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# Generalized Quark-Antiquark Potential in AdS/CFT:

## Strong Coupling Analysis via Semiclassical Strings

Masterarbeit zur Erlangung des akademischen Grades

**Master of Science**

im Fach Physik

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## Abstract

This master thesis deals with a generalized cusp anomaly $\Gamma(\lambda, \phi, \theta)$ for a cusped Wilson loop in $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, studied, in the spirit of the AdS/CFT correspondence, by means of a minimal surface computation in type IIB string theory on a $AdS_5 \times S^5$ background, where $\phi$ is the cusp angle and $\theta$ the opening angle on wide circles in $AdS_5$ and $S^5$ respectively.

Such generalization of the ultraviolet anomalous dimension of a Wilson loop with a cusp has proven to be an effective description of a wide variety of physical observables (generalized quark-antiquark potential, anomalous dimension of twist operators, etc) with various predictions on its strong coupling behavior formulated by means of gauge theoretical tools. In this thesis we work in sigma-model perturbation theory and confirm the predictions for configurations $(\phi \to i\infty\ ,\ \theta = 0)$ and $(\phi = 0\ ,\ \theta \to i\infty)$, explicitly working out corrections to the classical string results already present in the literature.



## Zusammenfassung

Diese Masterarbeit beschäftigt sich mit der verallgemeinerten Spitzenanomalie (cusp anomaly) $\Gamma(\lambda, \phi, \theta)$ einer gespitzten Wilson Schleife in $\mathcal{N} = 4$ super Yang-Mills-theorie in vier Dimensionen, untersucht im Sinne der $AdS/CFT$ Korrespondenz durch die Berechnung einer minimalen Fläche in IIB Stringtheorie auf dem Hintergrund $AdS_5 \times S^5$, wobei $\phi$ der Spitzenwinkel und $\theta$ Öffnungswinkel auf Großkreisen in $AdS_5$ und $S^5$ sind.

Solche Verallgemeinerung der ultravioletten anomalen Dimension einer gespitzten Wilson Schleife hat sich als eine effektive Beschreibung für diverse physikalische Observablen bewährt (wie z.B. verallgemeinertes Quark-Antiquark Potential, anomale Dimension von twist-Operatoren, etc), mit vielen Vorhersagen bezüglich des Verhaltens bei starker Kopplung formuliert mittels Eichtheoretischen Werkzeugs. In dieser Masterarbeit arbeiten wir in perturbativer sigma-Modell Theorie und bestätigen die Vorhersagen für Konfigurationen $(\phi \to i\infty\ ,\ \theta = 0)$ und $(\phi = 0\ ,\ \theta \to i\infty)$, indem wir explizit Korrekturen zu den in der Literatur vorhandenen klassischen Resultaten berechnen.

# Hilfsmittel

Diese Masterarbeit wurde mit LaTeX geschrieben. Die Rechnungen wurden mit Zuhilfenahme von *Mathematica* 10 ausgeführt. Informationen bezüglich einiger spezieller Funktionen und Reihendarstellungen wurden der **Wolfram functions** Website entnommen. ( http://functions.wolfram.com/ )

# Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Berlin, den 26. Juli 2013

Michael Zlotnikov

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# Contents

# 1 Introduction

String theory was first considered in physics when it was realized that the energy - angular momentum relation of a rotating relativistic string produced a similar kind of Regge trajectories as found in hadrons with increasing mass and spin [30]. However, the theory of quantum chromodynamics (QCD) was proven to be a more convenient description of the hadron physics, so that it was not until later that string theory received more attention. The discovery of excited massless superstring states with spin two (which could describe gravitons and therefore gravity)[31] alongside of a rich zoo of other states [32],[33], was a sign that string theory might be a real candidate for an eventual theory of everything (combining gravity with all other interactions in nature). Until now, no successful formulation of a theory of everything on the basis of string theory has been written down. However, during the past fifteen years string theory gained another aspect of importance. Namely, it was conjectured by J. Maldacena that certain conformal field theories are dual to certain string theories on Anti-deSitter spaces [4]. Conformal field theories (field theories with scale invariance) are mostly toy models studied to gain a better understanding of conventional field theories in general. If the correspondence conjecture holds true, string theory might turn out just as useful in this regard. However, since there is no formal proof of the conjectured duality, it is important to carry out explicit computations and "experimentally" find out to what extent the correspondence is valid. That is exactly the motivation of the thesis at hand.

A Wilson loop in field theories is a gauge invariant quantity that is fit to describe a wide range of observables.[1] For instance, a Wilson loop with a light-like cusp can be used to describe twist-two operators in $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions [2]. The cusp anomaly of this particular Wilson loop is essentially equivalent to the anomalous dimension of twist-two operator as both dimensions are governed, in certain limits [3], by the same scaling function of the t'Hooft coupling $\Gamma_{cusp}(\lambda)$ [7],[36],[37],[29].

By virtue of the Maldacena conjecture, a Wilson loop in $\mathcal{N} = 4$ SYM is dual to a certain minimal surface configuration in type IIB string theory on $AdS_5 \times S^5$ [3]. Therefore, if the duality holds, it should be possible to reproduce the same scaling function of twist-two operator anomalous dimensions on the

[1]For a short introduction to Wilson loops, see section 3.

[2] This is of course primarily true in the case of QCD [34],[35]. The twist operators we will refer to in this thesis are the supersymmetric counterpart of the QCD twist operators governing the operator product expansion in the description of deep inelastic scattering.

[3] These are the limit of large opening cusp angle for the Minkowskian continuation of the cusped Wilson loop, and the large spin limit for the twist operators.

string theory side using a minimal surface computation. Indeed, this result has been achieved in [5], where a minimal surface explicitly dual to a light-like cusped Wilson loop was considered. It is however of interest to regain this result – to one-loop accuracy in sigma-model perturbation theory – starting from a more general minimal surface dual to a space-like cusped Wilson loop and carrying out a certain analytic continuation to reach the light-like cusp configuration. This is an important consistency check for the identification of different objects in gauge theory, their dual objects in string theory and the respective geometric relations to each other.

In fact, one can consider the minimal surface dual to a cusped euclidean Wilson loop in $\mathcal{N} = 4$ super Yang-Mills gauge theory which also features a "jump" in the coupling to scalar fields at the cusp, and whose expectation value (which develops a logarithmic divergence) depends on the gauge coupling and on the angles, one geometric and one internal, $(\phi, \theta)$. Since the relevant cusped contour can be conformally mapped to a pair of antiparallel lines, the coefficient of its logarithmic (cusp) or linear (lines) divergence defines what is known as "generalized cusp" $\Gamma_{cusp}(\phi, \theta, \lambda)$, or "generalized quark-antiquark" potential [4].

In general, following the perturbative computations at weak and at strong coupling of [2], there has been a lot of recent progress in understanding the generalized cusp $\Gamma_{cusp}(\phi, \theta; \lambda)$ in various domains [5]. In a small angle limit and using localization techniques, an exact result was found in [28] (see also [42]) and shown to be in perfect agreement with the results of [2] both in perturbative gauge theory and at strong coupling. One interesting domain is obtained via a special scaling limit, taken on the parameter $\theta$ defining the internal orientation of the cusp rays [25] so that only certain Feynman diagrams (of ladder type [6]) contribute to the Wilson loop expectation value. A strong coupling (stringy) prediction is produced on the gauge theory side, by looking at the large $\lambda$ behavior of the infinite resummation of these diagrams, which should be possible to verify beyond leading classical string order by looking at fluctuations over a dual minimal surface in $AdS_5 \times S^5$ (see section 6).

The aim of this thesis is to explicitly carry out the tests to one-loop in the $AdS/CFT$ correspondence framework as mentioned above. To achieve that, the main task will be to acquire necessary knowledge on Wilson loops, string theory, classical minimal surface solutions of the sigma model and evaluation of quantum

[4] This function, defined in Section 3, is related to the $\Gamma_{cusp}(\lambda)$ previously mentioned via $\Gamma_{cusp}(\phi \to i\,\infty, \theta = 0, \lambda) = \Gamma_{cusp}(\lambda)$.

[5] Actually, after [3], one of the earliest studies of the effect of such relative internal space orientation for the Wilson lines is in [27] , in the context of the stringy derivation of a quark-antiquark potential for the gauge theory at finite-temperature.

[6] Ladder diagrams are Feynman diagrams with no internal vertices.

fluctuations over them evaluation with path integral formalism. For that end the body of the thesis will be concerned with reproduction of literature on the subject of Green-Schwarz superstring [18], publications dealing with the relevant minimal surface solution in $AdS_5 \times S^5$ [1],[2] and mathematical techniques required for evaluation of functional determinants [11],[12]. Subsequently, in a contribution of original work the necessary limits will be explicitly performed on the system to obtain the mentioned results to one-loop accuracy in sigma-model perturbation theory.

The thesis proceeds with the following structure.
Sections 2 and 3 contain introductions to $AdS/CFT$ correspondence and Wilson loops respectively. Section 4 is concerned with the main setup: classical minimal surface results, as well as the bosonic and fermionic fluctuation Lagrangians, are prepared for use and evaluation in subsequent sections. In section 5 we explicitly reproduce the scaling function of the twist-two operator anomalous dimension, starting from the minimal surface dual to a space-like cusped Wilson loop and up to one-loop accuracy. In section 6 we carry out the verification for the prediction on the cusp anomaly given by the ladder diagrams limit, again to one-loop accuracy in sigma-model perturbation theory. Section 7 contains conclusions and outlook. Appendices are devoted to preliminary and technical aspects such as anti-de Sitter geometry, Green-Schwarz formulation of the superstring action, and generalized $\zeta$-function approach to the computation of functional determinants.

# 2 The $AdS/CFT$ Correspondence

In [4], J. Maldacena considered the large $N$ behavior of certain conformal field theories ($N$ being the rank of the gauge group of the theory). In his investigation he found out that in certain cases the field content can be mapped to corresponding supergravity theories on Anti-deSitter spaces times spheres and/or other compact manifolds, making the theories essentially dual. This eventually led him to propose the conjecture of certain string theories on various Anti-deSitter spacetimes to be dual to various conformal field theories even without restriction to large $N$.

In this thesis we will consider the (most popular) special case of $AdS/CFT$ correspondence where $\mathcal{N} = 4$ *super Yang-Mills theory in four dimensions with gauge group* $SU(N)$ is proposed to be dual to *type IIB superstring theory on* $AdS_5 \times S^5$. The stringy coupling constant $g_s$ and Yang-Mills coupling constant $g_{YM}$ are then related in the following way:

$$g_s = \frac{g_{YM}^2}{4\pi}. \tag{2.1}$$

Apart from coupling constants, both theories feature one further parameter. The $AdS_5 \times S^5$ space[7] has a curvature radius $R$ and the super Yang-Mills theory can have gauge groups of different rank $N$. These two parameters are also cast into a relation to each other:

$$\lambda \equiv \frac{R^4}{\alpha'^2} = g_{YM}^2 N. \tag{2.2}$$

The constant $\alpha'$ is the so called slope parameter that originates from the energy-angular momentum relation of a rotating relativistic string.
Here we also conveniently defined the so called 't Hooft coupling constant $\lambda$. Since it is hard or even impossible to do computations in both involved theories at generic values of the parameters and coupling constants, certain simplifying limits are necessary to enable a perturbative approach. The 't Hooft coupling is especially useful when considering the limit $N \to \infty$ while simultaneously taking $g_{YM} \to 0$ such that $\lambda$ stays at a fixed value [8]. This relates a weakly coupled Yang-Mills theory with a gauge group of infinite rank to a weakly coupled string theory ($g_s = \lambda/N$) with variable curvature radius of spacetime. It is then remarkable to observe that perturbative computations on the string theory side are possible in the limit $\lambda \to \infty$ (large curvature radius corresponding to near flat space limit, obviously yielding simplifications), while on the gauge theory side conventional

[7] See appendix A.

[8] In this thesis we will be concerned only with the planar approximation which results from the leading term in this limit.

perturbative computations for $\lambda \to 0$ are convenient. Therefore, in the 't Hooft limit the $AdS/CFT$ correspondence relates two theories to each other, which are accessible in their respective framework at the two opposite regimes in their common coupling constant $\lambda$. In this particular thesis we will be working in the planar limit described above and on the string theory side, assuming then large $\lambda$.

To give a motivation for the $AdS/CFT$ correspondence in this special case, consider $N$ parallel $D3$-branes separated by a distance $d$ in type IIB string theory (with fixed coupling $g_s$). It is known that for low energies the theory on the $D3$ brane decouples. Instead of adjusting energies we can hold them fixed and take the *Maldacena limit* $\alpha' \to 0$ while also taking the separation $d \to 0$ such that $d/\alpha'$ =fixed. Then the resulting theory on the $D3$-branes will be $\mathcal{N} = 4$ SYM with gauge group $U(N)$ [4].
Correspondingly, considering a supergravity (low energy approximation) solution metric $\mathrm{d}s^2_{D3}$ carrying $D3$ brane charge (again taking the limits $\alpha' \to 0$ and $d \to 0$ such that $d/\alpha'$ =fixed as mentioned above) results in the metric of $AdS_5 \times S^5$ which remains constant in units of $\alpha'$. Since therefore $\mathcal{N} = 4$ SYM on the $D3$-branes and supergravity on $AdS_5 \times S^5$ are two viewpoints at the same limit of the same theory, they should be equivalent. Considering that supergravity on $AdS_5 \times S^5$ requires boundary conditions at infinity, it is expected that the gauge theory on the branes will provide these boundary conditions. Finally, undoing the limit $\alpha' \to 0$ and small energy constraint, the Maldacena conjecture assumes that the two theories are dual for any values of the parameters. On the grounds of the previous paragraph and from several other insights it follows that the $AdS/CFT$ correspondence should relate the string partition function with sources $J_s$ for string vertex operators fixed to the value $J$ at the boundary of $AdS_5$, to the $\mathcal{N} = 4$ SYM partition function with sources $J$ for local operators:

$$Z_{str}(J_s|_{\partial AdS} = J) \equiv Z_{\mathcal{N}=4\ \mathrm{SYM}}(J)\ . \tag{2.3}$$

Considering the symmetry groups of the dual theories is a sort of zeroth-order test of the correspondence. In appendix A we mention that the isometry groups of $AdS_5$ and $S^5$ spaces (being $SO(2,4)$ and $SO(6)$ respectively) combined together give the super conformal group $SU(2,2|4)$. In a similar fashion, in $\mathcal{N} = 4$ SYM in four dimensions the super Poincaré symmetry (translations, Lorentz transformations, dilatations and special conformal transformations) composes the symmetry group $SO(2,4)$. In addition to that, the so called R-symmetry describes transformations of the supercharges into each other (or equivalently rotations in the fermionic superspace variables), which contributes a $SO(6)$ symmetry group. Again, taking the direct product of these contributions, one ends up with the super conformal group $SU(2,2|4)$. That is a strong necessary feature, since two theories can only possibly be dual to each other if they share the same type and amount of symmetries.

A motivation for $AdS/CFT$ correspondence can get deeply mathematically involved [14]. However, despite all these realizations no mathematical proof for the conjecture exists as of now. Therefore, the next best thing we can do is to put the correspondence to explicit tests in certain computationally accessible configurations. Two such special tests are the main subject and aim of the thesis at hand.

# 3 Wilson Loops

A Wilson Line $U_P(z,y)$ is a *comparator* that relates the gauge transformation law of a field $A^\mu$ at spacetime point $z$ to the one at spacetime point $y$ for finite separations $z-y$ [9]:

$$U_P(z,y) = \exp\left[i\int_P \mathrm{d}x^\mu A_\mu(x)\right], \tag{3.1}$$

where $P$ is a path connecting $z$ and $y$.
If $A^\mu$ is for instance a Yang Mills gauge field, its local gauge transformation property is along the lines of $A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x)$, which corresponds to the desired transformation law for the comparator $U_P(z,y) \to \exp(i\alpha(z))U_P(z,y)\exp(-i\alpha(y))$.
It is important to note that the Wilson Line $U_P(z,y)$ is dependent on the path $P$. To see that immediately, we can consider propagating the comparator along a closed infinitesimal square spanned along spacetime directions $\vec{e}_1$ and $\vec{e}_2$:

$$\begin{aligned}\mathbf{U}(x) =& U(x, x+\epsilon\vec{e}_2)U(x+\epsilon\vec{e}_2, x+\epsilon\vec{e}_1+\epsilon\vec{e}_2)\times \\ &\times U(x+\epsilon\vec{e}_1+\epsilon\vec{e}_2, x+\epsilon\vec{e}_1)U(x+\epsilon\vec{e}_1, x),\end{aligned} \tag{3.2}$$

where $\epsilon$ is an infinitesimal parameter. Since each of the four steps is infinitesimal, the dependence of the field $A^\mu$ on spacetime in that small region is linear and we can exchange the integrations by middle values times range:

$$\begin{aligned}\mathbf{U}(x) =& \exp\Big(-i\epsilon\Big[-A_2\left(x+\frac{\epsilon}{2}\vec{e}_2\right) - A_1\left(x+\frac{\epsilon}{2}\vec{e}_1+\epsilon\vec{e}_2\right) \\ &+ A_2\left(x+\epsilon\vec{e}_1+\frac{\epsilon}{2}\vec{e}_2\right) + A_1\left(x+\frac{\epsilon}{2}\vec{e}_1\right)\Big] + O(\epsilon^3)\Big)\end{aligned} \tag{3.3}$$

$$=1 - i\epsilon^2\left(\partial_1 A_2(x) - \partial_2 A_1(x)\right) + O(\epsilon^3). \tag{3.4}$$

Therefore, we see that even though the closed contour returns to the same spacetime point $x$, the comparator is not exactly the identity but also contains a term proportional to the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and to the area $\epsilon^2$ spanned by the closed path. This is to be expected, if one considers a theoretical parallel between covariant derivatives on curved spacetime and covariant derivatives on internal spaces. As soon as a nontrivial covariant derivative is introduced, the *'curvature'* of spacetime and/or internal manifolds makes integrals between spacetime points path dependent.
The computation (3.3) can be generalized to a closed contour of finite size which is then called the Wilson Loop:

$$U_P(y,y) = \exp\left[i\oint_P \mathrm{d}x^\mu A_\mu(x)\right]. \tag{3.5}$$

[9] See for example [6].

As already became apparent in (3.4), the Wilson Loop can be expressed in terms of the field strength $F_{\mu\nu}$ (this can generally be achieved through use of Stokes theorem). Since almost all gauge invariant quantities of the theory can be constructed from $A_\mu$ and $F_{\mu\nu}$, we can think of the Wilson Loop as the most general building block for other invariant quantities (obtained through different closed integration paths), which makes the study of Wilson Loops a most important topic.

One prominent example for a Wilson loop, which is most relevant for the notion of Wilson loops as used in this thesis, is the quark-antiquark potential. Imagining that at some point in spacetime a quark and an antiquark are created, move apart and propagate in some way interacting with each other through exchange of gauge particles until they recombine and annihilate at some other point in spacetime, effectively creates a closed loop that can be described by a Wilson loop (the antiquark can be thought of as a quark propagating backwards in time). The expectation value of the rectangular loop with length T and width L, which in the limit $T \gg L$ can be seen as a pair of anti-parallel lines (the "quark" trajectories) at distance L, is

$$\langle W \rangle \propto e^{-V_{q\bar{q}}(L)\,T}\,, \qquad\qquad T \gg L \tag{3.6}$$

where the $V_{q\bar{q}}$ exhibits the famous area law. In QCD such a Wilson loop can provide a measure for confinement (the tendency of quarks to stay close to each other), which is a highly nontrivial problem still unsolved.

To include the propagation contours of quarks, the Wilson Loop can be generalized to the study of non-abelian gauge theory as follows:

$$W_P = \frac{1}{N}\mathrm{Tr}\; P\left\{\exp\left[i\oint \mathrm{d}s\frac{\mathrm{d}x^\mu}{\mathrm{d}s}A^a_\mu(x(s))t^a\right]\right\}, \tag{3.7}$$

where now a color index $a$ is added to $A^a_\mu$ and the group generators $t^a$ appear. The $1/N$ averages over all color degrees of freedom. Since the different $t^a$ do not necessarily commute at different spacetime points, the path-ordering operator $P\{\}$ is introduced to order the generator matrices such that the ones corresponding to higher values of the parameter along the path $s$ are to the left. Also, to retain gauge invariance the trace of the quantity which is now a matrix is taken as usual.

In a theory with more field content like $\mathcal{N} = 4$ SYM the Wilson Loop should feature an additional coupling to scalar fields $\Phi_I$ in a way which ensures the invariance of the Wilson Loop under a certain set of supersymmetry transformations. It is important to emphasize that locally (at one specific spacetime point) one can always find a supersymmetry transformation parameter so that the Maldacena Wilson loop is invariant, whereas globally such an invariance is realized only for some special contours. As was proposed by Maldacena in [2], such an extension of the coupling to scalars is realized by:

$$W_P = \frac{1}{N}\mathrm{Tr}\ P\left\{\exp\left[\oint \mathrm{d}s \left(i\frac{\mathrm{d}x^\mu}{\mathrm{d}s}A^a_\mu t^a + \Phi_I\Theta^I\left|\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right|\right)\right]\right\}, \tag{3.8}$$

where $\Theta^I$ is an internal vector that selects which scalar fields $\Phi_I$ should couple to the Wilson Loop.

According to the Maldacena proposal [3], the expectation value of a Wilson loop along a contour $C$ is given by the action of a string bounded by the loop at the boundary of $AdS$ space

$$\langle W_C\rangle = \int_{\partial X=C}\mathcal{D}X\, e^{-\sqrt{\lambda}S_{string}(X)}, \tag{3.9}$$

for some string action $S_{string}$. For large values of $\lambda$, and thus for classical strings (the area swept by them is exactly described by $S_{string}$), the path integral can be replaced by its saddle point approximation, and the expectation value of the Wilson loop is then related to the area $\mathcal{A}$ of the minimal surface bounded by $C$

$$\langle W_C\rangle \simeq \exp(-\sqrt{\lambda}\,\mathcal{A})\ . \tag{3.10}$$

As already mentioned earlier, different path choices for the Wilson Loop represent different physical observables. We want to turn our attention to a Wilson Loop with an Euclidean cusp (that is a non-differentiable change of direction at some particular spacetime point).

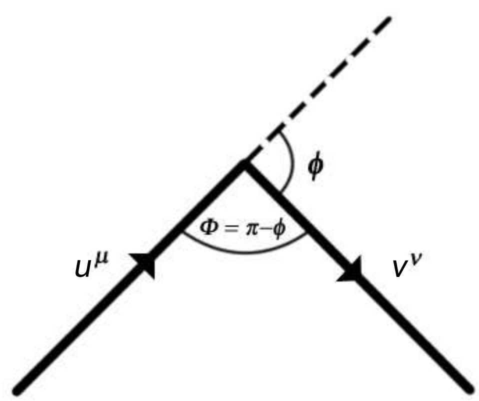


Figure 1: A cusp in the Wilson Loop parametrized by two vectors $u^\mu$ and $v^\nu$ with cusp angle $\phi$ and opening angle $\Phi = \pi - \phi$.

Considering the Wilson lines to be parametrized by vectors $u^\mu$ and $v^\nu$ near the cusp as in Figure 1, one can characterize the Euclidean cusp by the cusp angle straightforwardly obtained from:

$$\cos\phi = \frac{u \cdot v}{\sqrt{u^2}\sqrt{v^2}}. \tag{3.11}$$

It has been shown in [7] and expanded upon in [8],[9] that the expectation value of such a cusped Wilson Loop develops a logarithmic divergence:

$$\langle W \rangle \propto e^{-\Gamma_{\mathrm{cusp}}(\phi,\lambda)\log\frac{L_{\mathrm{IR}}}{\epsilon_{\mathrm{UV}}}}, \tag{3.12}$$

where $L_{\mathrm{IR}}$ and $\epsilon_{\mathrm{UV}}$ are infrared and ultraviolet cutoffs and $\Gamma_{\mathrm{cusp}}$ is the so called cusp anomaly which depends on the opening angle of the cusp $\phi$ and the coupling constant $\lambda$. The quantity $\Gamma_{\mathrm{cusp}}$ (or certain limits in its parameters) appears in a multitude of other physical observables and is therefore worthy of study (i.e. it governs the large spin limit of anomalous dimension for twist-two operators, the infrared behavior of gluon scattering amplitudes, or the energy of a static quark and anti-quark on a spatial three sphere at angle separation $\phi$ [2, 17]).

In particular it is possible to introduce a "jump" in $\Theta^I$ at the cusp, so that the coupling of the two Wilson Lines to scalar fields changes there. This jump in scalar coupling behavior at the cusp can be parametrized by an internal angle $\theta$ which enters the Wilson Loop expectation value as a further parameter :

$$\langle W_{cusp} \rangle \propto e^{-\Gamma_{\mathrm{cusp}}(\phi,\theta,\lambda)\log\frac{L_{\mathrm{IR}}}{\epsilon_{\mathrm{UV}}}}. \tag{3.13}$$

Via the exponential map, a cusped Wilson loop in flat euclidean space can be mapped into a pair of antiparallel lines on $\mathbb{S}^3 \times \mathbb{R}$. The loop is made of two lines, one going in the time future direction and one to the past, which are separated by an angle $\pi - \phi$ along a big circle on $\mathbb{S}^3$. For $\phi \to \pi$ the lines get very close together so to resemble antiparallel lines in flat space, and in fact from this picture and from (3.6) one gets

$$\langle W_{lines} \rangle \propto e^{-\Gamma_{\mathrm{cusp}}(\phi,\theta,\lambda)\,T}\ , \tag{3.14}$$

with the cutoffs of the two calculations being related by $\log\frac{L_{\mathrm{IR}}}{\epsilon_{\mathrm{UV}}} \sim T$.
As explained in the introduction, to the study of the so-called generalized cusp anomalous dimension $\Gamma_{\mathrm{cusp}}(\phi,\theta,\lambda)$ defined above much work has been devoted recently [2, 17, 25, 28, 42]. As relevant example, the presence of an additional parameter (the internal angle $\theta$) allows one to relate it to simpler observables for which exact results exist (this is the observation of Ref. [17]). Also, a way was found to describe the evolution of $\Gamma_{\mathrm{cusp}}(\phi,\theta,\lambda)$ with the coupling via integrability techniques [43, 28]. Various predictions exist on its strong coupling behavior,

which were formulated by means of gauge theoretical tools. By virtue of the $AdS/CFT$ correspondence, such a cusped Wilson Loop computed in $\mathcal{N} = 4$ SYM is dual to a certain minimal surface string configuration in type IIB string theory on $AdS^5 \times S_5$, where $\phi$ is the cusp angle and $\theta$ the opening angle on wide circles in $AdS_5$ and $S^5$ respectively. Actually, in this work we will concentrate on the string theory side of the duality and try to verify within this framework results and predictions obtained on the gauge theory side.

# 4 The Setup: Stringy Description of a Cusped Wilson Loop

This thesis deals with a specific minimal surface configuration in type $IIB$ string theory on $AdS_5 \times S^5$ that is dual via the $AdS/CFT$ correspondence to a cusped Wilson Loop in $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions with gauge group $SU(N)$ [2].

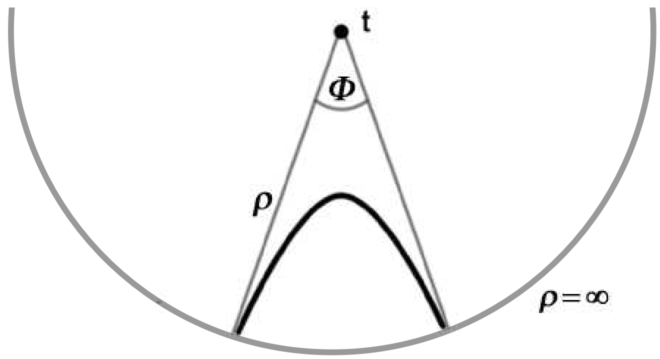


Figure 2: A tentative sketch of the minimal surface in the $(\rho,\Phi)$ plane while the $t$ coordinate is extending normal to the page.

As mentioned in Section 3, the cusped Wilson Loop of $\mathcal{N} = 4$ super Yang-Mills can be conformally mapped to two anti-parallel lines on $\mathbb{R} \times \mathbb{S}^3$ separated by an angle $\pi - \phi$ along a big circle on $\mathbb{S}^3$. According to $AdS/CFT$, $\mathbb{R} \times \mathbb{S}^3$ should be the boundary of the $AdS$ space where the relevant minimal surface has to end, which makes the global coordinates (A.3) the more appropriate parametrization of the $AdS$ space. The two anti-parallel lines of $\mathcal{N} = 4$ super Yang-Mills are then located at $\rho = \infty$, and the angle of the cusp can be parametrized by an angle $\varphi$ on a big circle of $AdS$ describing how far apart the two anti-parallel lines are located during their evolution with the time coordinate $t$ (see fig.2). Finally, the internal angle $\theta$ introduced in (3.8), which describes the jump in the coupling of the Wilson Loop to the scalar fields in $\mathcal{N} = 4$ super Yang-Mills, can be parametrized on the string theory side by a corresponding big circle angle $\vartheta$ in the $S^5$ part of $AdS_5 \times S^5$.
The picture above leads to the following reduction for the number of classically relevant coordinates in the $AdS_5 \times S^5$ metric (in units of $\alpha' = 1$):

$$\mathrm{d}s^2 = \sqrt{\lambda}(-\cosh^2\rho\;\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\;\mathrm{d}\varphi^2 + \mathrm{d}\vartheta^2), \tag{4.1}$$

where $\lambda$ is the 't Hooft coupling constant. The following symmetry properties are apparent. Since the two lines at the boundary do not locally move compared at different points in time, the system is invariant under time translations. Therefore, at any point in time $t$, the solution for the minimal surface extending into

the bulk of $AdS$ from the boundaries will be the same. Furthermore, the $\rho$ value of the solution will have to decrease from infinity at the first boundary to a certain minimal value $\rho_0$ in the bulk, and then increase again in a mirror-symmetric way until it reaches infinity at the second boundary. For the angle variables $\varphi$ and $\vartheta$ we can assume opening angles $\Phi$ and $\theta$ such that these variables take on values $-\Phi/2 < \varphi < \Phi/2$ and $-\theta/2 < \vartheta < \theta/2$ between the two boundaries.
Keeping the symmetries of the system in mind, a convenient choice for a world-sheet parametrization $(\tau, \sigma)$ of the world-surface in target space is the *static gauge*, in which $(\tau, \sigma)$ are identified with two target space coordinates (at the one-loop level demanding that there are no fluctuations in those directions). Defining $t = c\tau$ and keeping $\sigma$ undefined for the moment ($-\infty < \tau < \infty$ and $-\sigma_0 < \sigma < \sigma_0$ with some $c$ and $\sigma_0$ to be defined later), we have $\rho = \rho(\sigma)$, $\varphi = \varphi(\sigma)$, $\vartheta = \vartheta(\sigma)$, with boundary conditions:

$$\begin{aligned} \rho(-\sigma_0) = \infty \quad , \quad \rho(0) =& \rho_0 \quad , \quad \rho(\sigma_0) = \infty \\ \varphi(-\sigma_0) = -\Phi/2 \quad , \quad \varphi(0) =& 0 \quad , \quad \varphi(\sigma_0) = \Phi/2 \\ \vartheta(-\sigma_0) = -\theta/2 \quad , \quad \vartheta(0) =& 0 \quad , \quad \vartheta(\sigma_0) = \theta/2. \end{aligned} \tag{4.2}$$

The induced metric becomes

$$\mathrm{d}s^2 = \sqrt{\lambda}\Big( - \cosh^2\rho \ (\partial_\tau t)^2 \ \mathrm{d}\tau^2 + \Big((\partial_\sigma \rho)^2 + \sinh^2\rho \ (\partial_\sigma \varphi)^2 + (\partial_\sigma \vartheta)^2\Big)\mathrm{d}\sigma^2\Big). \tag{4.3}$$

Note that with this the opening angle $\Phi$ is related to the cusp angle $\phi$ as $\Phi = \pi - \phi$, just as shown in Figure 1. With these general ingredients we will now proceed writing down the explicit classical solution, considering then quantum fluctuations around it.

## 4.1 Bosonic Action

In this thesis we will use the Nambu-Goto form of the action for the bosonic string. The Lagrangian density in the Nambu-Goto action essentially consists of the area functional for the string world-sheet times a dimensionful constant that ensures the correct dimension of action and an appropriate normalization (for a more detailed treatment of bosonic strings by the author of this thesis see [15]). Although the Nambu-Goto action is highly non-linear, it has been successfully used in the framework of semiclassical quantization in the sense that in the quadratic approximation it is relatively easy to identify the operators of small fluctuations.
The Nambu-Goto action for the induced metric (4.3) is given by (in units of

$\alpha' = 1$)

$$\begin{aligned} S_{NG} =& \frac{1}{2\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-g} \\ =& \frac{\sqrt{\lambda}}{2\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \cosh\rho \ (\partial_\tau t) \ \sqrt{(\partial_\sigma \rho)^2 + \sinh^2\rho \ (\partial_\sigma \varphi)^2 + (\partial_\sigma \vartheta)^2}, \end{aligned} \tag{4.4}$$

where $g$ is the determinant of the induced metric. Since the $\tau$ (and therefore $t$) dependence is trivial, we can integrate it out immediately and are left with an integral over Lagrangian density:

$$\begin{aligned} S_{NG} =& \frac{\sqrt{\lambda}}{2\pi} T \int \mathrm{d}\sigma \mathcal{L}(\sigma) \\ =& \frac{\sqrt{\lambda}}{2\pi} T \int \mathrm{d}\sigma \cosh\rho \ \sqrt{(\partial_\sigma \rho)^2 + \sinh^2\rho \ (\partial_\sigma \varphi)^2 + (\partial_\sigma \vartheta)^2}, \end{aligned} \tag{4.5}$$

where $T$ is a cut-off in the $t$ variable. Recalling (3.14), the quantity we are interested in (the generalized cusp anomaly) is straightforwardly given by:

$$\Gamma_{\text{cusp}}(\lambda, \phi, \theta) = -\frac{S_{NG}}{T}. \tag{4.6}$$

Remembering our discussion of symmetries of the system, or just looking at (4.5), we realize that the conjugate angular momentum densities to $\varphi$ and $\vartheta$ in $\sigma$ translations will be conserved, since the Lagrangian density features a dependence on $\varphi' = \partial_\sigma \varphi$ and $\vartheta' = \partial_\sigma \vartheta$ but no explicit dependence on $\varphi$ and $\vartheta$. These conserved currents are given by [10]

$$P = \frac{\partial \mathcal{L}}{\partial \varphi'} = \frac{\sinh^2\rho \cosh\rho \ (\partial_\sigma \varphi)}{\sqrt{(\partial_\sigma \rho)^2 + \sinh^2\rho \ (\partial_\sigma \varphi)^2 + (\partial_\sigma \vartheta)^2}}, \tag{4.7}$$

$$J = \frac{\partial \mathcal{L}}{\partial \vartheta'} = \frac{\cosh\rho \ (\partial_\sigma \vartheta)}{\sqrt{(\partial_\sigma \rho)^2 + \sinh^2\rho \ (\partial_\sigma \varphi)^2 + (\partial_\sigma \vartheta)^2}}. \tag{4.8}$$

Actually, it is more convenient to work with a slightly different combination of these conserved quantities [2] given by

$$p = \frac{1}{P} \quad , \quad q = \frac{J}{P} = \frac{(\partial_\sigma \vartheta)}{\sinh^2\rho \ (\partial_\sigma \varphi)}. \tag{4.9}$$

Having determined the conserved quantities, we have essentially obtained the equations of motion for our system. The relation between $\varphi'$ and $\vartheta'$ is clear from

[10] It is important to emphasize that despite the formal use of terminology like *angular momentum* or *Hamiltonian*, the conserved charges in this system have no straightforward physical interpretation.

the second equation in (4.9). Considering both conserved quantities together we obtain

$$\rho'^2 = (p^2 \cosh^2 \rho \sinh^4 \rho - q^2 \sinh^4 \rho - \sinh^2 \rho)\varphi'^2. \tag{4.10}$$

Also, making use of (4.9) we can write the induced metric (4.3) as:

$$\mathrm{d}s^2 = \sqrt{\lambda} \cosh^2 \rho \Big( - (\partial_\tau t)^2 \, \mathrm{d}\tau^2 + p^2 \sinh^4 \rho \, (\partial_\sigma \varphi)^2 \, \mathrm{d}\sigma^2 \Big). \tag{4.11}$$

It is interesting to note that since we kept the possibility to choose any possible parametrization in $\sigma$, we still have an unfixed gauge degree of freedom. This kind of gauge invariance has certain implications. For instance, one can immediately see that the Hamiltonian in $\sigma$ translations obtained from (4.5) is identically zero (which is a direct consequence of unfixed gauge). However, this shall not concern us here since the total number of conserved quantities stays the same whether we fix the gauge or not. To see this, consider fixing the gauge freedom in $\sigma$ by the straightforward choice $\sigma \equiv \varphi$ with $\sigma_0 \equiv \Phi/2$. This effectively reduces the number of variables in the Lagrangian density by one. However, now the conserved quantities of the system are given by the angular momentum density conjugate to $\vartheta$ and the Hamiltonian density in $\varphi$ translations. The latter simply directly corresponds to equation (4.7), so that we actually end up with exactly the same setting but less freedom. Not fixing the gauge immediately, we can perform computations by choosing an explicit $\sigma$ parametrization of our liking 'on the go' which is a convenient feature to have.
For instance, making the very same choice mentioned above $\sigma \equiv \varphi$ with $\sigma_0 \equiv \Phi/2$, we can use (4.10) to determine the minimal value $\rho(0) = \rho_0$ in terms of the conserved quantities. Since $\rho(0)$ is a minimum, we have $\rho'(0) = 0$ and (4.10) gives the equation:

$$0 = p^2 \cosh^2 \rho_0 \sinh^4 \rho_0 - q^2 \sinh^4 \rho_0 - \sinh^2 \rho_0. \tag{4.12}$$

This now can be solved for i.e. $\cosh^2 \rho_0$ which gives:

$$\cosh^2 \rho_0 = \frac{1}{2p^2} \Big( p^2 + q^2 + \sqrt{(p^2 - q^2)^2 + 4p^2} \Big), \tag{4.13}$$

where we chose the nontrivial solution which cannot turn negative for real $p$ and $q$ to be consistent.

### 4.1.1 Classical Action

To be able to solve the classical system introduced above, we have to specify a particular $\sigma$ parametrization. The only requirement is that our choice should respect the boundary conditions and the symmetries of the system. Actually, one crucial thing to realize about the equations of motion is that they are elliptic. With this piece of information it is clear that the parametrization will be most conveniently expressed in terms of Jacobi elliptic functions. Building on that realization and following [2] we can demand

$$\cosh^2\rho = \frac{1+b^2}{b^2\mathrm{cn}^2(\sigma,k^2)}, \tag{4.14}$$

where we set $\sigma_0 = \mathbb{K}(k^2) \equiv \mathbb{K}$, so that $-\mathbb{K} < \sigma < \mathbb{K}$. Here $\mathrm{cn}(\sigma,k^2)$ is the Jacobi CN function with modulus $k^2$ and $\mathbb{K}(k^2)$ is the complete elliptic integral of the first kind evaluated at $k^2$. Also, the modulus $k^2$ appearing in the Jacobi elliptic function and the constant $b^2$ are defined as[11]:

$$b^2 = \frac{1}{2}(p^2 - q^2 + \sqrt{(p^2-q^2)^2+4p^2}) \quad , \quad k^2 = \frac{b^2(b^2-p^2)}{b^4+p^2}. \tag{4.15}$$

It is easy to check that the properties for $\rho$ listed in (4.2) with $\rho_0$ given in (4.13) are indeed satisfied by this choice. Having chosen the function itself, we can differentiate it to find $\rho'(\sigma)$ which we will require for the other equations of motion. A short computation yields

$$\rho' = \frac{\sqrt{1+b^2}\,\mathrm{dn}(\sigma,k^2)\,\mathrm{sn}(\sigma,k^2)}{\mathrm{cn}(s,k^2)\sqrt{1+b^2\,\mathrm{sn}^2(\sigma,k^2)}}, \tag{4.16}$$

where $\mathrm{sn}(\sigma,k^2)$ and $\mathrm{dn}(\sigma,k^2)$ are the Jacobi SN and Jacobi DN functions with modulus $k^2$ respectively. Making use of (4.9) and (4.10) we now can write down equations of motion for $\varphi$ and $\vartheta$ specific to this parametrization. After some straightforward manipulations involving addition theorems of Jacobi elliptic functions we find:

$$\vartheta'^2 = \frac{p^2(b^2+1)-b^4}{b^4+p^2} \quad , \quad \varphi'^2 = \frac{b^6\mathrm{cn}^4(\sigma,k^2)}{(b^4+p^2)(1+b^2\mathrm{sn}^2(\sigma,k^2))^2}. \tag{4.17}$$

Pushing onward, applying this parametrization to the induced metric (4.11) we obtain

$$\begin{aligned}\mathrm{d}s^2 =& \sqrt{\lambda}\frac{1+b^2}{b^2\mathrm{cn}^2(\sigma,k^2)}\Big(-(\partial_\tau t)^2\ \mathrm{d}\tau^2 + \frac{p^2b^2}{b^4+p^2}\ \mathrm{d}\sigma^2\Big) && (4.18)\\ =& \sqrt{\lambda}\frac{1-k^2}{\mathrm{cn}^2(\sigma,k^2)}\Big(-\frac{b^4+p^2}{p^2b^2}(\partial_\tau t)^2\ \mathrm{d}\tau^2 + \mathrm{d}\sigma^2\Big). && (4.19)\end{aligned}$$

[11] This is just one possible convenient choice of parametrization. One could expand on a lengthy motivation on how to derive it, but one always has to start with some sort of ansatz and essentially it would not give us more information other than that equations of motion are elliptic. Therefore, we can just as well take this convenient choice as our ansatz.

Here we realize that we can make the induced metric conformally flat if we choose the $\tau$ parametrization to be

$$t(\tau) = \frac{p\,b}{\sqrt{b^4+p^2}}\,\tau. \tag{4.20}$$

Therefore the conformally flat induced metric reads

$$\mathrm{d}s^2 = \sqrt{\lambda}\frac{1-k^2}{\mathrm{cn}^2(\sigma,k^2)}\Big(-\mathrm{d}\tau^2+\mathrm{d}\sigma^2\Big). \tag{4.21}$$

With this choice of parametrization, the equations of motion are put into a shape that makes their elliptic structure explicit and is convenient to deal with. Using an integral table or the computer program *Mathematica* we can now straightforwardly obtain the exact solutions for the motion of the system variables parametrized by $\sigma$. In fact, the equation for $\vartheta$ is trivially integrated:

$$\vartheta(\sigma) = \frac{\sqrt{p^2(b^2+1)-b^4}}{\sqrt{b^4+p^2}}\sigma + c_\theta. \tag{4.22}$$

Applying (4.2) to this, we realize that we should choose the integration constant $c_\theta = 0$, and therefore we obtain the following opening angle $\theta$ in terms of the conserved quantities of the system:

$$\vartheta(\sigma) = \frac{\sqrt{p^2(b^2+1)-b^4}}{\sqrt{b^4+p^2}}\sigma \quad , \quad \theta = 2\frac{\sqrt{p^2(b^2+1)-b^4}}{\sqrt{b^4+p^2}}\mathbb{K}(k^2). \tag{4.23}$$

To integrate the equation of motion for $\varphi$ in (4.17) we now have to use an integral table. (There is also no problem with extracting the square root, since all terms and functions stay positive on the interval $-\mathbb{K} < \sigma < \mathbb{K}$). The result of the integration is:

$$\varphi(\sigma) = \frac{b}{\sqrt{b^4+p^2}}\Big(-\sigma + (1+b^2)\Pi(-b^2, \mathrm{am}(\sigma,k^2),k^2)\Big), \tag{4.24}$$

where again any possible integration constant $c_\varphi$ has been set to zero due to (4.2). The function $\Pi(n, z(\sigma), k^2)$ is the elliptic integral of the third kind, and $\mathrm{am}(\sigma, k^2)$ is the Jacobi Amplitude with modulus $k^2$. The opening angle $\Phi$ is therefore given by:

$$\Phi = \frac{2\,b}{\sqrt{b^4+p^2}}\Big(-\mathbb{K}(k^2) + (1+b^2)\Pi(-b^2,k^2)\Big), \tag{4.25}$$

$\Pi(n, k^2)$ now being the complete elliptic integral of the third kind. Finally, we can integrate out the classical action (4.5) which in the chosen parametrization reads

$$\begin{aligned} S_{NG} =& \frac{\sqrt{\lambda}}{2\pi} T \int \mathrm{d}\sigma\, p \cosh^2 \rho \sinh^2 \rho\, \varphi' \\ =& \frac{\sqrt{\lambda} T}{2\pi} \frac{\sqrt{b^4+p^2}}{bp} \left[ \frac{(1+b^2)p^2}{b^4+p^2} \sigma - \mathbb{E}(\mathrm{am}(\sigma, k^2), k^2) + \frac{\mathrm{dn}(\sigma, k^2)\mathrm{sn}(\sigma, k^2)}{\mathrm{cn}(\sigma, k^2)} \right]_{-\mathbb{K}}^{\mathbb{K}}, \end{aligned} \tag{4.26}$$

where $\mathbb{E}(z(\sigma), k^2)$ is the elliptic integral of the second kind with modulus $k^2$. The last term on the right diverges at the boundaries. We can see that it behaves as

$$\left( \frac{\sqrt{b^4+p^2}}{bp} \frac{\mathrm{dn}(\sigma, k^2)\mathrm{sn}(\sigma, k^2)}{\mathrm{cn}(\sigma, k^2)} \right)_{\sigma \approx \mathbb{K}} \approx \frac{\sqrt{b^4+p^2}}{bp(\mathbb{K}-\sigma)} + O(\mathbb{K}-\sigma), \tag{4.27}$$

which exactly corresponds to the behavior of

$$(\sinh \rho)_{\sigma \approx \mathbb{K}} = \left( \frac{\sqrt{1+b^2 \mathrm{sn}^2(\sigma, k^2)}}{b\, \mathrm{cn}(\sigma, k^2)} \right)_{\sigma \approx \mathbb{K}} \approx \frac{\sqrt{b^4+p^2}}{bp(\mathbb{K}-\sigma)} + O(\mathbb{K}-\sigma). \tag{4.28}$$

This is a standard divergence for two lines along the boundary at infinity [2]. To obtain the finite part of the classical bosonic action that we are actually interested in, we subtract this divergence and obtain:

$$S_{NG} = \frac{\sqrt{\lambda} T}{\pi} \frac{\sqrt{b^4+p^2}}{bp} \left( \frac{(1+b^2)p^2}{b^4+p^2} \mathbb{K}(k^2) - \mathbb{E}(k^2) \right), \tag{4.29}$$

where $\mathbb{E}(k^2)$ is the complete elliptic integral of the second kind with modulus $k^2$. This concludes the classical bosonic computation since the exact behavior of the system is now made explicit through equations (4.14), (4.23), (4.24), (4.25) and (4.29). Taking different limits on the opening angles $\Phi$ and $\theta$, it is now possible to investigate the classical behavior of the cusped Wilson loop dual for diverse configurations. For instance, taking $\phi \to \pi$ while $\theta = 0$ (or expressed in opening angle $\Phi \to 0$) would yield the classical quark-antiquark potential (two anti-parallel lines) at strong coupling as described in [2]. In this thesis we will be interested in similar configurations.

### 4.1.2 Fluctuation Lagrangian

Having completed the classical computation, we can proceed to determine the quantum string correction to the Wilson loop expectation value. This is also referred to as a *one-loop* stringy calculation, having in mind the string two-dimensional loop expansion as a semiclassical ($1/\hbar$) expansion for which the role of the loop expansion parameter $\hbar$ is played by the (large) constant $\sqrt{\lambda} = R^2/\alpha'$ in front of the action [12]: the leading term coming from the classical action is proportional to $\sqrt{\lambda}$, the 1-loop correction is only a number, the 2-loop result is the coefficient of $1/\sqrt{\lambda}$ and so on. Since we have a stationary classical minimal surface solution at hand, bosonic quantum corrections can now be obtained following the standard background field method, namely deriving the action for small fluctuations near the classical string solution. Fermionic fluctuations are trivially decoupled from the bosonic ones at this order, their classical contribution is zero and their contribution can be independently evaluated starting from the quadratic part of the superstring action in $AdS_5 \times S^5$ background [19].

In other words, the classical saddle-point approximation of [3],[22] will be replaced by a functional integral over the fluctuations

$$W = \int \mathcal{D}\delta X \, \mathcal{D}\delta\theta \, e^{-S_{IIB}(X_{\rm cl}+\delta X, \delta\theta)} \tag{4.30}$$

where $\delta X, \delta\theta$ denote quantum fluctuations of the bosonic and fermionic coordinates, and the action in the exponential describes a collection of bosonic and fermionic fields. Therefore, following standard steps of the path integral formalism, the evaluation of the partition function is performed via the calculation of determinants of some bosonic and fermionic operators on a two-dimensional worldsheet (in $\tau$ and $\sigma$).

Since fluctuations are by definition allowed in the whole ten-dimensional target space, in the evaluation of the Nambu-Goto action for small fluctuations one should start from the complete $AdS_5 \times S^5$ metric

$$\begin{aligned} \mathrm{d}s^2 =& \sqrt{\lambda}\Big[ \cosh^2\rho \, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho \left(\mathrm{d}x_1^2 + \cos^2 x_1 \left(\mathrm{d}x_2^2 + \cos^2 x_2 \, \mathrm{d}\varphi^2\right)\right) \quad (4.31) \\ & + \mathrm{d}x_3^2 + \cos^2 x_3 \left(\mathrm{d}x_4^2 + \cos^2 x_4 \left(\mathrm{d}x_5^2 + \cos^2 x_5 \left(\mathrm{d}x_6^2 + \cos^2 x_6 \, \mathrm{d}\vartheta^2\right)\right)\right)\Big] \end{aligned}$$

with

$$\rho = \rho_{\rm cl} + \frac{\delta\rho}{\lambda^{\frac{1}{4}}} \quad , \quad x_j = \frac{\delta x_j}{\lambda^{1/4}} \quad , \quad j = 1, 2, ..., 6. \tag{4.32}$$

Here $x_1, x_2$ (in $AdS_5$) and $x_3, x_4, x_5, x_6$ (in $S^5$) are remaining angles with trivial classical part. The time direction $t$ has been Wick rotated as customary in the path integral formalism, to avoid oscillatory behavior in the time variable

[12] It is important to emphasize the crucial difference between such loop expansion in string theory, also known as sigma-model loop expansion or $\alpha'$-expansion, and the standard perturbative (genus) expansion of string scattering in target space (see for example Ch. 3, 5-7, 9 of [24]).

while computing functional determinants and ensure convergence. To go to static gauge we simply set the fluctuations to zero which are longitudinal to the worldsheet. This will automatically leave us with only physical degrees of freedom (longitudinal fluctuations can always be reabsorbed via diffeomorphisms), and the ghost contribution will be consequently trivial [16] [13].
We then proceed by denoting the complete fluctuated bosonic coordinate as $X^\mu = X^\mu_{cl} + \delta X^\mu/\lambda^{\frac{1}{4}}$ where the index assumes values from both subspaces $\mu = 0, 1, ..., 9$ and

$$X^\mu_{cl} =( t \ , \ \rho \ , \ 0 \ , \ 0 \ , \ \varphi \ , \ 0 \ , \ 0 \ , \ 0 \ , \ 0 \ , \ \vartheta), \tag{4.33}$$
$$\delta X^\mu =(\delta t \, , \, \delta\rho \, , \, \delta x_1 \, , \, \delta x_2 \, , \, \delta\varphi \, , \, \delta x_3 \, , \, \delta x_4 \, , \, \delta x_5 \, , \, \delta x_6 \, , \, \delta\vartheta). \tag{4.34}$$

Considering two vectors manifestly parallel to the classical worldsheet:

$$t^\mu_1 = \partial_\tau X^\mu_{cl} = \dot{X}^\mu_{cl} \quad , \quad t^\mu_2 = \partial_\sigma X^\mu_{cl} = X'^\mu_{cl}, \tag{4.35}$$

we should think of two classical unit vectors $\xi^\mu_7$ and $\xi^\mu_8$ which are orthogonal to each other, as well as to both vectors $t^\mu_1$ and $t^\mu_2$. If we find such vectors, we can project all four fluctuations of classical variables $\delta t$, $\delta\rho$, $\delta\varphi$ and $\delta\vartheta$ onto new fluctuations $\zeta_7$ and $\zeta_8$ along the vectors $\xi^\mu_7$ and $\xi^\mu_8$ respectively. This will reduce the total number of fluctuation fields by two, removing fluctuations longitudinal to the worldsheet exactly as we wish. One possible choice for classical unit vectors $\xi^\mu_7$ and $\xi^\mu_8$ orthogonal to each other and $t^\mu_1$, $t^\mu_2$ is given by:

$$\xi^\mu_7 =\frac{( 0 \ , \ \vartheta'\rho' \ , \ 0 \ , \ 0 \ , \ \vartheta'\varphi' \ , \ 0 \ , \ 0 \ , \ 0 \ , \ 0 \ , \ -(\rho'^2 + \sinh^2\rho\, \varphi'^2)\, )}{\sqrt{(\vartheta'^2 + \rho'^2 + \sinh^2\rho\vartheta'^2)(\rho'^2 + \sinh^2\rho\varphi'^2)}}, \tag{4.36}$$
$$\xi^\mu_8 =\frac{\sinh\rho}{\sqrt{\sinh^2\rho\,\varphi'^2 + \rho'^2}}( 0 \ , \ -\varphi' \ , \ 0 \ , \ 0 \ , \ \frac{\rho'}{\sinh^2\rho} \ , \ 0 \ , \ 0 \ , \ 0 \ , \ 0 \ , \ 0). \tag{4.37}$$

Setting $\delta x_i = 0$ with $i \in \{1, 2, ..., 6\}$ in (4.31), one recovers the classical metric on $AdS_5 \times S^5$. If we denote this classical metric in matrix notation as $G_{\mu\nu}$, then it is straightforward to check that all the required conditions are valid for the specific vectors above:

$$\begin{aligned} G_{\mu\nu}\xi^\mu_7\xi^\nu_7 =&1 \quad , \quad G_{\mu\nu}\xi^\mu_7\xi^\nu_8 = 0 \quad , \quad G_{\mu\nu}\xi^\mu_7 t^\nu_1 = 0 \quad , \quad G_{\mu\nu}\xi^\mu_8 t^\nu_1 = 0 \\ G_{\mu\nu}\xi^\mu_8\xi^\nu_8 =&1 \quad , \quad G_{\mu\nu}t^\mu_1 t^\nu_2 = 0 \quad , \quad G_{\mu\nu}\xi^\mu_7 t^\nu_2 = 0 \quad , \quad G_{\mu\nu}\xi^\mu_8 t^\nu_2 = 0. \end{aligned} \tag{4.38}$$

[13] A covariant, but less "physical", gauge choice could be the conformal gauge, more naturally used in quantizing the string action in Polyakov form. In this case Virasoro constraints and ghost contributions would have to be considered, with the determinant of the latter expected to be canceled by the determinant of the longitudinal modes [16]. On the basis of a comparison with the semiclassical quantization (in conformal gauge) of the solution corresponding to a folded string rotating in $AdS_5 \times S^5$, the advantage of the static gauge is also due to the considerably less involved form of the coupled fluctuations.

Now, all we need to do is project the fluctuation vector (4.34) onto $\xi_7^\mu$ and $\xi_8^\mu$ to obtain the relevant fluctuation fields $\zeta_7$ and $\zeta_8$. Also, we should use the equations

$$G_{\mu\nu} t_1^\mu \delta X^\nu \equiv 0 \quad , \quad G_{\mu\nu} t_2^\mu \delta X^\nu \equiv 0 \tag{4.39}$$

as constraints to simplify the result of the projection and explicitly build in the vanishing of longitudinal fluctuations. A short computation yields:

$$\zeta_7 = G_{\mu\nu}\xi_7^\mu \delta X^\nu = \frac{\vartheta'(\rho'\,\delta\rho + \sinh^2\rho\,\varphi'\,\delta\varphi) - (\rho'^2 + \sinh^2\rho\,\varphi'^2)\delta\vartheta}{\sqrt{(\vartheta'^2 + \rho'^2 + \sinh^2\rho\,\varphi'^2)(\rho'^2 + \sinh^2\rho\,\varphi'^2)}}, \tag{4.40}$$

$$\zeta_8 = G_{\mu\nu}\xi_8^\mu \delta X^\nu = \frac{\sinh\rho\,(\rho'\,\delta\varphi - \varphi'\,\delta\rho)}{\sqrt{\sinh^2\rho\,\varphi'^2 + \rho'^2}}. \tag{4.41}$$

Correspondingly, we can use (4.39), (4.40) and (4.41) to invert these relations, and write down the fluctuations of classical target space coordinates in terms of the new fluctuation fields $\zeta_7$ and $\zeta_8$:

$$\delta t = 0 \qquad , \qquad \delta\vartheta = -\zeta_7\sqrt{\frac{\rho'^2 + \sinh^2\rho\,\varphi'^2}{\vartheta'^2 + \rho'^2 + \sinh^2\rho\,\varphi'^2}}\ , \tag{4.42}$$

$$\delta\rho = \frac{\vartheta'\,\rho'}{\sqrt{(\rho'^2 + \sinh^2\rho\,\varphi'^2)(\vartheta'^2 + \rho'^2 + \sinh^2\rho\,\varphi'^2)}}\zeta_7 - \frac{\sinh\rho\,\varphi'}{\sqrt{\rho'^2 + \sinh^2\rho\,\varphi'^2}}\zeta_8\ ,$$

$$\delta\varphi = \frac{\rho'}{\sinh\rho\sqrt{\rho'^2 + \sinh^2\rho\,\varphi'^2}}\zeta_8 + \frac{\vartheta'\,\varphi'}{\sqrt{(\rho'^2 + \sinh^2\rho\,\varphi'^2)(\vartheta'^2 + \rho'^2 + \sinh^2\rho\,\varphi'^2)}}\zeta_7.$$

Since we chose the vectors $\xi_7^\mu$ and $\xi_8^\mu$ to be unit normalized, the kinetic terms for the fields $\zeta_7$ and $\zeta_8$ will be canonical. This means, these kinetic terms will be written in terms of bare covariant derivatives on the worldsheet in parameter space with no additional factors involved. The same is true for fluctuation fields $\delta x_i$ where $i = 3,\ 4,\ 5,\ 6$, since the classical norm for their direction vectors in $\delta X^\mu$ is also equal to one. However, the directions for fluctuations $\delta x_1$ and $\delta x_2$ are not yet unit normalized (due to the factor of $\sinh^2\rho$ from the classical metric). To have the same normalization for all the fields, we define:

$$\zeta_i = \delta x_i\,\sinh\rho \quad , \quad i = 1,\ 2 \tag{4.43}$$
$$\Rightarrow \quad \delta x_i = \frac{\zeta_i}{\sinh\rho}.$$

Finally, to have consistent notation we identify

$$\delta x_s = \zeta_s \quad , \quad s = 3,\ 4,\ 5,\ 6. \tag{4.44}$$

With (4.42), (4.43) and (4.44) obtained, all the preparatory work for the fluctuation fields is done and we can proceed to expand the fluctuated Lagrangian density.

It is important to note that the eight fluctuation fields $\zeta = \zeta(\tau, \sigma)$ are arbitrary functions of $\tau$ and $\sigma$ and have no relation to the classical equations of motion. Nevertheless, the boundary conditions and symmetry properties of the system hold for the quantum fluctuations just the same. So obeying Dirichlet conditions at the boundaries $\rho = \infty$, the fluctuation fields will have to vanish $\zeta(\tau, \sigma^*) = 0$ for $\sigma^* = \pm\mathbb{K}(k^2)$.

If we denote the fluctuated $AdS_5 \times S^5$ metric (4.31) in matrix notation as $G^{fl}_{\mu\nu}$, then the bosonic Nambu-Goto action (4.4) (in Euclidean signature) can be covariantly written as

$$S_{NG} = \frac{1}{2\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{\prod_{a\in\{\tau,\sigma\}} G^{fl}_{\mu\nu}(\partial_a X^\mu)(\partial_a X^\nu)}, \tag{4.45}$$

where the upper and lover $\mu$ and $\nu$ spacetime indices are contracted. This particular expression is valid since the induced worldsheet metric is diagonal. If we had off-diagonal elements, we should have included additional metric determinant terms with mixed derivatives in $\tau$ and $\sigma$. The next step in computation is purely technical. We treat the square root expression in (4.45) as a function in 't Hooft coupling $\lambda$ and expand it in a Taylor series around $\lambda$ going to infinity. With the leading term we recover as expected the classical action given in (4.4):

$$\begin{aligned} S^{(0)}_{NG} &= \frac{1}{2\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{\prod_{a\in\{\tau,\sigma\}} G_{\mu\nu}(\partial_a X^\mu_{cl})(\partial_a X^\nu_{cl})} \\ &= \frac{\sqrt{\lambda}}{2\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \cosh\rho \ (\partial_\tau t) \ \sqrt{(\partial_\sigma \rho)^2 + \sinh^2\rho \ (\partial_\sigma \varphi)^2 + (\partial_\sigma \vartheta)^2}. \end{aligned} \tag{4.46}$$

Further treatment of this term is given by the classical computation in the previous subsection.

The next expansion term proportional to $\lambda^{1/4}$ is linear in fluctuation fields $\zeta$ or their derivatives with respect to $\tau$ or $\sigma$, which appear equipped with some unwieldy factors involving classical variables and their derivatives. Evaluated on the equations of motion (4.9) and (4.10) the factors identically vanish, which is expected on general grounds but still constitutes a consistency check.

The next expansion term of order $\lambda^0$ gives the one-loop fluctuation Lagrangian that we are actually looking for. As argued earlier, the kinetic terms for all eight

$\zeta$ fields turn out to be canonical. Each of these fields features a 'mass term' (that is a $\zeta^2$ term with some factor). The fields $\zeta_7$ and $\zeta_8$ are found to be coupled (meaning, that mixed terms like $\zeta_7\zeta_8$ are present). After several partial integrations which bring all the terms into a shape appropriate for use in the path integral formalism [14], the one-loop fluctuation Lagrangian can be written as follows

$$\mathcal{L}_B^{(2)} = \frac{1}{2}\sqrt{g}\zeta_i \mathbb{O}_{ij}\zeta_j \quad , \quad i,j \in \{1,2,...,8\}, \tag{4.47}$$

where $g$ is the determinant of the induced metric and the matrix elements of the differential operator $\mathbb{O}$ are given by

$$\mathbb{O}_{ij} = -\nabla^2\delta_{ij} + A\partial_\sigma(\delta_{i\,8}\,\delta_{j\,7} - \delta_{i\,7}\,\delta_{j\,8}) + M_{ij}. \tag{4.48}$$

Here $\nabla^2 = g^{ab}\partial_a\partial_b$ with $a,b \in \{\tau,\sigma\}$ is the squared nabla operator on the parameter space. The factor $A$ in front of the first derivative coupling term for the fields $\zeta_7$ and $\zeta_8$ is given by:

$$A = \frac{2b^2\sqrt{b^4+p^2}\sqrt{-b^4+(1+b^2)p^2}\,\mathrm{cn}^4(\sigma,k^2)}{p(1+b^2)\Big((1+b^2)p^2 + (b^4-(1+b^2)p^2)\,\mathrm{cn}^2(\sigma,k^2)^2\Big)}. \tag{4.49}$$

The mass matrix $M_{ij}$ is almost diagonal, except for the following off-diagonal entries (symmetrical $M_{78} = M_{87}$):

$$M_{78} = \frac{2b^2\sqrt{-b^4+(1+b^2)p^2}\,\mathrm{cn}^3(\sigma,k^2)\sqrt{\mathrm{sn}^2(\sigma,k^2)\Big(b^4+p^2-b^2(b^2-p^2)\mathrm{sn}^2(\sigma,k^2)\Big)}}{p(1+b^2)\Big((1+b^2)p^2+(b^4-(1+b^2)p^2)\mathrm{cn}^2(\sigma,k^2)\Big)}. \tag{4.50}$$

[14]The matrix of the differential operator in our fluctuation Lagrangian should have strict symmetry properties when used in the path integral formalism to determine the partition function. Namely, it has to be symmetric in terms with an even amount of derivatives and anti-symmetric in terms with odd number of derivatives. This shape can be achieved unproblematically through use of partial integrations, since all boundary terms vanish thanks to Dirichlet boundary conditions.

Finally, the diagonal entries of the mass matrix $M_{ij}$ are:

$$\begin{aligned}
M_{ii} =& \frac{b^4-(1+b^2)p^2}{(1+b^2)p^2}\mathrm{cn}^2(\sigma,k^2)+2 \qquad , \quad i=1,\,2\\
M_{ss} =& \frac{b^4-(1+b^2)p^2}{(1+b^2)p^2}\mathrm{cn}^2(\sigma,k^2) \qquad\qquad , \quad s=3,\,4,\,5,\,6\\
M_{77} =& \frac{b^4\mathrm{cn}^4(\sigma,k^2)\Big(2(1+b^2)p^2+(b^4-(1+b^2)p^2)\mathrm{cn}^2(\sigma,k^2)\Big)}{(1+b^2)\Big((1+b^2)p^2+(b^4-(1+b^2)p^2)\mathrm{cn}^2(\sigma,k^2)\Big)^2}\\
&+\frac{b^4-(1+b^2)p^2}{(1+b^2)p^2}\mathrm{cn}^2(\sigma,k^2)+\frac{2b^2(p^2-b^2)\mathrm{cn}^4(\sigma,k^2)}{(1+b^2)p^2}\\
M_{88} =& 2-\frac{3b^4\mathrm{cn}^4(\sigma,k^2)}{(1+b^2)\Big((1+b^2)p^2+(b^4-(1+b^2)p^2)\mathrm{cn}^2(\sigma,k^2)\Big)}\\
&+\frac{b^4-(1+b^2)p^2}{(1+b^2)p^2}\mathrm{cn}^2(\sigma,k^2)+\frac{b^4p^2\mathrm{cn}^4(\sigma,k^2)}{\Big((1+b^2)p^2+(b^4-(1+b^2)p^2)\mathrm{cn}^2(\sigma,k^2)\Big)^2}.
\end{aligned} \tag{4.51}$$

It is immediately clear that the differential operator $\mathbb{O}$ is a rather complicated object and hopes are low that we will be able to obtain the exact one-loop partition function. However, to have a complete discussion let us ignore this and proceed to outline the general computational steps.
From the (Euclidean) path integral formalism in QFT we have the following relation for the bosonic effective action

$$e^{-S_{\textit{eff}}} = \int \mathcal{D}\zeta\ e^{-\frac{1}{2}\int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{g}\ \zeta\, \mathbb{O}\, \zeta} = \frac{1}{\sqrt{\det \mathbb{O}}}. \tag{4.52}$$

That gives

$$S_{\textit{eff}} = \frac{1}{2}\ln\det\mathbb{O} = \frac{1}{2}\mathrm{Tr}\ln\mathbb{O}. \tag{4.53}$$

The trace can be written making use of the complete sets of eigenvectors to the operator $\mathbb{O}$ in $\tau$ and $\sigma$:

$$\mathrm{Tr}\ln\mathbb{O} = \int \mathrm{d}\tau \int \mathrm{d}\sigma\ \langle\sigma|\langle\tau|\ln\mathbb{O}|\tau\rangle|\sigma\rangle. \tag{4.54}$$

The operator $\mathbb{O}$ is translation invariant in the $\tau$ direction, therefore we can switch to momentum space in $\tau$ by making proper insertions of the unity expansion in the momentum variable $i\partial_\tau \to \omega_\tau$:

$$\begin{aligned}\operatorname{Tr}\ln\mathbb{O} &= \int \mathrm{d}\tau \int \mathrm{d}\sigma \int \frac{\mathrm{d}\omega_\tau}{2\pi} \int \frac{\mathrm{d}\omega'_\tau}{2\pi}\ \langle\sigma|\langle\tau|\omega_\tau\rangle\langle\omega_\tau|\ln\mathbb{O}|\omega'_\tau\rangle\langle\omega'_\tau|\tau\rangle|\sigma\rangle \\ &= \int \mathrm{d}\tau \int \mathrm{d}\sigma \int \frac{\mathrm{d}\omega_\tau}{2\pi} \int \frac{\mathrm{d}\omega'_\tau}{2\pi}\ \langle\sigma|e^{i\omega_\tau\tau}\ln\mathbb{O}\langle\omega_\tau|\omega'_\tau\rangle e^{-i\omega'_\tau\tau}|\sigma\rangle \\ &= \int \mathrm{d}\tau \int \mathrm{d}\sigma \int \frac{\mathrm{d}\omega_\tau}{2\pi} \int \frac{\mathrm{d}\omega'_\tau}{2\pi}\ 2\pi\delta(\omega_\tau-\omega'_\tau)e^{i(\omega_\tau-\omega'_\tau)\tau}\ \langle\sigma|\ln\mathbb{O}|\sigma\rangle,\end{aligned} \tag{4.55}$$

where it is intended that, in the case of a matrix-valued differential operator, the ket $|\sigma\rangle$ includes matrix indices together with sigma, and together with integral over $\sigma$ a sum over those indices has to be considered. In the first line above the operator $\mathbb{O}$ is acted on by the eigenstate $\langle\omega_\tau|$ from the left which replaces all the appearing $i\partial_\tau$ operators by their eigenvalue $\omega_\tau$, so that we can drag the state $\langle\omega_\tau|$ past the operator on the next line. We also make use of the standard relations $\langle\tau|\omega_\tau\rangle = e^{i\omega_\tau\tau}$ and $\langle\omega_\tau|\omega'_\tau\rangle = 2\pi\delta(\omega_\tau-\omega'_\tau)$. Carrying out the $\omega'_\tau$ integration we obtain:

$$\operatorname{Tr}\ln\mathbb{O} = \mathcal{T}\int\frac{\mathrm{d}\omega_\tau}{2\pi}\int\mathrm{d}\sigma\ \langle\sigma|\ln\mathbb{O}|\sigma\rangle, \tag{4.56}$$

where we also integrated out $\int \mathrm{d}\tau = \mathcal{T}$, since all the $\tau$ dependence is gone from the operator on the right. Still the trace over the subspace of eigenstates in $\sigma$ appears in the initial shape. Since the operator $\mathbb{O}$ has no translation invariance in $\sigma$, we cannot repeat the same procedure as with $\tau$. It is more convenient to go back to the determinant notation and evaluate the determinant of the operator now in one dimension making use of functional techniques introduced in Appendix C. Therefore, we write

$$\operatorname{Tr}\ln\mathbb{O} = \mathcal{T}\int\frac{\mathrm{d}\omega_\tau}{2\pi}\ln\det\mathbb{O}, \tag{4.57}$$

where the determinant is now implied to be taken only over the $\sigma$ subspace of eigenstates. Back in equation (4.53) we obtain the following formal expression for the bosonic one-loop effective action:

$$S_{\mathit{eff}} = -\mathcal{T}\int\frac{\mathrm{d}\omega_\tau}{2\pi}\ln\frac{1}{\sqrt{\det\mathbb{O}}}. \tag{4.58}$$

This expression for the effective action will receive an additional term of the fermionic determinant from the fermionic action in the next section.

### 4.1.3 Fermionic Action

We start with the formal expression for the fermionic Lagrangian quadratic in the fields motivated in Appendix B

$$\mathcal{L}^{IIB}_{2F} = -i(\sqrt{g}g^{\alpha\beta}\delta_{IJ} - \epsilon^{\alpha\beta}s_{IJ})\bar{\Theta}^I\gamma_\alpha D_\beta\Theta^J, \tag{4.59}$$

where $\alpha$, $\beta$ run over the worldsheet indices $\tau$, $\sigma$. In type IIB string theory we have two supersymmetries ($\mathcal{N}$=2), therefore the fermionic spinor index takes on values $I, J = 1, 2$. Here $\delta_{I,J}$ is the Kronecker Delta, we have $s_{11} = -s_{22} = 1$ , $s_{12} = s_{21} = 0$ and $\epsilon^{\alpha\beta}$ is the Levi-Civita tensor in two dimensions. Different from the case of flat space discussed in the appendix, here the background is $AdS_5 \times S^5$. Therefore, the pullbacks of the ten dimensional gamma matrices are defined by:

$$\gamma_\alpha = \Gamma_a e^a_\mu \partial_\alpha X^\mu_{cl}, \tag{4.60}$$

where $e^a_\mu$ are the vielbeins for our metric and the flat space gamma matrices have been generalized to curved spacetime $\Gamma_\mu \to \Gamma_a e^a_\mu$ with $a = 1, 2, ..., 10$. Also the covariant derivative gets generalized to our background:

$$D_\beta \Theta^I = (\delta^{IJ}\mathcal{D}_\beta + \frac{1}{2}\epsilon^{IJ}\mathcal{U}\gamma_\beta)\Theta^J. \tag{4.61}$$

The $\mathcal{D}_\beta$ in this expression is the generic part of the covariant derivative that we would actually expect to see here. The additional term involving $\mathcal{U}$ is due to the so called Ramond-Ramond field strength. The derivation of this term is very involved and goes beyond the scope of this thesis. For a rigorous derivation of the type IIB superstring action in the $AdS_5 \times S^5$ background, see [19]. The term $\mathcal{U}$ can be written as:

$$\mathcal{U} = \frac{1}{\sqrt{g_{AdS}}}(e^a_t\Gamma_a)(e^a_\rho\Gamma_a)(e^a_\varphi\Gamma_a)(e^a_{x_1}\Gamma_a)(e^a_{x_2}\Gamma_a), \tag{4.62}$$

where $g_{AdS} = \cosh^2\rho \sinh^6\rho$ is the determinant of the (Wick rotated as with the bosons) $AdS_5$ part of the metric. Also, suggestive index notation has been used replacing explicit indices by the corresponding coordinate. For the regular covariant derivative $\mathcal{D}_\beta$ we have the more familiar expression:

$$\mathcal{D}_\beta = \partial_\beta + \frac{1}{4}(\partial_\beta X^\mu_{cl})\Omega^{a\,b}_\mu \Gamma_a \Gamma_b, \tag{4.63}$$

where $\Omega^{a\,b}_\mu$ is the spin connection. The spin connection is known from General Relativity and is defined in terms of vielbeins, inverse vielbeins and the Christoffel symbol $\Gamma^\lambda{}_{\mu\nu}$ as follows:

$$\Omega^{a\,b}_\mu = \delta^{a\,k}(e_k{}^\nu \Gamma^\lambda{}_{\mu\nu} e_\lambda{}^b - e_k{}^\nu \partial_\mu e_\nu{}^a), \tag{4.64}$$

where the Kronecker delta has been used to pull up one index since after Wick rotation the space is Euclidean. The Christoffel symbol is given in terms of the metric tensor by the more familiar formula

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}G^{\lambda\sigma}(\partial_\mu G_{\nu\sigma} + \partial_\nu G_{\mu\sigma} - \partial_\sigma G_{\mu\nu}). \tag{4.65}$$

To bring the Lagrangian into a more applicable shape, one should choose an explicit set of vielbeins. Being a noncoordinate basis, vielbeins can be chosen based on any set of spacetime vectors as long at they are metric compatible. (In our case metric compatible means $\delta_{ab}e^a_\mu e^b_\nu = G_{\mu\nu}$.) For instance, we can therefore choose our vielbeins inspired by the vectors $t^\mu_1$, $t^\mu_2$, $\xi^\mu_7$ and $\xi^\mu_8$. In a way this choice is attractive, since then the vielbein basis directions will correspond to the directions we chose to treat the bosonic fluctuations. With this choice the relevant vielbein basis vectors are given by:

$$\begin{aligned} e^0 &= \cosh\rho \mathrm{d}t \quad , \quad e^9 = \frac{\rho'\mathrm{d}\rho + \sinh^2\rho\,\varphi'\,\mathrm{d}\varphi + \vartheta'\,\mathrm{d}\vartheta}{\sqrt{\rho'^2 + \sinh^2\rho\,\varphi'^2 + \vartheta'^2}} \qquad (4.66)\\ e^8 &= \frac{\sinh\rho(\rho'\,\mathrm{d}\varphi - \varphi'\,\mathrm{d}\rho)}{\sqrt{\rho'^2 + \sinh^2\rho\,\varphi'^2 + \vartheta'^2}} \quad , \quad e^7 = \frac{\vartheta'(\rho'\,\mathrm{d}\rho + \sinh^2\rho\,\varphi'\,\mathrm{d}\varphi) - (\rho'^2 + \sinh^2\rho\,\varphi'^2)\mathrm{d}\vartheta}{\sqrt{(\rho'^2 + \sinh^2\rho\,\varphi'^2)(\rho'^2 + \sinh^2\rho\,\varphi'^2 + \vartheta'^2)}}. \end{aligned}$$

The indices are chosen such that $e^7$ and $e^8$ correspond to the directions of bosonic fluctuation fields $\zeta_7$ and $\zeta_8$. Also $e^0$ and $e^9$ correspond to directions parallel to the classical worldsheet $t^\mu_1$, $t^\mu_2$, the bosonic fluctuations in which directions have been set to zero in the static gauge. The remaining vielbein directions are straightforwardly given by:

$$e^i = \sinh\rho\,\mathrm{d}x_i \quad , \quad i = 1, 2 \qquad (4.67)$$

$$e^s = \mathrm{d}x_s \quad , \quad s = 3, 4, 5, 6. \qquad (4.68)$$

Actually, it has some formal advantages to perform an additional rotation on the vielbein directions $e^7$ and $e^8$, explicitly given by:

$$\begin{pmatrix} e^8 \\ e^7 \end{pmatrix} \to \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} e^8 \\ e^7 \end{pmatrix}, \qquad (4.69)$$

where the angle $\beta$ is given by:

$$\beta = \frac{b^4\,p}{b^2\sqrt{b^4 + p^2}\sqrt{-b^4 + (b^2+1)p^2}} \left(\sigma - \frac{(b^2+1)p^2}{b^4}\Pi\left(\frac{b^4 - b^2\,p^2 - p^2}{b^4}, \mathrm{am}(\sigma), k^2\right)\right), \qquad (4.70)$$

where the function $\Pi(n, z(\sigma), m)$ by now should be familiar. A similar rotation is also possible for the bosonic fluctuation fields (which would then eliminate the first order derivative terms from the fluctuation Lagrangian), however in case of bosons we chose not to bother since the advantage is purely of notational nature and does not contribute to an actual solution for the problem.[15]

[15] The idea behind such a rotation is to try and find similarities in the bosonic and fermionic coupled systems which might contribute to a solution. Unfortunately, this did not prove effective as of now.

Now one can go on to explicate all the indices in the fermionic action and multiply all the terms out. The terms become unwieldy at first and it is a notational nightmare, but in principle it is a straightforward process that can be automated. However, there remains one more point of physical thinking which simplifies the expressions at hand drastically. That is $\kappa$-symmetry, which we have to remove (fix) somehow to avoid ambiguities due to gauge freedom. We will make a fixing choice that is very convenient in type IIB superstring theory [2]. Since in type IIB both Majorana Weyl spinors $\Theta^1$ and $\Theta^2$ have the same chirality, their 16 components respectively occupy the same half of the 32 components in general spinor notation in ten dimensions. Since $\kappa$-symmetry removes half of the degrees of freedom of $\Theta^1$ and $\Theta^2$ put together as gauge freedom, it is actually possible to demand

$$\Theta^1 = \Theta^2 = \Theta \tag{4.71}$$

to obtain a valid fixing of $\kappa$-symmetry. (Note that this would not have been possible in type IIA, since there $\Theta^1$ and $\Theta^2$ are of opposite chirality.) Having made this choice from the very beginning, all the terms in the fermion Lagrangian (4.59) simplify drastically, such that only the following remains:

$$\mathcal{L}_{2F} = -2i\sqrt{g}\bar{\Theta}\left(g^{\alpha\beta}\gamma_\alpha\mathcal{D}_\beta + \frac{1}{2\sqrt{g}}s^{\alpha\beta}\gamma_\alpha\mathcal{U}\gamma_\beta\right)\Theta. \tag{4.72}$$

Starting from this expression and explicating all indices and making appropriate use of classical equations of motion it is then straightforward to arrive at the following:

$$\begin{aligned}\mathcal{L}_{2F} =& -2\sqrt{g}\bar{\Theta}\left(-\frac{i}{g^{1/4}}\Gamma_0\,\partial_\tau - \frac{i}{g^{1/4}}\Gamma_9\,\partial_\sigma - \frac{i}{g^{1/4}}m_F + M_F\right)\Theta && (4.73)\\ =& -2\sqrt{g}\bar{\Theta}\mathbb{O}_F\Theta, && (4.74)\end{aligned}$$

with fermionic differential operator $\mathbb{O}_F$, where the 'mass' terms are

$$\begin{aligned}m_F =& \Gamma_9\frac{\mathrm{sn}(\sigma,k^2)\mathrm{dn}(\sigma,k^2)}{2\mathrm{cn}(\sigma,k^2)} && (4.75)\\ M_F =& i\sqrt{1+\frac{b^4-(1+b^2)p^2}{(1+b^2)p^2}\mathrm{cn}^2(\sigma,k^2)}\;\Gamma_{12}\left(\Gamma_7\,\cos\beta+\Gamma_8\,\sin\beta\right). && (4.76)\end{aligned}$$

Having gotten our hands on the fermionic differential operator, we can again utilize the path integral formalism from QFT to obtain the effective fermionic action. In particular, the relevant equation corresponding to the bosonic case of

(4.52) reads[16]:

$$e^{-S_{F,\mathit{eff}}} = \int \mathcal{D}\bar{\Theta}\mathcal{D}\Theta \; e^{2\int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{g}\; \bar{\Theta}\,\mathbb{O}_F\,\Theta} = (\det \mathbb{O}_F)^{1/4}. \tag{4.77}$$

All the considerations regarding translation invariance in $\tau$ go through just the same as in the bosonic case. With this we then obtain the complete one-loop effective action containing both bosonic and fermionic contributions:

$$S_{\mathit{eff}} = -\frac{\mathcal{T}}{2}\int \frac{\mathrm{d}\omega_\tau}{2\pi} \ln \frac{\sqrt{\det \mathbb{O}_F}}{\det \mathbb{O}}, \tag{4.78}$$

where we pulled one power of 1/2 out of the logarithm as a factor in front for convenience.

In fact, the fermionic differential operator $\mathbb{O}_F$ in (4.74) can be further simplified in noticing that the mass term $m_F$ is proportional to the gamma matrix $\Gamma_9$, just as the 'kinetic' term $\partial_\sigma$. One can see that:

$$\sqrt{\mathrm{cn}(\sigma,k^2)}\partial_\sigma \frac{1}{\sqrt{\mathrm{cn}(\sigma,k^2)}} = \partial_\sigma + \frac{\mathrm{sn}(\sigma,k^2)\mathrm{dn}(\sigma,k^2)}{2\mathrm{cn}(\sigma,k^2)}. \tag{4.79}$$

Pulling out these factors left and right from the differential operator exactly gets rid of the term $m_F$:

$$\begin{aligned}\mathbb{O}_F =& \left(-\frac{\omega_\tau}{g^{1/4}}\Gamma_0 - \frac{i}{g^{1/4}}\Gamma_9\,\partial_\sigma - \frac{i}{g^{1/4}}m_F + M_F\right)\\ =&\sqrt{\mathrm{cn}(\sigma,k^2)}\left(-\frac{\omega_\tau}{g^{1/4}}\Gamma_0 - \frac{i}{g^{1/4}}\Gamma_9\,\partial_\sigma + M_F\right)\frac{1}{\sqrt{\mathrm{cn}(\sigma,k^2)}},\end{aligned} \tag{4.80}$$

where now the 'inner' differential operator can be called $\mathbb{O}_F'$:

$$\mathbb{O}_F' = \left(-\frac{\omega_\tau}{g^{1/4}}\Gamma_0 - \frac{i}{g^{1/4}}\Gamma_9\,\partial_\sigma + M_F\right). \tag{4.81}$$

Considering the determinant we then realize that $\det \mathbb{O}_F$ actually must give exactly the same result as $\det \mathbb{O}_F'$:

$$\det(\mathbb{O}_F) = \det\left(\sqrt{\mathrm{cn}(\sigma,k^2)}\right)\det(\mathbb{O}_F')\frac{1}{\det\left(\sqrt{\mathrm{cn}(\sigma,k^2)}\right)} = \det(\mathbb{O}_F'). \tag{4.82}$$

[16] Since the path integral is over Grassmann valued fermionic fields, the determinant appears in the numerator instead of denominator as was the case with bosons. Also due to the spinors being Majorana we get an extra square root.

Therefore, instead of $\mathbb{O}_F$ we can use $\mathbb{O}'_F$ in (4.78) to determine the effective action. Finally, to be able to solve for the determinant of the differential operator $\mathbb{O}'_F$ using the functional techniques introduced in Appendix C, one will have to choose a specific representation for the gamma matrices. It will turn out that in a special case (of our interest) we will be able to diagonalize the operator after squaring $\mathbb{O}'_F$ and choosing one specific basis of gamma matrices such that they have a two dimensional representation. In the general form (4.81), just as in the bosonic case, it is clear that the differential operator is too complicated to compute it's determinant. In the next section we will carry out the computation in the special case $q = 0$ which corresponds to vanishing opening angle $\theta = 0$.

### 4.1.4 Divergence Cancellation

To be precise we should actually address the issue of divergence cancellations before we proceed. Ratios of functional determinants of differential operators are not always finite, and there exists a mathematical machinery using so called Seeley coefficients to check whether divergence cancellations occur or not.
Considering the ratio of two second order Laplace type operators $\Delta_1$ and $\Delta_2$ put in the 'standard' form

$$\Delta = -g^{ab}\mathcal{D}_a\mathcal{D}_b + X, \tag{4.83}$$

where $\mathcal{D}_i$ is the covariant derivative and $X$ the mass term, certain *divergence coefficients* can be defined for each of them [26]:

$$b(\Delta) = Tr\left(\frac{1}{2}\mathbb{1}R^{(2)} - X\right), \tag{4.84}$$

where $R^{(2)}$ is the Ricci curvature scalar that we will mention explicitly in (6.15). Then the full divergence of the ratio of determinants is given by:

$$\mathcal{G} = \int_M \mathrm{d}^2\sigma\Big(b(\Delta_1) - b(\Delta_2)\Big), \tag{4.85}$$

where $M$ is the manifold (two dimensional parameter space in $\tau$ and $\sigma$ in our case). Considering the bosonic and fermionic operators as discussed above (squaring the fermionic operator to get it into standard form and therefore multiplying the coefficient by one half), we get up to a constant factor:

$$\frac{1}{2}b(\mathbb{O}_F^2) - b(\mathbb{O}) \sim R^{(2)}. \tag{4.86}$$

Therefore, in our case the full divergence is proportional to:

$$\mathcal{G} \sim \int_M \mathrm{d}^2\sigma R^{(2)}. \tag{4.87}$$

As already observed in [16, 2], this is reminiscent of the Gauss-Bonnet theorem – which for surfaces with boundary, as in this case at hand, is formally written as $\int_M R^{(2)} + \int_{\partial_M} K = 2\pi\,\chi(M)$, where $K$ is the so-called extrinsic curvature and $\chi(M)$ is the Euler number associated to the manifold. Namely, a more careful evaluation of (4.84) (following the arguments of [46]) would say that the divergence proportional to $R^{(2)}$ should be accompanied by a boundary term, thus promoting it to the Euler number. Since the worldsheet is a single strip with no holes in it the Euler characteristic vanishes, $\chi(M) = 0$. In other words it is expected that a proper evaluation of Seeley coefficients would lead to the exact cancellation of divergencies, and while this as yet an unproven result, we will just assume - as already done in [16, 2] - that it is true.

# 5 Space-like and Light-like Cusped Wilson Loops

In one of the early checks of certain aspects of AdS/CFT correspondence Gubser, Klebanov and Polyakov found out that so called twist-two operators[17] in $\mathcal{N} = 4$ SYM correspond to rotating strings in $AdS_5 \times S^5$ (with angular momentum on the $AdS_5$ part of space) for large value of the 't Hooft coupling $\lambda$ [20]. The correspondence relies on the observation that the logarithmic asymptotics for the energy of a string rotating in $AdS_5 \times S^5$ are qualitatively the same as in the ones exhibited from the anomalous dimensions of twist operators in the large spin limit, and which are known to be governed by the cusp anomaly $\Gamma_{cusp}(\lambda)$. The explicit study of the energy of rotating strings has thus lead to the first strong coupling results for the cusp anomaly, which in this context [18] is also known as "scaling function", whose leading and subleading strong coupling values read [20, 44]

$$f(\lambda)_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi} - \frac{3 \ln 2}{\pi} + .... \tag{5.1}$$

On the gauge theory side the same quantity appears as cusp anomaly of a light-like Wilson loop, which then by AdS/CFT correspondence can also be computed by a minimal surface configuration on $AdS_5 \times S^5$ related to the one discussed in this thesis. In [5], the scaling function coefficients (5.1) have been reproduced to one-loop accuracy in a sigma model computation for a minimal surface explicitly dual to a light-like cusped Wilson loop. However, starting from a minimal surface dual to a more general space-like cusped Wilson loop (which we discuss in this thesis) and performing an analytic continuation to reach the light-like configuration, the coefficient of the scaling function has been computed by Kruczenski in [1] only at the classical level (the term proportional to $\sqrt{\lambda}$). Here we will follow through his computation and reproduce the result. After that we will conduct a one-loop computation to also verify the second scaling function coefficient (proportional to $\lambda^0$) via this somewhat different approach. This explicit check shall strengthen our assumption that our understanding of the geometry of minimal surfaces, their geometric relations to each other and identifications with objects (Wilson loops) on the gauge theory side is correct.

[17] In this thesis we are concerned with computations on the string theory side. To find details regarding twist-two operators in $\mathcal{N} = 4$ SYM we refer the reader to [45].

[18] We notice that for the scaling function $f(\lambda)$ or cusp anomaly $\Gamma_{cusp}(\lambda)$ an all loop result has been derived, known as BES (Beisert-Eden-Staudacher) equation, with the use of integrability techniques. The result has been confirmed both in its weak and strong coupling expansion by independent calculations such as evaluation of gluon scattering amplitudes or results of the type derived in this section.

## 5.1 Classical Result

Since the quantity we are about to compute is dual to a simple light-like cusped Wilson loop there is no reason to maintain the jump in the coupling to scalar fields at the cusp. In our system, this corresponds to setting $q = 0$ which makes the $\vartheta$ dimension vanish from classical considerations. With this the relevant classical metric becomes:

$$ds^2 = \sqrt{\lambda}(-\cosh^2\rho\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\varphi^2). \tag{5.2}$$

For convenience, in this particular classical computation we will fix the parametrization as $\tau = t$ and $\sigma = \varphi$ with $\sigma_0 = \Phi/2$. Then the induced metric reads:

$$ds^2 = \sqrt{\lambda}(-\cosh^2\rho\,\mathrm{d}t^2 + ((\partial_\varphi\rho)^2 + \sinh^2\rho)\mathrm{d}\varphi^2). \tag{5.3}$$

The corresponding Nambu-Goto action is straightforwardly obtained to be:

$$S_{NG} = \frac{1}{2\pi}\int \mathrm{d}t\mathrm{d}\varphi\sqrt{-g} = \frac{\sqrt{\lambda}}{2\pi}T\int \mathrm{d}\varphi\sqrt{(\partial_\varphi\rho)^2 + \sinh^2\rho}. \tag{5.4}$$

Since the Lagrangian density has no explicit dependence on $\varphi$, the corresponding Hamiltonian is conserved. A short computation gives the Hamiltonian (which we choose to be positive definite):

$$E = \frac{\cosh\rho\,\sinh^2\rho}{\sqrt{(\partial_\varphi\rho)^2 + \sinh^2\rho}} = \cosh\rho_0\,\sinh\rho_0. \tag{5.5}$$

Here we made use of the symmetry property of our system $\rho'(0) = 0$ and expressed the Hamiltonian in terms of the minimal value $\rho_0$. With (5.5) we effectively obtained the classical equation of motion:

$$(\partial_\varphi\rho)^2 = \left(\frac{\cosh^2\rho\,\sinh^2\rho}{\cosh^2\rho_0\,\sinh^2\rho_0} - 1\right)\sinh^2\rho. \tag{5.6}$$

Starting from the equation of motion we can obtain an integral representation for the opening angle $\Phi$ as follows:

$$\frac{\Phi}{2} = \int_0^{\frac{\Phi}{2}}\mathrm{d}\varphi = \int_{\rho(0)}^{\rho(\Phi/2)}\frac{1}{(\partial_\varphi\rho)}\mathrm{d}\rho \tag{5.7}$$

$$= \int_{\rho_0}^{\infty}\frac{1}{\sqrt{\left(\frac{\cosh^2\rho\,\sinh^2\rho}{\cosh^2\rho_0\,\sinh^2\rho_0} - 1\right)\sinh^2\rho}}\mathrm{d}\rho. \tag{5.8}$$

To bring this integral into a shape that is more convenient to deal with, we perform the following variable substitution:

$$\sinh^2\rho = w^2 + \sinh^2\rho_0 \quad , \quad \mathrm{d}\rho = \frac{w}{\sqrt{(1+w^2+\sinh^2\rho_0)(w^2+\sinh^2\rho_0)}}\mathrm{d}w. \tag{5.9}$$

With this the integral in (5.8) can be unfolded to describe the complete opening angle $\Phi$ and not only a half of it:

$$\Phi = \int_{-\infty}^{\infty} \mathrm{d}w \frac{f_0\sqrt{f_0^2+1}}{(w^2+f_0^2)\sqrt{(w^2+f_0^2+1)(w^2+2f_0^2+1)}}, \tag{5.10}$$

where we used the abbreviation $f_0 = \sinh\rho_0$ to simplify notation. Making use of the same steps and variable substitutions, we can re-express the Nambu-Goto action (5.4) as follows:

$$S_{NG} = \frac{\sqrt{\lambda}}{2\pi} T \int_{-\infty}^{\infty} \mathrm{d}w \sqrt{\frac{w^2+f_0^2+1}{w^2+2f_0^2+1}}. \tag{5.11}$$

In this integral representation the standard divergence for two lines along the boundary (already discussed around (4.28)) becomes especially apparent. To obtain the finite worldsheet area we can explicitly subtract this divergence at $w \to \infty$ and write:

$$A = \frac{\sqrt{\lambda}}{2\pi} T \int_{-\infty}^{\infty} \mathrm{d}w \left( \sqrt{\frac{w^2+f_0^2+1}{w^2+2f_0^2+1}} - 1 \right). \tag{5.12}$$

Equations (5.10) and (5.12) are exactly the ones Kruczenski started with in his paper. From the way the classical system was set up in section 4 it is obvious that the cusp of the Wilson loop at hand is space-like. However, the duality exists for light-like cusped Wilson loops and rotating strings, so that to make connection to it from our system we have to find an analytic continuation to make the cusp light-like. The cusp becomes light-like if one takes the real part of the opening angle to be $\Phi = \pi$ (straight line with no cusp) and then adds an imaginary part going to infinity [29], so that

$$\Phi = \pi + i\gamma, \tag{5.13}$$

with $|\gamma| \to \infty$. This also corresponds to cusp angle $\phi = -i|\gamma|$. To see this, consider the $AdS_5$ space in Poincaré patch:

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(\mathrm{d}z^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2\right). \tag{5.14}$$

Here we have Euclidean signature. Now, consider going into polar coordinates, say, in $(x_1, x_2) \to (\rho, \varphi)$. With this we get the familiar relations $x_1^2 + x_2^2 = \rho^2$ and $x_2/x_1 = \tan\varphi$. At this point taking $\varphi \to \pi + i\varphi'$ corresponds to:

$$\frac{x_2}{x_1} = \tan(\pi + i\varphi') = -i\tanh\varphi', \tag{5.15}$$

so that one of the $x_1$ or $x_2$ must have become complex since the right hand side is purely imaginary. Lets say we take $x_2$ complex (which yields a negative $\mathrm{d}x_2^2$ in the metric), therefore the analytic continuation in $\varphi$ gives the passing to Minkowski signature as a result. Finally, taking $\varphi' \to \infty$ in (5.15) results in the ratio $ix_2/x_1 = 1$, so that not only the signature is Minkowski, but the combination of coordinates $x_1$, $x_2$ is also constrained to a light-cone which exactly gives a light-like system. In global coordinates the argumentation is less transparent but has effectively the same result.
Considering (5.10), this light-like continuation $\Phi = \pi + i\gamma$ (or $\phi = -i\gamma$) with $\gamma \to \infty$ can be reached by analytically continuing $f_0 \to if_0$ and then taking $f_0 \to 1/\sqrt{2}$. After the analytic continuation the integrand in the opening angle equation becomes purely imaginary. However, it also develops a residue which contributes the real part [19]:

$$\Phi = \int_{-\infty}^{\infty} \mathrm{d}w \frac{if_0\sqrt{1-f_0^2}}{(w^2 - f_0^2)\sqrt{(w^2 - f_0^2 + 1)(w^2 - 2f_0^2 + 1)}} \tag{5.16}$$

$$= \pi + iP.P.\int_{-\infty}^{\infty} \mathrm{d}w \frac{f_0\sqrt{1-f_0^2}}{(w^2 - f_0^2)\sqrt{(w^2 - f_0^2 + 1)(w^2 - 2f_0^2 + 1)}}, \tag{5.17}$$

where $P.P.$ denotes the principal part of the integral. To demonstrate that the $\gamma$ part of the opening angle indeed diverges for $f_0 \to 1/\sqrt{2}$, one can define $2\delta = 1 - 2f_0^2$, so that we get:

$$\gamma = P.P.\int_{-\infty}^{\infty} \mathrm{d}w \frac{\sqrt{1-2\delta}\sqrt{1+2\delta}}{(2w^2 - 1 + 2\delta)\sqrt{(w^2 + 2\delta)(w^2 + \frac{1}{2} + \delta)}}. \tag{5.18}$$

Now, for $\delta \to 0$ (which obviously corresponds to $f_0 \to 1/\sqrt{2}$) one can see that the biggest contribution stems from the term $\sqrt{w^2 + 2\delta}$ in the denominator around $w \approx 0$. Simplifying the integral to compute only this leading part, we get:

$$\gamma \approx \int_{-\epsilon}^{\epsilon} \mathrm{d}w \frac{1}{-\sqrt{(w^2 + 2\delta)^{\frac{1}{2}}}} = -2\sqrt{2}\mathrm{arcsinh}\left(\frac{\epsilon}{\sqrt{2\delta}}\right) \approx \sqrt{2}\ln\delta, \tag{5.19}$$

[19] Actually, considering that we are going to send the imaginary part to infinity, the finite real part is irrelevant since it will get dominated out anyway.

so that $\gamma$ indeed diverges for $f_0 \to 1/\sqrt{2}$. Performing the same analytic continuation on the area (5.12) we observe:

$$A = \frac{\sqrt{\lambda}}{2\pi} T \int_{-\infty}^{\infty} \mathrm{d}w \left( \sqrt{\frac{w^2 + \frac{1}{2} + \delta}{w^2 + 2\delta}} - 1 \right), \tag{5.20}$$

so that again the leading contribution is due to the $\sqrt{w^2 + 2\delta}$ term in the denominator around $w \approx 0$. The leading order result is then:

$$A \approx \frac{\sqrt{\lambda}}{2\pi} T \frac{1}{\sqrt{2}} \int_{-\epsilon}^{\epsilon} \mathrm{d}w \sqrt{\frac{1}{w^2 + 2\delta}} \approx -\frac{\sqrt{\lambda}}{2\pi} T \frac{1}{\sqrt{2}} \ln \delta = \frac{\sqrt{\lambda}}{4\pi} T\, |\gamma|. \tag{5.21}$$

Finally, considering that the anomalous dimension of the twist-two operator is given by $\gamma_s = -2\bar{\Gamma}^{(adj.)}_{cusp} \ln S$, where $S$ is the spin and $\bar{\Gamma}^{(adj.)}_{cusp} = 2\bar{\Gamma}_{cusp}$, we can obtain the reduced cusp anomaly $\bar{\Gamma}_{cusp}$ from the area through:

$$\bar{\Gamma}_{cusp} = \frac{-A}{|\gamma|\, T} = -\frac{\sqrt{\lambda}}{4\pi}. \tag{5.22}$$

With this the anomalous dimension of the twist-two operator is therefore given by $\gamma_s = \frac{\sqrt{\lambda}}{\pi} \ln S$, which reproduces the classical coefficient for the scaling function $f(\lambda)_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi}$. To sum up, we learned that with the classical scaling function coefficient $a_0 = 1/\pi$ the classical area in the light-like cusp limit is given by:

$$A = \mathcal{V} \sqrt{\lambda} a_0, \tag{5.23}$$

where $\mathcal{V} = T\, |\gamma|/4$. With this we expect the one-loop effective action in the next chapter eventually to result in:

$$S_{eff} = \mathcal{V} a_1, \tag{5.24}$$

with the same $\mathcal{V}$, but where now $a_1$ will be the second coefficient of the scaling function. Also, since we will go back to our previous notation for conserved charges to compute the one loop result we should express the analytic continuation and limit in terms of the parameter $p$. We realize making use of (4.13) that before the analytic continuation (having $q = 0$):

$$f_0^2 = \sinh^2 \rho_0 = \frac{1}{2} \left( -1 + \frac{1}{p^2} \sqrt{p^4 + 4p^2} \right). \tag{5.25}$$

Performing the analytic continuation $f_0 \to i f_0$ and then taking the value $f_0 = 1/\sqrt{2}$ effectively yields $f_0^2 \to -1/2$. Considering (5.25) it immediately becomes clear that this value is reached by analytically continuing $p \to ip$ and then taking

$p = 2$, or in other words $p^2 \to -4$. Expressed in the parameter $p$ (after analytic continuation $p \to ip$) the proportionality factor in (5.23) and (5.24) then reads:

$$\mathcal{V} = \frac{T\,|\gamma|}{4} = \frac{\sqrt{2}}{4} T \lim_{p \to 2} \ln \frac{2}{\sqrt{p-2}}, \tag{5.26}$$

where the summand containing $\ln 2$ actually is irrelevant, since it gets dominated out by the logarithmic divergence.
Having made ourselves familiar with the classical computation and result, we can now proceed to conduct the one-loop investigation and find out whether the second scaling function coefficient can also be found by use of minimal surfaces [20].

## 5.2 One-Loop Determinants

Back in section 4 where we discussed the problem setting, we found that the general bosonic and fermionic differential operators turn out to be very complicated, such that it is not possible to obtain exact functional determinants for them. In this section however, the situation improves since we consider the special case $q = 0$. With $q = 0$ the off-diagonal elements in the bosonic differential operator $\mathbb{O}$ vanish. All mass terms also experience a drastic simplification. In fact, the most convenient shape for the operators is reached by rescaling the bosonic operator $\mathbb{O}$ by $\sqrt{g}$, rescaling the fermionic operator $\mathbb{O}'_F$ by $g^{1/4}$ and then squaring the fermionic operator (which makes it diagonalizable) before computing the determinants. [21]

With the above rescalings and $q = 0$ the bosonic differential operator (4.48) can be written as follows (introducing new abbreviations to avoid notational

[20]Since this particular chapter was mostly a reproduction of the work of Kruczenski, it is kept very short in the style of a summary. For a more explicit classical treatment please refer to the original paper by Kruczenski [1]

[21]Such rescaling of differential operators can introduce extra finite parts or logarithmic divergences into the determinant ratio. These possible contributions can be computed making use of certain Seeley coefficients related to the ones discussed in Appendix A of [16]. For the operators and rescaling functions that appear here, the evaluation of the extra contributions formally should go along the lines of the explicit computation in [40], where it was found that they identically cancel. Therefore, the rescalings do not alter the determinant ratio.

clutter):

$$\mathbb{O}_{B,i} := \sqrt{g}\mathbb{O}_{ii} = -\,\partial_\sigma^2 + \omega_\tau^2 + \frac{2(1-k^2)}{\mathrm{cn}^2(\sigma,k^2)} \qquad , \quad i = 1,2 \tag{5.27}$$

$$\mathbb{O}_{B,s} := \sqrt{g}\mathbb{O}_{77} = \sqrt{g}\mathbb{O}_{ss} = -\,\partial_\sigma^2 + \omega_\tau^2 \qquad , \quad s = 3,4,5,6 \tag{5.28}$$

$$\mathbb{O}_{B,8} := \sqrt{g}\mathbb{O}_{88} = -\,\partial_\sigma^2 + \omega_\tau^2 + \frac{2(1-k^2)}{\mathrm{cn}^2(\sigma,k^2)} - 2k^2\mathrm{cn}^2(\sigma,k^2). \tag{5.29}$$

Since all eight matrix dimensions of the operator are decoupled from each other, their determinants can be computed separately and then multiplied together. The differential operator (5.28) obviously looks simple enough so that analytic solutions for its homogeneous differential equation can be computed. The other two types of operators (5.27) and (5.29) look more involved, but in fact they can be reshaped to so called single gap Lamé operators, for which solutions of homogeneous equations are also known in closed form.

A generic single gap Lamé differential operator is given by:

$$\mathbb{O}_\Lambda = -\partial_\sigma^2 - \Lambda + 2\,m\,\mathrm{sn}^2(\sigma,m), \tag{5.30}$$

where $\Lambda$ is an arbitrary constant and $m$ is the modulus. To bring the operators (5.27) and (5.29) into this form, one can for instance use several modulus and argument transformations for Jacobi elliptic functions. This would be rather tedious. Alternatively, one can make a general ansatz for the desired form of the mass term:

$$M_g = C_1 + 2C_2\mathrm{sn}^2(C_3\sigma + C_4, C_2), \tag{5.31}$$

and then compare the expansion-coefficients of this term with the ones of mass terms in (5.27) and (5.29) around, say, $\sigma = 0$. With an expansion to the fourth order all constants $C_j$ should be fixed. If then further expansion steps do not yield inconsistencies, we will have found the correct reshaping. Of course, having obtained the necessary constants $C_j$ it is straightforward to check the equality of the different representations numerically.

Applying the above procedure, we arrive at the following single gap Lamé shapes for the bosonic operators (5.27) and (5.29):

$$\mathbb{O}_{B,i} =(1-k^2)\Big(-\,\partial_{\sigma_1}^2 + \omega_1^2 + 2k_1^2\mathrm{sn}^2(\sigma_1 + i\mathbb{K}_1', k_1^2)\Big) \tag{5.32}$$

$$\mathbb{O}_{B,8} =(1-k^2)(1+k_1)^2\Big(-\,\partial_{\sigma_2}^2 + \omega_2^2 + 2k_2^2\mathrm{sn}^2(\sigma_2 + i\mathbb{K}_2', k_2^2)\Big), \tag{5.33}$$

with

$$k_1^2 = \frac{k^2}{k^2-1} \quad , \quad \mathbb{K}_j' = \mathbb{K}(1-k_j^2) \quad , \quad \sigma_1 = \sqrt{1-k^2}\sigma + \mathbb{K}_1 \quad , \quad \omega_1^2 = \frac{\omega_\tau^2}{1-k^2} \tag{5.34}$$

$$k_2^2 = \frac{4k_1}{(1+k_1)^2} \quad , \quad \sigma_2 = (1+k_1)(\sqrt{1-k^2}\sigma + \mathbb{K}_1) \quad , \quad \omega_2^2 = \frac{\omega_\tau^2}{(1-k^2)(1+k_1)^2} - k_2^2.$$

It will turn out that we will be able to also write the fermionic operator in the form of a single gap Lamé operator. For that end we start from the expression (4.81). With $q = 0$ this reads:

$$g^{1/4}\mathbb{O}'_F = -i\Gamma_9\,\partial_\sigma - \omega_\tau\Gamma_0 + i\frac{\sqrt{1-k^2}}{\mathrm{cn}(\sigma,k^2)}\Gamma_1\Gamma_2\Gamma_7. \tag{5.35}$$

(The angle $\beta$ vanishes for $q = 0$.) The reason we want to square this operator before computing the determinant lies in the number and position of different gamma matrices which appear. Having five gamma matrices in the operator distributed over all the appearing terms makes it clear that the matrix structure is not diagonalizable. After squaring we expect most terms to vanish due to anti-commutation relations of gamma matrices. Squaring the operator (5.35) we obtain:

$$(g^{1/4}\mathbb{O}'_F)^2 = \mathbb{1}\left(-\partial_\sigma^2 + \omega_\tau^2 + \frac{1-k^2}{\mathrm{cn}^2(\sigma,k^2)}\right) + \frac{\sqrt{1-k^2}\mathrm{sn}(\sigma,k^2)\mathrm{dn}(\sigma,k^2)}{\mathrm{cn}^2(\sigma,k^2)}\Gamma_9\Gamma_1\Gamma_2\Gamma_7. \tag{5.36}$$

Now we only have four gamma matrices left and all of them are gathered in one spot. At this point we can choose the following particular representation for the gamma matrices which will make the squared operator (5.36) diagonal:

$$\Gamma_9 = (\mathbb{1}\otimes\sigma_1) \quad , \quad \Gamma_1 = (\sigma_1\otimes\sigma_2) \quad , \quad \Gamma_2 = (\sigma_2\otimes\sigma_2) \quad , \quad \Gamma_7 = (\sigma_3\otimes\sigma_2), \tag{5.37}$$

where $\sigma_j$ with $j = 1, 2, 3$ are Pauli matrices and $\mathbb{1}$ is the two dimensional unity matrix. With this choice we straightforwardly find:

$$\Gamma_9\Gamma_1\Gamma_2\Gamma_7 = (\mathbb{1}\otimes\sigma_1)(\sigma_1\otimes\sigma_2)(\sigma_2\otimes\sigma_2)(\sigma_3\otimes\sigma_2) = -(\mathbb{1}\otimes\sigma_3). \tag{5.38}$$

Since $\sigma_3$ is diagonal, the whole combination of gamma matrices is therefore diagonal and we find only the following diagonal entries for the squared fermionic operator:

$$(g^{1/4}\mathbb{O}'_F)^2_{ll} = -\partial_\sigma^2 + \omega_\tau^2 + \frac{1-k^2 \pm \sqrt{1-k^2}\,\mathrm{sn}(\sigma,k^2)\mathrm{dn}(\sigma,k^2)}{\mathrm{cn}^2(\sigma,k^2)}, \tag{5.39}$$

where the $\pm$ is due to different entries of the $\sigma_3$ Pauli matrix. Due to the antisymmetry of $\mathrm{sn}(-\sigma,k^2) = -\mathrm{sn}(\sigma,k^2)$ while having symmetric $\mathrm{dn}(-\sigma,k^2) = \mathrm{dn}(\sigma,k^2)$, the two kinds of potentials are just reversed versions of each other, which ensures that the determinant is the same for any choice of the sign. Therefore, to find the fermionic determinant we will only have to compute the determinant for one of the components and take it to the necessary power.

Applying the same procedure as with the bosons to (5.39), we find the single gap Lamé shape of the fermionic operator for the sign choice of, say, plus (again introducing a new abbreviation to simplify notation):

$$\mathbb{O}_{F,1} := (g^{1/4}\mathbb{O}'_F)^2_{11} = \frac{(1-k^2)(1+k_1)^2}{4}\left(-\partial^2_{\sigma_3} + \omega_3^2 + 2k_2^2\mathrm{sn}^2(\sigma_3 + \mathbb{K}_2 + i\mathbb{K}'_2, k_2^2)\right), \tag{5.40}$$

where

$$\sigma_3 = \frac{\sqrt{1-k^2}(1+k_1)}{2}(\sigma+\mathbb{K}) \quad , \quad \omega_3^2 = k_2^2\left(\frac{\omega_\tau^2}{k_1(1-k^2)-1}\right). \tag{5.41}$$

Now that we have obtained all the necessary bosonic and fermionic operators in the single gap Lamé shape, we can proceed to compute the functional determinants through use of the Gel'fand Yaglom theorem described in appendix C. For that end we require the solution of the single gap Lamé differential equation:

$$[-\partial_\sigma^2 - \Lambda(\lambda) + 2\,m\,\mathrm{sn}^2(\sigma, m)]f(\sigma) = 0 \tag{5.42}$$

with initial conditions $f(-\mathbb{K}) = 0$ and $f'(-\mathbb{K}) = 1$. The determinant will then be given by $\det \mathbb{O} = f(\mathbb{K})_{\lambda=0}$. To construct such a solution, we can start from the following two linearly independent general solutions:

$$f^\pm(\sigma) = \frac{\vartheta_1\left(\frac{\pi(\sigma\pm\alpha)}{2\mathbb{K}}, q\right)}{\vartheta_4\left(\frac{\pi\sigma}{2\mathbb{K}}, q\right)} e^{\mp\sigma Z(\alpha)} \quad \text{with} \quad \mathrm{sn}(\alpha, m) = \sqrt{\frac{1}{m}(1+m-\Lambda)}, \tag{5.43}$$

where $\vartheta_1, \vartheta_4$ are Jacobi theta functions, $q = \exp\left(-\pi\frac{\mathbb{K}'}{\mathbb{K}}\right)$ and $Z(\alpha) := Z(\mathrm{am}(\alpha, m), m)$ is the Jacobi zeta function. Unfortunately, the solutions (5.43) diverge at the boundaries $\sigma_* \in \{-\mathbb{K}, \mathbb{K}\}$. Therefore, we will have to introduce a regulator $\epsilon$ and then consider the leading behavior of all the terms under $\epsilon \to 0$. Formally, a solution with the required initial conditions is straightforwardly constructed out of the general solutions to be:

$$f(\sigma) = \frac{1}{W(-\mathbb{K}+\epsilon)}\Big(f^+(-\mathbb{K}+\epsilon)f^-(\sigma) - f^-(-\mathbb{K}+\epsilon)f^+(\sigma)\Big), \tag{5.44}$$

where the so called Wronskian makes its appearance:

$$W(\sigma) = f^+(\sigma)(\partial_\sigma f^-(\sigma)) - f^-(\sigma)(\partial_\sigma f^+(\sigma)). \tag{5.45}$$

Then evaluating the leading order of $f(\mathbb{K}-\epsilon)_{\lambda=0}$ for $\epsilon \to 0$ should give us the functional determinants we are looking for. The whole business is purely technical and the results for the bosonic and fermionic operators above are[22]:

$$\det \mathbb{O}_{B,i} = -\frac{\sinh(2\mathbb{K}_1 Z(\alpha_1))}{\epsilon^2\sqrt{k^2-\omega_\tau^2}\sqrt{(\omega_\tau^2-k^2+1)(-\omega_\tau^2+2k^2-1)}}, \tag{5.46}$$

$$\det \mathbb{O}_{B,s} = \frac{\sinh(2\mathbb{K}\sqrt{\omega_\tau^2})}{\sqrt{\omega_\tau^2}}, \tag{5.47}$$

$$\det \mathbb{O}_{B,8} = -\frac{\sinh(2\mathbb{K}_2 Z(\alpha_2))}{\epsilon^2\sqrt{\omega_\tau^2(\omega_\tau^4+(2-4k^2)\omega_\tau^2+1)}}, \tag{5.48}$$

$$\det \mathbb{O}_{F,1} = -\frac{4i\sinh(\mathbb{K}_2 Z(\alpha_F))}{\epsilon\sqrt{1+8\omega_\tau^2(1-2k^2+2\omega_\tau^2)}}, \tag{5.49}$$

where according to (5.43)

$$\mathrm{sn}(\alpha_1,k_1^2) = \sqrt{\frac{1+k_1^2+\omega_1^2}{k_1^2}}\ ,\ \mathrm{sn}(\alpha_2,k_2^2) = \sqrt{\frac{1+k_2^2+\omega_2^2}{k_2^2}}\ ,\ \mathrm{sn}(\alpha_F,k_2^2) = \sqrt{\frac{1+k_2^2+\omega_3^2}{k_2^2}}.$$

We expect the resulting effective action to be finite. However, since we did not account carefully for boundary counterterms in our regularization scheme, extra divergences for large $\omega_\tau$ and small $\epsilon$ entered the expressions. Since these divergences are logarithmic, we can easily subtract them explicitly and obtain the following finite effective action:

$$S_{eff} = -\frac{\mathcal{T}}{2}\lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega_\tau}{2\pi}\ln\frac{\epsilon^2\omega_\tau^2\det^8\mathbb{O}_{F,1}}{\det^5\mathbb{O}_{B,s}\det^2\mathbb{O}_{B,i}\det\mathbb{O}_{B,8}}. \tag{5.50}$$

This integral over $\omega_\tau$ is too complicated to be evaluated exactly. However, what we are looking for is only the result for the light-like cusp limit $p\to 2i$. Taking this limit of the parameter $p$ in the integrand will yield a simplification so that we will be able to carry out the integration. This is the content of the next subsection.

## 5.3 One-Loop light-like Limit Computation

Before we attempt a limit evaluation on expression (5.50) it is of great benefit to first take a step back and consider whether the shape of our integral proves

[22] Since this is merely a reproduction of the work done in [2] we do not go into much detail on this part. The result for the fermionic determinant has been however reshaped into a simpler form with respect to the one featured in [2], by making use of the identities

$$\frac{\vartheta(2,0,q)}{\vartheta'(1,0,q)} = \frac{\pi}{2(1-k^2)^{1/4}\mathbb{K}} \qquad \text{and} \qquad \frac{\vartheta(4,\pi x/(2\mathbb{K}),q)}{\vartheta(3,\pi x/(2\mathbb{K}),q)} = \frac{(1-k^2)^{1/4}}{\mathrm{dn}(x,k^2)}.$$

convenient for limit considerations or not. We know that in the end we will have to divide the effective action by $\mathcal{V}$ in (5.26) to obtain our candidate for the one-loop scaling function coefficient. $\mathcal{V}$ basically contains a logarithmic divergence in $p$ and the time 'period' $T$. However, in (5.50) we find $\mathcal{T}$ instead of $T$ as a factor, so that with $q = 0$ the proportionality factor is:

$$\frac{\mathcal{T}}{T} = \frac{1}{\sqrt{1-2k^2}}. \tag{5.51}$$

After analytic continuation $p \to ip$, in the light-like cusp limit $p \to 2$ the parameter $k^2$ goes to infinity as $k^2 \propto 1/\sqrt{p-2}$. Therefore, we realize that without even considering the behavior of the logarithmic integrand in (5.50), the prefactor already gives a suppressing contribution $\sim \sqrt{p-2}$ to the effective action. Considering that we expect a logarithmic divergence and not zero, it is clear that such a suppressing prefactor is not a convenient feature to have at the beginning of this particular limit computation.

In order to resolve this problem we therefore perform the following rescaling on the integration variable $\omega_\tau$ in (5.50) before taking the light-like cusp limit:

$$\omega_\tau \to \sqrt{1-2k^2}\omega_t \equiv \sqrt{1-2k^2}\omega. \tag{5.52}$$

Considering that $\omega_\tau$ is the Fourier transform of $i\partial_\tau$, the rescaled variable $\omega_t$ is then simply a corresponding Fourier transform of $i\partial_t$ where $t$ is the target space time variable. Writing the integral for our one-loop effective action in terms of target space variables before taking limits will certainly cause no problems and will give a well defined result. Since it balances out the unwanted suppressing prefactor mentioned above, this rescaling actually yields a drastic simplification for the limit computation.[23]

[23] In some sense, one could argue that this rescaling is actually mandatory required before taking the limit, since the physical result we want to obtain originates in target space, while the map into parameter space variable $\tau$ becomes singular and non-differentiable in the limit at hand.

With this rescaling the determinants given in (5.46) through (5.49) become:

$$\det \mathbb{O}_{B,i} = -\frac{\sinh(2\mathbb{K}_1 Z(\alpha_1))}{\epsilon^2\sqrt{k^2-(1-2k^2)\omega^2}\sqrt{((1-2k^2)\omega^2-k^2+1)(2k^2-1)(\omega^2+1)}}, \tag{5.53}$$

$$\det \mathbb{O}_{B,s} = \frac{\sinh(2\mathbb{K}\sqrt{(1-2k^2)\omega^2})}{\sqrt{(1-2k^2)\omega^2}}, \tag{5.54}$$

$$\det \mathbb{O}_{B,8} = -\frac{\sinh(2\mathbb{K}_2 Z(\alpha_2))}{\epsilon^2\sqrt{(1-2k^2)\omega^2((1-2k^2)^2\omega^4+(2-4k^2)(1-2k^2)\omega^2+1)}}, \tag{5.55}$$

$$\det \mathbb{O}_{F,1} = -\frac{4i\sinh(\mathbb{K}_2 Z(\alpha_F))}{\epsilon\sqrt{1+8(1-2k^2)\omega^2(1-2k^2+2(1-2k^2)\omega^2)}}, \tag{5.56}$$

where we obviously also get the rescalings in:

$$\omega_1^2 = \frac{(1-2k^2)\omega^2}{1-k^2}, \quad \omega_2^2 = \frac{(1-2k^2)\omega^2}{(1-k^2)(1+k_1)^2} - k_2^2, \quad \omega_3^2 = k_2^2\frac{(1-2k^2)\omega^2}{k_1(1-k^2)-1}. \tag{5.57}$$

With this we can return to the equation for the effective action and proceed with the light-like cusp limit:

$$S_{eff} = -\frac{T}{2}\lim_{\epsilon\to 0}\int\frac{\mathrm{d}\omega}{2\pi}\ln\frac{\epsilon^2(1-2k^2)\omega^2\det^8\mathbb{O}_{F,1}}{\det^5\mathbb{O}_{B,s}\det^2\mathbb{O}_{B,i}\det\mathbb{O}_{B,8}}. \tag{5.58}$$

Note, that now $T$ appears in front of the integral as the time 'period' instead of $\mathcal{T}$, since the rescaling of d$\omega$ took care of the difference factor. To evaluate the limit on the integrand it is convenient to use the logarithm property and write two separate terms:

$$\begin{aligned}
&\ln\frac{\epsilon^2(1-2k^2)\omega^2\det^8\mathbb{O}_{F,1}}{\det^5\mathbb{O}_{B,s}\det^2\mathbb{O}_{B,i}\det\mathbb{O}_{B,8}} = \\
&\quad = \ln\Big(\frac{2^{16}(1-2k^2)^4\omega^6\sqrt{(1-2k^2)\omega^2}}{(1+8(1-2k^2)^2\omega^2(1+2\omega^2))^4}(1+\omega^2)(1-k^2+(1-2k^2)\omega^2)\cdot \\
&\qquad \cdot(k^2-(1-2k^2)\omega^2)\sqrt{\omega^2(1-2k^2+2(1-2k^2)^3\omega^2+(1-2k^2)^3\omega^4)}\Big) \\
&\qquad + \ln\frac{\sinh^8(\mathbb{K}_2 Z(\alpha_F))}{\sinh^5(2\mathbb{K}\sqrt{(1-2k^2)\omega^2})\ \sinh^2(2\mathbb{K}_1 Z(\alpha_1))\ \sinh(2\mathbb{K}_2 Z(\alpha_2))}.
\end{aligned} \tag{5.59}$$

The limit is easy to perform on the first logarithm containing just a rational function in $\omega$ and $k$. The result turns out to be finite:

$$\lim_{p\to 2i} \ln\left(\frac{2^{16}(1-2k^2)^4\omega^6\sqrt{(1-2k^2)\omega^2}}{(1+8(1-2k^2)^2\omega^2(1+2\omega^2))^4}(1+\omega^2)(1-k^2+(1-2k^2)\omega^2)\cdot\right. \tag{5.60}$$

$$\left.\cdot(k^2-(1-2k^2)\omega^2)\sqrt{\omega^2(1-2k^2+2(1-2k^2)^3\omega^2+(1-2k^2)^3\omega^4)}\right) = \ln\frac{4(1+\omega^2)\sqrt{\omega^2(2+\omega^2)}}{(1+2\omega^2)^2}.$$

Performing the $\omega$ integration on this result also yields a finite number:

$$\int \mathrm{d}\omega \ln\frac{4(1+\omega^2)\sqrt{\omega^2(2+\omega^2)}}{(1+2\omega^2)^2} = (2-\sqrt{2})\pi. \tag{5.61}$$

We expect the one-loop result to diverge with $|\gamma|\to\infty$ just as the classical result. Therefore, this contribution is probably not going to be of leading order. Still, we have the second logarithm term in (5.59) to evaluate.
To obtain the limit $p\to 2i$ on the second logarithm term in (5.59), we will have to find the leading contributions for Jacobi zeta functions in the case of modulus going to $m\to 1$. This could turn out to be a very non-trivial thing to do, especially if the first argument (integration range) also tends to some specific value. Fortunately, the first argument does not approach any specific value in the limit for all Jacobi zeta functions involved:

$$\lim_{p\to 2i}\mathrm{am}(\alpha_1,k_1^2) = \arcsin(\sqrt{2(1+\omega^2)}) \;\;,\;\; \lim_{p\to 2i}\mathrm{am}(\alpha_2,k_2^2) = \arcsin(\sqrt{1+\omega^2/2}) \;,$$
$$\lim_{p\to 2i}\mathrm{am}(\alpha_F,k_2^2) = \arcsin(\sqrt{1+2\omega^2}). \tag{5.62}$$

Therefore, the limit at hand does not force an evaluation of the Jacobi zeta functions at some specific point in the variable, modulus configuration, but rather just fixes the modulus without constraining the variable. With this the leading contribution to all the Jacobi zeta functions is obtained by simply taking $p=2i$. Since the modulus in each case becomes 1, the Jacobi zeta functions simplify as follows:

$$Z(\alpha_1)_{p=2i} = \sqrt{2(1+\omega^2)} \;\;,\;\; Z(\alpha_2)_{p=2i} = \sqrt{1+\omega^2/2} \;\;,\;\; Z(\alpha_F)_{p=2i} = \sqrt{1+2\omega^2}.$$

This great simplification shifts our attention to the complete elliptic integrals of the first kind which appear in the sinh functions alongside the Jacobi zetas. Series expansions for complete elliptic integrals of the first kind around the modulus values of our interest are readily available in the literature. $\mathbb{K}_1$ and $\mathbb{K}_2$ are logarithmically divergent, since their moduli $k_1^2$ and $k_2^2$ are going to 1. The modulus $k^2$ in $\mathbb{K}$ is diverging, which makes the function tend to zero, but the divergence

$\sqrt{(1-2k^2)\omega^2}$ next to it balances this out, so that again a net logarithmic divergence remains. In summary, after performing the analytic continuation $p \to ip$ we end up with the following leading contributions considering $p \to 2$:

$$\sinh(\mathbb{K}_2 Z(\alpha_F)) \approx \frac{1}{2} e^{\sqrt{1+2\omega^2}\ln\frac{2^3}{\sqrt{p-2}}}, \tag{5.63}$$

$$\sinh(2\mathbb{K}_1 Z(\alpha_1)) \approx \frac{1}{2} e^{\sqrt{2(1+\omega^2)}\ln\frac{2^3}{\sqrt{p-2}}}, \tag{5.64}$$

$$\sinh(2\mathbb{K}_2 Z(\alpha_2)) \approx \frac{1}{2} e^{2\sqrt{1+\omega^2/2}\ln\frac{2^3}{\sqrt{p-2}}}, \tag{5.65}$$

$$\sinh(2\mathbb{K}\sqrt{(1-2k^2)\omega^2}) \approx \frac{1}{2} e^{\sqrt{2\omega^2}\ln\frac{2^3}{\sqrt{p-2}}}. \tag{5.66}$$

Each term has the same kind of divergence and contributes to the leading order. These results combine to give the following leading order contribution for the second logarithm term in (5.59):

$$\begin{aligned}\ln\frac{\sinh^8(\mathbb{K}_2 Z(\alpha_F))}{\sinh^5(2\mathbb{K}\sqrt{(1-2k^2)\omega^2})\ \sinh^2(2\mathbb{K}_1 Z(\alpha_1))\ \sinh(2\mathbb{K}_2 Z(\alpha_2))} &= \\ = \left(8\sqrt{1+2\omega^2} - 5\sqrt{2\omega^2} - 2\sqrt{2(1+\omega^2)} - 2\sqrt{1+\omega^2/2}\right)\ln\frac{2^3}{\sqrt{p-2}}&.\end{aligned} \tag{5.67}$$

Since there is a divergence independent of $\omega$ integration, this part of the integrand dominates out the finite part (5.61). Therefore, the leading behavior of the effective action (5.58) is given by:

$$\begin{aligned}S_{eff} &= -\frac{T}{2}\ln\frac{2^3}{\sqrt{p-2}}\int\frac{\mathrm{d}\omega}{2\pi}\left(8\sqrt{1+2\omega^2} - 5\sqrt{2\omega^2} - 2\sqrt{2(1+\omega^2)} - 2\sqrt{1+\omega^2/2}\right) \\ &= -\frac{T}{\sqrt{2}}\frac{\ln 2^3}{2\pi}\ln\frac{2^3}{\sqrt{p-2}}.\end{aligned} \tag{5.68}$$

As expected the one-loop effective action features a logarithmic divergence in the light-like cusp limit. To compare this result with the second scaling function coefficient we should divide out the prefactor $\mathcal{V}$ given in (5.26). This straightforwardly leads to:

$$a_1 = \lim_{p\to 2}\frac{S_{eff}}{\mathcal{V}} = -\frac{3\ln 2}{\pi}, \tag{5.69}$$

which exactly reproduces the one-loop scaling function coefficient in (5.1). Therefore, we have shown that the minimal surface approach to space-like cusped Wilson loops on the string theory side confirms the scaling function up to one-loop accuracy. With this, we find our assumptions concerning minimal surfaces confirmed. In the next section we will deal with a somewhat related limiting behavior, aiming to find another kind of confirmation in a certain configuration.

# 6 A powerful scaling limit on $\Gamma_{cusp}(\phi,\theta,\lambda)$

As briefly explained in the introduction, much attention has been devoted in the past two years to the so-called generalized cusp anomalous dimension [2, 17, 25, 28, 42] $\Gamma_{cusp}(\phi,\theta,\lambda)$, with notable insights on the possibility to deriving exact formulas for it, either relating it to results given by localization techniques [17] or to objects for which integrability tools can come into play [28]. Additionally, in [25] a special scaling limit was identified, which involves the complexified angle $\theta$

$$i\,\theta \gg 1, \qquad \lambda \ll 1, \qquad \text{with} \qquad \hat{\lambda} = \lambda\, e^{i\,\theta}/4 \qquad \text{finite} \tag{6.1}$$

and which selects ladder diagrams (*i.e.* diagrams with no internal vertices) to be the only contribution to the generalized cusp anomaly $\Gamma_{cusp}(\phi,\theta,\lambda) \to \Gamma_{cusp}(\phi,\hat{\lambda})$ [24]. This particular restriction on the system is extremely powerful, in that it is known how to resum them at all orders [41], providing thus precise predictions at strong coupling. To reach this goal [25], all 2PI diagrams [26] can be considered for different orders in the new coupling $\hat{\lambda}$ (the number of these 2PI diagrams is finite), and a so-called Bethe-Salpeter equation can be used to generate all possible diagrams from the 2PI ones. Denoting two intervals along each of the Wilson lines starting at the cusp by $(0,T)$ and $(0,S)$, the Bethe-Salpeter equation can be written as an integro-differential equation:

$$\frac{\partial^2\Gamma}{\partial S\partial T} = \int_0^S \mathrm{d}s \int_0^T \mathrm{d}t K(S,s;T,t)\Gamma(s,t), \tag{6.2}$$

where $\Gamma(S,T)$ is the sum of all possible exchange diagrams between the two lines and $K$ is the Bethe-Salpeter kernel consisting of all 2PI diagrams. Performing change of variables, making a certain ansatz for $\Gamma$ in terms of a wave function $\psi_0$ times an exponential in the ladder limit case and observing a simplification for the kernel $K$ for small $\lambda$, the integro-differential equation is reduced to the following one dimensional Schrödinger type equation:

$$\left(\partial_y^2 + \frac{\hat{\lambda}}{8\pi^2(\cosh y + \cos\phi)}\right)\psi_0 = \frac{\alpha^2}{4}\psi_0, \tag{6.3}$$

[24] At the leading order in this limit, only the diagrams with the greatest power of $\cos\theta$ contribute at each $l$-loop order, which are diagrams where $l$ scalar lines end on each Wilson line. Since one takes simultaneously $\lambda \to 0$ the other possible contribution, given by gluon exchange, remains small.

[25] We sketch the derivation of such resummation as described in [25].

[26] Two particle irreducible diagrams are such diagrams which do not get cut in half by removing one internal leg. In QFT the exponentiated sum of 2PI diagrams equals the sum of all possible diagrams.

where $\phi$ is the cusp angle.
The ground state energy $\alpha_0$ is related to the cusp anomalous dimension in the scaling limit through $\Gamma_{cusp}(\phi,\hat{\lambda}) = -\alpha_0$.

As observed in [25], at strong coupling the potential becomes very deep, the energy can be approximated by the minimum of the potential at $y = 0$ from which

$$\frac{\hat{\lambda}}{2\pi^2(1+\cos\phi)} = \frac{\alpha_0^2}{4} \quad \longrightarrow \quad \Gamma_{cusp}(\phi,\hat{\lambda}) = -\alpha_0 = -\frac{\sqrt{\hat{\lambda}}}{2\pi\cos(\phi/2)}\,, \qquad \hat{\lambda} \gg 1\ . \tag{6.4}$$

It is not difficult (see next section) to verify that this is in agreement with the classical strong coupling computation of [2]. An interesting question is whether the resummation of diagrams in this limit can be also verified from a string theory computation beyond the classical level, and thus as a correction of order $(1/\sqrt{\hat{\lambda}})^0$. An explicit prediction, although in the special case $\phi \to \pi$ corresponding to the antiparallel lines limit or quark-antiquark potential, can be found in [21] [27], where at large $\hat{\lambda}$ it was obtained [28]

$$\frac{1}{\pi-\phi}\alpha_0 = \frac{1}{\pi-\phi}\Big(\frac{\sqrt{\hat{\lambda}}}{\pi} - 1\Big) + \mathcal{O}\big(1/\sqrt{\hat{\lambda}}\big)\ . \tag{6.5}$$

While the first term is clearly the $\phi \to \pi$ limit of (6.4) above, the second term is derived from the $\phi \to \pi$ limit of (6.3) as the energy of the zero-point fluctuations.

In this section our aim will be to verify, beyond the leading order in sigma-model perturbation theory, the prediction obtained from the resummation of ladder diagrams.

## 6.1 A first limit on the stringy results of [2]

As already noticed in [25], it is not difficult to verify the prediction on $\Gamma_{cusp}(\phi,\hat{\lambda})$ coming from the ladder diagrams resummation at leading order in a strong coupling expansion. Having realized that it is setting $q = -i\,r$ and sending $p \to 0$ that the expression for $\theta$ given in (4.23) becomes imaginary and large, one rewrite

[27] The primary goal of [21] was to establish a way to systematically compute corrections to the ladder limit. These are corrections of order $\lambda/\hat{\lambda}$, and as such they belong to the classical string regime.

[28] Note that there potentially might be an order of limits problem, since we first approximate the kernel at small $\lambda$ and only after that compose $\hat{\lambda}$ and take it to large values. Therefore, the string theory result where $\lambda$ itself is already large from the beginning is not necessarily expected to give the same result and it is actually somewhat astounding that it eventually does.

the classical expressions contained in Section **??** in terms of $p$ and $r$ and studies their behavior for small $p$. This way one can check that

$$\frac{e^{i\,\theta/2}}{2} = \frac{2\,r^2}{p\,\sqrt{1+r^2}} + \mathcal{O}(p) \tag{6.6}$$

$$\phi = \pi - 2\,\arctan r + \mathcal{O}(p^2) \tag{6.7}$$

where we used the relation $\Phi = \pi - \phi$ to obtain the result in terms of cusp angle $\phi$. Expanding in the limit the exact classical action (4.29) one gets

$$\Gamma_{cusp} \equiv \frac{S_{NG}}{T} = -\frac{\sqrt{\lambda}}{\pi}\frac{r}{p} + \mathcal{O}(p) \tag{6.8}$$

and therefore

$$\Gamma_{cusp} = -\frac{\sqrt{\hat{\lambda}}}{2\,\pi\,\cos\frac{\phi}{2}} + \mathcal{O}((1/\sqrt{\hat{\lambda}})^0)\ . \tag{6.9}$$

This leading order expression in the ladder limit verifies equation (6.4) and therefore (6.5).

Trying to evaluate the correction due to stringy fluctuations described by (6.5) is a highly non-trivial task, since it corresponds to the point $(\phi=\pi, \theta=i\infty)$ in parameter space. In fact, an *exact* solution for the one-loop partition function of $\Gamma_{cusp}(\phi,\theta,\lambda)$, as defined in the analysis of Section 4 is only known [2] in the $(\phi,\theta=0)$ and $(\phi=0,\theta)$ cases, corresponding to points of the parameter space in which the fluctuations decouple and the mass matrix is diagonal. The evaluation of the one-loop partition function in the case of elliptic classical solutions which are non trivial both in the $AdS_5$ and in the $S^5$ part of the string background is an as yet unsolved problem deserving future attention. Therefore, at one loop we cannot expand an exact result as was done above for the classical string theory, but we are forced to proceed performing first an expansion at the level of the geometry and then see whether from this expansion a solvable set of fluctuation operators would come out whose total determinant can then be integrated in the partition function. In other words, we have to exchange the order of the limit and integration with respect to the analysis done, for example, in Section 5 – where we consider a limit on the exact determinants and then integrated the partition function – and hope for the best. To get a test of the difficulties that one encounters following this approach and starting from the setting of Section 4, it is instructive to perform the same exchange of order of operations at the classical level. This amounts to expand the square root of the conformally flat induced metric (4.21) *before* integrating it to give the Nambu-Goto action

$$\begin{aligned} S_{NG} &= \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\,\frac{1-k^2}{\mathrm{cn}^2\sigma} = \frac{T\sqrt{\lambda}}{2\pi}\frac{\mathcal{T}}{T}\,(1-k^2)\int_{-\mathbb{K}}^{+\mathbb{K}} d\sigma\,\frac{1}{\mathrm{cn}^2\sigma} \\ &= \frac{T\sqrt{\lambda}}{2\pi}\,\frac{r}{p}\,\frac{p^2(r^2+1)}{r^4}\int_{-\mathbb{K}}^{\mathbb{K}} d\sigma\Big[\cosh^2\sigma + \frac{p^2(r^2+1)\sinh(2\sigma)(\sinh(2\sigma)-2\sigma)}{8r^4} + O(p^4)\Big] \\ &\simeq \frac{T\sqrt{\lambda}}{2\pi}\,\frac{p(r^2+1)}{r^3}\Big[(\mathbb{K}+\sinh\mathbb{K}\cosh\mathbb{K}) + \\ &\qquad + \mathbf{p^2}\,\frac{(r^2+1)(-4\mathbb{K}+4\sinh(2\mathbb{K})+\sinh(4\mathbb{K})-8\mathbb{K}\cosh(2\mathbb{K}))}{32r^4}\Big] \end{aligned}$$

where in the expansion between brackets one has still to substitute the leading behavior of the complete elliptic function in the ladder limit, which is $\mathbb{K} \simeq \ln\left(\frac{4r^2}{p\sqrt{r^2+1}}\right)$. Doing this, one gets, formally at order $\mathcal{O}(p^4)$,

$$S_{NG} \simeq \frac{T\sqrt{\lambda}}{2\pi}\,\frac{p(r^2+1)}{r^3}\left[\frac{4r^4}{(1+r^2)\,p^2} + \mathbf{p^2}\left(\frac{4r^4}{(1+r^2)\;p^4} + \frac{\ln\frac{p^2\left(r^2+1\right)}{16r^4}+1}{p^2}\right)\right]. \quad (6.10)$$

It is easy to explicitly check, however, that at each order $p^{2n}$ in the expansion of the integrand a new contribution always of type $\frac{4r^4}{(1+r^2)\,p^2}$ will be generated, in that a $\sinh((2n+2)\mathbb{K})$ (with a suitable coefficient) will emerge from the integration over $\sigma$ which has the following behavior in the ladder limit

$$p^{2n}\,\sinh(2(n+1)\mathbb{K}) \simeq p^{2n}\cdot\frac{1}{p^{2n+2}} \simeq \frac{1}{p^2}\;. \quad (6.11)$$

This way one gets apparently a breakdown of the ladder expansion. However, having noticed [29] that at each order the new contribution is always of the same type, one can still try to formally regularize the leading order in the expansion above as

$$S_{NG} \simeq \frac{T\sqrt{\lambda}}{2\pi}\,\frac{p(r^2+1)}{r^3}\sum_{\mathbf{n}}\frac{4r^4}{(1+r^2)\,p^2} \equiv \frac{T\sqrt{\lambda}}{\pi}\frac{2\,r}{p}\,\boldsymbol{\zeta_0} \equiv -\frac{T\sqrt{\lambda}}{\pi}\frac{r}{p} \quad (6.12)$$

where we have used $\zeta_0 = -\frac{1}{2}$. This is indeed the classical behavior at leading order in ladder limit in (6.8).

The description above has shown that, already at the classical level, it is only with a nontrivial resummation, which includes the effect of the ladder limit at

[29] The elliptic function $\mathrm{cn}(\sigma, k^2)$ satisfies a differential equation in the modulus (which can be found i.e. at the Wolfram functions website), such that the contributions can be shown to be the same to arbitrary order in the expansion.

the boundary of the $\sigma$-region, that we could safely exchange the limit on the geometry with the integration over it, reproducing the behavior for $\Gamma_{cusp}(\phi, \hat{\lambda})$ obtained expanding the exact (classical) result.

Clearly the hope that a similar resummation can be performed at the one loop level is little. We have in fact tried this option, and encountered several difficulties, starting for example from a consistent definition of perturbative solution for the Gelfand-Yaglom problem.

In fact the kind of difficulties described above is a strong indication that the underlying system is ill-defined on a more fundamental level. This can be understood explicitly considering the classical induced metric (4.11) written as:

$$\mathrm{d}s^2 = \sqrt{\lambda}\cosh^2\rho\Big( -\mathrm{d}t^2 + p^2 \sinh^4\rho (\partial_\sigma \varphi)^2 \mathrm{d}\sigma^2\Big) \tag{6.13}$$

$$= \sqrt{\lambda}\cosh^2\rho\Big( -\mathrm{d}t^2 + \frac{p^2}{q^2} (\partial_\sigma \vartheta)^2 \mathrm{d}\sigma^2\Big). \tag{6.14}$$

From (6.14) it already becomes apparent that the two dimensional induced metric would suffer a curvature singularity if one was to, say send $p \to 0$ while simultaneously having a complex $q$ and not sending it to zero at the same rate. This behavior becomes even more clear if we consider the Ricci scalar for (6.14) [30]:

$$R^{(2)} = -2\frac{q^2}{p^2}\frac{1}{\sqrt{\lambda}\cosh^4\rho\,\vartheta'^3}\Big(\vartheta'\,\rho'^2 + \sinh\rho\,\cosh\rho\,(\vartheta'\,\rho'' - \rho'\,\vartheta'')\Big). \tag{6.15}$$

Here it is important to realize, that in any valid parametrization the quantities $\cosh\rho,\ \vartheta',\ \rho',\ \vartheta'',\ \rho''$ have to be differentiable functions not identically zero (or infinity) on the interval $-\sigma_0/2 < \sigma < \sigma_0/2$ for any values of $p$ and $q$. Therefore, considering a configuration with $p = 0$ no valid parametrization exists where any of these quantities are proportional to some power of $p$ (since that would render them exactly zero different from what we expect considering the geometry of the minimal surface) [31]. As mentioned above, $p \to 0$ with $q$ not proportional to $p$ then directly yields a divergence for the Ricci scalar (6.15). Having a divergence in a curvature scalar directly indicates actual curvature singularities which cannot be undone by any clever choice of coordinate parametrization.

To reach the configuration ($\phi = \pi\ ,\ \theta = i\infty$), we could for instance try to fix ($\phi = \pi\ ,\ \theta$) or ($\phi\ ,\ \theta = i\infty$) first, compute the functional determinants *if* the system is regular and simplified, and take a second limit (respectively $\theta \to i\infty$ or $\phi \to \pi$ ) after that. Unfortunately, both these possibilities for intermediate

[30] Being a curvature scalar, the behavior of the Ricci scalar is the same in any parametrization.

[31] For instance, the ($\tau$,$\sigma$) parametrization as used throughout the thesis is exactly an example of parametrization which does not respect this at $p = 0$.

step configurations are of the singular type described above. For ($\phi = \pi$ , $\theta$) we have to take $p \to 0$ while keeping $q = \sqrt{p}c$ with i.e. $c = \sqrt{\frac{1-2k^2}{\sqrt{k^2(1-k^2)}}}$ to keep $k^2$ as a parameter. Since $q$ is then not going to zero at the same rate as $p$ we end up with a curvature singularity on parameter space as described above. For ($\phi$ , $\theta = i\infty$) we should take $p \to 0$ while having $q = ir$ with a real parameter $r$. This configuration is singular in the same way.

One could argue that a choice of gauge-fixing different from the one adopted in [2], in which already the ratio (4.20) between the worldsheet time and the target space time $\tau/t$ diverges (this ratio appears just in front of the effective action), would help in solving the problem. We have not addressed such question in this thesis, where our main problem is verifying the prediction (6.5) via the evaluation of functional determinants, because such alternative gauge choice would necessarily lead to an induced metric *non-conformally flat*, thus leading to a very complicated term for the kinetic term of the differential operators governing fluctuations.

In order to carry out safely an explicit check of the prediction following from the ladder diagrams resummation beyond the leading order in sigma- model perturbation theory, and in a spirit similar to the one adopted already in [2, 17] we have turned our attention to the simplifying $\phi = 0$ case, for which an analogue prediction can be extracted from the Schrödinger problem (6.3).

## 6.2 The ($\phi = 0$ , $\theta = i\infty$) configuration

As already mentioned, for small movement around the minimum of the potential in (6.3), we should consider $y \approx 0$. To do that systematically, we should make use of the large coupling constant $\hat{\lambda}$ provided by the equation and consider zero point fluctuations in $y \to y/\sqrt{\hat{\lambda}}$ to next to leading order in $\hat{\lambda} \to \infty$. Performing the necessary expansion in (6.3) the equation is approximated by:

$$\left(-\partial_y^2 + \frac{y^2}{16\hat{\lambda}\pi^2(\cos\phi + 1)^2}\right)\psi_0 = \left(\frac{1}{8\pi^2(\cos\phi + 1)} - \frac{1}{\hat{\lambda}}\frac{\alpha^2}{4}\right)\psi_0. \tag{6.16}$$

Equation (6.16) is very much reminiscent of the quantum mechanical harmonic oscillator energy equation $(-\partial_x^2 + \omega^2 x^2)\psi = 2E_n\psi$ where $E_n = \omega(n + 1/2)$. In our case we are interested in the lowest energy level $n = 0$. Equating the appropriate $2E_0$ with the right hand side of (6.16), we can solve for $\alpha$ and obtain:

$$\alpha = \sqrt{\frac{\hat{\lambda} - 2\pi\sqrt{\hat{\lambda}}}{2\pi^2(1 + \cos\phi)}} \approx \frac{1}{2\cos\frac{\phi}{2}}\left(\frac{\sqrt{\hat{\lambda}}}{\pi} - 1\right) + O(1/\sqrt{\hat{\lambda}}). \tag{6.17}$$

which is indeed a generalization of the prediction (6.5) (where $\phi \to \pi$) to generic values of the cusp angle $\phi$.

For the case of interest in this section, which is $\phi = 0$, the prediction for the $\Gamma_{cusp}(\phi,\theta,\lambda)$ in the ladder limit is thus

$$\Gamma_{cusp}(\phi = 0, \hat{\lambda}) = \frac{1}{2}\Big( - \frac{\sqrt{\hat{\lambda}}}{\pi} + 1\Big), \qquad \hat{\lambda} \gg 1 \tag{6.18}$$

From the point of view of the limit in the parameter space, we should take $p \to \infty$ while keeping $k$ finite which corresponds to the constraint $q = ipk/\sqrt{1-k^2}$. Clearly, if $p$ and $q$ diverge at the same rate, there will be no curvature singularity in (6.14), which is of course totally expected since there is nothing singular about a straight line limit $\phi = 0$.

Since the parameter $k^2$ does not become fixed if we take $q = ipk/\sqrt{1-k^2}$ and $p \to \infty$, it is easy to show (expanding only in terms involving $b$ and $p$ and keeping an unfixed $k^2$) that the opening angle (4.25) then indeed gives:

$$\Phi = \pi + O(1/p) \tag{6.19}$$

as we want it to. For the opening angle $\theta$ in (4.23) we find:

$$\theta = 2ik\mathbb{K}(k^2) + O(1/p^2). \tag{6.20}$$

Therefore, in a next step we just have to send $k \to 1$ to obtain the ladder limit $\theta \to i\infty$:

$$\theta_{k\to1} = i \ln \frac{2^3}{1-k} + O(1-k). \tag{6.21}$$

For the classical action (4.29) the straight line limit $\phi \to 0$ gives:

$$S_{NG} = -\frac{\sqrt{\lambda}T}{\sqrt{1-k^2}\pi}\Big(\mathbb{E}(k^2) - (1-k^2)\mathbb{K}(k^2)\Big) + O(1/p). \tag{6.22}$$

Subsequently, performing the ladder limit $k \to 1$ on the classical action results in:

$$S_{NG\,k\to1} = -\frac{\sqrt{\lambda}T}{\sqrt{2}\pi\sqrt{1-k^2}} + O(\sqrt{1-k}). \tag{6.23}$$

At this point we should switch our notation from $\lambda$ to $\hat{\lambda}$ in order to be able to compare with (6.18). Considering (6.1) and (6.21), we write the classical action (6.23) in the limit ($\phi = 0$ , $\theta = i\infty$) as:

$$-S_{NG\,k\to 1} = \frac{\sqrt{\hat{\lambda}}T}{2\pi}. \tag{6.24}$$

Since on the string theory side $\langle W \rangle \approx \exp(-S)$, we have multiplied the above with $(-1)$ to be able to compare our result with (6.18) directly. Evidently, removing the common factor of $T/2$, we recover exactly the classical contribution to $\alpha_0 = \sqrt{\hat{\lambda}}/\pi$ as given in (6.5). As mentioned earlier, the choice ($\phi = 0$ , $\theta = i\infty$) will be most useful when computing the first order fluctuation contribution.

Just as in section five, we can write an expectation for the structure of the one-loop result to be $S_{eff} = \mathcal{V}_{ll}\alpha_{fl}$, where $\alpha_{fl}$ will be the one-loop contribution to $\alpha_0$ and the (rescaled) linear time divergence $\mathcal{V}_{ll}$ is given by:

$$\mathcal{V}_{ll} = -\frac{T}{2}. \tag{6.25}$$

In the next subsection we will obtain the functional determinants of our bosonic and fermionic operators for the configuration ($\phi = 0$ , $\theta$).

## 6.3 One-Loop Determinants

Fortunately, in the $\phi = 0$ limit $p \to \infty$ while having $q = ipk/\sqrt{1-k^2}$ the bosonic and fermionic differential operators simplify to a great extent. For instance, all off-diagonal mass matrix entries and first derivative terms vanish. Just as in section five in the case of $\theta = 0$, the most convenient shape for the differential operators is reached rescaling the bosonic operator by $\sqrt{g}$ and rescaling the fermionic operator by $g^{1/4}$ and squaring it. The argumentation for why this is allowed and does not change the result stays the same.

With this, the bosonic operators in the $\phi = 0$ limit read:

$$\mathcal{O}_8 = \mathcal{O}_i = \sqrt{g}\mathbb{O}_{ii} = -\partial_\sigma^2 + \omega_\tau^2 + \frac{2(1-k^2)}{\mathrm{cn}^2(\sigma, k^2)} + k^2 \tag{6.26}$$

$$\mathcal{O}_s = \sqrt{g}\mathbb{O}_{ss} = -\partial_\sigma^2 + \omega_\tau^2 + k^2 \tag{6.27}$$

$$\mathcal{O}_7 = \sqrt{g}\mathbb{O}_{77} = -\partial_\sigma^2 + \omega_\tau^2 + 2k^2\mathrm{sn}^2(\sigma, k^2) - k^2. \tag{6.28}$$

Where we again introduced abbreviations to simplify notation, but this time chose a curly $\mathcal{O}$ to clearly distinguish the operators from the ones in section five. Just as in the case $\theta = 0$, the operator (6.27) is simple enough so that it's corresponding differential equation can be solved straightforwardly. In fact, it is clear from comparing (6.27) and (5.28) that the determinant will be of the same structure but with a substitution $\omega_\tau \to \omega_\tau + k^2$. The same is true for the operator (6.26) compared with the case (5.27). Finally, the operator (6.28) actually is already of the single gap Lamé operator type, so that no further adjusting is necessary.[32]

For the fermionic operator (4.81) in the $\phi = 0$ limit we find:

$$g^{1/4}\mathbb{O}'_F = -\, i\Gamma_9\partial_\sigma - \omega\Gamma_0 + i\frac{\mathrm{dn}(\sigma,k^2)}{\mathrm{cn}(\sigma,k^2)}\Gamma_1\Gamma_2\Gamma_7. \tag{6.29}$$

(Again the angle $\beta$ is zero in this limit.) Just as in section five, we consider the square of this operator to simplify the gamma matrix structure:

$$(g^{1/4}\mathbb{O}'_F)^2 = \mathbb{1}\Big(-\partial_\sigma^2 + \omega^2 + \frac{\mathrm{dn}^2(\sigma,k^2)}{\mathrm{cn}^2(\sigma,k^2)}\Big) + \frac{(1-k^2)\mathrm{sn}(\sigma,k^2)}{\mathrm{cn}(\sigma,k^2)}\Gamma_9\Gamma_1\Gamma_2\Gamma_7. \tag{6.30}$$

Now, choosing the same representation for the gamma matrices as in (5.37) we obtain a diagonalized form of the squared fermionic operator, where only the following components appear:

$$(g^{1/4}\mathbb{O}'_F)^2_{ll} = -\partial_\sigma^2 + \omega^2 + \frac{1 \pm k^2\mathrm{sn}(\sigma,k^2)}{1 \pm \mathrm{sn}(\sigma,k^2)}. \tag{6.31}$$

The $\pm$ is due to different entries of the $\sigma_3$ matrix appearing in the result of (5.38). Thanks to the antisymmetric property $\mathrm{sn}(-\sigma,k^2) = -\mathrm{sn}(\sigma,k^2)$, the potentials for both sign choices in (6.31) are just mirrored versions of each other so that the determinant is guaranteed to be the same. Therefore, we can choose one sign case, say plus, compute the determinant and take it to the appropriate power to obtain the complete functional determinant of the fermionic operator. Applying the same technique as in section five, a component of the squared fermionic operator can be written in the shape of single gap Lamé operator type. Introducing yet another abbreviation to simplify notation this reads:

$$\mathcal{O}_F = (g^{1/4}\mathbb{O}'_F)^2_{11} = \left(\frac{1+k}{2}\right)^2\left(-\partial_{\sigma_4}^2 + \omega_4^2 + 2k_4^2\mathrm{sn}^2(\sigma_4 - \frac{3}{4}\mathbb{K}_4 + i\mathbb{K}'_4, k_4^2) - k_4^2\right), \tag{6.32}$$

where we have used a curly $\mathcal{O}_F$ to distinguish this operator from the one in section five. Here the parameters are given by:

$$\sigma_4 = (1+k)\frac{\sigma}{2} \quad , \quad \omega_4^2 = \frac{4\omega_\tau^2}{(1+k)^2} \quad , \quad k_4^2 = \frac{4k}{(1+k)^2}, \tag{6.33}$$

[32] Actually, all the techniques required to obtain the functional determinants in this case are completely in parallel with section five, so that we could almost skip to the end immediately.

while also $\mathbb{K}_4 = \mathbb{K}(k_4^2)$ and $\mathbb{K}'_4 = \mathbb{K}(1-k_4^2)$.

Having obtained the bosonic and fermionic differential operators in the shape of single gap Lamé operators, we can now follow through the computation of functional determinants analogous to the computation in section five.[33] The results are:

$$\det \mathcal{O}_s^\epsilon = \frac{\sinh(2\mathbb{K}\sqrt{k^2+\omega_\tau^2})}{\sqrt{k^2+\omega_\tau^2}}, \tag{6.34}$$

$$\det \mathcal{O}_i^\epsilon = \det \mathcal{O}_8^\epsilon = \frac{\sinh(2\mathbb{K}_1 Z(\tilde{\alpha}_1))}{\epsilon^2\sqrt{\omega_\tau^2(\omega_\tau^2+1)(1-k^2+\omega_\tau^2)}}, \tag{6.35}$$

$$\det \mathcal{O}_7^\epsilon = -\frac{\sqrt{1-k^2+\omega_\tau^2}}{\sqrt{\omega_\tau^2(1+\omega_\tau^2)}}\sinh(2\mathbb{K}Z(\tilde{\alpha}_2)), \tag{6.36}$$

$$\det \mathcal{O}_F^\epsilon = -\frac{4i\sinh(\mathbb{K}_4 Z(\tilde{\alpha}_F))}{\epsilon\sqrt{k^4+2k^2(4\omega_\tau^2-1)+(4\omega_\tau^2+1)^2}}. \tag{6.37}$$

The tildes over the different $\tilde{\alpha}$ should help distinguish them from the ones in section five. The appearing $\tilde{\alpha}$ are explicated in the next subsection, where we take the ladder limit on these determinants.
Just as in section five, unwanted logarithmic divergences for large $\omega_\tau$ and small $\epsilon$ entered the determinants. Subtracting the divergences explicitly, the expression for the one-loop effective action reads:

$$S_{eff} = -\mathcal{T}\frac{1}{2}\lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega_\tau}{2\pi}\ln\frac{\epsilon^2\omega_\tau^2\det^8\mathcal{O}_F^\epsilon}{\det^2\mathcal{O}_i^\epsilon\det^4\mathcal{O}_s^\epsilon\det\mathcal{O}_7^\epsilon\det\mathcal{O}_8^\epsilon}. \tag{6.38}$$

In the next subsection we will perform the ladder limit on this expression to find the one-loop coefficient of $\alpha_0$.

## 6.4 Verifying the ladder diagrams resummation at one-loop in sigma-model perturbation theory

We start with the determinants as obtained in the previous chapter, but rescale $\omega_\tau \to \sqrt{1-k^2}\omega_t = \sqrt{1-k^2}\omega$ to simplify the procedure of taking the limit. Since with this the variable $\tau$ simply switches to target space variable $t$, this causes no problems. The argumentation is entirely the same as with the light-like cusp

[33] Again the results are reproductions of the work done in [2] and the fermionic operator has been reshaped into a more convenient form.

limit of section five:

$$\det \mathcal{O}_s^\epsilon = \frac{\sinh(2\mathbb{K}\sqrt{k^2+(1-k^2)\omega^2})}{\sqrt{k^2+(1-k^2)\omega^2}}, \tag{6.39}$$

$$\det \mathcal{O}_i^\epsilon = \det \mathcal{O}_8^\epsilon = \frac{\sinh(2\mathbb{K}_1 Z(\tilde{\alpha}_1))}{\epsilon^2(1-k^2)\sqrt{\omega^2((1-k^2)\omega^2+1)(1+\omega^2)}}, \tag{6.40}$$

$$\det \mathcal{O}_7^\epsilon = -\frac{\sqrt{1+\omega^2}}{\sqrt{\omega^2(1+(1-k^2)\omega^2)}}\sinh(2\mathbb{K}Z(\tilde{\alpha}_2)), \tag{6.41}$$

$$\det \mathcal{O}_F^\epsilon = -\frac{4i\sinh(\mathbb{K}_4 Z(\tilde{\alpha}_F))}{\epsilon\sqrt{k^4+2k^2(4(1-k^2)\omega^2-1)+(4(1-k^2)\omega^2+1)^2}}. \tag{6.42}$$

Where we should keep in mind several abbreviations:

$$k_1^2 = \frac{k^2}{k^2-1}, \qquad \mathbb{K}_1 = \mathbb{K}(k_1^2), \qquad k_4^2 = \frac{4k}{(1+k)^2}, \qquad \mathbb{K}_4 = \mathbb{K}(k_4^2), \tag{6.43}$$

and

$$\mathrm{sn}(\tilde{\alpha}_1, k_1^2) = \sqrt{\frac{1+k_1^2+\tilde{\omega}_1^2}{k_1^2}}, \quad \mathrm{sn}(\tilde{\alpha}_2, k^2) = \sqrt{\frac{1+(1-k^2)\omega^2}{k^2}}, \tag{6.44}$$

$$\mathrm{sn}(\tilde{\alpha}_F, k_4^2) = \sqrt{\frac{(k+1)^2+4(1-k^2)\omega^2}{4k}}, \tag{6.45}$$

with

$$\tilde{\omega}_1^2 = \frac{(1-k^2)\omega^2+k^2}{1-k^2}. \tag{6.46}$$

The one-loop effective action is then composed out of these determinants in the usual way as follows:

$$S_{eff} = -T\frac{1}{2}\lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega}{2\pi}\ln\frac{\epsilon^2(1-k^2)\omega^2\det^8\mathcal{O}_F^\epsilon}{\det^2\mathcal{O}_i^\epsilon\det^4\mathcal{O}_s^\epsilon\det\mathcal{O}_7^\epsilon\det\mathcal{O}_8^\epsilon}. \tag{6.47}$$

Note, that having rescaled $\omega$, we have $T$ as the time period in front of the integral instead of $\mathcal{T}$. Now we perform the Ladder limit $k \to 1$ on the integrand in (6.47). Again, it makes sense to tackle this task in bits. Using the property of the logarithm function we can separate:

$$\ln\frac{\epsilon^2(1-k^2)\omega^2\det^8\mathcal{O}_F^\epsilon}{\det^2\mathcal{O}_i^\epsilon\det^4\mathcal{O}_s^\epsilon\det\mathcal{O}_7^\epsilon\det\mathcal{O}_8^\epsilon} = \ln\frac{2^{16}\,\omega^6(1+\omega^2)(k^2+(1-k^4)\omega^2+(1-k^2)^2\omega^4)^2}{((1+4\omega^2)^2-k^2(1-4\omega^2)^2)^4}$$
$$+\ln\frac{-\sinh^8\big(\mathbb{K}_4 Z(\tilde{\alpha}_F)\big)}{\sinh^4\big(2\mathbb{K}\sqrt{k^2+(1-k^2)\omega^2}\big)\sinh^3\big(2\mathbb{K}_1 Z(\tilde{\alpha}_1)\big)\sinh\big(2\mathbb{K}Z(\tilde{\alpha}_2)\big)}. \tag{6.48}$$

On the first logarithm in (6.48), which contains only a simple rational function in $k$ and $\omega$, the limit is straightforwardly taken and gives:

$$\lim_{k\to 1} \ln \frac{2^{16}\,\omega^6(1+\omega^2)(k^2+(1-k^4)\omega^2+(1-k^2)^2\omega^4)^2}{((1+4\omega^2)^2-k^2(1-4\omega^2)^2)^4} = \ln \frac{1+\omega^2}{\omega^2}. \tag{6.49}$$

To obtain the limit of the terms contained in the second logarithm of (6.48) we will have to work much harder. In particular, the Jacobi Zeta functions again pose the greatest difficulty since now the expansions involve both of their arguments. Luckily, we will only have to obtain their leading contributions for our purposes which will prove to be accessible at the cost of some effort. Starting with the simpler cases we make our way towards the harder ones.

The first expression we want to consider is $\sinh^4(2\mathbb{K}\sqrt{k^2+(1-k^2)\omega^2})$. To determine its limit under $k \to 1$ we require the behavior of $\mathbb{K}(k^2)$, which is readily found in the literature or making use of the program *Mathematica*:

$$\mathbb{K}(k^2)_{k\to 1} = \ln\left(\frac{2^{3/2}}{\sqrt{1-k}}\right) + O(1-k). \tag{6.50}$$

Therefore, we learn that the leading contribution of $\mathbb{K}$ logarithmically goes to infinity. With this we obtain the leading contribution for the first term of interest:

$$\sinh^4(2\mathbb{K}\sqrt{k^2+(1-k^2)\omega^2})_{k\to 1} \approx \left(\frac{1}{2}\exp\left[2\ln\left(\frac{2^{3/2}}{\sqrt{1-k}}\right)\right]\right)^4 \tag{6.51}$$

$$=\frac{2^8}{(1-k)^4}. \tag{6.52}$$

That was indeed rather simple. However, one should keep in mind that the simplicity is largely due to the great pool of documented properties and expansions for elliptic integrals in the literature. Without this aid we probably would have to look at explicit integral representations every time to deduce a leading contribution.

The second term of interest is $\sinh^8(\mathbb{K}_4 Z(\tilde{\alpha}_F))$. It is second in simplicity, because the modulus $k_4^2 \approx 1-(1-k)^2/4+O((1-k)^3)$ approaches 1 faster than the moduli involved in the other Jacobi Zeta functions. Using the primary definition of the Jacobi Zeta function, we can write:

$$\mathbb{K}(m)Z(\mathrm{am}(\alpha,m),m) = \mathbb{K}(m)\mathbb{E}(\mathrm{am}(\alpha,m),m) - \mathbb{E}(m)F(\mathrm{am}(\alpha,m),m). \tag{6.53}$$

Therefore, to get the leading contribution of the term of interest we will mainly have to work on expanding incomplete elliptic integrals of the first and second kind. (It is important to realize, that an abbreviation $Z(\alpha)$ actually stands for $Z(\mathrm{am}(\alpha, m), m)$ with a corresponding modulus $m$.) As already mentioned, since the modulus $k_4^2$ in this case approaches 1 very fast (first correction being quadratic and not linear in $1-k$), we can treat the first argument as being still a variable while the modulus tends to 1. Then the leading contributions to all four functions appearing in (6.53) can be directly read off from the documented expansions of these functions around the modulus $k_4^2 \approx 1$. The contributions are as follows:

$$\begin{aligned}
\mathbb{K}(k_4^2)_{k\to 1} &= \ln \frac{2^3}{1-k} + O((1-k)^2) \quad , \quad F(\mathrm{am}(\tilde{\alpha}_F, k_4^2), k_4^2)_{k\to 1} = -\frac{i\pi}{2} + \ln \frac{-i\sqrt{2}}{\sqrt{\omega^2(1-k)}} + O((1-k)^2) \\
\mathbb{E}(k_4^2)_{k\to 1} &= 1 + O((1-k)^2) \quad , \quad \mathbb{E}(\mathrm{am}(\tilde{\alpha}_F, k_4^2), k_4^2)_{k\to 1} = 1 + O((1-k)^2).
\end{aligned} \tag{6.54}$$

Again we encounter logarithmic divergence, so that sinh can be approximated as in (6.51). Combining these findings, we obtain the following leading order expression for the second term of interest:

$$\sinh^8(\mathbb{K}_4 Z(\tilde{\alpha}_F))_{k\to 1} \approx \frac{2^{12}\omega^8}{(1-k)^4}. \tag{6.55}$$

It is interesting to find that the divergences in (6.52) and (6.55) exactly cancel. Since we expect the one-loop result to be finite in the Ladder limit, it was somewhat clear from the beginning that this must happen. However, we still have two more bosonic contributions to evaluate.

The next contribution we want to consider is $\sinh^3(2\mathbb{K}_1 Z(\tilde{\alpha}_1))$. It is special in the way that here the first argument of the Jacobi Zeta function (integration range) is going to zero while the modulus $k_1^2$ diverges at the same rate when we take $k \to 1$. Unfortunately, no expansions are known for this function for this particular simultaneous behavior of the two arguments. Or at least no known expansion could be found after a sensible amount of search. Therefore, in this case we will have to look at integral representations of incomplete elliptic integrals to extract the leading contribution. For the complete versions of the elliptic integrals we get the correct behavior from tables without a problem:

$$\mathbb{K}(k_1^2)_{k\to 1} = \frac{1}{\sqrt{2}} \ln\left(\frac{2^3}{1-k}\right)\sqrt{1-k} + O((1-k)^{3/2}) \tag{6.56}$$

$$\mathbb{E}(k_1^2)_{k\to 1} = \frac{1}{\sqrt{2(1-k)}} + O(\sqrt{1-k}). \tag{6.57}$$

One integral representation of the incomplete elliptic integral of the second kind is given by:

$$\mathbb{E}(\mathrm{am}(\tilde{\alpha}_1,k_1^2),k_1^2)=\int_0^{\sin(\mathrm{am}(\tilde{\alpha}_1,k_1^2))}\frac{\sqrt{1-k_1^2t^2}}{\sqrt{1-t^2}}\mathrm{d}t. \tag{6.58}$$

While the integration range goes to zero with $k\to 1$, the modulus $k_1^2$ goes to infinity with the same rate, so that these tendencies balance each other out for the $k_1^2t^2$ term in the numerator. In the denominator however, the $t^2$ becomes negligible in the limit and gets dominated out by the 1. Therefore, the leading contribution for $k\to 1$ is given by the simplified integral:

$$\begin{aligned}\mathbb{E}(\mathrm{am}(\tilde{\alpha}_1,k_1^2),k_1^2)_{k\to 1}\approx&\int_0^{\sin(\mathrm{am}(\tilde{\alpha}_1,k_1^2))}\sqrt{1-k_1^2t^2}\;\mathrm{d}t &(6.59)\\ \approx&\frac{1}{\sqrt{2}}\Big(\sqrt{(1+\omega^2)\omega^2}+i\arcsin(\sqrt{1+\omega^2}\,)\Big)\sqrt{1-k}. &(6.60)\end{aligned}$$

Analogously, one integral representation of the incomplete elliptic integral of the first kind is given by:

$$F(\mathrm{am}(\tilde{\alpha}_1,k_1^2),k_1^2)=\int_0^{\sin(\mathrm{am}(\tilde{\alpha}_1,k_1^2))}\frac{1}{\sqrt{(1-t^2)(1-k_1^2t^2)}}\mathrm{d}t. \tag{6.61}$$

With the same argumentation as above, the leading contribution for $k\to 1$ is then obtained by neglecting the lonely $t^2$:

$$\begin{aligned}F(\mathrm{am}(\tilde{\alpha}_1,k_1^2),k_1^2)_{k\to 1}\approx&\int_0^{\sin(\mathrm{am}(\tilde{\alpha}_1,k_1^2))}\frac{1}{\sqrt{1-k_1^2t^2}}\mathrm{d}t &(6.62)\\ \approx&\, i\sqrt{2(1-k)}\arcsin(\sqrt{1+\omega^2}). &(6.63)\end{aligned}$$

Again, collecting the results above to shape (6.53), we obtain the leading contribution for the third term of interest:

$$\sinh^3(2\mathbb{K}_1 Z(\tilde{\alpha}_1))_{k\to 1}\approx 2^3\Big(\omega^2(1+\omega^2)\Big)^{3/2}. \tag{6.64}$$

In this case, since no divergence appears, this is actually the exact limiting value for the term in the limit $k\to 1$. (To make sure that we do not miss any important contribution when performing our approximations, here and for all the other terms of interest the leading order results have been checked to be correct numerically.)

Finally, the behavior under $k\to 1$ of one more term of interest $\sinh(2\mathbb{K}Z(\tilde{\alpha}_2))$ remains to be computed. We consider this contribution to be the hardest to determine, since it demands the most work on our part. As with the second term of interest, the modulus $k^2$ is going to 1 and simultaneously the first argument is approaching $\pi/2$. However, the difference as compared to the fermionic case is that the two arguments approach their respective limiting value at the same rate. Therefore, we cannot treat any of the two as generic while expanding in the other one.

Looking back at (6.53), we again have no troubles with the complete versions of the elliptic integrals thanks to expansions available in the literature:

$$\mathbb{K}(k^2)_{k\to1} = \ln\left(\frac{2^{3/2}}{\sqrt{1-k}}\right) + O(1-k) \qquad , \qquad \mathbb{E}(k^2)_{k\to1} = 1 + O(1-k). \tag{6.65}$$

Actually, the incomplete elliptic integral of the second kind also poses no difficulty, since it is well behaved and regular at the limiting values of the arguments, so that we can simply evaluate it to obtain the leading contribution:

$$\mathbb{E}(\mathrm{am}(\tilde{\alpha}_2,k^2),k^2)_{k\to1} \approx \mathbb{E}\left(\frac{\pi}{2},1\right) = 1. \tag{6.66}$$

The nasty part, in comparison, is the incomplete elliptic integral of the first kind. It turns out, that to obtain a correct leading contribution we should start from the following exact sum representation that can be found i.e. at the Wolfram functions website:

$$\begin{aligned} F(z,m) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n^2}{(n!)^2}(m-1)^n \cdot \\ \cdot\left(\ln(\sec(z)+\tan(z)) + \frac{1}{2}\csc(z)\sum_{j=1}^{n}\frac{(-1)^j(j-1)!\tan^{2j}(z)}{\left(\frac{1}{2}\right)_j}\right), \end{aligned} \tag{6.67}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the so called Pochhammer function. In the limit of our interest $k\to1$, the arguments $z$ and $m$ behave as:

$$z = \mathrm{am}(\tilde{\alpha}_2,k^2) \approx \arcsin\Big(1+(1+\omega^2)(1-k)+O((1-k)^2)\Big)\,, \tag{6.68}$$

$$m = k^2 \approx 1-2(1-k)+O((1-k)^2). \tag{6.69}$$

With this we straightforwardly obtain the leading behavior under $k\to1$ of the relevant terms appearing in (6.67):

$$\log(\sec(z)+\tan(z)) \approx \ln\left(-i\frac{\sqrt{2}}{\sqrt{(1+\omega^2)(1-k)}}\right) + O(1-k)\,, \tag{6.70}$$

$$(m-1) \approx 2(k-1) + O((1-k)^2)\,, \tag{6.71}$$

$$\csc(z) \approx 1 + O(1-k)\,, \tag{6.72}$$

$$\tan^2(z) \approx -\frac{1}{2(1+\omega^2)(1-k)} - \frac{3}{4} + O(1-k)\,. \tag{6.73}$$

Looking at these expressions, we realize that the term (6.70) appearing in (6.67) will survive the limit $k \to 1$ only in the case $n = 0$, since for all other values of $n$ it will get suppressed by a polynomial power of (6.71) in front. For $n > 0$ the second sum in (6.67) kicks in. Here we see, that only terms with $j = n$ survive the limit, since only then $(m-1)^n \tan^{2j}(z)$ balances out the powers of $(1-k)$ and gives a non-vanishing contribution. Restricting $j = n$, the infinite sum over $n > 0$ can be computed exactly and gives (keeping only leading contributions in $k \to 1$):

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!\; 2\,n} \left(\frac{1}{1+\omega^2}\right)^n = \ln\left(\frac{2\sqrt{1+\omega^2}}{\sqrt{1+\omega^2}+\sqrt{\omega^2}}\right). \tag{6.74}$$

Combining the two parts (6.70) and (6.74), we then obtain the leading order term for the incomplete elliptic integral of the first kind:

$$F(\mathrm{am}(\tilde{\alpha}_2, k^2), k^2)_{k\to 1} \approx \ln\left(-i\frac{2\sqrt{2}}{\sqrt{1-k}(\sqrt{1+\omega^2}+\sqrt{\omega^2})}\right). \tag{6.75}$$

Collecting the above results to form (6.53), we observe a somewhat spectacular cancellation of the logarithmic divergence in these terms, such that we actually obtain the finite intermediate result:

$$\mathbb{K}(k^2)Z(\mathrm{am}(\tilde{\alpha}_2, k^2), k^2) = \frac{i\pi}{2} + \ln\left(\sqrt{\omega^2}+\sqrt{1+\omega^2}\right) + O(1-k). \tag{6.76}$$

With this the final leading contribution to the fourth term of interest turns out to be:

$$\sinh(2\mathbb{K}Z(\tilde{\alpha}_2)) \approx -2\sqrt{\omega^2(1+\omega^2)}. \tag{6.77}$$

So the contribution is of the same type as the third term of interest above, but with a different sign. (Which is a good thing to discover, considering that we had a stray minus sign inside the second logarithm back in (6.48)).

Now, putting together the results for all four terms of interest (6.52), (6.55), (6.64) and (6.77), we finally obtain the long sought explicit second logarithm term in (6.48) in the limit $k \to 1$:

$$\lim_{k\to 1} \ln \frac{-\sinh^8\!\left(\mathbb{K}_4 Z(\tilde{\alpha}_F)\right)}{\sinh^4\!\left(2\mathbb{K}\sqrt{k^2+(1-k^2)\omega^2}\right)\sinh^3\!\left(2\mathbb{K}_1 Z(\tilde{\alpha}_1)\right)\sinh\!\left(2\mathbb{K}Z(\tilde{\alpha}_2)\right)} = \ln\frac{\omega^4}{(1+\omega^2)^2}. \tag{6.78}$$

Taking (6.49) and (6.78) into account, the one-loop effective action (6.47) in the ladder limit results to be:

$$S_{eff} = -T\frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} \ln\frac{\omega^2}{1+\omega^2} = T\frac{1}{2}. \tag{6.79}$$

Finally, dividing out the expected (rescaled) linear time divergence factor $\mathcal{V}_{ll}$ given in (6.25), we obtain the one-loop coefficient $\alpha_{fl}$ to the function $\alpha_0$ in (6.5):

$$\alpha_{fl} = \frac{S_{eff}}{\mathcal{V}_{ll}} = -1. \tag{6.80}$$

That is exactly the expected result (6.25). Thus, we have explicitly verified the prediction (6.18) given by the resummation of ladder diagrams to one-loop accuracy in sigma-model perturbation theory.

# 7 Conclusions and outlook

Normally, perturbative gauge theory computations are accessible at a small value of the 't Hooft coupling $\lambda$. The powerful insight of the $AdS/CFT$ correspondence consists in setting a framework in which one particular gauge theory, the maximally supersymmetric $\mathcal{N}=4$ SYM theory, can have a strong coupling description in terms of string theory.

In this thesis we were concerned with that aspect of the $AdS/CFT$ correspondence which uses minimal surface solutions of type IIB string theory on the $AdS_5 \times S^5$ background as a way to describe Wilson loops in the dual gauge theory. In particular, we have considered the minimal surface dual to a euclidean Wilson loop, in $\mathcal{N}=4$ super Yang-Mills gauge theory, with cusp angle $\phi$ and cusp rays exhibiting a relative internal orientation parametrized by $\theta$. Its renormalization is governed by the so-called "generalized cusp" $\Gamma_{cusp}(\phi, \theta, \lambda)$ (or "generalized quark-antiquark potential").

Following [2], we have studied in details the general problem of quantum fluctuation on the relevant minimal surface, and, as original results, we found an analytic answer to two questions motivated by current research, which correspond to the study of two specific points of the parameter space $(\phi, \theta)$. The first question we have addressed – corresponding to the point $(\phi \to i\infty, \theta = 0)$ in parameter space - is the reproduction of the (leading and) subleading strong coupling result for the so-called cusp anomaly of $\mathcal{N}=4$ SYM, which appears also governing the logarithmic asymptotics of the large spin energy of rotating strings or their dual anomalous dimensions of twist operators, and the renomalization of light-like Wilson loops. The result, here obtained from a minimal surface dual to a space-like cusped Wilson loop via an analytic continuation to a light-like cusped Wilson loop configuration [1], is also in accordance with the minimal surface computations manifestly dual to a light-like cusped Wilson loop [5]. The answer obtained constitutes a preliminary step for the goal of establishing a direct relation between the BES (Beisert-Eden-Staudacher) equation for the cusp anomaly $\Gamma_{cusp}(\lambda)$ of $\mathcal{N}=4$ SYM' [38] and certain TBA-like equations [28] written down for $\Gamma_{cusp}(\phi, \theta; \lambda)$.

Another point of the parameter space – the $(\phi, \theta = i\,\infty)$ case – has revealed to be extremely rich in terms of predictivity from the gauge theory side. When the internal angle $\theta$ is taken to complex infinity while the combination $\hat{\lambda} = \lambda e^{i\theta}/4$ is held fixed and $\lambda \ll 1$, scalar ladder diagrams contributing to the expectation value of (the Wilson loop relevant for) $\Gamma_{cusp}(\phi, \theta, \lambda)$ dominate and can be in fact resumed to all orders thus leading to a precise strong coupling prediction for the generalized cusp. Our computation in Section 6 has verified that these coefficients are exactly confirmed by the minimal surface computation up to one-loop accuracy. It is interesting to notice, that such agreement was not necessarily expected due to a potential order of limits problem. The gauge theory computation first relies on an approximation due to $\lambda \ll 1$. Only later the rescaled coupling $\hat{\lambda}$ is composed and can be taken to large values. On the string theory side $\lambda$ and $\hat{\lambda}$ are always large. Despite this source for errors, an exact agreement up to one-loop is found.

Finally, a potentially interesting question to answer is whether these two points in the parameter space are at all connected. To describe the (standard) cusp anomaly, or light-like cusped Wilson loop, the opening angle $\Phi$ of the space-like cusped Wilson loop has to be taken to complex infinity. To reach the scalar ladder diagram limit one should take the internal angle $\theta$ to complex infinity. While on the gauge theory side both these angles have completely different origins and interpretations ($\Phi$ being a physical opening angle, and $\theta$ an angle on the internal vector space of fields), in string theory they parametrize angles of the ten dimensional spacetime of $AdS_5 \times S^5$, and it would be interesting to find a "duality" in some sense.

# A Appendix: The $AdS_5 \times S^5$ Space

The space $AdS_5 \times S^5$ is a direct product of five dimensional Anti-de Sitter space and a five dimensional sphere. This particular combination is of interest in the $AdS/CFT$ correspondence where a superstring theory is considered on $AdS_5 \times S^5$ and the space therefore should be ten dimensional.

## A.1 The $AdS_5 \times S^5$ Metric

Five dimensional Anti-de Sitter space is defined as the surface of a five dimensional hyperboloid with curvature radius $R$. The easiest way to write this in coordinates is to embed the hyperboloid in a flat six dimensional space

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2 = -R^2, \tag{A.1}$$

where the coordinates $X_0$ and $X_5$ are time-like and the rest space-like. There exist several useful parametrizations for (A.1) like the *Poincaré patch*. In this thesis we will be mainly working with the so called *Global Coordinates* obtained by the substitution

$$\begin{aligned} X_0 &= R\ \cosh\rho\ \cos t, \\ X_5 &= R\ \cosh\rho\ \sin t, \\ X_i &= R\ \sinh\rho\ y_i \quad , \quad i = 1,2,3,4, \end{aligned} \tag{A.2}$$

where $y_i$ denote Euclidean coordinates which are constrained to a 3-dimensional unit sphere embedded in 4-dimensional space $\sum_i y_i^2 = 1$. Expressing the $y_i$ in polar coordinates $\Omega_3$ reduces the amount of required coordinates by one, so that the metric of $AdS_5$ takes on the following shape:

$$\mathrm{d}s^2 = R^2 \left(-\cosh^2\rho\ \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\ \mathrm{d}\Omega_3\right). \tag{A.3}$$

In fact, if we choose to constrain the coordinates as $0 \leq \rho \leq \infty$ and $-\pi \leq t < \pi$, we already cover the entire hyperboloid of the $AdS_5$ space. However, to have a causality respecting spacetime we should not have a periodic time dimension. Therefore, we should unwrap the $S^1$ circle of the $t$-coordinate and take $-\infty \leq t \leq \infty$ with no identifications to obtain the so called universal cover of $AdS_5$ [10]. We can obtain a *Poincaré patch* like $AdS$ metric if we Wick rotate $t$ (which means to take it imaginary $t \to it$) and then use the substitution

$$\sinh\rho = \frac{r}{z} \quad , \quad \tanh t = \frac{r^2 + z^2 - 1}{r^2 + z^2 + 1}. \tag{A.4}$$

Inserting the corresponding differentials into (A.3) yields the metric

$$\mathrm{d}s^2 = \frac{R^2}{z^2}(\mathrm{d}z^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_3). \tag{A.5}$$

This is essentially a kind of mixing for the $\rho$ and $t$ variables where the explicit dependence of the metric on the variable $r$ still can be removed. This can be done by choosing $z = rv$ for some $v$ and then $r = \exp(t')$, where $t'$ can be understood as a new time variable. The resulting metric will then be non-diagonal, but can be particularly useful in special situations

$$\mathrm{d}s^2 = \frac{R^2}{v^2}\left((1+v^2)\ \mathrm{d}t'^2 + \mathrm{d}v^2 + 2\ v\ \mathrm{d}t\ \mathrm{d}v + \mathrm{d}\Omega_3\right). \tag{A.6}$$

The $S^5$ space is the familiar five sphere with the metric given by:

$$\mathrm{d}s^2 = R^2\Omega_5 = R^2(\mathrm{d}\theta_1^2 + \cos^2\theta_1(\mathrm{d}\theta_2^2 + \cos^2\theta_2(\mathrm{d}\theta_3^2 + \cos^2\theta_3(\mathrm{d}\theta_4^2 + \cos^2\theta_4\ \mathrm{d}\theta_5^2)))). \tag{A.7}$$

Here one should be careful to keep the radius $R$ of the five sphere the same as the curvature parameter for the $AdS_5$ space. Later this radius will be related to the coupling constant of our theory and therefore it should be the same on all the subspaces of the manifold in question.

## A.2 Space Isometry

The $AdS_5$ space and the $S_5$ are maximally symmetric spaces. That means the maximal number of Killing vectors are available according to their respective geometry. Therefore, $AdS_5$ inherits from it's embedding space the Lorentz like isometry group $SO(2,4)$ (two time directions) [14]. Similarly $S_5$, which can be embedded into six dimensional Euclidean space, inherits the isometry group $SO(6)$. Taking the direct product of these two groups exactly results in the same amount of degrees of freedom as the super conformal group $SO(2,4)\times SO(6) = SU(2,2|4)$. This is a very important fact for the $AdS/CFT$ correspondence, since the total amount of symmetries in $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions with gauge group $SU(N)$ is governed by the super conformal group $SU(2,2|4)$.

# B Appendix: Green-Schwarz Superstring

We will introduce the Green-Schwarz superstring action following [18]. Extending the bosonic notion of mapping the parameter worldsheet into target space via a mapping function $X^\mu$, we can introduce fermionic spinor coordinates for each supersymmetry and add a second mapping function from the parameter worldsheet to these superspace coordinates. Let us call the two mapping functions:

$$X^\mu(\tau,\sigma) \quad , \quad \Theta^{A\,a}(\tau,\sigma). \tag{B.1}$$

Having $\mathcal{N}$ supersymmetries in the system then means $A = 1, 2, ..., \mathcal{N}$. In our case of interest (type IIB superstring theory) we have $\mathcal{N} = 2$. Since we are talking about spinors, the index $a$ assumes values $a = 1, 2, ..., 2^{D/2}$ for even spacetime dimension $D$ (considering a generic Dirac spinor). Being fermionic coordinates means that the $\Theta^A$ are anti-commuting. After extending the target space to a superspace, we can write down global supersymmetry transformations:

$$\delta\Theta^{A\,a} = \epsilon^{A\,a} \qquad , \qquad \delta X^\mu = \bar{\epsilon}^A\Gamma^\mu\Theta^A, \tag{B.2}$$

where the $\epsilon^A$ are Majorana spinors consisting of Grassmann valued constants and $\Gamma^\mu$ are the Dirac gamma matrices in $D$ dimensions. We can check the action of these new transformations:

$$[\delta_1\,,\,\delta_2]\Theta^A = \delta_1\epsilon^2 - \delta_2\epsilon^1 = 0 \tag{B.3}$$

$$[\delta_1\,,\,\delta_2]X^\mu = \bar{\epsilon}^2\Gamma^\mu\epsilon^1 - \bar{\epsilon}^1\Gamma^\mu\epsilon^2 = a^\mu, \tag{B.4}$$

where $a^\mu$ denotes some constant vector. Therefore, the supersymmetry transformations generate an infinitesimal spacetime translation of $X^\mu$ by a constant $a^\mu$. Combining conventional Poincaré symmetry with supersymmetry we get the super-Poincaré symmetry. These are global symmetries independent of $\tau$ and $\sigma$.

## B.1 Degrees of Freedom

To have supersymmetry, the amounts of bosonic and fermionic degrees of freedom have to match.[34] While the bosonic degrees of freedom are all real and their number is equal to $D-2$ (transverse directions to target space worldsheet), a generic Dirac spinor has far too many independent components. In our case we have $D = 10$, so that the bosonic degrees of freedom are 8. But a Dirac spinor in ten dimensions has 32 complex components resulting in 64 fermionic degrees of freedom. Evidently we should try to reduce this number somehow if

[34]Having a cusp in the dual Wilson loop softly breaks the supersymmetry, but that does not affect the considerations for degrees of freedom.

we want our theory to be supersymmetric. Therefore, we choose to constrain the spinors to be Majorana. This makes all spinor components real and reduces the fermionic degrees of freedom by half. Additionally, in $D = 10$ it is possible to have spinors that simultaneously are Majorana and Weyl spinors (since the $\Gamma_{11}$ matrix used in the projection operator can be constructed purely real). Imposing the Weyl constraint on our Majorana spinor, we chop off one half of the spinor components, making it either left or right handed, resulting in the remainder of 16 real fermionic degrees of freedom. Still we have twice as many fermionic degrees of freedom as we require. Further reduction will be due to $\kappa$-symmetry, which we will consider next[35].

## B.2 $\kappa$-Symmetry

In the considerations of the previous chapter we realized that we have to search for some means to fix additional fermionic degrees of freedom to be able to match bosons and fermions. It turns out that the successful approach is to construct the supersymmetric action such that it be invariant under one more symmetry - the so called $\kappa$-symmetry. The coordinate variations under these symmetry transformations are given by:

$$\delta X^\mu = \bar{\Theta}^A \Gamma^\mu \delta\Theta^A \quad , \quad \delta\bar{\Theta}^1 = \bar{\kappa}^1 P_- \quad , \quad \delta\bar{\Theta}^2 = \bar{\kappa}^2 P_+, \tag{B.5}$$

where $\bar{\kappa}^1$ and $\bar{\kappa}^2$ are arbitrary (local, so $\tau$ and $\sigma$ dependent) Majorana Weyl spinors of fitting chirality and $P_\pm = \frac{1}{2}(1 \pm \gamma)$ are projection operators defined as follows. Consider

$$\gamma = -\frac{\epsilon^{\alpha\beta}\Pi^\mu_\alpha \Pi^\nu_\beta \Gamma_{\mu\nu}}{2\sqrt{-G}}, \tag{B.6}$$

where $\epsilon^{\alpha\beta}$ is the totally antisymmetric tensor in two dimensions, the abbreviation $\Gamma_{\mu\nu} = \frac{1}{2}(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu)$ denotes anti-symmetrization of two gamma matrices in ten dimensions, the expression $\Pi^\mu_\alpha$ stands for

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - \bar{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A, \tag{B.7}$$

and $G = \det(G_{\alpha\beta})$ is the determinant of $G_{\alpha\beta} = \Pi_\alpha \cdot \Pi_\beta$. It can be easily shown that $\gamma$ squares to one, so it is justified to build a projection operator from it. The other terms $G$ and $\Pi^\mu_\alpha$ will be motivated in the next section where we derive the Green-Schwarz action. These terms are mentioned for completeness, yet in

[35] As a side remark, since we have two Majorana Weyl spinors it appears as if their total amount of degrees of freedom is 32 and not 16. However, in the path integral formalism fermionic determinants from Majorana spinors get an additional square root, which balances things out.

principle we could just have given the first equation in (B.5) to define $\kappa$-symmetry and let $\delta\Theta^A$ unspecified. Then the appropriate remaining expressions could be found at the end after introducing the action and computing its variation.

## B.3 Green-Schwarz Action

We start from the familiar bosonic Nambu-Goto action (here in units of $\alpha' = 1/2$):

$$S_{NG} = -\frac{1}{\pi}\int \mathrm{d}^2\sigma\sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)}. \tag{B.8}$$

For the sake of simplicity of the introduction we consider flat $D$-dimensional spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$), thus the straightforward contraction over the index $\mu$ in the equation above. Extending the target space to superspace we should now alter the terms of our action such as to make it invariant under the whole super-Poincaré symmetry. The first supersymmetric guess we can make is:

$$S_1 = -\frac{1}{\pi}\int \mathrm{d}^2\sigma\sqrt{-G}, \tag{B.9}$$

where $G = \det(G_{\alpha\beta})$ with $G_{\alpha\beta} = \Pi_\alpha \cdot \Pi_\beta$ and

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - \bar{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A \tag{B.10}$$

are the terms already mentioned in the last chapter. Considering (B.2) and keeping in mind that the variation spinor $\epsilon^A$ is constant, it immediately becomes clear that we have the invariance $\delta\Pi^\mu_\alpha = 0$ under super-Poincaré transformations which makes $S_1$ invariant and therefore a good initial choice. However, as argued before, $S_1$ is not yet invariant under the $\kappa$-symmetry:

$$\delta S_1 = \frac{2}{\pi}\int \mathrm{d}^2\sigma\sqrt{-G}\, G^{\alpha\beta}\, \Pi^\mu_\alpha\, \delta\bar{\Theta}^A \Gamma_\mu \partial_\beta \Theta^A, \tag{B.11}$$

where $\delta\bar{\Theta}^A$ is given in (B.5). Therefore, one further supersymmetric action term $S_2$ has to be constructed to ensure this invariance.
To construct $S_2$ we take a general approach. Considering that $S_1$ has the structure of a supersymmetrized volume (in particular involves the worldsheet metric), we can choose $S_2$ to be an integral over a two-form (now independent of worldsheet metric) to immediately ensure manifest diffeomorphism invariance for $S_2$. So we can write

$$S_2 = \frac{1}{2}\int \mathrm{d}^2\sigma \epsilon^{\alpha\beta}\Omega_{\alpha\beta}, \tag{B.12}$$

where $\Omega_{\alpha\beta}$ are the components of the two-form. Then we can apply the following mathematical trick. Consider adding an auxiliary dimension (which has no physical meaning as the worldsheet) and formally take the three-form $\Omega_3 = \mathrm{d}\Omega_2$. We can think of the space where $\Omega_3$ lives as some three dimensional region $U$ with the boundary $M = \partial U$ (string worldsheet). Making use of Stokes Theorem we then have the relation:

$$S_2 = \int_M \Omega_2 = \int_U \Omega_3. \tag{B.13}$$

The slight advantage is that in $\Omega_3$ the symmetries are straightforwardly manifest while in $\Omega_2$ only up to a total derivative (which would also be fine with us).
Now, to construct $\Omega_3$ we should think of three supersymmetric one forms (the word supersymmetric implying that they should vanish under super-Poincaré variations). The three one forms with this property that immediately come to mind are:

$$\mathrm{d}\Theta^1 \quad , \quad \mathrm{d}\Theta^2 \quad , \quad \Pi^\mu = \mathrm{d}X^\mu - \bar{\Theta}^A \Gamma^\mu \mathrm{d}\Theta^A. \tag{B.14}$$

Since these one-forms are linearly independent, we can use them as a basis and therefore $\Omega_3$ has to be a Lorentz invariant three-form constructed out of these. The appropriate choice that should yield an $S_2$ which will match $S_1$ is:

$$\Omega_3 = \frac{2}{\pi}\left(\mathrm{d}\bar{\Theta}^1 \Gamma_\mu \mathrm{d}\Theta^1 - \mathrm{d}\bar{\Theta}^2 \Gamma_\mu \mathrm{d}\Theta^2\right)\Pi^\mu. \tag{B.15}$$

One can verify that $\Omega_3$ is closed[36] (so $\mathrm{d}\Omega_3 = 0$) which is necessary if we have $\Omega_3 = \mathrm{d}\Omega_2$. After a few computational steps the $\kappa$-symmetry variation of $\Omega_3$ is then given by:

$$\delta\Omega_3 = \mathrm{d}\left[\frac{4}{\pi}(\delta\bar{\Theta}^1 \Gamma_\mu \mathrm{d}\Theta^1 - \delta\bar{\Theta}^2 \Gamma_\mu \mathrm{d}\Theta^2)\Pi^\mu\right]. \tag{B.16}$$

From this we can directly read off the $\kappa$-symmetry variation of $\Omega_2$ (making use of $\Omega_3 = \mathrm{d}\Omega_2$):

$$\delta\Omega_2 = \frac{4}{\pi}(\delta\bar{\Theta}^1 \Gamma_\mu \mathrm{d}\Theta^1 - \delta\bar{\Theta}^2 \Gamma_\mu \mathrm{d}\Theta^2)\Pi^\mu. \tag{B.17}$$

Again, straightforwardly inserting this into ($B.12$) we obtain the variation of $S_2$ under $\kappa$-symmetry:

$$\delta S_2 = \frac{2}{\pi}\int \mathrm{d}^2\sigma \epsilon^{\alpha\beta}(\delta\bar{\Theta}^1 \Gamma_\mu \partial_\alpha \Theta^1 - \delta\bar{\Theta}^2 \Gamma_\mu \partial_\alpha \Theta^2)\Pi^\mu_\beta. \tag{B.18}$$

Now, after some reshaping steps which are a little involved, the $\kappa$-symmetry variation of the complete action $S = S_1 + S_2$ is given by:

$$\delta S = \delta S_1 + \delta S_2 = \frac{4}{\pi}\int \mathrm{d}^2\sigma \epsilon^{\alpha\beta}(\delta\bar{\Theta}^1 P_+ \Gamma_\mu \partial_\alpha \Theta^1 - \delta\bar{\Theta}^2 P_- \Gamma_\mu \partial_\alpha \Theta^2)\Pi^\mu_\beta, \tag{B.19}$$

[36] To do this, one will have to make use of the key identity for Majorana Weyl spinors $\Theta$ in ten dimensions $\Gamma^\mu \mathrm{d}\Theta \mathrm{d}\bar{\Theta} \Gamma_\mu \mathrm{d}\Theta = 0$.[18]

where $P_\pm$ are exactly the projection operators introduced in (B.5). From this equation we actually easily see why the fermionic $\kappa$-symmetry variations have the shapes given in (B.5): applying two projections orthogonal to each other on one object necessarily gives zero, so that the action becomes invariant.
What remains to be done now, is to write down an actual expression for $S_2$. Solving $\Omega_3 = \mathrm{d}\Omega_2$ for $\Omega_2$ gives:

$$\Omega_2 = \frac{2}{\pi}(\bar{\Theta}^1\Gamma_\mu \mathrm{d}\Theta^1 - \bar{\Theta}^2\Gamma_\mu \mathrm{d}\Theta^2)\mathrm{d}X^\mu - \frac{2}{\pi}(\bar{\Theta}^1\Gamma_\mu \mathrm{d}\Theta^1)(\bar{\Theta}^2\Gamma^\mu \mathrm{d}\Theta^2). \tag{B.20}$$

Using this in (B.12) then gives

$$S_2 = \frac{1}{\pi}\int \mathrm{d}^2\sigma \epsilon^{\alpha\beta}\Big(s_{IJ}\bar{\Theta}^I\gamma_\alpha\partial_\beta\Theta^J - (\bar{\Theta}^1\Gamma_\mu\partial_\beta\Theta^1)(\bar{\Theta}^2\Gamma^\mu\partial_\alpha\Theta^2)\Big), \tag{B.21}$$

where for the matrix $s$ we have $s_{12} = s_{21} = 0$ and $s_{11} = -s_{22} = 1$ and the abbreviation $\gamma_\alpha = \Gamma_\mu\partial_\alpha X^\mu$. Putting $S_1 + S_2$ together then gives the complete supersymmetric Green Schwarz string action in flat D-dimensional spacetime:

$$S = -\frac{1}{\pi}\int \mathrm{d}^2\sigma \left[\sqrt{-G} - \epsilon^{\alpha\beta}\Big(s_{IJ}\bar{\Theta}^I\gamma_\alpha\partial_\beta\Theta^J + (\bar{\Theta}^1\Gamma_\mu\partial_\beta\Theta^1)(\bar{\Theta}^2\Gamma^\mu\partial_\alpha\Theta^2)\Big)\right]. \tag{B.22}$$

This is not yet the shape of the action as we want to use it. In the next chapter we make an appropriate expansion.

## B.4 Quadratic Fermionic Fluctuations

In the problem setting at hand we have the 't Hooft coupling constant $\lambda$ involved in the metric. The large $\lambda$ underlines the classical nature of bosons at leading order for the minimal surface solution. Fermions (being operator valued anti-commuting numbers) enter as quantum fluctuations. Additionally, we are only interested in the first order fluctuation correction which comes from terms quadratic in fluctuation fields. Therefore, we will neglect all terms with more fluctuation field terms than two. Expanding (B.22) for large $\lambda$ we obtain:

$$\mathcal{L} = (\text{bosons}) - (\sqrt{g}g^{\alpha\beta}\delta_{IJ} - \epsilon^{\alpha\beta}s_{IJ})\bar{\Theta}^I\gamma_\alpha\partial_\beta\Theta^J + O(\text{fluc.flds.}^4), \tag{B.23}$$

where the term labeled 'bosons' summarizes the straightforward classical and first order (quadratic in fields) fluctuation bosonic contributions. Therefore, what (B.23) outlines is the fermionic Lagrangian quadratic in fluctuation fields. Generalizing to curved spacetime by introduction of a covariant derivative $\partial_\beta \to D_\beta$ and adding a conventional (but irrelevant) factor of $i$ we can write:

$$\mathcal{L}_{2F}^{IIB} = -i(\sqrt{g}g^{\alpha\beta}\delta_{IJ} - \epsilon^{\alpha\beta}s_{IJ})\bar{\Theta}^I\gamma_\alpha D_\beta\Theta^J, \tag{B.24}$$

which is the formal expression for the fermionic fluctuation Lagrangian we shall be concerned with. This Lagrangian, together with the bosonic contributions, is still perturbatively invariant under $\kappa$-symmetry up to quadratic terms in fluctuation fields.

# C Appendix: The Generalized $\zeta$ Function Approach

To be able to compute the effective action for one-loop fluctuations around a classical minimal surface solution, we require a way to compute the determinant of a differential operator. A technique that can manage this task is the generalized $\zeta$ function approach that will be introduced in this chapter.

## C.1 General Procedure

An invertible differential operator $\mathbb{O}$, like any other well behaved operator, has a set of eigenvalues and eigenvectors associated with it. If $\{\lambda_i\}$ with $0 < i \leq n$ is the set of all $n \in \mathbb{N}$ eigenvalues of $\mathbb{O}$, then obviously the determinant is given by $\det \mathbb{O} = \prod_i \lambda_i$. Sometimes the operator $\mathbb{O}$ might contain zero modes (eigenvalues that are zero), which would make $\mathbb{O}$ not invertible. In these cases it is sensible to remove the zero modes from the determinant product to obtain a still meaningful partial determinant result. However, in the following discussion we will not encounter such badly behaved operators, so that it is sufficient just to note that such a possibility exists.
Following [11] and [12] we can define the generalized zeta function for the differential operator $\mathbb{O}$ as

$$\zeta(s) := \sum_i \frac{1}{\lambda_i^s}. \tag{C.1}$$

It is called the generalized zeta function, since it is reminiscent of the Riemann-zeta function but with the positive integers replaced with $\{\lambda_i\}$. Differentiating (C.1) once obviously gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s) = \zeta'(s) = -\sum_i \frac{\ln(\lambda_i)}{\lambda_i^s}, \tag{C.2}$$

where upon directly follows

$$\zeta'(0) = -\ln\left(\prod_i \lambda_i\right). \tag{C.3}$$

Therefore, formally the determinant of the differential operator $\mathbb{O}$ can be written as

$$\det \mathbb{O} = \exp(-\zeta'(0)). \tag{C.4}$$

The above formal expression is only consistent if the generalized $\zeta$ function is convergent around $s = 0$. For instance, it has been shown in the past that for

second order elliptic operators (which we will be dealing with in this thesis) the $\zeta$ function converges only for $\Re(s) > d/2$, where $d$ is the dimension of the manifold [13]. Nevertheless, one can analytically continue the solution to $s = 0$ and still obtain a meaningful result.
In practice the set of all eigenvalues of the operator $\mathbb{O}$ is not known explicitly. However, it is often possible to find a function $\mathcal{F}(\lambda)$ that has the set $\{\lambda_i\}$ as its zeros. Or in other words:

$$\mathcal{F}(\lambda) = 0 \quad \forall \quad \lambda = \lambda_i \quad \text{and} \quad \mathcal{F}(\lambda) \neq 0 \quad \text{everywhere else.} \tag{C.5}$$

Considering such a function $\mathcal{F}$, one realizes that the expression

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathcal{F}(\lambda) = \frac{\mathcal{F}'(\lambda)}{\mathcal{F}(\lambda)} \tag{C.6}$$

features simple poles with residue equal to 1 at all $\lambda = \lambda_i$. This handy property can be exploited to gain access to all the eigenvalues and construct the generalized $\zeta$ function through straightforward use of Cauchy integral formula:

$$\zeta(s) = \frac{1}{2\pi i} \int_\gamma \mathrm{d}\lambda \; \lambda^{-s} \; \frac{\mathrm{d} \ln \mathcal{F}(\lambda)}{\mathrm{d}\lambda}, \tag{C.7}$$

where the contour $\gamma$ is chosen such as to closely encircle the positive real axis in the mathematically positive sense (anti-clockwise). The branch cut associated with the $\lambda^{-s}$ term is conventionally chosen along the negative real axis. Now, to be able to evaluate the contour integral more conveniently, we can deform the contour $\gamma \to \gamma'$ as to closely encircle the negative real axis instead. In doing that, we effectively substitute the $\lambda$ in the integrand for the upper half of the contour by $\lambda \to e^{i\pi}\lambda$ and in the integrand for the lower half of the contour by $\lambda \to e^{-i\pi}\lambda$. That yields:

$$\zeta(s) = \frac{1}{2\pi i} \left( e^{-i\pi s} \int_\infty^0 \mathrm{d}\lambda \; \lambda^{-s} \; \frac{\mathrm{d} \ln \mathcal{F}(e^{i\pi}\lambda)}{\mathrm{d}\lambda} + e^{i\pi s} \int_0^\infty \mathrm{d}\lambda \; \lambda^{-s} \; \frac{\mathrm{d} \ln \mathcal{F}(e^{-i\pi}\lambda)}{\mathrm{d}\lambda} \right) \tag{C.8}$$

$$= \frac{\sin(\pi s)}{\pi} \int_0^\infty \mathrm{d}\lambda \; \lambda^{-s} \; \frac{\mathrm{d} \ln \mathcal{F}(-\lambda)}{\mathrm{d}\lambda}. \tag{C.9}$$

Now we can differentiate (C.9) with respect to $s$ and set $s = 0$ to obtain:

$$-\zeta'(0) = -\ln \mathcal{F}(-\infty) + \ln \mathcal{F}(0). \tag{C.10}$$

This implies for the determinant:

$$\det \mathbb{O} = \exp(-\zeta'(0)) = \frac{\mathcal{F}(0)}{\mathcal{F}(-\infty)}. \tag{C.11}$$

The denominator evaluated at minus infinity generally has to be treated with care involving analytic continuations for $\mathcal{F}$ to ensure convergence. However, if we are interested in a ratio of operator determinants, then the functions $\mathcal{F}(-\infty)$ will cancel and the ratio will depend only on the different $\mathcal{F}(0)$ contributions. This has to do with how $\mathcal{F}(\lambda)$ is constructed. As we will see in a minute, it is composed of solutions for a homogeneous differential equation involving the operator $\mathbb{O}$ with certain boundary conditions. The $\lambda$ enters as a constant potential term such that for large $\lambda$ the other potential terms are suppressed and all the solutions become trivial independently of the specific potential terms of the operator. With this have:

$$\frac{\det \mathbb{O}_1}{\det \mathbb{O}_2} = \frac{\mathcal{F}_1(0)}{\mathcal{F}_2(0)}, \tag{C.12}$$

which is the relevant equation, since we will have to evaluate only ratios of determinants to obtain the effective action of the one-loop system.

Now we should address the construction of $\mathcal{F}(\lambda)$. For this sake consider the eigenvalue equation of the operator $\mathbb{O}$:

$$\mathbb{O}\vec{y}_j = \lambda_j \vec{y}_j, \tag{C.13}$$

where $\vec{y}_j$ are eigenvectors to the eigenvalues $\lambda_j$ with the same dimensionality as the operator $\mathbb{O}$. Obviously, necessary boundary conditions are provided by the physical system in question. Now we simplify this equation by replacing the eigenvectors $\vec{y}_j$ by some solution $\vec{y}$ and replacing the concrete eigenvalues $\lambda_i$ by the variable $\lambda$ which we already encountered above. This gives us a homogeneous differential equation where $\lambda$ appears as a constant potential term:

$$(\mathbb{O} - \lambda)\vec{y} = 0. \tag{C.14}$$

The next step is to compute the so called fundamental matrix $H_\lambda$ for the solutions of this differential equation. For this end (C.13) should be written in terms of a first order differential equation which doubles its matrix dimensionality and where the solution $\vec{y}$ receives twice as many components, which are now alternating the components of the previous solution vector and their corresponding first derivatives. The fundamental matrix then contains as column entries the components of the new first order equation solution $\vec{y}$. The initial conditions of the differential equation for these solutions are chosen such that when evaluated at the left border of the interval $I$ the fundamental matrix $H_\lambda(x_L)$ is equal to the identity matrix. This is the defining property of $H_\lambda$ which ensures that the evolution of the system starting from the left border of the interval $I$ is governed by the fundamental matrix. Therefore, any particular solution $\vec{y}(x)$ can be expressed as $\vec{y}(x) = H_\lambda(x)\vec{y}(x_L)$. Having established this formalism we can now conveniently express any boundary conditions we like if we introduce matrices $M$ (for conditions at left boundary) and $N$ (conditions at right boundary):

$$(M + NH_\lambda(x_R))\,\vec{y}(x_L) = 0. \tag{C.15}$$

The matrices $M$ and $N$ contain only 1 and 0 as their entries to pick certain components from the solution vector $\vec{y}$. Indeed, $M$ picks components from $\vec{y}(x_L)$ (corresponding to the left boundary) and $N$ picks components from $\vec{y}(x_R)$ (right boundary), since $\vec{y}(x_L)$ first hits $H_\lambda(x_R)$ which propagates it to $\vec{y}(x_R)$ before the solution reaches matrix $N$. Since the matrix dimensionality of the equation is now twice of the initial second order differential operator, this is enough to fix two boundary conditions per original field so that a relevant solution to the second order differential equation is fixed uniquely.
Now if we use $M$ and $N$ to impose the boundary conditions required by our physical system, the equation (C.14) can only hold, if the whole matrix combination on the left itself has eigenvalues equal to zero. but this automatically leads to the necessary scalar condition:

$$\det\left[M + NH_\lambda(x_R)\right] = 0. \tag{C.16}$$

Luckily, the fundamental matrix $H_\lambda(x_R)$ still features the variable $\lambda$ that can be used to ensure the equality (C.15). Indeed, considering how the solutions that $H_\lambda$ consists of have been obtained, the equality (C.15) will be true for exactly any value of $\lambda$ that is an eigenvalue of the differential operator $\mathbb{O}$ with boundary conditions specified by $M$ and $N$. Therefore, we have found a fitting candidate for the function $\mathcal{F}(\lambda)$, namely:

$$\mathcal{F}(\lambda) = \det\left[M + NH_\lambda(x_R)\right]. \tag{C.17}$$

With this the generalized $\zeta$ function treatment is complete and (C.11) is ready to be applied.

## C.2 Gel'fand Yaglom Theorem

The Gel'fand Yaglom theorem [11] is mainly based on the previous considerations of this chapter. The difference is only in the choice of the function $\mathcal{F}(\lambda)$, which in this case is specialized for uncoupled differential operators (acting on one single field). Also, the Gel'fand Yaglom theorem is only applicable in the case of homogeneous Dirichlet boundary conditions.
Again, consider the following eigenvalue problem on the interval $I = [x_L, x_R]$

$$\mathbb{O}y_i = \lambda_i y_i \quad , \quad y_i(x_L) = y_i(x_R) = 0, \tag{C.18}$$

where now the second order differential operator $\mathbb{O}$ is one dimensional and is therefore acting on a single eigenfield $y_i$. As before, instead of computing the explicit eigenvalues, we simplify the problem to

$$(\mathbb{O} - \lambda) y_\lambda = 0, \tag{C.19}$$

where we now impose the specific initial values $y_\lambda(x_L) = 0$ and $y'_\lambda(x_L) = a$ with a some positive real number. Having specified two conditions for a second order differential equation, the solution is uniquely fixed. However, the first condition demands the Dirichlet boundary condition at the left border of the interval, but the second condition merely fixes a normalization for the solution. This means that the Dirichlet boundary condition at the right border of the interval $y_\lambda(x_R) = 0$ will only be valid for such values of $\lambda$ which are eigenvalues of the original problem (C.17). Just as before, we therefore have found a fitting candidate for the $\mathcal{F}(\lambda)$ function

$$\mathcal{F}(\lambda) = y_\lambda(x_R). \tag{C.20}$$

Following the same argumentation as in the previous section, one concludes that the ratio of the determinants of two differential operators is given by the analog of equation (C.11)

$$\frac{\det \mathbb{O}_1}{\det \mathbb{O}_2} \propto \frac{\mathcal{F}_1(0)}{\mathcal{F}_2(0)}. \tag{C.21}$$

However, in this case an additional subtlety appears regarding the normalization of the two solutions via the initial condition $y'_\lambda(x_L) = a$. To avoid dealing with unnecessary complications, we agree to choose the same normalization for the solutions involving $\mathbb{O}_1$ and $\mathbb{O}_2$, namely $y'_\lambda(x_L) = 1$. With this (C.20) again becomes an exact equality

$$\frac{\det \mathbb{O}_1}{\det \mathbb{O}_2} = \frac{\mathcal{F}_1(0)}{\mathcal{F}_2(0)}. \tag{C.22}$$

The Gel'fand Yaglom theorem in this formulation will be always used later in the thesis where a decoupled system of differential operators will appear.

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