On AdS/CFT Correspondence beyond SUGRA: plane waves, free CFTs and double-trace deformations

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Abstract

This thesis deals with three corners of the AdS/CFT Correspondence that lie one step beyond the classical supergravity (SUGRA) approximation.

We first explore the BMN limit of the duality and study, in particular, the behavior of field theoretic propagators in the corresponding Penrose limit. We unravel the semiclassical (WKB-) exactness of the propagators in the resulting plane wave background metric.

Then, we address the limit of vanishing coupling of the conformal field theory (CFT) at large N. In the simplified scenario of Higher Spin/O(N) Vector Model duality, the conformal partial wave (CPW) expansion of scalar four-point functions are reorganized to make them suggestive of a bulk interpretation in term of a consistent truncated massless higher spin theory and their corresponding exchange Witten graphs. We also explore the connection to the interacting O(N) Vector Model at its infra-red fixed point, at leading large N.

Finally, coming back to the gauge theory, we study the effect of a relevant double-trace deformations of the boundary CFT on the partition function and its dual bulk interpretation. We show how the one-loop computation in the Anti-de Sitter (AdS) space correctly reproduces the partition function and conformal anomaly of the boundary theory. In all, we get a clean test of the duality beyond the classical SUGRA approximation in the AdS bulk and at the corresponding next-to-leading $1/N$ order of the CFT at the conformal boundary.

Keywords:
AdS/CFT Correpondence, Penrose limit, CPW, conformal anomaly

Zusammenfassung

Diese Arbeit beschäftigt sich mit drei Aspekten der AdS/CFT-Korrespondenz, die alle einen Schritt über die klassische SUGRA-Näherung hinausgehen.

Zuerst diskutieren wir den BMN Grenzfall der Korrespondenz und untersuchen insbesondere das Verhalten der quantenfeldtheoretischen Propagatoren. Dabei weisen wir nach, dass die Propagatoren im für den BMN Fall relevanten Hintergrund ebener Wellen semiklassisch (WKB) exakt beschrieben werden.

Danach wird im Rahmen der AdS/CFT-Korrespondenz der Grenzfall verschwindender Kopplung der konformen Feldtheorie betrachtet. Zur technischen Vereinfachung geschieht dies für das Beispiel des $O(N)$-Vektormodells. Dabei wird die OPE der Vierpunktfunktionen so umgeschrieben, dass sie strukturelle Ähnlichkeit mit Witten-Diagrammen einer korrespondierenden Theorie von Strömen mit höherem Spin hat. Außerdem wird das $O(N)$-Vektormodell bei großem N am wechselwirkenden Infrarot-Fixpunkt untersucht.

Im letzten Punkt wenden wir uns schließlich der ursprünglichen AdS/CFT-Dualität unter Mitnahme der nächstführenden Ordnung der $1/N$-Entwicklung zu. Für die Deformationen der CFT durch relevante Doppelspur-Operatoren finden wir bei Zustandssummen und konformen Anomalien exakte Übereinstimmung zwischen direkter und AdS-seitiger indirekter Rechnung. Damit wird ein nicht trivialer Test der Korrespondenz über die SUGRA-Näherung hinaus erbracht.

Schlagwörter:
AdS/CFT-Korrespondez, Penrose-Limes, CPW, konforme Anomalie

To $\mathcal{V}.\mathcal{A}.$

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Introduction

String theory[1] arose in attempts to understand strong interactions. As we know, this program was not fully successful and later abandoned in favor of Quantum Chromodynamics (QCD [2]) as the quantum field theory (*gauge theory!*) of the strong interaction, with asymptotic freedom and 'infrared slavery' playing a crucial role. Yet, string-like behavior arises in QCD as an emergent feature. For example, long flux tubes are well approximated by long strings whose dynamics is dictated by Nambu-Goto action. Little hope was left, however, for a fundamental QCD-string that could lead to, say, the spectra of light mesons and glueballs. String theory took a drastic turn after identifying the *graviton* with the massless spin-two excitation of the closed string spectrum and tuning the typical size of the string from that of a hadron ($\sim 10^{-13}cm$) to the *Planck length* ($\sim 10^{-32}cm$). It emerged then as the leading (if not the only!) prospect for *quantum gravity* and *unification* with the other interactions.

The long-standing belief in an exact *string/gauge duality*, on the other hand, was supported by 't Hooft *large-N* expansion of QCD. This is the only known way to turn QCD, with N colors, into a perturbative theory (in $1/N$) at all energies. To have well defined Green's functions in the limit, the coupling should scale as $g_{YM} \sim 1/\sqrt{N}$. Another argument in this direction comes from the RG beta function for pure SU(N) Yand-Mills at leading order in perturbation theory $\frac{d}{d\,log\,\mu} g_{YM} \equiv \beta(g_{YM}) = -\frac{3N}{16\pi^2} g_{YM}^3$. When taking $N \to \infty$, it results in a sensible beta function for the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$, namely $\frac{d}{d\,log\,\mu}\lambda \equiv \beta(\lambda) = -\frac{3}{8\pi^2}\lambda^2$. This has a remarkable effect in the Feynman diagrams (for simplicity, consider connected vacuum amplitudes) which in double-line notation can be viewed as (oriented) two-dimensional surfaces with holes, the colors loops. It was further conjectured that there should be an equivalent description in which the holes get filled up, leading to closed Riemann surfaces without boundaries. The $1/N$ expansion becomes

[1]Disclaimer: no attempt will be made to provide a comprehensive set of references for this introduction, we just refer to [1] and references therein.

[2]A list of abbreviations is included at the end.

then a *genus* expansion, suggestive of a closed string worldsheet expansion with string coupling $g_s \sim 1/N$.
Indeed, normalizing the gauge potential so that the coupling appears only as a factor $1/g^2_{YM}$ in front of the action, each Feynman diagram (with V vertices, E propagators and F color loops) carries a factor

$$(g^2_{YM})^{-V+E}\, N^F = (g^2_{YM})^{-V+E-F}\, (g^2_{YM} N)^F \equiv (g^2_{YM})^{-\chi}\, \lambda^F \tag{1}$$

where we recognize the Euler characteristic $\chi = 2 - 2g$ (g=genus of the Riemann surface, i.e. number of handles) and the 't Hooft coupling $\lambda \equiv g^2_{YM} N$ that remains finite in the limit. For any amplitude $\mathcal{A}$, the perturbative series of Feynman diagrams can be rewritten as a sum over topologies

$$\mathcal{A} = \sum_{g,F=0}^{\infty} \mathcal{F}_{g,F}\, (g^2_{YM})^{2g-2}\, \lambda^F = \sum_{g=0}^{\infty} \mathcal{F}_g(\lambda)\, (g^2_{YM})^{2g-2} = \sum_{g=0}^{\infty} \widetilde{\mathcal{F}}_g(\lambda)\, N^{2-2g} \tag{2}$$

where the coefficients $\mathcal{F}_{g,F}$ depend on the other coupling constants of the theory. The last two equalities are obtained after adding up the perturbative expansion at fixed genus and are believed to exist for *any* value of λ, in particular at strong coupling.
The large-N limit is dominated by diagrams with the topology of the sphere but, in contrast to vector models which are then soluble and where the large-N expansion has its roots, even in this *planar limit* the gauge theory is far from being solved (except in $d = 2$ dimensions).

AdS/CFT Correspondence

The arguments above do not give a direct construction of the string dual to a specific large-N gauge theory. In particular, concrete realizations for *four-dimensional* gauge theories had to wait until the advent of *AdS/CFT Correspondence*. Now there are many examples in which, in some *decoupling limit*, the dual to gauge theories realized on the worldvolume of *D-branes* are known to be none other than *type IIB superstrings* living in a warped 10-dim space. General gauge theories are supposed to be reached by suitable *deformations* of the boundary theory and, correspondingly, of the bulk geometry. In some cases, the duality is a *strong/weak* one in the sense that when the gauge theory is strongly coupled, the dual string background is weakly curved and the superstring theory may be well approximated by *supergravity* (SUGRA).

Let us roughly recall the circle of ideas that motivated *Maldacena's conjecture*. Of course, each one is a whole rich field by its own, so that we will omit the details and just highlight the relevant quantities (modulo numerical factors), in the maximally symmetric scenario, leading to the final mapping of parameters.

- IIB SUGRA as low energy limit (effective field theory as $\alpha' \equiv l_s^2 \to 0$) of type IIB strings: 10-dim Newton constant and therefore 10-dim Planck length in terms of string length l_s and string dimensionless coupling constant g_s,
$$l_P = l_s\, g_s^{1/4}. \tag{3}$$
- Tension (mass per unit volume) of a Dp-brane, stable (BPS) soliton-like membranes in theories of closed strings. Solitonic nature seen in the tension $\tau_p \sim 1/g_s$. Balance between gravitational attraction and RR repulsion, mass=charge. Preserve 16 SUSYs. Dimensional analysis fixes the rest,
$$\tau_p = \frac{1}{g_s\, l_s^{p+1}}. \tag{4}$$
- SYM realized on the worldvolume of Dp-branes, as massless spectrum of open strings ending on a stack of Dp-branes. Low energy limit dynamics dictated by DBI action. Once τ_p is known, g_{YM} follows as
$$g_{YM}^2 = \frac{1}{\tau_p\, l_s^4} = g_s\, l_s^{p-3}. \tag{5}$$
Notice that for D3-branes, the worldvolume is four-dimensional and $g_{YM}^2 = g_s$ is dimensionless, as expected.
- Classical SUGRA (p=3)-brane (extremal) solution, carrying RR-charge (quantized flux) and constant dilaton. Preserve 16 SUSYs, BPS condition mass=charge. Details encoded in a harmonic function of the transverse coordinates r: $H = 1 + \frac{L^4}{r^4}$. Harmonic superposition principle, which follows from the BPS condition, implies that for a stack of coincident branes L^4 is additive. So is the ADM mass, i.e. no binding energy, so that $M \sim L^4$ and the rest is fixed by dimensional analysis
$$M = \frac{L^4}{l_P^8}. \tag{6}$$
- Dual descriptions of the very same object. *Large* number N of copies source gravity and have an effective description in terms of classical metric, dilaton and RR field switched on:
$$M = N\, \tau_3\ . \tag{7}$$
This identification fixes $\frac{L^4}{l_s^4} = g_s\, N = g_{YM}^2\, N = \lambda$ and $\frac{L^4}{l_P^4} = N$.

- Near-horizon/decoupling/Maldacena's limit: $l_s \to 0$, keeping $U \equiv \frac{r}{l_s^2}$ and all other physical scales fixed. In this limit the geometry becomes that of the throat $AdS_5 \times S^5$, with AdS on its Poincare patch ($z \equiv \sqrt{\lambda}/U$), with equal radii L

$$ds^2 = \frac{L^2}{z^2}(dz^2 + dx_\mu^2) + L^2 d\Omega_5^2. \tag{8}$$

  This near-horizon region decouples from the asymptotically flat region of the D3-brane and at the same time, the SYM theory in the worldvolume decouples from the closed strings (i.e., from gravity!). The relation between the dimensionless parameters of the two theories, IIB Superstring in the near-horizon geometry and $\mathcal{N} = 4$ $SU(N)$ SYM_4, goes then as follows

$$g_s = g_{YM}^2 \tag{9a}$$

$$\frac{L^4}{l_s^4} = \lambda \tag{9b}$$

$$\text{or} \qquad \frac{L^4}{l_P^4} = N. \tag{9c}$$

In units of the radius [3] of AdS, we have that the Planck length decreases in the large-N limit. Quantum loop effects are therefore suppressed and one can trust classical string theory. In addition, the string size in the same units decreases when λ is large, so that at strong 't Hooft coupling one can rely on the classical SUGRA approximation. There are, in consequence, three stages of Maldacena conjecture:

- Strong form: duality between full quantum IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ SYM_4 for any value of N and g_{YM}.

- 't Hooft limit: duality between classical IIB string theory on $AdS_5 \times S^5$ and planar limit $\mathcal{N} = 4$ $SU(N)$ SYM_4 for any fixed λ. Perturbative expansion in g_s corresponds to the perturbative $1/N$ one.

[3] Most of the time we are going to be using this units, e.g. to recover the mass dimension $m \to mL$.

- Classical SUGRA approximation: duality between classical IIB SUGRA on $AdS_5 \times S^5$ and planar $\mathcal{N} = 4$ $SU(N)$ SYM_4 at strong coupling λ. The α' corrections to SUGRA correspond to $1/\sqrt{\lambda}$ corrections.

The 4-dim $\mathcal{N} = 4$ supersymmetric SU(N) gauge theory has adjoint matter consisting of 4 Majorana fermions and 6 scalars. All Yukawa and scalar self-couplings are given in terms of the gauge coupling. The theory is conformally invariant, i.e. has vanishing β-functions. This maximally symmetric version of the correspondence maps the isometry group of AdS_5 ($SO(2,4)$) to the conformal group in 1+3-dimensional Minkowski space and the isometries of the sphere S^5 ($SO(6) \sim SU(4)$) are mapped to the internal R-symmetry of the $\mathcal{N} = 4$ SYM. The supersymmetric completion of these bosonic symmetries results in $PSU(2,2 \mid 4)$.

There is in addition a one-to-one map between gauge invariant operators in the CFT at the boundary and fields (or extended configurations) in the bulk of AdS_5. The symmetries naturally lead to the identification between scaling dimension of (conformal primary) operators and mass of the dual bulk field. As an illustration, consider the boundary two-point function of an operator O with scaling dimension Δ and the corresponding bulk Green's function of its dual scalar field ϕ with a huge mass m. In this "classical" limit, the bulk propagator is given essentially by the classical action of a very massive particle $e^{-m\sqrt{\sigma}}$, where $\sqrt{\sigma}$ is the geodesic distance between the two spacetime points in consideration. Let them now approach the boundary: in the Poincare coordinates of before, small ϵ is a radial IR-cutoff and $\sqrt{\sigma} \sim \log\frac{|x-x'|^2}{\epsilon^2}$. That gives $e^{-m\sqrt{\sigma}} \sim (\frac{|x-x'|^2}{\epsilon^2})^{-m}$ and can be directly compared with the CFT correlator $\langle O(x)O(x')\rangle \sim (\frac{|x-x'|^2}{b^2})^{-\Delta}$ with an UV-cutoff b. We are then lead to the identification $m \sim \Delta$ (the precise relation being $m^2 = \Delta(\Delta - 4)$) and $\epsilon \sim b$.
The correspondence was further given a precise calculational prescription, involving the *generating functional* of correlation functions in the CFT and the *string partition function* with given boundary conditions. In the SUGRA approximation, correlators at $\lambda >> 1$ are obtained from the saddle point approximation, i.e. from the classical SUGRA action.

As said before, many other examples of the correspondence have been found from other brane systems containing many potential predictions rather than confirmation of somehow expected features.

Holography and IR/UV-connection

The generic picture is that of string/M-theory on a certain manifold being equivalent to a quantum field theory on its (conformal) boundary. This can

be rephrased as: (a candidate for) a consistent quantum theory of gravity whose fundamental degrees of freedom reside at the boundary of spacetime rather than in the bulk interior. This assertion entails part of the *holographic principle*, which had been earlier conjectured.
However, the claim that AdS/CFT correspondence is a realization of the holographic principle, even in the well understood cased of above, based only on the mapping of Green's functions and bulk-field/boundary-operator identifications is certainly incomplete. With some exaggeration, this is just as "holographic" as classical boundary value problems in electrostatics or the Cauchy problem in PDEs.

It remains to be verified the *holographic bound*, namely that there is at most one degree of freedom per Planck "area" on the boundary. The difficulty in establishing the bound being two-fold due to the divergent area of the boundary and the infinitely many degrees of freedom of the continuous gauge theory. On one hand, introducing an *UV-cutoff* on the gauge theory one can count the resulting finite number of degrees of freedom. Let us consider Poincare coordinates

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + dz^2 + d\overrightarrow{x}^2) \tag{10}$$

and at fixed time t and at fixed distance z from the boundary, divide a "surface" (a three dimensional volume, in fact) region V into elementary cells by introducing an UV-cutoff b. All points inside each individual cell of size b^3 are "coarse-grained" to a single one and we also have a finite amount of cells $\frac{V}{b^3}$. There are then roughly N^2 degrees of freedom associated to each cell, since the elementary field is matrix-valued. We wind up with $N_{dof} \sim \frac{V}{b^3} N^2$ degrees of freedom in the UV-regulated boundary theory. On the other hand, the infinite area of the conformal boundary of AdS can be rendered finite by an *IR-cutoff*, the distance ϵ to the boundary. At fixed time t and at a small distance ϵ from the boundary the area is given by the integral of the induced metric $Area = \int d^3\overrightarrow{x}\sqrt{h} = V\,\frac{L^3}{\epsilon^3}$. The "area" V is "red-shifted" by the warping factor L^2/z^2 in the Poincare metric. Taking then the ratio of the two regularized quantities

$$\frac{N_{dof}}{Area} \sim \frac{N^2}{L^3}\,\frac{\epsilon^3}{b^3} \sim \frac{L^5}{l_P^8}\,\frac{\epsilon^3}{b^3} \sim \frac{1}{l_{P_5}^3}\,\frac{\epsilon^3}{b^3} \tag{11}$$

the cut-off dependence goes away and one gets (modulo numerical factor) one degree of freedom per 5-dim Planck area[4] on the boundary,

$$\frac{N_{dof}}{Area/l_{P_5}^3} \sim 1 \tag{12}$$

provided one identifies the cut-offs $\frac{\epsilon}{b} \sim 1$. This identification that solves the issue of the entropy bound is known as the *IR/UV-connection* in AdS/CFT. This relation we already met before in the analysis of two-point functions and is supported by several other arguments. For example, recall the mapping of symmetry groups: a rigid scaling to localize objects in the CFT comes, e.g. in Poincare coordinates, with a corresponding scaling of the radial distance, so that the boundary gets closer. In all, the Bekenstein bound is justified in AdS/CFT.

RG flow and holographic c-theorem

Pushing forward the IR/UV-connection, the mapping of the *conformal anomaly* (*c-charge*) immediately follows at leading large-N. In the free field theory in a curved background, it is read from the "log-term" in the UV-regulated one-loop effective action. According to the AdS/CFT prescription, one looks at the classical gravity action on AdS (which is IR-divergent due to the infinite volume) to get the corresponding quantity at strong coupling. In the light of the IR/UV-connection, one should look then at the coefficient of the "log-term" in the IR-regulated version. It happens to coincide with the anomaly of the full $\mathcal{N} = 4$ SYM multiplet. A non-renormalization theorem, due to SUSY, lies behind this strong/weak matching.

Perturbing the original CFT by some *relevant deformation* the theory generically flows to another CFT or to a theory with a mass gap. In the first case, the SUGRA side is described by a domain wall solution interpolating to another AdS-like region which represents the IR theory. The RG evolution parameter (energy scale) becomes a coordinate in spacetime. For all these SUGRA-driven flows a *c-function* follows naturally, decreasing along the flow. The IR geometry is that of AdS with smaller radius. The central charge of $\mathcal{N} = 4$ SU(N)SYM at leading large-N being $c \sim N^2$, a measure of the number of (massless) degrees of freedom, is translated into $c \sim \frac{L^8}{l_P^8}$. Therefore, the decrease in the radius is translated in a decrease of the central charge at IR, in accord with generalizations of the c-theorem to four dimensions.

Our motivations

[4] That is, $Area/G_5$ with the effective 5-dim Newton constant and Planck length related to the 10-dim parents via the volume of the S^5: $G_5 \sim l_{P_5}^3 \sim \frac{l_P^8}{L^5}$.

Our aim in this thesis is to explore some limits of the AdS/CFT Correspondence which promise to be more tractable than the original formulation. One limit corresponds to a large quantum number regime in the gauge theory (BMN limit) which is translated in a Penrose limit of the background geometry, ending up in a plane wave. Progress in the construction of the relevant field theoretic propagators were restricted to the scalar one, where some similarities with flat space were noticed. Second motivation comes from considering the $\lambda = 0$ limit in the 't Hooft large-N expansion. Even in this limit, correlation functions for gauge invariant composite operators admit a topological expansion and several efforts have been done in trying to understand the AdS dual of such limit. The OPE analysis reveals the presence of higher spin currents, suggestive of a higher spin theory as the effective field description in the bulk. A simplified scenario is that of the free O(N) vector model at large N. One should be able to rewrite four point correlators in terms of Witten graphs in AdS, involving the exchange of an infinite tower of HS fields. As a bonus, some connections to the IR fixed point of the interacting theory may be found. Finally, this flow from UV to IR at large N fits in naturally in the generalized AdS/CFT prescription to incorporate double-trace deformations. In this context, there were impressive results mapping $1/N^2$ corrections to the conformal anomaly and evidence for the validity of the holographic c-theorem at this non-trivial level, but several issues were open stemming from the formal identity between partition functions.

Outline

We start with a preliminary first chapter reviewing the proper-time construction and its uses to compute conformal anomaly, effective action and free correlators. We also briefly review the holographic anomaly, the mapping of symmetries and the Fefferman-Graham construction as preparation for the subsequent chapters. The original contributions in the following chapters are based on joint work with my advisor Dr. Harald Dorn [34, 35, 36]. In the second chapter we study field theoretic propagators in the plane wave limit of the correspondence. Our main goal is to gain some understanding of the semiclassical nature of the limit, that is apparent in this large-quantum-number limit of the gauge theory. In the third chapter another corner, that of a free CFT, is addressed in the simplified scenario of the Higher Spin/O(N) Vector Model duality. As a natural generalization of the pattern found here, we further examine the effect of double-trace deformations back to gauge theory. In particular, the issues of partition functions and (correction to) conformal anomaly are addressed in the fourth chapter. We close with conclusion and outlook. Useful formulas and supporting material are collected in a series of appendices.

Chapter 1

Preliminaries

In this chapter we briefly review some basic constructions, focusing on the aspects that are relevant for the original contributions of the subsequent chapters.

1.1 Proper-time

One central technique underlying all issues addressed in this thesis is the proper-time representation of Green's functions. Most of this background material in contained in the classic work [30, 31].

1.1.1 Schwinger-DeWitt construction

Let us start with the scalar Feynman propagator in a curved background. It is the solution of the wave equation with a point-like source

$$\left(\Box - m^2\right) G(x,x') = \delta(x,x') \tag{1.1}$$

together with appropriate boundary conditions. Here $\delta(x,x')$ denotes the invariant δ-function. The Fock-Schwinger-DeWitt proper-time representation for the Feynman propagator, which incorporates the Feynman boundary conditions by the $i0^+$ prescription, is based on the formal solution

$$\frac{1}{\Box - m^2 + i0^+} = -i \int_0^\infty ds\, e^{-is\, m^2 - s0^+}\, e^{is\,\Box}. \tag{1.2}$$

The Schwinger-DeWitt kernel (the kernel of the exponentiated operator),

$$K(x\,, x' \mid s) \equiv \langle x \mid e^{is\,\Box} \mid x' \rangle = e^{is\,\Box}\delta(x,x') \tag{1.3}$$

satisfies the following “Schrödinger equation” and initial condition

$$(i\partial_s + \Box)\, K(x, x' \mid s) = 0 \tag{1.4a}$$

$$K(x, x' \mid 0) = \delta(x, x'). \tag{1.4b}$$

A WKB-inspired ansatz for the solution, meant to be only an asymptotic one, is

$$K(x, x' \mid s) = \frac{i}{(4\pi i s)^{\frac{d}{2}}}\, \triangle^{\frac{1}{2}} e^{i\sigma/2s}\, \Omega(x, x' \mid s) + ... \tag{1.5}$$

where $\sigma(x, x')$ is the geodetic interval (half the geodesic distance squared between the two points),

$$\triangle(x, x')[g(x)g(x')]^{\frac{1}{2}} \equiv -\det(-\frac{\partial^2\sigma}{\partial x^\mu \partial x'^\nu}) \tag{1.6}$$

is the Van Vleck-Pauli-Morette determinant (an important improvement of the WKB ansatz). $\Omega(x, x' \mid s)$ has a power expansion in the proper time s

$$\Omega(x, x' \mid s) = \sum_{n=0}^{\infty} (is)^n\, a_n(x, x'), \tag{1.7}$$

whose coefficients $a_n(x, x')$ are regular functions in the coincidence limit $x \to x'$, and finally the ellipsis stands for indirect geodesic contributions. The coefficients, sometimes referred to as HaMiDeW coefficients, must satisfy the recursion relation

$$(n + 1)\, a_{n+1} + \partial^\mu \sigma\, \partial_\mu a_{n+1} = \triangle^{-\frac{1}{2}}\, \Box\, (\triangle^{\frac{1}{2}} a_n) \tag{1.8}$$

starting with $\partial^\mu \sigma\, \partial_\mu a_0 = 0$ and $a_0(x, x) = 1$. For the present scalar case, the chain of HaMiDeW coefficients trivially starts with $a_0(x, x') = 1$.

1.1.2 Effective action

The 1-loop effective action resulting from integrating out the quadratic fluctuations of the scalar field is given by the logarithm of the fluctuation determinant

$$W = \frac{1}{2}\log\, \det(-\Box + m^2) = \frac{1}{2}\mathrm{tr}\, \log(-\Box + m^2). \tag{1.9}$$

Notice that $\frac{\partial}{\partial m^2}$ formally gives the trace of the propagator. Using the proper-time representation of above, the derivative can be undone to get

$$W = \frac{1}{2}\mathrm{tr}\int_0^\infty \frac{ds}{is}\, e^{-im^2 s} K(x,x' \mid s) + const \tag{1.10a}$$

$$= \frac{1}{2}\mathrm{tr}\int_0^\infty \frac{ds}{is}\, \frac{i}{(4\pi i s)^{\frac{d}{2}}}\, e^{i\sigma/2s - im^2 s}\, \Omega(x,x' \mid s) + const \tag{1.10b}$$

$$= \int w(x)\, \sqrt{g(x)}\, d^n x + const \tag{1.10c}$$

where we used the fact that the coincidence limit of the Van Vleck-Morette determinant is one.
The proper-time integral diverges at the lower limit for all positive values of the spacetime dimension n, as can be seen after inserting the expansion (1.7). Now, it can be regularized respecting the symmetries of the classical action. To this end, one interprets the dimensionality n of spacetime as a complex number instead of a positive integer and defines $w(x)$ by analytic continuation from the region of convergence in the complex n-plane to the vicinity of the actual physical dimension. For dimensional reasons, one has to introduce an arbitrary mass scale μ. One can therefore integrate by parts and afterwards come back to the physical dimension

$n \to 2$:

$$w(x) = -\frac{1}{4\pi}\left(\frac{1}{n-2} - \frac{1}{2}\right)\left\{a_1(x,x) - m^2\right\}$$
$$-\frac{i}{8\pi}\int_0^\infty ds \log(4\pi i \mu^2 s)(\frac{\partial}{i\partial s})^2 \left[e^{-im^2 s}\, \Omega(x,x \mid s)\right]. \tag{1.11}$$

$n \to 3$:

$$w(x) = \frac{i}{12\pi^{\frac{3}{2}}}\int_0^\infty ds\, (is)^{-\frac{1}{2}}(\frac{\partial}{i\partial s})^2 \left[e^{-im^2 s}\, \Omega(x,x \mid s)\right]. \tag{1.12}$$

$n \to 4$:

$$w(x) = -\frac{1}{32\pi^2}\left(\frac{1}{n-4} - \frac{3}{4}\right)\left\{2a_2(x,x) - 2m^2 a_1(x,x + m^4\right\}$$
$$-\frac{i}{64\pi^2}\int_0^\infty ds \log(4\pi i \mu^2 s)(\frac{\partial}{i\partial s})^3 \left[e^{-im^2 s}\, \Omega(x,x \mid s)\right]. \tag{1.13}$$

The divergences show up as poles in the complex n-plane in even dimensions. In odd dimensions, in turn, there is generally no divergence in one-loop order with DR. The terms containing the poles are real valued and have no effect on the vacuum persistence probability. They may be subtracted (renormalized) without affecting the physical content of the theory.

The diagonal coefficients of the SD expansion (1.7) are expressed in terms of local curvature invariants, valid as well for a nonminimally coupled scalar ($\xi \neq 0$), with no explicit dependence on the dimension. The first nontrivial ones are

$$a_1(x,x) = \left(\frac{1}{6} - \xi\right) R. \tag{1.14a}$$

$$\begin{aligned} a_2(x,x) = &\frac{1}{6}\left(\frac{1}{5} - \xi\right) \Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 \\ &- \frac{1}{180} Ric^2 + \frac{1}{180} Riem^2. \end{aligned} \tag{1.14b}$$

1.1.3 Conformal/trace/Weyl anomaly

Not all local invariances survive renormalization, conformal invariance is a typical example. Consider a massless scalar conformally coupled, that is, with $\xi = \frac{1}{4}\frac{n-2}{n-1}$ for which the DR-regularized W is

$n \to 2$:

$$W = W_{ren} - \frac{1}{4\pi}\left(\frac{1}{n-2} - \frac{1}{2}\right)\left[\frac{1}{6} - \frac{1}{4}\frac{n-2}{n-1}\right] \int R\sqrt{g} d^n x \tag{1.15}$$

$n \to 4$:

$$\begin{aligned} W = W_{ren} - &\frac{1}{16\pi^2}\left(\frac{1}{n-4} - \frac{3}{4}\right) \int \left\{ \frac{1}{6}\left(\frac{1}{5} - \frac{n-2}{4(n-1)}\right) \Box R \right. \\ &\left. + \frac{1}{2}\left[\frac{n-4}{12(n-1)}\right]^2 R^2 - \frac{1}{180} Ric^2 + \frac{1}{180} Riem^2 \right\} \sqrt{g} d^n x \end{aligned} \tag{1.16}$$

where W_{ren} is the finite remainder after subtraction of the local terms. Under a metric variation $\delta g_{\mu\nu}$ one gets for the renormalized trace of the stress-energy density

$$2g^{\mu\nu}\frac{\delta W_{ren}}{\delta g_{\mu\nu}} = \langle T_{ren\,\mu}{}^{\mu} \rangle \ , \tag{1.17}$$

after some computation,

$$
\begin{aligned}
& n \to 2: \\
& \langle T_{ren\mu}{}^{\mu} \rangle \to \frac{1}{24\pi} g^{\frac{1}{2}} R
\end{aligned}
\tag{1.18}
$$

$$
\begin{aligned}
& n \to 4: \\
& \langle T_{ren\mu}{}^{\mu} \rangle \to \frac{1}{16\pi^2} g^{\frac{1}{2}} \frac{1}{180} \left(\Box R - Ric^2 + Riem^2 \right) ,
\end{aligned}
\tag{1.19}
$$

where the invariance of the the classical action, and hence of W, was used. The variation of W_{ren} can be read off from that of the *local* counterterms. The anomaly arises then in DR due to a cancelation of the pole against a zero coming from the variation of these counterterms, which are certainly invariant at the physical integer dimension.

In even dimensional spacetime the renormalized stress-energy tensor is seen to have a nonvanishing trace, the *trace anomaly.* It can be expressed for all n in the form

$$
\langle T_{ren\mu}{}^{\mu} \rangle = \begin{cases} 0 & (n\, odd) \\ \frac{1}{(4\pi)^{n/2}}\, g^{1/2}(x)\, a_{n/2}(x,x) & (n\, even). \end{cases}
$$

Back to four dimensions, the geometric *local* contributions to the anomaly are exhausted by the following four conformal invariants

$$
\langle T_{ren\mu}{}^{\mu} \rangle = \frac{c}{16\pi^2} Weyl^2 - \frac{a}{16\pi^2} \widetilde{Riem}^2 + \alpha \Box R + \beta R^2, \tag{1.20}
$$

with the Weyl tensor squared and Euler density given by

$$
Weyl^2 = Riem^2 - 2Ric^2 + \frac{1}{3} R^2 \tag{1.21a}
$$

$$
\widetilde{Riem}^2 = Riem^2 - 4Ric^2 + R^2. \tag{1.21b}
$$

The "charges" c and a are renormalization-scheme independent. The β-term does not satisfy Wess-Zumino consistency which means that no effective action, local or non-local, can give rise to an R^2 trace. The α-term is scheme dependent (see e.g. [8]) and is the variation of the local term $\int d^4x \sqrt{g} R^2$. Usually, only the first two terms are considered as non-trivial anomalies.

Our previous result for a real free scalar corresponds to $c = \frac{1}{120}$ and $a = \frac{1}{360}$. The extension to N_0 scalars and inclusion of $N_{1/2}$ Dirac spinors and

N_1 gauge bosons generalize to

$$c = \frac{N_0 + 6N_{1/2} + 12N_1}{120} \tag{1.22a}$$

$$a = \frac{N_0 + 11N_{1/2} + 62N_1}{360}. \tag{1.22b}$$

The conformal anomaly is not an obstacle for consistent quantization, even if the background metric is allowed to become dynamical. The diffeomorphism invariance is the one that must be preserved by the renormalization procedure. This is similar to the flat spacetime pure YM theory where despite the conformal invariance of the classical action in four dimensions, gauge invariance is the one preserved by quantization.

1.2 Holographic anomaly

Let us consider the free $\mathcal{N} = 4$ $SU(N)$ SYM_4 and perform the counting of fields to get their contribution to the conformal anomaly. We have then to include the gauge field, 6 real scalars and 4 Majorana (or Weyl) fermions (counted as 2 Dirac spinors) and on top of that, all in the adjoint, $N^2 - 1$ copies of each. That makes $(N_1, N_{1/2}, N_0) = [N^2 - 1](1, 2, 6)$ and therefore

$$c = a = \frac{N^2 - 1}{4}. \tag{1.23}$$

Whenever $c = a$ the Riemann tensor squared goes away and one gets

$$\langle T_{ren\mu}{}^{\mu} \rangle = \frac{c}{8\pi^2} \left(Ric^2 - \frac{1}{3} R^2 \right). \tag{1.24}$$

The anomaly is robust with respect to the regularization scheme used to control the divergences. Had we chosen an UV-cutoff ϵ in the proper-time integral to control the divergences, the anomaly would have shown up as the coefficient of a $\log \epsilon$-term.

Motivated by the IR/UV-connection, let us look at the log-term in the gravitational action. Here we have to "fluctuate" the AdS_{n+1} geometry to have a source for the boundary stress tensor. This can be done in a restricted way, letting the metric to be that of a Poincare-Einstein manifold, i.e. $Ric(g_+) = -ng_+$ (the L^{-2} factor is absorbed in a redefinition of g). In this case, the action is proportional to the (infinite) volume of the space and it is supplemented by some boundary terms (e.g. Gibbons-Hawking), but they play no role in the anomaly calculation. The bulk geometry can be partially reconstructed by an asymptotic expansion, which is essentially the content of the Fefferman-Graham theorem [49], and all divergencies can be isolated. One can always find local coordinates near the boundary (at $r = 0$) to write the bulk metric as

$$g_+ = r^{-2}\{dr^2 + g_r\}. \tag{1.25}$$

Euclidean AdS_{n+1} corresponds to the choice $g_r = (1 - r^2)^2 g_0$ with $4\,g_0$ being the round metric on the sphere S^n. The "reconstruction" theorem leads to the asymptotics

$$\begin{aligned} &n\,odd: \\ &g_r = g^{(0)} + g^{(2)}r^2 + (even\,powers) + g^{(n)}r^n + ... \end{aligned} \tag{1.26}$$

$$\begin{aligned} &n\,even: \\ &g_r = g^{(0)} + g^{(2)}r^2 + (even\,powers) + g^{(n)}r^n + hr^n \log\, r + ... \end{aligned} \tag{1.27}$$

where $g^{(0)} = g$ is the chosen metric at the conformal boundary. For odd n, $g^{(j)}$ are tensors on the boundary and $g^{(n)}$ is trace-free. For $0 \leq j \leq n-1$, $g^{(j)}$ are locally formally determined by the conformal representative but $g^{(n)}$ is formally undetermined, subject to the trace-free condition. For even n, $g^{(j)}$ are locally determined for j even $0 \leq j \leq n-2$, h is locally determined and trace-free. The trace of $g^{(n)}$ is locally determined, but its trace-free part is formally undetermined. All this is dictated by Einstein equations.

The volume element has then an asymptotic expansion

$$\begin{aligned} dv_{g_+} &= \sqrt{\frac{detg_r}{detg}}\frac{dv_g\,dr}{r^{n+1}} \\ &= \{1 + v^{(2)}r^2 + (even\,powers) + v^{(n)}r^n + ...\}\frac{dv_g\,dr}{r^{n+1}}, \end{aligned} \tag{1.28}$$

where all coefficients $v^j, j = 1..n$ are locally determined in term of curvature invariants of the boundary metric and $v^n = 0$ if n is odd. The volume regularization can be carried out with an IR-cutoff ϵ and results in an asymptotic expansion in negative powers of ϵ

$$\begin{aligned} &n\,odd: \\ &\mathrm{Vol}_{g_+}(r > \epsilon) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + (odd\,powers) + c_{n-1}\epsilon^{-1} \\ &\qquad + \mathcal{V} + o(1) \end{aligned} \tag{1.29}$$

$$\begin{aligned} &n\,even: \\ &\mathrm{Vol}_{g_+}(r>\epsilon) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + (even\,powers) + c_{n-2}\epsilon^{-2} \\ &\qquad\qquad\qquad - \mathcal{L}\log\epsilon + \mathcal{V} + o(1). \end{aligned} \tag{1.30}$$

When n is odd, the constant term $\mathcal{V}$ in the expansion (renormalized volume) turns out to be independent of the conformal choice of the boundary metric. If n is even, in turn, $\mathcal{V}$ is no longer invariant and its variation under a Weyl transformaton of the boundary metric gives rise to the *conformal anomaly.* It is

$$\mathcal{L} = \int d^4x\sqrt{g}\,v^{(n)}\ , \tag{1.31}$$

the coefficient of the $\log\epsilon$-term, the invariant one in this case and it is given by the integral of a local curvature expression on the boundary. The variation of $\mathcal{V}$ happens to be connected to this invariant, $g \to e^{2\sigma}g$ for infinitesimal σ makes $\mathcal{V} \to \mathcal{V} + \int d^4x\sqrt{g}\,v^{(n)}\sigma$. In four dimensions, we have

$$v^{(4)} = \frac{1}{32}\left(Ric^2 - \frac{1}{3}R^2\right). \tag{1.32}$$

Back to physics, to fix the overall coefficient we only have to include the constant Lagrangian $R - \Lambda = \frac{-8}{L^2}$, the volume dimensions L^5, which were rescaled away, and Newton's 5-dim constant $\frac{1}{16\pi G_5} = \frac{\mathrm{Vol}(S^5)}{16\pi G_{10}} = \frac{N^2}{8\pi^2 L^3}$. Altogether, the anomaly comes with the same coefficient as the free CFT anomaly computed before, at leading large-N [69]. The mismatch N^2 instead of N^2-1 as in the field theory computation is an $O(1/N^2)$ correction that should be related, according to the correspondence, to a quantum loop in the SUGRA approximation.

1.3 Emergent AdS from free correlators

One important feature of the IR/UV-connection in the matching of the conformal anomaly is the analogy between the Schwinger parameter in the free field computation and the radial coordinate in AdS. As a remarkable outgrowth of this observation, Gopakumar [55, 56, 57] embarked on a broader project where the original hope was to start with Schwinger parametrization of the CFT correlators and, after some transformations on this moduli space, pursue the emergence of AdS geometry in the form of Witten graphs. The simplest setting is that of the free limit of the CFT where correlators of composite singlet operators still admit a topological expansion in powers of $1/N^2$.

Let us consider the single-trace bilinear $O = Tr\Phi^2$ in the free field theory limit where it carries the *naive* dimension $\Delta = d - 2$. Start with the two-point correlation function which is easily obtained after Wick contracting, only the free propagator is needed,

$$\langle O(x_1)\, O(x_2)\rangle \sim \frac{1}{r_{12}^{\Delta}}. \tag{1.33}$$

Pictorially, this planar contribution is represented by a loop with insertions at the location of the operators.

Let us nonetheless make a detor by deriving this simple result using the worldline formalism, i.e. the first quantized picture. The amplitude is described then by a path integral along the loop and forced to meet the insertion points

$$\Gamma(x_1, x_2) = \left\langle \int_0^T \prod_{i=1,2} d\tau_i \; \delta(x(\tau_i) - x_i) \right\rangle \tag{1.34}$$

where the average is taken with respect to the free particle vacuum amplitude

$$\langle ... \rangle = \int_0^\infty \frac{dT}{T} \int_{T-periodic} \mathcal{D}x(\tau)\; \exp\left\{ -\frac{1}{2}\int_0^T d\tau\; \dot{x}^2(\tau) \right\} ... \tag{1.35}$$

The easiest way to take the Gaussian path integral is to go to momentum space and it can be solved in terms of the classical path. The only input is the Green's function of the one-dimensional operator $-\frac{d^2}{d\tau^2}$ on the circle. In the present case one gets, in addition to an overall delta-function enforcing momentum conservation, after little effort

$$\widetilde{\Gamma}(p) = \int_0^\infty \frac{dT}{T^{\frac{d}{2}-1}} \int_0^1 d\alpha\; \exp\left\{ -\frac{\alpha(1-\alpha)T}{2} p^2 \right\}. \tag{1.36}$$

There has been already a gluing up here, two Schwinger parameters were traded by one. To see this more explicit in terms of an electric circuit analogy, let us introduce an auxiliary momentum integral to break up the exponential

$$\widetilde{\Gamma}(p) = \int_0^\infty dT\, T \int_0^1 d\alpha \int d^d q\; \exp\left\{ -\frac{\alpha T}{2} q^2 - \frac{(1-\alpha)T}{2}(p-q)^2 \right\}. \tag{1.37}$$

The overall proper-time integral brings the amplitude back to the usual Feynman parameter trick to compute the loop. But already in the above form the gluing up is more transparent: we have the power dissipated by the circuit where the "input current" p was split in the two legs carrying q and $p - q$ with "resistances" αT and $(1-\alpha)T$, respectively. The gluing up consists then in the substitution of the "resistors in parallel" by the "equivalent resistor"

$\alpha(1-\alpha)T$ carrying all the input current. The equivalent circuit has then a single resistance and the "power dissipated" is $\alpha(1-\alpha)Tp^2$.

Now we want to argue that this glued-up form naturally leads to a Witten graph in AdS, namely, the convolution of two bulk-to-boundary propagators of the dual bulk field

$$\Gamma(x_1, x_2) \sim \int_0^\infty \frac{dz}{z^{d+1}} \int d^d x_3 \; \left\{\frac{z}{z^2 + r_{13}}\right\}^\Delta \; \left\{\frac{z}{z^2 + r_{23}}\right\}^\Delta . \tag{1.38}$$

In going to position space, one introduces a "center-of-mass" coordinate to represent the overall momentum conservation. This integration, together with the overall proper-time T, results in the volume element of AdS in Poincare coordinates. The glued-up integrand is essentially the Schwinger parametrization of the denominators of the bulk-to-boundary propagators. Despite finding the correct functional dependence in both cases, there are subtleties concerning two point functions because of their poor convergence in the massless limit (CFT!) and the precise matching is spoiled by some formally divergent factors that we do not attempt to regularize. Instead, we turn to the better behaved case of the three-point function.

For brevity, we work directly in position space. The correlator is just the product of three correlators of the elementary fields Φ,

$$\langle O(x_1)\, O(x_2)\, O(x_3)\rangle \sim \frac{1}{r_{12}^{\Delta/2} r_{23}^{\Delta/2} r_{31}^{\Delta/2}} \sim \prod_{cyclic(1,2,3)} \int_0^\infty d\tau_i \, \tau_i^{\Delta/2-1} \, \exp\{-\tau_i\, r_{jk}\} \tag{1.39}$$

where we have introduced Schwinger parameters τ_i for each edge of the "triangular loop". Now we introduce an auxiliary point x_4 to beak-up the gaussian exponent

$$\prod_{cyclic(1,2,3)} \exp\{-\tau_i\, r_{jk}\} = (\sum_{i=1,2,3} \rho_i)^{d/2} \int d^d x_4 \prod_{i=1,2,3} \exp\{-\rho_i\, r_{i4}\}, \tag{1.40}$$

where $\tau_1 = \rho_2\, \rho_3 / \sum_{i=1,2,3} \rho_i$, etc. This is precisely the relation in electric networks between "delta-" and "star-" configurations of conductances. In momentum space the Schwinger parameter becomes the inverse of that used in position space and therefore the analogy is with resistances as in the previous case of the two-point function. The Jacobian of the transformation $\{\tau_i\} \to \{\rho_i\}$ is $\rho_1 \rho_2\, \rho_3 / (\sum_{i=1,2,3} \rho_i)^3$. An additional Schwinger parameter for the factor $(\sum_{i=1,2,3} \rho_i)^{d/2-3\Delta/2}$ casts the original correlator in the following form

$$\int_0^\infty \frac{dT}{T^{d/2+1}} \int d^d x_4 \prod_{i=1,2,3} \int d\rho_i \rho_i^{\Delta-1}\, T^{\Delta/2} \; \exp\{-\rho_i\, T - \rho_i\, r_{i4}\}. \tag{1.41}$$

Here we immediately recognize the Witten graph associated to the convolution over a common bulk point $(z, \overrightarrow{x}_4)$ of three bulk-to-boundary propagators

$$\langle O(x_1)\, O(x_2)\, O(x_3)\rangle \sim \int_0^\infty \frac{dz}{z^{d+1}} \int d^d x_4 \prod_{i=1,2,3} K_\Delta(z, \overrightarrow{x}_4; \overrightarrow{x}_i), \tag{1.42}$$

where $T \equiv z^2$ and three Schwinger parameters, as before, were used to exponentiate the denominator of

$$K_\Delta(z, \overrightarrow{x}; \overrightarrow{y}) = C_\Delta \frac{z^\Delta}{[z^2 + (\overrightarrow{x} - \overrightarrow{y})^2]^\Delta}. \tag{1.43}$$

The pattern is the same as before, coming from momentum space, the overall momentum conservation is traded by a center-of-mass integration that together with the overall proper-time combine in the measure for AdS and the integrand can easily be related, as expected, to the bulk-to-boundary propagators.

The effect of trading loops by trees can be seen as a degenerate version of *open/closed duality* transformation which is believed to be the dynamical principle behind all forms of gauge/string dualities [103]. The field theory is inherited from open strings in the limit $\alpha' \to 0$ and the worldlines are remnants of open string worldsheets. They close to form tree diagrams which are in turn a degenerate version of a planar closed worldsheet. That is, the gluing up is a realization of the filling of the holes in the 't Hooft expansion at large N. The hope was to start from the free field theory and reconstruct the closed string theory, or at least its effective field theory description at large N (the long-sought *master field*). In fact, the more ambitious program goes beyond the bilinears single trace operators and the planar limit by exploiting the electrical network analogy and, after a partial gluing up into "skeleton diagrams", one discovers the moduli space of punctured Riemann surfaces.

1.4 Isometries *vs.* conformal symmetries

Let us take a brief look at the matching of symmetries that played a leading role in the original formulation of the Maldacena's conjecture. We will make $AdS_{d+1} - isometry/\mathcal{M}_d - conformal - symmetry$ manifest by considering an *ambient* d+2-dimensional flat space. Let us choose the metric $\eta_{AB} = diag(-1, 1, ..., 1, -1)$

$$ds^2 = -dX_0^2 + d\overrightarrow{X}^2 - dX_{d+1}^2. \tag{1.44}$$

with $A, B = 0, 1, ..., d, d+1$.

Isometries of AdS_{d+1}

As is well-known, anti-de Sitter space AdS_{d+1} can be viewed as (the universal covering of) the embedded hyperboloid

$$- X_0^2 - X_{d+1}^2 + \overrightarrow{X}^2 = -L^2. \tag{1.45}$$

The isometries of AdS, coordinate transformations that leave the metric invariant, are then the homogeneous ("Lorentz") part of the isometries of the embedding flat space, that is, SO(2,d) (for those connected to the identity). They are linearly realized with generators

$$J_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}. \tag{1.46}$$

Being maximally symmetric, AdS_{d+1} has as many isometries as the Minkowski space $\mathcal{M}_{d+1}$, which add up to $(d+1)(d+2)/2$. This is precisely the counting of the J_{AB}.

There are several popular choices of coordinates, some of them cover only a portion of the full hyperboloid as is the case of the Poincare patch: consider local coordinates

$$X^\mu = \frac{u}{L} x^\mu \qquad (\mu = 0, 1, ..., d-1) \tag{1.47a}$$

$$X^+ := X^d + X^{d+1} = \frac{u}{L} \tag{1.47b}$$

$$X^- := X^d - X^{d+1} = -\frac{u^2\, x^\mu x_\mu + L^4}{uL}\,, \tag{1.47c}$$

the induced metric on the hyperboloid is then

$$ds^2 = \frac{L^2}{u^2} du^2 + \frac{u^2}{L^2} dx^\mu dx_\mu. \tag{1.48}$$

This Poincare patch (we had it before after $u/L \to L/z$) covers only half of the whole hyperboloid.

Conformal symmetries of $\mathcal{M}_d$

Somewhat less known is the fact that if in the same ambient (Dirac's *conformal space* [37]) one considers the embedding

$$- X_0^2 - X_{d+1}^2 + \overrightarrow{X}^2 = 0 \tag{1.49}$$

of a null hypercone, then conformal transformations of d-dim Minkowski space act linearly on the homogeneous coordinates X^A. The corresponding generators are precisely the J_{AB} of above! Consider the local coordinates of above adapted to the cone, just take $L \to 0$ and keep fixed $u/L := \gamma \neq 0$,

$$X^\mu = \gamma\, x^\mu \qquad (\mu = 0, 1, ..., d-1) \tag{1.50a}$$
$$X^+ = \gamma \tag{1.50b}$$
$$X^- = -\gamma\, x^\mu x_\mu\,. \tag{1.50c}$$

The "induced metric" on the cone is degenerate and conformal to Minkowski

$$ds^2 = \gamma^2\, dx^\mu dx_\mu. \tag{1.51}$$

The degeneracy is easy to understand, all points in the equivalence class $X^A \sim \lambda X^A$ with $\lambda \neq 0$ are mapped to the same cone. The intersection with the hyperplane $X^+ = \gamma = 1$ results in Minkowski space

$$ds^2 = \gamma^2\, dx^\mu dx_\mu. \tag{1.52}$$

The isometries of the original metric that leave the cone invariant form the group $O(2,d)$. Let us see their action on the above obtained Minkowski space:

$$(X^\mu, X^+, X^-) \to (\Lambda^\mu_\nu X^\nu, X^+, X^-) \tag{1.53a}$$
$$\Rightarrow\ (x^\mu, \gamma) \to (\Lambda^\mu_\nu x^\nu, \gamma), \tag{1.53b}$$

$$(X^\mu, X^+, X^-) \to (X^\mu + \Lambda^\mu_+ X^+, X^+, X^-) \tag{1.53c}$$
$$\Rightarrow\ (x^\mu, \gamma) \to (x^\mu + a^\mu, \gamma), \tag{1.53d}$$

where we set $\Lambda^\mu_+ := a^\mu$. Since they leave γ invariant, they are also isometries of Minkowski space and form the Poincare group that is linearly realized on the coordinates x^μ as seen above. They exhaust $d(d+1)/2$ of the $(d+1)(d+2)/2 = d(d+1)/2 + 1 + d$ continuous symmetries. Consider further

$$(X^\mu, X^+, X^-) \to (X^\mu, X^+/\lambda, \lambda X^-) \tag{1.54a}$$
$$\Rightarrow\ (x^\mu, \gamma) \to (\lambda x^\mu, \gamma/\lambda). \tag{1.54b}$$

This symmetry, *dilatation*, is still linearly realized but re-scales the Minkowski metric by the constant factor λ^2. Finally, consider the discrete transformation that exchanges the $+$ and $-$ directions

$$(X^\mu, X^+, X^-) \to (X^\mu, X^-, X^+) \tag{1.55a}$$
$$\Rightarrow\ (x^\mu, \gamma) \to (-\frac{x^\mu}{x^\nu x_\nu}, -\gamma x^\nu x_\nu). \tag{1.55b}$$

This is nothing but the inversion in Minkowski space that produces an additional (conformal) factor $1/(x^\nu x_\nu)^2$ in the metric. It is not a continuous symmetry, but one can consider the sandwich of two inversions and any of the previous transformations. The only nontrivial result comes from $inversion-translation-inversion$ that produces the last d *special conformal transformations*(SCT) to complete the list of continuous transformations.

1.5 Fefferman-Graham construction

The *reconstruction theorem* by Fefferman and Graham [49] that was used in the derivation of the holographic anomaly naturally generalizes Dirac's *conformal space* notion that we have just met. It is based in the observation (Euclidean version!) that the group of conformal automorphisms of the n-sphere (Möbius group) is essentially the same as the group of Lorentz transformations of $(n+2)$-dim Minkowski space. The conformal structure of the n-sphere can be obtained by viewing the sphere as a cross-section of the forward light-cone in Minkowski space. Fefferman and Graham attempted to embed an arbitrary conformal n-manifold into a $(n + 2)$-dim Ricci-flat Lorentzian manifold. The outcome is that such a Lorentzian metric (ambient metric) can be constructed by formal power series for any Riemannian metric, to infinite order when n is odd and to order $n/2$ when n is even (the obstruction to the power expansion for even n is contained in the so-called *FG obstruction tensor*). This formal metric is a conformal invariant of the original conformal structure, so that its pseudo-Riemannian invariants automatically give conformal invariants of the original conformal structure.

An equivalent construction in turn, exploits the fact that the Möbius group can be pictured as the set of isometries of the hyperbolic metric on the interior of the unit ball $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$. In fact, Fefferman and Graham showed that given a compact Riemannian manifold (M, g), the problem of finding a Ricci-flat ambient Lorentz metric for the conformal structure $[g]$ is equivalent to that of finding an asymptotically hyperbolic Einstein metric on the interior of an $(n + 1)$-dim manifold-with-boundary that has M as boundary and $[g]$ as conformal infinity.

These two alternative constructions are in correspondence with the fact that Lorentz transformations in Dirac's conformal space can be viewed as isometries of the embedded hyperboloid or as conformal transformation of Minkowski space, which was a key ingredient in Maldacena's conjecture.

Chapter 2

The plane wave limit

This first corner of AdS/CFT correspondence is accessed via a large R-charge subsector of the $\mathcal{N} = 4$ super Yang-Mills gauge field theory[10] that is translated in a certain limit (Penrose) of the background string solution. The logic behind this limit goes as follows: a major difficulty is posed by the quantization of the IIB superstring in a RR-background; however, whatever the spectrum may be it should approach the spectrum of the string in flat spacetime as the radius of AdS blows up. Now, it turns out that one can do better and take a combined limit (Penrose) by zooming in on the neighborhood of a null geodesic resulting in a plane-wave with nonvanishing RR-flux. In contrast to the original $AdS_5 \times S^5$, in these (plane wave) backgrounds the exact quantization of strings is known. This allows for tests of the correspondence including genuine stringy properties. Although the standard lore is that stringy excitations correspond to long operators with generic dimension of order $\lambda^{1/4}$ that decouple from SUGRA correlators, this is not the case for the sector at hand where the operators carry a very large bare dimension (R-charge). The bulk SUGRA approximation for describing these operators is inadequate even at small curvature of AdS (corresponding to large λ). The subsector of gauge invariant operators dual to SUGRA and other string modes is identified and a successful matching of energy/scale-dimension is achieved at small λ/J^2 (J is the length of such operators).

The field theoretical properties of plane wave backgrounds become relevant. In particular the propagators, both bulk-to-bulk and bulk-to-boundary, for $AdS_5 \times S^5$ and for the plane wave arising in the Penrose limit should play a crucial role in understanding the degeneration of the holographic picture from a 4- to a 1-dimensional boundary. In spite of several attempts [75, 28, 80, 11, 27, 118, 89, 43], no complete understanding of holography in the plane wave geometry has emerged.

We will focus on the propagators, which are essential in implementing the AdS/CFT calculational prescription. The scalar propagator in the relevant plane wave has been constructed for generic mass values via direct mode summation in [92]. In addition, there have been observed structural similarities with the flat space propagator and their possible role in guessing the propagators for higher spin fields was stressed. The alternative route via the limiting behavior of the $AdS_5 \times S^5$ propagator was taken in [42] for the conformally coupled scalar.

We want to study the propagators in the plane wave background along the line of the Schwinger-DeWitt construction [30, 31] introduced in the previous chapter. This technique is based on an expansion near the light-cone. It has been successfully applied to the propagator construction in various specific backgrounds as well as to issues related to near light cone and anomaly problems in generic backgrounds. It lies at the heart of most regularization techniques of QFT in curved spaces (see, e.g. [14]). Our aim here is to explain the above mentioned structural similarities to the flat space case by the termination of the underlying WKB expansion and to make progress in the explicit construction for higher spin cases. We will also explore the alternative approach to derive the plane wave propagators as a limiting case of propagators in spaces which in a Penrose limit yield the plane wave under discussion. For this we relate our results to information on propagators in Einstein Static Universe (ESU) available in the literature.
As we will see, the Penrose limit is nothing but an expansion near a null geodesic; the SD-technique, being an expansion near the light-cone, is the natural tool to address this limit.

2.1 Penrose limit

The particular plane wave background to be considered is the conformally flat one obtained as a Penrose limit of $AdS_5 \times S^5$ [15, 16, 10] with equal radii, although at some stages the results can be adapted to other dimensions by just varying the number of transverse directions $\vec{x}$. The line element is given by

$$ds^2 = 2dudv - \vec{x}^2 du^2 + d\vec{x}^2. \tag{2.1}$$

As noticed by Penrose [101], this limit is nothing but an adaptation to pseudo-Riemannian manifolds of the standard procedure of taking tangent space limit, the main difference being that when applied to a null geodesic it results in a curved spacetime, namely a plane wave. One could as well end up with flat space, but the generic situation is a plane wave. It is this zooming into the neighborhood of the null geodesic what gives the Penrose limit a local character.

Usually electromagnetic and gravitational plane wave solutions of general relativity are discussed in the context of the linearized theory. However, the plane waves we consider here are solutions of the full nonlinear Einstein equations.

Recently, Penrose limits of a whole variety of space-times has been thoroughly studied (see, e.g. [75] and reference[13] therein). The particular plane wave metric (2.1) together with a RR-flux correspond to a maximally supersymmetric solution of Type II-B SUGRA, as first found in [15]. This very same Type II-B SUGRA background can also be obtained as a Penrose limit of the less supersymmetric $AdS_5 \times T^{1,1}$ [73, 54, 100], and surely from many other backgrounds. Now, as far as one is interested only in the metric, the spacetime with the same Penrose limit, which ought to be considered the conceptually simplest one, is the Einstein Static Universe ESU_{10}. In parts of the following discussion we will benefit from this fact.

2.1.1 Anti-de Sitter x Sphere

Let us start with AdS_{p+1} in global coordinates and with the $(q+1)$-sphere parametrized in terms of a $(q-1)$-sphere (a is the common radius of AdS and the sphere)

$$\begin{aligned} ds^2_{AdS_{p+1}\times S^{q+1}} =& a^2(-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho\, d\Omega^2_{p-1} \\ &+ \cos^2\theta d\psi^2 + d\theta^2 + \sin^2\theta\, d\Omega'^2_{q-1}). \end{aligned} \tag{2.2}$$

Now one focuses on the immediate neighborhood of a null geodesic that remains at the center of AdS_{p+1} while it wraps an equator of S^{q+1}, say $t = \psi = u$ (affine parameter along the null ray) and $\rho = \theta = 0$. Introducing local coordinates

$$t = u \qquad \psi = u + \frac{v}{a^2} \qquad \rho = \frac{x}{a} \qquad \theta = \frac{y}{a} \tag{2.3}$$

and expanding in inverse powers of the radius, one gets

$$ds^2_{AdS_{p+1}\times S^{q+1}} = 2dudv - (x^2+y^2)du^2 + dx^2 + x^2 d\Omega^2_{p-1} + dy^2 + y^2 d\Omega'^2_{q-1} + O(a^{-2}) \tag{2.4}$$

so that in the limit $a \to \infty$, blowing up the neighborhood and collecting the flat transverse directions in $\vec{x}$, one ends up with the plane wave metric (2.1).

2.1.2 Einstein Static Universe

We want to stress that the same plane wave results from ESU_n, topologically $R \times S^{n-1}$. Let us conveniently parametrize the $(n-1)$-sphere in terms of a

$(n-3)$-sphere

$$ds^2_{ESU_n} = a^2(-dt^2 + d\Omega^2_{n-1}) \tag{2.5a}$$

$$= a^2\left(-dt^2 + d\alpha^2 + \cos^2\alpha\, d\beta^2 + \sin^2\alpha\, d\Omega^2_{n-3}\right). \tag{2.5b}$$

The null geodesic will be the one given by $t = \beta = u$ (affine parameter along the null ray) and $\alpha = 0$, and the local coordinates in its neighborhood

$$t = u \qquad \beta = u + \frac{v}{a^2} \qquad \alpha = \frac{r}{a}. \tag{2.6}$$

Then, expanding the line element

$$ds^2_{ESU_n} = 2dudv - r^2du^2 + dr^2 + r^2d\Omega^2_{n-3} + O(a^{-2}) \tag{2.7}$$

and letting $a \to \infty$ one gets the plane wave metric (2.1).

That both Penrose limits give the same metric can be easily understood if one remembers that there is a conformal map that allows for a Penrose diagram for $AdS_{p+1} \times S^{q+1}$. Defining $\tan\vartheta \equiv \sinh\rho$ in (2.2), one obtains that both metrics are related by[1]

$$ds^2_{AdS_{p+1}\times S^{q+1}} = \frac{1}{\cos^2\vartheta} ds^2_{ESU_{p+q+2}}. \tag{2.8}$$

Now, in the local coordinates (2.3) near the null geodesic at the center of Ad_{p+1} we have $\vartheta = \frac{x}{a} + O(a^{-3})$ and the conformal factor $\cos^{\pm 2}\vartheta = 1 + O(a^{-2})$, therefore up to $O(a^{-2})$ both metric are equivalent, i.e. the RHS of (2.4) holds for both backgrounds. Consequently, in the limit $a \to \infty$ the resulting metrics coincide. This is again a manisfestation of the inherent locality of the Penrose limit.

2.2 Propagators in the plane wave

The scalar Feynman propagator in the plane wave background (2.1) has already been obtained by explicit summation of the eigenmodes in recent works [75, 92]. Here we will treat it differently using the Schwinger-DeWitt technique which admits a readily generalization to the spinor and vector fields.

[1] Obviously, there is an obstruction to this argument if the null geodesic, on which one focuses in the Penrose limit, reaches the boundary of $AdS \times S$ where the conformal factor becomes singular. It is precisely in this situation when the null geodesic is totally contained in AdS and the Penrose limit of $AdS \times S$ gives just Minkowski space.

2.2.1 Scalar propagator

Now we are in position to apply this construction to the plane wave background. From the geodetic interval between two generic points (see appendix A.3) one obtains the Van Vleck-Morette determinant. The important ingredients are

$$g(x) = -1 \tag{2.9a}$$
$$\frac{\sigma(x,x')}{u-u'} = [v - v' + \frac{\vec{x}^2 + \vec{x}'^2}{2} \cot{(u-u')} - \vec{x}\cdot\vec{x}' \csc{(u-u')} \tag{2.9b}$$
$$\triangle(x,x') = \left[\frac{u-u'}{\sin{(u-u')}}\right]^{d-2} . \tag{2.9c}$$

With this at hand, one can check that $\triangle^{\frac{1}{2}}(x,\cdot)$ is harmonic, i.e. $\Box\,\triangle^{\frac{1}{2}}(x,\cdot) = 0$, because $\triangle(x,\cdot)$ is a function only of u and the inverse metric has $g^{uu} = 0$, so that the recurrence relations (1.8) are satisfied by $a_n(x,x') = \delta_{0,n}$. Thus the only non-zero coefficient in the expansion (1.7) is just the first one. That is why we say that the scalar Schwinger-DeWitt kernel in the plane wave background is leading-WKB exact. The kernel and the Green's function, after performing the proper time integral, are then given by[2]

$$K(x,x' \mid s) = \frac{i\triangle^{\frac{1}{2}}}{(4\pi i s)^{\frac{d}{2}}}\; e^{i\sigma/2s} \tag{2.10a}$$
$$G(x,x') = \frac{-i\pi\triangle^{\frac{1}{2}}}{(4\pi i)^{\frac{d}{2}}} \left(\frac{2m^2}{\sigma}\right)^{\frac{d-2}{4}} H^{(2)}_{\frac{d}{2}-1}\left(\left[-2m^2\sigma\right]^{\frac{1}{2}}\right). \tag{2.10b}$$

One can get Minkowski space by rescaling $u \to \mu u, v \to v/\mu$ and letting μ go to zero. The effect of this in (2.9, 2.10) is $\triangle \to 1$ and $2\sigma \to 2(u - u')(v-v') + (\vec{x}-\vec{x}')^2$ and

$$K_M(x,x' \mid s) = \frac{i}{(4\pi i s)^{\frac{d}{2}}}\; e^{i\sigma/2s} \tag{2.11a}$$
$$G_M(x,x') = \frac{-i\pi}{(4\pi i)^{\frac{d}{2}}} \left(\frac{2m^2}{\sigma}\right)^{\frac{d-2}{4}} H^{(2)}_{\frac{d}{2}-1}\left(\left[-2m^2\sigma\right]^{\frac{1}{2}}\right). \tag{2.11b}$$

The difference between the two results, apart from the fact that the geodetic interval is of course different, is that for the plane wave we get a nontrivial

[2] As usually, the Feynman Green's function should be understood as the boundary value of a function which is analytic in the upper-half σ plane, so that in fact $\sigma + i0^+$ is meant in what follows.

Van Vleck-Morette determinant. The analogy with the Minkowski case observed in [92] is thus fully explained by the leading-WKB exactness of the plane wave background. The coincidence limit of our results, where the coefficients become local functions of curvature invariants [30, 25], is consistent with the fact that for the plane wave background there are no non-vanishing curvature invariants [72].

Finally, for the massless scalar one can take the massless limit in both expressions to get[3]

$$D(x,x') = \frac{-i\,\Gamma(d/2-1)}{2(2\pi)^{d/2}} \triangle^{\frac{1}{2}} \left(\frac{1}{\sigma}\right)^{\frac{d-2}{2}} = \frac{-i\,\Gamma(d/2-1)}{2(2\pi)^{d/2}} \left(\frac{1}{\Phi}\right)^{\frac{d-2}{2}} \tag{2.12}$$

$$D_M(x,x') = \frac{-i\,\Gamma(d/2-1)}{2(2\pi)^{d/2}} \left(\frac{1}{\sigma}\right)^{\frac{d-2}{2}}. \tag{2.13}$$

2.2.2 Spinor field: leading-WKB exactness

One might guess that the similarity with Minkowski space kernel and Green's function still holds for higher spin fields. Now we will turn our attention to the spin$-\frac{1}{2}$ case.

The spinor Green's function is now a bi-spinor which satisfies the Dirac equation with a point-like source

$$[\gamma^\mu(x)\nabla_\mu + m]\,\mathbb{S}(x,x') = \delta(x,x')\mathbb{I}, \tag{2.14}$$

where $\gamma^\mu(x)$ are the curved space Dirac matrices and ∇_μ is the spinor covariant derivative (see appendix A.2).

To apply the Schwinger-DeWitt technique one introduces an auxiliary bi-spinor $\mathbb{G}(x,x')$ defined by

$$\mathbb{S}(x,x') = (\gamma^\mu(x)\nabla_\mu - m)\,\mathbb{G}(x,x') \tag{2.15}$$

to obtain the following wave equation for $\mathbb{G}(x,x')$

$$(\Box - \frac{R}{4} - m^2)\mathbb{G}(x,x') = \delta(x,x')\mathbb{I}, \tag{2.16}$$

where R is the scalar curvature.

Now one can apply the Schwinger-DeWitt construction as in the scalar case, but this time the auxiliary Green's function $\mathbb{G}(x,x')$, the kernel $\mathbb{K}(x,x' \mid s)$ as well as the HaMiDeW coefficients $\mathbb{A}_n(x,x')$ are bi-spinors and the recurrence relations (1.8) involve the spinor covariant derivative. One starts

[3] The geometrical meaning of the quantity Φ is explained in appendix A.1.

with $\mathbb{A}_0(x,x') = \mathbb{U}(x,x')$, the spinor parallel transporter along the geodesic connecting the two points (appendix A.2).

For the plane wave background (2.1) one can check that again the recurrence relations are satisfied by $\mathbb{A}_n(x,x') = \delta_{n,0}\mathbb{U}(x,x')$, the reason being that $\triangle^{\frac{1}{2}}(x,\cdot)\mathbb{U}(x,\cdot)$ is harmonic, with respect to the spinor D'Alembertian (i.e. the square of the Dirac operator).

Therefore, the spinor kernel and the spinor auxiliary Green's function are leading-WKB exact and can be written in terms of the respective scalar quantities

$$\mathbb{K}(x,x' \mid s) = K(x,x' \mid s)\,\mathbb{U}(x,x'), \tag{2.17a}$$

$$\mathbb{G}(x,x') = G(x,x')\,\mathbb{U}(x,x'). \tag{2.17b}$$

Flat space results can also be recovered as in the scalar case, taking into account that in the limit $\mathbb{U}(x,x') \to \mathbb{I}$. The similarity with flat space result is still present, the only additional nontrivial piece being the spinor geodesic parallel transporter, and is better appreciated in terms of the kernel and the auxiliary Green's function, so we do not show the explicit expression for $\mathbb{S}$.

2.2.3 Vector field: next-to-leading-WKB exactness

Let us examine the Maxwell field. Now we have additional complications due to the gauge freedom, so we add a gauge fixing term $-\frac{1}{2\xi}(\nabla_\mu A^\mu)^2$ in the action to get an invertible differential operator

$$\Big[g_{\mu\rho}\Box - R_{\mu\rho} - (1-\xi^{-1})\nabla_\mu\nabla_\rho\Big]\, G^\rho_{\nu'}(x,x') = \delta(x,x')g_{\mu\nu'}(x), \tag{2.18}$$

where the Ricci tensor $R_{\mu\nu}$ arises from the commutator of the covariant derivatives. Its only non-vanishing component in the plane wave geometry is $R_{uu} = d-2$. This can be easily obtained from the Christoffel symbols (A.22).

In the Feynman gauge $\xi = 1$, corresponding to a "minimal" wave operator in the sense of Barvinsky and Vilkovisky [9], one can work out a Schwinger-DeWitt construction[4] and this time we have to deal with bi-vectors. The recurrences are slightly changed to

$$(n+1)\,a_{n+1\,\mu\nu'} + \partial^\rho\sigma\;\nabla_\rho a_{n+1\,\mu\nu'} = \triangle^{-\frac{1}{2}}\,\Box\,(\triangle^{\frac{1}{2}}a_{n\,\mu\nu'}) - R_\mu^{\ \rho}a_{n\,\rho\nu'}, \tag{2.19}$$

[4] As shown by R. Endo [48], having the vector kernel in Feynman gauge one can easily go to any other covariant gauge.

and the chain of HaMiDeW coefficients starts with the vector geodesic parallel transporter $a_{0\,\mu\nu'}(x,x') = P_{\mu\nu'}(x,x')$ (appendix A.3) .

What one can show in this case is that the recurrences are solved by

$$a_{n\,\mu\nu'}(x,x') = \begin{cases} P_{\mu\nu'}(x,x'), & n=0, \\ (2-d)\delta_{u\mu}\delta_{u'\nu'}\frac{\tan\frac{u-u'}{2}}{\frac{u-u'}{2}}, & n=1, \\ 0, & n\geq 2. \end{cases} \tag{2.20}$$

The vector kernel is then

$$K^{\mu}_{\ \nu'}(x,x'\mid s) = \frac{i\triangle^{\frac{1}{2}}}{(4\pi i s)^{\frac{d}{2}}}\, e^{i\sigma/2s}\left(\delta^{\mu}_{\ \rho} - is\frac{\tan\frac{u-u'}{2}}{\frac{u-u'}{2}}R^{\mu}_{\ \rho}\right)P^{\rho}_{\ \nu'}(x,x'), \tag{2.21}$$

and the Green's function can be written in terms of the massless scalar Green's function as

$$G^{\mu}_{\ \nu'}(x,x') = \left(D(x,x')\,\delta^{\mu}_{\ \rho} - \frac{1}{4\pi\cos^2\frac{u-u'}{2}}Q(x,x')R^{\mu}_{\ \rho}\right)P^{\rho}_{\ \nu'}(x,x'), \tag{2.22}$$

where the functional dependence of Q on $u-u'$ and σ is precisely the same as in D but in two dimensions less (see 2.12), i.e.

$$\begin{aligned} Q(x,x') &= \frac{-i\,\Gamma(d/2-2)}{2(2\pi)^{d/2-1}}\left[\frac{u-u'}{\sin{(u-u')}}\right]^{\frac{d-4}{2}}\left(\frac{1}{\sigma}\right)^{\frac{d-4}{2}} \\ &= \frac{-i\,\Gamma(d/2-2)}{2(2\pi)^{d/2-1}}\left(\frac{1}{\Phi}\right)^{\frac{d-4}{2}} \end{aligned} \tag{2.23}$$

That we should not expect leading-WKB exactness this time can be seen by examining the coincidence limit $x \to x'$, where general results [30, 25] are available. In particular for the plane wave under consideration, one must have $a_{1\,\mu\nu}(x,x) = -R_{\mu\nu}(x)$, and this can be readily checked in (2.21) remembering that the coincidence limit of the vector parallel transporter is just the metric tensor, $P_{\mu\nu}(x,x) = g_{\mu\nu}(x)$.

After all, we obtained the minimal departure: next-to-leading WKB-exactness. This time, the similarity with flat space results is still present although obscured by an additional term. The flat space limit can be taken as in the preceding two cases, this time the vector parallel transporter goes to the metric tensor and the a_1 coefficient together with the Ricci tensor go to zero to end up with the usual Minkowski space results in Feynman gauge, that is, the metric tensor times the massless scalar propagator.

2.3 Resummation and Penrose limit in ESU

One can take advantage of the fact that the ESU has the same Penrose limit and try to take the limit directly in the Green's functions for ESU where some results are available in the literature. How does the limit work directly on the propagators is apparently not easy to see in the mode summation form. But, after a resummation things might get clearer. The resummation we will explore is the one implicit in the so called "duality spectrum-geodesics". That is, the kernel can be written either as an eigenfunction expansion or as a "sum over classical paths" [20].

2.3.1 Scalar field in ESU

The resummation is implicit in the following form for the scalar Green's function in ESU_4 obtained by Dowker and Critchley [46](see also [20]) based on the Schwinger-DeWitt technique. The Schwinger-DeWitt kernel, as well as the heat kernel, factorizes for a product space and since ESU_4 is nothing but $R \times S^3$ one just needs the free kernel for the time direction $K_R(t, t' \mid s) = \frac{i}{(4\pi i s)^{1/2}}\ e^{-ia^2(t-t')^2/4s}$ and the kernel for the 3-sphere. The whole problem reduces to finding K_{S^3} and one can show that the 3-sphere is leading-WKB exact[5]. The only complication is that, due to the compactness of the sphere, one has multiple geodesics in addition to the direct one so that one has to include indirect geodesic contributions which restore the periodicity on the sphere

$$K_{S^3}(q, q' \mid s) = \sum_{n=-\infty}^{\infty} K^0_{S^3}(\chi + 2\pi n a \mid s), \tag{2.24}$$

where χ is the length of the shortest arc connecting the two points q, q' on the 3-sphere and

$$K^0_{S^3}(\chi \mid s) = \frac{1}{(4\pi i s)^{\frac{3}{2}}} \triangle^{\frac{1}{2}}\ e^{i\chi^2/4s + is/a^2}, \tag{2.25}$$

with the Van Vleck-Morette determinant for the sphere resulting in $\triangle^{\frac{1}{2}} = \frac{\chi/a}{\sin(\chi/a)}$. The corresponding Green's function for ESU_4 is also given by direct plus indirect geodesic contributions

$$G_{ESU_4}(x, x') = \sum_{n=-\infty}^{\infty} G^0_{ESU_4}(t - t', \chi + 2\pi n a), \tag{2.26}$$

[5] The odd-dimensional spheres turn out to be WKB exact after factorizing a constant phase involving the scalar curvature. This phase can in turn be absorbed in the definition of the differential operator and its effect in the Green's function is just a shift in the mass. This must be taken into account when comparing the results in [46] with those in [20].

where

$$G^0_{ESU_4}(t-t',\chi) = \frac{i\triangle^{\frac{1}{2}}}{8\pi}\left(\frac{m^2-a^{-2}}{2\sigma}\right)^{\frac{1}{2}} H_1^{(2)}\left(\left[-2(m^2-a^{-2})\sigma\right]^{\frac{1}{2}}\right) \quad (2.27)$$

and the direct geodetic interval is $\sigma = \frac{-a^2(t-t')^2+\chi^2}{2}$. Now one can take the Penrose limit (see appendix A.1), and the result is that only the direct geodesic contribution survives the limit to give precisely the plane wave results (2.10) for $d = 4$. The indirect geodesic terms become rapidly oscillating or exponentially decaying. Therefore they vanish as a distribution for $a \to \infty$. This is similar to the flat space limit of ESU_4 discussed in [46].

This construction can be generalized to higher dimensional ESU_n. For odd-dimensional spheres the Schwinger-DeWitt kernel is WKB exact [20] and for even-dimensional spheres one only has an asymptotic expansion, but in all cases the only term that survives the Penrose limit is the first coefficient in the direct geodesic contribution and this can be seen in the asymptotic expansion, all other terms are suppressed by inverse powers of the radius or are rapidly oscillating.

2.3.2 Spinor field in ESU

For ESU_4, Altaie and Dowker [45] obtained the spinor S-D kernel and the spinor Green's function. To our purposes it suffices to take a look at the spinor S-D kernel, which due to the compactness of the 3-sphere is again a sum over all geodesics connecting the two points, with the direct term

$$\mathbb{K}^0_{ESU_4}(x,x'\mid s) = \frac{i}{(4\pi i s)^2}\triangle^{\frac{1}{2}}\, e^{i(\chi^2-a^2(t-t')^2)/4s}\left(1-\frac{is\tan(\chi/a)}{a\chi}\right)\mathbb{U}(x,x'). \quad (2.28)$$

In the Penrose limit (see appendix A.1) one gets again the same behavior, i.e. only the first coefficient in the direct geodesic term survive and everything else is suppressed as in the scalar case.

One can follow this construction using the spinor kernel for the higher-dimensional spheres, already calculated by Camporesi [22], and one gets again agreement with our previous results from direct computation in the plane wave background. In all cases, the relevant information is contained in the S-D asymptotic (S-D stands either for Schwinger-DeWitt or for short-distance), the rest is just scaled away in the Penrose limit. This is precisely the resummation we were looking for.

2.4 Penrose limit of AdS x S propagators

The key tool for the previous results was the resummation implicit in the Schwinger-DeWitt asymptotics. So, this could be the recipe to obtain the limiting values of the Green's functions.

Let us first explore for some cases where closed results are still available before drawing conclusions for the generic case.

2.4.1 Three dimensions and equal radii

The kernel for AdS_3 can be obtained from the heat kernel for H^3 [20] by analytic continuation, for spacelike intervals both must coincide. For timelike intervals in AdS_3, which are the relevant ones for the Penrose limit since the null geodesic is always spacelike on the sphere so that it must be timelike in AdS_3 [43], one has the continuation

$$K_{AdS_3}(\zeta \mid s) = \frac{i}{(4\pi i s)^{\frac{3}{2}}} \triangle^{\frac{1}{2}}\, e^{i\zeta^2/4s - is/a^2}, \tag{2.29}$$

where $\frac{\zeta^2}{2}$ is the geodetic interval and $\triangle^{\frac{1}{2}} = \frac{\zeta/a}{\sinh(\zeta/a)}$. This kernel gives the standard Green's function corresponding to Dirichlet boundary conditions, which can be expressed in terms of hypergeometric functions (see, e.g. [32]).

This allows us to write the exact kernel for $AdS_3 \times S^3$, given again by a sum to produce the periodicity on the 3-sphere, with the direct geodesic term

$$K^0_{AdS_3\times S^3}(\zeta, \chi \mid s) = \frac{i}{(4\pi i s)^3}\, \frac{\zeta/a}{\sinh(\zeta/a)}\, \frac{\chi/a}{\sin(\chi/a)}\, e^{i(\zeta^2+\chi^2)/4s}. \tag{2.30}$$

In the Penrose limit, the indirect geodesic contributions are suppressed, $\zeta^2 + \chi^2 \to 2\sigma$, $\frac{\zeta/a}{\sinh(\zeta/a)}$ and $\frac{\chi/a}{\sin(\chi/a)}$ both $\to \frac{u-u'}{\sin(u-u')}$ and one recovers the plane wave results (2.10) for $d = 6$.

2.4.2 Conformal coupling

In $AdS_{p+1} \times S^{q+1}$ with equal radii which is then conformally flat, for the conformally coupled scalar one gets a powerlike function in the total chordal distance when mapping to the massless scalar in flat space. This can also be obtained by a direct summation of the harmonics on the sphere as shown in [42][6]. The limit agrees with the plane wave result for the massless case

[6] In fact, in [42] one also obtains a powerlike function for a particular mass in the case where the radii are different, when no conformal map to flat space is possible. We have also managed to reproduce this result using the kernels and a nice relation to the conformal situation was found in terms of a contour integral, as we will see.

where the Green's function is an inverse power of Φ (see appendixA.1).

We can accommodate this case in our scheme. Start with $AdS^3 \times S^3$ and use the whole kernel, that is

$$K_{AdS_3\times S^3}(\zeta,\chi \mid s) = \sum_{n=-\infty}^{\infty} K^0_{AdS_3\times S^3}(\zeta,\chi+2\pi n a \mid s). \tag{2.31}$$

For the Weyl invariant scalar, corresponding in this case to $m = 0$ one can take the proper time integral and perform the sum of all direct and indirect geodesics to get

$$\begin{aligned} G_{AdS_3\times S^3}(\zeta,\chi \mid s) &\sim \frac{1}{[\cos(\chi/a)-\cos(\zeta/a)]^2} \\ &\sim \frac{1}{[\text{total squared chordal distance}]^2}. \end{aligned} \tag{2.32}$$

Now one can take the Penrose limit at any of the two stages, in this final expression or first in the kernel.

The Weyl coupling case for higher dimension can now be generated by the "intertwining" technique [20]. Applied to the kernel one obtains a kernel that produces the desired power in the total chordal distance for the Green's function. Alternative, the intertwining can be applied directly to the Green's function. The intertwining technique reduces basically to the fact when one can obtain the kernel or the Green's function for the conformally coupled scalar by just taking derivatives with respect to the chordal distances. One can start with $AdS_3 \times S^1$, taking partial derivative with respect the chordal distance in AdS one gets the results for the product space with two dimensions higher in AdS and taking partial derivative with respect the chordal distance in the sphere one gets the results for the product space with two dimensions higher in the sphere[7]. In this way one generates the higher negative powers in the total distance for the conformally coupled scalar [42]. Again, in the Penrose limit only the leading term of the direct geodesic survives the limit.

[7]To cover he whole range of dimensions for the product space $AdS \times S$ ([odd,odd], [odd,even], [even,odd] and [even,even]) one needs in addition the S^2 and AdS_2 results, see [20].

2.4.3 Miscellany: non-conformally flat background

Let us consider the Euclidean version $H^3 \times S^3$ with different radii (say, a and αa). Up to normalization factors, the kernel $K^*_{H^3\times S^3}(\zeta, \chi \mid s)$ is given by

$$\frac{1}{s^3}\,\frac{\zeta/a}{\sinh(\zeta/a)}\,\frac{1}{\sin(\chi/\alpha a)}\;e^{\,is/a^2(1/\alpha^2-1)}\sum_{n=-\infty}^{\infty}(\chi/\alpha a\,+\,2\pi n)\;e^{\,i[\zeta^2+(\chi+2\pi n\alpha a)^2]/4s}. \tag{2.33}$$

The kernel this time has a remaining "s" dependent term in the exponent that can only be eliminated by a special value of the mass, $m_*^2 = \frac{1}{a^2}(1-\frac{1}{\alpha^2})$. This value of the mass is precisely the one used in [42] to get a closed expression for the Green's function. What one can see is that for this value one can perform the integral to get for the Green's function

$$\frac{\zeta/a}{\sinh(\zeta/a)}\,\frac{1}{\sin(\chi/\alpha a)}\sum_{n=-\infty}^{\infty}\frac{\chi/\alpha a\,+\,2\pi n}{[\zeta^2+(\chi+2\pi n\alpha a)^2]^2} \tag{2.34}$$

and the resulting series can be exactly computed with the aid of a Poisson summation (after taking partial derivative with respect to x to relate both sums)

$$\sum_{n=-\infty}^{\infty}\frac{y}{y^2+(x+n)^2} = \frac{1-e^{-4\pi y}}{1-2\cos(2\pi x)e^{-2\pi y}+e^{-4\pi y}} \tag{2.35}$$

to get

$$G^*_{H^3\times S^3}(\zeta,\chi) \sim \frac{\sinh(\zeta/\alpha a)}{\sinh(\zeta/a)}\,\frac{1}{[\cosh(\zeta/\alpha a)-\cos(\chi/\alpha a)]^2}. \tag{2.36}$$

Now, when the two radii are equal ($\alpha = 1$) one gets of course the conformally coupled scalar in the conformally flat background. The conformally flat case is periodic on the arc in the sphere χ with period $2\pi a$ while the period in the non-conformally flat is $2\pi\alpha a$. The interesting thing to notice is that the kernels as well as the Green's functions are related by a contour integral due to Sommerfeld that restores the appropriate periodicity (see, e.g. [44]). This can be explicitly checked for the special mass above [90]

$$G^*_{H^3\times S^3}(\zeta,\chi) = G_{H^3\times S^3}(\zeta,\chi) + \frac{i}{4\pi\alpha}\int_\Gamma dw\,\cot\left(\frac{w}{2\alpha}\right)\,G_{H^3\times S^3}(\zeta,\chi+wa), \tag{2.37}$$

where the contour Γ consists of two vertical lines from $(-\pi+i\infty)$ to $(-\pi-i\infty)$ and from $(\pi-i\infty)$ to $(\pi+i\infty)$ and intersecting the real axis between the poles of $\cot(\frac{w}{2\alpha})$: $-2\pi\alpha, 0$ and $0, 2\pi\alpha$, respectively.

This very same formula gives the heat kernel for the cone starting with the one for the plane. This is a remarkable property since equal radii corresponds to a conformally flat situation and different radii is conformal to a singular background with a tip, a conical singularity.

2.5 Discussion

Our main result is the explicit construction of the spinor and vector propagator in the plane wave background (2.1) arising in a Penrose limit from $AdS_{p+1} \times S^{q+1}$. The spinor propagator is constructed for generic mass values, the vector propagator for massless gauge fields in Feynman gauge.

The construction was based on the Schwinger-DeWitt technique. In general backgrounds via this method one gets only an asymptotic WKB series with respect to the approach to the light cone. Global issues for the propagators remain open. In fact, we know that the correct propagator beyond the caustic ($u - u' = \pi$) should pick up an additional phase (Maslov index $e^{-i\pi/4}$) for each transverse coordinate, each time one goes beyond the cautics. This is the analog of the full Feynman-Soriau formula for the harmonic oscillator (see e.g. [38]). Now, the Penrose construction breaks down whenever caustics (conjugate points) come into game, so that the above observation would be helpful in sewing together different patches where the Penrose limit is well defined. Fortunatelly, for the case under consideration there are eight transverse oscillators and the issue of the Maslov index is irrelevant.

For the background under discussion we could show that the series terminates with its leading or next-to-leading term. This then strongly suggests that the resulting expressions are indeed the correct propagators. We checked this by reproducing the scalar propagator already constructed in the literature by different methods. In this check we also explained by the WKB exactness the structural similarity with the flat space scalar propagator pointed out in [92]. The propagator in both cases is given by the same function of the respective geodesic distances up to an additional factor generated by the nontrivial Van Vleck-Morette determinant of the plane wave background. This ordinary determinant for the plane wave can be shown to be equal to the functional determinant of the quadratic fluctuations in the path integral formalism [98], where leading-WKB-exactness amounts to the exactness of the Gaussian approximation for the path integral.

Besides the explicit construction in the plane wave geometry, we made some observations on the relation between both propagators and kernels to those in spaces from which the plane wave arises in a Penrose limit. After remarking that the plane wave under study can also be obtained from ESU, we discussed the limit starting from known explicit expressions both for the scalars and spinors in ESU. It turned out that only the leading term in the direct geodesic contribution survives the limit. This nicely corresponds with the local nature of the Penrose limit. This picture was supported by similar observations starting from some special $AdS \times S$ cases. In addition for the $AdS \times S$ propagators we were able to explain the distinguished role of a special mass value for non-Weyl invariant coupling in spaces with different radii for AdS_{p+1} and S^{q+1} [42]. Just for this value in the exponent of the Schwinger-DeWitt kernel the term linear in the proper time cancels and one can explicitly perform the sum. A contour integral relates the kernels and propagators for this special non-conformally flat (conformal to a spacetime with a conical singularity) case to a conformally flat spacetime.

Further study should clarify whether there is a general theorem behind. Given a generic plane wave arising in a Penrose limit from some other spacetime, does then the information on the first few coefficients of the direct geodesic contribution in the original spacetime always contain enough information to get the plane wave propagators? Is the WKB-exactness a generic feature of the Penrose limit? We believe the answer is in the affirmative. The generic plane wave after a Penrose limit can be casted in Brinkmann coordinates

$$ds^2 = 2dudv - A_{ab}(u)x^a x^b du^2 + d\vec{x}^2. \tag{2.38}$$

We can repeat the steps undertaken before, the only technical difficulty is obtaining the classical action for the resulting coupled system of time-dependent harmonic oscillators (see the discussion above A.23) in order to compute the geodetic interval and the Van Vleck-Morette determinant. The caustic structure (determined by the zeros of the VVM-determinant) will of course be much more complicated and explicit expressions will in general not be available. Still, the WKB-exactness would be a feature of the propagator since, at any rate, the geodetic interval will be quadratic in the transverse variables x^a and the VVM-determinant will again be only a function of u so that the recursion relations will be satisfied as before.

Finally, let us stress that the semiclassical nature of the limit is the central feature here. On the gauge side, it is related to the notion of large quantum numbers (Bohr's correspondence principle in Quantum Mechanics). On the AdS side, we have unraveled the semiclassical exactness of field theoretic propagators. In addition, it was found out that the strings are indeed semiclassical in the sense that the spectrum is reproduced by a 1-loop sigma-model computation, i.e. quadratic fluctuations, around certain solitonic configuration [62, 112]. This semiclassical expansion is more general than starting directly with the plane-wave background and allows for tests of the energy/dimension relation beyond the leading 1-loop order. It also opened new avenues regarding a whole "fauna" of classical and semiclassical string configurations with large quantum numbers.

Chapter 3

Towards AdS duals of free CFTs

As already mentioned, duality between strongly coupled SYM in the large N limit and weakly coupled SUGRA is the form of Maldacena's conjecture where many interesting results have been obtained and most of the tests have been performed. However, much less is known about the bulk dual to perturbative gauge theories or even to free theory. This constitutes the second corner of the correspondence that we want to address. It has been conjectured that the dual bulk theory to large N free gauge theory is a HS theory of Fradkin-Vasiliev type [115, 107]. A simpler scenario for testing these ideas has been proposed by Klebanov and Polyakov [77], concerning the bulk dual of the critical O(N) vector model. Vector models have always been useful in understanding features that arise in the more complicated case of gauge theories. Here one can use the vast experience in large-N limit of O(N) vector models to reconstruct the bulk theory.

An analogous attempt has been started by Gopakumar [55, 56, 57] for the singlet bilinear sector of the gauge theory involving only the scalars Φ^I; but despite the initial success in casting two and three point function of scalar bilinears into AdS amplitudes as we saw in the preliminary chapter, four point correlators have remained a challenge [1]. The technical difficulty has been that one should include the whole tower of HS fields, dual to the HS conserved currents of the CFT, in the exchange graphs since the OPE structure of the free field correlators involve the whole tower of conserved currents. A bulk theory consistently truncated to massless fields should be reflected somehow in a closure of the corresponding dual sector of CFT oper-

[1]See however [2] for recent progress and for a large set of references to proposals for string dual of free large N gauge theories. These include string bit models, tensionless strings, strings in highly curved space, etc.

ators. We study the free four point function of the scalar bilinear by means of a conformal partial wave expansion as alternative "gluing up" and reorganize it so as to involve only the minimal twist sector [2], i.e. the conserved HS currents and their descendants, as required by the correspondence. The fusion coefficients are analytically checked to factorize in terms of two- and three-point function coefficients. A comparison with the correlators at the IR fixed point can be made by means of the amputation procedure that realizes the Legendre transformation connecting the two conformal theories at leading order in the large-N expansion, in close analogy with the effect of double-trace deformations in the gauge theory (see e.g. [64] and references therein). We pursue the view that both CFTs are on equal footing, related by Legendre transformation, and that one can compute directly the Witten graphs with either branch $\Delta_{+/-}$ [3]. Our aim is to have an autonomous way to compute directly in the free theory, having in mind a possible extension to free gauge theories, with no need of Legendre transforming from a conjugate CFT that arises at leading large N but whose existence is otherwise uncertain. As a consequence, at $d = 3$ one has a vanishing three-point function for the scalar bilinear at IR. On the other hand, at UV the three-point function is nonzero due to the compensation of the vanishing coupling by a divergence of the corresponding Witten graph. This is similar to the case of extremal correlators (see e.g. [32]). The underlying assumption of a common bulk theory, degeneracy of the holographic image, is also consistent with the CPWE of the four-point correlators. Progress in the bulk side of the correspondence is considerably more difficult due to the complicated nature of the interacting HS theories on AdS. We use the CPWs to mimic the effect of the corresponding bulk exchange graphs, even though the CPW is generically only a part of the Witten graph and one can only hope that after including the whole tower of HS exchange the additional terms cancel out. In this direction, we study the scalar exchange in AdS and relate it to the corresponding CPW.

3.1 The Klebanov-Polyakov Conjecture

Let us briefly review the essentials of the conjecture. The singlet sector of the critical 3-dim O(N) vector model with the $(\vec{\varphi}^2)^2$ interaction is conjectured to be dual, in the large N limit, to the minimal bosonic theory in AdS_4 containing massless gauge fields of even spin. There is a one-to-one correspondence between the spectrum of currents and that of massless higher-spin

[2]Here we follow a suggestion in [107] for the free gauge theory case and turn it into a quantitative result.

[3]A recent paper by Hartman and Rastelli [67] also stresses this view.

fields. In addition we have a scalar bilinear J mapped to a bulk scalar ϕ. The AdS/CFT correspondence working in the standard way (conventional dimension Δ_+ for J) produces the correlation functions of the singlet currents in the interacting large N vector model at its IR critical point from the bulk action in AdS_4 by identifying the boundary term ϕ_0 of the field ϕ with a source in the dual field theory (cf. appendix B.6). At the same time, the correlators in the free theory are obtained by Legendre transforming the generating functional with respect to the source that couples to the scalar bilinear J; this corresponds on the AdS side to the procedure for extracting the correlation functions working with the unconventional branch Δ_- [77, 78]. However, we want to stress that one can directly compute the bulk graphs with the Δ_- branch, by using the boundary term A (cf. appendix B.6) as source in the boundary theory, and that the boundary correlator obtained is precisely related by Legendre transformation to the one computed with the standard Δ_+ branch.

3.2 Free O(N) Vector Model

We start by considering N elementary real fields φ^a in d-dim Minkowski space, vectors under the global O(N) symmetry and Lorentz scalars with canonical scaling dimension $\delta = d/2-1$(in what follows we switch to Euclidean space). They satisfy the free equation of motion $\partial^2\varphi^a = 0$. We normalize their two point function as

$$\langle\varphi^a(x_1)\varphi^b(x_2)\rangle = \frac{\delta^{ab}}{r_{12}^{\delta}}\,, \quad a,b = 1,...,N\,, \tag{3.1}$$

where $\quad r_{ij} = |x_i - x_j|^2 = |x_{ij}|^2$.

3.2.1 HS Conserved Currents

In this free theory there is an infinite tower of higher-spin currents, bilinear in the elementary fields, which are totally symmetric, traceless and conserved. These three properties fix their form, their precise expression can be found in [6, 82]. We will only need them in the following form (assuming normal order and omitting free indices)

$$J_l = \sum_{k=0}^{l} a_k\, \partial^k\overrightarrow{\varphi}\cdot\partial^{l-k}\overrightarrow{\varphi}\ - traces, \tag{3.2}$$

with [4]

$$a_k = a_{l-k} = \frac{1}{2}(-1)^k \binom{l}{k} \frac{(\delta)_l}{(\delta)_k(\delta)_{l-k}}. \tag{3.3}$$

Note that this convention means

$$J_l = \overrightarrow{\varphi} \cdot \partial^l \overrightarrow{\varphi} \ + .., \tag{3.4}$$

where the ellipsis stands for terms involving derivatives of both fields. They are conformal quasi-primaries, minimal twist operators with scaling dimension

$$\Delta_l = d - 2 + l = 2\delta + l. \tag{3.5}$$

The AdS/CFT Correspondence relates them to massless HS bulk field since the canonical dimension Δ_l precisely saturates the unitarity bound for totally symmetric traceless rank l tensors ($l > 0$, even).

3.2.2 Two- and Three-Point Functions

The singlet-bilinear sector is completed by adding to the above list the scalar bilinear $J = \overrightarrow{\varphi}^2$ ("spin-zero current") with canonical dimension $\Delta_J = d-2 = 2\delta$. At $d = 3$ its bulk partner is a conformally coupled scalar.

Let us compute the two point function of the HS currents and the three point function of two spin-zero and a HS current.

The conformal symmetry fixes the form of the two point function up to a constant (B.3),

$$\langle J_{l\ \mu_1\ldots\mu_l}(x) J_{l\ \nu_1\ldots\nu_l}(0)\rangle = C_{J_l}\ r^{-2\delta-l}\ sym\{I_{\mu_1\nu_1}(x)\ldots I_{\mu_l\nu_l}(x)\}. \tag{3.6}$$

To find the coefficient it is then sufficient to look at the term $2^l \frac{x\ldots x}{r^l}$ involving x $2l$ times. By Wick contracting we get

$$\langle J_l(x) J_l(y)\rangle = \sum_{k,s=0}^{l} a_k\, a_s \left\{\partial_x^k\, \partial_y^s \langle\varphi^a(x)\varphi^b(y)\rangle\ \partial_x^{l-k}\, \partial_y^{l-s}\langle\varphi^a(x)\varphi^b(y)\rangle\right.$$

$$\left. + \ (s \leftrightarrow l-s)\right\} - \text{traces}. \tag{3.7}$$

Using now the symmetry $a_k = a_{l-k}$, trading ∂_y by $-\partial_x$ and taking $y = 0$ we get, up to trace terms,

$$2N \sum_{k,s=0}^{l} a_k\, a_s (\partial^{k+s} r^{-\delta})(\partial^{2l-k-s} r^{-\delta}) = 2^{2l+1} N \frac{x..x}{r^{2\delta+2l}} \sum_{k,s=0}^{l} a_k\, a_s\ (\delta)_{k+s}\ (\delta)_{2l-k-s}. \tag{3.8}$$

[4] The Pochhammer symbol $(q)_r = \frac{\Gamma(q+r)}{\Gamma(q)}$.

The double summation is done using generalized hypergeometric series in appendix B.2 giving $\frac{1}{4}l!(2\delta-1+l)_l$. The coefficient of the two point function is then ($l>0$)

$$C_{J_l} = 2^{l-1}Nl!(2\delta-1+l)_l \tag{3.9}$$

that coincides with the extrapolated result reported by Anselmi [6]. Analogously, the three point function form is dictated by the conformal symmetry (see B.1)

$$\langle J(x_1)J(x_2)J_{l\,\nu_1\ldots\nu_l}(x_3)\rangle = C_{JJJ_l}\;(r_{12}\,r_{23}\,r_{31})^{-\delta}\;\lambda^{x_3}_{\mu_1..\mu_l}(x_1,x_2). \tag{3.10}$$

Now we focus on the coefficient of $2^l\frac{x_{31}\ldots x_{31}}{r^l}$ involving x_{31} $2l$ times after Wick-contracting,

$$\langle J(x_1)J(x_2)J_l(x_3)\rangle = \langle\varphi^a(x_1)\varphi^b(x_2)\rangle\langle\varphi^b(x_2)\varphi^c(x_3)\rangle\partial^l_{x_3}\langle\varphi^c(x_3)\varphi^a(x_1)\rangle$$

$$+permutations + \ldots = 4N(r_{12}r_{23})^{-\delta}2^l(\delta)_l\frac{x_{31}\ldots x_{31}}{r_{31}^{\delta+l}} + \ldots\,. \tag{3.11}$$

Finally, we get for the coefficient of the three-point function ($l>0$)

$$C_{JJJ_l} = 2^{l+2}N(\delta)_l\,. \tag{3.12}$$

The corresponding values for the scalar are $C_J = 2N$ and $C_{JJJ} = 8N$.

3.2.3 Scalar Four-Point Function

The four-point function contains much more dynamical information encoded in a function of two conformal invariant cross-ratios which is not fixed by conformal symmetry. Still, its form is constrained by the OPE of any two fields and therefore the contributions of operators of arbitrary spin, including their derivative descendants, can be unveiled. The connected part of the spin-zero "current" four point function is obtained by Wick contractions

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle_{free,conn} = \frac{16N}{(r_{12}r_{34})^{2\delta}}\left\{u^\delta + (\frac{u}{v})^\delta + u^\delta(\frac{u}{v})^\delta\right\}, \tag{3.13}$$

where $u=\frac{r_{12}\,r_{34}}{r_{13}\,r_{24}}$ and $v=\frac{r_{14}\,r_{23}}{r_{13}\,r_{24}}$. Diagrammatically, it is given by the three boxes in fig. 3.1.

Following Klebanov and Polyakov [77] we notice that the leading term in the box diagram

$$\frac{1}{(r_{12}r_{23}r_{34}r_{41})^{1/2}} \sim \frac{1}{(r_{12}r_{34})^{1/2}}\frac{1}{r_{13}} \tag{3.14}$$

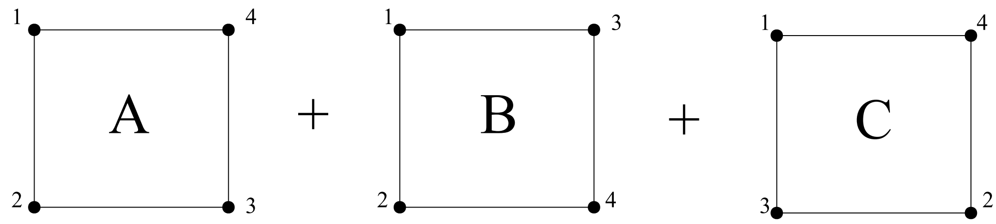

Figure 3.1: Connected part of the free four point function of the spin-zero current J.

in the direct channel limit $1 \to 2, 3 \to 4$ correctly reproduces the contribution of the scalar J with dimension $\Delta = \Delta_J = 1$ to the double OPE (see, e.g. [32]), which in general reads

$$\langle J(x_1)J(x_2)J(x_3)J(x_4)\rangle \sim \frac{1}{(r_{12}r_{34})^{\Delta_J - \Delta/2}} \frac{1}{(r_{13})^{\Delta}}. \tag{3.15}$$

Sub-leading terms in the expansion of the box diagram should correspond to the contribution of the currents $J_l \sim \overrightarrow{\varphi} \cdot \partial^l \overrightarrow{\varphi}, l > 0$. This structure is precisely what we want to study in detail and the best way to identify all these contributions is via a conformal partial wave expansion (CPWE); i.e., decomposing into eigenfunctions of the quadratic Casimir of the conformal group SO(1,d+1) in Euclidean space $\mathbb{R}^d$ [41].

3.3 From Free Fields to AdS via CPWE

The attempts to cast the box diagrams into AdS amplitudes via Schwinger parametrization have not succeeded so far [56]. From the previous analysis of the OPE, it becomes apparent that the whole tower of HS field exchange has to be taken into account. Even though some progress has been made in obtaining bulk-to-bulk propagators for the HS fields in AdS [81, 85], there is no closed analytic form that could be used to include all the infinite tower. We will content ourselves with the CPW amplitude to mimic the effect of the corresponding exchange Witten graph. In general, the CPW is contained in the exchange Witten graph but there appear additional terms that cannot be precisely identified as CPWs [87, 86, 70, 71, 40].

Let us first quote the essentials of the CPWE (see, e.g., [40] and references therein). The contribution of a quasi-primary $O^{(l)}_{\mu_1 \ldots \mu_l}$ of scale dimension Δ and spin l, and its derivative descendants, to the OPE of two scalar operators

ϕ_i of dimension Δ_i,

$$\phi_1(x)\phi_2(y) \sim \frac{C_{\phi_1\phi_2 O^{(l)}}}{C_{O^{(l)}}} \frac{1}{| x-y |^{\Delta_1+\Delta_2-\Delta+l}} C^{(l)}(x-y, \partial_y)_{\mu_1..\mu_l} O^{(l)}_{\mu_1...\mu_l}(y). \tag{3.16}$$

The derivative operator is fixed by requiring consistency of the OPE with the two- and three-point functions of the involved fields. Based on these constraints one can work out the contribution of the conformal block corresponding to the quasi-primary $O^{(l)}$, and its derivative descendants, to the four-point function. This is given by the Conformal Partial Wave (see appendix B.3)

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle \sim \frac{C_{\phi_1\phi_2 O^{(l)}} C_{\phi_3\phi_4 O^{(l)}}}{C_{O^{(l)}}} (\frac{r_{24}}{r_{14}})^{\Delta_{12}/2} (\frac{r_{14}}{r_{13}})^{\Delta_{34}/2}$$
$$\times \frac{u^{(\Delta-l)/2}}{(r_{12}r_{34})^{\Delta_\phi}} G^{(l)}(\frac{\Delta-\Delta_{12}-l}{2}, \frac{\Delta+\Delta_{34}-l}{2}, \Delta; u, v), \tag{3.17}$$

which depends on the two cross-ratios

$$u = \frac{r_{12}\, r_{34}}{r_{13}\, r_{24}}, \qquad v = \frac{r_{14}\, r_{23}}{r_{13}\, r_{24}}. \tag{3.18}$$

The CPWs have been obtained as double series in the direct channel limit $(u, 1-v) \to 0$ by several authors (see, e.g. [79] and references therein). They can also be shown to satisfy the recurrence relation (B.16), obtained by Dolan and Osborn [40].

3.3.1 HS currents and CPWs

We are interested in the singlet-bilineares, minimal twist operators, in the free O(N) vector model. These are the "spin-zero"($J \sim \varphi^a\varphi^a$) and the higher spin conserved currents ($J_l \sim \varphi^a\partial^l\varphi^a$) with canonical dimension $\Delta_l = d-2+l$. We can consider this limiting case in the recurrences (B.16), by first setting $e = (\Delta-l)/2 = d/2-1 = \delta$ and in the end $S = 2\delta+l, b = \delta$. A crucial simplification occurs in the recurrence relation, only the third line in (B.16) survives:

$$G^{(l)}(b,\delta,S;u,v) = \frac{1}{2}\frac{S+l-1}{\delta+l-1}\left\{G^{(l-1)}(b,\delta,S;u,v) - G^{(l-1)}(b+1,\delta,S;u,v)\right\} \tag{3.19}$$

The iteration can then be easily done for the coefficients of the double expansion

$$G^{(l)}(b,e,S;u,v) = \sum_{m,n=0}^{\infty} a^{(l)}_{nm}(b,S)\, \frac{u^n}{n!}\frac{(1-v)^m}{m!}\ . \tag{3.20}$$

Pascal's triangle coefficients $\binom{l}{k}$ arise to get

$$a_{nm}^{(l)}(b,S)=\frac{1}{2^l}\frac{(S)_l}{(\delta)_l}\sum_{k=0}^{l}(-1)^k\binom{l}{k}a_{nm}^{(0)}(b+k,S), \tag{3.21}$$

where $\delta=\mu-1=d/2-1$ and we start with the scalar exchange (B.17)

$$a_{nm}^{(0)}(b,S)=\frac{(\delta)_{m+n}}{(S)_{m+2n}}(S-b)_n(b)_{m+n}. \tag{3.22}$$

This can be summed up into a closed form involving a terminating generalized hypergeometric of unit argument

$$a_{nm}^{(l)}(b,S)=\frac{1}{2^l}\frac{(\delta+l)_{m+n-l}}{(S+l)_{m+2n-l}}(b)_{m+n}(S-b)_n\;{}_3F_2\left(\begin{matrix}-l,1+b-S,b+m+n\\ b,1+b-S-n\end{matrix}\right). \tag{3.23}$$

With these conventions, the normalization is fixed by

$$a_{0l}^{(l)}=(-\frac{1}{2})^l l!. \tag{3.24}$$

Now, using twice an identity ([5], pp.141), obtained as a limiting case of a result due to Whipple for balanced ${}_4F_3$ series, one can rewrite the coefficients as a terminating (after $n+1$ terms) series. This coincides with the result from the "Master Formula" in [83] [5] for $b=\delta$ and $S=2\delta+l$

$$\begin{aligned}a_{nm}^{(l)}&=a_{0l}^{(l)}\binom{m+n}{l}\frac{(\delta+l)_{m+n-l}^2}{(2\delta+2l)_{m+n-l}}\;{}_3F_2\left(\begin{matrix}-n,1+m+n,\delta+m+n\\ 1+m+n-l,2\delta+m+n+l\end{matrix}\right)\\ &=a_{0l}^{(l)}\sum_{s=0}^{n}(-1)^s\binom{n}{s}\binom{m+n+s}{l}\frac{(\delta+l)_{m+n-l}(\delta+l)_{m+n-l+s}}{(2\delta+2l)_{m+n-l+s}}.\end{aligned} \tag{3.25}$$

In this form one can easily recognize a triangular structure of the coefficients, i.e. $a_{nm}^{(l>m+2n)}=0$, which has proved useful in performing computer symbolic algebraic manipulations [83].

3.3.2 Modified CPWE and Closure

Now we compute the contribution of the singlet bilinear sector to the four-point function by summing the CPWs with the corresponding fusion coef-

[5] In their normalization, $a_{0l}^{(l)}$ are set to 1.

ficients (γ_l^{uv}) in terms of those of the two and three point functions [6] as (see 3.17)

$$(\gamma_l^{uv})^2 \;=\; C^2_{JJJ_l}/C_{J_l}. \tag{3.26}$$

Using our previous results (3.9) and (3.12) we have

$$(\gamma_l^{uv})^2 \;=\; 16N\,\frac{2^l}{l!}\,\frac{2\,(\delta)_l^2}{(2\delta-1+l)_l}. \tag{3.27}$$

The result of the direct channel summation (appendix B.4) is the observation that we can expand the first two terms (boxes) of (3.13) in partial waves of the bilinears, in the s-channel, as (B.27)

$$16N\,\frac{u^\delta}{(r_{12}r_{34})^{2\delta}}\left\{1+v^{-\delta}\right\} = \frac{1}{(r_{12}r_{34})^{2\delta}}\sum_{l\geq 0,\,even}(\gamma_l^{uv})^2 u^{(\Delta_l-l)/2}G^{(l)}(\delta,\delta,\Delta_l;u,v). \tag{3.28}$$

The full connected correlator is obtained then by crossing symmetry, since the three box diagrams A,B,C transform under crossing symmetry in the following way,

$$(2\rightarrow 4, t-channel)(u,v)\rightarrow(v,u):(A,B,C)\rightarrow(A,C,B) \tag{3.29}$$

$$(2\rightarrow 3, u-channel)(u,v)\rightarrow(1/u,v/u):(A,B,C)\rightarrow(C,B,A). \tag{3.30}$$

What we have found amounts to the diagrammatic identity in fig. 3.2.

Our rewriting is different from the standard CPWE where the whole crossing symmetric result is reproduced in each channel. The OPE of two scalar bilinear J contains the contributions of the identity, of the conformal blocks of the bilinears (minimal twist) and also of the “double-trace” (higher twist) operators starting with $(\overrightarrow{\varphi}^2)^2$. In the large N analysis, one can see that when the OPE is inserted in the four-point function, the identity produces only one piece of the disconnected part (which goes as N^2) and the completion comes precisely form the double-traces (their fusion coefficients squared also goes as $N^2+O(N)$) [77, 7]. It is also easy to see that the bilinear sector only contributes to the connected part (which goes as N, just like the fusion coefficients squared of this minimal twist sector, eq. 3.27).

[6] We can check the consistency of our conventions by comparing for the energy-momentum tensor ($l = 2$). To keep track of the normalization coming from Ward identities we use $\phi = \frac{1}{\sqrt{2N}}J$ and the canonically normalized energy-momentum tensor $T = -\frac{1}{2(d-1)S_d}J_2$ [40], where $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$, to have $C_{\phi\phi T} = -\frac{\Delta_\phi d}{d-1} = -\frac{(d-2)d}{d-1}$. The fusion coefficient in eq.(4.13) gets multiplied by $(\frac{1}{\sqrt{2N}})^4$. Then one gets from eq.(4.12) for the coefficient of the energy-momentum two point function $\frac{C_T}{S_d^2}$, with the well known result for the free O(N) vector model $C_T = \frac{d}{d-1}N$.

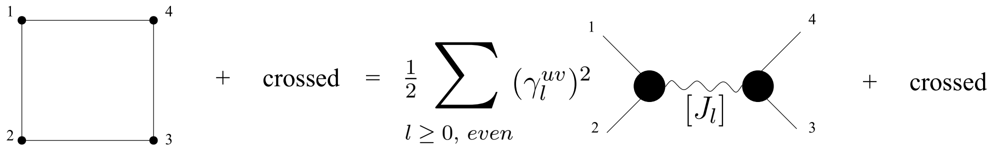

Figure 3.2: Free four point function as a sum partial waves of minimal twist operators.

Now the full disconnected piece is obtained from the Witten graphs containing two disconnected lines, where the three channels are included and with no need of additional fields in the bulk of AdS. One would then expect that for the connected part, something similar might happen. At leading $1/N$, tree approximation for the bulk theory, we have a classical field theory where one has to consider the exchange graphs in each channel separately; trading bulk exchanges by the corresponding CPW, one should then expect to write the connected part in terms of only CPWs of the minimal twist sector in the three channels, with no explicit reference to contributions from higher-twist/double-trace operators. To our surprise, this is precisely what we have obtained above!

In this way, we rescue the closure of the minimal twist sector that is in correspondence with a consistent truncation to the massless sector of the dual HS bulk theory. This result is valid as well for the bilinear single trace sector of free gauge theories considered in [56, 115, 107, 93], and amounts to a closure of the twist-two sector of scalars (without the double-trace operators this time!) in $d = 4$ by including the crossed channels, in conformity with the expectations for a consistent truncation of the bulk theory [107, 93].

3.4 Degeneracy of the Hologram: IR CFT at d=3

Now we examine a peculiarity of this O(N) vector model at $d = 3$, which mimics the effect of double-trace deformations of the free gauge theory (see e.g. [64] and references therein).

The canonical dimension of the scalar J is $\Delta = 1$. This value Δ_- is mapped, via AdS/CFT Correspondence, to a conformally coupled bulk scalar. However,there is a conjugate dimension $\Delta_+ = 2$ which agrees (at leading $1/N$ order) with the known result for the dimension of $\vec{\varphi}^2$ at the interacting IR critical point. This led Klebanov and Polyakov to conjecture that the minimal bosonic HS gauge theory with even spins and symmetry group $hs(4)$ is related, via standard AdS/CFT methods with the "conventional" branch Δ_+, to the interacting large N vector model at its IR critical point. The free theory, UV fixed point, corresponds then to the other branch Δ_-.

The existence of this IR stable critical point of the O(N) vector model below four dimension is a well established fact [7]. Standard approaches are the ϵ-expansion in $4-\epsilon$ dimensions which leads to the Wilson-Fisher fixed point and the large-N expansion which reveals a fixed point at $2 < d < 4$. Our analysis will be restricted to the leading $1/N$ results. An efficient way to perform the large-N expansion is introducing an auxiliary field α coupled to the vector field via a triple vertex $\alpha\,\varphi^a\varphi^a$ and then integrate out φ^a which appears now quadratically, to get the effective action for α. The diagrammatic expansion in $1/N$ involves skeleton graphs with the field φ^a running along internal lines and the triple vertices of two φ's with the auxiliary field [114].

3.4.1 IR Two- and Three-Point Functions

At leading $1/N$ we keep the free two point function of the elementary fields φ^a, they acquire anomalous dimension of order $1/N$, and for the auxiliary field α with dimension $\Delta_+ = 2$ one can set [8]

$$\langle \alpha(x)\alpha(0)\rangle = r^{-2}, \tag{3.31}$$

absorbing the normalizations in the vertex, which becomes [84] (see appendix B.5 for notations)

$$(\frac{z_1}{N})^{1/2} \quad , \quad z_1 = -2p(2). \tag{3.32}$$

Analogously, the three point function form is dictated by the conformal symmetry

$$\langle \alpha(x_1)\alpha(x_2)\alpha(x_3)\rangle = C_{\alpha\alpha\alpha}\,(r_{12}r_{23}r_{31})^{-1} \tag{3.33}$$

and

$$\langle \alpha(x_1)\alpha(x_2)J_{l\,\nu_1\ldots\nu_l}(x_3)\rangle = C_{\alpha\alpha J_l}\; r_{12}^{-2+\delta}(r_{23}r_{31})^{-\delta}\;\lambda^{x_3}_{\mu_1\ldots\mu_l}(x_1x_2). \tag{3.34}$$

[7]In four dimensions the IR fixed point merges with the UV one and the duality is no longer valid in the way we have just presented. Still one can modify the O(N) Vector Model (by gauging) to have a similar holographic scenario [105].

[8]In what follows and in an abuse of notation, a correlator involving α is understood to be computed at the IR fixed point, while the same correlator at UV contains J instead.

They are related to the respective free correlators by amputation relations (Legendre transform). For the two point function

$$\langle J(x)J(0)\rangle = -\frac{2N}{p(2)}\langle\alpha(x)\alpha(0)\rangle^{-1}. \tag{3.35}$$

With our choice of normalizations, we have to amputate with $\langle\alpha(x)\alpha(0)\rangle^{-1}$ and multiply by the factor $(-\frac{2N}{p(2)})^{\frac{1}{2}}$ for each scalar leg that is amputated to go from IR to UV at leading N, while the legs corresponding to the HS current remain the same. To compare this normalization with that of Klebanov and Witten [78] and appendix F, we denote their Legendre transformation by $\Gamma[A] = W[\phi_0] - (2\Delta_+ - d)\int A\phi_0$. Then at $d = 3$ the source for α is ϕ_0/π and that for J is $A/(2\pi\sqrt{N})$. The amputation is done with the D'EPP formula and its generalization (see appendix B.5). For the three point function of the scalars one gets

$$C_{\alpha\alpha\alpha} = N(-\frac{2}{Np(2)})^{\frac{3}{2}}v^2(2,\delta,\delta)v(2,1,2\delta-1) \tag{3.36}$$

as obtained in [84, 102]. There is a factor $\frac{1}{\Gamma(d-3)}$ that forces the vanishing of the IR three point function at $d = 3$ in correspondence with the vanishing of the bulk coupling in the HS AdS_4 theory [102, 108].

We extend this amputation procedure to the other three point functions to get [9]

$$C_{\alpha\alpha J_l} = 2^{l+1}\frac{l!\,(2\delta-1)\,(\delta)_l}{(2\delta-1)_l} \tag{3.37}$$

which agrees with what was obtained in [82] [10] by a different procedure, namely computing the four point function first of the two scalars with two elementary fields and then forming the HS current by contracting the two legs of the elementary fields acting with derivatives and letting their argument to coincide at the end. This computation was done at $d = 3$, however we corroborate the validity for any $2 < d < 4$. This has the surprising implication that a graph contributing to the four point function above mentioned, which vanishes at $d = 3$, does not contribute to the HS current correlator at generic $2 < d < 4$ as well.

3.4.2 Scalar Four-Point Function at IR

Let us now examine the implications of the degeneracy of the hologram for the four point function at the IR critical point. The AdS amplitude should

[9] This time the amputation is done with the generalization B.32 of the D'EPP formula.

[10] There is a relative factor of 2 due to normalization of the HS current and a missing factor 2^l, by misprint, in equation (97) of this paper.

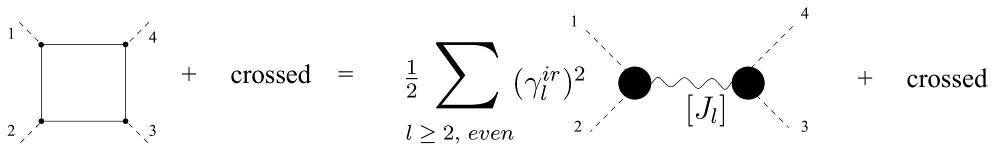

Figure 3.3: IR four point function as a sum of partial waves of minimal twist operators at $d = 3$.

involve the same bulk exchange graphs, only the scalar bulk-to-boundary and and bulk-to-bulk propagators are switched to the ones with Δ_+. We trade them by CPWs with the appropriate fusion coefficients that follow from the amputation program. Therefore we guess the modified CPWE in the interacting theory as indicated in fig. 3.3.

The fusion coefficients

$$(\gamma_l^{ir})^2 \;=\; C^2_{\alpha\alpha J_l}/C_{J_l}, \tag{3.38}$$

using our previous results from the amputations, are given by

$$(\gamma_l^{ir})^2 \;=\; \frac{1}{N}\,\frac{2^l}{l!}\,\frac{8(l!)^2\,(\delta)_l^2}{(2\delta)_{l-1}\,(2\delta)_{2l-1}}. \tag{3.39}$$

That the four point function at the IR critical point at leading $1/N$ has precisely this expansion has been shown by Rühl [104], by explicit computations at the IR critical point and the fusion coefficients obtained by extrapolation of computer algebraic manipulations. What we have analytically found confirms those results and prove their validity for the whole range $2 < d < 4$ where the scalar contribution accounts for the one-line-reducible graph, both of them being now non-vanishing. The shadow contribution in the one-line-reducible graph is canceled by contributions from the box as shown in [77] for the leading singular term and in [83] for the full CPW. The quotient $\gamma_l^{ir}/\gamma_l^{uv}$ for $d = 3$ is (5.10) $\gamma_l^{ir}/\gamma_l^{uv} = l/(2N)$, which is valid even for $l = 0$ since $\gamma_l^{ir} = 0$. Note that in the other normalization for fields with sources ϕ_0 and A, the ratio turns out to be equal to $2l$.

3.5 CPW vs. AdS Exchange Graph at d=3

The two and three point functions considered before can be reproduced from a bulk action, being relevant only up to cubic terms of the bulk Lagrangian.

These have been obtained in [82], assuming a bulk coupling of the HS field with a bulk current, bilinear in the scalar bulk field and involving up to l derivatives [11]. In their scheme there are two different couplings of the HS bulk field to two bulk scalars, one to reproduce the UV correlators and another for the IR case, and therefore two different bulk Lagrangians. We however adopt the view of a unique bulk Lagrangian, as expected from double trace deformations. It is not difficult then to realize that the bulk graphs corresponding to the coupling of the HS fields to the AdS current, bilinear in the bulk scalar, obtained in [82] lead to boundary three-point functions which are precisely related by amputation of the scalar legs. This is done in the boundary theory with the generalized D'EPP formula (B.32) and in the bulk graph this amounts to changing the dimension $\Delta_- \leftrightarrow \Delta_+$ of the bulk-to-boundary propagator of the scalar legs [39].

For the four point function, the CPW expansion obtained is indeed a step in the ambitious program of bottom to top approach, in which one uses the knowledge of the boundary CFT to reconstruct the bulk theory. This is a formidable task, but a Witten graph is certainly closer to a CPW as we know since the early days of AdS/CFT, although the precise correspondence has always been elusive and tricky.

Here we will study the scalar exchange and see what happens when one considers the boundary scalar bilinear to have canonical dimension $\Delta_- = d-2 = 1$. Let us start with the free 3-pt function. Despite the success of predicting the vanishing of the scalar three point function at the IR critical point and matching with the HS bulk theory [102, 108], the non-vanishing result for the free correlator cannot be obtained from a null result via the proposed Legendre transformation. One is forced to make a regularization, and the appropriate way turns out to be that the bulk coupling goes like $g \sim (d-3)$. Here we propose to compute directly with the canonical dimension and to control the divergence of the Witten graph by dimensional regularization. The graph is divergent at $d = 3$, but a cancellation against the vanishing coupling gives the correct result for the free correlator[12]. Starting with the free correlator and following Gopakumar [56] in bringing it to an AdS Witten graph, one gets the identity sketched [13] in fig. 3.4 .

[11] However, what one obtains from the HS theory for $l = 2$ is a bulk energy-momentum involving infinitely many derivatives. It is still an open issue to see whether both formulations are equivalent via some field redefinition. This we believe must first be clarified before trying to explore HS bulk exchange graphs, the coupling to the scalar is still ambiguous although their should be fixed by the conformal symmetry.

[12] Essentially the same cancellation that occurs for extremal correlators in standard AdS/CFT.

[13] In this section the equality sign is to be understood modulo finite factors that we omit for simplicity. The precise relation can be read from (E.7).

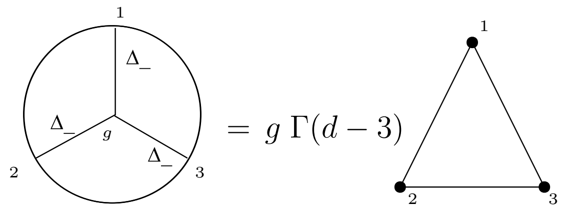

Figure 3.4: Free three point function and star Witten graph at $d \to 3$.

The divergence of the star graph is then controlled by the zero in $1/\Gamma(d-3)$, rendering the correct result for the free correlator.

Now we move on and examine the scalar exchange Witten graph, with the external legs having canonical dimension $\Delta_- = d-2 \to 1$ and coupling vanishing as $g \sim (d-3) \to 0$. A suitable evaluation of this graph, worked out in [71] using Mellin-Barnes representation and performing a contour integral, is given by a double series expansion involving three coefficients $a^{(1)}_{nm}$, $b^{(1)}_{nm}$ and $c^{(1)}_{nm}$ (in [71], eq.23-26). In the limit $\Delta = \widetilde{\Delta} = d-2$ and $d \to 3$, they all become divergent but only $c^{(1)}_{nm}$ and the first term in $b^{(1)}_{nm}$ develop a double pole, the rest being less singular. They cancel against the double zero from g^2 and the final result can be precisely casted into the CPW of the free scalar J plus its shadow, a scalar of dimension $\Delta_+ = 2$. The piece coming from the $c^{(1)}_{nm}$ coefficient goes to the $c_{nm}(1)$ coefficient of the CPW and the contribution from $b^{(1)}_{nm}$ produces the shadow term with coefficient $c_{nm}(2)$ ([71], eq.35-36). We end up with a precise identification in term of CPWs as sketched in fig. 3.5.

When one continues the crossed channel expansions to get their contribution in the direct channel one gets $log\, u$ terms but no non-analytic terms in $(1-v)$ [14]. This happens both for a Witten graph and for the combined CPW , i.e. direct plus shadow. The mechanisms that prevent the appearance of such terms are different [70, 71], in one case is due to some nontrivial hypergeometric identities and in the second case is due to the presence of the shadow field contribution. In the above case, both mechanisms coincide and the identification in terms of CPW is precise (this identification is in general incomplete, as mentioned before). That is, there is more structure in the scalar exchange graph than in a generic one and we take this as a good sign that after all, when computing the infinite tower of exchange diagrams, many delicate cancellations of additional terms take place to end up with

[14] This is a simple way to see that the bold identification of the scalar exchange Witten graph with the CPW, as originally proposed in [87], was certainly not correct. In our case we bypass this difficulty due to the shadow term, which makes the whole expression manifestly "shadow-symmetric".

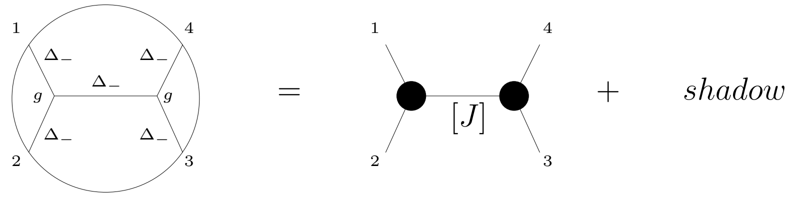

Figure 3.5: Scalar exchange Witten graph *vs.* scalar CPWs as $d \to 3$

just the sum of CPWs as obtained before. In particular, the additional term in the scalar case is a "shadow" contribution which are indeed absent in any full physical amplitude.

3.6 Discussion

We have re-organized the CPWE of the free four-point function of the scalar singlet bilinear in the natural way one would expect from AdS/CFT Correspondence; that is, by explicit inclusion of the crossed channels and involving only CPWs of the minimal twist sector which is holographically dual to the bulk HS gauge fields. This result is applicable as well to the corresponding sector of singlet bilinears in the scalars of the free gauge theory. Kinematically, double-trace operators are dual to two-particle bulk states; however, it is hard to see how such bulk states arise in the tree bulk computation that one has to perform at leading large-N. We guess that the double-trace operators arise indirectly, just in the way they show up in the free O(N) vector model.

In $2 < d < 4$ dimensions, one can flow (at leading large-N) into the IR fixed point of the O(N) vector model by just Legendre transforming. In this way, we have completed the program initiated in [102] for the three-point functions. In addition, under the assumption of a degenerate hologram, i.e. same bulk content but different asymptotics for the scalar bulk field, the modified CPWE of the four-point function was also obtained at IR. All two- and three-point function coefficients as well as fusion coefficients were analytically obtained, in some cases corroborating extrapolations from computer algebraic manipulations.

For the scalar exchange Witten graph with canonical dimensions, a funny cancellation occurs and the result can be precisely identified in terms of CPWs of the corresponding scalar and its shadow. This reveals more structure than the generic case, and we hope that such "accidents" are indeed needed to obtain the full four-point correlator if one were able to sum the infinite tower of HS field exchanges.

Chapter 4

Double-trace deformations

Let us recall that the Maldacena's conjecture and its calculational prescription [88, 63, 116] entail the equality between the partition function of String/M-theory (with prescribed boundary conditions) in the product space $AdS_{d+1} \times X$, where X is certain compact manifold, and the generating functional of gauge invariant single-trace composite operators of the dual CFT_d at the boundary. As said, it has been fairly well tested at the level of classical SUGRA in the bulk and at the corresponding leading order at large N on the boundary.
One of the most remarkable tests is the mapping of the conformal anomaly [69]. Since the rank N of the group measures the size of the geometry in Planck units, quantum corrections correspond to subleading terms in the large N limit. Corrections of order $O(N)$ have also been obtained [3, 17, 97, 99], but they rather correspond to tree-level corrections after inclusion of open or unoriented closed strings. Truly quantum corrections face the notorious difficulty of RR-backgrounds and only few examples, besides semiclassical limits of the correspondence, have circumvented it and corroborated the conjecture at this nontrivial level [13, 91]. These results rely on whole towers of KK-states and SUSY. The regimes in which the bulk and boundary computations can be done do not overlap and some sort of non-renormalization must be invoked.

In this chapter we deal with a universal AdS/CFT result, not relying on SUSY or any other detail encoded in the compact space X, concerning an $O(1)$ correction to the conformal anomaly under a flow produced by a double-trace deformation. This was first computed in the bulk of AdS [65] and confirmed shortly after by a field theoretic computation on the dual boundary theory [64] (see also [61]).

Let us roughly recapitulate the sequence of developments leading to this remarkable success. It starts with a scalar field ϕ with "tachyonic" mass in the window $-\frac{d^2}{4} \leq m^2 < -\frac{d^2}{4} + 1$ where two AdS-invariant quantizations are known to exist [19]. The conformal dimensions of the dual CFT operators, given by the two roots Δ_+ and Δ_- of the AdS/CFT relation $m^2 = \Delta(\Delta - d)$:

$$\Delta_\pm \; = \; \frac{d}{2} \pm \nu \; , \qquad \nu \; = \; \sqrt{\frac{d^2}{4} + m^2} \tag{4.1}$$

are then ($0 \leq \nu < 1$) both above the unitarity bound. The modern AdS/CFT interpretation [78] assigns the same bulk theory to two different CFTs at the boundary, whose generating functionals are related to each other by Legendre transformation at leading large N. The only difference is the interchange of the roles of boundary operator/source associated to the asymptotic behavior of the bulk scalar field near the conformal boundary.
The whole picture fits into the generalized AdS/CFT prescription to incorporate boundary multi-trace operators [117, 12, 96, 94]. The two CFTs are then the end points of a RG flow triggered by the relevant perturbation $f\, O_\alpha^2$ of the $\alpha-$CFT, where the operator O_α has dimension Δ_- (so that $\Delta_- + \Delta_+ = d$, $\Delta_- \leq \Delta_+ \Rightarrow 2\Delta_- \leq d$). The $\alpha-$theory flows into the $\beta-$theory which now has an operator O_β with dimension $\Delta_+ = d - \Delta_-$, conjugate to Δ_-. The rest of the operators remains untouched at leading large N, which suggests that the metric and the rest of the fields involved should retain their background values, only the dual bulk scalar changes its asymptotics [1].

The crucial observation in [65] is the following: since the only change in the bulk is in the asymptotics of the scalar field, the effect on the partition function cannot be seen at the classical gravity level in the bulk, i.e. at leading large N, since the background solution has $\phi = 0$; but the quantum fluctuations around this solution, given by the functional determinant of the kinetic term (inverse propagator), are certainly sensitive to the asymptotics since there are two different propagators G_Δ corresponding to the two different AdS-invariant quantizations. The partition function including the one-loop correction is

$$Z^{\pm}_{grav} = Z^{class}_{grav} \cdot \left[\det{}_\pm(-\Box + m^2)\right]^{-\frac{1}{2}} \, , \tag{4.2}$$

where Z^{class}_{grav} refers to the usual saddle point approximation. Notice that the functional determinant is independent of N, this makes the scalar one-loop quantum correction an $O(1)$ effect. The 1-loop computation turns out to be very simple for even dimension d and is given by a polynomial in Δ.

[1] The simplest realization of this behavior being the $O(N)$ vector model in $2 < d < 4$ studied in the previous chapter (see also [77, 35]).

No infinities besides the IR one, related to the volume of AdS, show up in the relative change Z^+_{grav}/Z^-_{grav}, since the UV-divergences can be controlled exactly in the same way for both propagators. From this correction to the classical gravitational action one can read off an $O(1)$ contribution to the *holographic conformal anomaly* [69].

The question whether this $O(1)$ correction to the anomaly could be recovered from a pure CFT_d calculation was answered shortly after in the affirmative [64]. Using the Hubbard-Stratonovich transformation (or auxiliary field trick) and large N factorization of correlators, the Legendre transformation relation at leading large N is shown. An extra $O(1)$ contribution, the fluctuation determinant of the auxiliary field, is also obtained. Turning the sources to zero, the result for the CFT partition function can be written as

$$Z_\beta = Z_\alpha \ \cdot [\det(\Xi)]^{-\frac{1}{2}} \,, \tag{4.3}$$

where the kernel $\Xi \equiv \mathbb{I} + f\,G$ in position space in $\mathbb{R}^d$ is given by $\delta^d(x,x') + \frac{f}{|x-x'|^{2\Delta_-}}$. The $\beta-$CFT is reached in the limit $f \to \infty$.
From the CFT point of view, the conformal invariance of this functional determinant has then to be probed. Putting the theory on the sphere $\mathbb{S}^d$ and expanding in spherical harmonics, using Stirling formula for large principal quantum number l and zeta-function regularization, the coefficient of the log-divergent term is isolated. It happily coincides (for the explored cases $d = 2, 4, 6, 8$) with the AdS_{d+1} prediction for the anomaly.

Despite the successful agreement, there are several issues in this derivation that ought to be further examined. No track is kept on the overall coefficient in the CFT computation, in contrast to the mapping at leading order [69] that matches the overall coefficient as well. For odd dimension d, the CFT determinant has no anomaly, whereas there is a nonzero AdS result that could be some finite term in field theory not computed so far [64]. From a computational point of view the results are quite different. The AdS answer is a polynomial for generic even dimension, whereas for odd d only numerical results are reported. The CFT answer, on the other hand, is obtained for few values of the dimension d, a proof for generic d is lacking. Yet, the very same $O(1)$ nature of the correction on both sides of the correspondence calls for a full equivalence between the relative change in the partition functions, and not only just the conformal anomaly. This poses a new challenge since in the above derivation there are several (divergent) terms that were disregarded, for they do not contribute to the anomaly.

As we have seen, it all boils down to computing functional determinants. In a more recent work [67], a "kinematical" understanding of the agreement between the bulk and boundary computations was achieved based on the equality between the determinants. The key is to explicitly separate the transverse coordinates in AdS, expand the bulk determinant in this basis inserting the eigenvalues of the transverse Laplacian weighted with their degeneracies. In this way, one gets a weighted sum/integral of effective radial (one-dimensional) determinants which are then evaluated via a suitable generalization of the Gel'fand-Yaglom formula. The outcome turns out to coincide with the expansion of the auxiliary field fluctuation determinant of [64]. However, this procedure is known to be rather formal (see, e.g., [47] and references therein) and the result to be certainly divergent. No further progress is done on either side of this *formal* equality and the issue of reproducing the full bulk result from a field theoretic computation at the boundary remains open.

We will show in this chapter that all above open questions can be thoroughly clarified or bypassed if one uses dimensional regularization to control all the divergences. Both IR and UV divergencies are now on equal footing, which is precisely the essence of the IR-UV connection [110]: the key to the holographic bound is that an IR regulator for the boundary area becomes an UV regulator in the dual CFT. The bulk effective potential times the infinite AdS volume, i.e. the effective action, and the boundary sum, using Gauß's "proper-time representation" for the digamma function to perform it, are shown to coincide in dimensional regularization.

We start with the bulk partition function and compute the regularized effective action. Here one needs to compute separately the volume and the effective potential. Then we move to the boundary to compute the change induced by the double-trace deformation. Having established the equivalence for dimensionally regularized quantities we go back to the physical dimensions and extract the relevant results for the renormalized partition functions and the conformal anomaly. We further discuss some technical details.

4.1 The bulk computation: one-loop effective action

Let us start with the Euclidean action for gravity and a scalar field

$$S_{d+1}^{class} = \frac{-1}{2\kappa^2}\int dvol_{d+1}\,[R-\Lambda] \quad + \quad \int dvol_{d+1}\,[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2]. \quad (4.4)$$

For negative Λ the Euclidean version of AdS_{d+1}, i.e. the Lobachevsky space $\mathbb{H}^{d+1}$, is a classical solution. There are, of course, additional terms like the Gibbons-Hawking surface term and contributions from other fields, but they will play no role in what follows, nor will the details of the leading large N duality. This is an indication of universality of the results.

We are interested in the quantum one-loop correction from the scalar field with the Δ_+ or Δ_- asymptotic behavior

$$S^{\pm}_{d+1} = \frac{1}{2} \log \det{}_{\pm}(-\Box + m^2) = \frac{1}{2} \operatorname{tr}_{\pm} \log(-\Box + m^2) \ . \tag{4.5}$$

It will prove simpler to consider instead the quantities

$$\frac{\partial}{\partial m^2} S^{\pm}_{d+1} = \frac{1}{2} \operatorname{tr}_{\pm} \frac{1}{-\Box + m^2} \ . \tag{4.6}$$

This rather symbolic manipulation, casted into a concrete form [2], reads

$$\frac{\partial}{\partial m^2} S^{\pm}_{d+1} = \frac{1}{2} \int dvol_{\mathbb{H}^{d+1}} \, G_{\Delta_{\pm}}(z, z). \tag{4.7}$$

There are two kinds of divergencies here, one is the infinite volume of the hyperbolic space (IR) and the other is the short distance singularity of the propagator (UV). The latter is conventionally controlled by taking the difference of the $\pm$-versions; this produces a finite result and was the crucial observation in [65]. Then one gets for the difference of the one-loop corrections for the $\Delta_{\pm}$ asymptotics

$$\frac{\partial}{\partial m^2} \left(S^{+}_{d+1} - S^{-}_{d+1} \right) = \frac{1}{2} \int dvol_{\mathbb{H}^{d+1}} \left\{ G_{\Delta_+}(z, z) - \ G_{\Delta_-}(z, z) \right\} \ . \tag{4.8}$$

One might be tempted to factorize away the volume (usual procedure) and work further only with the effective potential. However, the perfect matching with the boundary computation will require keeping track of the volume as well. In the spirit of the IR-UV connection we now use dimensional regularization to control both the IR divergencies in the bulk as well as the UV divergencies on the boundary.

4.1.1 Dimensionally regularized volume

Starting from the usual representation of $\mathbb{H}^{d+1}$ in terms of a unit ball with metric $ds^2 = 4(1-x^2)^{-2} dx^2$ one gets, after the substitution $r = (1-|x|)/(1+$

[2] We have been a little cavalier here since the Breitenlohner-Freedman analysis is done in Lorentzian signature. However, for computational purposes is easier to consider the Euclidean formulation of the CFT and the volume renormalization of Riemannian manifolds, so that a Wick rotation should be performed. The Feynman propagator for the regular modes (Δ_+) in AdS_{d+1} becomes the resolvent in $\mathbb{H}^{d+1}$, whereas the continuation to hyperbolic space of the propagator for the irregular modes is only achieved via the continuation of the resolvent from Δ_+ to Δ_- [21].

$|x|$), the metric

$$G = r^{-2}[(1-r^2)^2 g_0 + dr^2]\,, \tag{4.9}$$

with $4g_0$ being the usual round metric on $\mathbb{S}^d$. Then

$$\left(\frac{\det G}{\det g_0}\right)^{\frac{1}{2}} = r^{-1-d}\,(1-r^2)^d, \tag{4.10}$$

and the volume is then given by

$$\int dvol_{\mathbb{H}^{d+1}} \;=\; 2^{-d}\,vol_{\mathbb{S}^d}\int_0^1 dr\,r^{-d-1}\,(1-r^2)^d. \tag{4.11}$$

Up to this point, we have just followed [60] to compute the volume. From here on, there are two standard ways to proceed in the mathematical literature, namely Hadamard or Riesz regularization (see, e.g. [4]). We will use none of them, although our choice of dimensional regularization is closer to Riesz's scheme. This IR-divergent volume will now be controlled with DR: set $d \to D = d-\epsilon$ and perform the integral to get, after some manipulations,

$$vol_{\;\mathbb{H}^{D+1}} \;=\; \pi^{\frac{D}{2}}\,\Gamma(-\frac{D}{2})\;. \tag{4.12}$$

Let us now send ϵ to zero:

$$vol_{\;\mathbb{H}^{D+1}} \;=\; \frac{\mathcal{L}_{d+1}}{\epsilon} + \mathcal{V}_{d+1} + o(1)\;. \tag{4.13}$$

For *even* d we find the "integrated conformal anomaly" (integral of Branson's Q-curvature, a generalization of the scalar curvature, see e.g. [59]) and renormalized volume given by

$$\mathcal{L}_{d+1} = (-1)^{\frac{d}{2}}\,\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d+2}{2})}\;, \tag{4.14a}$$

$$\mathcal{V}_{d+1} \;=\; \frac{1}{2}\,\mathcal{L}_{d+1}\cdot\left[\psi(1+\frac{d}{2}) - \log\pi\right]\;. \tag{4.14b}$$

For *odd* d in turn, $\mathcal{L}_{d+1}$ vanishes and the renormalized volume is given by

$$\mathcal{V}_{d+1} = (-1)^{\frac{d+1}{2}}\,\frac{\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d+2}{2})}\;. \tag{4.15}$$

The *conformal invariants* $\mathcal{L}_{d+1}$ and $\mathcal{V}_{d+1}$ for $d = even$ and $d = odd$ respectively, coincide with those obtained by Hadamard regularization [60]. For even d in turn, the regularized volume fails to be conformal invariant and its integrated infinitesimal variation under a conformal change of representative metric on the boundary is precisely given by $\mathcal{L}$ [60]. In all, the presence of the pole term is indicative of an anomaly under conformal transformation of the boundary metric. As usual in DR, the pole corresponds to the logarithmic divergence in a cutoff regularization. In the pioneering work of Henningson and Skenderis [69], an IR cutoff is used and the (integrated) anomaly turns out to be the coefficient of the $\log \epsilon$ after the radial integration is performed.

4.1.2 Dimensionally regularized one-loop effective potential

As effective potential we understand the integrand in (4.8), i.e.

$$\frac{\partial}{\partial m^2}\left(V^+_{d+1} - V^-_{d+1}\right) = \frac{1}{2}\left\{G_{\Delta_+}(z,z) - G_{\Delta_-}(z,z)\right\}, \tag{4.16}$$

where the propagator at coincident points, understood as analytically continued [74] from $D = d - \epsilon$, is given by ($m^2 = \Delta(\Delta - d)$, $\Delta_\pm = d/2 \pm \nu$)

$$G_\Delta(z,z) = \frac{\Gamma(\Delta)}{2^{1+\Delta}\pi^{\frac{D}{2}}\Gamma(1+\Delta-\frac{D}{2})}\, F(\frac{\Delta}{2}, \frac{1+\Delta}{2}; 1+\Delta-\frac{D}{2}; 1). \tag{4.17}$$

Using now Gauß's formula for the hypergeometric with unit argument and Legendre duplication formula for the gamma function (appendix C.3), the dimensionally regularized version of (4.16) can be written as

$$\frac{\partial}{\partial m^2}\left(V^+_{D+1} - V^-_{D+1}\right) = \frac{1}{2^{D+2}\,\pi^{\frac{D+1}{2}}}\,\Gamma(\frac{1-D}{2})\left[\frac{\Gamma(\nu+\frac{D}{2})}{\Gamma(1+\nu-\frac{D}{2})} - (\nu \to -\nu)\right]. \tag{4.18}$$

Letting now $\epsilon \to 0$, the limit is trivial to take when $d = \textit{even}$ since all terms are finite; for $d = \textit{odd}$ however, care must be taken to cancel the pole of the gamma function with the zero coming from the expression in square brackets in that case. This is in agreement with general results of QFT in curved space; using heat kernel and dimensional regularization one can show that in odd-dimensional spacetimes the dimensionally regularized effective potential is finite, whereas in even dimensions the UV singularities show up as a pole at the physical dimension (see, e.g., [31]), which cancel in the difference taken above. Ultimately, one gets a finite result valid for both even and odd d

$$\frac{\partial}{\partial m^2}\left(V^+_{d+1} - V^-_{d+1}\right) = \frac{1}{2\nu}\,\frac{1}{2^d\,\pi^{\frac{d}{2}}}\,\frac{(\nu)_{\frac{d}{2}}\,(-\nu)_{\frac{d}{2}}}{(\frac{1}{2})_{\frac{d}{2}}} \equiv \mathcal{A}_d(\nu). \tag{4.19}$$

We used the last equation also to introduce an abbreviation $\mathcal{A}_d(\nu)$ for later convenience. That this formula comprises both even and odd d can be better appreciated in the derivation given in appendix C.1. Written in the form (4.19), this result coincides with that of Gubser and Mitra (eq. 24 in [65]) but now valid for d odd as well. We have to keep in mind to undo the derivative at the end. Interestingly, for the corresponding integral the integrand is essentially the *Plancherel measure* [3] for the hyperbolic space at imaginary argument $i\,\nu$ (see appendix C.1).

4.1.3 Dimensionally regularized one-loop effective action

The product of the regularized volume (4.12) and the regularized one-loop potential (4.18) yields the dimensionally regularized one-loop effective action

$$\begin{aligned} \frac{\partial}{\partial m^2}\left(S^+_{D+1}-S^-_{D+1}\right) &= \frac{1}{2}\,\Gamma(-D)\left[\frac{\Gamma(\nu+\frac{D}{2})}{\Gamma(1+\nu-\frac{D}{2})}-(\nu\to-\nu)\right] \\ &= \frac{\sin\pi\nu}{2\sin\pi D/2}\,\frac{\Gamma(\frac{D}{2}+\nu)\Gamma(\frac{D}{2}-\nu)}{\Gamma(1+D)}\;. \end{aligned} \tag{4.20}$$

The poles of $\Gamma(-D)$ are deceiving. For $D\to odd$, the pole is canceled against a zero from the square bracket. Only at $D\to even$ there is a pole.

The claim now is that this full result can be recovered from the dual boundary theory computation if we use the same regularization procedure, namely dimensional regularization.

4.2 The boundary computation: deformed partition function

Let us first take a brief look at the way the RG-flow picture is exploited to get the $O(1)$ contribution on the boundary theory. Start by turning on the deformation $f\,O_\alpha^2$ in the $\alpha-$theory. Then use the Hubbard-Stratonovich transformation (i.e. auxiliary field trick) to linearize in O_α

$$\langle e^{-\frac{f}{2}\int O_\alpha^2}\rangle \sim \int \mathcal{D}\sigma\, e^{\frac{1}{2f}\int\sigma^2}\,\langle e^{\int\sigma O_\alpha}\rangle\;. \tag{4.21}$$

[3] Presumably, the easiest way to see this is via the spectral representation in terms of spherical functions (see e.g. [21]), it pick ups the residue at $i\,\nu$. But the construction is valid only for the Δ_+ propagator, Δ_- is only reached at the end by suitable continuation. These details will be presented elsewhere.

Now make use of the large-N factorization, which means that the correlators are dominated by the product of two-point functions, to write

$$\langle e^{\int \sigma O_\alpha} \rangle_{N \gg 1} \approx e^{\frac{1}{2} \int \int \sigma \langle O_\alpha O_\alpha \rangle \sigma} \ . \tag{4.22}$$

Finally, integrate back the auxiliary field which produces its fluctuation determinant $\Xi^{-1/2} = (\mathbb{I} + f \langle O_\alpha O_\alpha \rangle)^{-1/2}$. We have therefore the relation between partition functions

$$Z_\beta = Z_\alpha \ \cdot [\det(\Xi)]^{-1/2} \tag{4.23}$$

as $f \to \infty$.

Putting the CFT on the sphere $\mathbb{S}^d$ with radius R, the kernel Ξ becomes [64]

$$\Xi = \delta^d(x, x') + \frac{f}{s^{2\Delta_-}(x, x')} \ , \tag{4.24}$$

where s is the chordal distance on the sphere. The quotient of the partition functions in the $\alpha-$ and $\beta-$theory is then given by

$$W_d^+ - W_d^- \ \equiv \ -\log \frac{Z_\beta}{Z_\alpha} = \frac{1}{2} \lim_{f \to \infty} \log \det \Xi = \lim_{f \to \infty} \frac{1}{2} \sum_{l=0}^{\infty} \deg(d, l) \log(1 + f \, g_l) \ , \tag{4.25}$$

where

$$g_l = \pi^{\frac{d}{2}} 2^{2\nu} \frac{\Gamma(\nu)}{\Gamma(\frac{d}{2} - \nu)} \frac{\Gamma(l + \frac{d}{2} - \nu)}{\Gamma(l + \frac{d}{2} + \nu)} \ R^{2\nu} \ , \tag{4.26}$$

$$\deg(d, l) = (2l + d - 1) \frac{(l + d - 2)!}{l! \, (d - 1)!} \equiv \frac{2l + d - 1}{d - 1} \frac{(d - 1)_l}{l!} \ . \tag{4.27}$$

Here g_l is the coefficient [4] of the expansion of $s^{-2\Delta}(x, x')$ in spherical harmonic and $\deg(d, l)$ counts the degeneracies. For large l one finds (our interest concerns $0 \leq \nu < 1$, see (4.1))

$$\deg(D, l) \sim l^{D-1} \ , \qquad g_l \sim l^{-2\nu} \ , \tag{4.28}$$

implying convergence of the sum in (4.25) for $D < 2\nu$. To define (4.25) for the physically interesting positive integers d we favor dimensional regularization and use analytical continuation from the save region $D < 0$. There, in addition, the limit $f \to \infty$ can be taken under the sum.

For this limit an amusing property of the sum of the degeneracies $\deg(D, l)$ alone turns out to be crucial. After short manipulations it can be casted into

[4] There is a missing factor of $2^{-\Delta}$ in eqs. (19) and (24) of [64]. It can be traced back to the chordal distance in term of the azimuthal angle $s^2 = 2(1 - cos\theta)$.

the binomial expansion of $(1-1)^{-D}$ (see (C.26)) which is zero for negative D, i.e.

$$\sum_{l=0}^{\infty} \deg(D,l) \;=\; 0 \; . \tag{4.29}$$

As a consequence all factors in g_l not depending on l have no influence on the limit $f \to \infty$ and we arrive at

$$W_D^+ - W_D^- \;=\; -\frac{1}{2}\sum_{l=0}^{\infty} \deg(D,l) \log\frac{\Gamma(l+\frac{D}{2}+\nu)}{\Gamma(l+\frac{D}{2}-\nu)} \; . \tag{4.30}$$

Here we want to stress that this is our full answer, whereas it is just a piece in [64] where zeta-function regularization was preferred. Although the zeta-function regularization of the sum of the degeneracies alone vanish in odd dimensions, in even dimension it is certainly nonzero.

To make contact with the mass derivative of the effective action of the previous section we take the derivative $\frac{\partial}{\partial m^2} = \frac{1}{2\nu}\frac{\partial}{\partial \nu}$

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_D^+ - W_D^-\right) = -\frac{1}{4\nu}\sum_{l=0}^{\infty} \deg(D,l)\left(\psi(l+\frac{D}{2}+\nu) + \psi(l+\frac{D}{2}-\nu)\right) \; . \tag{4.31}$$

The task is now to compute the sum. For this we want to exploit Gauß's integral representation for $\psi(z)$ (C.25). However, since it requires $z > 0$ we first keep untouched the $l = 0$ term and get for $2\nu - 2 < D < 0$

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_D^+ - W_D^-\right) = -\frac{1}{4\nu}\left(\psi(\frac{D}{2}+\nu) + \psi(\frac{D}{2}-\nu)\right)$$

$$-\frac{1}{4\nu}\sum_{l=1}^{\infty} \deg(D,l) \int_0^{\infty} dt \left(2\,\frac{e^{-t}}{t} \;-\; \frac{e^{-t(l+d/2)}}{1-e^{-t}}(e^{-t\nu}+e^{t\nu})\right) \; . \tag{4.32}$$

Now the sum of the l independent term under the integral can be performed with (4.29). The other sums via (C.26) can be reduced to $\sum_{l=1}^{\infty}\frac{(D-1)_l}{l!}e^{-tl} = (1-e^{-t})^{1-D}-1$. Then with $\psi(z) = \psi(1+z)-1/z$ and the Gauß representation for $\psi(D/2+1\pm\nu)$ we arrive at

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_D^+ - W_D^-\right) = \frac{D}{\nu(D-2\nu)(D+2\nu)}$$

$$+\frac{1}{4\nu}\int_0^1 du\; u^{\frac{D}{2}-1}(u^{\nu}+u^{-\nu})\Big((1-u)^{-D-1}(1+u)-1\Big) \; . \tag{4.33}$$

In identifying the remaining integral as a sum of Euler's beta functions a bit of caution is necessary since we are still confined to the convergence region $2\nu - 2 < D < 0$. But using both the standard representation and the subtracted version (C.27) we finally get

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_D^+ - W_D^-\right) = \frac{1}{2}\Gamma(-D)\left[\frac{\Gamma(\nu+\frac{D}{2})}{\Gamma(1+\nu-\frac{D}{2})} - (\nu\to-\nu)\right]. \quad (4.34)$$

This has been derived by allowed manipulations of convergent sums and integrals in the region $2\nu - 2 < D < 0$. From there we analytically continue and a comparison with (4.20) gives now for all D

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_D^+ - W_D^-\right) = \frac{\partial}{\partial m^2}\left(S_{D+1}^+ - S_{D+1}^-\right). \quad (4.35)$$

Let us just mention that one can, in principle, choose Hadamard regularization, i.e. subtract as many terms of the Taylor expansion in ν of the sum (4.30) as necessary to guarantee convergence. This results in the renormalized bulk result plus a polynomial in ν of degree d. This polynomial is just an artifact of the regularization scheme and is of no physical meaning. The question whether in this framework there is a subtraction scheme on the boundary that exactly reproduces the bulk result seems to find an answer in a generalization of Weierstrass formula for the multigamma functions [113]. Surprisingly, the effective potential in AdS can be written in terms the multigamma functions [74, 26]. We refrain from pursuing this *Weierstrass regularization* here and stick to DR for simplicity.

4.3 Back to the physical dimensions

Let us now send $\epsilon \to 0$ in the dimensionally regularized partition functions (eqs. 4.20 and 4.35) and see what happens in odd and even dimensions.

4.3.1 d odd: renormalized partition functions

Let us assume a minimal subtraction scheme to renormalize and establish the holographic interpretation of the boundary result. In this case we have

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_D^+ - W_D^-\right) = \frac{\pi}{2\nu}\frac{(-1)^{\frac{d+1}{2}}}{\Gamma(1+d)}(\nu)_{\frac{d}{2}}(-\nu)_{\frac{d}{2}} + o(1). \quad (4.36)$$

Now, the renormalized value is exactly the renormalized volume times the renormalized effective potential [5]

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_d^+ - W_d^+\right) = \mathcal{V}_{d+1} \cdot \mathcal{A}_d(\nu) = \frac{\partial}{\partial m^2}\left(S_{d+1}^+ - S_{d+1}^-\right) . \qquad (4.37)$$

This completes the matching in [64], the finite nonzero bulk result being indeed a finite contribution in the CFT computation which has not been computed before. At the same time, it is not a contribution to the conformal anomaly, this being absent for d odd as expected on general grounds.

Holography (AdS/CFT correspondence) in this case matches the renormalized partition functions at $O(1)$ order in CFT_d and at one-loop quantum level in AdS_{d+1}.

4.3.2 d even: anomaly and renormalized partition functions

Following the same steps as above, we get for this case

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_D^+ - W_D^-\right) = \frac{1}{\epsilon} \cdot \frac{1}{\nu}\frac{(-1)^{\frac{d}{2}}}{\Gamma(1+d)}(\nu)_{\frac{d}{2}}(-\nu)_{\frac{d}{2}}$$

$$+ \frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_d^+ - W_d^-\right) + o(1). \qquad (4.38)$$

Here we can identify the factorized form of the term containing the pole, the contribution to the conformal anomaly,

$$\text{Res}\left[\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_D^+ - W_D^-\right), D = d\right] = \mathcal{L}_{d+1} \cdot \mathcal{A}_d(\nu) . \qquad (4.39)$$

Note that according to (4.19) $\mathcal{A}_d(\nu)$ is just the derivative of the difference of the renormalized effective potentials for the $\alpha-$ and $\beta-$CFT.

This is the proof to generic even dimension of the matching between bulk [65] and boundary [64] computations concerning the correction to the *conformal anomaly*, including the overall coefficient.

[5] This exact agreement can be upset if different regularization/renormalization procedures were chosen, but in any case this ambiguity would show up only as a polynomial in ν of degree d at most. This is related to the fact that if one differentiate enough times with respect to ν (equivalently, m^2) the result is no longer divergent and therefore "reg.-scheme"-independent.

However, there is apparently a puzzle here concerning the finite remnant. The renormalized value

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_d^+ - W_d^-\right) = \frac{\mathcal{L}_{d+1}\ \cdot \mathcal{A}_d(\nu)}{2}\left\{2\,\psi(1+d) - \psi(\frac{d}{2}+\nu) - \psi(\frac{d}{2}-\nu)\right\} \tag{4.40}$$

is certainly non-polynomial in ν.

Had we computed only the renormalized effective potential, then after subtraction of the pole we would end up with the finite result $\mathcal{V}_{d+1}\ \cdot \mathcal{A}_d(\nu)$. But $\mathcal{A}_d(\nu)$ is polynomial in ν and therefore it could have been renormalized away. Yet, the CFT computation renders the non-polynomial finite result of above that cannot be accounted for by the renormalized effective potential, which is only polynomial in ν.

Here is that IR-UV connection enters in a crucial way, and the non-polynomial result is obtained by the cancellation of the pole term in the regularized volume (IR) with the $O(\epsilon)$ term in the regularized effective potential (UV). Only in this way is the naive factorization bypassed. In fact, one can check that the coefficient of the non-polynomial part in the CFT computation is precisely the $\mathcal{L}$ factor, rather than the regularized volume $\mathcal{V}$.

That is, we have to keep track of the $O(\epsilon)$ term in the expansion of the regularized effective potential (4.18, 4.19)

$$\frac{\partial}{\partial m^2}\left(V_{D+1}^+ - V_{D+1}^-\right) \;=\; \mathcal{A}_d(\nu) \;+\; \epsilon\,\cdot \mathcal{B}_d(\nu) \;+\; o(\epsilon), \tag{4.41}$$

where

$$\mathcal{B}_d(\nu) = \frac{\mathcal{A}_d(\nu)}{2}\left\{\log(4\pi) + \psi(\frac{1}{2}-\frac{d}{2}) - \psi(\frac{d}{2}+\nu) - \psi(\frac{d}{2}-\nu)\right\}\ , \tag{4.42}$$

is almost the non-polynomial part of above.

After using two identities for $d = even$, $\psi(\frac{1}{2}-\frac{d}{2}) = \psi(\frac{1}{2}+\frac{d}{2})$ and then $2\psi(1+d) = 2\,\log 2 + \psi(\frac{1}{2}+\frac{d}{2}) + \psi(1+\frac{d}{2})$ -which are the "log-derivatives" of Euler's reflection and Legendre duplication formula respectively (C.23), one can finally write the renormalized CFT_d result (4.40) in terms of the bulk quantities (4.14b, 4.42) for $d = even$ as

$$\frac{1}{2\nu}\frac{\partial}{\partial\nu}\left(W_d^+ - W_d^-\right) = \mathcal{V}_{d+1}\ \cdot \mathcal{A}_d(\nu) \;+\; \mathcal{L}_{d+1}\ \cdot \mathcal{B}_d(\nu) \;=\; \frac{\partial}{\partial m^2}\left(S_{d+1}^+ - S_{d+1}^-\right)\ . \tag{4.43}$$

4.4 Miscellany

4.4.1 Breitenlohner-Freedman mass

Our main results (4.35, 4.37, 4.43) still contain a mass derivative (equivalently, derivative with respect to ν). Integrating these equations introduces an integration constant which cannot be fixed without further input. Equivalently, so far we only know (see (4.25))

$$W_d(\nu) - W_d(\nu_0) \;=\; -\log\left(\frac{Z_\beta(\nu)}{Z_\alpha(\nu)}\,\frac{Z_\alpha(\nu_0)}{Z_\beta(\nu_0)}\right)\;. \qquad (4.44)$$

Beyond dimensional regularization, in the framework of general renormalization theory, there appear free polynomials in ν anyway. Hence fixing this constant should be part of the physically motivated normalization conditions.

It was argued in [65] that both Z_α and Z_β at the BF mass, i.e. $\nu = 0$, should coincide; the argument given was shown in [67] to apply to the vacuum energy rather than to the effective potential and the equality was argued in a different way, replacing the BF mass by infinity as a reference point. We just want to point out that this procedure also has a potential loophole, namely the integration range exceeds the window in which the two CFTs are defined $m^2_{BF} \leq m^2 < 1 + m^2_{BF}$.

Drawing attention by the last comment to the case $\nu = 0$, another remark is in order. Then in (4.25) the product $f\,g_l$ is ill defined if f is assumed to be $\nu-$independent, as in [64, 67]. However, if one chooses

$$\widetilde{f} = f\;\pi^{\frac{d}{2}}\,\frac{\Gamma(1-\nu)}{\nu\,\Gamma(\frac{d}{2}-\nu)} \qquad (4.45)$$

as the true $\nu-$independent quantity, then the product $f(\nu)\,g_l(\nu) \equiv \widetilde{f}k_l(\nu)$ is well defined at $\nu = 0$. The relative factor between f and $\widetilde{f}$ can be traced back to the conventional normalization of the two point functions. In addition, while f is the coefficient of the relevant perturbation of the α-CFT, $\widetilde{f}$ appears in the parametrization of the boundary behavior of the bulk theory [64, 67]. However, fortunately, a switch from f to $\widetilde{f}$ has no effect on the limit $f \to \infty$ in (4.25) and the conclusions drawn from it in the previous section. This follows from the observation stated after eq. (4.29): any rescaling of f by a factor independent of l does not affect the limit.

The issue of the integration constant discussed above leading to (4.44) has still another aspect concerning the treatment of (4.25). Starting from the formal expression for $W_d(\nu) - W_d(\nu_0)$ on the r.h.s. we would get $\log(\frac{1+fg_l(\nu)}{1+fg_l(\nu_0)})$ instead of $\log(1+fg_l(\nu))$. Now the limit $f \to \infty$ is well defined for each l. The definition of the regularized sum over l could then be done directly with the summands referring to the difference of the two limiting conformal theories. Finally, differentiation with respect to ν would reproduce all our results of section 3.

4.4.2 Generalized Gel'fand-Yaglom formula

Recently, in ref. [67] a "kinematic explanation" of the equivalence of the bulk and boundary computation has been given . There a polar basis in $\mathbb{H}^{d+1}$ was used to compute the bulk fluctuation determinants. After inserting the eigenvalues of the angular Laplacian one ends up with a sum over spherical harmonics of effective radial determinants which are now one-dimensional. Using then a proposed generalization of the Gel'fand-Yaglom formula [52], it results in the same expansion as obtained on the boundary (4.25). Since, especially in their reasoning, it should be crucial to have a well defined limit $f \rightarrow \infty$ *b*efore the sum over l is taken, we would prefer to consider the cross-ratios

$$\frac{\det_{\widetilde{f_1}}(-\Delta_{rad} + \mathcal{V}_{eff}(\nu))}{\det_{\widetilde{f_1}}(-\Delta_{rad} + \mathcal{V}_{eff}(\nu_0))} \cdot \frac{\det_{\widetilde{f_2}}(-\Delta_{rad} + \mathcal{V}_{eff}(\nu_0))}{\det_{\widetilde{f_2}}(-\Delta_{rad} + \mathcal{V}_{eff}(\nu))} \tag{4.46}$$

instead of the single ratios obtained by dropping the ν_0 determinants. Besides giving a well defined limit for $\widetilde{f_1} \rightarrow \infty$, $\widetilde{f_2} \rightarrow 0$ this has the additional benefit that no generalization of the Gel'fand-Yaglom formula beyond that in [76] is needed to handle the ratio for operators with different boundary conditions; each of the two ratios in (4.46) refer to the same boundary condition. Even though the above recipe makes finite the quotient of the effective radial determinants, the inclusion of the infinite tower of harmonics makes the sum divergent. This remaining divergence is then the only source for IR divergence in the bulk and UV ones on the boundary. The *formal* equality calls for a more ambitious program including generic dimension and not only the matching of the anomalous part. There is nothing in the derivation that picks out $d = even$ in preference to $d = odd$. What we have shown in the previous section is that the equality can indeed be made rigorous if interpreted in the sense of dimensional regularization.

4.5 Discussion

4.5.1 Physical relevance

The relevant double-trace deformation of a CFT and its AdS dual picture provide a satisfactory test of the correspondence. The regimes in which the bulk and boundary computations are legitimate fully overlap and the mapping goes beyond the original correction to the conformal anomaly. Rather the full change in the partition functions is correctly reproduced on either side of the correspondence; on the boundary being subleading $O(1)$ in the large N limit, and in the bulk being a 1-loop quantum correction to the classical gravity action. Dimensional regularization proved to be the simplest and most transparent way to control the divergences on both sides, UV and IR infinities are then on equal footing, in accord with the IR-UV connection.

The anomaly turns out to be the same computed in DR and in zeta-function regularization, confirming its stability with respect to changes in the regularization method used [31]. They differ, however, in the regularized value of the boundary determinant for even dimension; this is also known to be the case in free CFTs in curved backgrounds when computing the regularized effective potential [31]. Going back to the anomaly, we recall that it arises in DR due to a cancellation of the pole against a zero, in fact minus the variation of the counterterms is the variation of the renormalized effective action [31, 106]. The pole we already had from the volume regularization and the zero comes from the invariance of $\mathcal{L}$ [60].

At odd d, the finite non-zero bulk change in the effective potential is reproduced from a finite remnant in the boundary computation, confirming the suspicion in [64]. But there is no anomaly in this case, just a conformally invariant renormalized contribution to the partition functions. At even d, in turn, the boundary change in the partition function is obtained only after a subtle cancellation of the pole in regularized volume (IR-div.) against a zero from the change in the effective potential (UV-div.). This mixing of IR and UV effects on the same side of the correspondence has no precedent in the leading order computation [69]. In that case the bulk computation is a tree level one, where no UV problems show up; i.e. the AdS answer is obtained from the classical SUGRA action.

We can contemplate several extensions of the program carried out. One can try to access to an intermediate stage of the RG flow, that is, finite f. Being away from conformality, the factorization of the volume breaks downs, the propagators at coincidence points depend on the radial position; this makes the task of regularization more difficult. Extensions to other bulk geometries seems, naively, immediate in terms of the Plancherel measure, it admits a readily generalization to symmetric spaces [68]. It would be interesting to explore whether this construction admits a holographic interpretation. In the other direction [6], one can trade the round sphere by a "squashed" one,

[6]We end up this section with a plausible connection to Polyakov formulas. Since these formulas are related to extremal problems and sharp inequalities, it would also be a chal-

conformal boundary of Taub-Nut-AdS and Taub-Bolt-AdS spacetimes.

4.5.2 IR-UV connection once more

One may not be fully satisfied by the dimensional regularization used in our derivation [7]. It was certainly the most economic way to get full agreement between bulk and boundary computations. However, to have a hint to which regularization on the boundary should reproduce the IR-cutoff bulk computation let us illustrate how the infinite volume of AdS is hidden in the sum over degeneracies [8]. Consider an UV-cutoff $1 \ll l_c < \infty$ in the sum over spherical harmonics [9]

$$\sum_{l=0}^{l_c} \deg(d, l) \ \sim \ (l_c)^d. \tag{4.47}$$

Now compare this term with the leading divergent term in the volume of AdS after introducing an IR-cutoff ϵ

$$\int dvol_{\mathbb{H}^{d+1}} \ = \ 2^{-d}\, vol_{\mathbb{S}^d} \int_{\epsilon}^{1} dr\, r^{-d-1}\, (1-r^2)^d \ \sim \ (\epsilon)^{-d}. \tag{4.48}$$

Last thing to do, according to the IR-UV connection, is to relate the two cutoffs as $l_c\, \epsilon \sim 1$.

To see this more clearly, consider the same counting on the 3-sphere. Now, $N^2\, l_c^3$ essentially counts the degree of freedom of the UV-cutoff gauge theory. On the other hand, the area of this embedded surface grows as $L^3\, \epsilon^{-3}$. As we know from the AdS/CFT dictionary $N^2/L^3 \sim 1/A_{P_5}$, where A_{P_5} is the 5-dim Planck area. Therefore, the number of degree of freedom per unit Planck area turns out to be

$$\frac{\mathcal{N}_{dof}}{A/A_{P_5}} \sim (l_c\, \epsilon)^3. \tag{4.49}$$

So that the holographic bound and the IR-UV connection are in conformity with the identification of the cutoffs as $\epsilon \sim 1/l_c$. Therefore, the leading divergence in the sum corresponds to the leading divergent term in the volume.

lenge to find the physical interpretation of these inequalities. Probably, the first two physically motivated inequalities that come to mind are the c-theorem and the positive mass conjecture.

[7] Mathematicians are always skeptic with respect to these formal manipulations.

[8] I am indebted to H.S. Yang for elucidation of this point.

[9] This construction readily admits a generalization to any compact manifold without boundaries in terms of the Weyl asymptotics (see e.g. [24]). The counting function of eigenstates of the Laplacian is estimated as $\lambda_c^{d/2}$, where λ_c is the largest eigenvalue. In our case, we simply have $\lambda = l(l+d-1)$.

4.5.3 Connection to mathematics

Finally, on the basis of the impressive agreement, one may wonder whether there is a parallel computation in the mathematical literature. If one is willing to allow for a continuation in Δ_- so that it becomes $d/2 - k$ ($k = 1, 2, ..., d/2$), then $\Xi \sim \langle O_\alpha O_\alpha \rangle \sim 1/s^{2\Delta_-}$ can be thought of as the inverse of the k-th GJMS conformal Laplacian [58]. These are conformally covariant differential operators whose symbol is given the k-th power of the Laplacian; for $k = 1$ one has just the conformal Laplacian (Yamabe operator), $k = 2$ corresponds to the Paneitz operator, etc. For even $d + 1$, we find then an analogous result in theorem 1.4 of [66] for a generalized notion of determinant of the k-th GJMS conformal Laplacian. The absence of anomaly for odd d is consistent with this determinant being a conformal invariant of the conformal infinity of the even dimensional asymptotically hyperbolic manifold. Unfortunately, "the delicate case of $d+1$ odd where things do not renormalize correctly", is still to be understood in this mathematical setting. We anticipate, by analogy with our results, that a proper analysis in this case should unravel a conformal anomaly which can be read off from quotient formulas (generalized Polyakov formulas, see e.g. [18]) of determinants of GJMS operators at conformally related metrics, which involve the higher-dimensional Q-curvatures.

We expect the AdS/CFT recipe to treat double-trace deformations and its bulk interpretation to be a way into these constructions in conformal geometry. The leading large-N anomaly matching already hinted in this direction, the relation between Q-curvature and volume renormalization emerges there; but higher-dimensional Q-curvatures in connection with generalized Polyakov formulas for GJMS operators had not shown up so far [10].

[10] In fact, one can even argue that the AdS/CFT prescription entails a holographic derivation of these Polyakov formulas and correctly reproduces the Q-curvature term [33].

Conclusion and Outlook

AdS/CFT correspondence relates two seemingly different theories, gauge theory and gravity, via two deeply rooted ideas in physics, namely, the string emerging at the large-N limit of the gauge theory and the holographic principle. Ten years after its birth and in spite of the many successful tests, AdS/CFT remains in the status of "true but not proven". However, this does not prevent it from "laying golden eggs" and there are plenty of new developments ranging from connections to mathematics to heavy-ion physics.

In this thesis we have explored three aspects of the AdS/CFT correspondence. In the BMN/plane-wave limit, we obtained the spinor and vector propagators. The important aspect emerging from the analysis carried out is the (WKB) semiclassical exactness [11] of the Green's functions in the plane wave obtained as Penrose limit of a given spacetime. The Schwinger-DeWitt construction and the Penrose limit are naturally connected since they both share the property of being based on a near light-cone expansion. We also learned how to take the limit directly on the propagators of the original spacetime, the resummation implicit in the SD-expansion was the crucial ingredient. Finally, for different radii of AdS and the sphere the geometry is no longer conformally flat. However, it is conformal to a spacetime with a conical singularity; intuitively one expects that this should play some role and, indeed, in this case the propagators were found to be related by Sommerfeld's formula, the very same that relates the heat kernel on the plane with that on the cone. Yet, string theory in the plane wave background is understood as being dual to a limit of the gauge theory but it is widely anticipated that there is some smaller CFT, not yet identified, which is precisely dual to the string side. Our understanding of propagators in the plane wave may give some insight into the nature of the CFT itself.

The program starting with free fields and ending up in AdS was pushed forward for the nontrivial case of four-point functions and we achieved "half-the-way" to AdS by an alternative "gluing up" rewriting the correlators as

[11] Not to be confused with WKB exactness of pp-waves, which is only true at the linearized level where the details encoded in the VVM-determinant are washed away.

CPW expansion in the three channels involving only the singlet bilinears. This is reminiscent of the infinite tower of exchange Witten graphs that should be included in the bulk computation and might facilitate the comparison with this technically challenging calculation. This also hints at the existence of a decoupled theory of massless HS fields in the bulk of AdS (consistent truncation). Various useful results involving the two and three point functions of the HS currents were derived. In addition, the connection to the IR fixed point made possible to obtain the analog CPW expansion and check fusion coefficients as well as two and three point functions via the amputation procedure (Legendre transformation). With the insight gained in the vector model, we should go back to gauge theory to parallel, at least in a restricted (closed) sector, the results at hand.

Finally, the effect of double trace deformations provided a clean test of the AdS/CFT prescription for the partition functions involving a 1-loop quantum correction in the bulk and a subleading contribution in the large N limit at the boundary. We fully mapped the conformal anomaly and renormalized partition functions. The IR-UV connection was crucial to find agreement with the pure CFT computation. With the work of Henningson and Skenderis on the holographic anomaly, new developments in conformal geometry were triggered by the discovery of new purely geometric invariants of conformally compact Einstein manifolds. As we saw in the preliminary chapter, the gravitational action on the bulk turns out to be proportional to the volume for an Einstein metric and an appropriate volume renormalization was carried out in the context of AdS/CFT correspondence which predicts that the volume anomaly is a particular linear combination of functional determinant anomalies of the Laplacian on scalars, spinors and 1-forms. The volume anomaly is closely related to the higher-dimensional Q-curvature which in turn is the universal term in the functional determinant anomaly for GJMS operators. But these operators don't play any role in the leading large-N boundary computation of Henningson and Skenderis and the original excitement seems to be fading away. However, this two-sided nature of the Q-curvature seems to be precisely what lies behind the matching of conformal anomalies that we have in the case of double-trace deformations and we hope this set up to be a new bridge between AdS/CFT correspondence and conformal geometry where higher-dimensional Q-curvature, GJMS operators and generalized Polyakov formulas arise in a natural way.

Appendix A

Appendix to chapter 2

A.1 Geodesic and chordal distances in ESU

Embedding the n-sphere in $(n+1)$-Euclidean space, a point on the sphere is given by the vector $a\,\hat{\Omega}$, with

$$\hat{\Omega} = (\cos\alpha\ \cos\beta\,,\ \cos\alpha\ \sin\beta\,,\ \sin\alpha\ \hat{\omega}) \tag{A.1}$$

where $\hat{\omega}$ is a unit vector on the $(n-2)$-sphere and the parametrization is as in equation (2.4). The chordal distance squared $\mu_n(x,x')$ between two points x and x' is related to the arc $\chi_n(x,x')$ (direct geodesic distance) by

$$1 - \frac{\mu_n}{2a^2} = \cos\frac{\chi_n}{a} = \cos\alpha\ \cos\alpha'\ \cos(\beta-\beta') + \sin\alpha\ \sin\alpha'\ \cos\frac{\chi_{n-2}}{a} \tag{A.2}$$

where $\cos\frac{\chi_n}{a} \equiv \hat{\Omega}\cdot\hat{\Omega}'$ and $\cos\frac{\chi_{n-2}}{a} \equiv \hat{\omega}\cdot\hat{\omega}'$, being χ_{n-2} the arc along the $(n-2)$-sphere. Let us take for simplicity x' to be at the origin.

Going to the local coordinates (equation 2.6) and expanding in inverse powers of the radius

$$\cos\frac{\chi_n}{a} = \cos\alpha\ \cos\beta = \cos u - \frac{\Phi}{a^2} + O(a^{-4}) \tag{A.3}$$

$$\frac{\chi_n}{a} = u + \frac{\Psi}{a^2} + O(a^{-4}), \tag{A.4}$$

where

$$\Phi = v\sin u + \frac{\vec{x}^2}{2}\ \cos u \tag{A.5}$$

$$\Psi = v + \frac{\vec{x}^2}{2}\ \cot u. \tag{A.6}$$

These two quantities naturally arise in the plane wave, $u\Psi$ is the geodetic interval [75, 92] and 2Φ is the limiting value of the total chordal distance squared in $AdS \times S$ as elucidated in [42]. Going back to our ESU, it is easy to see that this also holds provided one compactifies the time into a circle so that t becomes an angle. That is, as $a \to \infty$ one has

$$\text{geodetic interval} = \frac{-a^2t^2 + \chi^2}{2} \to u\Psi \tag{A.7}$$

$$\frac{\text{total chordal distance squared}}{2} = -a^2[1 - \cos t] + \frac{\mu}{2} \to \Phi. \tag{A.8}$$

A.2 Spinor Geodesic Parallel Transporter

Let us go to the frame given by

$$ds^2 = 2\theta^+\theta^- + \vec{\theta} \cdot \vec{\theta} \equiv \eta_{ab}\theta^a\theta^b \tag{A.9}$$

$$\theta^+ = du, \qquad \theta^- = dv - \frac{1}{2}\vec{x}^2 du, \qquad \vec{\theta} = d\vec{x}. \tag{A.10}$$

The spin-connection components can be read off from the first Cartan structure equation

$$d\theta^a + \omega^a_b \wedge \theta^b = 0 \tag{A.11}$$

(tangent indices $a, b = +, -, i$ with $i = 1, ..., d-2$ being the transverse ones) and the only nonvanishing ones are

$$\omega^{i-} = -\omega^{-i} = x^i\, du. \tag{A.12}$$

The covariant derivative on spinors

$$\nabla_\mu \equiv \partial_\mu + \Gamma_\mu = \partial_\mu + \frac{1}{4}\omega^{ab}_\mu \gamma_a\gamma_b\,, \tag{A.13}$$

where the $\gamma's$ fulfill the Clifford algebra in tangent space

$$\{\gamma_a,\, \gamma_b\} = 2\eta_{ab}\,\mathbb{I}, \tag{A.14}$$

is found to be

$$\nabla_\mu = \begin{cases} \partial_u - \frac{1}{2}\gamma_-\vec{\gamma}\cdot\vec{x} \\ \partial_v \\ \partial_i \end{cases} \tag{A.15}$$

that is, only Γ_u is nonzero. An important property is that $(\Gamma_u)^2 = 0$ because $(\gamma_-)^2 = \mathbb{I}\,\eta_{--} = 0$, i.e. Γ_u is nilpotent.

The spinor D'Alembertian can be written in terms of the scalar one as

$$g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) + 2\Gamma_u\partial_v\,. \tag{A.16}$$

The spinor parallel transporter is a bi-spinor that parallel transports a spinor along a given path and the path we need is the geodesic connecting the two points. This spinor geodesic parallel transporter must satisfy the parallel transport equation and the initial condition

$$\partial^\mu\sigma\ \nabla_\mu\mathbb{U}(x,x') = 0\,, \qquad \mathbb{U}(x,x) = \mathbb{I} \tag{A.17}$$

One can write a Dyson-type representation for it (see, e.g. [22]), integrating along the geodesic emanating from x' [43]

$$\mathbb{U}(t) = \mathbf{P}\exp-\int_0^t\Gamma_\mu(\tau)dx^\mu(\tau). \tag{A.18}$$

But for the plane wave metric, due to the nilpotency of Γ_μ, one can drop the path ordering symbol $\mathbf{P}$ because the matrices in the exponent commute, therefore one can perform the integration to get

$$\mathbb{U}(x,x') = \exp\frac{1}{2}\gamma_-\vec{\gamma}\cdot(\vec{x}+\vec{x}')\tan\frac{u-u'}{2} = \mathbb{I} + \frac{1}{2}\gamma_-\vec{\gamma}\cdot(\vec{x}+\vec{x}')\tan\frac{u-u'}{2}. \tag{A.19}$$

Finally, one can easily check that $\Box\,\mathbb{U}(x,x') = 0$.

A.3 Vector Geodesic Parallel Transporter

The Christoffel symbols for the plane wave metric can be directly read off from the geodesic equations which in turn can be derived from the Lagrangian

$$L(\dot{u},\dot{v},\dot{\vec{x}},\vec{x}) = \frac{1}{2}\dot{x}^\mu\dot{x}_\mu = \dot{u}\dot{v} + \frac{1}{2}\dot{\vec{x}}^2 - \frac{1}{2}\dot{u}^2\,\vec{x}^2, \tag{A.20}$$

where the dots are derivatives with respect to an affine parameter along the geodesic. The geodesic equations read

$$\ddot{u} = 0 \tag{A.21a}$$

$$\ddot{v} - 2\vec{x}\cdot\dot{\vec{x}}\dot{u} = 0 \tag{A.21b}$$

$$\ddot{\vec{x}} + \dot{u}^2\vec{x} = 0 \tag{A.21c}$$

and therefore the only nonzero Christoffels are

$$(\Gamma_u)^i_{\ u} = x^i\,, \qquad (\Gamma_u)^v_{\ i} = (\Gamma_i)^v_{\ u} = -x^i\,. \tag{A.22}$$

There are two types of geodesics [43]: type-A when $\dot{u}=0$ and the null ones in this category are parallel to the propagation direction of the wave, and type-B when one can take u as the affine parameter which is the generic situation. For this generic case, the Lagrangian (A.20) is $\frac{1}{2}\dot{x}^{\mu}\dot{x}_{\mu}=const$ and reproduces the one for a harmonic oscillator of unit mass and unit frequency plus an extra $\dot{v}$ term. Then, it is not difficult to see that the recipe to get the geodetic interval between two generic points is just the replacing by the classical action for the oscillator between two points $\vec{x}$ and $\vec{x}'$ followed by the shifts $u\to u-u'$ and $v\to v-v'$, so that

$$\begin{aligned}\text{geodetic interval} &= (u-u')(v-v')+(u-u')\\ &\times\left[\frac{\vec{x}^{\,2}+\vec{x}'^{\,2}}{2}\cot\left(u-u'\right)-\vec{x}\cdot\vec{x}'\,\csc\left(u-u'\right)\right]\end{aligned} \tag{A.23}$$

and for type-A, one just has to let $u\to 0$ which simply produces

$$\text{geodetic interval}=\frac{(\vec{x}-\vec{x}')^2}{2}. \tag{A.24}$$

This recipe also works for the quantities Ψ,Φ and $\triangle$, previously defined.

The vector parallel transporter is a bi-vector that parallel transports a vector along a given path, and the path we need is the geodesic connecting the two points. This vector geodesic parallel transporter must satisfy the parallel transport equation and the initial condition

$$\partial^{\rho}\sigma\;\nabla_{\rho}P_{\mu\nu'}(x,x')=0\,,\qquad P_{\mu\nu}(x,x)=g_{\mu\nu}(x). \tag{A.25}$$

One can also write a Dyson-type representation for it (see, e.g. [23]), integrating along the geodesic emanating from x' [43]

$$P^{\mu}_{\nu'}(x,x')=\mathbf{P}\exp-\int_0^t\left(\Gamma_{\rho}\right)^{\mu}_{\nu'}(\tau)\,dx^{\rho}(\tau). \tag{A.26}$$

But for the plane wave metric one can check that Γ_{ρ} as a matrix, with μ and ν' labeling its rows and columns respectively, commutes with itself at different points. One can therefore drop the path ordering symbol and perform the integration to get

$$P^{\mu}_{\nu'}(x,x') = \exp\begin{pmatrix} 0 & 0 & \vec{0} \\ \frac{\vec{x}^2-\vec{x}'^{\,2}}{2} & 0 & (\vec{x}+\vec{x}')\tan\frac{u-u'}{2} \\ -(\vec{x}^{\,t}+\vec{x}'^{\,t})\tan\frac{u-u'}{2} & \vec{0}^{\,t} & \mathbb{O} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \vec{0} \\ \frac{\vec{x}^2-\vec{x}'^{\,2}}{2} - \frac{|\vec{x}+\vec{x}'|^2}{2}\tan^2\frac{u-u'}{2} & 1 & (\vec{x}+\vec{x}')\tan\frac{u-u'}{2} \\ -(\vec{x}+\vec{x}')\tan\frac{u-u'}{2} & \vec{0}^{\,t} & \mathbb{I} \end{pmatrix}. \qquad \text{(A.27a)}$$

Appendix B

Appendix to chapter 3

B.1 Restrictions from Conformal Invariance

Conformal invariance dictates the form the three-point function of two scalars, of dimension Δ_i, with a totally symmetric traceless rank l tensor, of dimension Δ, to be (see e.g. [50])

$$\langle \phi_1(x_1)\phi_2(x_2)O^{(l)}_{\mu_1\ldots\mu_l}(x_3)\rangle =$$

$$C_{\phi_1\phi_2O^{(l)}} \frac{1}{r_{12}^{(\Delta_1+\Delta_2-\Delta+l)/2} r_{13}^{(\Delta+\Delta_{12}-l)/2} r_{23}^{(\Delta-\Delta_{12}-l)/2}} \lambda^{x_3}_{\mu_1\ldots\mu_l}(x_1,x_2), \tag{B.1}$$

where

$$\lambda^{x_3}_{\mu_1\ldots\mu_l}(x_1,x_2) = \lambda^{x_3}_{\mu_1}(x_1,x_2)\ldots\lambda^{x_3}_{\mu_l}(x_1,x_2) - traces, \tag{B.2a}$$

$$\lambda^{x_3}_{\mu}(x_1,x_2) = (\frac{x_{13}}{r_{13}} - \frac{x_{23}}{r_{23}})_\mu, \tag{B.2b}$$

and $\Delta_{ij} = \Delta_i - \Delta_j$.
Also the form of the two-point function of the symmetric traceless rank l tensor, which defines an orthogonality relation with respect to spin and conformal dimension, is required to be (see e.g. [50])

$$\langle O^{(l)}_{\mu_1\ldots\mu_l}(x_1)O^{(l)}_{\nu_1\ldots\nu_l}(x_2)\rangle = C_{O^{(l)}} \frac{1}{r_{12}^{\Delta}} \; sym\{I_{\mu_1\nu_1}(x)\ldots I_{\mu_l\nu_l}(x)\} \tag{B.3}$$

where

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{r} \tag{B.4}$$

is the inversion tensor, related to the Jacobian of the inversion $x_\mu \to x_\mu/r$, and *sym* means symmetrization and removal of traces.

The structure of the general four point conformal correlator is required to be

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \prod_{i<j,1}^{4} (r_{ij})^{(\Sigma/3-\Delta_i-\Delta_j)/2} F(u,v), \tag{B.5}$$

where $\Sigma = \Delta_1 + ... + \Delta_4$ and F is an arbitrary function of the invariant ratios.

B.2 HS Two-Point Function Coefficient

The double sum can be cast into the form

$$\frac{1}{4}\sum_{k=0}^{l}(-1)^k\binom{l}{k}\frac{(\delta)_l\,(\delta)_{2l-k}}{(\delta)_{l-k}}\sum_{s=0}^{l}(-1)^s\binom{l}{s}\frac{(\delta+k)_s\,(\delta+2l-k)_{-s}}{(\delta)_s\,(\delta+l)_{-s}}. \tag{B.6}$$

The last sum can be transformed, by means of elementary identities such as $(-1)^k\binom{n}{k} = \frac{(-n)_k}{k!}$ and $(-z)_n = (-1)^n\frac{1}{(1+z)_{-n}}$, in a terminating generalized hypergeometric series ${}_3F_2$ of unit argument

$$\sum_{s=0}^{l}\frac{1}{s!}(-l)_s\frac{(\delta+k)_s\,(1-\delta-l)_s}{(\delta)_s\,(1-\delta-2l+k)_s} = {}_3F_2\left(\begin{matrix}-l,\delta+k,1-l-\delta\\ \delta,1-2l-\delta-k\end{matrix}\right). \tag{B.7}$$

The evaluation of ${}_3F_2$ can be done by applying twice the same identity used in eq.(4.8),

$${}_3F_2\left(\begin{matrix}-n,a,b\\ d,e\end{matrix}\right) = \frac{(e-a)_n}{(e)_n}\,{}_3F_2\left(\begin{matrix}-n,a,d-b\\ d,1+a-n-e\end{matrix}\right) \tag{B.8}$$

to get

$$\begin{aligned}&\frac{(k-l)_l}{(1-2l-\delta+k)_l}\frac{(1-k+l)_k}{(1-k)_k}\,{}_3F_2\left(\begin{matrix}-k,-l,2\delta+l-1\\ \delta,-l\end{matrix}\right)\\ &= \frac{(k-l)_l}{(1-2l-\delta+k)_l}\frac{(1-k+l)_k}{(1-k)_k}\frac{(1-\delta-l)_k}{\delta_k} = (-1)^k\frac{l!}{(\delta)_k}(\delta+l)_{l-k}\end{aligned} \tag{B.9}$$

where the ${}_3F_2$ reduced to an ordinary ${}_2F_1$ of unit argument evaluated with the Chu-Vandermonde formula [5].

The sum that remains to be done reduces then again to a terminating ordinary hypergeometric of unit argument that is evaluated as before

$$\begin{aligned}\frac{1}{4}\,l!\,(\delta)_l\sum_{k=0}^{l}\binom{l}{k}\frac{1}{(\delta)_k\,(\delta+l)_{-k}} &= \frac{1}{4}\,l!\,(\delta)_l\,{}_2F_1\left(\begin{matrix}-l,1-\delta-l\\ \delta\end{matrix}\right)\\ &= \frac{1}{4}\,l!\,(2\delta-1+l)_l.\end{aligned} \tag{B.10}$$

B.3 CPW Recurrences

Inserting the OPE [1] (3.16) and using the orthogonality relation (B.3) we have for the action of the derivative operator

$$\langle\phi_1(x_1)\phi_2(x_2)\, O^{(l)}_{\mu_1\ldots\mu_l}(x_3)\rangle$$
$$= \frac{C_{\phi_1\phi_2 O^{(l)}}}{C_{O^{(l)}}} \frac{1}{r_{12}^{(\Delta_1+\Delta_2-\Delta+l)/2}} C^{(l)}(x_{12},\partial_{x_2})_{\nu_1\ldots\nu_l} \langle O^{(l)}_{\nu_1\ldots\nu_l}(x_2)\, O^{(l)}_{\mu_1\ldots\mu_l}(x_3)\rangle. \tag{B.11}$$

Inserting now the OPE (3.16) in the scalar four-point function one gets for the contribution of $O^{(l)}$ and its descendants

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle \sim \frac{C_{\phi_1\phi_2 O^{(l)}}}{C_{O^{(l)}}} \frac{1}{r_{12}^{(\Delta_1+\Delta_2-\Delta+l)/2}}$$
$$\times C^{(l)}(x_{12},\partial_{x_2})_{\mu_1\ldots\mu_l} \langle O^{(l)}_{\mu_1\ldots\mu_l}(x_2)\, \phi_3(x_3)\phi_4(x_4)\rangle. \tag{B.12}$$

In order to be able to act as before with the derivative operator on a two-point function, one has to re-write the x_2-dependence in $\langle O^{(l)}_{\mu_1\ldots\mu_l}(x_2)\, \phi_3(x_3)\phi_4(x_4)\rangle$ in a suitable way. This is achieved by introducing the 'shadow' operator (conformal partner) $O^{*(l)}$, a 'conventional' operator with labels $(\Delta^*, l) = (d-\Delta, l)$

$$O^{(l)}_{\mu_1\ldots\mu_l}(x_2) = \int d^d x\, \langle O^{(l)}_{\mu_1\ldots\mu_l}(x_2)\, O^{(l)}_{\nu_1\ldots\nu_l}(x)\rangle\, O^{*(l)}_{\nu_1\ldots\nu_l}(x). \tag{B.13}$$

Inserting this relation and using (B.11) one gets

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle \sim \int d^d x \langle\phi_1(x_1)\phi_2(x_2)\, O^{(l)}_{\mu_1\ldots\mu_l}(x)\rangle$$
$$\times\langle O^{*(l)}_{\mu_1\ldots\mu_l}(x)\, \phi_3(x_3)\phi_4(x_4)\rangle. \tag{B.14}$$

In all, one has just inserted the projection operator [77, 87]

$$\mathcal{P}_l = \int d^d x\, O^{(l)}_{\mu_1\ldots\mu_l}(x)|0\rangle\, \langle 0|O^{*(l)}_{\mu_1\ldots\mu_l}(x). \tag{B.15}$$

The integrand can be cast into a form involving Gegenbauer polynomials after contraction of Lorentz indices, and using their recurrence relations one

[1] The OPE involves the sum over the complete set of quasi-primaries. We consider no degeneracies for simplicity, ie. no additional labels apart from (Δ, l).

gets the following recurrences[2] [40]

$$
\begin{aligned}
&G^{(l)}(b,e,S;u,v) = \frac{1}{2}\frac{S+l-1}{d-S+l-2} \\
&\times \left\{ \frac{d/2-e-1}{f+l-1} \Big(v\, G^{(l-1)}(b+1,e+1,S;u,v) - G^{(l-1)}(b,e+1,S;u,v) \Big) \right. \\
&\left. + \frac{d/2-f-1}{e+l-1} \Big(G^{(l-1)}(b,e,S;u,v) - G^{(l-1)}(b+1,e,S;u,v) \Big) \right\} \\
&- \frac{1}{4}\frac{(S+l-1)(S+l-2)}{(d-S+l-2)(d-S+l-3)} \frac{(d/2-e-1)(d/2-f-1)}{(f+l-1)(e+l-1)} \\
&\times \frac{(l-1)(d+l-4)}{(d/2+l-2)(d/2+l-3)}\, u\, G^{(l-2)}(b+1,e+1,S;u,v), \qquad \text{(B.16)}
\end{aligned}
$$

with $S = e+f+l$. The starting point is the scalar result that in the direct channel limit, $u, 1-v \sim 0$ is given by the double power expansion

$$
G^{(0)}(b,e,S;u,v) = \sum_{m,n=0}^{\infty} \frac{(S-b)_n (S-e)_n}{(S+1-d/2)_n} \frac{(b)_{n+m}(e)_{n+m}}{(S)_{2n+m}} \frac{u^n}{n!} \frac{(1-v)^m}{m!}. \qquad \text{(B.17)}
$$

B.4 UV Fusion Coefficients

Using the double expansion in the direct channel limit $u, 1-v \sim 0$, the sum we have to perform is

$$
\sum_{l \geq 0,\, even} (\gamma_l^{uv})^2 \, a_{nm}^{(l)} \,, \qquad \text{(B.18)}
$$

where

$$
a_{nm}^{(l)} = a_{0l}^{(l)} \sum_{s=0}^{n} (-1)^s \binom{n}{s} \binom{m+n+s}{l} \frac{(\delta+l)_{m+n-l} (\delta+l)_{m+n-l+s}}{(2\delta+2l)_{m+n-l+s}} \qquad \text{(B.19)}
$$

and $(\gamma_l^{uv})^2$ given in 3.27.

First perform the sum over l, due to the triangular structure the sum is up to $m+2n$, and sum over all $l's$ writing

$$
(\gamma_l^{uv})^2 = [1+(-1)^l] \frac{{(\delta)_l}^2}{(2\delta+l-1)_l} \frac{16N}{a_{0l}^{(l)}}. \qquad \text{(B.20)}
$$

[2]In fact, by the previous procedure one gets in addition the contribution from the shadow operator. What follows is valid for the direct contribution.

Now make straightforward manipulations[3] to rewrite the sum in terms of terminating well-poised generalized hypergeometric ${}_3F_2$ of argument ± 1 as

$$16N\,(\delta)_{m+n}\sum_{s=0}^{n}(-1)^s\binom{n}{s}\frac{(\delta)_{m+n+s}}{(2\delta)_{m+n+s}}$$

$$\times\left\{{}_3F_2^-\begin{pmatrix}-m-n-s,\delta+\frac{1}{2},2\delta-1\\ 2\delta+m+n+s,\delta-\frac{1}{2}\end{pmatrix}+{}_3F_2^+\begin{pmatrix}-m-n-s,\delta+\frac{1}{2},2\delta-1\\ 2\delta+m+n+s,\delta-\frac{1}{2}\end{pmatrix}\right\}. \tag{B.21}$$

The evaluation at -1, with the particular case of a corollary of Dougall's formula ([5], pp.148)

$${}_3F_2^-\begin{pmatrix}a,1+\frac{a}{2},b\\ \frac{a}{2},1+a-b\end{pmatrix}=\frac{(1+a)_{-b}}{(\frac{1}{2}+\frac{a}{2})_{-b}}\,, \tag{B.22}$$

gives

$$\frac{(2\delta)_{m+n+s}}{(\delta)_{m+n+s}} \tag{B.23}$$

so that the first part is

$$16N\,(\delta)_{m+n}\sum_{s=0}^{n}(-1)^s\binom{n}{s}=16N\,(\delta)_{m+n}\,\delta_{n,0}=16N\,(\delta)_m\,\delta_{n,0}\,. \tag{B.24}$$

The evaluation at $+1$ is done with Dixon's identity ([5], pp.72), which can also be derived from Dougall's formula,

$${}_3F_2^+\begin{pmatrix}a,b,c\\ 1+a-b,1+a-c\end{pmatrix}=\frac{(1+a)_{-b}\,(1+a)_{-c}\,(1+\frac{a}{2})_{-b-c}}{(1+\frac{a}{2})_{-b}\,(1+\frac{a}{2})_{-c}\,(1+a)_{-b-c}}\,, \tag{B.25}$$

produces a factor $(0)_{m+n+s}$ which vanishes for $m+n+s\neq 0$. At $m=n=0$ (this forces s=0) one gets 1, so that the second part contributes

$$16N\,(\delta)_{m+n}\,\delta_{n,0}\,\delta_{m,0}\;=\;16N\,\delta_{n,0}\,\delta_{m,0}. \tag{B.26}$$

Finally, we find the equality

$$16N\,\delta_{n,0}\,\{\delta_{m,0}+(\delta)_m\}\;=\;\sum_{l\geq 0,\,even}(\gamma_l^{uv})^2\,a_{nm}^{(l)}. \tag{B.27}$$

[3]A term $(2\delta+2l)_{-1}$ involving $2l$ must be casted into $\frac{1}{2}(\delta+l+\frac{1}{2})_{-1}$.

B.5 D'EPP formula and star Witten graph

The inverse kernels are defined according to

$$p(\lambda)\int d^dx_3\, r_{13}^{-\lambda}\, r_{23}^{-d+\lambda} \;=\; \delta^d(x_{12}), \tag{B.28}$$

where

$$p(\lambda)=p(d-\lambda)=\pi^{-d}\frac{\Gamma(\lambda)\,\Gamma(d-\lambda)}{\Gamma(\frac{d}{2}-\lambda)\,\Gamma(\lambda-\frac{d}{2})}. \tag{B.29}$$

The D'Eramo-Parisi-Peliti formula [29, 111, 50, 102] reads

$$\int d^dx_4\, r_{14}^{-\delta_1}\, r_{24}^{-\delta_2}\, r_{34}^{-\delta_3} \;=\; v(\delta_1,\delta_2,\delta_3)\, r_{12}^{-\frac{d}{2}+\delta_3}\, r_{23}^{-\frac{d}{2}+\delta_1}\, r_{13}^{-\frac{d}{2}+\delta_2}\;, \tag{B.30}$$

where $\delta_1+\delta_2+\delta_3=d$ and

$$v(\delta_1,\delta_2,\delta_3) \;=\; \pi^{\frac{d}{2}}\frac{\Gamma(\frac{d}{2}-\delta_1)\,\Gamma(\frac{d}{2}-\delta_2)\,\Gamma(\frac{d}{2}-\delta_3)}{\Gamma(\delta_1)\,\Gamma(\delta_2)\,\Gamma(\delta_3)}. \tag{B.31}$$

We also need a generalization of D'EPP [50], obtained by differentiation,

$$\int d^dx_4\, r_{14}^{-\delta_1}\, r_{24}^{-\delta_2}\, r_{34}^{-\delta_3}\, \lambda^{x_1}_{\mu_1\ldots\mu_s}(x_4,x_2) =$$

$$\frac{(\frac{d}{2}-\delta_2)_s}{(\delta_1)_s}\, v(\delta_1,\delta_2,\delta_3)\, r_{12}^{-\frac{d}{2}+\delta_3}\, r_{23}^{-\frac{d}{2}+\delta_1}\, r_{13}^{-\frac{d}{2}+\delta_2}\, \lambda^{x_1}_{\mu_1.\mu_s}(x_3,x_2)\;. \tag{B.32}$$

The star Witten graph with scalar legs of generic dimensions Δ_i ($i=1,2,3$) is given by (see e.g. [32])

$$\frac{a(\Delta_1,\Delta_2,\Delta_3)}{r_{12}^{(\Delta_1+\Delta_2-\Delta_3)/2} r_{13}^{(\Delta_1+\Delta_3-\Delta_2)/2} r_{23}^{(\Delta_2+\Delta_3-\Delta_1)/2}}, \tag{B.33}$$

where

$$a(\Delta_1,\Delta_2,\Delta_3)=\frac{\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-d}{2})}{2\pi^d}\frac{\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_1+\Delta_3-\Delta_2}{2})\Gamma(\frac{\Delta_2+\Delta_3+\Delta_1}{2})}{\Gamma(\Delta_1-\frac{d}{2})\Gamma(\Delta_2-\frac{d}{2})\Gamma(\Delta_3-\frac{d}{2})}. \tag{B.34}$$

B.6 Regularized kernels

The aim of this part is to fix notation and thereby to summarize the facts concerning the reconstruction of the bulk fields out of its two types of asymptotics along the line of [78, 95]. Our presentation contains some new elements, insofar as we exclusively rely on convergent position space integrals. From them we will be able to *derive* the analytic continuation rules which usually appear a posteriori to give meaning to naively divergent integrals.

In AdS_{d+1} the scalar on shell bulk field $\phi(x)$, with $x = (z, \vec{x}),\ z \geq 0$ denoting Poincaré coordinates, has the near boundary asymptotics [4], see e.g. [1, 78]

$$\phi(x) \;=\; z^{\Delta_-}(\phi_0(\vec{x}) + \mathcal{O}(z^2)) \;+\; z^{\Delta_+}(A(\vec{x}) + \mathcal{O}(z^2)) \;, \tag{B.35}$$

where $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$. The standard bulk to bulk propagators obey $(\Delta = \Delta_\pm)$

$$\begin{aligned} (\Box_x - m^2) G_\Delta(x, x') &= -g^{-\frac{1}{2}}\ \delta(x, x') \;, \\ G_\Delta(x, x') &= z'^{\Delta} G^0_\Delta(x, \vec{x'}) + \mathcal{O}(z'^{\Delta+2}) \;. \end{aligned} \tag{B.36}$$

Using (B.35),(B.36) and Gauss theorem one gets with fixed $z' > 0$ [42, 53, 109]

$$\begin{aligned} \phi(x) = \int d^d\vec{x'} \{ &(\Delta - \Delta_-)\phi_0(\vec{x'})\ G^0_\Delta(x, \vec{x'})\ z'^{\Delta_- + \Delta - d} + \mathcal{O}(z'^{\Delta_- + \Delta - d + 2}) \\ &+ (\Delta - \Delta_+) A(\vec{x'})\ G^0_\Delta(x, \vec{x'})\ z'^{\Delta_+ + \Delta - d} + \mathcal{O}(z'^{\Delta_+ + \Delta - d + 2}) \} \;. \end{aligned} \tag{B.37}$$

Since always $\Delta_+ \geq \frac{d}{2}$, for the choice $\Delta = \Delta_+$ both $\mathcal{O}$-terms go to zero for $z' \to 0$. Choosing instead $\Delta = \Delta_-$, the vanishing of both $\mathcal{O}$-terms requires $\Delta_- > \frac{d-2}{2}$, i.e. just the unitarity bound. Altogether for $\frac{d-2}{2} < \Delta_- < \frac{d}{2} < \Delta_+$ one gets

$$\phi(x) \;=\; \int d^d\vec{x'}\ \phi_0(\vec{x'})\ K_{\Delta_+}(x, \vec{x'}) \;=\; \int d^d\vec{x'}\ A(\vec{x'})\ K_{\Delta_-}(x, \vec{x'}) \;, \tag{B.38}$$

with

$$\begin{aligned} K_{\Delta_\pm}(x, \vec{x'}) &= (2\Delta_\pm - d) \lim_{z' \to 0} z'^{-\Delta_\pm}\ G_{\Delta_\pm}(x, x') \\ &= \frac{\Gamma(\Delta_\pm)}{\pi^{\frac{d}{2}} \Gamma(\Delta_\pm - \frac{d}{2})}\ \frac{z^{\Delta_\pm}}{(z^2 + (\vec{x} - \vec{x'})^2)^{\Delta_\pm}} \;. \end{aligned} \tag{B.39}$$

[4] In the following we assume a suitable rapid falloff of A and ϕ_0 for $|\vec{x}| \to \infty$.

The reconstruction of the asymptotics (B.35) from the first eq. in (B.38) is given by

$$\int d^d\vec{x'}\ \phi_0(\vec{x'})\ K_{\Delta_+}(x,\vec{x'})\ =\ z^{\Delta_-}\ \phi_0(\vec{x})\ (1+\mathcal{O}(z^{2k}))+\frac{\Gamma(\Delta_+)}{\pi^{\frac{d}{2}}\Gamma(\Delta_+-\frac{d}{2})}z^{\Delta_+}$$
$$\times\left(\int d^d\vec{x'}\ \frac{\phi_0(\vec{x'})-\phi_0(\vec{x})-\cdots-\frac{((\vec{x'}-\vec{x})\vec{\partial})^{2k}}{(2k)!}\phi_0(\vec{x})}{|\vec{x'}-\vec{x}|^{2\Delta_+}}\ +\ \mathcal{O}(z^2)\right), \tag{B.40}$$

where k is the largest integer smaller than $\Delta_+-\frac{d}{2}$. Similarly one finds from the second representation of $\phi(x)$ in (B.38) for $\frac{d-2}{2}<\Delta_-<\frac{d}{2}$

$$\int d^d\vec{x'}\ A(\vec{x'})\ K_{\Delta_-}(x,\vec{x'})\ =\ z^{\Delta_+}\ A(\vec{x})\ (1+\mathcal{O}(z^{2(\Delta_--\frac{d}{2}+1)}))$$
$$+\ \frac{\Gamma(\Delta_-)}{\pi^{\frac{d}{2}}\Gamma(\Delta_--\frac{d}{2})}z^{\Delta_-}\times\left(\int d^d\vec{x'}\ \frac{A(\vec{x'})}{|\vec{x'}-\vec{x}|^{2\Delta_-}}\ +\ \mathcal{O}(z^2)\right). \tag{B.41}$$

We are mainly interested in the situation where *both* $\Delta_\pm$ are above the unitarity bound, then $k=0$ and A and ϕ_0 are related via the convergent position space integrals

$$A(\vec{x})=\frac{\pi^{-\frac{d}{2}}\Gamma(\Delta_+)}{\Gamma(\Delta_+-\frac{d}{2})}\int d^d\vec{x'}\frac{\phi_0(\vec{x'})-\phi_0(\vec{x})}{|\vec{x'}-\vec{x}|^{2\Delta_+}}\ , \tag{B.42a}$$

$$\phi_0(\vec{x})=\frac{\pi^{-\frac{d}{2}}\Gamma(\Delta_-)}{\Gamma(\Delta_--\frac{d}{2})}\int d^d\vec{x'}\frac{A(\vec{x'})}{|\vec{x'}-\vec{x}|^{2\Delta_-}}\ . \tag{B.42b}$$

Comparing the first formula in (B.42b), containing a subtraction, with the analytic continuation from $\Delta<\frac{d}{2}$ of the corresponding formula without subtraction, we find for $\frac{d}{2}<\Delta<\frac{d}{2}+1$

$$\int d^d\vec{x'}\frac{\phi_0(\vec{x'})-\phi_0(\vec{x})}{|\vec{x'}-\vec{x}|^{2\Delta}}\ =\ \left(\int d^d\vec{x'}\frac{\phi_0(\vec{x'})}{|\vec{x'}-\vec{x}|^{2\Delta}}\right)_{\text{continued}}\ . \tag{B.43}$$

To check (B.43) one has to split the integral in two parts $|\vec{x'}-\vec{x}|<K$ or $>K$, use the falloff property of ϕ_0 at $|\vec{x'}|\to\infty$ and to send the arbitrary auxiliary scale K to infinity *after* the continuation. Remarkably, the singularity of the r.h.s. for $\Delta\to\frac{d}{2}-0$ due to the short distance behavior is reproduced on the l.h.s. for $\Delta\to\frac{d}{2}+0$ via the infrared behavior of the subtraction term.

Appendix C

Appendix to chapter 4

C.1 Convolution of bulk-to-boundary propagators

The bulk-to-bulk propagator and bulk-to-boundary propagators in AdS (see, e.g., [78]), in Poincare coordinates

$$ds^2 = \frac{1}{z_0^2}(dz_0^2 + d\,\overrightarrow{x}^2), \tag{C.1}$$

are given, respectively, by

$$G_\Delta(z,w) = C_\Delta\,\frac{2^{-\Delta}}{2\Delta - d}\,\xi^\Delta\; F(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta - \frac{d}{2} + 1; \xi^{-2}), \tag{C.2}$$

in term of the hypergeometric function, and

$$K_\Delta(z,\overrightarrow{x}) = C_\Delta\left(\frac{z_0}{z_0^2 + (\overrightarrow{z} - \overrightarrow{x})^2}\right)^\Delta, \tag{C.3}$$

with the normalization constant

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\,\Gamma(\Delta - \frac{d}{2})}. \tag{C.4}$$

The quantity

$$\xi = \frac{2\,z_0\,w_0}{z_0^2 + w_0^2 + (\overrightarrow{z} - \overrightarrow{w})^2} \tag{C.5}$$

is related to the geodesic distance $d(z,w) = \log\frac{1+\sqrt{1-\xi^2}}{\xi}$.

There is a natural way to get precisely the difference of the two bulk-to-bulk propagators of conjugate dimension in AdS/CFT. It is based on the observation that the convolution along a common boundary point of two bulk-to-boundary propagator of conjugate dimensions, that is Δ and $d-\Delta$, results in the difference of the corresponding two bulk-to-bulk propagators [67, 81]. More precisely, the result is in fact the sum

$$\int_{\mathbb{R}^d} d^d\overrightarrow{x}\ K_\Delta(z,\overrightarrow{x})\ K_{d-\Delta}(w,\overrightarrow{x}) = (2\Delta-d)G_\Delta(z,w) + [\Delta \leftrightarrow d-\Delta]. \quad \text{(C.6)}$$

The coincidence limit $w \to z$ can be taken before the convolution, on both bulk-to-boundary propagators, to get

$$(2\Delta-d)\left\{G_\Delta(z,z) - G_{d-\Delta}(z,z)\right\} = C_\Delta C_{d-\Delta}\int_{\mathbb{R}^d} d^d\overrightarrow{x}\,\frac{z_0^d}{[z_0^2+(\overrightarrow{z}-\overrightarrow{x})^2]^d}. \quad \text{(C.7)}$$

The z_0 dependence in the integral is just illusory, the result is just $\frac{2\pi^{\frac{d}{2}}}{2^d\Gamma(\frac{d+1}{2})}$. As noted by Dobrev [39], the product of the two bulk-to-boundary normalization factors $C_\Delta\, C_{d-\Delta}$ coincides, modulo factors independent of $\Delta = \frac{d}{2}+x$, with the Plancherel measure for the d+1-dimensional hyperbolic space evaluated at imaginary argument $i\,x$. After putting all together, equation (4.19) is confirmed.

C.2 GJMS operators and Q-curvature (vulgarized)

To give a glimpse of these constructions in conformal geometry, let us go back to our "Dirichlet to Neumann map" (B.43) in the Poincare patch and examine the analytic continuation to $\Delta > d/2$. The kernel

$$\frac{C_\Delta}{|\overrightarrow{x}'-\overrightarrow{x}|^{2\Delta}} \quad \text{(C.8)}$$

will have single poles at $\Delta = d/2+k, k\in\mathbb{N}$, since in the neighborhood of these values (see e.g. [51])

$$\lim_{\Delta\to d/2+k}\frac{\Delta-d/2-k}{|\overrightarrow{x}|^{2\Delta}} = -c_k\,\Box^k\,\delta^{(d)}(\overrightarrow{x}) \quad \text{(C.9)}$$

where

$$c_k = \frac{1}{2^{2k}\,k!\,(k-1)!}. \quad \text{(C.10)}$$

Therefore, the relation between $A(\overrightarrow{x})$ and $\phi_0(\overrightarrow{x})$ for these "resonant values" is given by the k-th power of the Laplacian $\Box^k$, a conformal invariant (covariant) differential operator [1].

The generalization of this observation [59] for a filling Poincare metric associated to a given conformal structure involves P_k, the conformally invariant operators of GJMS [58].

GJMS operators

The GJMS operators P_k built using the Fefferman-Graham ambient construction have, among others, the following properties in a d-dim Riemannian manifold (M, g)

- On flat $\mathbb{R}^d$, $P_k = \Box^k$
- $P_k \ \exists\, k \in \mathbb{N}$ and $k - d/2 \neq \mathbb{Z}^+$
- $P_k = \Box^k + LOT$
- P_k is formally self-adjoint
- for $f \in C^\infty(M)$, under a conformal change of metric $\widehat{g} = e^{2\sigma} g$, $\sigma \in C^\infty(M)$, conformal covariance: $\widehat{P}_k f = e^{-\frac{d+2k}{2}\sigma} P_k(e^{\frac{d-2k}{2}\sigma} f)$
- P_k has a polynomial expansion in ∇ and the Riemann tensor (actually the Ricci tensor) in which all coefficients are rational in the dimension d
- P_k has the form $\nabla \cdot (S_k \nabla) + \frac{d-2k}{2} Q_k^d$, where $S_k = \Box^{k-1} + LOT$ and Q_k^d is a local scalar invariant.

Q-Curvature

The $Q-$*curvature* generalizes in many ways the 2-dim scalar curvature R. It original derivation tries to mimic the derivation of the *prescribed Gaussian curvature equation* (PGC) in 2-dim starting from the *Yamabe equation* in higher dimension and analytically continuing to $d = 2$.

Start with the conformal transformation of the scalar curvature at $d \geq 3$

$$e^{2\sigma} \widehat{R} = R - 2(d-1)\Box\sigma - (d-1)(d-2)\nabla\sigma \cdot \nabla\sigma \tag{C.11}$$

[1] There is a factor $(-1)^k$ hanging around, just because in the mathematical literature the *positive Laplacian* is preferred.

and absorb the quadratic term

$$\Box\sigma + (d/2-1)\nabla\sigma\cdot\nabla\sigma = \frac{2}{d-2}\, e^{-(d/2-1)\sigma}\,\Box\, e^{(d/2-1)\sigma}, \qquad \text{(C.12)}$$

to get for the Schouten scalar $J := \frac{R}{2(d-1)}$ and $u := e^{(d/2-1)\sigma}$ the Yamabe equation

$$[-\Box + (d/2-1)J]\, u = (d/2-1)\, \widehat{J}\, u^{\frac{d+2}{d-2}}. \qquad \text{(C.13)}$$

The trick (due to T. Branson) is now to slip in a 1 to rewrite as

$$-\Box(e^{(d/2-1)\sigma} - 1) + (d/2-1)\, J\, e^{(d/2-1)\sigma} \;=\; (d/2-1)\, \widehat{J}\, e^{(d/2+1)\sigma} \qquad \text{(C.14)}$$

and take now the limit $d \to 2$ that results in the PGC eqn.

$$e^{2\sigma}\, \widehat{J} \;=\; J - \Box\,\sigma. \qquad \text{(C.15)}$$

The very same trick applied now to the higher-order Yamabe eqn. based on the GJMS operators

$$P_k\, u \;=\; \nabla\cdot S_k\nabla\,(u-1) + (d/2-k)\, Q^d_k\, u \;=\; (d/2-k)\, \widehat{Q^d_k}\, u^{\frac{d+2k}{d-2k}} \qquad \text{(C.16)}$$

with $u = e^{(d/2-k)\sigma}$, in the limit $d \to 2k$ renders the higher (even-)dimensional generalization of the PGC eqn.

$$e^{d\sigma}\widehat{Q} \;=\; Q + P\,\sigma\ , \qquad \text{(C.17)}$$

where $Q := Q^d_{d/2}$ and $P := P_{d/2}$.

Among the properties of the Q-curvature, the conformal invariance of its volume integral easily follows. To conclude, let us mention two more relevant features (switch to the density-valued $\mathbf{Q} = \sqrt{g}\, Q$):

- For A being an operator with "decent" elliptic and conformal behavior (e.g. Yamabe, Dirac-squared or any of the GJMS operators) in low dimensions $d = 2, 4, 6$, and *conjecturally* in all even dimensions, the functional determinant quotient within a conformal class can be generically written as

$$-\log\frac{\det\widehat{A}}{\det A} = c\int_{\mathcal{M}} \sigma(\widehat{\mathbf{Q}} + \mathbf{Q}) + \int_{\mathcal{M}} (\widehat{\mathbf{F}} - \mathbf{F}) + (\text{global term}) \qquad \text{(C.18)}$$

  for some (universal) constant c, where $\mathbf{F}$ is some density-valued local invariant (which varies depending on A). These are called *generalized Polyakov formulas* (see e.g. [18, 24]).

- In addition, it provides one of the important terms in volume renormalization asymptotics (4.13) at conformal infinity [59]

$$\mathcal{L} = 2\, c_{d/2}\int_{\mathcal{M}} \mathbf{Q}\ . \qquad \text{(C.19)}$$

C.3 Useful formulas

Here we collect some formulas that have been used throughout the paper. They can all be found e.g. in [5].

$$(Euler's\ reflection) \quad \Gamma(z)\,\Gamma(1-z) \;=\; \frac{\pi}{\sin(\pi z)} \tag{C.20}$$

$$(Pochhammer\ symbol) \quad (z)_n \;=\; \frac{\Gamma(z+n)}{\Gamma(z)} \tag{C.21}$$

$$(1+z)_{-n} \;=\; \frac{(-1)^n}{(-z)_n} \tag{C.22}$$

$$(Legendre\ duplication) \quad \Gamma(z)\,\Gamma(z+\frac{1}{2}) \;=\; 2^{1-2z}\,\Gamma(\frac{1}{2})\,\Gamma(2z) \tag{C.23}$$

$$(Gauß's\ hypergeometric\ theorem,\ Re(c{-}a{-}b)>0) \quad F(a,b;c;1) = \frac{(c-b)_{-a}}{(c)_{-a}} \tag{C.24}$$

$$(Gauß's\ integral\ representation) \quad \psi(z) \;=\; \int_0^\infty dt\,\left(\frac{e^{-t}}{t} - \frac{e^{-t\,z}}{1-e^{-t}}\right) \tag{C.25}$$

$$(Binomial\ expansion) \quad (1-x)^a = \sum_{n=0}^{\infty} \frac{(-a)_n}{n!}\,x^n \tag{C.26}$$

$$B(a,b) - B(a,c) \;=\; \int_0^1 du\ (1-u)^{a-1}(u^{b-1} - u^{c-1})\ , \quad a>-1, \quad b,c>0\ . \tag{C.27}$$

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Abbreviations

ADM	after Arnowitt, Deser and Misner
AdS	anti-de Sitter space
BF	Breitenlohner-Freedman
BMN	after Berenstein, Maldacena and Nastase
BPS	after Bogomol'nyi, Prasad and Sommerfield
CFT	conformal field theory
CPW(E)	conformal partial wave (expansion)
DBI	after Dirac, Born and Infeld
D'EPP	after D'Eramo,Parisi and Peliti
Dp-brane	Dirichlet p-dimensional membrane
DR	dimensional regularization
ESU	Einstein static universe
FG	Fefferman-Graham
GJMS	after Graham, Jensen, Mason and Sparling
HS	higher spin
IR	infra-red
KK	Kaluza-Klein
OPE	operator product expansion
PDE	partial differential equation
PGC	prescribed Gaussian curvature
QCD	quantum chromodynamics
QFT	quantum field theory
RG	renormalization group
RR	Ramond-Ramond
SCT	special conformal transformations
SD	Schwinger-DeWitt
SUGRA	supergravity
SUSY	supersymmetry
(S)YM	(super) Yand-Mills
UV	ultra-violet
VVM	Van Vleck-Morette
WKB	after Wenzel, Kramers and Brillouin; semiclassical

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to be taken to Mass. Mima will probably pay a promise to the "Virgen de La Caridad". Papa will probably have a line of rum. Cury and Nene will probably have some rum or smoke a cigar. Danielito, Henry and Brian probably won't care at all and will continue playing whatever they were playing. Mary and Arelis will probably have a beer and Monica's family at the opposite shore will celebrate as well (a barbecue?).

Now, there was a certain girl with whom I used to share the same desk at high-school. I was really impressed by a piece of advice I got from her at that time: instead of keeping trying to "prove" some formula involving odd numbers by "experiment" (I was probably very advanced in my proof... close to 11, I recollect), write the number as $2n - 1$ with n natural! It was NOT in the middle of an examination, by the way. Today, some 20 years later, I do not regret to thank her for whatever impact this "revelation" might have had in this work being theoretical rather than experimental.

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfasst und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Danilo Eduardo Díaz Vázquez
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