix and via -$e^{\xi t}$ for the past helix

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\oint dt\left(iA_\mu\dot{x}^\mu \pm e^{\xi t}\Phi^1\right)\,. \tag{6.10}$$

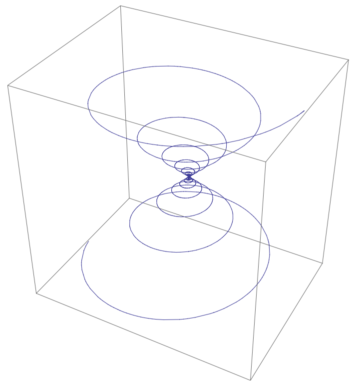


Figure 6.1: A helix on the lightcone; in the plot we set $\xi = 1/10$

After performing the supersymmetry variation we obtain the same constraints for the past as well as the future helix

$$(i\gamma_{01} + \rho^1\gamma^5\gamma_2)\epsilon_1 = 0, \qquad \epsilon_0 = 0\,. \tag{6.11}$$

A helix destroys all Poincaré supersymmetries and half of the superconformal symmetries, consequently it is a 1/4 BPS operator. For the future helix parametrized in (6.8) the expectation value up to g^2_{YM} is given by

$$\langle W\rangle_{Helix} = 1 + \frac{\lambda}{(4\pi)^2}\int dt_1 dt_2 \frac{1+\xi^2}{2} + O(g^4)\,. \tag{6.12}$$

It seems, that the scalar coupling for the light cone helix cannot be derived by considering a similar curve on $\mathbb{H}_3$ and then taking the light cone limit. Since the propagator is equal to a constant, the helix might lead to a general supersymmetry construction for curves on the light cone.

6.2.3 An interesting Interpolation

We saw that the hyperbolic line using the prescription (6.2) gives a 1/4 BPS operator, but additionaly we have seen in the previous chapter that when considering a hyperbolic line coupling to one scalar we get a 1/2 BPS operator. This naturally induces the question if there is a solution in between. The same question for the circle has been considered in [23]. In other words we want to take a look at a hyperbolic line and use the following coupling to the scalars (with an interpolation parameter α)

$$\vec{\Theta} = (i\cos\alpha\sinh t, \cos\alpha\cosh t, \sin\alpha)\,. \tag{6.13}$$

Supersymmetry then requires the two constraints

$$\begin{aligned} \cos\alpha(i\rho^{12} + \gamma_{01})\epsilon_1 &= 0 \\ \sin\alpha\rho^3\gamma^5\epsilon_0 &= (i\gamma_{01} + i\cos\alpha\rho^1\gamma^5\gamma_1)\epsilon_1\,. \end{aligned} \tag{6.14}$$

For $\alpha = \pi/2$ the first equation of (6.14) disappears and the second equation reduces to the constraint known for the 1/2 BPS hyperbolic line. We realize that there exists a solution for a generic α, which is 1/4 BPS. Taking a look at the vacuum expectation value up to lowest order we find

$$\langle W \rangle = 1 + \frac{g_{YM}^2}{(4\pi)^2} \int dt_1 dt_2 \frac{-\sin^2\alpha}{2} + O(g^4)\,. \tag{6.15}$$

Finally we want to mention, that we could also consider an interpolation of the following form (with interpolation parameter κ_0)

$$\vec{\Theta} = (i \sinh \kappa_0 \cosh t, \sinh \kappa_0 \sinh t, \cosh \kappa_0)\,. \tag{6.16}$$

It is easy to see that the second interpolation leads to the moved hyperbolic line studied in section (5.2.7).

VII

Wilson Loops in String Theory

In this chapter we present the dual string solutions to some of the Wilson loops which were discussed before in the gauge theory. To calculate the string solutions in the supergravity approximation we have to find the appropriate minimal surfaces under the side condition that the string worldsheet ends on the contour of the Wilson loop. In addition we evaluate the classical supergravity action and compare it to the expectation values calculated in the gauge theory. To compute the AdS duals of the loops in Minkowskian signature we have to perform analytic continuations. In the case of Wilson loops in Minkowskian signature these haven already been considered in [46]. The necessity to perform an analytic continuation when studying a Wilson loop with insertions of local operators has been contemplated in [47]. An interpretation of analytic continuations as a tunneling phenomena in AdS/CFT has been given in [48].

7.1 The circular Wilson Loop

The minimal surface ending on a circular loop was first presented in [49] and [25]. We shortly want to review the calculation in the conformal gauge and present some of the techniques that will be needed throughout the chapter. Since the circular loop in the gauge theory couples to a single scalar, we do not have to consider motion on S^5. We start our calculation with the metric on an AdS_3 subspace of AdS_5

$$ds^2 = \frac{L^2}{y^2}(dy^2 + dx_1^2 + dx_2^2) \tag{7.1}$$

and change to polar coordinates (r, ϕ) in the (x_1, x_2) plane

$$ds^2 = \frac{L^2}{y^2}(dy^2 + dr^2 + r^2 d\phi^2)\,. \tag{7.2}$$

We make the following ansatz and choose the static gauge

$$y = y(\sigma), \qquad r = r(\sigma), \qquad \phi = \tau\,. \tag{7.3}$$

The range of the worldsheet coordinates (τ, σ) is given by $0 \le \tau \le 2\pi$ and $0 \le \sigma \le \infty$. The Lagrangian in conformal gauge takes the form

$$\mathcal{L}_{Pol} = \frac{L^2}{y^2}(y'^2 + r'^2 + r^2) \tag{7.4}$$

whereas the Virasoro constraint reads

$$y'^2 + r'^2 = r^2\,. \tag{7.5}$$

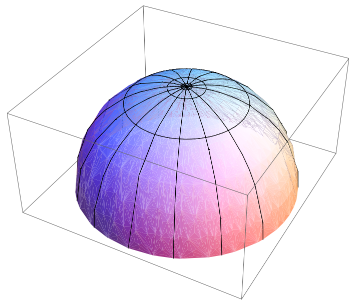


Figure 7.1: Minimal surface ending on a circular contour

The equations of motion

$$\begin{aligned} yy'' - y'^2 + r'^2 + r^2 &= 0 \\ r'' - r - 2\frac{r'y'}{y} &= 0 \end{aligned} \tag{7.6}$$

are solved by

$$y = \tanh\sigma, \qquad r = \frac{1}{\cosh\sigma}\,. \tag{7.7}$$

It is easy to see that the solution satisfies the Virasoro constraint. Evaluating the solution to the equation of motion on the classical supergravity action we find

$$\mathcal{S}_{Pol} = \frac{L^2}{2\pi\alpha'}\int d\sigma d\tau \frac{1}{\sinh^2\sigma}\,. \tag{7.8}$$

The τ integration is trivial; in order to evaluate the σ integration we have to regularize the action by including a boundary term [25]. The regularized action then reads

$$\mathcal{S}_{reg} = \sqrt{\lambda}\int_0^\infty d\sigma\left(\frac{1}{\cosh^2\sigma\sinh^2\sigma} - \frac{1}{\sinh^2\sigma}\right) = -\sqrt{\lambda}\,. \tag{7.9}$$

Note that through the regularization process the action has become negative. Using the Witten prescription we can finally write down the expectation value for the circular loop from string theory

$$\langle W\rangle = e^{\sqrt{\lambda}}\,. \tag{7.10}$$

This result is in agreement with the expectation value of the circular Wilson loop in the gauge theory extrapolated to the strong coupling regime.

7.2 Latitude

In this section we present the string theory dual to the latitude, which was first considered in [23] and [42]. Taking into account the scalar couplings from the gauge theory for the case of the latitude we use the metric on an $AdS_4 \times S^2$ subspace of $AdS_5 \times S^5$

$$ds^2 = \frac{L^2}{y^2}(dy^2 + dr^2 + r^2 d\phi^2 + dx_3^2) + L^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\,. \tag{7.11}$$

with polar coordinates (r, ϕ) in the (x_1, x_2) plane. The string should end along the curve $r = \sin\theta_0$ and constant $x_3 = \cos\theta_0$. In order to calculate the action in the conformal gauge we make the ansatz

$$y = y(\sigma), \qquad r = r(\sigma), \qquad \vartheta = \vartheta(\sigma), \qquad \phi = \varphi + \pi = \tau\,. \tag{7.12}$$

The phase difference of π in the ansatz between the ϕ and φ coordinates is a consequence of the supersymmetry construction from the gauge theory. The Lagrangian in conformal gauge assumes the shape

$$\mathcal{L}_{Pol} = \frac{L^2}{y^2}(y'^2 + r'^2 + r^2) + L^2(\vartheta'^2 + \sin^2\vartheta) \tag{7.13}$$

and the Virasoro constraint acquires the form

$$\frac{1}{y^2}(y'^2 + r'^2) + \vartheta'^2 = \sin^2\vartheta + \frac{r^2}{y^2}\,. \tag{7.14}$$

The AdS part of the Lagrangian can be integrated easily as

$$y = \sin\theta_0 \tanh\sigma, \qquad r = \frac{\sin\theta_0}{\cosh\sigma}\,. \tag{7.15}$$

Using the Virasoro constraint we find the equation of motion for ϑ

$$\sin^2\vartheta = \vartheta'^2\,, \tag{7.16}$$

which can be integrated as

$$\sin\vartheta(\sigma) = \frac{1}{\cosh(\sigma_0 \pm \sigma)}, \qquad \cos\vartheta(\sigma) = \tanh(\sigma_0 \pm \sigma)\,. \tag{7.17}$$

The boundary condition at $\sigma = 0$ reads

$$\sin\vartheta_0 = \cos\theta_0 = \frac{1}{\cosh\sigma_0}, \qquad \tanh^2\sigma_0 = \sin^2\theta_0\,. \tag{7.18}$$

The boundary term will be the same as in the case of the circular loop. Therefore the classical action acquires the form

$$\begin{aligned}\mathcal{S}_{reg} &= \sqrt{\lambda}\int_0^\infty d\sigma\left(\frac{1}{\cosh^2\sigma\sinh^2\sigma} - \frac{1}{\sinh^2\sigma} + \frac{1}{\cosh^2(\sigma_0 \pm \sigma)}\right) \\ &= \mp\sin\theta_0\sqrt{\lambda}\,.\end{aligned} \tag{7.19}$$

If we would consider the 1/4 BPS circle coupling to two scalars given by Zarembo's construction, the boundary condition would be $\sigma_0 = 0$, which means that the AdS part of the action cancels against the S^5 part leading to a trivial expectation value [5]. For the latitude the ambiguity in the sign in front of $\sin\theta_0$ should be chosen that way, that the classical action is minimized. Consequently the expectation value of the Wilson loop at strong coupling is given by

$$\langle W\rangle = e^{\sin\theta_0\sqrt{\lambda}}\,. \tag{7.20}$$

The other solution corresponds to an unstable instanton, which is exponentially supressed at a large value of λ.

7.3 The lightlike Cusp

Wilson loops in the lightcone seem to be very special. We already saw in the gauge theory that lightlike lines preserve a large amount of supersymmetry. Before turning to the hyperbolic line let us shortly review the case of the lightlike cusp, meaning we consider two semi infinite lightlike lines meeting at a point. This case was first considered in [50], but recently there is a growing interest in these objects since one can compute gluon scattering amplitudes at strong coupling with the help of lightlike Wilson loops [30, 31]. Even before the actual interest it was known that lightlike Wilson loops have a lot of interesting features, see for example [51, 45].

In the gauge theory lightlike lines are supersymmetric without any coupling to the scalars, therefore we use an AdS_3 subspace of AdS_5

$$ds^2 = \frac{L^2}{y^2}\left(dy^2 - dx_0^2 + dx_1^2\right) . \tag{7.21}$$

We choose the following parametrization

$$x_0 = e^\tau \cosh\sigma, \qquad x_1 = e^\tau \sinh\sigma, \qquad y = e^\tau w(\tau) \tag{7.22}$$

and using the Nambu-Goto action we get the following Lagrangian

$$\mathcal{L}_{NG} = \frac{\sqrt{1-(w+\dot{w})^2}}{w^2} . \tag{7.23}$$

The equation of motion for w then follows as usual from the Euler-Lagrange equation

$$0 = \frac{d}{d\tau}\left(\frac{-(\dot{w}+w)}{w^2\sqrt{1-(w+\dot{w})^2}}\right) + \left(\frac{2}{w^3}\sqrt{1-(w+\dot{w})^2} + \frac{\dot{w}+w}{w^2\sqrt{1-(w+\dot{w})^2}}\right) \tag{7.24}$$

and it is easy to realize that it is solved by $w(\tau) = \sqrt{2}$. Consequently, the surface is given by

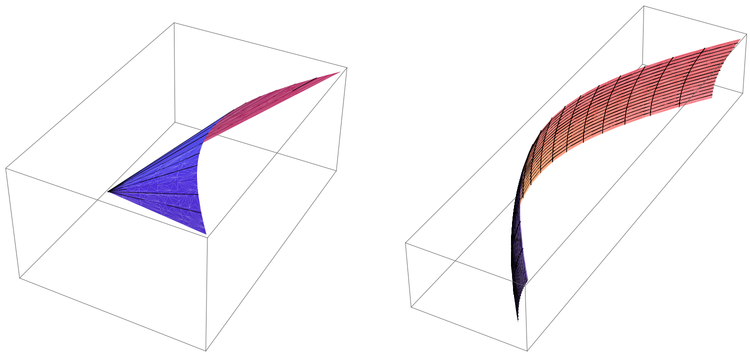
$$y = \sqrt{2}\sqrt{x_0^2 - x_1^2} . \tag{7.25}$$

Evaluating the action for the lightlike cusp requires to introduce a cutoff, which we will also have to introduce in the AdS calculation of the hyperbolic line.

7.4 The hyperbolic Line

The main difference between the hyperbolic line and the lightlike cusp considered before is the fact that the hyperbolic line does not have cusp at the origin, it is smooth. We know from the gauge theory that we do not have to consider motion on S^5, therefore we again take the metric on AdS_3

$$ds^2 = \frac{L^2}{y^2}\left(dy^2 - dx_0^2 + dx_1^2\right) . \tag{7.26}$$



(a) Minimal surface ending on a hyperbolic line

(b) Zooming into the origin one recognizes that the hyperbolic line is smooth

We choose hyperbolic coordinates in the (x_0, x_1) plane

$$x_0 = u \cosh t, \qquad x_1 = u \sinh t \tag{7.27}$$

and the metric in these coordinates is then given by

$$ds^2 = \frac{L^2}{y^2}\left(dy^2 - du^2 + u^2 dt^2\right) . \tag{7.28}$$

As in the case of the circular loop we make the following ansatz and choose the static gauge

$$y = y(\sigma), \qquad u = u(\sigma), \qquad t = \tau . \tag{7.29}$$

To find the string solution let us write down the Lagrangian in conformal gauge

$$\mathcal{L}_{Pol} = \frac{L^2}{y^2}(y'^2 - u'^2 + u^2) \tag{7.30}$$

together with the Virasoro constraint

$$y'^2 - u'^2 = u^2 . \tag{7.31}$$

The equations of motion

$$\begin{aligned} y''y - y'^2 - u'^2 + u^2 &= 0, \\ u'' - \frac{2y'}{y}u' + u &= 0 \end{aligned} \tag{7.32}$$

can be integrated as

$$y = \tan\sigma, \qquad u = \frac{1}{\cos\sigma} \tag{7.33}$$

and it is easy to see that the solution satisfies the Virasoro constraint and gives the right behaviour at the boundary.

The range of the world sheet coordinate σ is given by $0 \leq \sigma \leq \pi/2$, whereas τ varies between $-\infty$ and ∞. We conclude that the minimal surface ending on a hyperbolic line is given by

$$x_0 = \frac{\cosh\tau}{\cos\sigma}, \qquad x_1 = \frac{\sinh\tau}{\cos\sigma}, \qquad y = \tan\sigma . \tag{7.34}$$

and can be written in the form

$$y = \frac{1}{u}\sqrt{u^2-1}\sqrt{x_0^2-x_1^2}\,. \tag{7.35}$$

Before calculating the classical action let us take a look at the metric induced on the minimal surface, which is euclidian up to a conformal factor

$$ds^2 = \frac{1}{\sin^2\sigma}(d\sigma^2+d\tau^2)\,. \tag{7.36}$$

The bulk part of the action reads

$$\mathcal{S}_{bulk} = \frac{L^2}{2\pi\alpha'}\int d\sigma d\tau \frac{1}{\sin^2\sigma} \tag{7.37}$$

and the boundary term is given by

$$\mathcal{S}_{boundary} = \frac{L^2}{2\pi\alpha'}\int d\sigma d\tau \frac{1}{\sin^2\sigma\cos^2\sigma}\,, \tag{7.38}$$

so that the regularized action takes the following form

$$\mathcal{S}_{reg} = -\frac{L^2}{2\pi\alpha'}\int d\sigma d\tau \frac{1}{\cos^2\sigma}\,. \tag{7.39}$$

By adding the appropriate boundary term we have removed the divergence that appears from being close to the boundary. In the case of the hyperbolic line there is another kind of divergence originating from the infinite length of the line. One way of regularizing the surface is to set a cutoff on both σ and τ, which gives the following result

$$\begin{aligned} \mathcal{S}_{reg} &= -\frac{L^2}{2\pi\alpha'}\int_{-\Lambda_\tau}^{\Lambda_\tau} d\tau \int_0^{\Lambda_\sigma} d\sigma \frac{1}{\cos^2\sigma} \\ &= -\frac{L^2}{2\pi\alpha'}(2\Lambda_\tau \tan\Lambda_\sigma)\,. \end{aligned} \tag{7.40}$$

Since this result is hard to interprete we want to apply a different regularization scheme and use a physical cutoff on the diverging integral. Using the solution to the equations of motion, we can express the worldsheet coordinates in terms of the original coordinates

$$\cosh\tau = \frac{x_0}{\sqrt{x_0^2-x_1^2}}, \qquad \cos\sigma = \frac{1}{\sqrt{x_0^2-x_1^2}}\,. \tag{7.41}$$

Changing from (σ,τ) to (x_0,x_1) gives the following Jacobian

$$d\sigma d\tau = \frac{1}{\sqrt{x_0^2-x_1^2-1}(x_0^2-x_1^2)}dx_0 dx_1\,, \tag{7.42}$$

hence we can rewrite the action in terms of the original coordinates

$$\mathcal{S}_{reg} = -\frac{L^2}{2\pi\alpha'}\int dx_0 dx_1 \frac{1}{\sqrt{x_0^2-x_1^2-1}}\,. \tag{7.43}$$

For fixed x_0 the variable x_1 varies between the two roots of $x_1^2 = x_0^2 - 1$. Afterwards we perform the integration over x_0 from its minimal value 1 to a cutoff Λ

$$\mathcal{S}_{reg} = -\frac{L^2}{2\pi\alpha'} \int_1^{\Lambda} dx_0 \int_{-\sqrt{x_0^2-1}}^{\sqrt{x_0^2-1}} dx_1 \frac{1}{\sqrt{x_0^2 - x_1^2 - 1}} \,. \tag{7.44}$$

The first integral gives a factor of π and the second integration becomes trivial. Finally the action takes the simple form

$$\mathcal{S}_{reg} = -\frac{1}{2}\sqrt{\lambda}(\Lambda - 1) \,. \tag{7.45}$$

The expectation value for the hyperbolic using the physical cutoff is therefore given by

$$\langle W \rangle = e^{\frac{1}{2}\sqrt{\lambda}(\Lambda-1)} \,. \tag{7.46}$$

From (7.45) we realize that the action is diverging linearly in the cutoff Λ. Although the second way of regularizing the integral seems to be more natural from a physical point of view it is still hard to give a reasonable interpretation of the result. Comparing the cutoff used in the string theory to the cutoff used in the gauge theory we realize that in the latter the only variable to set a cutoff on is x_0. In the string theory we have to set a cutoff on both x_0 and x_1, which makes it hard to compare the result to the gauge theory.

Additionally, it would be nice to relate the result obtained for the hyperbolic line to the lightlike cusp studied before.

7.5 Circle in hyperbolic Space

In this section we want to find a minimal surface in AdS ending on a circle with radius $\rho = \sinh\theta_0$. In contrast to the latitude we will have to use a different ansatz for the S^5 part of the action. Again, we use the metric

$$ds^2 = \frac{L^2}{y^2}(dy^2 - dx_0^2 + dr^2 + r^2 d\phi^2) + L^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{7.47}$$

with polar coordinates (r, ϕ) in the (x_1, x_2) plane. In order to calculate the action in the conformal gauge we make the ansatz

$$y = y(\sigma), \qquad r = r(\sigma), \qquad \vartheta = \vartheta(\sigma), \qquad \phi = \varphi = \tau \tag{7.48}$$

leading to the Lagrangian

$$\mathcal{L} = \frac{L^2}{y^2}(y'^2 + r'^2 + r^2) + L^2(\vartheta'^2 + \sin^2\vartheta) \tag{7.49}$$

and the Virasoro constraint

$$\frac{1}{y^2}(y'^2 + r'^2) + \vartheta'^2 = \sin^2\vartheta + \frac{r^2}{y^2} \,. \tag{7.50}$$

The AdS part of the ansatz can be integrated easily as

$$y = \sinh\theta_0 \tanh\sigma, \qquad r = \frac{\sinh\theta_0}{\cosh\sigma}\,. \tag{7.51}$$

We are left with a differential equation for the S^5 part

$$\vartheta'^2 = \sin^2\vartheta\,. \tag{7.52}$$

Up to this point the calculation has been essentially the same as in the case of the latitude. To evaluate the S^5 part of the ansatz we take a look at the supersymmetry construction in the gauge theory.
The curve describing the coupling to the scalars is given by a circle in de Sitter space dS_2. To support this statement, we remember that the manifold dS_2 is defined by the constraint

$$-\,x_1^2 + x_2^2 + x_3^2 = 1 \tag{7.53}$$

and the curve we are interested in looks like

$$x_1 = \sinh\vartheta, \qquad x_2 = \cosh\vartheta\cos t, \qquad x_3 = \cosh\vartheta\sin t\,. \tag{7.54}$$

Here we interprete the analytic continuation performed in the gauge theory as a Wick rotation on the coordinate x_1.

We conclude that we have to substitute $\vartheta \to \left(\frac{\pi}{2} + i\vartheta\right)$ into (7.52) giving the following equation of motion

$$-\,\vartheta'^2 = \cosh^2\vartheta \tag{7.55}$$

that can be integrated as

$$\cosh^2\vartheta(\sigma) = \frac{1}{\cosh^2(\sigma_0 \pm \sigma)}\,. \tag{7.56}$$

After the analytic continuation we get the following boundary condition

$$\cosh\vartheta_0 = \cosh\theta_0 = \frac{1}{\cosh\sigma_0}\,, \tag{7.57}$$

which can be rewritten as

$$\tanh^2\sigma_0 = -\sinh^2\theta_0\,. \tag{7.58}$$

At this point are ready to evaluate the action, the boundary term is the same as in the case of the circular Wilson loop

$$\mathcal{S}_{reg} = \sqrt{\lambda}\int_0^\infty d\sigma\left(\frac{1}{\sinh^2\sigma\cosh^2\sigma} - \frac{1}{\sinh^2\sigma} + \frac{1}{\cosh^2(\sigma_0 \pm \sigma)}\right)\,. \tag{7.59}$$

Evaluating the integrals gives the result

$$\mathcal{S} = \frac{L^2}{2\pi\alpha'}(-1 + 1 \mp \tanh\sigma_0) = \mp\sqrt{\lambda}\,i\sinh\theta_0\,. \tag{7.60}$$

Finally, the expectation value from string theory is given by

$$\langle W\rangle = e^{\pm i\sqrt{\lambda}\sinh\theta_0}\,. \tag{7.61}$$

We realize that the AdS calculation leads to the same oscillating behaviour in the vacuum expectation value as the calculation on the gauge theory side extrapolated to strong coupling (5.57). We conclude that the hyperbolic circle is unstable, since it does not minimize the action.

7.6 Two Longitudes/hyperbolic Rays with a Cusp

The strategy for finding the minimal surfaces in AdS that correspond to two longitudes on S^2 and two intersecting rays on $\mathbb{H}_2$ is to first solve a system of two lines in the plane with a cusp at the origin. Afterwards we perform conformal transformations to S^2 and $\mathbb{H}_2$. The coupling to the scalars in the gauge theory is reflected by turning on one coordinate on S^5. When looking at the two longitudes and the two intersecting rays in the gauge theory we find both of them to have the same scalar couplings. Furthermore, we notice that the scalar couplings are not changed under the conformal transformation.

7.6.1 Nambu-Goto Action

We start with the metric of $AdS_3 \times S^1$. Here, we choose polar coordinates (r, ϕ) in the plane and we will also use them as worldsheet coordinates

$$ds^2 = \frac{L^2}{y^2}(dy^2 + dr^2 + r^2 d\phi^2) + L^2 d\varphi^2 \,. \tag{7.62}$$

Our ansatz for a system of two lines with a cusp in the plane looks like

$$y = rv(\phi), \qquad \varphi = \varphi(\phi) \,. \tag{7.63}$$

We realize that at $y = 0$ either r or v has to vanish, so the ansatz is quite natural for a system of two lines with a cusp in the origin. The Nambu-Goto action is given by

$$\mathcal{S}_{NG} = \frac{\sqrt{\lambda}}{2\pi} \int dr d\phi \sqrt{-(X' \cdot \dot{X})^2 + (\dot{X})^2 (X')^2} \,. \tag{7.64}$$

In our case prime represents differentiation with respect to ϕ, whereas a dot refers to differentiation with respect to r. Inserting the ansatz into the Nambu-Goto action we find

$$\mathcal{S}_{NG} = \frac{\sqrt{\lambda}}{2\pi} \int dr d\phi \frac{1}{rv^2} \sqrt{v'^2 + (1+v^2)(1+v^2\varphi'^2)} \,. \tag{7.65}$$

Since the Nambu-Goto action does not depend on ϕ explicitly we immediately find one conserved quantity, namely the energy E. Additionally φ is cyclic, therefore the canonical momentum conjugate to φ, which we will denote by J, will be conserved as well

$$E = \frac{1+v^2}{v^2\sqrt{v'^2 + (1+v^2)(1+v^2\varphi'^2)}}, \qquad J = \frac{(1+v^2)\varphi'}{\sqrt{v'^2 + (1+v^2)(1+v^2\varphi'^2)}} \,. \tag{7.66}$$

Using the Euler-Lagrange equations we can write down the equation of motion for v

$$0 = \left(\frac{v'}{v^2\sqrt{v'^2 + (1+v^2)(1+v^2\varphi'^2)}} \right)' - \frac{\partial}{\partial v}\left(\frac{1}{v^2}\sqrt{v'^2 + (1+v^2)(1+v^2\varphi'^2)} \right) \,. \tag{7.67}$$

To derive the BPS condition we have to consider the Legendre transformation of the original Lagrangian and add it to the action

$$\mathcal{S}_{LT} = \frac{\sqrt{\lambda}}{2\pi}\int drd\phi (yp_y)' = \frac{\sqrt{\lambda}}{2\pi}\int drd\phi \frac{-v'^2 - 2 - v^2(1+\varphi'^2)}{rv^2\sqrt{v'^2 + (1+v^2)(1+v^2\varphi'^2)}} \,. \tag{7.68}$$

In the last equality we made use of the equation of motion for v. Requiring that the total Lagrangian vanishes locally leads to the BPS condition

$$v^4\varphi'^2 - 1 = 0\,. \tag{7.69}$$

Note that the BPS condition can also be expressed in terms of the conserved quantities as $E^2 = J^2$. Now it is easy to find the equation of motion for v

$$v' = \frac{1+v^2}{v^2}\sqrt{p^2 - v^2}, \qquad p = \frac{1}{E}\,. \tag{7.70}$$

We can solve this equation with inverse trigonometric functions

$$\phi = \arcsin\frac{v}{p} - \frac{1}{\sqrt{1+p^2}}\arcsin\sqrt{\frac{1+1/p^2}{1+1/v^2}}\,. \tag{7.71}$$

Since the function arcsin is only defined for arguments less or equal to one, this expression is valid on half of the worldsheet, afterwards we have to make an analytic continuation

$$\phi = \pi - \arcsin\frac{v}{p} - \frac{1}{\sqrt{1+p^2}}\left(\pi - \arcsin\sqrt{\frac{1+1/p^2}{1+1/v^2}}\right)\,. \tag{7.72}$$

ϕ reaches its final value on the boundary again

$$\phi_f = \pi\left(1 - \frac{1}{\sqrt{1+p^2}}\right)\,. \tag{7.73}$$

The equation of motion for φ can be derived from the BPS condition

$$\varphi' = \pm\frac{1}{v^2} = \pm\frac{v'}{(1+v^2)\sqrt{p^2-v^2}} \tag{7.74}$$

and the solution is given by

$$\varphi = \frac{1}{\sqrt{1+p^2}}\arcsin\sqrt{\frac{1+1/p^2}{1+1/v^2}}\,. \tag{7.75}$$

Again this equation is only valid on half of the worldsheet. After performing the same analytic continuation the final value for φ is given by

$$\varphi_f = \frac{\pi}{\sqrt{1+p^2}}\,. \tag{7.76}$$

When we take the positive sign in (7.74) the relation $\phi_f + \varphi_f = \pi$ is satisfied. Now let us write down the induced metric on the minimal surface

$$g_{rr} = \frac{1+v^2}{r^2 v^2}, \qquad g_{rv} = \frac{1}{rv}, \qquad g_{vv} = \frac{p^2(1+v^2) - v^4}{v^2(v^2-p^2)(1+v^2)}\,. \tag{7.77}$$

Inserting the solutions to the equations of motion into the Nambu-Goto action yields

$$\mathcal{S}_{NG} = \frac{\sqrt{\lambda}}{2\pi}\int dr d\phi \frac{p}{r}\frac{(v^2+1)}{v^4} = \frac{\sqrt{\lambda}}{2\pi}\int dr dv \frac{p}{r}\frac{1}{v^2\sqrt{p^2-v^2}}\,. \tag{7.78}$$

Using the dictionary of the AdS/CFT correspondence a conformal transformation in the gauge theory is reflected by an isometry in AdS_5. Since we want to stereographically project the cusp solution from the plane to S^2 and $\mathbb{H}_2$, we have to find the appropriate isometries in AdS that reduce to the stereographic projections on the boundary.

7.6.1.1 Stereographic Projection to S^2

To map the solution to S^2 we use the following isometry

$$\begin{aligned} \tilde{x}_1 &= \frac{2r}{1+r^2+y^2}\cos\phi, & \tilde{x}_2 &= \frac{2r}{1+r^2+y^2}\sin\phi, \\ \tilde{x}_3 &= \frac{-1+r^2+y^2}{1+r^2+y^2}, & \tilde{y} &= \frac{2y}{1+r^2+y^2}\,. \end{aligned} \tag{7.79}$$

Note that

$$\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2 + \tilde{y}^2 = 1\,. \tag{7.80}$$

Let us take a look at the surface in global AdS, which we parametrize by

$$\begin{aligned} Y_0 &= \cosh\rho, & Y_1 &= \sinh\rho\sin\theta\cos\phi, \\ Y_2 &= \sinh\rho\sin\theta\sin\phi, & Y_3 &= \sinh\rho\cos\theta\,. \end{aligned} \tag{7.81}$$

Choosing this parametrization the metric on $AdS_4 \times S^1$ is given by

$$ds^2 = L^2(d\rho^2 + \sinh^2\rho(d\theta^2 + \sin\theta^2 d\phi^2) + d\varphi^2)\,. \tag{7.82}$$

In the global coordiantes the isometry (7.79) then looks like

$$\begin{aligned} Y_1 &= \frac{r}{y}\cos\phi, & Y_2 &= \frac{r}{y}\sin\phi, \\ Y_3 &= \frac{-1+r^2+y^2}{2y}, & Y_{-1} &= \frac{1+r^2+y^2}{2y}\,. \end{aligned} \tag{7.83}$$

We realize that the minimal surface in global AdS is given by

$$\cosh\rho = \frac{1+y^2+r^2}{2y}, \qquad \sinh\rho\sin\theta = \frac{r}{y}\,. \tag{7.84}$$

The coordinates ϕ and φ are mapped to themselves, whereas the relation between (r, v) and (ρ, θ) is then given by

$$\begin{aligned} r &= \frac{\sin\theta\sinh\rho}{1+\sin^2\theta\sinh^2\rho}\left(\cosh\rho \pm \sqrt{\cosh^2\rho - 1 - \sin^2\theta\sinh^2\rho}\right), \\ v &= \frac{1}{\sinh\rho\sin\theta}. \end{aligned} \tag{7.85}$$

The plus sign corrpesponds to the two longitudes on S^2, whereas the minus sign does not have any physical relevance. After the conformal transformation to S^2 the Nambu-Goto action for the two longitudes acquires the following form (the Jacobian is rather lengthy, so we only give the final result)

$$\mathcal{S}_{NG} = \frac{\sqrt{\lambda}}{2\pi}\int d\rho\, d\theta \frac{p\sinh^2\rho\sin\theta}{\sqrt{p^2\sinh^2\rho\sin^2\theta - 1}}. \tag{7.86}$$

First of all we want to perform the integral over θ. If we take ρ to be fixed then the domain of integration for θ is determined by the roots of $p^2\sin^2\theta\sinh^2\rho = 1$. Afterwards we are left with the integral over ρ, which we perform by integrating from the minimal value $\sinh\rho = 1/p$ to a cutoff ρ_0

$$\mathcal{S}_{NG} = \sqrt{\lambda}\int d\rho\sinh\rho = \sqrt{\lambda}\left(\cosh\rho_0 - \sqrt{1+\frac{1}{p^2}}\right). \tag{7.87}$$

In this case regulating the action is very subtle. We cannot simply neglect the divergent part. To find the appropriate boundary term, we have to consider the Legendre transform of the original Lagrangian with respect to the coordinates orthogonal to the boundary. In global AdS it is given by

$$\mathcal{L}_{Boundary} = -\coth\rho_0 p_\rho = -\coth\rho_0\rho'\frac{\delta\mathcal{L}_{NG}}{\delta\rho'}. \tag{7.88}$$

In the last equality ρ' is the derivative of ρ with respect to the worldsheet coordinate orthogonal to the boundary. We notice that $\coth\rho_0$ can be set equal to one for large values of ρ_0. To explicitely calculate the boundary term one has to introduce ρ' into the Nambu-Goto action (7.86), leading to

$$\begin{aligned} S_{Boundary} = -\frac{\sqrt{\lambda}}{2\pi}\int d\theta \frac{\sqrt{p^2\sinh^2\rho_0\sin^2\theta - 1}}{p\sinh^2\rho_0\sin\theta} \\ \times\left[\sinh^2\rho_0(1+\sin^2\theta(\partial_\theta\phi)^2 + (\partial_\theta\varphi)^2)\right]. \end{aligned} \tag{7.89}$$

Using the solutions to the equations of motion for ϕ and φ we evaluate the boundary term as

$$S_{Boundary} = -\frac{\sqrt{\lambda}}{2\pi}\int d\theta\sin\theta\frac{p^2\sinh^2\rho_0(\sinh^2\rho_0\sin^2\theta + 1) - \cosh^2\rho_0}{p(\sinh^2\rho_0\sin^2\theta+1)\sqrt{p^2\sinh^2\rho_0\sin^2\theta - 1}}. \tag{7.90}$$

We realize that the first part of the boundary term gives $2\pi \sinh \rho_0$ by the same arguments as in the case of the unregularized action, whereas the second term can be integrated to

$$S_{Boundary} \simeq -\sqrt{\lambda}\left(\sinh \rho_0 - \frac{\coth \rho_0}{p\sqrt{1+p^2}}\right). \tag{7.91}$$

Again, for large ρ_0 we can set $\coth \rho_0$ equal to one. We realize that the divergent parts cancel each other, so we are left with the finite part of the action

$$\mathcal{S} = -\frac{p}{\sqrt{1+p^2}}\sqrt{\lambda} = -\frac{\sqrt{\phi_f(2\pi - \phi_f)}}{\pi}\sqrt{\lambda}\,. \tag{7.92}$$

To express p in terms of ϕ we used (7.73). Finally, the expectation value for the two longitudes from string theory is given by

$$\langle W \rangle = e^{\frac{\sqrt{\lambda}\sqrt{\phi_f(2\pi-\phi_f)}}{\pi}}\,. \tag{7.93}$$

We note that for $\Phi_f = \pi$ the expectation value for the two longitudes reduces to the expectation value of the circular loop. This is in perfect agreement with the gauge theory: In the case of the angle between the two halfcircles beeing equal to π the cusp disappears and we regain the large circle.

7.6.1.2 Stereographic Projection to $\mathbb{H}_2$

To transform the solution to $\mathbb{H}_2$ we write down the isometry in AdS again expressed in the Poincaré patch

$$\tilde{x}_0 = \frac{1+r^2+y^2}{1-r^2-y^2}, \qquad \tilde{x}_1 = \frac{2r}{1-r^2-y^2}\cos\phi, \tag{7.94}$$
$$\tilde{x}_2 = \frac{2r}{1-r^2-y^2}\sin\phi, \qquad \tilde{y} = \frac{2y}{1-r^2-y^2}$$

with

$$-\tilde{x}_0^2 + \tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{y}^2 = -1\,. \tag{7.95}$$

Consequently the surface which ends on two rays with a cusp in the Poincaré patch is given by

$$r = \frac{-1 \pm \sqrt{1+\tilde{r}^2+\tilde{r}^2 v^2}}{\tilde{r}+\tilde{r}v^2}, \qquad v = \tilde{v}\,. \tag{7.96}$$

Inserting into the Nambu-Goto action (7.78) yields (we will drop the ˜ superscript)

$$\mathcal{S}_{NG} = \frac{\sqrt{\lambda}}{2\pi}\int dr dv \frac{1}{r\sqrt{1+r^2(1+v^2)}}\frac{p}{v^2\sqrt{p^2-v^2}}\,. \tag{7.97}$$

Evaluating the Nambu-Goto action leads to a complicated expression in terms of elliptic integrals. Therefore it seems likely that we should change to a different set of coordinates more suited for the calculation.

7.6.2 Calculation in conformal Gauge

The basic idea for the calculation in conformal gauge is the same as before. First we consider a cusp in the plane and afterwards use stereographic projections to S^2 and $\mathbb{H}_2$. To derive the solution in the plane we start with an $AdS_3 \times S^1$ subspace of $AdS_5 \times S^5$ and the following parametrization

$$y = \rho \sin \nu , \qquad x_1 = \rho \cos \nu \cos \phi , \qquad x_2 = \rho \cos \nu \sin \phi \,. \tag{7.98}$$

In this parametrization the metric reads

$$ds^2 = L^2 \left(\frac{d\rho^2}{\rho^2 \sin^2 \nu} + \frac{d\nu^2}{\sin^2 \nu} + \cot^2 \nu \, d\phi^2 + d\varphi^2 \right) \tag{7.99}$$

and we make the following ansatz

$$\rho = \rho(\tau), \qquad \nu = \nu(\sigma), \qquad \phi = \phi(\sigma), \qquad \varphi = \varphi(\sigma) \,. \tag{7.100}$$

The Lagrangian in conformal gauge is then given by

$$\mathcal{L}_{Pol} = L^2 \left(\frac{\dot\rho^2}{\rho^2 \sin^2 \nu} + \frac{\nu'^2}{\sin^2 \nu} + \cot^2 \nu \phi'^2 + \varphi'^2 \right) , \tag{7.101}$$

and the Virasoro constraint reads

$$\frac{\dot\rho^2}{\rho^2} = \nu'^2 + \cos^2 \nu \phi'^2 + \sin^2 \nu \varphi'^2 \,. \tag{7.102}$$

By setting $\rho = e^{a\tau}$ the Virasoro constraint simplifies

$$a^2 = \nu'^2 + \cos^2 \nu \phi'^2 + \sin^2 \nu \varphi'^2 \,. \tag{7.103}$$

Like before we find two conserved quantities, since ϕ and φ are cyclic. In the BPS case the conserved charges should be equal, which enables us to reduce the Virasoro constraint down to

$$a^2 = \nu'^2 + \frac{p^2}{\tan^2 \nu} \,. \tag{7.104}$$

For $p^2 < 1$ we can set $a^2 = 1 - p^2$ and get

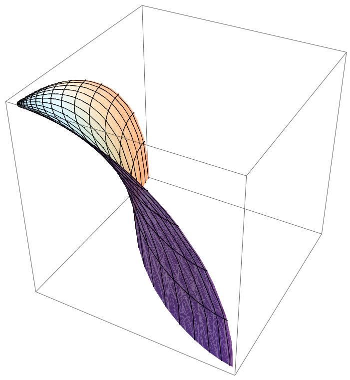
$$\nu'^2 = 1 - \frac{p^2}{\cos^2 \nu} \,. \tag{7.105}$$

From this equation we realize that ν starts at the boundary of AdS and reaches a maximal value when $\cos \nu_0 = p$, since then the right hand side of (7.105) vanishes. It is easy to integrate (7.105)

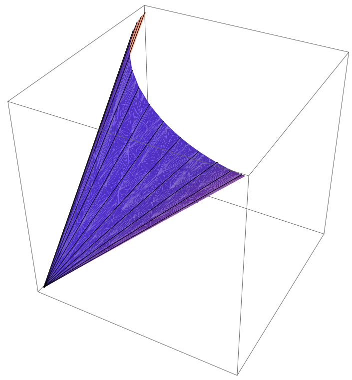
$$\sin \nu = \sqrt{1 - p^2} \sin \sigma = a \sin \sigma \,. \tag{7.106}$$

The equation of motion for ϕ can be derived from the original Lagrangian (7.101)

$$\phi' = p \tan^2 \nu = p \frac{(1 - p^2) \sin^2 \sigma}{\cos^2 \sigma + p^2 \sin^2 \sigma} \tag{7.107}$$



(c) Stereographic projection to S^2



(d) Stereographic projection to $\mathbb{H}_2$

Figure 7.2: Minimal surfaces belonging to two longitudes/intersecting rays

and can be integrated as

$$\tan(\phi + p\sigma) = p \tan\sigma \,. \tag{7.108}$$

The equation of motion for φ is simply $\varphi' = p$ and is solved by $\varphi = p\sigma$. Just for completeness let us write down the classical action

$$\mathcal{S} = \frac{L^2}{4\pi\alpha'} \int d\sigma d\tau \frac{2}{\sin^2\sigma} \,. \tag{7.109}$$

Having arrived at this point we are ready to carry out stereographic projections to S^2 and $\mathbb{H}_2$ respectively to find the desired minimal surfaces.
Projecting the solution to S^2 we find

$$\begin{aligned} x_1 &= \frac{p\sin\sigma\sin p\sigma + \cos\sigma\cos p\sigma}{\cosh a\tau}, \\ x_2 &= \frac{p\sin\sigma\cos p\sigma - \cos\sigma\cos p\sigma}{\cosh a\tau}, \\ x_3 &= \tanh a\tau, \end{aligned} \tag{7.110}$$

whereas on $\mathbb{H}_2$ the minimal surface looks like

$$\begin{aligned} x_0 &= -\coth a\tau, \\ x_1 &= -\frac{p\sin\sigma\sin p\sigma + \cos\sigma\cos p\sigma}{\sinh a\tau}, \\ x_2 &= -\frac{p\sin\sigma\cos p\sigma - \cos\sigma\cos p\sigma}{\sinh a\tau}\,. \end{aligned} \tag{7.111}$$

For an explicit from of the stereographic projections the reader might take a look at the appendix.

7.6.3 Non-supersymmetric Case

We shortly consider the non-supersymmetric case for two lines in the plane with a cusp at the origin. Here, the conserved charges are not equal but will be proportional to each other. In the following we will denote the ratio of the conserved charges by q

$$v^2\varphi' = q\,. \tag{7.112}$$

Again we get a differential equation for v, but this time it is more complicated

$$v'^2 = \frac{1+v^2}{v^4}(p^2 + (p^2 - q^2)v^2 - v^4), \qquad p = \frac{1}{E}\,. \tag{7.113}$$

To bring the above equation into the standard form of an elliptic equation we define the quantities

$$\begin{aligned} z &= \sqrt{\frac{v^2(1+b^2)}{b^2(1+v^2)}}, \qquad b^2 = \frac{1}{2}\left(p^2 - q^2 + \sqrt{(p^2-q^2)^2 + 4p^2}\right), \\ k^2 &= \frac{b^2(p^2-b^2)}{p^2(1+b^2)}\,. \end{aligned} \tag{7.114}$$

Note that b^2 solves the following equation

$$-b^4 + b^2(p^2 - q^2) + p^2 = 0\,. \tag{7.115}$$

We are now ready to write down a differential equation for z

$$z'^2 = \frac{p^2(1+b^2)}{b^2}\left(1 - \frac{1+b^2}{b^2z^2}\right)^2 (1-z^2)(1-k^2z^2)\,. \tag{7.116}$$

The equation can be solved via elliptic integrals

$$\phi = \frac{b}{p\sqrt{1+b^2}}\left[F(\arcsin z; k) - \Pi\left(\frac{b^2}{1+b^2}, \arcsin z; k\right)\right]\,. \tag{7.117}$$

Here, F is an elliptic integral of the first kind, whereas Π is an elliptic integral of the third kind. At the boundary we have $v = 0$ which also means $z = 0$. As in the supersymmetric case after reaching a maximal value of $z = 1$ another copy of the surface continues with

$$\begin{aligned} \phi = \frac{b}{p\sqrt{1+b^2}}\Bigg[&2K(k) - 2\Pi\left(\frac{b^2}{1+b^2}; k\right) \\ &- F(\arcsin z; k) + \Pi\left(\frac{b^2}{1+b^2}, \arcsin z; k\right)\Bigg]\,. \end{aligned} \tag{7.118}$$

ϕ takes its final value when we reach the boundary again and can then be expressed by complete elliptic integrals

$$\phi_f = \frac{2b}{p\sqrt{1+b^2}}\left[K(k) - \Pi\left(\frac{b^2}{1+b^2}; k\right)\right]\,. \tag{7.119}$$

In the last formula K denotes the complete elliptic integral of the first kind, whereas Π is the complete elliptic integral of the third kind. The equation of motion for φ can be derived from (7.112) and gives

$$\varphi'^2 = \frac{q\,b}{p\sqrt{1+b^2}}\frac{1}{(1-z^2)(1-k^2z^2)}\,. \tag{7.120}$$

It can again be integrated via elliptic integrals

$$\varphi = \frac{q\,b}{p\sqrt{1+b^2}} F(\arcsin z; k)\,. \tag{7.121}$$

The final value of φ is then given by a complete elliptic integral again

$$\varphi_f = 2\frac{q\,b}{p\sqrt{1+b^2}} K(k)\,. \tag{7.122}$$

Evaluating the classical action is rather technical due to the presence of the elliptic integrals [8]. We are only giving the final result

$$\mathcal{S}_{NG} = \frac{\sqrt{\lambda}}{2\pi} \int \frac{2\sqrt{1+b^2}}{b\,r\,z_0}\,. \tag{7.123}$$

In the last equation z_0 is a cutoff for small values of z. This is the standard divergence when considering a cusp in the plane.

7.7 Moved hyperbolic Line

To find the string theory dual of the moved hyperbolic line, which has been studied in the gauge theory in section (5.2.7), we take the following metric

$$ds^2 = \frac{L^2}{y^2}\left(dy^2 - du^2 + u^2 dt^2 + dx^3\right) + L^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\,. \tag{7.124}$$

The string should then end along the curve $u = \cosh\kappa_0$ and constant $x_3 = \sinh\kappa_0$. To find the string solution we take the following ansatz

$$y = y(\sigma), \qquad u = u(\sigma), \qquad \vartheta = \vartheta(\sigma), \qquad t = \tau \tag{7.125}$$

so that the Lagrangian in conformal gauge is given by

$$\mathcal{L} = \frac{L^2}{y^2}(y'^2 - u'^2 + u^2) + L^2(\vartheta'^2 + \sin^2\vartheta) \tag{7.126}$$

and the Virasoro constraint takes the following form

$$\frac{1}{y^2}(y'^2 - u'^2) + \vartheta'^2 = \sin^2\vartheta + \frac{u^2}{y^2}\,. \tag{7.127}$$

First of all, let us only consider the AdS part. Taking care of the boundary conditions we can integrate the equations of motion for u and y as

$$y = \cosh\kappa_0 \tan\sigma, \qquad u = \frac{\cosh\kappa_0}{\cos\sigma}, \qquad 0 \le \sigma \le \frac{\pi}{2}\,. \tag{7.128}$$

As in the case of the 1/2 BPS hyperbolic line we express the action in terms of the original coordinates

$$\cosh t = \frac{x_0}{\sqrt{x_0^2 - x_1^2}}, \qquad \cos\sigma = \frac{\cosh\kappa_0}{\sqrt{x_0^2 - x_1^2}} \tag{7.129}$$

and changing from (σ, τ) to (x_0, x_1) gives the following Jacobian

$$d\sigma d\tau = \frac{\cosh \kappa_0}{(x_0^2 - x_1^2)\sqrt{x_0^2 - x_1^2 - \cosh^2 \kappa_0}} \,. \tag{7.130}$$

After subtracting the boundary divergence the action in terms of (σ, τ) reads

$$\mathcal{S}_{AdS} = -\frac{L^2}{2\pi\alpha'} \int d\sigma d\tau \frac{1}{\cos^2 \sigma} \tag{7.131}$$

or equivalently in terms of (x_0, x_1)

$$\mathcal{S}_{AdS} = -\frac{L^2}{2\pi\alpha'} \int dx_0 dx_1 \frac{1}{\cosh \kappa_0 \sqrt{x_0^2 - x_1^2 - \cosh^2 \kappa_0}} \,. \tag{7.132}$$

To determine the integration limits we note that for fixed x_0 the variable x_1 varies between the two roots of $x_0^2 - \cosh^2 \kappa_0 = x_1^2$

$$\mathcal{S}_{AdS} = -\frac{L^2}{2\pi\alpha'} \int_{\cosh \kappa_0}^{\Lambda \cosh \kappa_0} dx_0 \int_{-\sqrt{x_0^2 - \cosh^2 \kappa_0}}^{\sqrt{x_0^2 - \cosh^2 \kappa_0}} dx_1 \times \frac{1}{\cosh \kappa_0 \sqrt{x_0^2 - x_1^2 - \cosh^2 \kappa}} \,. \tag{7.133}$$

The first integral gives a factor of π and the second integration becomes trivial

$$\mathcal{S}_{AdS} = -\frac{\sqrt{\lambda}}{2}(\Lambda - 1) \,. \tag{7.134}$$

Now let us turn to the S^5 part of the action. In the gauge theory the coupling to the scalars is decribed by a line

$$x_1 = \sinh\vartheta \cosh t, \qquad x_2 = \sinh\vartheta \sinh t, \qquad x_3 = \cosh\vartheta \tag{7.135}$$

which is constrained to de Sitter space dS_2

$$-x_1^2 + x_2^2 + x_3^2 = 1 \,. \tag{7.136}$$

We already observed in the case of the hyperbolic circle that the curve describing the coupling to the scalars is inside de Sitter space. By a change of coordinates it is possible to map the curve associated to the hyperbolic circle, denoted by the coordinates $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, to the line (7.135) we are studying at the moment. Written explicitly, the transformation takes the following form

$$\sqrt{1 + \tilde{x}_1^2} = x_3, \qquad \arctan \frac{\tilde{x}_3}{\tilde{x}_2} = \operatorname{arctanh} \frac{x_2}{x_1} \,. \tag{7.137}$$

A review of different coordinate systems in de Sitter space is given by [52]. Apart from the change of coordinates we could have also performed the double analytic continuation $\vartheta \to i\vartheta$ and $\varphi \to i\tau$ in the original S^5 coordinates.

With the help of the Virasoro constraint we derive the following equation of motion for ϑ

$$\vartheta'^2 = -\sinh^2\vartheta\,, \tag{7.138}$$

which can be integrated as

$$-\sinh^2\vartheta(\sigma) = \frac{1}{\sin^2(\sigma_0 \pm \sigma)}\,. \tag{7.139}$$

The integration constant σ_0 is fixed by the boundary condition

$$\sinh\vartheta_0 = \sinh\kappa_0\,. \tag{7.140}$$

To regularize the S^5 part of the action we can use the same methods that we used to regularize the AdS part of the action. Rewriting the S^5 part

$$\mathcal{S}_{S^5} = \frac{L^2}{2\pi\alpha'}\int d\sigma d\tau \frac{1}{\sin^2(\sigma_0 \pm \sigma)} \tag{7.141}$$

in terms of physical coordinates (x_1, x_2) leads to a complicated expression which cannot be integrated easily. Therefore it seems likely that we have to use a different regularization scheme.

We want to finish this chapter by making some general observations. First of all, we observe that calculating the classical action and in addition its regularization for the various hyberbolic lines is hard to interprete from a physical perspective. By considering the hyperbolic circle and the moved hyperbolic line, we found out that the S^5 part of the action has to be analytically continued in such a way that we are dealing with de Sitter space dS_2. It seems likely that this fact applies also to general curves.

VIII
Conclusions and Outlook

8.1 Summary

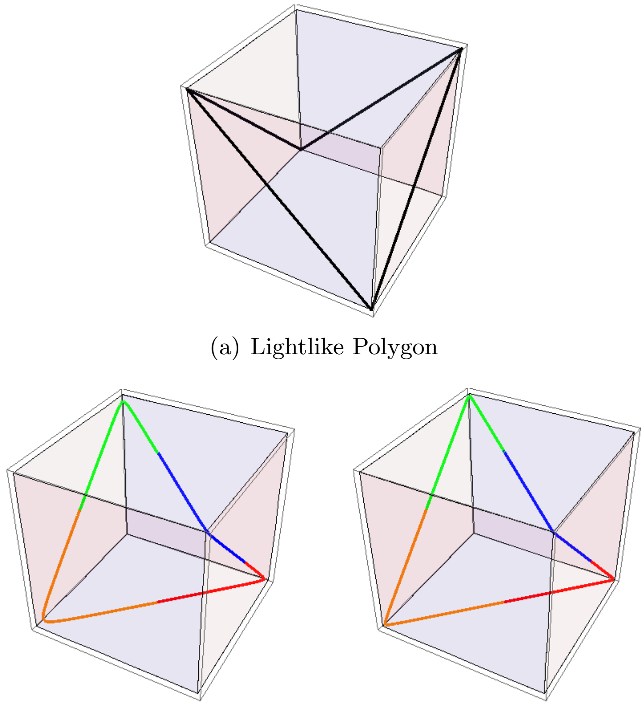
In this thesis we considered several classes of supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM. For some special loops we could give explicit results in the string as well as the gauge theory in the context of the AdS/CFT duality. After reviewing loops which are restricted to flat space and to S^3 we first presented a new class of supersymmetric Wilson loops. In this class the Wilson loops are restricted to hyperbolic three space $\mathbb{H}_3$. We found out that a general curve preserves 1/16 of the original supersymmetries. In addition we put forward some special loops which preserve 1/8, 1/4 and 1/2 of the original supersymmetries. We investigated the perturbative behaviour of the hyperbolic line by a one loop calculation. Afterwards we examined Wilson loops in Minkowskian signature and found some special examples in flat Minkowski space and in the light cone as well. In the last chapter we took a look at the dual Wilson loops in string theory and calculated their expectation value.

8.2 Outlook

There are still a lot of open issues concerning this new class of Wilson loops on $\mathbb{H}_3$. First of all, it would be nice to have a general construction in the string theory for the loops which are restricted to $\mathbb{H}_2$ in the gauge theory. In addition, it would be desireable to derive a general formula for the pertubative expansion in the gauge theory.

Furthermore, it is necessary to better understand the results obtained for the hyperbolic line and its relation to cusped Wilson loops. In particular the different regularizations of the minimal surface in AdS should be further investigated.

Nevertheless, the hyperbolic line may be useful in the following way: If we consider the scattering of two particles into two particles, then according to the Alday/Maldacena prescription we have to calculate a minimal surface in AdS ending on a light-like polygon. It is still an open question how the classical action for the polygon has to be regularized. We want to sketch an approach different from the ones that have been considered up to now. The idea is to smooth out the cusps in the polygon with the help of hyperbolic lines (with curvature radius R). Thereafter we can study the limit $R \to 0$. In the plot one can clearly see that by decreasing the curvature radius R we regain the lightlike polygon. Since we have a description of the hyperbolic line in AdS it maybe useful as a tool to regularize the action.



(a) Lightlike Polygon

(b) R=1/20

(c) R=1/100

Figure 8.1: Lightlike Polygon as a limit of hyperbolic lines

Finally we want to emphasize that all Wilson loops for which we have a description in gauge and string theory can be regarded as a further test of the AdS/CFT duality.

Appendix A
Clifford Algebras

A.1 Pauli Matrices

The generators of $SU(2)$ are the three Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.1}$$

The satisfy the following properties

$$\tau_i \tau_j = \delta_{ij} + i\epsilon_{ijk}\tau_k, \qquad [\tau_i, \tau_j] = 2i\epsilon_{ijk}\tau_k, \qquad \{\tau_i, \tau_j\} = 2\delta_{ij} . \tag{A.2}$$

A.2 Clifford Algebra belonging to $SO(4)$

Euclidian gamma matrices are defined by (with $j = 1, 2, 3$)

$$\gamma_j = \begin{pmatrix} 0 & -i\tau_j \\ i\tau_j & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{A.3}$$

They satisfy the Clifford Algebra in flat space

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} . \tag{A.4}$$

In Euclidian signature the chirality matrix γ^5 is defined as $\gamma^5 = -\gamma_1\gamma_2\gamma_3\gamma_4$. γ^5 satisfies the properties

$$\{\gamma^5, \gamma_\mu\} = 0, \qquad (\gamma^5)^2 = \mathbb{1} . \tag{A.5}$$

A.3 Clifford Algebra belonging to $SO(1, 3)$

The gamma matrices belonging to $SO(1, 3)$ are given by (with $j = 1, 2, 3$)

$$\gamma_0 = -i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma_j = \begin{pmatrix} 0 & -i\tau_j \\ i\tau_j & 0 \end{pmatrix} . \tag{A.6}$$

They satisfy the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{A.7}$$

with $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$. The chirality matrix γ^5 is defined as $\gamma^5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$. Again γ^5 satisfies the properties

$$\{\gamma^5, \gamma_\mu\} = 0, \qquad (\gamma^5)^2 = \mathbb{1} . \tag{A.8}$$

A.4 Clifford Algebra belonging to $SO(6)$

The matrices $\rho^I, I = 1, \ldots, 6$ belonging to the Clifford algebra of $SO(6)$ obey the commutation relations

$$\{\rho^I, \rho^J\} = 2\delta^{IJ}\,. \tag{A.9}$$

A.5 Conventions

In the text we use the following conventions

$$\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \qquad \rho^{IJ} = \frac{1}{2}(\rho^I \rho^J - \rho^J \rho^I)\,. \tag{A.10}$$

Appendix B
Stereographic Projections

A stereographic projection is a conformal map, therefore it preserves angles.

B.1 Stereographic Projection between S^2 and $\mathbb{R}^2$

Let (X, Y) denote coordinates on the plane whereas (x_1, x_2, x_3) are coordinates on S^2 given by the constraint $x_1^2 + x_2^2 + x_3^2 = 1$. The stereographic projection from S^2 to $\mathbb{R}^2$ is given by

$$(X, Y) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right) . \tag{B.1}$$

The inverse map is given by

$$(x_1, x_2, x_3) = \left(\frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{-1 + X^2 + Y^2}{1 + X^2 + Y^2} \right) . \tag{B.2}$$

B.2 Stereographic Projection between $\mathbb{H}_2$ and the unit Disc

Here (X, Y) are coordinates on the unit disc whereas (x_0, x_1, x_2) are coordinates on $\mathbb{H}_2$ satisfying $-x_0^2 + x_1^2 + x_2^2 = -1$. The stereographic projection from $\mathbb{H}_2$ onto the unit disc is given by

$$(X, Y) = \left(\frac{x_1}{1 + x_0}, \frac{x_2}{1 + x_0} \right) . \tag{B.3}$$

The inverse transformation reads

$$(x_0, x_1, x_2) = \left(\frac{1 + X^2 + Y^2}{1 - X^2 - Y^2}, \frac{2X}{1 - X^2 - Y^2}, \frac{2Y}{1 - X^2 - Y^2} \right) . \tag{B.4}$$

B.3 Half angle Formulas

In the context of the sterographic projections we often make use of the formulas

$$\frac{\sin x}{1 + \cos x} = \tan \frac{x}{2}, \qquad \frac{\sinh x}{1 + \cosh x} = \tanh \frac{x}{2} . \tag{B.5}$$

Appendix C

Aspects of $SU(N)$ Gauge Theories

We want to shortly summarize the aspects of $SU(N)$ gauge theories and Feynman integrals/parameters that are needed throughout the thesis.

C.1 Propagators

First of all let us write down the needed propagators in Feynman gauge in four dimensions

$$\left\langle A_\mu^a(x) A_\nu^b(y) \right\rangle = \frac{g^2 \delta^{ab} \eta_{\mu\nu}}{4\pi^2 (x-y)^2}, \qquad \left\langle \Phi_I^a(x) \Phi_J^b(y) \right\rangle = \frac{g^2 \delta_{IJ} \delta^{ab}}{4\pi^2 (x-y)^2}. \tag{C.1}$$

In 2ω dimensions the propagator is modified in the following way

$$\Delta(x) = \frac{\Gamma(\omega-1)}{4\pi^\omega} \frac{1}{(x^2)^{\omega-1}}. \tag{C.2}$$

The generators of the gauge group $SU(N)$ are denoted by T^a; there defining property is the so called structure equation,

$$[T^a, T^b] = i f^{abc} T_c \tag{C.3}$$

here f^{abc} is the structure constant of $SU(N)$. The generators are normalized in such a way that $T^a T^b = \delta^{ab} N^2/2$.

C.2 Feynman Parameters

The general Feynman parameter formula is given by

$$\prod_{i=1} A_i^{-n_i} = \frac{\Gamma(\Sigma n_i)}{\Pi_i(\Gamma(n_i))} \int_0^1 dx_1 \dots dx_k \,.x_1^{n_1-1} \dots x_k^{n_k-1} \frac{\delta(1-\Sigma_i x_i)}{(\Sigma_i A_i x_i)^{\Sigma n_i}}. \tag{C.4}$$

C.3 Gamma Function Identities

The Euler Beta function and the gamma function are defined by

$$\Gamma(n+1) = \int_0^\infty dt\; t^n e^{-t}, \qquad \beta(\mu,\nu) = \int_0^1 x^{\mu-1}(1-x)^{\nu-1} dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}. \tag{C.5}$$

The gamma functions satisfies the following properties [53]

$$\Gamma(n+1) = n!, \qquad \Gamma\left(\frac{2n+1}{2}\right) = (n-1/2)\,(n-3/2)\dots(1/2)\,\sqrt{\pi},$$
$$\Gamma(n)\Gamma(1/2) = 2^{n-1}\Gamma\,(n/2)\,\Gamma\left(\frac{n+1}{2}\right). \tag{C.6}$$

C.4 Feynman Integrals in 2ω dimensions

Feynman integrals in 2ω dimensions can be found in standard quantum field theory books, for example [16]:

$$\int d^{2\omega}k(k^2+2p\cdot k+m^2)^{-s} = \pi^\omega \frac{\Gamma(s-\omega)}{\Gamma(s)}(m^2-p^2)^{\omega-s} \tag{C.7}$$

$$\int d^{2\omega}k k_\mu(k^2+2p\cdot k+m^2)^{-s} = -p_\mu \pi^\omega \frac{\Gamma(s-\omega)}{\Gamma(s)}(m^2-p^2)^{\omega-s}$$

$$\int d^{2\omega}k k_\mu k_\nu(k^2+2p\cdot k+m^2)^{-s} = \pi^\omega \frac{1}{\Gamma(s)}(m^2-p^2)^{\omega-s}$$

$$\times\left[p_\mu p_\nu \Gamma(s-\omega) - \frac{1}{2}\eta_{\mu\nu}\Gamma(s-\omega-1)(p^2+m^2)\right]$$

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