# Yangian Symmetry of Maldacena-Wilson Loops 

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## Chapter 1

## Introduction

At present, we know of three fundamental forces which govern the microscopic world: The electromagnetic, the weak nuclear and the strong nuclear force, all of which can be accurately described by gauge quantum field theories. The easiest of these is quantum electrodynamics with Abelian gauge group $U(1)$. The combination of the electromagnetic and weak nuclear force is described by a non-Abelian gauge theory with (broken) gauge symmetry $S U(2) \times U(1)$. The strong nuclear force, which is responsible for the formation of quarks into hadrons, is described by quantum chromodynamics (QCD) which has the colour gauge group $S U(3)$. These theories are combined into the Standard model of particle physics, which has been tested to great accuracy in scattering experiments and has recently received another confirmation by the discovery of the Higgs boson at the Large Hadron Collider. Of course, we know that the Standard Model is incomplete, as it does not contain gravity or provide candidate particles which could constitute the dark matter hypothesized in astrophysics. However, also the theories beyond the Standard model are typically gauge theories.

Despite their fundamental role, our understanding of gauge theories is not as advanced as one could hope. This concerns both higher-order calculations of scattering amplitudes as well as an understanding of non-perturbative effects such as the confinement of quarks. The consideration of a similar but simpler gauge theory could lead to novel insights, which might in turn be used to improve our understanding of other gauge theories such as QCD. This approach has been successful in many branches of physics. A brillant example is our understanding of atomic spectra which is based on the exact quantum mechanical solution of the Schrödinger equation for the hydrogen atom.

The simplicity of a quantum field theory may be connected to its degree of symmetry. In this regard, $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory with gauge group $S U(N)$ stands out among other four-dimensional gauge theories, as it possesses maximally extended supersymmetry and a conformal symmetry, which is also present at the quantum level. The supersymmetry is said to be maximally extended in the sense that a theory with a higher degree of supersymmetry would necessarily contain particles of spin $s>1$, for which no renormalizable theory is known. With a view to its high degree of symmetry and the hope that an exact solution could be within
reach, $\mathcal{N}=4$ SYM has been called the hydrogen atom of gauge theories.
For example, it has been possible to derive certain tree-level amplitudes for QCD from the respective amplitudes in $\mathcal{N}=4$ SYM [1]. The calculation of the scattering amplitudes in the latter theory is greatly simplified by their superconformal symmetry and in fact it has been possible to derive all tree-level amplitudes in $\mathcal{N}=4$ SYM [2]. Apart from the superconformal symmetry, the scattering amplitudes in $\mathcal{N}=4$ SYM also possess a higher Yangian symmetry, which has been discussed in [3]. The Yangian algebra is an infinte-dimensional algebra, which extends the underlying symmetry algebra. Its appearance is often related to an underlying integrability of the theory. In the case of $\mathcal{N}=4$ SYM, integrable structures have been found for different observables, an important example are the two-point functions of local operators, see [4] for a review.

But there are more aspects that raise attention towards $\mathcal{N}=4$ SYM. In 1998, Juan Maldacena proposed the now famous correspondence between $\mathcal{N}=4 \mathrm{SYM}$ and a type IIB superstring theory on the background space $A d S_{5} \times S^{5}$ [5]. In a weaker form, the conjecture is restricted to the so-called planar limit [6], where the rank of the gauge group $S U(N)$ is sent to infinity, while the 't Hooft coupling constant $\lambda=g^{2} N$ is kept fixed. Interestingly, the conjecture relates a strongly coupled Yang-Mills theory to a weakly coupled string theory. The correspondence, if it holds, thereby allows to gain insights into the parts of both theories which are inaccessible to perturbation theory. Similar dualities for theories with less symmetry have been formulated and applied. The case of $\mathcal{N}=4 \mathrm{SYM}$ is however best tested and understood. For a review see 77 as well as (4).

The Wilson loop is an interesting observable which can be considered in any gauge theory. It was first introduced by Kenneth Wilson [8] in the study of the confinement problem in QCD, where it arises because the calculation of the expectation value of the Wilson loop over certain contours allows to derive the force between two static quarks. This computation can be performed to all orders in pure quantum electrodynamics, providing a derivation of the Coulomb potential. In $\mathcal{N}=4$ SYM there is a natural extension of the Wilson loop, which was suggested by Juan Maldacena [9]. The Maldacena-Wilson loop is in many respects easier to handle than the ordinary Wilson loop. It has a finite expectation value, which is conformally symmetric and (locally) invariant under certain linear combinations of the supersymmetry generators. Using the conformal invariance of the expectation value, it has been possible to derive exact results for special contours [10]. It also has a dual string description [9], which relates it to the area of a minimal surface in $A d S_{5}$. The consideration of scattering amplitudes at strong coupling [11] has led to the proposal of a duality between scattering amplitudes and (Maldacena)-Wilson loop\& ${ }^{1}$, which has been put forward also at weak coupling (12].

The combination of this duality with the Yangian symmetry of scattering amplitudes has led to the idea ${ }^{2}$ that also the Maldacena-Wilson loop possesses a Yangian

[^0]symmetry. This possible higher symmetry of the Maldacena-Wilson loop is studied in this thesis. We show that a supersymmetric extension of the Maldacena-Wilson loop, which includes also fermionic fields, indeed exhibits the proposed symmetry. The results of this thesis have been incorporated in [13], where it is also shown that the strong coupling description of the Maldacena-Wilson loop is invariant under a Yangian symmetry. Interestingly, at strong coupling it is not necessary to include fermionic degrees of freedom.

## Outline

Before we embark on the study of the Yangian symmetry of the Maldacena-Wilson loop, we introduce the necessary preliminaries for this discussion:

In chapter 2 we discuss spinors in four- and ten-dimensional Minkowski space as well as six-dimensional Euclidean space. The concepts and notation introduced here will be needed at various instances. The four- and six-dimensional spinor indices are used in labelling the fields of $\mathcal{N}=4$ SYM, the ten-dimensional spinors will be employed in the construction of the action of $\mathcal{N}=4$ SYM and in the discussion of the local supersymmetry of the Maldacena-Wilson loop.

In chapter 3 we describe the different symmetries which are applied or studied in this thesis. We give a brief introduction to conformal symmetry, focusing largely on the conformal algebra. We also introduce the $\mathcal{N}=4$ superconformal algebra and provide a representation in terms of differential operators acting in the superspace employed in the supersymmetric extension of the Maldacena-Wilson loop. Furthermore, we discuss the Yangian over a simple Lie algebra.

In chapter 4 we discuss $\mathcal{N}=4$ supersymmetric Yang-Mills theory. We derive the action by performing a dimensional reduction from $\mathcal{N}=1$ supersymmetric Yang-Mills theory in ten dimensions and briefly discuss its symmetries. We also derive the sclar, gluon and gluino propagators which are needed to compute the expectation value of the extended Maldacena-Wilson loop. Furthermore, we discuss scattering amplitudes and their symmetries in $\mathcal{N}=4 \mathrm{SYM}$.

In chapter 5 we introduce the Maldacena-Wilson loop. For this, we first discuss Wilson loops in general gauge theories and then show how the Maldacena-Wilson loop arises by putting a constraint on the ten-dimensional Wilson loop in $\mathcal{N}=1$ SYM. We show how this constraint is related to supersymmetry and that it leads to a finite expectation value for well-behaved curves. Furthermore, we discuss the conformal symmetry of the expectation value and briefly review the conjectured duality to scattering amplitudes.

In chapter 6, we then turn to the Yangian symmetry of the Maldacena-Wilson loop. Building upon the conformal symmetry, we propose a possible level- 1 momentum generator of the Yangian algebra and study whether it is possible to modify it in such a way that it annihilates the vacuum expectation value of the Maldacena-Wilson loop to first order in perturbation theory. Although we find that this cannot be done, the form of the result leads us to construct an extension of the Maldacena-Wilson loop which includes also the fermion fields of $\mathcal{N}=4$ SYM. This extension is (to the orders
we compute it) entirely fixed by supersymmetry. We then show that the modified level-1 momentum generator does annihilate the expectation value of the extended Maldacena-Wilson loop.

An outlook on possible subsequent works will be given in chapter 7 .

## Chapter 2

## Spinors in Various Dimensions

In this section we will introduce spinors in four, six and ten dimensions. These are needed in many parts of this thesis, for example to construct the action of $\mathcal{N}=4$ supersymmetric Yang-Mills theory by dimensional reduction from $\mathcal{N}=1$ supersymmetric Yang-Mills theory in ten dimensions. Apart from introducing our conventions, we also provide several technical identities. We follow the conventions of [14] and [15]. For details concerning the derivation of technical identities which are not proven below, the reader is referred to the latter.

### 2.1 Four-dimensional Minkowski Space

We consider Minkowski space with the metric $\eta_{\mu \nu}=\operatorname{diag}(+1,-1,-1,-1)$. Four-dimensional Dirac spinors can be written in the following form:

$$
\psi=\binom{\psi_{\alpha}}{\tilde{\psi}^{\dot{\alpha}}}
$$

The indices $\alpha \in\{1,2\}$ and $\dot{\alpha} \in\{\dot{1}, \dot{2}\}$ are raised or lowered by contracting with the two-dimensional epsilon tensors $\epsilon^{\alpha \beta}$ and $\epsilon^{\dot{\alpha} \dot{\beta}}$,

$$
\begin{equation*}
\epsilon^{12}=\epsilon_{12}=1, \quad \epsilon^{\mathrm{i} \dot{2}}=\epsilon_{\mathrm{i} \dot{2}}=-1 \quad \Rightarrow \epsilon^{\alpha \beta} \epsilon_{\gamma \beta}=\delta_{\gamma}^{\alpha}, \quad \epsilon^{\dot{\alpha} \dot{\beta}} \epsilon_{\dot{\gamma} \dot{\beta}}=\delta_{\dot{\gamma}}^{\dot{\alpha}}, \tag{2.1}
\end{equation*}
$$

in the following way:

$$
\begin{equation*}
\psi^{\alpha}=\epsilon^{\alpha \beta} \psi_{\beta}, \quad \psi_{\alpha}=\psi^{\beta} \epsilon_{\beta \alpha}, \quad \tilde{\psi}_{\dot{\alpha}}=\epsilon_{\dot{\alpha} \dot{\beta}} \tilde{\psi}^{\dot{\beta}}, \quad \tilde{\psi}^{\dot{\alpha}}=\tilde{\psi}_{\dot{\beta}} \epsilon^{\dot{\beta} \dot{\alpha}} \tag{2.2}
\end{equation*}
$$

To give a representation of the Clifford algebra in four dimensions, we introduce the following two bases of hermitian matrices:

$$
\begin{equation*}
\sigma^{\mu \dot{\alpha} \beta}:=(\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma}_{\alpha \dot{\beta}}^{\mu}:=(\mathbb{1},-\vec{\sigma}), \tag{2.3}
\end{equation*}
$$

where $\vec{\sigma}$ are the Pauli matrices

$$
\sigma^{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \quad \sigma^{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), \quad \sigma^{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
$$

The sigma matrices can be identified as follows:

$$
\sigma^{\mu \dot{\alpha} \beta}=\epsilon^{\beta \gamma} \bar{\sigma}_{\gamma \dot{\delta}}^{\mu} \epsilon^{\dot{\delta} \dot{\alpha}}=\bar{\sigma}^{\mu \beta \dot{\alpha}}, \quad \bar{\sigma}_{\alpha \dot{\beta}}^{\mu}=\epsilon_{\dot{\beta} \dot{\gamma}} \sigma^{\mu \dot{\gamma} \delta} \epsilon_{\delta \alpha}=\sigma_{\mu \dot{\beta} \alpha} .
$$

Furthermore they satisfy the following identity:

$$
\begin{equation*}
\bar{\sigma}_{\alpha \dot{\gamma}}^{\mu} \sigma^{\nu \dot{\gamma} \beta}+\bar{\sigma}_{\alpha \dot{\gamma}}^{\nu} \sigma^{\mu \dot{\gamma} \beta}=2 \eta^{\mu \nu} \delta_{\beta}^{\alpha}, \quad \sigma^{\mu \dot{\alpha} \gamma} \sigma_{\gamma \dot{\beta}}^{\nu}+\sigma^{\nu \dot{\alpha} \gamma} \sigma_{\gamma \dot{\beta}}^{\mu}=2 \eta^{\mu \nu} \delta_{\dot{\beta}}^{\dot{\alpha}} . \tag{2.4}
\end{equation*}
$$

Thus the following matrices form a representation of the Clifford algebra:

$$
\gamma^{\mu}=\left(\begin{array}{cc}
0 & \bar{\sigma}_{\alpha \dot{\beta}}^{\mu}  \tag{2.5}\\
\sigma^{\mu \dot{\alpha} \beta} & 0
\end{array}\right), \quad \gamma^{5}=i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}=\left(\begin{array}{cc}
\mathbb{1} & 0 \\
0 & -\mathbb{1}
\end{array}\right) .
$$

This representation is called the chiral representation. In any dimension a representation of the Clifford algebra induces a representation of the universal cover $S L(2, \mathbb{C})$ of the Lorentz group $S O(1,3)$. In the given representation the reducibility of the induced representation of $S L(2, \mathbb{C})$ is manifest as the generators

$$
\gamma^{\mu \nu}:=\frac{1}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right]=\frac{1}{4}\left(\begin{array}{cc}
\bar{\sigma}_{\alpha \dot{\beta}}^{\mu} \sigma^{\nu \dot{\beta} \gamma}-\bar{\sigma}_{\alpha \dot{\beta}}^{\nu} \sigma^{\mu \dot{\beta} \gamma} & 0 \\
0 & \sigma^{\mu \dot{\alpha} \beta} \bar{\sigma}_{\beta \dot{\gamma}}^{\nu}-\sigma^{\nu \dot{\alpha} \beta} \bar{\sigma}_{\beta \dot{\gamma}}^{\mu}
\end{array}\right)
$$

are block-diagonal. Of course, the question of reducibility is not related to a concrete representation of the Clifford algebra. The irreducible representations constituting the above representation act on the left- and right-handed Weyl spinors

$$
\binom{\psi_{\alpha}}{0}=P_{L} \psi=\frac{1}{2}\left(\mathbb{1}+\gamma^{5}\right) \psi, \quad\binom{0}{\tilde{\psi}^{\dot{\alpha}}}=P_{R} \psi=\frac{1}{2}\left(\mathbb{1}-\gamma^{5}\right) \psi .
$$

Apart from left- or right-handed Weyl spinors, one often also considers Majorana spinors ${ }^{17}$. The Majorana condition can be viewed as a Lorentz invariant reality condition. To impose such a condition, we need a charge conjugation matrix $C_{(1,3)}$, which satisfies

$$
\begin{equation*}
C_{(1,3)}^{\dagger} C_{(1,3)}=\mathbb{1}, \quad C_{(1,3)}^{\dagger} \gamma^{\mu} C_{(1,3)}=-\left(\gamma^{\mu}\right)^{*} \tag{2.6}
\end{equation*}
$$

These properties make sure that the charge conjugated spinor

$$
\psi^{(c)}:=C_{(1,3)} \psi
$$

transforms like $\psi$ under Lorentz transformations. To see this, consider the commutator of $C_{(1,3)}$ with a representation matrix

$$
S[\Lambda]=\exp \left(\frac{1}{2} \omega_{\mu \nu} \gamma^{\mu \nu}\right)
$$

By (2.6) we have $C_{(1,3)} \gamma^{\mu \nu}=\left(\gamma^{\mu \nu}\right)^{*} C_{(1,3)}$ and hence

$$
C_{(1,3)} S[\Lambda]^{*}=S[\Lambda] C_{(1,3)} .
$$

[^1]The charge conjugated spinor then transforms as

$$
\psi^{(c)} \rightarrow C_{(1,3)} S[\Lambda]^{*} \psi^{*}=S[\Lambda] C_{(1,3)} \psi^{*}=S[\Lambda] \psi^{(c)} .
$$

Therefore the Majorana condition

$$
\begin{equation*}
\psi^{(c)}=\psi \tag{2.7}
\end{equation*}
$$

is Lorentz invariant, meaning that it holds in any frame if it holds in one. A possible choice for a charge conjugation matrix is the following:

$$
C_{(1,3)}=i \gamma^{2}=\left(\begin{array}{cc}
0 & -i \sigma^{2}  \tag{2.8}\\
i \sigma^{2} & 0
\end{array}\right)=\left(\begin{array}{cc}
0 & \epsilon_{\dot{\alpha} \dot{\beta}} \\
\epsilon^{\alpha \beta} & 0
\end{array}\right) .
$$

It is easy to see that this satisfies the conditions (2.6). The Majorana condition then becomes

$$
\binom{\psi_{\alpha}}{\tilde{\psi}^{\dot{\alpha}}}=\left(\begin{array}{cc}
0 & \epsilon_{\dot{\alpha} \dot{\beta}} \\
\epsilon^{\alpha \beta} & 0
\end{array}\right)\binom{\left(\psi_{\alpha}\right)^{*}}{\left(\tilde{\psi}^{\dot{\alpha}}\right)^{*}}=\binom{\left(\tilde{\psi}_{\dot{\alpha}}\right)^{*}}{\left(\psi^{\alpha}\right)^{*}} .
$$

We hence see, that left- or right-handed Weyl spinors cannot satisfy the Majorana condition ${ }^{2}$. This is not the case for ten-dimensional Minkowski space, where we consider Majorana-Weyl spinors.

To any four-vector in Minkowski space we can assign a bi-spinor by contracting it with the sigma matrices introduced above. To be able to translate back and forth between bi-spinors and four-vectors, the following identities for contractions of spacetime or spinor indices will be crucial:

$$
\begin{equation*}
\bar{\sigma}_{\alpha \dot{\beta}}^{\mu} \bar{\sigma}_{\mu \gamma \dot{\delta}}=-2 \epsilon_{\alpha \gamma} \epsilon_{\dot{\beta} \dot{\delta}}, \quad \sigma_{\dot{\alpha} \dot{\beta}}^{\mu} \sigma_{\mu \dot{\gamma} \delta}=-2 \epsilon_{\beta \delta \delta} \epsilon_{\dot{\alpha} \dot{\gamma}}, \quad \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} \bar{\sigma}^{\nu \alpha \dot{\alpha}}=2 \eta^{\mu \nu}=\sigma_{\dot{\alpha} \alpha}^{\mu} \sigma^{\nu \dot{\alpha} \alpha} . \tag{2.9}
\end{equation*}
$$

The assignment of a bi-spinor to a four vector $p^{\mu}$ is given by the following contraction:

$$
p^{\alpha \dot{\alpha}}:=\bar{\sigma}^{\mu \alpha \dot{\alpha}} p_{\mu}=\sigma^{\mu \dot{\alpha} \alpha} p_{\mu}=: p^{\dot{\alpha} \alpha} .
$$

This can be inverted making use of (2.9):

$$
\begin{equation*}
p^{\mu}=\frac{1}{2} \bar{\sigma}^{\mu \alpha \dot{\alpha}} p_{\alpha \dot{\alpha}}=\frac{1}{2} \sigma^{\mu \dot{\alpha} \alpha} p_{\dot{\alpha} \alpha}, \quad p_{\alpha \dot{\alpha}} k^{\alpha \dot{\alpha}}=2 p_{\mu} k^{\mu} . \tag{2.10}
\end{equation*}
$$

There are different possibilities in defining bi-spinor derivatives. Following [15], we define them by the translation we use for other Minkowski four-vectors, i.e.:

$$
\begin{equation*}
\partial^{\dot{\alpha} \alpha}=\frac{\partial}{\partial x_{\dot{\alpha} \alpha}}:=\sigma^{\mu \dot{\alpha} \alpha} \frac{\partial}{\partial x^{\mu}} . \tag{2.11}
\end{equation*}
$$

This definition leads to the following relation:

$$
\begin{equation*}
\partial^{\dot{\alpha} \alpha} x_{\dot{\beta} \beta}=2 \delta_{\beta}^{\alpha} \delta_{\dot{\beta}}^{\dot{\alpha}} . \tag{2.12}
\end{equation*}
$$

[^2]Note also that in our conventions the chain rule receives a factor of one half:

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} s} F(x(s))=\frac{\partial F}{\partial x^{\mu}} \dot{x}^{\mu}(s)=\frac{1}{2} \frac{\partial F}{\partial x^{\alpha \dot{\alpha}}} \dot{x}^{\alpha \dot{\alpha}}(s) . \tag{2.13}
\end{equation*}
$$

We also assign bi-spinors to antisymmetric tensors of rank two. For this, we define:

$$
\begin{align*}
\sigma^{\mu \nu \alpha \beta} & :=\frac{i}{2}\left(\bar{\sigma}^{\mu \alpha \dot{\delta}} \sigma^{\nu \dot{\gamma} \beta}-\bar{\sigma}^{\nu \alpha \dot{\delta}} \sigma^{\mu \dot{\gamma} \beta}\right) \epsilon_{\dot{\gamma} \dot{\delta}},  \tag{2.14}\\
\bar{\sigma}^{\mu \nu \dot{\alpha} \dot{\beta}} & :=\frac{i}{2}\left(\sigma^{\mu \dot{\alpha} \gamma} \bar{\sigma}^{\nu \delta \dot{\beta}}-\sigma^{\nu \dot{\alpha} \gamma} \bar{\sigma}^{\mu \delta \dot{\beta}}\right) \epsilon_{\delta \gamma} . \tag{2.15}
\end{align*}
$$

Then we assign the following bi-spinors to an antisymmetric tensor $F_{\mu \nu}$ by:

$$
\begin{equation*}
F^{\alpha \beta}:=F_{\mu \nu} \sigma^{\mu \nu \alpha \beta}, \quad F^{\dot{\alpha} \dot{\beta}}:=F_{\mu \nu} \bar{\sigma}^{\mu \nu \dot{\alpha} \dot{\beta}} . \tag{2.16}
\end{equation*}
$$

The two bi-spinors associated to $F_{\mu \nu}$ can be related to

$$
F^{\alpha \dot{\alpha} \dot{\beta} \dot{\beta}}:=F_{\mu \nu} \bar{\sigma}^{\mu \alpha \dot{\alpha}} \bar{\sigma}^{\nu \beta \dot{\beta}}
$$

by the following identity:

$$
\begin{equation*}
F^{\alpha \dot{\alpha} \beta \dot{\beta}}=\frac{i}{2} \epsilon^{\dot{\alpha} \dot{\beta}} F^{\alpha \beta}+\frac{i}{2} \epsilon^{\alpha \beta} F^{\dot{\alpha} \dot{\beta}} . \tag{2.17}
\end{equation*}
$$

As this differs from the one given in [15] we will go through the proof. To check the above statements, it suffices to contract the equation with the epsilon tensors $\epsilon_{\alpha \beta}$ and $\epsilon_{\dot{\alpha} \dot{\beta} \dot{\beta}}$. Using the symmetry properties of the bi-spinors defined above, $F^{\alpha \beta}=F^{\beta \alpha}$ and $F^{\dot{\alpha} \dot{\beta}}=F^{\dot{\beta} \dot{\alpha}}$, we have:

$$
\epsilon_{\alpha \beta}\left(\frac{i}{2} \epsilon^{\dot{\alpha} \dot{\beta}} F^{\alpha \beta}+\frac{i}{2} \epsilon^{\alpha \beta} F^{\dot{\alpha} \dot{\beta}}\right)=i F^{\dot{\alpha} \dot{\beta}}, \quad \epsilon_{\dot{\alpha} \dot{\beta}}\left(\frac{i}{2} \epsilon^{\dot{\alpha} \dot{\beta}} F^{\alpha \beta}+\frac{i}{2} \epsilon^{\alpha \beta} F^{\dot{\alpha} \dot{\beta}}\right)=i F^{\alpha \beta} .
$$

On the other hand we find

$$
\begin{aligned}
\epsilon_{\alpha \beta} F^{\alpha \dot{\alpha} \beta \dot{\beta}} & =\epsilon_{\alpha \beta} F_{\mu \nu} \bar{\sigma}^{\mu \alpha \dot{\alpha}} \bar{\sigma}^{\nu \beta \dot{\beta}}=\frac{1}{2} F_{\mu \nu}\left(\bar{\sigma}^{\mu \alpha \dot{\alpha}} \bar{\sigma}^{\nu \beta \dot{\beta}}-\bar{\sigma}^{\nu \alpha \dot{\alpha}} \bar{\sigma}^{\mu \beta \dot{\beta}}\right) \epsilon_{\alpha \beta} \\
& =i F_{\mu \nu} \frac{i}{2}\left(\sigma^{\mu \dot{\alpha} \alpha} \bar{\sigma}^{\nu \dot{\beta}}-\sigma^{\nu \dot{\alpha} \alpha} \bar{\sigma}^{\mu \beta \dot{\beta}}\right) \epsilon_{\beta \alpha}=i F^{\dot{\alpha} \dot{\beta}},
\end{aligned}
$$

and in the same way

$$
\epsilon_{\dot{\alpha} \dot{\beta}} F^{\alpha \dot{\alpha} \beta \dot{\beta}}=i F^{\alpha \beta}
$$

which concludes the proof.
The bi-spinors associated to an antisymmetric tensor via (2.16) are symmetric. Bi-spinors which do not have this property can be decomposed into a symmetric and an antisymmetric piece:

$$
\Lambda_{\alpha \beta}=\Lambda_{(\alpha \beta)}+\Lambda_{[\alpha \beta]}=\Lambda_{(\alpha \beta)}+C_{1} \epsilon_{\alpha \beta}, \quad \Lambda_{\dot{\alpha} \dot{\beta}}=\Lambda_{(\dot{\alpha} \dot{\beta})}+C_{2} \epsilon_{\dot{\alpha} \dot{\beta}} .
$$

Contracting both sides with $\epsilon^{\alpha \beta}$ or $\epsilon^{\dot{\alpha} \dot{\beta}}$ respectively we find the general symmetry properties:

$$
\begin{align*}
& \Lambda_{(\alpha \beta)}=\Lambda_{\alpha \beta}+\frac{1}{2} \epsilon_{\alpha \beta} \Lambda^{\gamma}{ }_{\gamma},  \tag{2.18}\\
& \Lambda_{(\dot{\alpha} \dot{\beta})}=\Lambda_{\dot{\alpha} \dot{\beta}}+\frac{1}{2} \epsilon_{\dot{\alpha} \dot{\beta}} \Lambda_{\dot{\gamma}}^{\dot{\gamma}} . \tag{2.19}
\end{align*}
$$

As a direct consequence of (2.9), we find the Fierz identity

$$
\begin{equation*}
\tilde{\xi}^{\tilde{\alpha}^{\beta}} \xi^{\beta}=\frac{1}{2} \sigma^{\mu \dot{\alpha} \beta}\left(\tilde{\xi}^{\dot{\gamma}} \sigma_{\mu \dot{\gamma} \delta} \xi^{\delta}\right), \tag{2.20}
\end{equation*}
$$

and the trace identity

$$
\begin{equation*}
\frac{1}{2} \operatorname{Tr}\left(\bar{\sigma}^{\mu} \sigma^{\nu}\right)=\eta^{\mu \nu} . \tag{2.21}
\end{equation*}
$$

For our further calculations, we also need the following higher trace identities:

$$
\begin{align*}
& \frac{1}{2} \operatorname{Tr}\left(\bar{\sigma}^{\mu} \sigma^{\nu} \bar{\sigma}^{\rho} \sigma^{\kappa}\right)=\eta^{\mu \nu} \eta^{\rho \kappa}+\eta^{\nu \rho} \eta^{\mu \kappa}-\eta^{\mu \rho} \eta^{\nu \kappa}-i \epsilon^{\mu \nu \rho \kappa},  \tag{2.22}\\
& \frac{1}{2} \operatorname{Tr}\left(\sigma^{\mu} \bar{\sigma}^{\nu} \sigma^{\rho} \bar{\sigma}^{\kappa}\right)=\eta^{\mu \nu} \eta^{\rho \kappa}+\eta^{\nu \rho} \eta^{\mu \kappa}-\eta^{\mu \rho} \eta^{\nu^{\kappa}}+i \epsilon^{\mu \nu \rho \kappa} . \tag{2.23}
\end{align*}
$$

These identities can be derived similarly to the derivation of trace identities for fourdimensional gamma matrices. Using the Clifford relation in terms of the sigma matrices (2.4), we have:

$$
\begin{aligned}
\operatorname{Tr}\left(\bar{\sigma}^{\mu} \sigma^{\nu} \bar{\sigma}^{\rho} \sigma^{\kappa}\right) & =2 \eta^{\mu \nu} \operatorname{Tr}\left(\bar{\sigma}^{\rho} \sigma^{\kappa}\right)-\operatorname{Tr}\left(\bar{\sigma}^{\nu} \sigma^{\mu} \bar{\sigma}^{\rho} \sigma^{\kappa}\right)= \\
& =4\left(\eta^{\mu \nu} \eta^{\rho \kappa}-\eta^{\mu \rho} \eta^{\nu \kappa}+\eta^{\mu \kappa} \eta^{\nu \rho}\right)-\operatorname{Tr}\left(\bar{\sigma}^{\nu} \sigma^{\rho} \bar{\sigma}^{\kappa} \sigma^{\mu}\right) .
\end{aligned}
$$

Using the cyclicity of the trace, we arrive at

$$
\operatorname{Tr}\left(\bar{\sigma}^{\mu} \sigma^{\nu} \bar{\sigma}^{\rho} \sigma^{\kappa}\right)+\operatorname{Tr}\left(\sigma^{\mu} \bar{\sigma}^{\nu} \sigma^{\rho} \bar{\sigma}^{\kappa}\right)=4\left(\eta^{\mu \nu} \eta^{\rho \kappa}-\eta^{\mu \rho} \eta^{\nu \kappa}+\eta^{\mu \kappa} \eta^{\nu \rho}\right) .
$$

On the other hand, one can easily assure oneself that

$$
\operatorname{Tr}\left(\bar{\sigma}^{\mu} \sigma^{\nu} \bar{\sigma}^{\rho} \sigma^{\kappa}\right)-\operatorname{Tr}\left(\sigma^{\mu} \bar{\sigma}^{\nu} \sigma^{\rho} \bar{\sigma}^{\kappa}\right)
$$

is totally antisymmetric in all indices. A calculation with fixed indices then shows that

$$
\operatorname{Tr}\left(\bar{\sigma}^{\mu} \sigma^{\nu} \bar{\sigma}^{\rho} \sigma^{\kappa}\right)-\operatorname{Tr}\left(\sigma^{\mu} \bar{\sigma}^{\nu} \sigma^{\rho} \bar{\sigma}^{\kappa}\right)=-4 i \epsilon^{\mu \nu \rho \kappa} .
$$

The combination of these two identities proves (2.22) and (2.23).

### 2.2 Six-dimensional Euclidean Space

We consider $\mathbb{R}^{6}$ with negative definite metric $\eta_{I J}=\operatorname{diag}(-, \ldots,-)$. The reason for this choice of metric is, that we will later combine this space with four-dimensional Minkowski space to give the ten-dimensional Minkowski space $\mathbb{R}^{(1,9)}$. The gamma matrices for the six-dimensional Euclidean space can be written as

$$
\hat{\gamma}^{I}=\left(\begin{array}{cc}
0 & \Sigma^{I A B}  \tag{2.24}\\
\bar{\Sigma}_{A B}^{I} & 0
\end{array}\right), \quad \hat{\gamma}^{7}=i \prod_{I=1}^{6} \hat{\gamma}^{I}=\left(\begin{array}{cc}
\mathbb{1} & 0 \\
0 & -\mathbb{1}
\end{array}\right) .
$$

Here, $I$ runs from 1 to 6 while the upper or lower indices $A, B$ take values in $\{1,2,3,4\}$. The sigma matrices are defined by

$$
\begin{align*}
\left(\Sigma^{1 A B}, \ldots, \Sigma^{6 A B}\right) & =\left(\eta_{1 A B}, \eta_{2 A B}, \eta_{3 A B}, i \bar{\eta}_{1 A B}, i \bar{\eta}_{2 A B}, i \bar{\eta}_{3 A B}\right),  \tag{2.25}\\
\left(\bar{\Sigma}_{A B}^{1}, \ldots, \bar{\Sigma}_{A B}^{6}\right) & =\left(\eta_{1 A B}, \eta_{2 A B}, \eta_{3 A B},-i \bar{\eta}_{1 A B},-i \bar{\eta}_{2 A B},-i \bar{\eta}_{3 A B}\right), \tag{2.26}
\end{align*}
$$

where $\eta_{i A B}$ and $\bar{\eta}_{i A B}$ denote the 't Hooft symbols

$$
\begin{equation*}
\eta_{i A B}:=\epsilon_{i A B 4}+\delta_{i A} \delta_{4 B}-\delta_{i B} \delta_{4 A}, \quad \bar{\eta}_{i A B}:=\epsilon_{i A B 4}-\delta_{i A} \delta_{4 B}+\delta_{i B} \delta_{4 A} \tag{2.27}
\end{equation*}
$$

Here, $\epsilon_{A B C D}$ denotes the four-dimensional epsilon tensor normalised by

$$
\epsilon_{1234}=\epsilon^{1234}=1 .
$$

We note the following contraction:

$$
\begin{equation*}
\epsilon_{D A B C} \epsilon^{D K L M}=\delta_{A B C}^{K L M}+\delta_{A B C}^{M K L}+\delta_{A B C}^{L M K}-\delta_{A B C}^{L K M}-\delta_{A B C}^{M L K}-\delta_{A B C}^{K M L} . \tag{2.28}
\end{equation*}
$$

Note also, that the sigma matrices are antisymmetric. In terms of the sigma matrices the Clifford relation, $\hat{\gamma}^{I} \hat{\gamma}^{J}+\hat{\gamma}^{J} \hat{\gamma}^{I}=-2 \delta^{I J} \mathbb{1}$, takes the following form:

$$
\begin{equation*}
\Sigma^{I A B} \bar{\Sigma}_{B C}^{J}+\Sigma^{J A B} \bar{\Sigma}_{B C}^{I}=-2 \delta^{I J} \delta_{C}^{A}, \quad \bar{\Sigma}_{A B}^{I} \Sigma^{J B C}+\bar{\Sigma}_{A B}^{J} \Sigma^{I B C}=-2 \delta^{I J} \delta_{A}^{C} . \tag{2.29}
\end{equation*}
$$

Proving the above involves a straightforward but lengthy calculation.
Similar to the four-dimensional case, we assign (4×4)-matrices to a vector $\phi_{I} \in \mathbb{R}^{6}$ by the prescription

$$
\begin{equation*}
\phi^{A B}:=\frac{1}{\sqrt{2}} \Sigma^{I A B} \phi^{I}, \quad \bar{\phi}_{A B}:=\frac{1}{\sqrt{2}} \bar{\Sigma}_{A B}^{I} \phi^{I} . \tag{2.30}
\end{equation*}
$$

These matrices are related by

$$
\bar{\phi}_{A B}=\frac{1}{2} \epsilon_{A B C D} \phi^{C D}, \quad \phi^{A B}=\frac{1}{2} \epsilon^{A B C D} \bar{\phi}_{C D}
$$

For the scalar product we find the following identity:

$$
\begin{equation*}
X^{A B} \bar{Y}_{A B}=\bar{X}_{A B} Y^{A B}=-2 X^{I} Y_{I} \tag{2.31}
\end{equation*}
$$

In our later calculations we will need the following identity for a unit vector $n_{I}$ satisfying $n^{I} n_{I}=-1$ :

$$
\begin{equation*}
\bar{n}_{A B} n^{C B}=\frac{1}{2} \delta_{A}^{C} . \tag{2.32}
\end{equation*}
$$

The proof of this can be reduced to using the Clifford relation (2.29),

$$
\begin{aligned}
\bar{n}_{A B} n^{C B} & =\frac{1}{2} \bar{\Sigma}_{A B}^{I} \Sigma^{J C B} n_{I} n_{J}=-\frac{1}{4}\left(\bar{\Sigma}_{A B}^{I} \Sigma^{J B C}+\bar{\Sigma}_{A B}^{J} \Sigma^{I B C}\right) n_{I} n_{J} \\
& =\frac{1}{2} \delta_{A}^{C} \delta^{I J} n_{I} n_{J}=\frac{1}{2} \delta_{A}^{C} .
\end{aligned}
$$

### 2.3 Ten-dimensional Minkowski Space

We consider $\mathbb{R}^{(1,9)}$ with the metric $\eta^{M N}=\operatorname{diag}(+,-, \ldots,-)$. A representation of the Clifford algebra can be constructed from the respective representations we have used for the six-dimensional Euclidean space and the four-dimensional Minkowski space. It takes the following form:

$$
\Gamma^{M}=\left\{\begin{array}{ll}
\mathbb{1}_{8} \otimes \gamma^{\mu} & \text { for } M=\mu \in\{0,1,2,3\} \\
\hat{\gamma}^{I} \otimes \gamma^{5} & \text { for } M=I+3 \in\{4,5,6,7,8,9\}
\end{array}, \quad \Gamma^{11}=\hat{\gamma}^{7} \otimes \gamma^{5} .\right.
$$

Using the Clifford relations in four and six dimensions, one easily shows that the above matrices satisfy $\left\{\Gamma^{M}, \Gamma^{N}\right\}=2 \eta^{M N}$. The spinors in $\mathbb{C}^{32} \simeq \mathbb{C}^{8} \otimes \mathbb{C}^{4}$ can be written in a similar way as

$$
\begin{equation*}
\xi=\binom{\xi^{A}}{\xi_{A}} \otimes\binom{\xi_{\alpha}}{\tilde{\xi}^{\dot{\alpha}}} . \tag{2.33}
\end{equation*}
$$

Note however, that this is not the most general form of an element of $\mathbb{C}^{32}$, as not every element of a tensor product space can be written as the tensor product of two vectors. To avoid writing linear combinations of vectors, we will write the spinors in the following form:

$$
\xi=\left(\xi_{\alpha}^{A}, \tilde{\xi}^{A \dot{\alpha}}, \xi_{A \alpha}, \tilde{\xi}_{A}^{\dot{\alpha}}\right)^{T}
$$

How to multiply the gamma matrices with these spinors can be read off from (2.33) and it is actually clear by the position of the indices. However, let us be very explicit and write out how an arbitrary matrix

$$
A:=\left(\begin{array}{cc}
A^{A}{ }_{B} & B^{A B} \\
C_{A B} & D_{A}{ }^{B}
\end{array}\right) \otimes\left(\begin{array}{cc}
a_{\alpha}{ }^{\beta} & b_{\alpha \dot{\beta}} \\
c^{\dot{\alpha} \beta} & d^{\dot{\alpha}}{ }_{\beta}
\end{array}\right)
$$

acts on such a spinor. From

$$
A\left[\binom{\xi^{B}}{\xi_{B}} \otimes\binom{\xi_{\beta}}{\xi^{\dot{\beta}}}\right]=\binom{A^{A}{ }_{B} \xi^{B}+B^{A B} \xi_{B}}{C_{A B} \xi^{B}+D_{A}{ }^{B} \xi_{B}} \otimes\binom{a_{\alpha}{ }^{\beta} \xi_{\beta}+b_{\alpha \dot{\beta}} \tilde{\xi}^{\dot{\beta}}}{c^{\alpha \beta} \xi_{\beta}+d^{\alpha} \tilde{\xi}_{\dot{\beta}} \tilde{\xi}^{\dot{\beta}}}
$$

we read off that

For the ten-dimensional $\mathcal{N}=1$ gauge theory, we will need Majorana-Weyl spinors in ten dimensions. A left-handed Weyl spinor satisfies the following condition:

$$
\Gamma^{11} \xi=\left(\xi_{\alpha}^{A},-\tilde{\xi}^{A \dot{\alpha}},-\xi_{A \alpha}, \tilde{\xi}_{A}^{\dot{\alpha}}\right)^{T}=\xi
$$

Thus, a left-handed Weyl-spinor has the form

$$
\xi=\left(\xi_{\alpha}^{A}, 0,0, \tilde{\xi}_{A}^{\dot{\alpha}}\right)^{T}
$$

For the Majorana condition we need to define a charge conjugation matrix $C_{1,9}$ satisfying $\Gamma^{M} C_{1,9}=-C_{1,9}\left(\Gamma^{M}\right)^{*}$. A possible choice is

$$
C_{1,9}=\left(\begin{array}{cc}
0 & \mathbb{1}_{4}  \tag{2.34}\\
\mathbb{1}_{4} & 0
\end{array}\right) \otimes C_{1,3}, \quad \text { where } \quad C_{1,3}=i \gamma^{2}=\left(\begin{array}{cc}
0 & -\epsilon \\
\epsilon & 0
\end{array}\right) .
$$

The Majorana condition $C_{1,9} \xi^{*}=\xi$ takes the following form for a left-handed Weyl spinor (note (2.1) for signs):

$$
\begin{equation*}
C_{1,9} \xi^{*}=\left(\epsilon_{\dot{\alpha} \dot{\beta}} \tilde{\xi}_{A}^{\dot{\beta}}, 0,0, \epsilon^{\alpha \beta} \xi_{\beta}^{A}\right)^{T}=\xi . \tag{2.35}
\end{equation*}
$$

This imposes the restrictions $\left(\tilde{\xi}_{A \dot{\alpha}}\right)^{*}=\xi_{\alpha}^{A}$ and $\left(\xi^{A \alpha}\right)^{*}=\tilde{\xi}_{A}^{\dot{\alpha}}$. The 32 complex degrees of freedom of a Dirac spinor in ten dimensions are hence reduced to 16 real degrees of freedom for a Majorana-Weyl spinor in ten dimensions.

The conjugate spinor is as usual defined by $\bar{\xi}:=\xi^{\dagger} \Gamma^{0}$. In order to perform the dimensional reduction from $\mathcal{N}=1$ gauge theory in ten dimensions to $\mathcal{N}=4$ gauge theory in four dimensions, we are interested in writing out the expression $\bar{\xi} \Gamma^{M} \Psi$ in terms of the components of $\xi$ and $\Psi$. We note that

$$
\xi^{\dagger}=\left(\tilde{\xi}_{A \dot{\alpha}}, 0,0, \xi^{A \alpha}\right)
$$

For $M=\mu \in\{0,1,2,3\}$ we have

$$
\Gamma^{0} \Gamma^{\mu} \Psi=\left(\mathbb{1}_{8} \otimes \gamma^{0} \gamma^{\mu}\right) \Psi=\left(\sigma^{\mu \dot{\alpha} \beta} \psi_{\beta}^{A}, 0,0, \bar{\sigma}_{\alpha \dot{\beta}}^{\mu} \tilde{\psi}_{A}^{\dot{\beta}}\right)^{T}
$$

Hence we have

$$
\begin{equation*}
\bar{\xi} \Gamma^{\mu} \Psi=\tilde{\xi}_{A \dot{\alpha}} \sigma^{\mu \dot{\alpha} \beta} \psi_{\beta}^{A}+\xi^{A \alpha} \bar{\sigma}_{\alpha \dot{\beta}}^{\mu} \tilde{\psi}_{A}^{\dot{\beta}} . \tag{2.36}
\end{equation*}
$$

For $M=I+3 \in\{4, \ldots, 9\}$, we get

$$
\Gamma^{0} \Gamma^{I+3} \Psi=\left(\hat{\gamma}^{I} \otimes \gamma^{0} \gamma^{5}\right) \Psi=\left(-\Sigma^{I A B} \tilde{\psi}_{B}^{\dot{\alpha}}, 0,0, \bar{\Sigma}_{A B}^{I} \psi_{\alpha}^{B}\right)
$$

from which we conclude that

$$
\begin{equation*}
\bar{\xi} \Gamma^{I+3} \Psi=-\tilde{\xi}_{A \dot{\alpha}} \Sigma^{I A B} \tilde{\psi}_{B}^{\dot{\alpha}}+\xi^{A \alpha} \bar{\Sigma}_{A B}^{I} \psi_{\alpha}^{B} . \tag{2.37}
\end{equation*}
$$

## Chapter 3

## Symmetries

Symmetry has served as an important guiding principle in the construction of new theories or objects as well as a powerful tool to perform calculations otherwise impossible. In this chapter, we discuss the different symmetry transformations and algebras which are applied in this thesis.

### 3.1 Conformal Symmetry

We begin by introducing the conformal transformations of Minkowski space $\mathbb{R}^{(1,3)}$. We focus on infinitesimal conformal transformations, deriving the Lie algebra $\mathfrak{s o}(2,4)$ of the conformal group, but also give the form of large conformal transformations and comment on the notion of the conformal group. The background for this concerning differential geometry may be found in [17], the discussion of the conformal group and algebra is based on [18] as well as 19.

The concept of conformal transformations enhances the notion of isometries. While an isometry preserves scalar products, conformal transformations change the scale leaving only the angle between two vectors invariant. As it will turn out that certain conformal transformations cannot be defined on the whole Minkowski space, we will consider an open subset $U \subset \mathbb{R}^{(1,3)}$ of Minkowski space in the definition of them. We then define a conformal transformation of Minkowski space to be a smooth map $f: U \rightarrow \mathbb{R}^{(1,3)}$ which satisfies $f^{*} \eta=e^{2 \sigma} \eta$, or equivalently

$$
\begin{equation*}
\frac{\partial f(x)^{\rho}}{\partial x^{\mu}} \frac{\partial f(x)^{\sigma}}{\partial x^{\nu}} \eta_{\rho \sigma}=e^{2 \sigma(x)} \eta_{\mu \nu} \quad \forall x \in U \tag{3.1}
\end{equation*}
$$

where $\sigma: U \rightarrow \mathbb{R}$ is a smooth function. In finding all conformal transformations it can be helpful to consider the vector fields which generate them. Vector fields generate transformations in the following sense: To each vector field we can assign integral curves, whose tangent vectors are given by the vector field. We will denote the maximal integral curve to the vector field $X$ through a given point $x_{0}$ by $\gamma_{x_{0}}^{X}(\tau)$. This curve satisfies

$$
\begin{equation*}
\gamma_{x_{0}}^{X}(0)=x_{0}, \quad \dot{\gamma}_{x_{0}}^{X}(\tau)=X\left(\gamma_{x_{0}}^{X}(\tau)\right) \quad \forall \tau \in I_{x_{0}} . \tag{3.2}
\end{equation*}
$$

Here, $I_{x_{0}}$ denotes the maximal interval to which the solution of the ordinary differential equation (3.2) can be extended. This allows us to define the flow which the vector field $X$ generates. The space on which the flow can be defined is related to the maximal intervals defined above,

$$
\mathcal{D}:=\left\{\left(\tau, x_{0}\right) \in \mathbb{R} \times \mathbb{R}^{(1,3)} \mid \tau \in I_{x_{0}}\right\} .
$$

The flow of the vector field $X$ is then the smooth map $\sigma^{X}: \mathcal{D} \rightarrow \mathbb{R}^{(1,3)}$ defined by

$$
\sigma^{X}\left(\tau, x_{0}\right)=\gamma_{x_{0}}^{X}(\tau), \quad \forall\left(\tau, x_{0}\right) \in \mathcal{D} .
$$

For fixed $\tau$ the flow gives rise to a diffeomorphism between open subsets of Minkowski space. Vector fields for which these diffeomorphisms are conformal transformations are called conformal Killing fields. One can show that these vector fields satisfy the conformal Killing equation $\mathcal{L}_{X} \eta=\frac{1}{2}\left(\partial_{\mu} X^{\mu}\right) \eta$, which has the following coordinate expression in Minkowski space:

$$
\partial_{\mu} X_{\nu}+\partial_{\nu} X_{\mu}=\frac{1}{2}\left(\partial_{\rho} X^{\rho}\right) \eta_{\mu \nu}
$$

On general grounds it is clear that the set of vector fields satisfying the conformal Killing equation forms a Lie algebra with the Lie bracket given by the vector field commutator. We will call this Lie algebra the conformal algebra.

Finding the most general solution of the conformal Killing equation allows to derive all conformal transformations of Minkowski space which are continuously connected to the identity. One can show that any conformal Killing field can be expressed in the following way:

$$
X(x)=\left(a^{\mu} P_{\mu}(x)+\frac{1}{2} \omega^{\mu \nu} M_{\mu \nu}(x)+s D(x)+c^{\mu} K_{\mu}(x)\right)
$$

Here, we defined the following basis of the conformal algebra:

$$
\begin{align*}
P_{\mu}(x) & =\partial_{\mu}, & M_{\mu \nu}(x) & =\left(x_{\mu} \partial_{\nu}-x_{\nu} \partial_{\mu}\right),  \tag{3.3}\\
D(x) & =x^{\mu} \partial_{\mu}, & K_{\mu}(x) & =\left(x^{2} \eta_{\mu \nu}-2 x_{\mu} x_{\nu}\right) \partial^{\nu} .
\end{align*}
$$

The vector fields $P^{\mu}$ and $M^{\mu \nu}$ generate translations and Lorentz transformations and form the Lie algebra of the Poincaré group. The conformal algebra is given by the following commutators:

$$
\begin{array}{rlrl}
{\left[M_{\mu \nu}, M_{\rho \sigma}\right]} & =\eta_{\mu \sigma} M_{\nu \rho}+\eta_{\nu \rho} M_{\mu \sigma}-\eta_{\mu \rho} M_{\nu \sigma}-\eta_{\nu \sigma} M_{\mu \rho} \\
{\left[P_{\mu}, P_{\nu}\right]} & =0 & {\left[M_{\mu \nu}, P_{\lambda}\right]} & =\eta_{\nu \lambda} P_{\mu}-\eta_{\mu \lambda} P_{\nu} \\
{\left[D, P_{\mu}\right]} & =-P_{\mu} & {\left[D, K_{\mu}\right]} & =K_{\mu}  \tag{3.5}\\
{\left[D, M_{\mu \nu}\right]} & =0 & {\left[M_{\mu \nu}, K_{\rho}\right]} & =\eta_{\nu \rho} K_{\mu}-\eta_{\mu \rho} K_{\nu} \\
{\left[P_{\mu}, K_{\nu}\right]} & =2 M_{\mu \nu}-2 \eta_{\mu \nu} D & {\left[K_{\mu}, K_{\nu}\right]} & =0
\end{array}
$$

One can show that the Lie algebra given by the above relations is isomorphic to the Lie algebra $\mathfrak{s o}(2,4)$. For this reason, we will refer to the conformal algebra as $\mathfrak{s o}(2,4)$.

We now turn to the discussion of large conformal transformations. The vectorfield $D$ generates dilatations, $x \mapsto e^{2 s} x$. The vector fields $K_{\mu}$ generate so-called special conformal transformations,

$$
\begin{equation*}
K_{c}(x)=\frac{x+x^{2} c}{1+2 c \cdot x+c^{2} x^{2}} . \tag{3.6}
\end{equation*}
$$

Both statements may be checked upon expanding the transformation for small $s$ or $c$ respectively. In dealing with special conformal transformations, it can be helpful to know that they may be expressed as the composition of two inversions and a translation. The inversion map is given by

$$
S(x)=\frac{x}{x^{2}} .
$$

The special conformal transformation $K_{c}$ can then be expressed as

$$
K_{c}=S \circ T_{c} \circ S,
$$

where $T_{c}$ denotes a translation by $c$. From (3.6) we infer that special conformal transformations become singular at a light cone centered at $-c / c^{2}$. Hence they are not well-defined on the whole Minkowski space. To have globally well-defined conformal transformations, one can define a conformal compactification of Minkowski space, on which the conformal transformations can be analytically continued to become diffeomorphisms. One can show that the group of diffeomorphisms so defined is isomorphic to the group $S O(2,4) /\{ \pm \mathbb{1}\}$. For details concerning this construction the reader is referred to [18].

## $3.2 \boldsymbol{N}=4$ Superconformal Symmetry

In this section, we discuss the $\mathcal{N}=4$ supersymmetric extension of the conformal algebra $\mathfrak{s o}(2,4)$. For a general introduction to supersymmetry, the reader is referred to [20] or [21], which gives many explicit calculations. A more formal introduction to Lie super algebras may be found in [22]. An interesting account of superconformal algebras can be found in [23] as well as [24].

The commutation relations of the superconformal algebra can be written in a more compact way, if we write the generators of the conformal algebra as bispinors,

$$
P_{\alpha \dot{\alpha}}=\bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} P_{\mu}, \quad K_{\alpha \dot{\alpha}}=\bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} K_{\mu}, \quad M_{\alpha \beta}=\sigma_{\alpha \beta}^{\mu \nu} M_{\mu \nu}, \quad \bar{M}_{\dot{\alpha} \dot{\beta}}=\bar{\sigma}_{\dot{\alpha} \dot{\beta}}^{\mu \nu} M_{\mu \nu}
$$

The commutation relations of these generators are given by:

$$
\begin{align*}
{\left[M_{\alpha \beta}, M_{\gamma \delta}\right] } & =-2 i\left(\epsilon_{\alpha \gamma} M_{\beta \delta}+\epsilon_{\alpha \delta} M_{\beta \gamma}+\epsilon_{\beta \gamma} M_{\alpha \delta}+\epsilon_{\beta \delta} M_{\alpha \gamma}\right) \\
{\left[\bar{M}_{\dot{\alpha} \dot{\beta}}, \bar{M}_{\dot{\gamma} \dot{\delta}}\right] } & =2 i\left(\epsilon_{\dot{\alpha} \dot{\gamma}} \bar{M}_{\dot{\beta} \dot{\delta}}+\epsilon_{\dot{\alpha} \dot{\delta}} \bar{M}_{\dot{\beta} \dot{\gamma}}+\epsilon_{\dot{\beta} \dot{\gamma}} \bar{M}_{\dot{\alpha} \dot{\delta}}+\epsilon_{\dot{\beta} \dot{\delta}} \bar{M}_{\dot{\alpha} \dot{\gamma}}\right) \\
{\left[M_{\alpha \beta}, \bar{M}_{\dot{\alpha} \dot{\beta}}\right] } & =0 \quad\left[P_{\alpha \dot{\alpha}}, P_{\beta \dot{\beta}}\right]=0 \quad\left[K_{\alpha \dot{\alpha}}, K_{\beta \dot{\beta}}\right]=0 \\
{\left[M_{\alpha \beta}, P_{\gamma \dot{\gamma}}\right] } & =-2 i\left(\epsilon_{\alpha \gamma} P_{\beta \dot{\gamma}}+\epsilon_{\beta \gamma} P_{\alpha \dot{\gamma}}\right) \quad\left[M_{\alpha \beta}, K_{\gamma \dot{\gamma}}\right]=-2 i\left(\epsilon_{\alpha \gamma} K_{\beta \dot{\gamma}}+\epsilon_{\beta \gamma} K_{\alpha \dot{\gamma}}\right)  \tag{3.7}\\
{\left[\bar{M}_{\dot{\alpha} \dot{\beta}}, P_{\gamma \dot{\gamma}}\right] } & =-2 i\left(\epsilon_{\dot{\alpha} \dot{\gamma}} P_{\gamma \dot{\beta}}+\epsilon_{\dot{\beta} \dot{\gamma}} P_{\gamma \dot{\alpha}}\right) \quad\left[\bar{M}_{\dot{\alpha} \dot{\beta}}, K_{\gamma \dot{\gamma}}\right]=-2 i\left(\epsilon_{\dot{\alpha} \dot{\gamma}} K_{\gamma \dot{\beta}}+\epsilon_{\dot{\beta} \dot{\gamma}} K_{\gamma \dot{\alpha}}\right) \\
{\left[P_{\alpha \dot{\alpha}}, K_{\beta \dot{\beta}}\right] } & =i \epsilon_{\alpha \beta} \bar{M}_{\dot{\alpha} \dot{\beta}}+i \epsilon_{\dot{\alpha} \dot{\beta}} M_{\alpha \beta}+4 \epsilon_{\alpha \beta} \epsilon_{\dot{\alpha} \dot{\beta}} D
\end{align*}
$$

The dilatation generator $D$ satisfies the commutation relations

$$
[D, J]=\operatorname{dim}(J) J,
$$

thus assigning dimensions to the other generators of the conformal group. We have

$$
\begin{equation*}
\operatorname{dim}(K)=1, \quad \operatorname{dim}(M)=\operatorname{dim}(\bar{M})=0, \quad \operatorname{dim}(P)=-1 \tag{3.8}
\end{equation*}
$$

The conformal algebra is augmented by sixteen supercharges $Q_{A}^{\alpha}$ and $\bar{Q}^{A \dot{\alpha}}$, which satisfy the key commutation relation of supersymmetry ${ }^{11}$

$$
\begin{equation*}
\left\{Q_{A}^{\alpha}, \bar{Q}^{B \dot{\alpha}}\right\}=2 i \delta_{A}^{B} P^{\alpha \dot{\alpha}} \tag{3.9}
\end{equation*}
$$

We consider an extension without central charges $Z^{A B}$ :

$$
\begin{equation*}
\left\{Q_{A}^{\alpha}, Q_{B}^{\beta}\right\}=0, \quad\left\{\bar{Q}^{A \dot{\alpha}}, \bar{Q}^{B \dot{\beta}}\right\}=0 . \tag{3.10}
\end{equation*}
$$

The supersymmetry generators are spinors transforming in the left- or right-handed representation of $S L(2, \mathbb{C})$ respectively,

$$
\begin{array}{lll}
{\left[M_{\alpha \beta}, Q_{A \gamma}\right]} & =-2 i\left(\epsilon_{\alpha \gamma} Q_{A \beta}+\epsilon_{\beta \gamma} Q_{A \alpha}\right), & {\left[\bar{M}_{\dot{\alpha} \dot{\beta}}, Q_{A \gamma}\right]=0,} \\
{\left[\bar{M}_{\dot{\alpha} \dot{\beta}}, \bar{Q}_{\dot{\gamma}}^{A}\right]} & =-2 i\left(\epsilon_{\dot{\alpha} \dot{\gamma}} \bar{Q}_{\dot{\beta}}^{A}+\epsilon_{\dot{\beta} \dot{\gamma}} \bar{Q}_{\dot{\alpha}}^{A}\right), & {\left[M_{\alpha \beta}, \bar{Q}_{\dot{\gamma}}^{A}\right]=0 .} \tag{3.12}
\end{array}
$$

Furthermore, they have half the dimension of $P$,

$$
\begin{equation*}
\operatorname{dim}(Q)=\operatorname{dim}(\bar{Q})=-\frac{1}{2} \tag{3.13}
\end{equation*}
$$

Additionally there are sixteen superconformal generators $S_{\alpha}^{A}$ and $\bar{S}_{A \dot{\alpha}}$, which follow from the commutation relations

$$
\begin{equation*}
\left[K_{\alpha \dot{\alpha}}, Q_{A}^{\beta}\right]=2 \delta_{\alpha}^{\beta} \bar{S}_{A \dot{\alpha}}, \quad\left[K_{\alpha \dot{\alpha}}, \bar{Q}^{A \dot{\beta}}\right]=2 \delta_{\dot{\alpha}}^{\dot{\beta}} S_{\alpha}^{A} . \tag{3.14}
\end{equation*}
$$

[^3]They satisfy a relation analogous to (3.9):

$$
\begin{equation*}
\left\{S_{\alpha}^{A}, \bar{S}_{B \dot{\alpha}}\right\}=-2 i \delta_{B}^{A} K_{\alpha \dot{\alpha}} \tag{3.15}
\end{equation*}
$$

Thus they have half the dimension of $K$,

$$
\begin{equation*}
\operatorname{dim}(S)=\operatorname{dim}(\bar{S})=\frac{1}{2} \tag{3.16}
\end{equation*}
$$

Also the superconformal generators $S$ and $\bar{S}$ transform in the respective representation of $S L(2, \mathbb{C})$,

$$
\begin{align*}
{\left[M_{\alpha \beta}, S_{\gamma}^{A}\right] } & =-2 i\left(\epsilon_{\alpha \gamma} S_{\beta}^{A}+\epsilon_{\beta \gamma} S_{\alpha}^{A}\right) & {\left[\bar{M}_{\dot{\alpha} \dot{\beta}}, S_{\gamma}^{A}\right] } & =0  \tag{3.17}\\
{\left[\bar{M}_{\dot{\alpha} \dot{\beta}}, \bar{S}_{A \dot{\gamma}}\right] } & =-2 i\left(\epsilon_{\dot{\alpha} \dot{\gamma}} \bar{S}_{A \dot{\beta}}+\epsilon_{\dot{\beta} \dot{\gamma}} \bar{S}_{A \dot{\alpha}}\right) & & {\left[M_{\alpha \beta}, \bar{S}_{A \dot{\gamma}}\right] } \tag{3.18}
\end{align*}
$$

Furthermore, we have the commutation relations:

$$
\begin{array}{rlrl}
{\left[P^{\alpha \dot{\alpha}}, S_{\beta}^{A}\right]} & =2 \delta_{\beta}^{\alpha} \bar{Q}^{A \dot{\alpha}} & {\left[P^{\alpha \dot{\alpha}}, \bar{S}_{A \dot{\beta}}\right]=2 \delta_{\dot{\beta}}^{\dot{\alpha}} Q_{A}^{\alpha}} \\
\left\{\bar{Q}^{A \dot{\alpha}}, S_{\alpha}^{B}\right\} & =0 & \left\{Q_{A}^{\alpha}, \bar{S}_{B \dot{\alpha}}\right\}=0 \\
\left\{Q_{A}^{\alpha}, S_{\beta}^{B}\right\} & =\delta_{A}^{B} M_{\beta}^{\alpha}+\delta_{\beta}^{\alpha} R_{A}^{B}+2 i \delta_{A}^{B} \delta_{\beta}^{\alpha}(D+C) \\
\left\{\bar{Q}^{A \dot{\alpha}}, \bar{S}_{B \dot{\beta}}\right\} & =-\delta_{B}^{A} \bar{M}^{\dot{\alpha}}{ }_{\dot{\beta}}-\delta_{\dot{\beta}}^{\dot{\alpha}} R_{B}^{A}+2 i \delta_{B}^{A} \delta_{\dot{\beta}}^{\dot{\alpha}}(D-C) \tag{3.22}
\end{array}
$$

Here, we introduced the central charge $C$, which commutes with all generators of the superconformal algebra, and the $R$-symmetry generators $R_{B}^{A}$. There are only 15 linearly independent $R$-symmetry generators as $\sum_{A} R_{A}^{A}=0$. They generate the $R$-symmetry group $S U(4)$, which transforms the different supercharges into another. The generators satisfy the following canonical commutation relations:

$$
\begin{array}{rlr}
{\left[R_{B}^{A}, Q_{C}^{\alpha}\right]} & =4 i \delta_{C}^{A} Q_{B}^{\alpha}-i \delta_{B}^{A} Q_{C}^{\alpha} & {\left[R_{B}^{A}{ }_{B} \bar{Q}^{C \dot{\alpha}}\right]=-4 i \delta_{B}^{C} \bar{Q}^{A \dot{\alpha}}+i \delta_{B}^{A} \bar{Q}^{C \dot{\alpha}}} \\
{\left[R_{B}^{A}, \bar{S}_{C \dot{\alpha}}\right]} & =4 i \delta_{C}^{A} \bar{S}_{B \dot{\alpha}}-i \delta_{B}^{A} \bar{S}_{C \dot{\alpha}} & {\left[R^{A}{ }_{B}, S_{\alpha}^{C}\right]=-4 i \delta_{B}^{C} S_{\alpha}^{A}+i \delta_{B}^{A} S_{\alpha}^{C}} \\
{\left[R_{B}^{A}, R_{D}^{C}\right]} & =4 i \delta_{D}^{A} R_{B}^{C}-4 i \delta_{B}^{C} R_{D}^{A} & \tag{3.25}
\end{array}
$$

In a super Poincaré algebra, the $R$-symmetry is an outer automorphism of the algebra, which may or may not be a symmetry of the action. For a superconformal algebra however, the relation (3.21) shows that the generators of the $R$-symmetry are elements of the algebra and hence they have to be symmetries of the action as well [23].

The $\mathcal{N}=4$ superconformal algebra described here is called $\mathfrak{s u}(2,2 \mid 4)$. As the central charge $C$ commutes with all generators, it is possible to consider representations where $C=0$. In this case, the resulting Lie super algebra is called $\mathfrak{p s u}(2,2 \mid 4)$. For a representation of the superconformal algebra in terms of complex (4|4) supermatrices, the reader is referred to 25]. The supermatrix representation also clarifies the naming of the algebra.

In the discussion of the supersymmetric extension of the Maldacena-Wilson loop, we will need a representation of the superconformal algebra in terms of differential operators acting in a non-chiral superspace spanned by the variables

$$
\left(x_{\alpha \dot{\alpha}}, \theta_{\alpha}^{A}, \bar{\theta}_{A \dot{\alpha}}\right) .
$$

For the Graßmann odd coordinates $\theta_{\alpha}^{A}$ and $\bar{\theta}_{A \dot{\alpha}}$ we define the following derivatives:

$$
\begin{equation*}
\frac{\partial}{\partial \theta_{\alpha}^{A}} \theta_{\beta}^{B}=\delta_{A}^{B} \delta_{\beta}^{\alpha}, \quad \frac{\partial}{\partial \bar{\theta}_{A \dot{\alpha}}} \bar{\theta}_{B \dot{\beta}}=\delta_{B}^{A} \delta_{\dot{\beta}}^{\dot{\alpha}} . \tag{3.26}
\end{equation*}
$$

As before, we define the derivatives to obey the same transformation rules as the other spinors, which is reflected in the following derivatives:

$$
\begin{equation*}
\frac{\partial}{\partial \theta^{A \alpha}} \theta^{B \beta}=-\delta_{A}^{B} \delta_{\alpha}^{\beta}, \quad \frac{\partial}{\partial \bar{\theta}_{A}^{\dot{\alpha}}} \bar{\theta}_{B}^{\dot{B}}=-\delta_{B}^{A} \delta_{\dot{\alpha}}^{\dot{\beta}} . \tag{3.27}
\end{equation*}
$$

A representation of the superconformal algebra in a chiral superspace has been given in [3]. However, when extending this representation to include the variables $\bar{\theta}_{A \dot{\alpha}}$ one finds it impossible to obtain the commutation relations of the superconformal algebra. Rather, we need to introduce another bosonic variable $y_{A}{ }^{B}$, where $A$ and $B$ take values in $\{1,2,3,4\}$. For this, we define the derivative

$$
\begin{equation*}
\frac{\partial y_{A}^{B}}{\partial y_{C}{ }^{D}}=\delta_{A}^{C} \delta_{D}^{B} \tag{3.28}
\end{equation*}
$$

We will now give the explicit form of the generators of the superconformal algebra satisfying the commutation relations stated above. To write the generators in a more compact form, we introduce the following abbreviations:

$$
\partial_{A}^{\alpha}=\frac{\partial}{\partial \theta_{\alpha}^{A}}, \quad \partial^{A \dot{\alpha}}=\frac{\partial}{\partial \bar{\theta}_{A \dot{\alpha}}}, \quad \partial_{B}{ }^{A}=\frac{\partial}{\partial y_{A}{ }^{B}} .
$$

The generators of the superconformal algebra $\mathfrak{s u}(2,2 \mid 4)$ are then given by:

$$
\begin{align*}
& M_{\alpha \beta}=2 i x_{\dot{\gamma}(\alpha} \partial_{\beta)}{ }^{\dot{\gamma}}+4 i \theta_{(\alpha}^{A} \partial_{\beta) A} \quad \bar{M}_{\dot{\alpha} \dot{\beta}}=2 i x^{\gamma}{ }_{(\dot{\alpha} \dot{\beta}) \gamma} \partial_{\dot{\beta}}-4 i \bar{\theta}_{A(\dot{\alpha}} \partial_{\dot{\beta})}^{A}  \tag{3.29}\\
& D=\frac{1}{2} x_{\alpha \dot{\alpha}} \partial^{\alpha \dot{\alpha}}+\frac{1}{2} \theta_{\beta}^{B} \partial_{B}^{\beta}+\frac{1}{2} \bar{\theta}_{B \dot{\beta}} \partial^{B \dot{\beta}} \quad P^{\alpha \dot{\alpha}}=\partial^{\alpha \dot{\alpha}}  \tag{3.30}\\
& K_{\alpha \dot{\alpha}}=-x_{\alpha \dot{\gamma}} x_{\dot{\alpha} \gamma} \partial^{\gamma \dot{\gamma}}-2 x_{\dot{\alpha} \gamma} \theta_{\alpha}^{C} \partial_{C}^{\gamma}-2 x_{\alpha \dot{\gamma}} \bar{\theta}_{C \dot{\alpha}} \partial^{C \dot{\gamma}}+4 i \theta_{\alpha}^{A} \bar{\theta}_{B \dot{\alpha}} \partial_{A}{ }^{B}  \tag{3.31}\\
& Q_{A}^{\alpha}=-\partial_{A}^{\alpha}+y_{A}{ }^{B} \partial_{B}^{\alpha}+i \bar{\theta}_{A \dot{\alpha}} \partial^{\alpha \dot{\alpha}} \quad \bar{Q}^{A \dot{\alpha}}=\partial^{A \dot{\alpha}}+y_{B}{ }^{A} \partial^{B \dot{\alpha}}-i \theta_{\alpha}^{A} \partial^{\alpha \dot{\alpha}}  \tag{3.32}\\
& S_{\alpha}^{A}=\left(\delta_{B}^{A}+y_{B}{ }^{A}\right)\left(x_{\alpha \dot{\gamma}} \partial^{B \dot{\gamma}}+2 i \theta_{\alpha}^{C} \partial_{C}{ }^{B}\right)-i x_{\alpha \dot{\gamma}} \theta_{\beta}^{A} \partial^{\beta \dot{\gamma}}-2 i \theta_{\beta}^{A} \theta_{\alpha}^{C} \partial_{C}^{\beta}  \tag{3.33}\\
& \bar{S}_{A \dot{\alpha}}=\left(-\delta_{A}^{B}+y_{A}{ }^{B}\right)\left(x_{\dot{\alpha} \gamma} \partial_{B}^{\gamma}-2 i \bar{\theta}_{D \dot{\alpha}} \partial_{B}{ }^{D}\right)+i x_{\dot{\alpha} \gamma} \bar{\theta}_{A \dot{\beta}} \partial^{\gamma \dot{\beta}}+2 i \bar{\theta}_{A \dot{\beta}} \bar{\theta}_{C \dot{\alpha}} \partial^{C \dot{\beta}}  \tag{3.34}\\
& R^{\prime A}{ }_{B}=2 i\left(-\delta_{B}^{D}+y_{B}{ }^{D}\right)\left(\delta_{C}^{A}+y_{C}{ }^{A}\right) \partial_{D}{ }^{C}+2 i\left(-\delta_{B}^{C}+y_{B}{ }^{C}\right) \theta_{\gamma}^{A} \delta_{C}^{\gamma} \\
& +2 i\left(\delta_{C}^{A}+y_{C}{ }^{A}\right) \bar{\theta}_{B \dot{\alpha}} \partial^{C \dot{\alpha}}+2 \bar{\theta}_{B \dot{\alpha}} \theta_{\alpha}^{A} \partial^{\alpha \dot{\alpha}} \\
& R^{A}{ }_{B}=R^{\prime A}{ }_{B}-\frac{1}{4} \delta_{B}^{A} R^{\prime C}{ }_{C}  \tag{3.35}\\
& C=\frac{1}{4}\left(\theta_{\alpha}^{D} \partial_{D}^{\alpha}-\bar{\theta}_{C \dot{\alpha}} \partial^{C \dot{\alpha}}+i \theta_{\alpha}^{A} \bar{\theta}_{A \dot{\alpha}} \partial^{\alpha \dot{\alpha}}-\partial_{A}{ }^{A}\right. \\
& \left.+y_{A}{ }^{B} \theta_{\alpha}^{A} \partial_{B}^{\alpha}+y_{A}{ }^{B} \bar{\theta}_{B \dot{\alpha}} \partial^{A \dot{\alpha}}+y_{A}{ }^{C} y_{C}{ }^{B} \partial_{B}{ }^{A}\right) \tag{3.36}
\end{align*}
$$

### 3.3 Yangian Symmetry

The Yangian $Y(\mathfrak{g})$ over a simple Lie algebra $\mathfrak{g}$ was first introduced by Drinfeld in [26]. In this chapter we will introduce it in a way which is known as Drinfeld's first realization of the Yangian. Our discussion is based on [27] as well as [28]. In appendix B. we provide the algebraic preliminaries which are needed in this section.

In the construction of the Yangian algebra, we start with a simple Lie algebra $\mathfrak{g}$ spanned by the generators $\left\{J_{a}^{(0)}\right\}$ satisfying

$$
\begin{equation*}
\left[J_{a}^{(0)}, J_{b}^{(0)}\right]=f_{a b}^{c} J_{c}^{(0)} . \tag{3.37}
\end{equation*}
$$

Additionally, we consider a second set of generators $\left\{J_{a}^{(1)}\right\}, a=1, \ldots, \operatorname{dim}(\mathfrak{g})$ and we define commutation relations between the $J_{a}^{(0)}$ and the $J_{a}^{(1)}$ :

$$
\begin{equation*}
\left[J_{a}^{(0)}, J_{b}^{(1)}\right]=f_{a b}{ }^{c} J_{c}^{(1)} . \tag{3.38}
\end{equation*}
$$

If one interprets the above commutator as the definition of a representation of $\mathfrak{g}$ on $\operatorname{span}\left\{J_{a}^{(1)}\right\}$, one reads off from (3.38) that the $\left\{J_{a}^{(1)}\right\}$ transform in the adjoint representation of $\mathfrak{g}$. Note that we have not defined a Lie algebra structure on $\mathfrak{g} \oplus \operatorname{span}\left\{J_{a}^{(1)}\right\}$ as we did not specify commutation relations between the $\left\{J_{a}^{(1)}\right\}$. The Yangian $Y(\mathfrak{g})$ is now defined to be the enveloping algebra of $\mathfrak{g} \oplus \operatorname{span}\left\{J_{a}^{(1)}\right\}$, such that

$$
\begin{align*}
& {\left[J_{a}^{(0)}, J_{b}^{(0)}\right]=f_{a b}{ }^{c} J_{c}^{(0)},}  \tag{3.39}\\
& {\left[J_{a}^{(1)},\left[J_{b}^{(1)}, J_{c}^{(0)}\right]\right]-\left[J_{a}^{(0)},\left[J_{b}^{(1)}, J_{c}^{(1)}\right]\right]=f_{a b}{ }^{c} J_{c}^{(1)},}  \tag{3.40}\\
& \alpha^{2} a_{a b c}{ }^{\operatorname{deg}}\left\{J_{d}^{(0)}, J_{e}^{(0)}, J_{g}^{(0)}\right\} .
\end{align*}
$$

The last equation is sometimes referred to as the Serre relation. For $\mathfrak{g}=\mathfrak{s u}(2)$ it is trivially satisfied and needs to be replaced by another relation, which is otherwise implied, see [27]. Let us explain the notation in the above equations. The enveloping algebra $Y(\mathfrak{g})$ is an associative algebra with unit and we omit writing out the algebra product explictly,

$$
\begin{equation*}
m_{Y(\mathfrak{g})}(x \otimes y)=: x y \quad \text { for } x, y \in Y(\mathfrak{g}) . \tag{3.41}
\end{equation*}
$$

Also, we define the commutator on $Y(\mathfrak{g})$ as usual by $[x, y]:=x y-y x$. Furthermore, we have a one-to-one map $i: \mathfrak{g} \oplus \operatorname{span}\left\{J_{a}^{(1)}\right\} \rightarrow Y(\mathfrak{g})$ and to shorten our notation, we write $J_{a}^{(0)}=i\left(J_{a}^{(0)}\right), J_{a}^{(1)}=i\left(J_{a}^{(1)}\right)$. Equation (3.39) is then just the statement, that the enveloping algebra should be such that $i$ preserves the structure we have defined on $\mathfrak{g} \oplus \operatorname{span}\left\{J_{a}^{(1)}\right\}$. In (3.40), we used the following definitions ${ }^{2}$;

$$
\begin{align*}
a_{a b c}{ }^{d e g}:= & f_{a r}{ }^{d} f_{b s}{ }^{e} f_{c t}{ }^{g} f^{r s t},  \tag{3.42}\\
\left\{J_{d}^{(0)}, J_{e}^{(0)}, J_{g}^{(0)}\right\}:= & J_{d}^{(0)} J_{e}^{(0)} J_{g}^{(0)}+J_{e}^{(0)} J_{g}^{(0)} J_{d}^{(0)}+J_{g}^{(0)} J_{d}^{(0)} J_{e}^{(0)} \\
& +J_{e}^{(0)} J_{d}^{(0)} J_{g}^{(0)}+J_{d}^{(0)} J_{g}^{(0)} J_{e}^{(0)}+J_{g}^{(0)} J_{e}^{(0)} J_{d}^{(0)} . \tag{3.43}
\end{align*}
$$

[^4]The constant $\alpha \in \mathbb{C}$ is arbitrary.
One can define a Hopf algebra structure on $Y(\mathfrak{g})$ by setting

$$
\begin{align*}
\Delta\left(J_{a}^{(0)}\right) & =J_{a}^{(0)} \otimes \mathbb{1}+\mathbb{1} \otimes J_{a}^{(0)}, & & \varepsilon\left(J_{a}^{(0)}\right)=0,  \tag{3.44}\\
\Delta\left(J_{a}^{(1)}\right) & =J_{a}^{(1)} \otimes \mathbb{1}+\mathbb{1} \otimes J_{a}^{(1)}+\frac{\alpha}{2} f_{a}{ }^{c b} J_{b}^{(0)} \otimes J_{c}^{(0)}, & & \varepsilon\left(J_{a}^{(1)}\right)=0, \\
s\left(J_{a}^{(1)}\right) & =-J_{a}^{(1)}+\frac{\alpha}{2} f_{a}{ }^{c b} J_{b}^{(0)} J_{c}^{(0)}, & & s\left(J_{a}^{(0)}\right)=-J_{a}^{(0)} . \tag{3.45}
\end{align*}
$$

Additionally, we set

$$
\begin{equation*}
\Delta(\mathbb{1})=\mathbb{1} \otimes \mathbb{1}, \quad \varepsilon(\mathbb{1})=1, \quad s(\mathbb{1})=\mathbb{1} . \tag{3.47}
\end{equation*}
$$

At this point, we have only defined how $\Delta, \varepsilon$ and $s$ act on the generators $J_{a}^{(0)}$ and $J_{a}^{(1)}$. For elements of $Y(\mathfrak{g})$ which are given as linear combinations of products of the $J_{a}^{(0)}$ and $J_{a}^{(1)}$, the action of coproduct, counit and antipode are defined by the requirement that these maps be linear algebra morphisms. The constraints (3.39) and (3.40) are designed in such a way that this can be done consistently, see also [29]. In [30] it is shown in detail that when (3.39) are enforced, we also need to require (3.40) in order for the coproduct to become an algebra morphism. If one has established, that both $\Delta$ and $\varepsilon$ are algebra morphisms, it is sufficient to check the defining properties of a Hopf algebra on a set that generates this Hopf algebra, in our case $\left\{J_{a}^{(0)}, J_{a}^{(1)}\right\}$. It is then easy to go through the properties listed in appendix B. The antipode $s$ should satisfy the relation

$$
\begin{equation*}
(m \circ(\mathrm{id} \otimes s) \circ \Delta)\left(J_{a}^{(1)}\right)=(u \circ \varepsilon)\left(J_{a}^{(1)}\right)=0 \tag{3.48}
\end{equation*}
$$

This can easily be seen from an explicit calculation using the definitions (3.44) - 3.46):

$$
\begin{aligned}
(m \circ(\mathrm{id} \otimes s) \circ \Delta)\left(J_{a}^{(1)}\right) & =(m \circ(\mathrm{id} \otimes s))\left(J_{a}^{(1)} \otimes \mathbb{1}+\mathbb{1} \otimes J_{a}^{(1)}+\frac{\alpha}{2} f_{a}{ }^{c b} J_{b}^{(0)} \otimes J_{c}^{(0)}\right) \\
& =m\left(J_{a}^{(1)} \otimes \mathbb{1}-\mathbb{1} \otimes J_{a}^{(1)}+\frac{\alpha}{2} f_{a}{ }^{c b} \mathbb{1} \otimes J_{b}^{(0)} J_{c}^{(0)}-\frac{\alpha}{2} f_{a}{ }^{c b} J_{b}^{(0)} \otimes J_{c}^{(0)}\right) \\
& =0 .
\end{aligned}
$$

The coassociativity of the coproduct is also checked easily:

$$
\begin{aligned}
((\Delta \otimes \mathrm{id}) \circ \Delta)\left(J_{a}^{(1)}\right)= & (\Delta \otimes \mathrm{id})\left(J_{a}^{(1)} \otimes \mathbb{1}+\mathbb{1} \otimes J_{a}^{(1)}+\frac{\alpha}{2} f_{a}{ }^{c b} J_{b}^{(0)} \otimes J_{c}^{(0)}\right) \\
= & J_{a}^{(1)} \otimes \mathbb{1} \otimes \mathbb{1}+\mathbb{1} \otimes J_{a}^{(1)} \otimes \mathbb{1}+\mathbb{1} \otimes \mathbb{1} \otimes J_{a}^{(1)} \\
& +\frac{\alpha}{2} f_{a}^{c b}\left(J_{b}^{(0)} \otimes \mathbb{1} \otimes J_{c}^{(0)}+\mathbb{1} \otimes J_{b}^{(0)} \otimes J_{c}^{(0)}+J_{b}^{(0)} \otimes J_{c}^{(0)} \otimes \mathbb{1}\right) \\
= & ((\mathrm{id} \otimes \Delta) \circ \Delta)\left(J_{a}\right) .
\end{aligned}
$$

The other relations can be checked similarly.
The Yangian $Y(\mathfrak{g})$ can be thought of as a graded algebra spanned by a set of generators $\left\{J_{a}^{(0)}, J_{a}^{(1)}, J_{a}^{(2)}, \ldots\right\}$. The superscripts denote the level of the generators,

$$
\operatorname{deg}\left(J_{a}^{(0)}\right)=0, \quad \operatorname{deg}\left(J_{a}^{(1)}\right)=1
$$

Higher-grade elements may be constructed from the commutators of lower-grade elements. In this regard, (3.40) can be viewed as a constraint on the construction of them. If we define a grade-two element by

$$
J_{a}^{(2)}:=\frac{1}{c_{A}} f_{a}{ }^{b c}\left[J_{c}^{(1)}, J_{d}^{(1)}\right],
$$

the commutator of the level-1 generators can be reexpressed as

$$
\left[J_{b}^{(1)}, J_{c}^{(1)}\right]=f_{b c}{ }^{d} J_{d}^{(2)}+X_{b c}, \quad \text { where } \quad f_{a}{ }^{b c} X_{b c}=0
$$

Imposing (3.40) then uniquely determines $X_{b c}$, see [27].
Yangian symmetries have been encountered in many different contexts. In the case of multi-site spaces, the level zero and one generators are given by ${ }^{3}$ :

$$
\begin{align*}
J_{a}^{(0)} & =\sum_{i=1}^{n} J_{a, i},  \tag{3.49}\\
J_{a}^{(1)} & =\sum_{i<j} f_{a}^{b c} J_{c, i} J_{b, j}+\sum_{k=1}^{n} c_{k} J_{a, k}, \tag{3.50}
\end{align*}
$$

Here, the $J_{a, i}$ form a representation of the Lie algebra $\mathfrak{g}$. The Yangian symmetry generators for tree-level scattering amplitudes [3] are of this type. The generators we will consider can be thought of as a continuum limit of the above generators. Formally, they are more similar to the Yangian symmetry generators encountered in the study of two-dimensional integrable field theories, which have the form

$$
\begin{equation*}
J_{a}^{(0)}=\int_{-\infty}^{\infty} \mathrm{d} x j_{0 a}(x), \quad J_{a}^{(1)}=\int_{-\infty}^{\infty} \mathrm{d} x j_{1 a}(x)+f_{a}^{b c} \int_{x<y} \mathrm{~d} x \mathrm{~d} y j_{0 c}(x) j_{0 b}(y) . \tag{3.51}
\end{equation*}
$$

Here, $j_{\mu}(t, x), \mu=0,1$, is a Lie algebra valued current with components $j_{\mu a}, j_{\mu}=j_{\mu a} T^{a}$, which satisfies

$$
\begin{equation*}
\partial^{\mu} j_{\mu}=0, \quad \partial_{0} j_{1}-\partial_{1} j_{0}+\left[j_{0}, j_{1}\right]=0 \tag{3.52}
\end{equation*}
$$

More details can be found in [27] and references therein.

[^5]
## Chapter 4

## $\mathcal{N}=4$ Supersymmetric Yang-Mills Theory

In this chapter, we introduce $\mathcal{N}=4$ supersymmetric Yang-Mills theory, which is interesting in many respects: It is the most symmetric four-dimensional gauge theory possessing $\mathcal{N}=4$ superconformal symmetry also at the quantum level. A theory with $\mathcal{N}>4$ would necessarily contain particles with spin $s>1$, for which no renormalizable theory is known. In this sense, the degree of supersymmetry is maximal for $\mathcal{N}=4$ SYM. It is also the corresponding gauge theory in the most prominent example of the AdS/CFT correspondence, which connects it to a type IIB superstring theory on $A d S_{5} \times S^{5}$.

We derive the action of $\mathcal{N}=4$ SYM by considering a dimensional reduction from a ten-dimensional $\mathcal{N}=1$ supersymmetric Yang-Mills theory to four dimensions and briefly describe the symmetries of the action. This way of deriving the action of $\mathcal{N}=4$ SYM was first described in [32]. The Feynman rules of $\mathcal{N}=4$ SYM will not be needed in this thesis, as the computation of the expectation value of the MaldacenaWilson loop to first order in perturbation theory only requires the knowledge of the propagators. We derive them in section 4.2 .

As the Yangian symmetry of scattering amplitudes is a key motivation for this thesis, we will shortly discuss them in section 4.3 . This is also of interest for us because the Yangian generators found for the Maldacena-Wilson loop are structurally similar to those for the scattering amplitudes.

### 4.1 Derivation of the Action

The field content of $\mathcal{N}=1$ supersymmetric Yang-Mills theory in ten dimensions is given by the gauge field $A_{M}$ and a Majorana-Weyl spinor $\Psi$. All fields are Lie-algebra valued in $\mathfrak{s u}(N)$, and we expand them as

$$
\Psi=\Psi^{a} T^{a}, \quad A_{M}=A_{M}^{a} T^{a}
$$

For the basis $\left\{T^{a}\right\}$ of $\mathfrak{s u}(N)$, we fix the following convention:

$$
\begin{equation*}
\operatorname{Tr}\left(T^{a} T^{b}\right)=\frac{1}{2} \delta^{a b} \tag{4.1}
\end{equation*}
$$

The field strength and covariant derivative are given by

$$
\begin{align*}
F_{M N} & =\partial_{M} A_{N}-\partial_{N} A_{M}-i\left[A_{M}, A_{N}\right],  \tag{4.2}\\
D_{M} \Psi & =\partial_{M} \Psi-i\left[A_{M}, \Psi\right] . \tag{4.3}
\end{align*}
$$

The action of $\mathcal{N}=1$ supersymmetric Yang-Mills theory in ten dimensions reads:

$$
\begin{equation*}
S=\frac{1}{g_{10}^{2}} \int \mathrm{~d}^{10} x \mathcal{L}_{10}=\frac{1}{g_{10}^{2}} \int \mathrm{~d}^{10} x \operatorname{Tr}\left(-\frac{1}{2} F_{M N} F^{M N}+i \bar{\Psi} \Gamma^{M} D_{M} \Psi\right) . \tag{4.4}
\end{equation*}
$$

It is invariant under the gauge transformations

$$
\begin{equation*}
A_{M} \rightarrow U(x)\left(A_{M}+i \partial_{M}\right) U(x)^{\dagger}, \quad \Psi \rightarrow U(x) \Psi U(x)^{\dagger} \tag{4.5}
\end{equation*}
$$

and the supersymmmetry transformations

$$
\begin{equation*}
\delta_{\varepsilon} \Psi=\frac{i}{2} F_{M N} \Gamma^{M N} \varepsilon, \quad \delta_{\varepsilon} A_{M}=-i \bar{\varepsilon} \Gamma_{M} \Psi . \tag{4.6}
\end{equation*}
$$

Here, $\varepsilon$ is a constant Majorana-Weyl spinor and $\Gamma_{M N}:=\frac{i}{2}\left[\Gamma_{M}, \Gamma_{N}\right]$. The invariance of the action under the above transformations is shown in [15].

For the dimensional reduction, we demand that the fields $A_{M}$ and $\Psi$ only depend on the first four coordinates $x^{\mu} \in \mathbb{R}^{(1,3)} \subset \mathbb{R}^{(1,9)}$, i.e. that

$$
\begin{equation*}
\partial_{M} A_{N}=0=\partial_{M} \Psi \quad \text { for } M \in\{4,5, \ldots, 9\} . \tag{4.7}
\end{equation*}
$$

Furthermore, we prescribe:

$$
\begin{equation*}
\phi_{I}:=A_{I+3} \quad \text { for } I \in\{1,2, \ldots, 6\} . \tag{4.8}
\end{equation*}
$$

This allows us to write the Lagrangian density in terms of the fields $A_{\mu}, \phi_{I}, \psi_{\alpha}^{A}$ and $\tilde{\psi}_{A}^{\dot{\alpha}}$. For the bosonic fields we can rewrite:

$$
-\frac{1}{2} F_{M N} F^{M N}=-\frac{1}{2} F_{\mu \nu} F^{\mu \nu}-F_{\mu, I+3} F^{\mu, I+3}-\frac{1}{2} F_{I+3, J+3} F^{I+3, J+3} .
$$

Using the prescriptions for the dimensional reduction, we have

$$
F_{\mu, I+3}=\partial_{\mu} \Phi_{I}-i\left[A_{\mu}, \phi_{I}\right]=D_{\mu} \phi_{I}, \quad F_{I+3, J+3}=-i\left[\phi_{I}, \phi_{J}\right] .
$$

Thus it follows that

$$
\begin{equation*}
-\frac{1}{2} F_{M N} F^{M N}=-\frac{1}{2} F_{\mu \nu} F^{\mu \nu}+\frac{1}{2}\left(D_{\mu} \bar{\phi}_{A B}\right)\left(D^{\mu} \phi^{A B}\right)+\frac{1}{8}\left[\bar{\phi}_{A B}, \bar{\phi}_{C D}\right]\left[\phi^{A B}, \phi^{C D}\right] . \tag{4.9}
\end{equation*}
$$

For the fermionic fields, we employ (2.36) to rewrite:

$$
\bar{\Psi} \Gamma^{\mu} D_{\mu} \Psi=\tilde{\psi}_{A \dot{\alpha}} \sigma^{\mu \dot{\alpha} \beta} D_{\mu} \psi_{\beta}^{A}+\psi^{A \alpha} \bar{\sigma}_{\alpha \dot{\beta}}^{\mu} D_{\mu} \tilde{\psi}_{A}^{\dot{\beta}}=\tilde{\psi}_{A \dot{\alpha}} \sigma^{\mu \dot{\alpha} \beta} D_{\mu} \psi_{\beta}^{A}-\left(D_{\mu} \tilde{\psi}_{A \dot{\alpha}}\right) \sigma^{\mu \dot{\alpha} \beta} \psi_{\beta}^{A} .
$$

[^6]When considering the action we may perform an integration by parts on the second term. This is also possible for the covariant derivative as

$$
\begin{aligned}
\operatorname{Tr}\left(\left[A_{\mu}, \tilde{\psi}_{A \dot{\alpha}}\right] \psi_{\beta}^{A}\right) & =A_{\mu}^{a} \tilde{\psi}_{A \dot{\alpha}}^{b} \psi_{\beta}^{A c} \operatorname{Tr}\left(\left[T^{a}, T^{b}\right] T^{c}\right) \\
& =-A_{\mu}^{a} \tilde{\psi}_{A \dot{\alpha}}^{b} \psi_{\beta}^{A c} \operatorname{Tr}\left(T^{b}\left[T^{a}, T^{c}\right]\right)=-\operatorname{Tr}\left(\tilde{\psi}_{A \dot{\alpha}}\left[A_{\mu}, \psi_{\beta}^{A}\right]\right) .
\end{aligned}
$$

Hence we have:

$$
\int \mathrm{d}^{4} x \operatorname{Tr}\left(\bar{\Psi} \Gamma^{\mu} D_{\mu} \Psi\right)=\int \mathrm{d}^{4} x \operatorname{Tr}\left(2 \tilde{\psi}_{A \dot{\alpha}} \sigma^{\mu \dot{\alpha} \beta} D_{\mu} \psi_{\beta}^{A}\right) .
$$

For $M=I+3 \in\{4, \ldots, 9\}$, we apply (2.37), which yields

$$
\begin{align*}
\bar{\Psi} \Gamma^{I+3} D_{I+3} \Psi & =-\tilde{\psi}_{A \dot{\alpha}} \Sigma^{I A B} D_{I+3} \tilde{\psi}_{B}^{\dot{\alpha}}+\psi^{A \alpha} \bar{\Sigma}_{A B}^{I} D_{I+3} \psi_{\alpha}^{B} \\
& =i \tilde{\psi}_{A \dot{\alpha}} \Sigma^{I A B}\left[\phi_{I}, \tilde{\psi}_{B}^{\dot{\alpha}}\right]-i \psi^{A \alpha} \bar{\Sigma}_{A B}^{I}\left[\phi_{I}, \psi_{\alpha}^{B}\right] \\
& =-\sqrt{2} i \tilde{\psi}_{A \dot{\alpha}}\left[\phi^{A B}, \tilde{\psi}_{B}^{\dot{\alpha}}\right]+\sqrt{2} i \psi^{A \alpha}\left[\bar{\phi}_{A B}, \psi_{\alpha}^{B}\right] . \tag{4.10}
\end{align*}
$$

Hence, the action of $\mathcal{N}=4$ supersymmetric Yang-Mills theory is given by:

$$
\begin{align*}
S=\frac{1}{g^{2}} \int & \mathrm{~d}^{4} x \operatorname{Tr}\left(-\frac{1}{2} F_{\mu \nu} F^{\mu \nu}+\frac{1}{2}\left(D_{\mu} \bar{\phi}_{A B}\right)\left(D^{\mu} \phi^{A B}\right)+\frac{1}{8}\left[\bar{\phi}_{A B}, \bar{\phi}_{C D}\right]\left[\phi^{A B}, \phi^{C D}\right]\right. \\
& \left.+2 i \tilde{\psi}_{A \dot{\alpha}} \sigma^{\mu \dot{\alpha} \beta} D_{\mu} \psi_{\beta}^{A}+\sqrt{2} \tilde{\psi}_{A \dot{\alpha}}\left[\phi^{A B}, \tilde{\psi}_{B}^{\dot{\alpha}}\right]-\sqrt{2} \psi^{A \alpha}\left[\bar{\phi}_{A B}, \psi_{\alpha}^{B}\right]\right) . \tag{4.11}
\end{align*}
$$

Here, we have absorbed the remaining volume integrals $V=\int \mathrm{d} x^{4} \ldots \mathrm{~d} x^{9}$ into the coupling constant $g=V^{-1 / 2} g_{10}$, which is now dimensionless.

### 4.1.1 Symmetries of the Action

Of course, the action (4.11) is still invariant under the supersymmetry transformations given in (4.6). They act on the fields of $\mathcal{N}=4$ SYM in the following way:

$$
\begin{align*}
\delta_{\varepsilon} A^{\mu} & =-i \varepsilon^{A \alpha} \bar{\sigma}_{\alpha \dot{\beta}}^{\mu} \tilde{\psi}_{A}^{\dot{\beta}}-i \tilde{\varepsilon}_{A \dot{\alpha}} \sigma^{\mu \dot{\alpha} \beta} \psi_{\beta}^{A},  \tag{4.12}\\
\delta_{\varepsilon} \phi^{I} & =i \tilde{\varepsilon}_{A \dot{\alpha}} \Sigma^{I A B} \tilde{\psi}_{B}^{\dot{\alpha}}-i \varepsilon^{A \alpha} \bar{\Sigma}_{A B}^{I} \psi_{\alpha}^{B},  \tag{4.13}\\
\delta_{\varepsilon} \Psi_{\alpha}^{A} & =\frac{i}{2} F_{\mu \nu} \sigma^{\mu \nu}{ }_{\alpha}{ }^{\beta} \varepsilon_{\beta}^{A}-\sqrt{2}\left(D_{\mu} \phi^{A B}\right) \bar{\sigma}_{\alpha \dot{\beta}}^{\mu} \varepsilon_{B}^{\dot{\beta}}+i\left[\phi^{A B}, \bar{\phi}_{B C}\right] \varepsilon_{\alpha}^{C},  \tag{4.14}\\
\delta_{\varepsilon} \tilde{\Psi}_{A}^{\dot{\alpha}} & =\frac{i}{2} F_{\mu \nu} \bar{\sigma}^{\mu \dot{\alpha} \dot{\alpha}}{ }_{\dot{\beta}} \tilde{\varepsilon}_{A}^{\dot{\beta}}+\sqrt{2}\left(D_{\mu} \bar{\phi}_{A B}\right) \sigma^{\mu \dot{\alpha} \beta} \varepsilon_{\beta}^{B}+i\left[\bar{\phi}_{A B}, \phi^{B C}\right] \tilde{\varepsilon}_{C}^{\dot{\alpha}} . \tag{4.15}
\end{align*}
$$

The first two equations are trivially obtained by combining the ten-dimensional supersymmetry transformations (4.6) with (2.36) and (2.37). The supersymmetry transformations of the fermion fields follow from writing out $F_{M N} \Gamma^{M N}$ for the ten-dimensional gamma matrices introduced in chapter 2. We define the supersymmetry generators $\mathcal{Q}_{A}^{\alpha}$ and $\overline{\mathcal{Q}}^{A \dot{\alpha}}$ by stripping off the spinor indices, i.e. for any field $\mathcal{O}$ and any MajoranaWeyl spinor $\varepsilon$ we demand that

$$
\delta_{\varepsilon} \mathcal{O}=\varepsilon_{\alpha}^{A} \mathcal{Q}_{A}^{\alpha}(\mathcal{O})+\tilde{\varepsilon}_{A \dot{\alpha}} \overline{\mathcal{Q}}^{A \dot{\alpha}}(\mathcal{O})
$$

Clearly, the supersymmetry generators so defined also annihilate the action as their contraction with any Majorana-Weyl spinor $\varepsilon$ does. From (4.12) - 4.15), we can compute how the generators act on the fields:

$$
\begin{array}{lr}
\mathcal{Q}_{A}^{\alpha}\left(A^{\beta \dot{\beta}}\right)=2 i \epsilon^{\alpha \beta} \tilde{\psi}_{A}^{\dot{\beta}} & \overline{\mathcal{Q}}^{A \dot{\alpha}}\left(A^{\beta \dot{\beta}}\right)=-2 i \epsilon^{\dot{\alpha} \dot{\beta}} \psi^{A \beta} \\
\mathcal{Q}_{A}^{\alpha}\left(\bar{\phi}_{B C}\right)=\sqrt{2} i \epsilon_{A B C D} \psi^{D \alpha} & \overline{\mathcal{Q}}^{A \dot{\alpha}}\left(\bar{\phi}_{B C}\right)=-\sqrt{2} i\left(\tilde{\psi}_{B}^{\dot{\alpha}} \delta_{C}^{A}-\tilde{\psi}_{C}^{\dot{\alpha}} \delta_{B}^{A}\right) \\
\mathcal{Q}_{A}^{\alpha}\left(\psi^{B \beta}\right)=\frac{i}{2} F^{\alpha \beta} \delta_{A}^{B}+i \epsilon^{\beta \alpha}\left[\bar{\phi}_{A C}, \phi^{B C}\right] & \overline{\mathcal{Q}}^{A \dot{\alpha}}\left(\psi^{B \beta}\right)=-\sqrt{2} D^{\beta \dot{\alpha}} \phi^{A B} \\
\mathcal{Q}_{A}^{\alpha}\left(\tilde{\psi}_{B}^{\dot{\beta}}\right)=-\sqrt{2} D^{\dot{\beta} \alpha} \bar{\phi}_{A B} & \overline{\mathcal{Q}}^{A \dot{\alpha}}\left(\tilde{\psi}_{B}^{\dot{\beta}}\right)=-\frac{i}{2} F^{\dot{\alpha} \dot{\beta}} \delta_{B}^{A}+i \epsilon^{\dot{\alpha} \dot{\beta}}\left[\phi^{A C}, \bar{\phi}_{B C}\right]
\end{array}
$$

Apart from the Poincaré symmetries and supersymmetry, the action is also invariant under the scale transformation

$$
x \rightarrow c^{-1} x, \quad A \rightarrow c A, \quad \phi \rightarrow c \phi, \quad \psi \rightarrow c^{3 / 2} \psi .
$$

This scale invariance points at a conformal invariance of the theory, which together with super Poincaré invariance combines into superconformal invariance.

However, the classical symmetries of the action might not survive quantization. The introduction of a renormalization scale $\mu$ in the quantized theory typically breaks scale and thus also conformal invariance. The scale dependence of observables is then described through the scale dependence of the parameters of the theory. In the case of $\mathcal{N}=4$ super Yang-Mills theory, the only parameter is the coupling constant $g$, the scale dependence of which is described by the beta function

$$
\beta=\mu \frac{\partial g}{\partial \mu} .
$$

Perturbative calculations have shown that the beta function vanishes,

$$
\beta=0 .
$$

This behaviour is believed to extend to all orders in perturbation theory, see for example 33. In this case the superconformal symmetry of the action is preserved at the quantum level.

Often, one considers the planar limit, where the rank of the gauge group $\operatorname{SU}(N)$ is sent to infinity while the gauge coupling constant $g$ is sent to zero in such a way that the 't Hooft coupling

$$
\lambda=g^{2} N
$$

is kept fixed. In this limit, new symmetries can arise which are not classical symmetries of the action. In our study of the Yangian symmetries of the Maldacena-Wilson loop, it will not be necessary to consider the planar limit as we will be working to first order in perturbation theory, where no subleading contributions in $1 / N$ to the vacuum expectation value of the Maldacena-Wilson loop exist.

### 4.2 Propagators

We now turn to the derivation of the propagators in $\mathcal{N}=4$ SYM. In order to derive the scalar propagator, we note the relevant part of the action in component notation,

$$
-\frac{1}{2 g^{2}}\left(\partial_{\mu} \phi_{I}^{a}\right)\left(\partial^{\mu} \phi^{a I}\right) .
$$

Using the Green's function

$$
\begin{equation*}
G(x-y):=-\int \frac{\mathrm{d}^{4} p}{(2 \pi)^{4}} \frac{e^{-i p(x-y)}}{p^{2}+i \varepsilon}, \quad \partial_{\rho} \partial^{\rho} G(x-y)=\delta^{(4)}(x-y), \tag{4.20}
\end{equation*}
$$

we can perform a shift

$$
\phi_{I}^{a}(x) \rightarrow \phi_{I}^{\prime a}(x)=\phi_{I}^{a}(x)+g^{2} \int \mathrm{~d}^{4} y G(x-y) J_{I}^{a}(y)
$$

in the path integral such that the $J$-dependent terms can be factorized ${ }^{2}$ Then, we find for the correlation function of two scalar fields:

$$
\left\langle\phi_{I}^{a}\left(x_{1}\right) \phi_{J}^{b}\left(x_{2}\right)\right\rangle=-i g^{2} \delta^{a b} \delta_{I J} G\left(x_{1}-x_{2}\right) .
$$

The Green's function $G(x)$ takes the following form, see for example 35:

$$
\begin{equation*}
G(x)=-\frac{i}{4 \pi^{2}} \frac{1}{x^{2}} . \tag{4.21}
\end{equation*}
$$

Thus the scalar propagator is given by

$$
\begin{equation*}
\left\langle\phi_{I}^{a}\left(x_{1}\right) \phi_{J}^{b}\left(x_{2}\right)\right\rangle=-\frac{g^{2}}{4 \pi^{2}} \frac{\delta^{a b} \delta_{I J}}{\left(x_{1}-x_{2}\right)^{2}} . \tag{4.22}
\end{equation*}
$$

If the scalar fields are expressed as antisymmetric $(4 \times 4)$-matrices, the respective expression reads:

$$
\begin{equation*}
\left\langle\phi^{a A B}\left(x_{1}\right) \phi^{b C D}\left(x_{2}\right)\right\rangle=-\frac{g^{2}}{4 \pi^{2}} \frac{\delta^{a b} \epsilon^{A B C D}}{\left(x_{1}-x_{2}\right)^{2}} . \tag{4.23}
\end{equation*}
$$

For the fermion fields, the relevant part of the action is given by

$$
\frac{i}{g^{2}} \tilde{\psi}_{A \dot{\alpha}}^{a} \sigma^{\mu \dot{\alpha} \beta} \partial_{\mu} \psi_{\beta}^{a A}
$$

The correlation function for two fermion fields can be derived similarly to the scalar case, noting that

$$
\begin{equation*}
g(x-y)_{\alpha \dot{\alpha}}:=-\bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} \partial_{\mu} \int \frac{\mathrm{d}^{4} p}{(2 \pi)^{4}} \frac{e^{-i p(x-y)}}{p^{2}+i \varepsilon} \tag{4.24}
\end{equation*}
$$

[^7]satisfies
$$
\sigma^{\mu \dot{\alpha} \gamma} \partial_{\mu} g(x-y)_{\gamma \dot{\beta}}=\delta_{\dot{\beta}}^{\dot{\alpha}} \delta^{(4)}(x-y),
$$
which is a simple consequence of 4.20 and (2.4). Then we find that
$$
\left\langle\psi_{\alpha}^{a A}\left(x_{1}\right) \tilde{\psi}_{B \dot{\alpha}}^{b}\left(x_{2}\right)\right\rangle=g^{2} \delta^{a b} \delta_{B}^{A} g\left(x_{1}-x_{2}\right)_{\alpha \dot{\alpha}} .
$$

Inserting (4.21), we arrive at

$$
\begin{equation*}
\left\langle\psi_{\alpha}^{a A}\left(x_{1}\right) \tilde{\psi}_{B \dot{\alpha}}^{b}\left(x_{2}\right)\right\rangle=\frac{i g^{2}}{2 \pi^{2}} \delta^{a b} \delta_{B}^{A} \frac{\left(x_{1}-x_{2}\right)_{\alpha \dot{\alpha}}}{\left(x_{1}-x_{2}\right)^{4}} . \tag{4.25}
\end{equation*}
$$

For the gauge field propagator, we note the relevant term of the gauge fixed action in Feynman gauge,

$$
\frac{1}{2 g^{2}} A_{\mu}^{a} \eta^{\mu \nu} \partial_{\rho} \partial^{\rho} A_{\nu}^{a}
$$

In the Feynman gauge, the calculation of the gauge field propagator works in exactly the same way as that for the scalar propagator and we reach the result

$$
\begin{equation*}
\left\langle A_{\mu}^{a}\left(x_{1}\right) A_{\nu}^{b}\left(x_{2}\right)\right\rangle=\frac{g^{2}}{4 \pi^{2}} \frac{\eta_{\mu \nu} \delta^{a b}}{\left(x_{1}-x_{2}\right)^{2}} . \tag{4.26}
\end{equation*}
$$

### 4.3 Scattering Amplitudes

Scattering amplitudes are a good example for the possibility to extract valuable information about gauge theories from $\mathcal{N}=4$ supersymmetric Yang-Mills theory as well as for the apperance of higher symmetries for certain observables in this theory. Indeed, it was possible to derive the form of all tree-level scattering amplitudes in $\mathcal{N}=4$ SYM [2]. This information could in turn be used to derive certain tree-level amplitudes for QCD [1.

Although scattering amplitudes are not the subject of this thesis, their Yangian symmetry motivates to look for a Yangian symmetry of Maldacena-Wilson loops by the conjectured duality. The duality will be discussed in chapter 5.3. Here, we will introduce the color-ordered superamplitudes in $\mathcal{N}=4$ SYM and briefly review their symmetries. The account is based on [3, 36]

To make contact with [3] we alter our conventions in this section. The difference to the conventions discussed in 2.1 lies in defining $\sigma_{\alpha \dot{\alpha}}^{\mu}:=(\mathbb{1}, \vec{\sigma}), \bar{\sigma}^{\mu \dot{\alpha} \alpha}:=(\mathbb{1},-\vec{\sigma})$ and in the definitions for raising and lowering spinor indices. In this section,

$$
\begin{equation*}
\lambda_{\alpha}=\epsilon_{\alpha \beta} \lambda^{\beta}, \quad \lambda^{\alpha}=\epsilon^{\alpha \beta} \lambda_{\beta}, \quad \lambda_{\dot{\alpha}}=\epsilon_{\dot{\alpha} \dot{\beta}} \lambda^{\dot{\beta}}, \quad \lambda^{\dot{\alpha}}=\epsilon^{\dot{\alpha} \dot{\beta}} \lambda_{\dot{\beta}}, \tag{4.27}
\end{equation*}
$$

where the two-dimensional epsilon tensors are given by

$$
\begin{equation*}
\epsilon_{12}=\epsilon^{21}=1=\epsilon_{\mathrm{i} \dot{2}}=\epsilon^{2 \mathrm{i}} . \tag{4.28}
\end{equation*}
$$

In the calculation of scattering amplitudes we assign bi-spinors

$$
p_{i}^{\alpha \dot{\alpha}}:=\sigma_{\alpha \dot{\alpha}}^{\mu} p_{i \mu}
$$

to the external momenta $p_{i}$. For massless particles, $\operatorname{det}\left(p_{i}^{\alpha \dot{\alpha}}\right)=p_{i}^{2}=0$ and one may write

$$
p_{i}^{\alpha \dot{\alpha}}=\lambda_{i}^{\alpha} \tilde{\lambda}_{i}^{\dot{\alpha}} .
$$

The reality of the external momenta translates to the condition that $\left(\lambda_{i}^{\alpha}\right)^{*}= \pm \tilde{\lambda}_{i}^{\dot{\alpha}}$, where the free sign depends on the sign of $p_{i}^{0}$. The helicity spinors $\lambda$ and $\tilde{\lambda}$ are thus only defined up to a simultaneous rescaling

$$
\lambda_{i} \rightarrow e^{i \alpha} \lambda_{i}, \quad \tilde{\lambda}_{i} \rightarrow e^{-i \alpha} \tilde{\lambda}_{i} .
$$

Defining the contractions

$$
\begin{equation*}
\langle i j\rangle=\left\langle\lambda_{i} \lambda_{j}\right\rangle:=\lambda_{i}^{\alpha} \lambda_{j \alpha}, \quad[i j]=\left[\tilde{\lambda}_{i} \tilde{\lambda}_{j}\right]:=\tilde{\lambda}_{i \dot{\alpha}} \tilde{\lambda}_{i}^{\dot{\alpha}}, \tag{4.29}
\end{equation*}
$$

we note that $p_{i} \cdot p_{j}=\langle i j\rangle[j i]$. The use of helicity spinors $\lambda$ and $\tilde{\lambda}$ simplifies the calculation of scattering amplitudes because apart from the momenta also the information about the helicities of the external particles can be stored in them. For more information on the use of helicity spinors, see for example [37].

The evaluation of scattering amplitudes can be further simplified by separating the kinematical data from the calculation of the gauge group factors. This leads to the introduction of so-called color-ordered amplitudes, which no longer contain information on the color structure. For example, a pure-gluon tree-level amplitude in $S U(N)$ gauge theory may be shown to take the following form:

$$
\begin{aligned}
A_{\mathrm{n} \text {-gluon }}^{\text {tree }}\left(\left\{a_{i}, p_{i}, h_{i}\right\}\right)=g^{n-2} \sum_{\sigma \in S_{n-1}} & \operatorname{Tr}\left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n-1)}} T^{a_{n}}\right) \\
& \mathcal{A}_{\mathrm{n-gluon}}^{\text {tree }}\left(p_{\sigma(1)}, h_{\sigma(1)} ; \ldots ; p_{\sigma(n-1)}, h_{\sigma(n-1)} ; p_{n}, h_{n}\right) .
\end{aligned}
$$

Here, the helicities of the gluons are denoted by $h_{i} \in\{ \pm 1\}$. The color-ordered amplitudes are given by the coefficients $\mathcal{A}_{n-\mathrm{g} \text { giun }}^{\text {tree }}\left(\left\{p_{i}, h_{i}\right\}\right)$. For a general treatment of color-ordered amplitudes the reader is referred to [37] and references therein.

To describe scattering amplitudes in $\mathcal{N}=4$ SYM it is convenient to assemble the on-shell states of the theory into an on-shell superfield making use of four Graßmann odd parameters $\eta^{A}$ :

$$
\begin{align*}
\Phi(p, \eta)= & g_{+}(p)+\eta^{A} \tilde{g}_{A}(p)+\frac{1}{2!} \eta^{A} \eta^{B} S_{A B}(p)+\frac{1}{3!} \eta^{A} \eta^{B} \eta^{C} \epsilon_{A B C D} \overline{\tilde{g}}^{D}(p)  \tag{4.30}\\
& +\frac{1}{4!} \eta^{A} \eta^{B} \eta^{C} \eta^{D} \epsilon_{A B C D} g_{-}(p) . \tag{4.31}
\end{align*}
$$

Here, $g_{+}(p)\left(g_{-}(p)\right)$ describes a gluon of positive (negative) helicity, $\tilde{g}_{A}(p)\left(\overline{\tilde{g}}^{D}(p)\right)$ describes a gluino (anti-gluino) of helicity $+1 / 2(-1 / 2)$ and $S_{A B}(p)$ describes a scalar
particle, which has helicity 0 . Making use of the above superfield one considers the color-ordered superamplitudes

$$
\begin{equation*}
\mathcal{A}_{n}\left(\left\{\lambda_{i}, \tilde{\lambda}_{i}, \eta_{i}\right\}\right)=\mathcal{A}\left(\Phi\left(p_{1}, \eta_{1}\right) \ldots \Phi\left(p_{n}, \eta_{n}\right)\right) \tag{4.32}
\end{equation*}
$$

The amplitudes for a given set of states may then be extracted from the superamplitude by extracting the corresponding coefficient of the expansion in the $\eta$-variables.

The superconformal symmetry of $\mathcal{N}=4$ SYM is also manifest for the scattering amplitudes. In terms of the kinematical variables $\left\{\lambda_{i}, \tilde{\lambda}_{i}, \eta_{i}\right\}$, a representation of the superconformal algebra $\mathfrak{s u}(2,2 \mid 4)$ can be given by the following set of generators:

$$
\begin{align*}
p^{\alpha \dot{\alpha}} & =\sum_{i=1}^{n} \lambda_{i}^{\alpha} \tilde{\lambda}_{i}^{\dot{\alpha}} & k_{\alpha \dot{\alpha}} & =\sum_{i=1}^{n} \partial_{i \alpha} \tilde{\partial}_{i \dot{\alpha}}  \tag{4.33}\\
m_{\alpha \beta} & =\sum_{i=1}^{n} \lambda_{i(\alpha} \partial_{i \beta)} & \bar{m}_{\dot{\alpha} \dot{\beta}} & =\sum_{i=1}^{n} \tilde{\lambda}_{i(\dot{\alpha}} \tilde{\partial}_{i \dot{\beta})} \\
d & =\sum_{i=1}^{n}\left(\frac{1}{2} \lambda_{i}^{\alpha} \partial_{i \alpha}+\frac{1}{2} \tilde{\lambda}_{i}^{\dot{\alpha}} \tilde{\partial}_{i \dot{\alpha}}+1\right) & r_{B}^{A} & =\sum_{i=1}^{n}\left(-\eta_{i}^{A} \partial_{i B}+\frac{1}{4} \delta_{B}^{A} \eta_{i}^{C} \partial_{i C}\right)  \tag{4.34}\\
q^{A \alpha} & =\sum_{i=1}^{n} \lambda_{i}^{\alpha} \eta_{i}^{A} & \bar{q}_{A}^{\dot{\alpha}} & =\sum_{i=1}^{n} \tilde{\lambda}_{i}^{\dot{\alpha}} \partial_{i A}  \tag{4.35}\\
s_{A \alpha} & =\sum_{i=1}^{n} \partial_{i \alpha} \partial_{i A} & \bar{s}_{\dot{\alpha}}^{A} & =\sum_{i=1}^{n} \eta_{i}^{A} \tilde{\partial}_{i \dot{\alpha}} \\
c & =\sum_{i=1}^{n}\left(1+\frac{1}{2} \lambda_{i}^{\alpha} \partial_{i \alpha}-\frac{1}{2} \tilde{\lambda}_{i}^{\alpha} \tilde{\partial}_{i \dot{\alpha}}-\frac{1}{2} \eta_{i}^{A} \partial_{i A}\right) & & \tag{4.36}
\end{align*}
$$

Here, we use the following short-hand notation for derivatives:

$$
\partial_{i \alpha}:=\frac{\partial}{\partial \lambda_{i}^{\alpha}}, \quad \tilde{\partial}_{i \dot{\alpha}}:=\frac{\partial}{\partial \tilde{\lambda}_{i}^{\dot{\alpha}}}, \quad \partial_{i A}:=\frac{\partial}{\partial \eta_{i}^{A}} .
$$

The superamplitudes 4.32) are invariant under all these generators,

$$
\begin{equation*}
\left\{m_{\alpha \beta}, \bar{m}_{\dot{\alpha} \dot{\beta}}, p^{\alpha \dot{\alpha}}, d, k_{\alpha \dot{\alpha}}, r_{B}^{A}, q^{A \alpha}, \bar{q}_{A}^{\dot{\alpha}}, s_{A \alpha}, \bar{s}_{\dot{\alpha}}^{A}, c\right\} \mathcal{A}_{n}=0 . \tag{4.39}
\end{equation*}
$$

This enforces that the superamplitudes have the following form:

$$
\begin{equation*}
\mathcal{A}_{n}\left(\left\{\lambda_{i}, \tilde{\lambda}_{i}, \eta_{i}\right\}\right)=\frac{\delta^{(4)}(p) \delta^{(8)}(q)}{\langle 12\rangle\langle 23\rangle \ldots\langle n 1\rangle} \mathcal{P}_{n} . \tag{4.40}
\end{equation*}
$$

Note that $\delta^{(8)}(q)$ is a Graßmann delta-function,

$$
\delta^{(8)}(q)=\prod_{\alpha=1}^{2} \prod_{A=1}^{4}\left(\sum_{i=1}^{n}\left(\lambda_{i}^{\alpha} \eta_{i}^{A}\right)\right)
$$

such that the $\eta$-expansion of the amplitude starts at order $\eta^{8}$. Furthermore $\mathcal{P}_{n}$ has the following expansion:

$$
\mathcal{P}_{n}=\mathcal{P}_{n}^{(0)}+\mathcal{P}_{n}^{(4)}+\ldots+\mathcal{P}_{n}^{(4 n-16)}
$$

where $\mathcal{P}_{n}^{(j)}$ is of order $\eta^{j}$.
In [38] it was found that tree-level scattering amplitudes possess an additional symmetry, dual superconformal symmetry. This symmetry can be expressed in terms of the dual variables $x_{i}$ and $\theta_{i}$, which are defined by the relations

$$
\begin{equation*}
\left(x_{i}-x_{i+1}\right)_{\alpha \dot{\alpha}}=\lambda_{i \alpha} \tilde{\lambda}_{i \dot{\alpha}}, \quad\left(\theta_{i}-\theta_{i+1}\right)_{\alpha}^{A}=\lambda_{i \alpha} \eta_{i}^{A} . \tag{4.41}
\end{equation*}
$$

The dual variables define a chiral superspace. Note that they are only fixed up to an arbitrary reference point. A representation of the superconformal algebra $\mathfrak{s u}(2,2 \mid 4)$ in terms of the dual variables is given in [38]. This specific representation respects the constraints 4.41). Introducing the abbreviations

$$
\partial_{i A \alpha}:=\frac{\partial}{\partial \theta_{i}^{A \alpha}}, \quad \partial_{i \alpha \dot{\alpha}}:=\frac{\partial}{\partial x_{i}^{\alpha \dot{\alpha}}},
$$

it is given by:

$$
\begin{align*}
& P_{\alpha \dot{\alpha}}=\sum_{i} \partial_{i \alpha \dot{\alpha}} \quad Q_{A \alpha}=\sum_{i} \partial_{i A \alpha} \quad \bar{Q}_{\dot{\alpha}}^{A}=\sum_{i}\left(\theta_{i}^{A \alpha} \partial_{i \alpha \dot{\alpha}}+\eta_{i}^{A} \tilde{\partial}_{i \dot{\alpha}}\right)  \tag{4.42}\\
& M_{\alpha \beta}= \sum_{i}\left(x_{(\alpha}^{\dot{\alpha}} \partial_{i \beta) \dot{\alpha}}+\theta_{i(\alpha}^{A} \partial_{i \beta) A}+\lambda_{i(\alpha} \partial_{i \beta)}\right)  \tag{4.43}\\
& \bar{M}_{\dot{\alpha} \dot{\beta}}= \sum_{i}\left(x_{i(\dot{\alpha}}{ }^{\alpha} \partial_{i \dot{\beta}) \alpha}+\tilde{\lambda}_{i(\dot{\alpha}} \tilde{\partial}_{i \dot{\beta})}\right)  \tag{4.44}\\
& D= \sum_{i}\left(-x_{i}^{\dot{\alpha} \alpha} \partial_{i \alpha \dot{\alpha}}-\frac{1}{2} \theta_{i}^{A \alpha} \partial_{i A \alpha}-\frac{1}{2} \lambda_{i}^{a} \partial_{i \alpha}-\frac{1}{2} \tilde{\lambda}_{i}^{\dot{\alpha}} \partial_{i \dot{\alpha}}\right)  \tag{4.45}\\
& K_{\alpha \dot{\alpha} \dot{\alpha}}=\sum_{i}\left(x_{i \dot{\alpha} \beta} x_{i \dot{\alpha} \beta} \partial_{i}^{\beta \dot{\beta}}-x_{i \dot{\alpha} \beta} \theta_{i \alpha}^{B} \partial_{i B}^{\beta}-x_{i \dot{\alpha} \beta} \lambda_{i \alpha} \partial_{i}^{\beta}\right. \\
&\left.\quad-x_{i+1 \dot{\beta} \dot{\alpha}} \tilde{\lambda}_{i \dot{\alpha}} \tilde{\partial}_{i}^{\dot{\beta}}+\tilde{\lambda}_{i \dot{\alpha}} \theta_{i+1 \alpha}^{B} \partial_{i B}\right)  \tag{4.46}\\
& R_{B}^{A}=\sum_{i}\left(\theta_{i}^{A \alpha} \partial_{i B \alpha}+\eta_{i}^{A} \partial_{i B}-\frac{1}{4} \delta_{B}^{A} \theta_{i}^{C \alpha} \partial_{i C \alpha}-\frac{1}{4} \delta_{B}^{A} \eta_{i}^{C} \partial_{i C}\right)  \tag{4.47}\\
& C= \sum_{i}\left(-\frac{1}{2} \lambda_{i}^{\alpha} \partial_{i \alpha}+\frac{1}{2} \tilde{\lambda}_{i}^{\dot{\alpha}} \partial_{i \dot{\alpha}}+\frac{1}{2} \eta_{i}^{A} \partial_{i A}\right)  \tag{4.48}\\
& S_{\alpha}^{A}=\sum_{i}\left(x_{i \alpha \dot{\beta}} \theta_{i \beta}^{A} \partial_{i \beta \dot{\beta}}-\theta_{i \alpha}^{B} \theta_{i}^{A \beta} \partial_{i B \beta}+\lambda_{i \alpha} \theta_{i}^{A \gamma} \partial_{i \gamma}\right. \\
&\left.\quad+x_{i+1 \alpha}{ }^{\dot{\beta}} \eta_{i}^{A} \partial_{i \dot{\beta}}-\theta_{i+1 \alpha}^{B} \eta_{i}^{A} \partial_{i B}\right)  \tag{4.49}\\
& \bar{S}_{A \dot{\alpha}}=\sum_{i}\left(x_{i \dot{\alpha}}{ }^{\beta} \partial_{i A \beta}+\tilde{\lambda}_{i \dot{\alpha}} \partial_{i A}\right) \tag{4.50}
\end{align*}
$$

If one imposes momentum conservation and supersymmetry in the definition (4.41) of the dual variables, the dual variables become cyclic, $\left(x_{n+1}, \theta_{n+1}\right)=\left(x_{1}, \theta_{1}\right)$. However, if one wants to act on the amplitudes in the distributional sense stated in (4.40), the dual variables cannot be considered cyclic from the start. In this case, the summation in the above generators extends from 1 to $n+1$ for the terms involving derivatives in the dual variables.

The original superconformal symmetry and the dual superconformal symmetry combine into a Yangian symmetry, which is described in [3]. The level-1 momentum generator of the Yangian symmetry algebra is given by

$$
\begin{equation*}
p_{\alpha \dot{\alpha}}^{(1)}=\sum_{i>j}\left[\left(m_{i}{ }^{\gamma}{ }_{\alpha} \delta_{\dot{\alpha}}^{\dot{\gamma}}+\bar{m}_{i}{ }^{\dot{\gamma}}{ }_{\dot{\alpha}} \delta_{\alpha}^{\gamma}-d_{i} \delta_{\alpha}^{\gamma} \delta_{\dot{\alpha}}^{\dot{\gamma}}\right) p_{j \gamma \dot{\gamma}}+\bar{q}_{i \dot{\alpha} C} q_{j \alpha}^{C}-(i \leftrightarrow j)\right] . \tag{4.51}
\end{equation*}
$$

This generator has the structure described in (3.50) and we will use this information to make an ansatz for the level-1 momentum generator we study in chapter 6.

## Chapter 5

## Maldacena-Wilson Loops

In this chapter we introduce the Maldacena-Wilson loop operator in $\mathcal{N}=4$ supersymmetric Yang-Mills theory. We begin by discussing Wilson loops in general gauge theories, where we follow [39] and 40]. The Maldacena-Wilson loop, which was first introduced in [9], is an extension that arises naturally in $\mathcal{N}=4 \mathrm{SYM}$. As a preparation for the study of Yangian symmetries, we study its conformal symmetry both for large and infinitesimal transformations. Furthermore, we discuss the conjectured duality between Wilson loops and maximally helicity violating scattering amplitudes in $\mathcal{N}=4$ SYM.

### 5.1 Wilson Loops in Gauge Theories

In a general gauge theory one considers quantities which depend on a path in spacetime as opposed to local operators $\mathcal{O}(x)$ which depend on a point in spacetime. For a path $C$ connecting the points $x$ and $y$ in spacetime, given in a parametrization

$$
x:[a, b] \rightarrow \mathbb{R}^{(1,3)}, x(a)=x, x(b)=y,
$$

we define the Wilson line operator ${ }^{\text {1 }}$

$$
\begin{equation*}
V(y, x, C)=\mathcal{P} \exp \left(i \int_{C} A_{\mu} \mathrm{d} x^{\mu}\right) . \tag{5.1}
\end{equation*}
$$

In the above definition $\mathcal{P} \exp$ denotes the path-ordered exponential ${ }^{2}$, which implies that in the expansion of the exponential function the matrix-valued fields $A_{\mu}$ are ordered by the parametrization parameter $\tau$. To be concrete, we spell out the pathordered product of two gauge fields:

$$
\mathcal{P}\left(A_{\mu}\left(x\left(\tau_{1}\right)\right) A_{\nu}\left(x\left(\tau_{2}\right)\right)\right)=\left\{\begin{array}{ll}
A_{\nu}\left(x\left(\tau_{2}\right)\right) A_{\mu}\left(x\left(\tau_{1}\right)\right) & \text { if } \tau_{1}<\tau_{2} \\
A_{\mu}\left(x\left(\tau_{1}\right)\right) A_{\nu}\left(x\left(\tau_{2}\right)\right) & \text { if } \tau_{2}<\tau_{1}
\end{array} .\right.
$$

[^8]We will now turn to the discussion of gauge transformations of the Wilson line operator. Here, we will consider the gauge group $\operatorname{SU}(N)$. In order to compute the gauge transform of $V(y, x, C)$, we slice the curve into small pieces

$$
x \rightarrow x_{1} \rightarrow x_{2} \rightarrow \ldots \rightarrow x_{N-1} \rightarrow x_{N} \rightarrow y
$$

The Wilson line operator can then be written as

$$
\begin{equation*}
V(y, x, C)=V\left(y, x_{N}\right) V\left(x_{N}, x_{N-1}\right) \ldots V\left(x_{2}, x_{1}\right) V\left(x_{1}, x\right) . \tag{5.2}
\end{equation*}
$$

For a curve of infinitesimal length, we expand the Wilson line operator as

$$
V(x+\varepsilon, x)=\mathcal{P} \exp \left(i \int_{x}^{x+\varepsilon} A_{\mu} \mathrm{d} x^{\mu}\right)=\mathbb{1}+i \varepsilon^{\mu} A_{\mu}(x)+\mathcal{O}\left(\varepsilon^{2}\right) .
$$

Under a gauge transformation

$$
A_{\mu}(x) \rightarrow U(x) A_{\mu}(x) U(x)^{\dagger}-i\left(\partial_{\mu} U(x)\right) U(x)^{\dagger}
$$

we then have:

$$
\begin{align*}
V(x+\varepsilon, x) & \rightarrow \mathbb{1}+i \varepsilon^{\mu}\left(U(x) A_{\mu}(x) U(x)^{\dagger}-i\left(\partial_{\mu} U(x)\right) U(x)^{\dagger}\right)+\mathcal{O}\left(\varepsilon^{2}\right) \\
& =U(x+\varepsilon) V(x+\varepsilon, x) U(x)^{\dagger}+\mathcal{O}\left(\varepsilon^{2}\right) \tag{5.3}
\end{align*}
$$

Inserting this into (5.2), we find the gauge transform of the Wilson line operator:

$$
\begin{equation*}
V(y, x, C) \rightarrow U(y) V(y, x) U(x)^{\dagger} . \tag{5.4}
\end{equation*}
$$

This behaviour under gauge transformations may be proven more elegantly and rigorously by showing that the Wilson line operator as a function of the end point of the curve satisfies a first-order ordinary differential equation. One can then show that the gauge transformed Wilson line operator and the expression given in (5.4) are both solutions to this equation. The uniqueness theorem for ODEs then allows one to conclude that the two expressions are equal, consider for example [34] for a more detailed discussion. Here, we have taken a more intuitive approach.

The gauge transformation (5.4) shows that for a closed loop $C$ one can define a gauge invariant non-local operator by

$$
\mathrm{W}(C):=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \oint_{C} A_{\mu} \mathrm{d} x^{\mu}\right)
$$

This observable is called the Wilson loop operator. One very interesting application of it is the possibility to compute the force between two static charged particles in a gauge theory. Consider a rectangular contour $C_{T, R}$ with side length $T$ in the time direction and $R \ll T$ in some spatial direction. For large $T$, we have

$$
\left\langle\mathrm{W}\left(C_{T, R}\right)\right\rangle \simeq e^{-V(R) T},
$$

where $V(R)$ is the potential between the two static charges. Consider [39] for a detailed explanation. An interesting application is the derivation of the Coulomb potential in pure quantum electrodynamics, where the absence of interactions allows one to compute the expectation value of the Wilson loop to all orders in perturbation theory.

### 5.2 The Maldacena-Wilson Loop

Similarly to the construction of $\mathcal{N}=4$ SYM itself, we can construct the MaldacenaWilson loop [9] from the the Wilson loop in the ten-dimensional gauge theory, which is given by

$$
\begin{equation*}
U(C)=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \oint_{C} A_{M} \mathrm{~d} x^{M}\right) . \tag{5.5}
\end{equation*}
$$

Inserting (4.8) for the gauge field $A_{M}$, we can rewrite this using the field content of the four-dimensional theory:

$$
\begin{equation*}
U(C)=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \int \mathrm{~d} \tau\left(A_{\mu}(x) \dot{x}^{\mu}+\phi_{I}(x) \dot{y}^{I}\right)\right), \tag{5.6}
\end{equation*}
$$

where we have combined the six extra components of the ten-dimensional vector to a six-dimensional vector $y^{I}$ to which the scalar fields couple. We now impose the constraint that the ten-dimensional curve has light-like tangent vectors,

$$
\begin{equation*}
\dot{x}^{\mu} \dot{x}_{\mu}+\dot{y}^{I} \dot{y}_{I}=0 . \tag{5.7}
\end{equation*}
$$

We can realize this constraint by setting $\dot{y}^{I}=|\dot{x}| n^{I}$, where $n^{I}$ is a unit six vector which satisfies $n^{I} n_{I}=-1$. We will assume $n^{I}$ to be constant along the loop. Note also that we define

$$
|\dot{x}|:=\left\{\begin{array}{lll}
\|\dot{x}\| & \text { if } & \dot{x}^{2}>0 \\
i\|\dot{x}\| & \text { if } & \dot{x}^{2}<0
\end{array}, \quad \text { where } \quad\|\dot{x}\|:=\sqrt{\left|\dot{x}^{2}\right|},\right.
$$

such that $|\dot{x}|^{2}=\dot{x}^{2}$ for any $\dot{x}$. Satisfying the constraint (5.7), we define the MaldacenaWilson loop operator by

$$
\begin{equation*}
W(C):=\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \int_{C} \mathrm{~d} \tau\left(A_{\mu}(x) \dot{x}^{\mu}+\phi_{I}(x)|\dot{x}| n^{I}\right)\right) . \tag{5.8}
\end{equation*}
$$

The gauge invariance of this operator is clear because the ten-dimensional Wilson loop operator (5.5) is gauge invariant for any closed curve in ten dimensions. As the gauge transformations in $\mathcal{N}=4$ SYM only depend on the four-vector $x^{\mu}$, it suffices that the curve described by $x^{\mu}(\tau)$ is closed.

The constraint (5.7) can be related to supersymmetry in a sense which we will now discuss. Consider the supersymmetry variation of the Maldacena-Wilson loop operator. By (4.6) we find:

$$
\begin{align*}
\delta_{\varepsilon} W(C)= & \frac{1}{N} \operatorname{Tr} \mathcal{P}\left\{i \int \mathrm{~d} \tau \delta_{\varepsilon}\left(A_{\mu} \dot{x}^{\mu}+\phi_{I}(x)|\dot{x}| n^{I}\right)\right. \\
& \left.\exp \left(i \int \mathrm{~d} \tau\left(A_{\mu}(x) \dot{x}^{\mu}+\phi_{I}(x)|\dot{x}| n^{I}\right)\right)\right\} \\
= & \frac{1}{N} \operatorname{Tr} \mathcal{P}\left\{i \int \mathrm{~d} \tau \bar{\Psi}(x(\tau))\left(\Gamma_{\mu} \dot{x}^{\mu}+\Gamma_{I+3}|\dot{x}| n^{I}\right) \varepsilon\right. \\
& \left.\exp \left(i \int \mathrm{~d} \tau\left(A_{\mu}(x) \dot{x}^{\mu}+\phi_{I}(x)|\dot{x}| n^{I}\right)\right)\right\} . \tag{5.9}
\end{align*}
$$

The supersymmetry variation vanishes as a function of the fields if and only if

$$
\begin{equation*}
A(\tau) \varepsilon=0, \quad A(\tau):=\Gamma_{\mu} \dot{x}^{\mu}(\tau)+\Gamma_{I+3}|\dot{x}(\tau)| n^{I} . \tag{5.10}
\end{equation*}
$$

Although the spinor $\varepsilon$ in the supersymmetry variation has to be constant, it is interesting to see if 5.10 can be solved locally, i.e. whether there is a parametrization dependent function $\varepsilon(\tau)$, such that it holds. There is room for a non-trivial solution, as

$$
\begin{equation*}
\left(\Gamma_{\mu} \dot{x}^{\mu}(\tau)+\Gamma_{I+3}|\dot{x}(\tau)| n^{I}\right)^{2}=0 \tag{5.11}
\end{equation*}
$$

which follows easily from the ten-dimensional Clifford relations. This property of $A(\tau)$ is of course intimately related to the constraint 5.7). If one would not impose it, none of the supersymmetry would be preserved locally, as

$$
\left(x_{M} \Gamma^{M}\right)^{2}=x^{2} \mathbb{1} \quad \text { implies that } \quad \operatorname{det}\left(x_{M} \Gamma^{M}\right)=\left(x^{2}\right)^{16} .
$$

For an invertible matrix, there are of course no non-trivial solutions of (5.10). If the constraint is imposed, we conclude from (5.11) that the dimension of the kernel of $A(\tau)$ is at least sixteen, and in appendix $D$ we show that there are 8 linearly independent left-handed Majorana-Weyl spinors, which are mapped to zero by $A(\tau)$ if $\dot{x}(\tau)$ is a time-like or light-like vector. If $\dot{x}(\tau)$ is a space-like vector, there are still 8 linearly independent left-handed Weyl spinors satisfying (5.10), but the Majorana condition cannot be satisfied.

The difference between time-like and space-like tangent vectors may seem surprising, but there is a simple reason for it: It is impossible to extend a space-like vector in Minkowski space to a light-like vector in the ten-dimensional space $\mathbb{R}^{(1,9)}$ as we are adding six space-like directions. We only reach a light-like ten-vector ( $\left.\dot{x}^{\mu}+|\dot{x}| n^{I}\right)$ by setting $|\dot{x}|=i\|\dot{x}\|$ for space-like $\dot{x}$. This changes the commutation properties of $A(\tau)$ with the charge conjugation matrix, which are a crucial point in proving the possibility to pick Majorana spinors in $\operatorname{ker}(A(\tau))$. In both cases, we find eight different linear combinations of the supersymmetry variations, which leave the action invariant. The Maldacena-Wilson loop is therefore called $1 / 2$ BPS, in analogy to the BPS states encountered in supersymmetric theories. See e.g. [23] for an explanation.

If one requires that $n^{I}$ is constant along the loop, the only curves for which it is possible to pick a constant spinor $\varepsilon$, which satisfies (5.10), are straight lines. For these it is always possible to pick a parametrization, in which $\dot{x}$ is constant such that $A(\tau)$ is also constant, and hence half of the supersymmetry is preserved globally. This matches well with the finding that the expectation value for a straight line $L$ satisfies

$$
\langle W(L)\rangle=1,
$$

which has been checked perturbatively. If one allows $n^{I}$ to vary along the loop, it is possible to preserve some of the supersymmetry globally for special curves. This construction is described in 41. The class of loops, which are globally invariant also becomes larger if one considers general superconformal transformations, see for example [42].

Another interesting aspect of the Maldacena-Wilson loop is that it has a dual string description via the AdS/CFT correspondence, which was first described in [9|. The correspondence relates it to the area $A_{\min }(C)$ of a minimal surface in $A d S_{5}$, for which the boundary condition is given by the loop. To be more precise, one needs to consider the conformal compactifications and boundaries of Minkowski as well as Anti-de Sitter space $A d S_{5}$. In this construction one finds that the conformal compactification of Minkowski space $\mathbb{R}^{(1,3)}$ can be viewed as the conformal boundary ${ }^{3}$ of $A d S_{5}$, for details see e.g. [7]. One then considers the minimal surface which ends on the image of the loop on the conformal boundary of $A d S_{5}$. The area of this surface is divergent, as the $A d S$-metric becomes divergent upon approaching the conformal boundary. The divergence can be regulated by shifting the boundary loop slightly into $A d S$. Upon subtracting the divergent part, which is proportional to the length of the loop, one arrives at the renormalized area $A_{\min }^{\text {ren }}(C)$ of the minimal surface, which is related to the expectation value of the Maldacena-Wilson loop at strong coupling by

$$
\begin{equation*}
\langle W(C)\rangle=\exp \left(-\frac{\sqrt{\lambda}}{2 \pi} A_{\min }^{\text {ren }}(C)\right) . \tag{5.12}
\end{equation*}
$$

In this thesis, we focus on the vacuum expectation value of the Maldacena-Wilson loop to first order in perturbation theory. Expanding the exponential in (5.8) leads to:

$$
\begin{aligned}
\langle W(C)\rangle=1-\frac{1}{2 N} \int & \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \operatorname{Tr}\left(\dot{x}_{1}^{\mu} \dot{x}_{2}^{\nu}\left\langle\mathcal{P}\left(A_{\mu}\left(x_{1}\right) A_{\nu}\left(x_{2}\right)\right)\right\rangle\right. \\
& \left.+\left|\dot{x}_{1} \| \dot{x}_{2}\right| n^{I} n^{J}\left\langle\mathcal{P}\left(\phi_{I}\left(x_{1}\right) \phi_{J}\left(x_{2}\right)\right)\right\rangle\right)+\ldots
\end{aligned}
$$

Here we abbreviated $x_{i}:=x\left(\tau_{i}\right), i=1,2$. The expectation value is to be understood within the path integral formalism. Writing explicitly

$$
A_{\mu}(x)=A_{\mu}^{a}(x) T^{a}, \quad \phi_{I}(x)=\phi_{I}^{a}(x) T^{a},
$$

we realize that the functions $A_{\mu}^{a}(x)$ and $\phi_{I}^{a}(x)$ are automatically time-ordered in the path integral, while the generators of the Lie algebra $\mathfrak{s u}(N)$ remain path ordered. Nonetheless, since the trace is cyclic this is irrelevant here. Thus we get:

$$
\begin{aligned}
&\langle W(C)\rangle=1-\frac{1}{2 N} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left\{\dot{x}_{1}^{\mu} \dot{x}_{2}^{\nu}\left\langle A_{\mu}^{a}\left(x_{1}\right) A_{\nu}^{b}\left(x_{2}\right)\right\rangle \operatorname{Tr}\left(T^{a} T^{b}\right)\right. \\
&\left.+\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right| n^{I} n^{J}\left\langle\phi_{I}^{a}\left(x_{1}\right) \phi_{J}^{b}\left(x_{2}\right)\right\rangle \operatorname{Tr}\left(T^{a} T^{b}\right)\right\}+\ldots
\end{aligned}
$$

Inserting the propagators (4.22) and (4.26) and using the normalization

$$
\operatorname{Tr}\left(T^{a} T^{b}\right)=\frac{1}{2} \delta^{a b}
$$

[^9]of the generators of the gauge group, we arrive at ${ }^{4}$
\[

$$
\begin{equation*}
\langle W(C)\rangle=1-\frac{g^{2} N}{16 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \cdot \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}}+\ldots \tag{5.13}
\end{equation*}
$$

\]

For brevity, we introduce the following notation for the first-order approximation of the vacuum expectation value of the Maldacena-Wilson loop:

$$
\begin{equation*}
\langle W(C)\rangle_{(1)}:=-\frac{\lambda}{16 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} I\left(\tau_{1}, \tau_{2}\right), \quad \text { where } \quad I\left(\tau_{1}, \tau_{2}\right):=\frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}} \tag{5.14}
\end{equation*}
$$

Here we have used the 't Hooft coupling $\lambda=g^{2} N$. Having the vacuum expectation value at hand, we can discuss another important implication of the constraint (5.7), the finiteness of $\langle W(C)\rangle$. We will discuss this at first order in perturbation theory, an argument for all orders may be found in [43]. As we are working in Minkowski space, this is a little more subtle than for the Euclidean case. We need to assume that for any two distinct $\tau_{1}$ and $\tau_{2}$ the points $x\left(\tau_{1}\right)$ and $x\left(\tau_{2}\right)$ are space-like separated, consider appendix C for more details. Under these assumptions the integrand of (5.14) can only become divergent if $\tau_{1} \simeq \tau_{2}$. We discuss this limit using a parametrization by arc-length, $\dot{x}^{2}=-1$, by setting $\tau_{2}=\tau_{1}+\varepsilon$ and expanding for small $\varepsilon$. This gives:

$$
I(\tau, \tau+\varepsilon)=\frac{\dot{x}(\tau) \dot{x}(\tau+\varepsilon)+1}{(x(\tau)-x(\tau+\varepsilon))^{2}}=\frac{\varepsilon \dot{x}(\tau) \ddot{x}(\tau)+\varepsilon^{2} \dot{x}(\tau) x^{(3)}(\tau)+\mathcal{O}\left(\varepsilon^{3}\right)}{-\varepsilon^{2}+\mathcal{O}\left(\varepsilon^{3}\right)}
$$

Making use of the identities (C.1), we get:

$$
I(\tau, \tau+\varepsilon)=\ddot{x}^{2}+\mathcal{O}(\varepsilon)
$$

This shows that the integrand of (5.14) is bounded and hence that the first-order approximation of $\langle W(C)\rangle$ is finite.

We now turn to the discussion of the symmetries of the Maldacena-Wilson loop. Here, we consider the invariance of $\langle W(C)\rangle_{(1)}$ under conformal transformations. Of course, the finiteness of $\langle W(C)\rangle$ is crucial for this because otherwise we would have to consider a renormalization procedure which might spoil the conformal invariance. We focus on the infinitesimal conformal invariance, for which we consider a representation of the conformal algebra in terms of curve integrals over functional derivatives. It is, however, illustrative to also consider large conformal transformations.

### 5.2.1 Global Conformal Symmetry

It is easy to see that already the integrand $I\left(\tau_{1}, \tau_{2}\right)$ of $\langle W(C)\rangle_{(1)}$ is invariant under translations, Lorentz transformations and dilatations. To consider special conformal transformations, recall that they can be expressed as the composition of inversions and a translation. Hence, it suffices to study the inversion map

$$
S(x)=\frac{x}{x^{2}}
$$

[^10]Under this transformation we have:

$$
\begin{aligned}
& \dot{x} \rightarrow \partial_{\tau} \frac{x}{x^{2}}=\frac{\dot{x}}{x^{2}}-\frac{2(x \cdot \dot{x}) x}{x^{4}} \Rightarrow|\dot{x}| \rightarrow \frac{|\dot{x}|}{x^{2}}, \\
& \frac{1}{\left(x_{1}-x_{2}\right)^{2}} \rightarrow \frac{x_{1}^{2} x_{2}^{2}}{\left(x_{1}-x_{2}\right)^{2}} .
\end{aligned}
$$

From this transformation behaviour we read off that the part of the integrand which stems from the scalar propagator is invariant under an inversion and hence under all conformal transformations:

$$
\frac{\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}} \rightarrow \frac{\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}} .
$$

The whole integrand $I\left(\tau_{1}, \tau_{2}\right)=: I_{12}$ transforms as

$$
I_{12} \rightarrow \hat{I}_{12}=I_{12}-2 \frac{x_{1}^{2} x_{2}^{2}}{\left(x_{1}-x_{2}\right)^{2}}\left(\frac{\left(x_{1} \dot{x}_{1}\right)\left(x_{1} \dot{x}_{2}\right)}{x_{1}^{4} x_{2}^{2}}+\frac{\left(x_{2} \dot{x}_{2}\right)\left(x_{2} \dot{x}_{1}\right)}{x_{2}^{4} x_{1}^{2}}-2 \frac{\left(x_{1} x_{2}\right)\left(x_{1} \dot{x}_{1}\right)\left(x_{2} \dot{x}_{2}\right)}{x_{2}^{4} x_{1}^{4}}\right) .
$$

We will now abbreviate $\partial_{\tau_{i}}=\partial_{i}, x_{1}-x_{2}=x_{12}$. For the transformed integrand, we rewrite:

$$
\begin{aligned}
\Delta I_{12}: & =\hat{I}_{12}-I_{12}= \\
& =-\frac{1}{x_{12}^{2} x_{1}^{2} x_{2}^{2}}\left(\left(\partial_{1} x_{1}^{2}\right) x_{2}^{2} \partial_{2}\left(x_{1} \cdot x_{2}\right)+\left(\partial_{2} x_{2}^{2}\right) x_{1}^{2} \partial_{1}\left(x_{2} \cdot x_{1}\right)-\left(\partial_{1} x_{2}^{2}\right)\left(\partial_{2} x_{1}^{2}\right)\left(x_{1} x_{2}\right)\right) \\
& =-\frac{\left(\partial_{1} x_{1}^{2}\right) \partial_{2}\left(x_{1} x_{2}\right)}{x_{1}^{2} x_{12}^{2}}-\frac{\left(\partial_{2} x_{2}^{2}\right) \partial_{1}\left(x_{1} x_{2}\right)}{x_{2}^{2} x_{12}^{2}}+\frac{\left(\partial_{1} x_{1}^{2}\right)\left(\partial_{2} x_{2}^{2}\right)\left(x_{1} x_{2}\right)}{x_{1}^{2} x_{2}^{2} x_{12}^{2}} \\
& =\frac{1}{2}\left(\frac{\partial_{1} x_{1}^{2}}{x_{1}^{2}} \frac{\partial_{2} x_{12}^{2}}{x_{12}^{2}}+\frac{\partial_{2} x_{2}^{2}}{x_{2}^{2}} \frac{\partial_{1} x_{12}^{2}}{x_{12}^{2}}-\frac{\partial_{1} x_{1}^{2}}{x_{1}^{2}} \frac{\partial_{2} x_{2}^{2}}{x_{2}^{2}}\right) .
\end{aligned}
$$

We realize that this can be written as a total derivative unless the curve crosses the light-cone $x^{2}=0$. In this case, the curve is mapped to a curve which is only closed via infinity. In such cases the vacuum expectation value of the MaldacenaWilson loop need not be invariant and could receive a contribution which stems from integrating over the respective singularity. For Euclidean signature this anomaly has been considered in [10], where they studied an inversion mapping a straight line to a circle. This allowed to compute the vacuum expectation value of the MaldacenaWilson loop to all orders in $\lambda$ and $N$. For Minkowski signature, anomalies like the above are discussed in [44].

Assuming now, that for our curve $x_{12}^{2}<0$ and $x_{1}^{2}<0$ for all points on the loop, we rewrite:

$$
\begin{aligned}
\Delta I_{12}=\frac{1}{2} & \left(\partial_{2}\left(\partial_{1} \ln \left[-x_{1}^{2}\right] \ln \left[-x_{12}^{2}\right]\right)+\partial_{1}\left(\partial_{2} \ln \left[-x_{2}^{2}\right] \ln \left[-x_{12}^{2}\right]\right)\right. \\
& \left.-\partial_{2}\left(\partial_{1} \ln \left[-x_{1}^{2}\right] \ln \left[-x_{2}^{2}\right]\right)\right) .
\end{aligned}
$$

This is a total derivative and hence we find invariance.

### 5.2.2 Infinitesimal Conformal Symmetry

In order to discuss the invariance under infinitesimal conformal transformations, we consider a representation of the conformal algebra in terms of curve integrals over functional derivatives. In analogy to (3.3) and (3.4), we have the following basis:

$$
\begin{align*}
P_{\mu} & =\int d s p_{\mu}(s)=\int \mathrm{d} s \frac{\delta}{\delta x^{\mu}(s)},  \tag{5.15}\\
M_{\mu \nu} & =\int d s m_{\mu \nu}(s)=\int \mathrm{d} s\left(x_{\mu}(s) \frac{\delta}{\delta x^{\nu}(s)}-x_{\nu}(s) \frac{\delta}{\delta x^{\mu}(s)}\right),  \tag{5.16}\\
D & =\int d s d(s)=\int \mathrm{d} s x^{\mu}(s) \frac{\delta}{\delta x^{\mu}(s)},  \tag{5.17}\\
K_{\mu} & =\int d s k_{\mu}(s)=\int \mathrm{d} s\left(x^{2}(s) \frac{\delta}{\delta x^{\mu}(s)}-2 x_{\mu}(s) x^{\nu}(s) \frac{\delta}{\delta x^{\nu}(s)}\right) . \tag{5.18}
\end{align*}
$$

This can be viewed as the continuum limit of a sum of the respective generators acting on different points in space-time. In applying the above generators to curve integrals, we assume that the generator is given in the parametrization of the curve integral. However, it is important that one does not restrict the parametrizations of the curve integral. 5 . The functional derivatives are defined by

$$
\frac{\delta x_{\nu}(\tau)}{\delta x^{\mu}(s)}=\eta_{\mu \nu} \delta(\tau-s)
$$

and the common rules for derivatives of products and the chain rule. Assuming also that the functional derivative commutes with time derivatives, we note the following derivatives, which will be used frequently:

$$
\begin{array}{r}
\frac{\delta \dot{x}_{\mu}(\tau)}{\delta x^{\nu}(s)}=\eta_{\mu \nu} \partial_{\tau} \delta(\tau-s), \quad \frac{\delta|\dot{x}(\tau)|}{\delta x^{\nu}(s)}=\frac{\dot{x}_{\nu}(\tau)}{|\dot{x}(\tau)|} \partial_{\tau} \delta(\tau-s), \\
\frac{\delta}{\delta x^{\mu}(s)}\left(x\left(\tau_{1}\right)-x\left(\tau_{2}\right)\right)^{2}=2\left(x_{\mu}\left(\tau_{1}\right)-x_{\mu}\left(\tau_{2}\right)\right)\left(\delta\left(\tau_{1}-s\right)-\delta\left(\tau_{2}-s\right)\right) . \tag{5.20}
\end{array}
$$

Using the above rules we show that the first-order correction of the MaldacenaWilson loop is translation invariant by acting on it with $P_{\mu}$ :

$$
\begin{aligned}
P_{\mu} I_{12}= & \int \mathrm{d} s \frac{\delta}{\delta x^{\mu}(s)} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}} \\
= & \int \mathrm{d} s \frac{1}{\left(x_{1}-x_{2}\right)^{2}}\left[\dot{x}_{1 \mu} \partial_{\tau_{2}} \delta\left(\tau_{2}-s\right)+\dot{x}_{2 \mu} \partial_{\tau_{1}} \delta\left(\tau_{1}-s\right)-\frac{\left|\dot{x}_{1}\right|}{\left|\dot{x}_{2}\right|} \dot{x}_{2 \mu} \partial_{\tau_{2}} \delta\left(\tau_{2}-s\right)\right. \\
& \left.-\frac{\left|\dot{x}_{2}\right|}{\left|\dot{x}_{1}\right|} \dot{x}_{1 \mu} \partial_{\tau_{1}} \delta\left(\tau_{1}-s\right)-2 \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}}\left(x_{1}-x_{2}\right)_{\mu}\left(\delta\left(\tau_{1}-s\right)-\delta\left(\tau_{2}-s\right)\right)\right] .
\end{aligned}
$$

By virtue of

$$
\partial_{\tau_{i}} \int \mathrm{~d} s \delta\left(\tau_{i}-s\right)=0 \quad \text { and } \quad \int \mathrm{d} s\left(\delta\left(\tau_{1}-s\right)-\delta\left(\tau_{2}-s\right)\right)=0
$$

[^11]we indeed find $P_{\mu} I\left(\tau_{1}, \tau_{2}\right)=0$. A similar computation shows that also $M_{\mu \nu}$ and $D$ annihilate $I\left(\tau_{1}, \tau_{2}\right)$. For the generators $K_{\mu}$ of special conformal transformations, the computation is more involved. In our discussion of large conformal transformations we have seen that the part of $\langle W(C)\rangle_{(1)}$ which stems from the scalar propagator is invariant under conformal transformations and indeed we find that ${ }^{6}$
$$
K_{\mu} \frac{\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}}=0 .
$$

This simplifies the calculation of the action of $K_{\mu}$ on the integrand $I_{12}$ :

$$
\begin{aligned}
K_{\mu} I_{12}= & \int \mathrm{d} s\left(x^{2}(s) \delta_{\mu}^{\nu}-2 x_{\mu}(s) x^{\nu}(s)\right) \frac{\delta}{\delta x^{\nu}(s)} \frac{\dot{x}_{1} \dot{x}_{2}}{\left(x_{1}-x_{2}\right)^{2}} \\
= & \int \mathrm{d} s\left(x^{2}(s) \delta_{\mu}^{\nu}-2 x_{\mu}(s) x^{\nu}(s)\right) \frac{1}{x_{12}^{2}}\left[\dot{x}_{1 \nu} \partial_{\tau_{2}} \delta\left(\tau_{2}-s\right)+\dot{x}_{2 \nu} \partial_{\tau_{1}} \delta\left(\tau_{1}-s\right)\right. \\
& \left.-2 \frac{\dot{x}_{1} \dot{x}_{2}}{x_{12}^{2}} x_{12 \nu}\left(\delta\left(\tau_{1}-s\right)-\delta\left(\tau_{2}-s\right)\right)\right] \\
= & \frac{2}{x_{12}^{2}}\left[\dot{x}_{1 \mu} \dot{x}_{2}\left(x_{2}-x_{1}\right)+\dot{x}_{2 \mu} \dot{x}_{1}\left(x_{1}-x_{2}\right)-\dot{x}_{1} \dot{x}_{2}\left(x_{1}+x_{2}\right)_{\mu}+\dot{x}_{1} \dot{x}_{2}\left(x_{1}+x_{2}\right)_{\mu}\right] \\
= & 2 \dot{x}_{1 \mu} \frac{\dot{x}_{2}\left(x_{2}-x_{1}\right)}{\left(x_{1}-x_{2}\right)^{2}}+2 \dot{x}_{2 \mu} \frac{\dot{x}_{1}\left(x_{1}-x_{2}\right)}{\left(x_{1}-x_{2}\right)^{2}} \\
= & \dot{x}_{1 \mu} \partial_{\tau_{2}} \ln \left[-\left(x_{1}-x_{2}\right)^{2}\right]+\dot{x}_{2 \mu} \partial_{\tau_{1}} \ln \left[-\left(x_{1}-x_{2}\right)^{2}\right] \\
= & \partial_{\tau_{2}}\left(\dot{x}_{1 \mu} \ln \left[-\left(x_{1}-x_{2}\right)^{2}\right]\right)+\partial_{\tau_{1}}\left(\dot{x}_{2 \mu} \ln \left[-\left(x_{1}-x_{2}\right)^{2}\right]\right) .
\end{aligned}
$$

For the curves under consideration we have $\left(x_{1}-x_{2}\right)^{2}<0$, and the above is a total derivative. Therefore we have:

$$
K_{\mu}\langle W(C)\rangle_{(1)}=0
$$

### 5.3 The Correspondence Between Scattering Amplitudes and Wilson Loops

In this section we briefly review the conjectured duality between MHV gluon amplitudes and the expectation value of Wilson loops over specific contours in $\mathcal{N}=4$ super Yang-Mills theory. The account is based on [45], [46] and [47], where the reader will find detailed explanations and calculations. This duality is a strong motivation for our attempt at finding a Yangian symmetry for Maldacena-Wilson loops, as the scattering amplitudes possess such a symmetry.

The first hints for the duality came from the consideration of gluon amplitudes at strong coupling by Alday and Maldacena in [11. There, it was shown that the computation of the amplitudes at leading order reduces to the computation of the

[^12]area of a minimal surface that ends on a lightlike polygon related to the momenta of the gluon. It was also pointed out that the computation is formally equivalent to that of a Wilson loop on this contour at strong coupling. In [12] it was argued that the duality also holds at the weak coupling side of the AdS/CFT correspondence. Evidence for this duality has been provided in [48 and 49].

Planar MHV amplitudes have the property that their loop corrections can be written as multiples of the tree-level amplitudes,

$$
\begin{equation*}
\mathcal{A}_{n}^{\mathrm{MHV}}=\mathcal{A}_{n}^{\mathrm{MHV}, \text { tree }} M_{n}^{\mathrm{MHV}} . \tag{5.21}
\end{equation*}
$$

Here, the function $M_{n}$ depends only on the momentum invariants $p_{i} \cdot p_{j}$. The amplitudes are infrared divergent and can be regulated by dimensional regularization using $D=4-2 \varepsilon_{I R}$ with $\varepsilon_{I R}<0$. Moreover, the loop corrections factorize into divergent and finite parts. It is therefore natural to consider the logarithm of the loop correction, which is known to have the following form:

$$
\begin{equation*}
\ln \left(M_{n}^{\mathrm{MHV}}\right)=-\frac{1}{4} \sum_{l \geq 1} a^{l}\left(\frac{\Gamma_{\text {cusp }}^{(l)}}{\left(l \varepsilon_{I R}\right)^{2}}+\frac{G^{(l)}}{l \varepsilon_{I R}}\right) \sum_{i=1}^{n}\left(-\frac{\mu_{I R}}{s_{i, i+1}}\right)^{l \varepsilon_{I R}}+F_{n}^{\mathrm{MHV}}+\mathcal{O}\left(\varepsilon_{I R}\right) . \tag{5.22}
\end{equation*}
$$

Here, $s_{i, i+1}=\left(p_{i}+p_{i+1}\right)^{2}$ are the Mandelstam invariants of two adjacent gluons $\left(p_{n+1}=\right.$ $\left.p_{1}\right), \mu_{I R}^{2}$ is an infrared cut-off related to the dimensional regularization scale and $a$ is related to the 't Hooft coupling by $a=\lambda /\left(8 \pi^{2}\right)$. The finite part $F_{n}^{\mathrm{MHV}}$ is independent of the regulator $\varepsilon$ and $\mu_{I R}$. The coefficients $\Gamma_{\text {cusp }}^{(l)}$ and $G^{(l)}$ are the coefficents of the cusp and collinear anomalous dimensions,

$$
\Gamma_{\text {cusp }}(a)=\sum_{l \geq 1} \Gamma_{\text {cusp }}^{(l)} a^{l}, \quad G(a)=\sum_{l \geq 1} G^{(l)} a^{l},
$$

which start out as

$$
\Gamma_{\text {cusp }}(a)=2 a-2 \zeta_{2} a^{2}+\mathcal{O}\left(a^{3}\right), \quad G(a)=-\zeta_{3} a^{2}+\mathcal{O}\left(a^{3}\right) .
$$

The duality states that $M_{n}^{\mathrm{MHV}}$ is related to the expectation value of a Wilson loop over a polygon with cusp points defined by the gluon momenta via

$$
\begin{equation*}
x_{i}-x_{i+1}=p_{i} . \tag{5.23}
\end{equation*}
$$

The cusp points are only defined up to a reference point, which is irrelevant as the Wilson loop is translation invariant. By momentum conservation, the cusp points $x_{i}$ define an n-sided polygon $C_{n}$ with light-like edges, such that the Wilson and the Maldacena-Wilson loop agree on it. Due to the cusps, the Wilson loop is ultraviolet divergent. The divergence can be regulated by dimensional regularization using $D=$ $4-2 \varepsilon_{U V}$ with $\varepsilon_{U V}>0$. The associated ultraviolet cut-off is called $\mu_{U V}^{2}$. The vacuum expectation value of the Wilson loop for this contour is known to have the following form:

$$
\begin{align*}
\ln \left(\left\langle W\left(C_{n}\right)\right\rangle\right)=- & \frac{1}{4} \sum_{l \geq 1} a^{l}\left(\frac{\Gamma_{\text {cusp }}^{(l)}}{\left(l \varepsilon_{U V}\right)^{2}}+\frac{\Gamma^{(l)}}{l \varepsilon_{U V}}\right) \sum_{i=1}^{n}\left(-\left(x_{i-1}-x_{i+1}\right)^{2} \mu_{U V}^{2}\right)^{l \varepsilon_{U V}} \\
& +F_{n}^{\mathrm{WL}}+\mathcal{O}\left(\varepsilon_{U V}\right) . \tag{5.24}
\end{align*}
$$

Here, $\Gamma^{(l)}$ are the coefficients of the collinear anomalous dimension,

$$
\Gamma(a)=\sum_{l \geq 1} \Gamma^{(l)} a^{l}=-7 \zeta_{3} a^{2}+\mathcal{O}\left(a^{3}\right) .
$$

The regularisation and renormalization parameters can be matched in such a way that the divergent parts of $\ln \left(M_{n}^{\mathrm{MHV}}\right)$ and $\ln \left(\left\langle W\left(C_{n}\right)\right\rangle\right)$ match, compensating also for the difference between the collinear anomalous dimensions. This behaviour is not a special property of $\mathcal{N}=4 \mathrm{SYM}$, it actually holds in any gauge theory. The conjectured duality now states, that in the planar limit the finite parts are equal up to an additive constant, which does not depend on the kinematical data,

$$
\begin{equation*}
F_{n}^{\mathrm{MHV}}\left(p_{1}, \ldots, p_{n} ; a\right)=F_{n}^{\mathrm{WL}}\left(x_{1}, \ldots, x_{n} ; a\right)+d_{n}(a) . \tag{5.25}
\end{equation*}
$$

A variety of checks of this duality relation can be found in 46.

## Chapter 6

## Yangian Symmetry of Maldacena-Wilson Loops

The combination of the duality discussed above and the Yangian symmetry of scattering amplitudes points toward a Yangian symmetry of Wilson loops over light-like polygons. Such a symmetry has been discussed in [50]. As we have seen, the cusps in these contours lead to divergences, which complicate the study of symmetries. To avoid divergences, we consider smooth space-like curves for which the expectation value of the Maldacena-Wilson loop is finite. As for the conformal invariance of the loop, we discuss the invariance of the vacuum expectation value under variational derivative operators acting in the curve space. For simplicity, we do this at first order in perturbation theory. In the study of the conformal symmetry of the MaldacenaWilson loop, the symmetry generators are given by

$$
\begin{equation*}
J_{a}^{(0)}=\int \mathrm{d} s j_{a}^{(0)}(s) \tag{6.1}
\end{equation*}
$$

where $j_{a}^{(0)}(s)$ collectively denotes the densities $\left\{m_{\mu \nu}, p_{\mu}, d, k_{\mu}\right\}$ of the conformal algebra, which are introduced in (5.15) - (5.18). In analogy to the generators (3.51) found in integrable two-dimensional field theories, we write the level-1 generators as:

$$
\begin{equation*}
J_{a}^{(1)}=\int \mathrm{d} s j_{a}^{(1)}(s)+f_{a}^{c b} \int_{s_{1}<s_{2}} \mathrm{~d} s_{1} \mathrm{~d} s_{2} j_{b}^{(0)}\left(s_{1}\right) j_{c}^{(0)}\left(s_{2}\right) \tag{6.2}
\end{equation*}
$$

The knowledge of the dual structure constants $f_{a}{ }^{c b}$ of $\mathfrak{s o}(2,4)$ allows us to determine the bilocal part of $J_{a}^{(1)}$. The idea is then to first calculate the action of the non-local part of the level-1 generator on $\langle W(C)\rangle_{(1)}$ to find out, whether it is possible to define a level- 1 density $j_{a}^{(1)}(s)$ in such a way that

$$
\begin{equation*}
J_{a}^{(1)}\left(1+\langle W(C)\rangle_{(1)}\right)=0 . \tag{6.3}
\end{equation*}
$$

In particular, if $j_{a}^{(1)}(s)$ is of order $\lambda$ and contains no differential operators, we have:

$$
\begin{equation*}
J_{a}^{(1)}\left(1+\langle W(C)\rangle_{(1)}\right)=\int \mathrm{d} s j_{a}^{(1)}(s)+f_{a}^{c b} \int_{s_{1}<s_{2}} \mathrm{~d} s_{1} \mathrm{~d} s_{2} j_{b}^{(0)}\left(s_{1}\right) j_{c}^{(0)}\left(s_{2}\right)\langle W(C)\rangle_{(1)} . \tag{6.4}
\end{equation*}
$$

It is thus possible to reach invariance in the sense of (6.3), if the action of the non-local part of $J_{a}^{(1)}$ on $\langle W(C)\rangle_{(1)}$ turns out to be a single curve integral. We will check this for the level-1 momentum generator

$$
\begin{equation*}
P_{\mathrm{bos}, \mathrm{nl}}^{(1) \mu}=\int_{s_{1}<s_{2}} \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left\{\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} . \tag{6.5}
\end{equation*}
$$

While the result of this calculation turns out to be remarkably simple, it is not a single curve integral. We then consider a supersymmetric extension $\mathcal{W}(C)$ of the MaldacenaWilson loop employing a non-chiral superspace parametrized by the variables

$$
x^{\alpha \dot{\alpha}}(\tau)=\bar{\sigma}^{\mu \alpha \dot{\alpha}} x_{\mu}(\tau), \quad \theta_{\alpha}^{A}(\tau), \quad \bar{\theta}_{A \dot{\alpha}}(\tau), \quad y_{A}^{B}(\tau)
$$

Enlarging the underlying symmetry from $\mathfrak{s o}(2,4)$ to $\mathfrak{s u}(2,2 \mid 4)$ extends the non-local part of $P^{(1) \mu}$ to include

$$
\begin{equation*}
P_{\mathrm{ferm}, \mathrm{nl}}^{(1) \mu}=\frac{i}{4} \int \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left\{\bar{q}^{A \dot{\alpha}}\left(s_{1}\right) \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} q_{A}^{\alpha}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} \theta\left(s_{2}-s_{1}\right) . \tag{6.6}
\end{equation*}
$$

The supersymmetry densities $q_{A}^{\alpha}(s)$ and $\bar{q}^{A \dot{\alpha}}(s)$ are introduced in chapter 6.2. We show, that the action of the above generator on $\langle\mathcal{W}(C)\rangle$ cancels the problematic terms in $P_{\text {bos }, \mathrm{ll}}^{(1) \mu}\langle W(C)\rangle$, leaving the result to be a single curve integral. This determines the level-1 momentum density $p^{(1) \mu}(s)$.

We thus establish the invariance of the extended Maldacena-Wilson loop $\mathcal{W}(C)$ under the level-1 generator $P^{(1) \mu}$ to first order in perturbation theory and in the first non-trivial order in an extension in the Graßmann variables which describe our superspace. It is in principle possible to conclude the invariance under all level-1 generators from the invariance under the level-1 momentum generator making use of the commutation relation (3.38) of the Yangian algebra. This is however obstructed by the extension in the Graßmann variables.

### 6.1 Maldacena-Wilson Loop

We begin our study of the Yangian invariance of the Maldacena-Wilson loop by deriving the non-local part of the level-1 momentum generator. In order to do this, we read off the following dual structure constants from (A.10):

$$
\begin{equation*}
-f_{\hat{P}^{\mu}} \hat{P}^{\rho} \hat{D}=f_{\hat{P}^{\mu}}{ }^{\hat{D} \hat{P}^{\rho}}=\frac{1}{8} \delta_{\mu}^{\rho}, \quad-f_{\hat{P}^{\mu}} \hat{P}^{\lambda} \hat{M}^{\nu \rho}=f_{\hat{P}^{\mu}} \hat{M}^{\nu \rho \hat{P}^{\lambda}}=\frac{1}{8}\left(\eta^{\nu \lambda} \delta_{\mu}^{\rho}-\eta^{\rho \lambda} \delta_{\mu}^{\nu}\right), \tag{6.7}
\end{equation*}
$$



$$
\begin{aligned}
& f_{\hat{P}^{\mu}}{ }^{c b} j_{b}^{(0)}\left(s_{1}\right) j_{c}^{(0)}\left(s_{2}\right)=-f_{\hat{P}^{\mu}} \hat{D}^{\hat{P}} \hat{P}^{\rho} d\left(s_{1}\right) p_{\rho}\left(s_{2}\right)-f_{\hat{P}^{\mu}}{ }^{\hat{P}^{\rho}} \hat{D} p_{\rho}\left(s_{1}\right) d\left(s_{2}\right) \\
&-\sum_{\nu<\rho}\left(f_{\hat{P}^{\mu}} \hat{M}^{\nu \rho} \hat{P}^{\lambda}\right. \\
& m_{\nu \rho}\left(s_{1}\right) p_{\lambda}\left(s_{2}\right)-f_{\hat{P}^{\mu}}{ }^{\hat{P}^{\lambda}} \hat{M}^{\nu \rho} \\
&\left.p_{\lambda}\left(s_{1}\right) m_{\nu \rho}\left(s_{2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
= & -\frac{1}{8}\left(d\left(s_{1}\right) p_{\mu}\left(s_{2}\right)-p_{\mu}\left(s_{1}\right) d\left(s_{2}\right)+\frac{1}{2}\left(\eta^{\nu \lambda} \delta_{\mu}^{\rho}-\eta^{\rho \lambda} \delta_{\mu}^{\nu}\right) m_{\nu \rho}\left(s_{1}\right) p_{\lambda}\left(s_{2}\right)\right. \\
& \left.-\frac{1}{2}\left(\eta^{\nu \lambda} \delta_{\mu}^{\rho}-\eta^{\rho \lambda} \delta_{\mu}^{\nu}\right) p_{\lambda}\left(s_{1}\right) m_{\nu \rho}\left(s_{2}\right)\right) \\
= & -\frac{1}{8}\left(d\left(s_{1}\right) p_{\mu}\left(s_{2}\right)-d\left(s_{2}\right) p_{\mu}\left(s_{1}\right)+\left[d\left(s_{2}\right), p_{\mu}\left(s_{1}\right)\right]\right. \\
& \left.-m_{\mu}{ }^{\nu}\left(s_{1}\right) p_{\nu}\left(s_{2}\right)+m_{\mu}{ }^{\nu}\left(s_{2}\right) p_{\nu}\left(s_{1}\right)-\left[m_{\mu}{ }^{\nu}\left(s_{2}\right), p_{\nu}\left(s_{1}\right)\right]\right) \\
= & -\frac{1}{8}\left(\left(d\left(s_{1}\right) p_{\mu}\left(s_{2}\right)-m_{\mu}{ }^{\nu}\left(s_{1}\right) p_{\nu}\left(s_{2}\right)\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right) \\
& +\frac{1}{2} \delta\left(s_{1}-s_{2}\right) p_{\mu}\left(s_{1}\right) .
\end{aligned}
$$

Upon integration over $s_{1}$ and $s_{2}$ the commutator term will lead to a contribution proportional to $P_{\mu}^{(0)}$, which annihilates $\langle W(C)\rangle$. Moreover, the Yangian algebra is invariant under $J_{a}^{(1)} \rightarrow J_{a}^{(1)}+\beta J_{a}^{(0)}$, as one can easily deduce from (3.37) and (3.38). We therefore drop this contribution to the level-1 momentum generator. Furthermore, we scale the level-1 generator by a factor of 8 to reach a more convenient form. This merely corresponds to a different choice of $\alpha$ in the Yangian algebra and affects neither the algebraic structure nor the question of invariance. We thus have the following expression for the non-local part of the level-1 momentum generator $P^{(1) \mu}$ :

$$
\begin{equation*}
P_{\mathrm{bos}, \mathrm{nl}}^{(1) \mu}=\int \mathrm{d} s_{1} \mathrm{~d} s_{2}\left\{\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} \theta\left(s_{2}-s_{1}\right) . \tag{6.8}
\end{equation*}
$$

Before we can start the calculation, we need to discuss a regularisation for the level-1 generator (6.8). This is necessary as the evaluation of $P_{\text {bos }, \mathrm{nl}}^{(1)}$ on $\langle W(C)\rangle_{(1)}$ will give rise to several divergent terms such as $\delta(0)$. These divergences appear when the two functional derivative operators in $P_{\text {bos, } n l}^{(1) \mu}$ act at the same point on the loop. Hence we can regulate them by replacing the ordering condition $s_{1}<s_{2}$ in (6.8) by $s_{1}<s_{2}-\varepsilon$, thus introducing a regulator $\varepsilon$ in parameter space. However, we still need to make sure that the regulated level-1 momentum generator will give reparametrization invariant curve integrals when applied to such. Therefore, we regulate by the condition

$$
\begin{equation*}
s_{1}<s_{2}-d\left(s_{2}, \varepsilon\right), \quad \text { where } \quad \int_{s_{2}-d\left(s_{2}, \varepsilon\right)}^{s_{2}} \mathrm{~d} s\|\dot{x}(s)\|=\varepsilon \tag{6.9}
\end{equation*}
$$

The implicit definition of $d\left(s_{2}, \varepsilon\right)$ encodes that the distance along the curve between the points $x\left(s_{1}\right)$ and $x\left(s_{2}\right)$ must always be greater than $\varepsilon$, which is an explicitly reparametrization invariant condition. The regulated level-1 momentum generator is then given by

$$
\begin{equation*}
P_{\mathrm{bos}, \mathrm{nl}, \varepsilon}^{(1) \mu}=\int_{s_{1}<s_{2}-d\left(s_{2}, \varepsilon\right)} \mathrm{d} s_{1} \mathrm{~d} s_{2}\left\{\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} . \tag{6.10}
\end{equation*}
$$

Before turning to the calculation of $P_{\text {bos }, \text { n }}^{(1) \mu}\langle W(C)\rangle_{(1)}$, we rewrite the generator in a form that is better suited for computational purposes. In order to simplify the calculation it is very convenient to use the arc-length parametrization where $\dot{x}^{2} \equiv-1$. Then the

[^13]regulating function takes an especially simple form,
$$
\dot{x}^{2} \equiv-1 \quad \Rightarrow \quad d\left(s_{2}, \varepsilon\right)=\varepsilon \quad \forall s_{2} .
$$

We point out the use of such a parametrization by writing explicit boundaries 0 and $L$ for the curve integral. As discussed before, one is not allowed to fix a certain parametrization before having applied the functional derivatives and we will switch to arc-length parametrization only after that, then setting $d\left(s_{2}, \varepsilon\right)=\varepsilon$. Keeping this in mind, we may rewrite

$$
\begin{aligned}
P_{\text {bos }, \mathrm{ln}, \varepsilon}^{(1) \mu}= & \int \mathrm{d} s_{1} \mathrm{~d} s_{2}\left\{\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} \theta\left(s_{2}-s_{1}-\varepsilon\right) \\
= & \int \mathrm{d} s_{1} \mathrm{~d} s_{2}\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right)\left(\theta\left(s_{2}-s_{1}-\varepsilon\right)+\theta\left(s_{2}-s_{1}+\varepsilon\right)\right) \\
& -\int \mathrm{d} s_{1}\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) \int \mathrm{d} s_{2} p_{\nu}\left(s_{2}\right),
\end{aligned}
$$

where we have used $\theta(s)=1-\theta(s)$. The second term factorizes to $\left(M^{\mu \nu}-D \eta^{\mu \nu}\right) P_{\nu}$ and we know already that it annihilates $I_{12}$ defined in (5.14). Hence we only need to study the action of the generator

$$
\begin{equation*}
P_{\text {bos }, \text { nl }, \varepsilon}^{(1) \mu}=\int \mathrm{d} s_{1} \mathrm{~d} s_{2}\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right) \theta\left(s_{2}-s_{1}-\varepsilon\right)+(\varepsilon \rightarrow-\varepsilon), \tag{6.11}
\end{equation*}
$$

which we further rewrite:

$$
\begin{aligned}
& P_{\mathrm{bos}, \mathrm{nl}, \varepsilon}^{(1)}=\int \mathrm{d} s_{1} \mathrm{~d} s_{2} N^{\mu \nu \rho \sigma} x_{\nu}\left(s_{1}\right) \frac{\delta}{\delta x^{\rho}\left(s_{1}\right)} \frac{\delta}{\delta x^{\sigma}\left(s_{2}\right)}\left(\theta\left(s_{2}-s_{1}-\varepsilon\right)+\theta\left(s_{2}-s_{1}+\varepsilon\right)\right), \\
& N^{\mu \nu \rho \sigma}:=\eta^{\mu \nu} \eta^{\rho \sigma}-\eta^{\mu \rho} \eta^{\nu \sigma}-\eta^{\mu \sigma} \eta^{\nu \rho} .
\end{aligned}
$$

Using this form, we start by calculating the double functional derivative of $I_{12}$ multiplied with $x_{\nu}\left(s_{1}\right)$. By virtue of (5.19) and (5.20) we find:

$$
\begin{aligned}
& x_{\nu}\left(s_{1}\right) \frac{\delta}{\delta x^{\rho}\left(s_{1}\right)} \frac{\delta}{\delta x^{\sigma}\left(s_{2}\right)} \frac{\dot{x}\left(\tau_{1}\right) \dot{x}\left(\tau_{2}\right)-\left|\dot{x}\left(\tau_{1}\right)\right|\left|\dot{x}\left(\tau_{2}\right)\right|}{\left(x\left(\tau_{1}\right)-x\left(\tau_{2}\right)\right)^{2}}= \\
& =\frac{x_{\nu}\left(s_{1}\right)}{\left(x_{1}-x_{2}\right)^{2}}\left[\left(\eta_{\rho \sigma}-\frac{\dot{x}_{\sigma}\left(\tau_{2}\right) \dot{x}_{\rho}\left(\tau_{1}\right)}{\left|\dot{x}\left(\tau_{2}\right)\right|\left|\dot{x}\left(\tau_{1}\right)\right|}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{2}\right)\right. \\
& +\left(\eta_{\rho \sigma}-\frac{\dot{x}_{\sigma}\left(\tau_{1}\right) \dot{x}_{\rho}\left(\tau_{2}\right)}{\left|\dot{x}\left(\tau_{1}\right)\right|\left|\dot{x}\left(\tau_{2}\right)\right|}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{1}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right) \\
& +\left(\frac{\dot{x}_{\sigma}\left(\tau_{1}\right) \dot{x}_{\rho}\left(\tau_{1}\right)\left|\dot{x}\left(\tau_{2}\right)\right|}{\left|\dot{x}\left(\tau_{1}\right)\right|^{3}}-\eta_{\rho \sigma} \frac{\left|\dot{x}\left(\tau_{2}\right)\right|}{\left|\dot{x}\left(\tau_{1}\right)\right|}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right) \\
& \left.+\left(\frac{\dot{x}_{\sigma}\left(\tau_{2}\right) \dot{x}_{\rho}\left(\tau_{2}\right)\left|\dot{x}\left(\tau_{1}\right)\right|}{\left|\dot{x}\left(\tau_{2}\right)\right|^{3}}-\eta_{\rho \sigma} \frac{\left|\dot{x}\left(\tau_{1}\right)\right|}{\left|\dot{x}\left(\tau_{2}\right)\right|}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{1}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{2}\right)\right] \\
& -\frac{2 x_{\nu}\left(s_{1}\right)}{\left(x_{1}-x_{2}\right)^{4}}\left\{( x _ { \rho } ( \tau _ { 1 } ) - x _ { \rho } ( \tau _ { 2 } ) ) ( \delta ( \tau _ { 1 } - s _ { 1 } ) - \delta ( \tau _ { 2 } - s _ { 1 } ) ) \left[\dot{x}_{\sigma}\left(\tau_{1}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{2}\right)\right.\right. \\
& \left.+\dot{x}_{\sigma}\left(\tau_{2}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right)-\frac{\left|\dot{x}\left(\tau_{1}\right)\right|}{\left|\dot{x}\left(\tau_{2}\right)\right|} \dot{x}_{\sigma}\left(\tau_{2}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{2}\right)-\frac{\left|\dot{x}\left(\tau_{2}\right)\right|}{\left|\dot{x}\left(\tau_{1}\right)\right|} \dot{x}_{\sigma}\left(\tau_{1}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\left(x_{\sigma}\left(\tau_{1}\right)-x_{\sigma}\left(\tau_{2}\right)\right)\left(\delta\left(\tau_{1}-s_{2}\right)-\delta\left(\tau_{2}-s_{2}\right)\right)\left[\dot{x}_{\rho}\left(\tau_{1}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{1}\right)\right. \\
& \left.+\dot{x}_{\rho}\left(\tau_{2}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right)-\frac{\left|\dot{x}\left(\tau_{1}\right)\right|}{\left|\dot{x}\left(\tau_{2}\right)\right|} \dot{x}_{\rho}\left(\tau_{2}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{1}\right)-\frac{\left|\dot{x}\left(\tau_{2}\right)\right|}{\left|\dot{x}\left(\tau_{1}\right)\right|} \dot{x}_{\rho}\left(\tau_{1}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right)\right] \\
& \left.+\left(\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|\right) \eta_{\rho \sigma}\left(\delta\left(\tau_{1}-s_{2}\right)-\delta\left(\tau_{2}-s_{2}\right)\right)\left(\delta\left(\tau_{1}-s_{1}\right)-\delta\left(\tau_{2}-s_{1}\right)\right)\right\} \\
& +\frac{8 x_{\nu}\left(s_{1}\right)}{\left(x_{1}-x_{2}\right)^{6}}\left(\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|\right)\left(x_{\sigma}\left(\tau_{1}\right)-x_{\sigma}\left(\tau_{2}\right)\right)\left(x_{\rho}\left(\tau_{1}\right)-x_{\rho}\left(\tau_{2}\right)\right) . \\
& \cdot\left(\delta\left(\tau_{1}-s_{2}\right)-\delta\left(\tau_{2}-s_{2}\right)\right)\left(\delta\left(\tau_{1}-s_{1}\right)-\delta\left(\tau_{2}-s_{1}\right)\right) .
\end{aligned}
$$

We order the above result by the structure of the delta functions and derivatives that appear in it. To abbreviate, we use that the Wilson loop integral is symmetric under $\left(\tau_{1} \leftrightarrow \tau_{2}\right)$. Additionally, we now fix the parametrization to be of unit-speed, demanding that $|\dot{x}|=i$. Then we get the following expression (where by $\hat{=}$ we mean that the expression on the right-hand side gives the same result when integrated over $\tau_{1}$ and $\tau_{2}$ in parametrization by arc-length):

$$
\begin{align*}
& x_{\nu}\left(s_{1}\right) \frac{\delta}{\delta x^{\rho}\left(s_{1}\right)} \frac{\delta}{\delta x^{\sigma}\left(s_{2}\right)} \frac{\dot{x}\left(\tau_{1}\right) \dot{x}\left(\tau_{2}\right)-\left|\dot{x}\left(\tau_{1}\right)\right|\left|\dot{x}\left(\tau_{2}\right)\right|}{\left(x\left(\tau_{1}\right)-x\left(\tau_{2}\right)\right)^{2}} \hat{=} \\
& \hat{=} x_{\nu}\left(s_{1}\right)\left\{F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right)+F_{\rho \sigma}^{(2)}\left(\tau_{1}, \tau_{2}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{2}\right)\right. \\
& \quad+F_{\rho \sigma}^{(3)}\left(\tau_{1}, \tau_{2}\right)\left(\delta\left(\tau_{1}-s_{1}\right)-\delta\left(\tau_{2}-s_{1}\right)\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right) \\
& \quad+F_{\rho \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right)\left(\delta\left(\tau_{1}-s_{2}\right)-\delta\left(\tau_{2}-s_{2}\right)\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right) \\
& \left.\quad+F_{\rho \sigma}^{(4)}\left(\tau_{1}, \tau_{2}\right)\left(\delta\left(\tau_{1}-s_{2}\right)-\delta\left(\tau_{2}-s_{2}\right)\right)\left(\delta\left(\tau_{1}-s_{1}\right)-\delta\left(\tau_{2}-s_{1}\right)\right)\right\} . \tag{6.12}
\end{align*}
$$

Here, we defined:

$$
\begin{align*}
& F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right):=-\frac{2}{x_{12}^{2}}\left(\dot{x}_{1 \rho} \dot{x}_{1 \sigma}+\eta_{\rho \sigma}\right), \quad F_{\rho \sigma}^{(2)}\left(\tau_{1}, \tau_{2}\right):=\frac{2}{x_{12}^{2}}\left(\dot{x}_{1 \rho} \dot{x}_{2 \sigma}+\eta_{\rho \sigma}\right),  \tag{6.13}\\
& F_{\rho \sigma}^{(3)}\left(\tau_{1}, \tau_{2}\right):=\frac{4}{x_{12}^{4}} x_{12 \rho} \dot{x}_{12 \sigma}, \quad F_{\rho \sigma}^{(4)}\left(\tau_{1}, \tau_{2}\right):=2 \frac{\dot{x_{1}} \dot{x_{2}}+1}{x_{12}^{4}}\left(4 \frac{x_{12 \rho} x_{12 \sigma}}{x_{12}^{2}}-\eta_{\rho \sigma}^{2}\right) . \tag{6.14}
\end{align*}
$$

We denote the contribution to $P_{\text {bos }, \mathrm{nl}, \varepsilon}^{(1) \mu}\langle W(C)\rangle_{(1)}$ of the above terms by $C_{i}^{\mu}$,

$$
\begin{equation*}
C_{i}^{\mu}:=\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left\{\left(\theta\left(s_{2}-s_{1}-\varepsilon\right) c_{i}^{\mu}\left(\tau_{1}, \tau_{2}, s_{1}, s_{2}\right)\right)+(\varepsilon \rightarrow-\varepsilon)\right\}, \tag{6.15}
\end{equation*}
$$

and we define:

$$
\begin{align*}
& c_{1}^{\mu}\left(\tau_{1}, \tau_{2}, s_{1}, s_{2}\right):=N^{\mu \nu \rho \sigma} x_{\nu}\left(s_{1}\right) F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right),  \tag{6.16}\\
& c_{2}^{\mu}\left(\tau_{1}, \tau_{2}, s_{1}, s_{2}\right):=N^{\mu \nu \rho \sigma} x_{\nu}\left(s_{1}\right) F_{\rho \sigma}^{(2)}\left(\tau_{1}, \tau_{2}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right) \partial_{\tau_{2}} \delta\left(\tau_{2}-s_{2}\right),  \tag{6.17}\\
& c_{3}^{\mu}\left(\tau_{1}, \tau_{2}, s_{1}, s_{2}\right):=N^{\mu \nu \rho \sigma} x_{\nu}\left(s_{1}\right) F_{\rho \sigma}^{(3)}\left(\tau_{1}, \tau_{2}\right)\left(\delta\left(\tau_{1}-s_{1}\right)-\delta\left(\tau_{2}-s_{1}\right)\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right),  \tag{6.18}\\
& c_{4}^{\mu}\left(\tau_{1}, \tau_{2}, s_{1}, s_{2}\right):=N^{\mu \nu \rho \sigma} x_{\nu}\left(s_{1}\right) F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right)\left(\delta\left(\tau_{1}-s_{2}\right)-\delta\left(\tau_{2}-s_{2}\right)\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right),  \tag{6.19}\\
& c_{5}^{\mu}\left(\tau_{1}, \tau_{2}, s_{1}, s_{2}\right):=N^{\mu \nu \rho \sigma} x_{\nu}\left(s_{1}\right) F_{\rho \sigma}^{(5)}\left(\tau_{1}, \tau_{2}\right)\left(\delta\left(\tau_{1}-s_{2}\right)-\delta\left(\tau_{2}-s_{2}\right)\right) . \\
& \cdot\left(\delta\left(\tau_{1}-s_{1}\right)-\delta\left(\tau_{2}-s_{1}\right)\right) . \tag{6.20}
\end{align*}
$$

With these definitions, we have:

$$
P_{\mathrm{bos}, \mathrm{nl}, \varepsilon}^{(1) \mu}\langle W(C)\rangle_{(1)}=-\frac{\lambda}{16 \pi^{2}} \sum_{i=1}^{5} C_{i}^{\mu} .
$$

We first discuss these terms separately, integrating out the delta-functions. For explicitness, we spell out the calculation for $C_{1}^{\mu}$. Consider the integral ${ }^{2}$

$$
\begin{aligned}
& \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \int_{0}^{L} \mathrm{~d} s_{1} \mathrm{~d} s_{2} \theta\left(s_{2}-s_{1}-\varepsilon\right) x_{\nu}\left(s_{1}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{1}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s_{2}\right) \\
= & -\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \int_{0}^{L} \mathrm{~d} s_{1} x_{\nu}\left(s_{1}\right) \delta\left(\tau_{1}-s_{1}-\varepsilon\right) \partial_{s_{1}} \delta\left(\tau_{1}-s_{1}\right) \\
= & \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \int_{0}^{L} \mathrm{~d} s_{1} \delta\left(\tau_{1}-s_{1}\right) \partial_{s_{1}}\left(x_{\nu}\left(s_{1}\right) \delta\left(\tau_{1}-s_{1}-\varepsilon\right)\right) \\
= & \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{\nu}\left(\tau_{1}\right) \delta(-\varepsilon) \\
& +\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \int_{0}^{L} \mathrm{~d} s_{1} x_{\nu}\left(s_{1}\right) \delta\left(\tau_{1}-s_{1}\right) \partial_{\varepsilon} \delta\left(\tau_{1}-s_{1}-\varepsilon\right) \\
= & \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{\nu}\left(\tau_{1}\right) \delta(\varepsilon)+\partial_{\varepsilon} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) x_{\nu}\left(\tau_{1}\right) \delta(\varepsilon) .
\end{aligned}
$$

Using that $\delta(\varepsilon)=\delta(-\varepsilon)$ and $\partial_{\varepsilon} \delta(\varepsilon)=-\partial_{-\varepsilon} \delta(-\varepsilon)$ we get:

$$
\begin{equation*}
C_{1}^{\mu}=2 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{\nu}\left(\tau_{1}\right) \delta(\varepsilon) \tag{6.21}
\end{equation*}
$$

In a similar fashion we get the following results:

$$
\begin{align*}
C_{2}^{\mu}= & \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(2)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{\nu}\left(\tau_{1}\right)\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\delta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right), \\
& +\partial_{\varepsilon} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(2)}\left(\tau_{1}, \tau_{2}\right) x_{\nu}\left(\tau_{1}\right)\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)-\delta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right),  \tag{6.22}\\
C_{3}^{\mu}= & 2 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(3)}\left(\tau_{1}, \tau_{2}\right) x_{\nu}\left(\tau_{1}\right) \delta(\varepsilon), \\
& -\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(3)}\left(\tau_{1}, \tau_{2}\right) x_{\nu}\left(\tau_{1}\right)\left(\delta\left(\tau_{1}-\tau_{2}-\varepsilon\right)+\delta\left(\tau_{1}-\tau_{2}+\varepsilon\right)\right), \tag{6.23}
\end{align*}
$$

[^14]\[

$$
\begin{align*}
C_{4}^{\mu}= & \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{\nu}\left(\tau_{1}\right)-2 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right) x_{\nu}\left(\tau_{1}\right) \delta(\varepsilon) \\
& +\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right) x_{\nu}\left(\tau_{1}\right)\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\delta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right) \\
& -\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{\nu}\left(\tau_{1}\right)\left(\theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\theta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right)  \tag{6.24}\\
C_{5}^{\mu}= & -2 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(4)}\left(\tau_{1}, \tau_{2}\right)\left(x_{\nu}\left(\tau_{1}\right)-x_{\nu}\left(\tau_{2}\right)\right) \theta\left(\tau_{2}-\tau_{1}-\varepsilon\right) \tag{6.25}
\end{align*}
$$
\]

Taking into account that $N^{\mu \nu \rho \sigma} A_{[\rho \sigma] \nu}=0$, we can simplify $C_{3}^{\mu}+C_{4}^{\mu}$ to get

$$
\begin{align*}
C_{3}^{\mu}+C_{4}^{\mu}= & \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{\nu}\left(\tau_{1}\right) \\
& -\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{\nu}\left(\tau_{1}\right)\left(\theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\theta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right) \\
= & -\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} N^{\mu \nu \rho \sigma} F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right)\left(\dot{x}_{\nu}\left(\tau_{1}\right)-\dot{x}_{\nu}\left(\tau_{2}\right)\right)\left(\theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)\right) . \tag{6.26}
\end{align*}
$$

Inserting the contractions

$$
\begin{align*}
& N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{1 \nu}=-6 \frac{\dot{x}_{1}^{\mu}}{x_{12}^{2}}, \quad N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(2)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{1 \nu}=2 \frac{2 \dot{x}_{1}^{\mu}+\dot{x}_{2}^{\mu}}{x_{12}^{2}},  \tag{6.27}\\
& N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(2)}\left(\tau_{1}, \tau_{2}\right) x_{1 \nu}=\frac{2}{x_{12}^{2}}\left(2 x_{1}^{\mu}+x_{1}^{\mu}\left(\dot{x}_{1} \dot{x}_{2}\right)-\dot{x}_{1}^{\mu}\left(x_{1} \dot{x}_{2}\right)-\dot{x}_{2}^{\mu}\left(x_{1} \dot{x}_{1}\right)\right),  \tag{6.28}\\
& N^{\mu \nu \rho \sigma} F_{\sigma \rho}^{(3)}\left(\tau_{1}, \tau_{2}\right) \dot{x}_{12 \nu}=8 \frac{\dot{x}_{1} \dot{x}_{2}+1}{x_{12}^{4}} x_{12}^{\mu},  \tag{6.29}\\
& N^{\mu \nu \rho \sigma} F_{\rho \sigma}^{(4)}\left(\tau_{1}, \tau_{2}\right) x_{12 \nu}=-12 \frac{\dot{x}_{1} \dot{x}_{2}+1}{x_{12}^{4}} x_{12}^{\mu}, \tag{6.30}
\end{align*}
$$

into (6.21) - 6.26) and using the $\left(\tau_{1} \leftrightarrow \tau_{2}\right)$-symmetry of the integral we arrive at:

$$
\begin{align*}
& P_{\text {bos, }, \mathrm{l}, \varepsilon}^{(1)}\langle W(C)\rangle_{(1)}=-\frac{\lambda}{16 \pi^{2}} \sum_{i=1}^{5} C_{i}^{\mu}=-\frac{\lambda}{16 \pi^{2}}\left\{-6 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1}^{\mu}+\dot{x}_{2}^{\mu}}{\left(x_{1}-x_{2}\right)^{2}} \delta(\varepsilon)\right. \\
& +6 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1}^{\mu}}{\left(x_{1}-x_{2}\right)^{2}}\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\delta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right)+ \\
& +2 \partial_{\varepsilon} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{2 x_{1}^{\mu}+x_{1}^{\mu}\left(\dot{x}_{1} \dot{x}_{2}\right)-\dot{x}_{1}^{\mu}\left(x_{1} \dot{x}_{2}\right)-\dot{x}_{2}^{\mu}\left(x_{1} \dot{x}_{1}\right)}{\left(x_{1}-x_{2}\right)^{2}}\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)-\delta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right) \\
& \left.+16 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}+1}{\left(x_{1}-x_{2}\right)^{4}}\left(x_{1}-x_{2}\right)^{\mu} \theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)\right\} . \tag{6.31}
\end{align*}
$$

We now expand the two middle terms. At this point the arc-length parametrization turns out to be very useful, not only because of the absence of expressions like $\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|$, but also because it provides the following identities, which we will use frequently:

$$
\dot{x}^{2} \equiv-1 \Rightarrow \dot{x} \ddot{x} \equiv 0 \Rightarrow \ddot{x}^{2}+\dot{x} x^{(3)} \equiv 0 \Rightarrow 3 \ddot{x} x^{(3)}+\dot{x} x^{(4)} \equiv 0 .
$$

First, we expand the common denominator of the two terms up to order $\varepsilon^{0}$, which will turn out to suffice for our purposes. Noting that

$$
\begin{aligned}
(x(\tau+\varepsilon)-x(\tau))^{2} & =\varepsilon^{2}\left(-1+\frac{1}{4} \varepsilon^{2} \ddot{x}(\tau)^{2}+\frac{1}{3} \varepsilon^{2} \dot{x}(\tau) \cdot x^{(3)}(\tau)+\mathcal{O}\left(\varepsilon^{3}\right)\right) \\
& =-\varepsilon^{2}\left(1+\frac{1}{12} \varepsilon^{2} \ddot{x}(\tau)^{2}+\mathcal{O}\left(\varepsilon^{3}\right)\right)
\end{aligned}
$$

we find:

$$
\begin{equation*}
(x(\tau+\varepsilon)-x(\tau))^{-2}=-\varepsilon^{-2}+\frac{1}{12} \ddot{x}(\tau)^{2}+\mathcal{O}(\varepsilon) \tag{6.32}
\end{equation*}
$$

Thus we immediately find:

$$
\begin{align*}
6 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} & \frac{\dot{x}_{1}^{\mu}}{\left(x_{1}-x_{2}\right)^{2}}\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\delta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right)= \\
& =-\frac{12}{\varepsilon^{2}} \int \mathrm{~d} \tau \dot{x}^{\mu}(\tau)+\int_{0}^{L} \mathrm{~d} \tau \dot{x}^{\mu}(\tau) \ddot{x}(\tau)^{2}+\mathcal{O}(\varepsilon) \tag{6.33}
\end{align*}
$$

The expansion of the second term is more involved. Although the form it takes in (6.31) seems to be convenient for an expansion, it is actually easier to expand the antisymmetrized version

$$
\begin{gathered}
\Omega:=\partial_{\varepsilon} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{2\left(x_{1}-x_{2}\right)^{\mu}+\left(x_{1}-x_{2}\right)^{\mu}\left(\dot{x}_{1} \dot{x}_{2}\right)-\dot{x}_{1}^{\mu}\left(x_{1}-x_{2}\right) \dot{x}_{2}-\dot{x}_{2}^{\mu}\left(x_{1}-x_{2}\right) \dot{x}_{1}}{\left(x_{1}-x_{2}\right)^{2}} \\
\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)-\delta\left(\tau_{2}-\tau_{1}+\varepsilon\right)\right) .
\end{gathered}
$$

We expand the numerator up to order $\varepsilon^{3}$. To abbreviate, we use the short-hand notation $x=x_{1}=x(\tau), x_{2}=x(\tau+\varepsilon)$. We find the following expansion:

$$
\begin{align*}
& 2\left(x_{1}-x_{2}\right)^{\mu}+\left(x_{1}-x_{2}\right)^{\mu}\left(\dot{x}_{1} \dot{x}_{2}\right)-\dot{x}_{1}^{\mu}\left(x_{1}-x_{2}\right) \dot{x}_{2}-\dot{x}_{2}^{\mu}\left(x_{1}-x_{2}\right) \dot{x}_{1}= \\
& =-2\left(\varepsilon \dot{x}^{\mu}+\frac{1}{2} \varepsilon^{2} \ddot{x}^{\mu}+\frac{1}{6} \varepsilon^{3} x^{(3) \mu}\right)-\left(\varepsilon \dot{x}^{\mu}+\frac{1}{2} \varepsilon^{2} \ddot{x}^{\mu}+\frac{1}{6} \varepsilon^{3} x^{(3) \mu}\right) \dot{x} \dot{x}_{2}+\mathcal{O}\left(\varepsilon^{4}\right) \\
& \quad+\dot{x}^{\mu}\left(\varepsilon \dot{x}+\frac{1}{2} \varepsilon^{2} \ddot{x}+\frac{1}{6} \varepsilon^{3} x^{(3)}\right) \dot{x}_{2}+\left(\dot{x}^{\mu}+\varepsilon \ddot{x}^{\mu}+\frac{1}{2} \varepsilon^{2} x^{(3) \mu}\right) \dot{x}\left(\varepsilon \dot{x}+\frac{1}{2} \varepsilon^{2} \ddot{x}+\frac{1}{6} \varepsilon^{3} x^{(3)}\right) \\
& =-2\left(\varepsilon \dot{x}^{\mu}+\frac{1}{2} \varepsilon^{2} \ddot{x}^{\mu}+\frac{1}{6} \varepsilon^{3} x^{(3) \mu}\right)-\left(\frac{1}{2} \varepsilon^{2} \ddot{x}^{\mu}+\frac{1}{6} \varepsilon^{3} x^{(3) \mu}\right) \dot{x}(\dot{x}+\varepsilon \ddot{x})+\mathcal{O}\left(\varepsilon^{4}\right) \\
& \quad+\dot{x}^{\mu}\left(\frac{1}{2} \varepsilon^{2} \ddot{x}+\frac{1}{6} \varepsilon^{3} x^{(3)}\right)(\dot{x}+\varepsilon \ddot{x})-\left(\dot{x}^{\mu}+\varepsilon \ddot{x}^{\mu}+\frac{1}{2} \varepsilon^{2} x^{(3) \mu}\right)\left(\varepsilon+\frac{1}{6} \varepsilon^{3} \ddot{x}^{2}\right) \\
& =-3 \varepsilon \dot{x}^{\mu}-\frac{3}{2} \varepsilon^{2} \ddot{x}^{\mu}-\frac{2}{3} \varepsilon^{3} x^{(3) \mu}+\frac{1}{6} \varepsilon^{3} \dot{x}^{\mu} \ddot{x}^{2}+\mathcal{O}\left(\varepsilon^{4}\right) \tag{6.34}
\end{align*}
$$

Combining (6.32) with (6.34), we find that

$$
\begin{equation*}
\Omega=-\frac{6}{\varepsilon^{2}} \int \mathrm{~d} \tau \dot{x}^{\mu}(\tau)+\frac{4}{3} \int_{0}^{L} \mathrm{~d} \tau x^{(3) \mu}(\tau)-\frac{5}{6} \int_{0}^{L} \mathrm{~d} \tau \dot{x}^{\mu}(\tau) \ddot{x}^{2}(\tau)+\mathcal{O}(\varepsilon) . \tag{6.35}
\end{equation*}
$$

Inserting (6.33) and (6.35) into (6.31) and using that

$$
\int \mathrm{d} \tau \dot{x}^{\mu}(\tau)=0=\int \mathrm{d} \tau x^{(3) \mu}(\tau)
$$

which certainly holds for smooth periodic curves, we get:

$$
\begin{align*}
P_{\text {bos, nl, } \varepsilon}^{(1) \mu}\langle W(C)\rangle_{(1)}= & -\frac{\lambda}{16 \pi^{2}}\left\{\frac{1}{6} \int_{0}^{L} \mathrm{~d} \tau \dot{x}^{\mu}(\tau) \ddot{x}^{2}(\tau)-6 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1}^{\mu}+\dot{x}_{2}^{\mu}}{\left(x_{1}-x_{2}\right)^{2}} \delta(\varepsilon)\right. \\
& \left.+16 \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}+1}{\left(x_{1}-x_{2}\right)^{4}}\left(x_{1}-x_{2}\right)^{\mu} \theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\mathcal{O}(\varepsilon)\right\} . \tag{6.36}
\end{align*}
$$

The second term is a double curve integral over a delta function which has a strictly positive argument for any finite value of the regulator $\varepsilon$ and we therefore drop this term.

Moreover in the above result, the parametrization of the curve is still fixed to arc-length, which makes explicit evaluations for example curves difficult. Hence we are interested in a reparametrization invariant form of the above result. This is also necessary to consider functional variations of the result. For the last term in 6.36) this is obvious, but for the first term it is more difficult to find. Fortunately, we can employ an analogy to the theory of spatial curves to rewrite

$$
\int_{0}^{L} \mathrm{~d} \tau \dot{x}^{\mu}(\tau) \ddot{x}^{2}(\tau)
$$

as a proper curve integral. For a spatial curve in $\mathbb{R}^{3}$ which is parametrized by arclength, the function $\kappa(\tau)=\|\ddot{x}(\tau)\|$ describes the planar curvature. From elementary differential geometry, there is another formula available for $\kappa(\tau)$ which holds for any parametrization,

$$
\kappa(\tau)=\frac{\|\dot{x}(\tau) \times \ddot{x}(\tau)\|}{\|\dot{x}(\tau)\|^{3}} .
$$

Here, $\times$ denotes the vector-product in $\mathbb{R}^{3}$. To generalize to arbitrary dimensions we rewrite:

$$
\kappa(\tau)^{2}=\frac{\dot{x}(\tau)^{2} \ddot{x}(\tau)^{2}-(\dot{x}(\tau) \cdot \ddot{x}(\tau))^{2}}{\dot{x}(\tau)^{6}}
$$

This leads us to the following claim:

$$
\int_{0}^{L} \mathrm{~d} \tau \dot{x}^{\mu}(\tau) \ddot{x}^{2}(\tau)=\int \mathrm{d} \tau \dot{x}^{\mu}(\tau) \frac{\dot{x}(\tau)^{2} \ddot{x}(\tau)^{2}-(\dot{x}(\tau) \cdot \ddot{x}(\tau))^{2}}{\dot{x}(\tau)^{6}}
$$

Having guessed the right expression, it is easy to prove the above result. As $\dot{x}^{2} \equiv-1$ implies $\dot{x} \cdot \ddot{x} \equiv 0$, the statement clearly holds true if the right-hand-side is also given in parametrization by arc-length. Thus, one only has to show that the right-hand-side is really a reparametrization invariant curve integral. We do this in appendix C.2.

We can hence lift the constraint of arc-length parametrization, finding the following reparametrization invariant result:

$$
\begin{align*}
P_{\text {bos, } \text {,l, }, \varepsilon}^{(1) \mu}\langle W(C)\rangle_{(1)} & =-\frac{\lambda}{16 \pi^{2}}\left\{\frac{1}{6} \int \mathrm{~d} \tau \dot{x}^{\mu}(\tau)\left(\frac{\ddot{x}^{2}}{\dot{x}^{4}}-\frac{(\dot{x} \cdot \ddot{x})^{2}}{\dot{x}^{6}}\right)+\mathcal{O}(\varepsilon)\right. \\
& \left.+16 \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{4}}\left(x_{1}-x_{2}\right)^{\mu} \theta\left(\tau_{2}-\tau_{1}-d\left(\tau_{2}, \varepsilon\right)\right)\right\} . \tag{6.37}
\end{align*}
$$

From the discussion of the complete level-1 momentum generator at the beginning of this chapter, we recall that we still have the freedom to choose the level-1 density $p^{(1) \mu}(s)$. However, this will not compensate for the second term of the above result. We therefore conclude that the Maldacena-Wilson loop is not invariant under the

Yangian symmetry $Y(\mathfrak{s o}(2,4))$ we have discussed. At this point, the reader may ask, how one can be sure that it is impossible to rewrite the second term of (6.37) as a single curve integral. After all, also the double integral assigns a four-vector to a given curve. Before turning to this question, let us discuss the structure of our result in some detail.

### 6.1.1 Discussion of the Result

As the second term in (6.37) is the one that prohibits the Yangian symmetry of the Maldacena-Wilson loop, we start with it. At first, we notice that it is actually divergent in the limit $\varepsilon \rightarrow 0$ for a large class of curves. This can be seen by taking the derivative with respect to $\varepsilon$. Reverting again to arc-length parametrization, we find:

$$
\partial_{\varepsilon} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}+1}{\left(x_{1}-x_{2}\right)^{2}}\left(x_{1}-x_{2}\right)^{\mu} \theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)=-\frac{1}{2 \varepsilon} \int_{0}^{L} \mathrm{~d} \tau \dot{x}^{\mu}(\tau) \ddot{x}(\tau)^{2}+\mathcal{O}\left(\varepsilon^{0}\right) .
$$

This shows that the result (6.37) is logarithmically divergent,

$$
\begin{aligned}
& 16 \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{4}}\left(x_{1}-x_{2}\right)^{\mu} \theta\left(\tau_{2}-\tau_{1}-d\left(\tau_{2}, \varepsilon\right)\right)= \\
& =-8 \ln \varepsilon \int \mathrm{~d} \tau \dot{x}^{\mu}(\tau)\left(\frac{\ddot{x}^{2}}{\dot{x}^{4}}-\frac{(\dot{x} \cdot \ddot{x})^{2}}{\dot{x}^{6}}\right)+\alpha^{\mu}(C)+\mathcal{O}(\varepsilon),
\end{aligned}
$$

The constant term $\alpha^{\mu}(C)$ lies beyond the scope of our above discussion. Using this form of our result, we are now in a position to answer the legitimate question, whether we could just have regulated in parameter space which would have simplified our formulas. While this clearly would not have given reparametrization invariant curve integrals for finite $\varepsilon$ one could have hoped that the limit $\varepsilon \rightarrow 0$ would be unaffected. To answer this question, consider a scaling reparametrization

$$
\tau:[\lambda a, \lambda b] \rightarrow[a, b], \quad \tau(s)=\frac{s}{\lambda}
$$

As the integrand in

$$
\int \mathrm{d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{4}}\left(x_{1}-x_{2}\right)^{\mu} \theta\left(\tau_{2}-\tau_{1}-d\left(\tau_{2}, \varepsilon\right)\right)
$$

is invariant, the scaling only affects the boundaries of the integral, which results in $\varepsilon \rightarrow \lambda \varepsilon$. Expanding then as before, we get:

$$
\alpha^{\mu}(C) \rightarrow \alpha^{\mu}(C)-8 \ln \lambda \int \mathrm{~d} \tau \dot{x}^{\mu}(\tau)\left(\frac{\ddot{x}^{2}}{\dot{x}^{4}}-\frac{(\dot{x} \cdot \ddot{x})^{2}}{\dot{x}^{6}}\right) .
$$

Hence, the reparametrization invariance would have been violated at order $\varepsilon^{0}$.
To understand the coefficient of the divergence better, consider the following example curve:

$$
x(\tau)=(0, \cos \tau+\cos 2 \tau, \sin \tau, 0)
$$



Figure 6.1: Sketch of the example curve $x(\tau)=(0, \cos \tau+\cos 2 \tau, \sin \tau, 0)$. We have indicated the tangent vector $\dot{x}$ and $\ddot{x}$ in arc-length parametrization at two points related to each other by the mirror symmetry.

Numerically, one finds the following result:

$$
\int \mathrm{d} \tau \dot{x}^{\mu}(\tau)\left(\frac{\ddot{x}^{2}}{\dot{x}^{4}}-\frac{(\dot{x} \cdot \ddot{x})^{2}}{\dot{x}^{6}}\right)=\left(0,-8 \cdot 10^{-14},-148.3,0\right)
$$

The (numerical) zero in the second entry reflects that our example curve has a mirror symmetry about the $x^{1}$-axis, see figure 6.1. For any curve that has a mirror symmetry about some axis, the component of the curve integral parallel to this axis will vanish: Consider the curve in parametrization by arc-length and match the points related to each other by the mirror symmetry. The expression $\ddot{x}^{2}$ at the related points must be the same, while the components of the velocity vectors parallel to the symmetry axis have opposite signs. Hence this component of the local term vanishes. As a consequence, we find that for all curves which are contained in a two-dimensional spatial subspace of $\mathbb{R}^{(1,3)}$ and have two symmetry axes - e.g. an ellipsis - the local term vanishes and thence that our result (6.37) is not divergent as $\varepsilon \rightarrow 0$.
The constant term $\alpha^{\mu}(C)$ is difficult to evaluate for a general curve. This becomes easier for curves for which the coefficient of the divergence vanishes. The easiest example is a circle in a spacelike subspace of Minkowski space. It will be instructive to work out the bosonic result for this curve. Consider the parametrization

$$
x(\tau)=(0, \cos \tau, \sin \tau, 0) .
$$

As the result is non-divergent, we can safely take the limit $\varepsilon \rightarrow 0$. Abbreviating $c_{i}:=\cos \tau_{i}, s_{i}:=\sin \tau_{i}$, we have:

$$
\lim _{\varepsilon \rightarrow 0} P_{\text {bos }, \varepsilon}^{(1) 1}\langle W(C)\rangle_{1}=-\frac{\lambda}{\pi^{2}} \int_{0}^{2 \pi} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \theta\left(\tau_{2}-\tau_{1}\right) \frac{c_{1}-c_{2}}{4\left(1-c_{1} c_{2}-s_{1} s_{2}\right)}=
$$

$$
\begin{aligned}
= & \int_{0}^{\pi} \mathrm{d} \tau_{1} \mathrm{~d} \tau_{2} \theta\left(\tau_{2}-\tau_{1}\right) \frac{c_{1}-c_{2}}{4\left(1-c_{1} c_{2}-s_{1} s_{2}\right)}+\int_{\pi}^{2 \pi} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \theta\left(\tau_{2}-\tau_{1}\right) \frac{c_{1}-c_{2}}{4\left(1-c_{1} c_{2}-s_{1} s_{2}\right)} \\
& +\int_{0}^{\pi} \mathrm{d} \tau_{1} \int_{\pi}^{2 \pi} \mathrm{~d} \tau_{2} \frac{c_{1}-c_{2}}{4\left(1-c_{1} c_{2}-s_{1} s_{2}\right)} \\
= & \int_{0}^{\pi} \mathrm{d} \tau_{1} \mathrm{~d} \tau_{2} \theta\left(\tau_{2}-\tau_{1}\right) \frac{c_{1}-c_{2}}{4\left(1-c_{1} c_{2}-s_{1} s_{2}\right)}+\int_{0}^{\pi} \mathrm{d} \tau_{1}^{\prime} \mathrm{d} \tau_{2}^{\prime} \theta\left(\tau_{2}^{\prime}-\tau_{1}^{\prime}\right) \frac{c_{2}^{\prime}-c_{1}^{\prime}}{4\left(1-c_{1}^{\prime} c_{2}^{\prime}-s_{1}^{\prime} s_{2}^{\prime}\right)} \\
& +\int_{0}^{\pi} \mathrm{d} \tau_{1} \mathrm{~d} \tau_{2}^{\prime} \frac{c_{1}+c_{2}^{\prime}}{4\left(1+c_{1} c_{2}^{\prime}+s_{1} s_{2}^{\prime}\right)} .
\end{aligned}
$$

In the last step, we substituted $\tau_{i}^{\prime}=\tau_{i}-\pi$. The first two terms cancel and we are left with

$$
\lim _{\varepsilon \rightarrow 0} P_{\text {bos }, \mathrm{nl}, \varepsilon}^{(1) \mu}\langle W(C)\rangle_{(1)}=\int_{0}^{\pi} \mathrm{d} \tau_{1} \mathrm{~d} \tau_{2} \frac{c_{1}+c_{2}}{4\left(1+c_{1} c_{2}+s_{1} s_{2}\right)}
$$

By performing the substitution $\tau_{i}^{\prime}=\pi-\tau_{i}$, we see that

$$
\lim _{\varepsilon \rightarrow 0} P_{\text {bos }, \mathrm{nl}, \varepsilon}^{(1) 1}\langle W(C)\rangle_{(1)}=-\lim _{\varepsilon \rightarrow 0} P_{\text {bos }, \mathrm{nl}, \varepsilon}^{(1) 1}\langle W(C)\rangle_{(1)} \quad \Rightarrow \lim _{\varepsilon \rightarrow 0} P_{\text {bos }, \mathrm{nl}, \varepsilon}^{(1) 1}\langle W(C)\rangle_{(1)}=0 .
$$

Based on symmetry arguments one might now expect, that also $P_{\text {bos }, \varepsilon}^{(1) 2}\langle W(C)\rangle_{1}$ vanishes. But this is not the case. We find:

$$
\begin{aligned}
\lim _{\varepsilon \rightarrow 0} P_{\text {bos, nl, },}^{(1) 2}\langle W(C)\rangle_{(1)} & =-\frac{\lambda}{\pi^{2}} \int_{0}^{2 \pi} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \theta\left(\tau_{2}-\tau_{1}\right) \frac{s_{1}-s_{2}}{4\left(1-c_{1} c_{2}-s_{1} s_{2}\right)} \\
& =-\frac{\lambda}{4 \pi^{2}} \int_{0}^{\pi} \mathrm{d} \tau_{1} \mathrm{~d} \tau_{2} \frac{s_{1}+s_{2}}{1+c_{1} c_{2}+s_{1} s_{2}} .
\end{aligned}
$$

To reach the last line, we performed the same substitutions as for the other component. The above result must be different from zero as

$$
\frac{\sin \tau_{1}+\sin \tau_{2}}{1+\cos \tau_{1} \cos \tau_{2}+\sin \tau_{1} \sin \tau_{2}}=\frac{\sin \tau_{1}+\sin \tau_{2}}{1+\cos \left(\tau_{1}-\tau_{2}\right)}>0 \quad \forall \tau_{1}, \tau_{2} \in(0, \pi) .
$$

A numerical evaluation shows that

$$
\lim _{\varepsilon \rightarrow 0} P_{\mathrm{bos}, \mathrm{nl}, \varepsilon}^{(1) 2}\langle W(C)\rangle_{(1)} \simeq-\frac{\lambda}{\pi} .
$$

The reason for this peculiar behaviour of the result is that in choosing a certain parametrization for the curve, we have picked a starting point. Due to the path-ordering in the double curve integral, this breaks the symmetry of the problem. Indeed, if we choose a parametrization with a different starting point, the result will be different. Thus it is clear that it is impossible to express $\alpha^{\mu}(C)$ as a single curve integral, because they have the property to give the same result when evaluated using parametrizations with different starting points. Note that this property is more general than the reparametrization invariance introduced in appendix C. It is thus clear that it is impossible to reach invariance by choosing a level- 1 density $p^{(1) \mu}(s)$.

Not having found Yangian invariance, one could raise the question, whether there exist curves, for which $\alpha^{\mu}(C)=0$. This is a rather difficult task and as it has already
failed for the very simple example of a circle, we do not pursue it further. It should also be pointed out that such curves would only correspond to extremal points under the transformation generated by the level- 1 momentum generator.

In the following section we pursue a different ansatz. The disturbing term in $P_{\mathrm{bos}, \varepsilon}^{(1) \mu}\langle W(C)\rangle_{1}$ is structurally similar to the gluino propagator 4.25). Hence there is hope that the Maldacena-Wilson loop operator can be extended to include also the fermionic fields in such a way that the action of the non-local part of the level-1 momentum generator gives a single curve integral. This extension is the topic of the next section.

### 6.2 The Fermionic Extension of the MaldacenaWilson Loop

The above result points at a possible Yangian invariance of an extended MaldacenaWilson loop that includes also the gluino fields. It is natural to use supersymmetry as a guiding principle in the construction of such an extension. By the inclusion of supersymmetry generators, the conformal algebra $\mathfrak{s o}(2,4)$ is extended to the superconformal algebra $\mathfrak{s u}(2,2 \mid 4)$ discussed in chapter 3.2 . The new underlying symmetry algebra implies an extension of the non-local part of the level-1 momentum generator $P^{(1) \mu}$. Let us fix the form of this generator before turning to the extension of the Maldacena-Wilson loop.

The non-local part of the generator is fixed by the dual structure constants of the superconformal algebra, which have been computed in [3]. Although they were using different conventions, the structural form of the level-1 momentum generator can be inferred from their result 4.51). This leads to the following ansatz:

$$
\begin{align*}
P_{\mathrm{nl}}^{(1) \mu}=\int_{s_{1}<s_{2}} \mathrm{~d} s_{1} \mathrm{~d} s_{2} & \left\{\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right)\right. \\
& \left.+c \bar{q}^{A \dot{\alpha}}\left(s_{1}\right) \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} q_{A}^{\alpha}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} . \tag{6.38}
\end{align*}
$$

Here, $q_{A}^{\alpha}(s)$ and $\bar{q}^{A \dot{\alpha}}(s)$ denote the densities of the supersymmetry generators in some superspace. Their exact form will be discussed later. The constant $c$ can be fixed from the commutation relations (3.38) of the Yangian algebra, which imply that

$$
\begin{equation*}
\left[Q_{A}^{\alpha}, P_{\mathrm{nl}}^{(1) \mu}\right]=0 . \tag{6.39}
\end{equation*}
$$

From the commutation relations given in chapter 3.2, we read off:

$$
\begin{align*}
{\left[q_{A}^{\alpha}(\tau), m^{\mu \nu}\left(s_{1}\right)\right] } & =-\frac{i}{2} \sigma^{\mu \nu \alpha \beta} \delta\left(s_{1}-\tau\right) q_{A \beta}\left(s_{1}\right),  \tag{6.40}\\
{\left[q_{A}^{\alpha}(\tau), d\left(s_{1}\right)\right] } & =\frac{1}{2} \delta\left(s_{1}-\tau\right) q_{A}^{\alpha}\left(s_{1}\right),  \tag{6.41}\\
\left\{q_{A}^{\alpha}(\tau), \bar{q}^{B \dot{\beta}}\left(s_{1}\right)\right\} & =2 i \delta_{A}^{B} \delta\left(s_{1}-\tau\right) \bar{\sigma}^{\nu \alpha \dot{\alpha}} p_{\nu}\left(s_{1}\right) . \tag{6.42}
\end{align*}
$$

Making use of these commutators, we get:

$$
\begin{aligned}
{\left[Q_{A}^{\alpha}, P_{\mathrm{nl}}^{(1) \mu}\right]=} & \int_{s_{1}<s_{2}} \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left\{\left(-\frac{i}{2} \sigma^{\mu \nu \alpha \beta} q_{A \beta}\left(s_{1}\right)-\frac{1}{2} q_{A}^{\alpha}\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right)\right. \\
& \left.+2 i c \bar{\sigma}^{\nu \alpha \dot{\beta}} p_{\nu}\left(s_{1}\right) \bar{\sigma}_{\beta \dot{\beta}}^{\mu} q_{A}^{\beta}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} \\
= & \int_{s_{1}<s_{2}} \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left\{\left(-\frac{i}{2} \sigma^{\mu \nu \alpha \beta} \epsilon_{\gamma \beta}-\frac{1}{2} \eta^{\mu \nu} \delta_{\gamma}^{\alpha}\right.\right. \\
& \left.\left.-2 i c \bar{\sigma}^{\nu \alpha \dot{\beta}} \sigma_{\dot{\beta} \gamma}^{\mu}\right) q_{A}^{\gamma}\left(s_{1}\right) p_{\nu}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} .
\end{aligned}
$$

Making use of (2.4) and 2.15), we find:

$$
\begin{equation*}
\bar{\sigma}^{\nu \alpha \dot{\beta}} \sigma_{\dot{\beta} \gamma}^{\mu}=\bar{\sigma}^{[\nu \alpha \dot{\beta}} \sigma_{\dot{\beta} \gamma}^{\mu]}+\bar{\sigma}^{(\nu \alpha \dot{\beta}} \sigma_{\dot{\beta} \gamma}^{\mu)}=i \sigma^{\mu \nu \alpha \beta} \epsilon_{\gamma \beta}+\eta^{\mu \nu} \delta_{\gamma}^{\alpha} . \tag{6.43}
\end{equation*}
$$

Thus we have:

$$
\left[Q_{A}^{\alpha}, P_{\mathrm{n} 1}^{(1) \mu}\right]=\int_{s_{1}<s_{2}} \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left\{\left(\left(2 c-\frac{i}{2}\right)\left(\sigma^{\mu \nu \alpha \beta} \epsilon_{\gamma \beta}-i \eta^{\mu \nu} \delta_{\gamma}^{\alpha}\right)\right) q_{A}^{\gamma}\left(s_{1}\right) p_{\nu}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} .
$$

Demanding that 6.39 holds then enforces that $c=\frac{i}{4}$. Hence we have the following level-1 momentum generator in $Y(\mathfrak{s u}(2,2 \mid 4)$ :

$$
\begin{align*}
P_{\mathrm{nl}, \varepsilon}^{(1) \mu} & =P_{\mathrm{bos}, \mathrm{nl}, \varepsilon}^{(1)}+P_{\text {ferm, } \mathrm{l}, \varepsilon}^{(1) \mu},  \tag{6.44}\\
P_{\text {ferm }, \mathrm{nl}, \varepsilon}^{(1) \mu} & =\frac{i}{4} \int \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left\{\bar{q}^{A \dot{\alpha}}\left(s_{1}\right) \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} q_{A}^{\alpha}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} \theta\left(s_{2}-s_{1}-d\left(s_{2}, \varepsilon\right)\right) . \tag{6.45}
\end{align*}
$$

To include the gluino fields in the Maldacena-Wilson loop operator, we extend the loop to a full non-chiral superspace, which we describe by the variables

$$
x^{\alpha \dot{\alpha}}(\tau)=\bar{\sigma}^{\mu \alpha \dot{\alpha}} x_{\mu}(\tau), \quad \theta_{\alpha}^{A}(\tau), \quad \bar{\theta}_{A \dot{\alpha}}(\tau), \quad y_{A}{ }^{B}(\tau)
$$

The use of a non-chiral superspace parametrized by both $\theta$ and $\bar{\theta}$ variables is necessary, if one hopes to find corrections to 6.37 from the action of $P_{\text {ferm }, \varepsilon}^{(1) \mu}$ on the expectation value $\langle\mathcal{W}(C)\rangle$ of the extended loop. Being a commuting variable, $\langle\mathcal{W}(C)\rangle$ will only contain terms which are of even order in the Graßmann variables. Corrections to the bosonic result can then only occur, if the supersymmetry densities $q$ and $\bar{q}$ both contain Graßmann derivatives. The closure of the supersymmetry algebra enforces the introduction of the bosonic variable $y_{A}{ }^{B}$. However, the supersymmetry generators

$$
\begin{align*}
Q_{A}^{\alpha} & =\int \mathrm{d} s q_{A}^{\alpha}(s)=\int \mathrm{d} s\left(-\frac{\delta}{\delta \theta_{\alpha}^{A}(s)}+y_{A}{ }^{B}(s) \frac{\delta}{\delta \theta_{\alpha}^{B}(s)}+i \bar{\theta}_{A \dot{\alpha}}(s) \frac{\delta}{\delta x_{\alpha \dot{\alpha}}(s)}\right),  \tag{6.46}\\
\bar{Q}^{A \dot{\alpha}} & =\int \mathrm{d} s \bar{q}^{A \dot{\alpha}}(s)=\int \mathrm{d} s\left(\frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}(s)}+y_{B}{ }^{A}(s) \frac{\delta}{\delta \bar{\theta}_{B \dot{\alpha}}(s)}-i \theta_{\alpha}^{A}(s) \frac{\delta}{\delta x_{\alpha \dot{\alpha}}(s)}\right), \tag{6.47}
\end{align*}
$$

do not contain $y$-derivatives such that the extension in the $y$-variables will not give corrections to the bosonic result. We will therefore set $y=0$ in the remainder of this
thesis, working with the supersymmetry generators

$$
\begin{align*}
Q_{A}^{\alpha} & =\int \mathrm{d} s q_{A}^{\alpha}(s)=\int \mathrm{d} s\left(-\frac{\delta}{\delta \theta_{\alpha}^{A}(s)}++i \bar{\theta}_{A \dot{\alpha}}(s) \frac{\delta}{\delta x_{\alpha \dot{\alpha}}(s)}\right),  \tag{6.48}\\
\bar{Q}^{A \dot{\alpha}} & =\int \mathrm{d} s \bar{q}^{A \dot{\alpha}}(s)=\int \mathrm{d} s\left(\frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}(s)}+-i \theta_{\alpha}^{A}(s) \frac{\delta}{\delta x_{\alpha \dot{\alpha}}(s)}\right) . \tag{6.49}
\end{align*}
$$

While the corrections to the bosonic result for the action of the level- 1 momentum generator are unaffected by this, restricting to the subspace $y=0$ will not allow us to check the invariance under the generators $K, S, \bar{S}, R$ and $C$, which contain $y$ derivatives.

We make the following ansatz for the extended Maldacena-Wilson loop:

$$
\begin{align*}
\mathcal{W}(C) & =\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp (i I[A, \psi, \tilde{\psi}, \phi ; x, \theta, \bar{\theta}]),  \tag{6.50}\\
I[A, \psi, \tilde{\psi}, \phi ; x, \theta, \bar{\theta}] & =\oint_{C} \mathrm{~d} \tau\left(\mathcal{I}_{0,0}+\mathcal{I}_{1,0}+\mathcal{I}_{0,1}+\mathcal{I}_{1,1}+\mathcal{I}_{2,0}+\mathcal{I}_{0,2}+\ldots\right),  \tag{6.51}\\
I_{i, j} & =\oint_{C} \mathrm{~d} \tau \mathcal{I}_{i, j} . \tag{6.52}
\end{align*}
$$

Here, the terms $\mathcal{I}_{i, j}$ are of order $\theta^{i} \bar{\theta}^{j}$. As we are constructing an extension of the Maldacena-Wilson loop, we demand that

$$
\begin{equation*}
\mathcal{I}_{0,0}=A_{\mu}(x) \dot{x}^{\mu}+\Phi_{I}(x)|\dot{x}| n^{I}=\frac{1}{2} \dot{x}_{\alpha \dot{\alpha}} A^{\alpha \dot{\alpha}}-\frac{1}{2}|\dot{x}| \bar{n}_{A B} \phi^{A B} . \tag{6.53}
\end{equation*}
$$

It is natural to require that the expectation value of the extended Maldacena-Wilson loop is invariant under supersymmetry transformations in superspace. To implement this requirement, we demand that

$$
\begin{equation*}
\mathcal{Q}_{A}^{\alpha}(I)=Q_{A}^{\alpha}(I) \quad \overline{\mathcal{Q}}^{A \dot{\alpha}}(I)=\bar{Q}^{A \dot{\alpha}}(I) . \tag{6.54}
\end{equation*}
$$

Here, $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are the supersymmetry transformations of the fields which we discussed in chapter 4.1. This requirement leads to an expectation value which is invariant under supersymmetry transformations, since

$$
\begin{aligned}
Q_{A}^{\alpha}\langle\mathcal{W}(C)\rangle & =\frac{i}{N}\left\langle\operatorname{Tr} \mathcal{P}\left(Q_{A}^{\alpha}(I) \exp (i I)\right)\right\rangle= \\
& =\frac{i}{N}\left\langle\operatorname{Tr} \mathcal{P}\left(\mathcal{Q}_{A}^{\alpha}(I) \exp (i I)\right)\right\rangle=\left\langle\mathcal{Q}_{A}^{\alpha} \mathcal{W}(C)\right\rangle=0 .
\end{aligned}
$$

The last equation follows from the invariance of the vacuum state under supersymmetry transformations. Equation (6.54) allows us to construct the terms in (6.52) order by order. For the moment, we are only interested in the corrections to the expectation value which are of order $\theta \bar{\theta}$. Yet, to be consistent to second order in the Graßmann variables, we will also derive the form of $\mathcal{I}_{2,0}$ and $\mathcal{I}_{0,2}$. Furthermore, we will restrict ourselves to terms which are linear in the fields, since we are only interested in the one-loop expectation value $\langle\mathcal{W}(C)\rangle_{(1)}$ of the extended Maldacena-Wilson loop. This is


Figure 6.2: Depiction of the structure of $I$. Spinor and $R$-symmetry indices are suppressed, $\stackrel{i, j}{=}$ denotes equality at order $\theta^{i} \bar{\theta}^{j}$.
taken care of by using the linearized supersymmetry field variations $\mathcal{Q}_{A \text { lin }}^{\alpha}$ and $\overline{\mathcal{Q}}_{\text {lin }}^{A \dot{\alpha}}$, which are given by:

$$
\begin{align*}
& \mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(A^{\beta \dot{\beta}}\right)=2 i \epsilon^{\alpha \beta} \tilde{\psi}_{A}^{\dot{\beta}} \quad \overline{\mathcal{Q}}_{\operatorname{lin}}^{A \dot{\alpha}}\left(A^{\beta \dot{\beta}}\right)=-2 i \epsilon^{\dot{\alpha} \dot{\beta}} \psi^{A \beta}  \tag{6.55}\\
& \mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(\bar{\phi}_{B C}\right)=\sqrt{2} i \epsilon_{A B C D} \psi^{D \alpha} \quad \overline{\mathcal{Q}}_{\operatorname{lin}}^{A \dot{\alpha}}\left(\bar{\phi}_{B C}\right)=-\sqrt{2} i\left(\tilde{\psi}_{B}^{\dot{\alpha}} \delta_{C}^{A}-\tilde{\psi}_{C}^{\dot{\alpha}} \delta_{B}^{A}\right)  \tag{6.56}\\
& \mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(\psi^{B \beta}\right)=\frac{i}{2} F_{\operatorname{lin}}^{\alpha \beta} \delta_{A}^{B}  \tag{6.57}\\
& \overline{\mathcal{Q}}_{\text {lin }}^{A \dot{\alpha}}\left(\psi^{B \beta}\right)=-\sqrt{2} \partial^{\beta \dot{\alpha}} \phi^{A B} \\
& \mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(\tilde{\psi}_{B}^{\dot{\beta}}\right)=-\sqrt{2} \partial^{\dot{\alpha} \alpha} \bar{\phi}_{A B}  \tag{6.58}\\
& \overline{\mathcal{Q}}_{\text {lin }}^{A \dot{\alpha}}\left(\tilde{\psi}_{B}^{\dot{\beta}}\right)=-\frac{i}{2} F_{\text {lin }}^{\dot{\alpha} \dot{\beta}} \delta_{B}^{A}
\end{align*}
$$

The organization of this calculation is depicted in Figure 6.2. We will start by determining $I_{1,0}$ and $I_{0,1}$ from the equations

$$
\mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(I_{0,0}\right) \stackrel{0,0}{=} Q_{A}^{\alpha}\left(I_{1,0}\right), \quad \overline{\mathcal{Q}}_{\operatorname{lin}}^{A \dot{\alpha}}\left(I_{0,0}\right) \stackrel{0,0}{=} \bar{Q}^{A \dot{\alpha}}\left(I_{0,1}\right) .
$$

In these and the further equations of this type $\xlongequal{i, j}$ denotes equality at order $\theta^{i} \bar{\theta}^{j}$. Making use of 6.55 - 6.58) we find:

$$
\begin{align*}
\mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(\mathcal{I}_{0,0}\right) & =i \dot{x}^{\alpha}{ }_{\dot{\beta}} \tilde{\psi}_{A}^{\dot{B}}-\sqrt{2} i|\dot{x}| \bar{n}_{A B} \psi^{\alpha B}  \tag{6.59}\\
\overline{\mathcal{Q}}_{\operatorname{lin}}^{A \dot{\alpha}}\left(\mathcal{I}_{0,0}\right) & =i \dot{x}_{\beta}{ }^{\dot{\alpha}} \psi^{\beta A}-\sqrt{2} i|\dot{x}| n^{A B} \tilde{\psi}_{B}^{\dot{\alpha}} . \tag{6.60}
\end{align*}
$$

From this we read off $\mathcal{I}_{1,0}$ and $\mathcal{I}_{0,1}$ :

$$
\begin{align*}
& \mathcal{I}_{1,0}=-i \theta_{\alpha}^{A} \dot{x}^{\alpha}{ }_{\dot{\beta}} \tilde{\psi}_{A}^{\dot{\beta}}+\sqrt{2} i \theta_{\alpha}^{A}|\dot{x}| \bar{n}_{A B} \psi^{\alpha B},  \tag{6.61}\\
& \mathcal{I}_{0,1}=i \bar{\theta}_{A \dot{\alpha}} \dot{x}_{\beta}{ }^{\dot{\alpha}} \psi^{\beta A}-\sqrt{2} i \bar{\theta}_{A \dot{\alpha}}|\dot{x}| n^{A B} \tilde{\psi}_{B}^{\dot{\alpha}} . \tag{6.62}
\end{align*}
$$

The knowledge of $\mathcal{I}_{1,0}$ and $\mathcal{I}_{0,1}$ suffices to determine all terms in quadratic order. In order to do so, we need to know the action of the supersymmetry field transformations
on $\mathcal{I}_{1,0}$ and $\mathcal{I}_{0,1}$. With the knowledge of (6.55) - (6.58), these are easily determined:

$$
\begin{align*}
\mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(\mathcal{I}_{1,0}\right) & =-\sqrt{2} i \theta_{\beta}^{B} \dot{x}^{\beta}{ }_{\dot{\beta}} \partial^{\dot{\beta} \alpha} \bar{\phi}_{A B}-\frac{1}{\sqrt{2}} \theta_{\beta}^{B}|\dot{x}| \bar{n}_{A B} F_{\operatorname{lin}}^{\alpha \beta},  \tag{6.63}\\
\overline{\mathcal{Q}}_{\operatorname{lin}}^{A \dot{\alpha}}\left(\mathcal{I}_{1,0}\right) & =\frac{1}{2} \theta_{\alpha}^{A} \dot{x}_{\dot{\beta}}^{\alpha} F_{\operatorname{lin}}^{\dot{\alpha} \dot{\beta}}-2 i \theta_{\beta}^{C}|\dot{x}| \bar{n}_{B C} \partial^{\beta \dot{\alpha}} \phi^{A B},  \tag{6.64}\\
\mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(\mathcal{I}_{0,1}\right) & =\frac{1}{2} \bar{\theta}_{A \dot{\beta}} \dot{x}_{\beta}^{\dot{\beta}} F_{\operatorname{lin}}^{\alpha \beta}+2 i \bar{\theta}_{C \dot{\beta}}|\dot{x}| n^{B C} \partial^{\dot{\beta} \alpha} \bar{\phi}_{A B},  \tag{6.65}\\
\overline{\mathcal{Q}}_{\operatorname{lin}}^{A \dot{\alpha}}\left(\mathcal{I}_{0,1}\right) & =\sqrt{2} i \bar{\theta}_{B \dot{\beta}} \dot{x}_{\beta}^{\dot{\beta}} \partial^{\beta \dot{\alpha}} \phi^{A B}-\frac{1}{\sqrt{2}} \bar{\theta}_{B \dot{\beta}}|\dot{x}| n^{A B} F_{\operatorname{lin}}^{\dot{\alpha} \dot{\beta}} . \tag{6.66}
\end{align*}
$$

Here, $F_{\operatorname{lin}}^{\alpha \beta}$ and $F_{\operatorname{lin}}^{\dot{\alpha} \dot{\beta}}$ denote the parts of $F^{\alpha \beta}$ and $F^{\dot{\alpha} \dot{\beta}}$, which are linear in the gauge fields $A_{\mu}$,

$$
F_{\mathrm{lin}}^{\alpha \beta}=\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right) \sigma^{\mu \nu \alpha \beta}, \quad F_{\operatorname{lin}}^{\dot{\alpha} \dot{\beta}}=\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right) \bar{\sigma}^{\mu \nu \dot{\alpha} \dot{\beta}} .
$$

We can now apply (6.63) to determine $\mathcal{I}_{2,0}$ by imposing

$$
\begin{equation*}
\mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(I_{1,0}\right) \stackrel{1,0}{=} Q_{A}^{\alpha}\left(I_{2,0}\right) . \tag{6.67}
\end{equation*}
$$

This gives the following result:

$$
\begin{equation*}
\mathcal{I}_{2,0}=\frac{i}{\sqrt{2}} \theta_{\gamma}^{C} \theta_{\beta}^{B} \dot{x}_{\dot{\beta}}^{\beta} \partial^{\dot{\beta} \gamma} \bar{\phi}_{C B}+\frac{1}{2 \sqrt{2}} \theta_{\gamma}^{C} \theta_{\beta}^{B}|\dot{x}| \bar{n}_{C B} F_{\operatorname{lin}}^{\gamma \beta}+\sqrt{2} i \epsilon^{\gamma \beta} \theta_{\gamma}^{C} \dot{\theta}_{\beta}^{B} \bar{\phi}_{C B} . \tag{6.68}
\end{equation*}
$$

As the calculation, which shows that (6.67) is indeed satisfied, is a little more involved, we will go through the details explicitly. We have to calculate the action of $Q_{A}^{\alpha}$ on $I_{2,0}$. Since we are only interested in the linear order in $\theta$ this reduces to calculating the (functional) $\theta$-derivative of $I_{2,0}$. We find:

$$
\begin{aligned}
Q_{A}^{\alpha}\left(I_{2,0}\right) \stackrel{1,0}{=}-\int \mathrm{d} \tau & \left\{\frac{i}{\sqrt{2}}\left(\theta_{\beta}^{B} \dot{x}_{\dot{\beta}}{ }_{\dot{\beta}}\left(\partial^{\dot{\beta} \alpha} \bar{\phi}_{A B}\right)-\theta_{\gamma}^{C} \dot{x}_{\dot{\beta}}^{\alpha}\left(\partial^{\dot{\beta} \gamma} \bar{\phi}_{C A}\right)\right)\right. \\
& \left.+\frac{1}{2 \sqrt{2}}\left(\theta_{\beta}^{B}|\dot{x}| \bar{n}_{A B} F_{\mathrm{lin}}^{\alpha \beta}-\theta_{\gamma}^{C}|\dot{x}| \bar{n}_{C A} F_{\mathrm{lin}}^{\gamma \alpha}\right)+\sqrt{2} i \epsilon^{\alpha \beta} \dot{\theta}_{\beta}^{B} \bar{\phi}_{A B}\right\} \\
=-\int \mathrm{d} \tau & \left\{\sqrt{2} i \theta_{\beta}^{B} \dot{x}_{\dot{\beta}}^{(\beta} \partial^{\alpha) \dot{\beta}} \bar{\phi}_{A B}+\frac{1}{\sqrt{2}} \theta_{\beta}^{B}|\dot{x}| \bar{n}_{A B} F_{\mathrm{lin}}^{\alpha \beta}+\sqrt{2} i \dot{\theta}^{B \alpha} \bar{\phi}_{A B}\right\} .
\end{aligned}
$$

Employing the spinor identity $\Lambda^{(\alpha \beta)}=\Lambda^{\alpha \beta}+\frac{1}{2} \epsilon^{\alpha \beta} \Lambda^{\gamma}{ }_{\gamma}$ and using integration by parts in the last term, we arrive at (note the form (2.13) of the chain rule):

$$
\begin{aligned}
Q_{A}^{\alpha}\left(I_{2,0}\right) \stackrel{1,0}{=}-\int \mathrm{d} \tau & \left\{\sqrt{2} i \theta_{\beta}^{B} \dot{x}_{\dot{\beta}}^{\beta} \partial^{\dot{\beta} \alpha} \bar{\phi}_{A B}+\frac{i}{\sqrt{2}} \theta^{B \alpha} \dot{x}_{\beta \dot{\beta}} \partial^{\beta \dot{\beta}} \bar{\phi}_{A B}+\frac{1}{\sqrt{2}} \theta_{\beta}^{B}|\dot{x}| \bar{n}_{A B} F_{\mathrm{lin}}^{\alpha \beta}\right. \\
& \left.-\frac{i}{\sqrt{2}} \theta^{B \alpha} \dot{x}_{\beta \dot{\beta}} \partial^{\beta \dot{\beta}} \bar{\phi}_{A B}\right\}=\int \mathrm{d} \tau \mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(\mathcal{I}_{1,0}\right)
\end{aligned}
$$

A similar calculation shows that

$$
\begin{equation*}
\mathcal{I}_{0,2}=\frac{i}{\sqrt{2}} \bar{\theta}_{A \dot{\alpha}} \bar{\theta}_{B \dot{\beta}} \dot{x}_{\beta}^{\dot{\beta}} \partial^{\beta \dot{\alpha}} \phi^{A B}-\frac{1}{2 \sqrt{2}} \bar{\theta}_{A \dot{\alpha}} \bar{\theta}_{B \dot{\beta}}|\dot{x}| n^{A B} F_{\operatorname{lin}}^{\dot{\alpha} \dot{\beta}}-\sqrt{2} i \epsilon^{\dot{\alpha} \dot{\beta}} \bar{\theta}_{A \dot{\alpha}} \dot{\bar{\theta}}_{B \dot{\beta}} \phi^{A B} \tag{6.69}
\end{equation*}
$$

gives a solution of

$$
\begin{equation*}
\overline{\mathcal{Q}}_{\operatorname{lin}}^{A \dot{\alpha}}\left(I_{0,1}\right) \stackrel{0,1}{=} \bar{Q}^{A \dot{\alpha}}\left(I_{0,2}\right) . \tag{6.70}
\end{equation*}
$$

We will now turn to the calculation that determines $\mathcal{I}_{1,1}$. As can be seen in figure 6.2, we need to satisfy two equations:

$$
\begin{gather*}
\overline{\mathcal{Q}}_{\operatorname{lin}}^{A \dot{\alpha}}\left(I_{1,0}\right)-\bar{Q}^{A \dot{\alpha}}\left(I_{0,0}\right) \stackrel{1,0}{=} \bar{Q}^{A \dot{\alpha}}\left(I_{1,1}\right),  \tag{6.71}\\
\mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(I_{0,1}\right)-Q_{A}^{\alpha}\left(I_{0,0}\right) \stackrel{0,1}{=} Q_{A}^{\alpha}\left(I_{1,1}\right) . \tag{6.72}
\end{gather*}
$$

We start by calculating the left-hand side of these equations to inspire an educated guess for $\mathcal{I}_{1,1}$. The supersymmetry field transformations of $I_{0,1}$ and $I_{1,0}$ are given in (6.64) and (6.65). Hence we only have to calculate how $Q_{A}^{\alpha}$ acts on $I_{0,0}$ :

$$
\begin{aligned}
Q_{A}^{\alpha}\left(I_{0,0}\right)= & \int \mathrm{d} s \mathrm{~d} \tau i \bar{\theta}_{A \dot{\alpha}}(s) \partial^{\alpha \dot{\alpha}}(s)\left(\frac{1}{2} A^{\beta \dot{\beta}}(x(\tau)) \dot{x}_{\beta \dot{\beta}}(\tau)-\frac{1}{2}|\dot{x}(\tau)| \bar{n}_{B C} \phi^{B C}(x(\tau))\right) \\
= & \frac{i}{2} \int \mathrm{~d} \tau\left\{2 \dot{\bar{\theta}}_{A \dot{\alpha}} A^{\alpha \dot{\alpha}}+\bar{\theta}_{A \dot{\alpha}} \dot{x}_{\beta \dot{\beta}} \partial^{\alpha \dot{\alpha}} A^{\beta \dot{\beta}}-\dot{\bar{\theta}}_{A \dot{\alpha}} \frac{\dot{x}^{\alpha \dot{\alpha}}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C}\right. \\
& \left.\quad-\bar{\theta}_{A \dot{\alpha}}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C}\right\} \\
= & \frac{i}{2} \int \mathrm{~d} \tau\left\{\bar{\theta}_{A \dot{\alpha}} \dot{x}_{\beta \dot{\beta}}\left(\partial^{\alpha \dot{\alpha}} A^{\beta \dot{\beta}}-\partial^{\beta \dot{\beta}} A^{\alpha \dot{\alpha}}\right)-\dot{\bar{\theta}}_{A \dot{\alpha}} \frac{\dot{x}^{\alpha} \dot{\alpha}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C}\right. \\
& \left.\quad-\bar{\theta}_{A \dot{\alpha}}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C}\right\} .
\end{aligned}
$$

In the last step, we used integration by parts to simplify. Making use of (2.17),

$$
F^{\alpha \dot{\alpha} \dot{\beta} \dot{\beta}}=\frac{i}{2} \epsilon^{\dot{\alpha} \dot{\beta}} F^{\alpha \beta}+\frac{i}{2} \epsilon^{\alpha \beta} F^{\dot{\alpha} \dot{\beta}},
$$

for the linear part of $F_{\mu \nu}$, we arrive at

$$
\begin{gather*}
Q_{A}^{\alpha}\left(I_{0,0}\right)=\frac{i}{2} \int \mathrm{~d} \tau\left\{\frac{i}{2} \bar{\theta}_{A \dot{\alpha}}\left(\dot{x}_{\dot{\beta}}^{\alpha} F_{\operatorname{lin}}^{\dot{\alpha} \dot{\beta}}-\dot{x}_{\beta}^{\dot{\alpha}} F_{\mathrm{lin}}^{\alpha \beta}\right)-\dot{\bar{\theta}}_{A \dot{\alpha}} \frac{\dot{x}^{\alpha \dot{\alpha}}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C}\right. \\
\left.-\bar{\theta}_{A \dot{\alpha}}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C}\right\} . \tag{6.73}
\end{gather*}
$$

Combinig this with 6.65), we have the following condition for $I_{1,1}$ :

$$
\begin{align*}
&-\int \mathrm{d} s \frac{\delta}{\delta \theta_{A}^{\alpha}(s)}\left(I_{1,1}\right)= \mathcal{Q}_{A \operatorname{lin}}^{\alpha}\left(I_{0,1}\right)-Q_{A}^{\alpha}\left(I_{0,0}\right)= \\
&=\int \mathrm{d} \tau\left\{\frac{1}{4} \bar{\theta}_{A \dot{\alpha}}\left(\dot{x}^{\alpha}{ }_{\dot{\beta}} F_{\text {lin }}^{\dot{\alpha} \dot{\beta}}+\dot{x}_{\beta}{ }^{\dot{\alpha}} F_{\text {lin }}^{\alpha \beta}\right)+\frac{i}{2} \bar{\theta}_{A \dot{\alpha}}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C}\right. \\
&\left.\quad+2 i \bar{\theta}_{C \dot{\alpha}}|\dot{x}| n^{B C} \partial^{\alpha \dot{\alpha}} \bar{\phi}_{A B}+\frac{i}{2} \dot{\bar{\theta}}_{A \dot{\alpha}} \frac{\dot{x}^{\alpha \dot{\alpha}}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C}\right\} . \tag{6.74}
\end{align*}
$$

In the same fashion we find the second condition for $I_{1,1}$ :

$$
\begin{align*}
& \int \mathrm{d} s \frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}(s)}\left(I_{1,1}\right)= \overline{\mathcal{Q}}_{\mathrm{lin}}^{A \dot{\alpha}}\left(I_{0,1}\right)-\bar{Q}^{A \dot{\alpha}}\left(I_{0,0}\right)= \\
&=\int \mathrm{d} \tau\left\{\frac{1}{4} \theta_{\alpha}^{A}\left(\dot{x}^{\alpha}{ }_{\dot{\beta}} F_{\mathrm{lin}}^{\dot{\alpha} \dot{\alpha}}+\dot{x}_{\beta}^{\dot{\alpha}} F_{\mathrm{lin}}^{\alpha \beta}\right)-\frac{i}{2} \theta_{\alpha}^{A}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C}\right. \\
&\left.\quad-2 i \theta_{\beta}^{C}|\dot{x}| \bar{n}_{B C} \partial^{\beta \dot{\alpha}} \phi^{A B}-\frac{i}{2} \dot{\theta}_{\dot{\alpha}}^{A} \frac{\dot{x}^{\alpha \dot{\alpha}}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C}\right\} . \tag{6.75}
\end{align*}
$$

With regard to (6.74), we construct the following ansatz:

$$
\begin{align*}
\mathcal{I}_{1,1} & =-\frac{1}{4} \theta_{\alpha}^{A} \bar{\theta}_{A \dot{\alpha}}\left(\dot{x}^{\alpha}{ }_{\beta} F_{\mathrm{lin}}^{\dot{\alpha} \dot{\beta}}+\dot{x}_{\beta}{ }^{\dot{\alpha}} F_{\mathrm{lin}}^{\alpha \beta}\right)-\frac{i}{2} \theta_{\alpha}^{A} \bar{\theta}_{A \dot{\alpha}}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C} \\
& -2 i \theta_{\alpha}^{A} \bar{\theta}_{C \dot{\alpha}}|\dot{x}| n^{B C} \partial^{\alpha \dot{\alpha}} \bar{\phi}_{A B}-\frac{i}{2} \theta_{\alpha}^{A} \dot{\theta}_{A \dot{\alpha}} \frac{\dot{x}^{\alpha \dot{\alpha}}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C}+\frac{i}{2} \dot{\theta}_{\dot{\alpha}}^{A} \bar{\theta}_{A \dot{\alpha}} \frac{\dot{x}^{\alpha \dot{\alpha}}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C} . \tag{6.76}
\end{align*}
$$

This obviously satisfies (6.74) and we now show that it also satisfies (6.75). Taking a closer look, one realizes that only the two middle terms do not immediately give the respective expressions in (6.75). For these, we have:

$$
\begin{aligned}
& \int \mathrm{d} s \frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}(s)}\left(-\frac{i}{2} \theta_{\alpha}^{D} \bar{\theta}_{D \dot{\beta}}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\beta}} \phi^{B C}-2 i \theta_{\alpha}^{D} \bar{\theta}_{C \dot{\beta}}|\dot{x}| n^{B C} \partial^{\alpha \dot{\beta}} \bar{\phi}_{D B}\right)= \\
& =\frac{i}{2} \theta_{\alpha}^{A}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C}+2 i \theta_{\alpha}^{D}|\dot{x}| n^{B A} \partial^{\alpha \dot{\alpha}} \bar{\phi}_{D B} \\
& =\frac{i}{2} \theta_{\alpha}^{A}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C}+\frac{i}{2} \theta_{\alpha}^{D}|\dot{x}| \epsilon^{B A K L} \bar{n}_{K L} \partial^{\alpha \dot{\alpha}} \phi^{M N} \epsilon_{D B M N} \\
& =-\frac{i}{2} \theta_{\alpha}^{A}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C}-2 i \theta_{\alpha}^{D}|\dot{x}| \bar{n}_{B D} \partial^{\alpha \dot{\alpha}} \phi^{A B} .
\end{aligned}
$$

Hence (6.75) is also satisfied. This concludes the construction of the exponent of the supersymmetrically extended Maldacena-Wilson loop. For purposes of clarity, we recall our results:

$$
\begin{align*}
& \mathcal{I}_{0,0}=\frac{1}{2} \dot{x}_{\alpha \dot{\alpha}} A^{\alpha \dot{\alpha}}-\frac{1}{2}|\dot{x}| \bar{n}_{A B} \phi^{A B},  \tag{6.77}\\
& \mathcal{I}_{1,0}=-i \theta_{\alpha}^{A} \dot{x}^{\alpha}{ }_{\dot{\beta}} \tilde{\psi}_{A}^{\dot{\beta}}+\sqrt{2} i \theta_{\alpha}^{A}|\dot{x}| \bar{n}_{A B} \psi^{\alpha B},  \tag{6.78}\\
& \mathcal{I}_{0,1}=i \bar{\theta}_{A \dot{\alpha}} \dot{x}_{\beta}^{\dot{\alpha}} \psi^{\beta A}-\sqrt{2} i \bar{\theta}_{A \dot{\alpha}}|\dot{x}| n^{A B} \tilde{\psi}_{B}^{\dot{\alpha}},  \tag{6.79}\\
& \mathcal{I}_{2,0}=\frac{i}{\sqrt{2}} \theta_{\gamma}^{C} \theta_{\beta}^{B} \dot{x}^{\beta}{ }_{\dot{\beta}} \partial^{\dot{\beta} \gamma} \bar{\phi}_{C B}+\frac{1}{2 \sqrt{2}} \theta_{\gamma}^{C} \theta_{\beta}^{B}|\dot{x}| \bar{n}_{C B} F_{\text {lin }}^{\gamma \beta}+\sqrt{2} i \epsilon^{\gamma \beta} \theta_{\gamma}^{C} \dot{\theta}_{\beta}^{B} \bar{\phi}_{C B},  \tag{6.80}\\
& \mathcal{I}_{0,2}=\frac{i}{\sqrt{2}} \bar{\theta}_{A \dot{\alpha}} \bar{\theta}_{B \dot{\beta}} \dot{x}_{\beta}^{\dot{\beta}} \partial^{\beta \dot{\alpha}} \phi^{A B}-\frac{1}{2 \sqrt{2}} \bar{\theta}_{A \dot{\alpha}} \bar{\theta}_{B \dot{\beta}}|\dot{x}| \bar{n}_{A B} F_{\operatorname{lin}}^{\dot{\alpha} \dot{\beta}}-\sqrt{2} i \epsilon^{\dot{\alpha} \dot{\beta}} \bar{\theta}_{A \dot{\alpha}} \dot{\bar{\theta}}_{B \dot{\beta}} \phi^{A B},  \tag{6.81}\\
& \mathcal{I}_{1,1}=-\frac{1}{4} \theta_{\alpha}^{A} \bar{\theta}_{A \dot{\alpha}}\left(\dot{x}^{\alpha}{ }_{\dot{\beta}} F_{\text {lin }}^{\dot{\alpha} \dot{\beta}}+\dot{x}_{\beta}{ }^{\dot{\alpha}} F_{\operatorname{lin}}^{\alpha \beta}\right)-\frac{i}{2} \theta_{\alpha}^{A} \bar{\theta}_{A \dot{\alpha}}|\dot{x}| \bar{n}_{B C} \partial^{\alpha \dot{\alpha}} \phi^{B C} \\
& -2 i \theta_{\alpha}^{A} \bar{\theta}_{C \dot{\alpha}}|\dot{x}| n^{B C} \partial^{\alpha \dot{\alpha}} \bar{\phi}_{A B}-\frac{i}{2} \theta_{\alpha}^{A} \dot{\bar{\theta}}_{A \dot{\alpha}} \frac{\dot{x}^{\alpha \dot{\alpha}}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C}+\frac{i}{2} \dot{\theta}_{\dot{\alpha}}^{A} \bar{\theta}_{A \dot{\alpha}} \frac{\dot{x}^{\alpha \dot{\alpha}}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C} \text {. } \tag{6.82}
\end{align*}
$$

Using the above results, we can now calculate the vacuum expectation value $\langle\mathcal{W}(C)\rangle$ of the extended loop to first order in perturbation theory. We have:

$$
\begin{align*}
\langle\mathcal{W}(C)\rangle= & \left\langle\frac{1}{N} \operatorname{Tr} \mathcal{P} \exp \left(i \oint \mathrm{~d} \tau\left(\mathcal{I}_{0,0}+\mathcal{I}_{1,0}+\mathcal{I}_{0,1}+\mathcal{I}_{1,1}+\ldots\right)\right)\right\rangle \\
= & 1-\frac{1}{2 N} \operatorname{Tr}\left\{\int \mathrm { d } \tau _ { 1 } \mathrm { d } \tau _ { 2 } \left(\left\langle\mathcal{I}_{0,0}\left(\tau_{1}\right) \mathcal{I}_{0,0}\left(\tau_{2}\right)\right\rangle+\left\langle\mathcal{I}_{1,0}\left(\tau_{1}\right) \mathcal{I}_{0,1}\left(\tau_{2}\right)\right\rangle\right.\right. \\
& \left.\left.+\left\langle\mathcal{I}_{0,1}\left(\tau_{1}\right) \mathcal{I}_{1,0}\left(\tau_{2}\right)\right\rangle+\left\langle\mathcal{I}_{0,0}\left(\tau_{1}\right) \mathcal{I}_{1,1}\left(\tau_{2}\right)\right\rangle+\left\langle\mathcal{I}_{1,1}\left(\tau_{1}\right) \mathcal{I}_{0,0}\left(\tau_{2}\right)\right\rangle\right)\right\}+\mathcal{O}\left(\theta^{2}\right) \\
& +\mathcal{O}\left(\bar{\theta}^{2}\right)+\mathcal{O}\left(\lambda^{2}\right) \\
= & 1-\frac{1}{2 N} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \operatorname{Tr}\left\langle\mathcal{I}_{0,0}\left(\tau_{1}\right) \mathcal{I}_{0,0}\left(\tau_{2}\right)\right\rangle-\frac{1}{N} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \operatorname{Tr}\left\langle\mathcal{I}_{1,0}\left(\tau_{1}\right) \mathcal{I}_{0,1}\left(\tau_{2}\right)\right\rangle \\
& -\frac{1}{N} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \operatorname{Tr}\left\langle\mathcal{I}_{0,0}\left(\tau_{1}\right) \mathcal{I}_{1,1}\left(\tau_{2}\right)\right\rangle+\mathcal{O}\left(\theta^{2}\right)+\mathcal{O}\left(\bar{\theta}^{2}\right)+\mathcal{O}\left(\lambda^{2}\right) \tag{6.83}
\end{align*}
$$

In the last step, we performed a change of variables $\left(\tau_{1} \leftrightarrow \tau_{2}\right)$, used the cyclicity of the trace and the fact that the $\mathcal{I}_{i, j}$ are commuting variables. From the discussion in
chapter 5, we already know that

$$
\begin{equation*}
\operatorname{Tr}\left\langle\mathcal{I}_{0,0}\left(\tau_{1}\right) \mathcal{I}_{0,0}\left(\tau_{2}\right)\right\rangle=\frac{g^{2} N^{2}}{8 \pi^{2}} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}} . \tag{6.84}
\end{equation*}
$$

For the next term, we calculate:

$$
\begin{aligned}
& \operatorname{Tr}\left\langle\mathcal{I}_{1,0}\left(\tau_{1}\right) \mathcal{I}_{0,1}\left(\tau_{2}\right)\right\rangle=\operatorname{Tr}\left(T^{a} T^{b}\right)\langle \left(-i \theta_{1}{ }_{\alpha}^{A} \dot{x}_{1}{ }_{\dot{\beta}} \tilde{\psi}_{A}^{\dot{\beta} a}\left(x_{1}\right)+\sqrt{2} i \theta_{1}{ }_{\alpha}^{A}\left|\dot{x}_{1}\right| \bar{n}_{A B} \psi^{\alpha B a}\left(x_{1}\right)\right) \\
&\left.\left(i \bar{\theta}_{2 C \dot{\alpha}} \dot{x}_{2 \beta}{ }^{\dot{\alpha}} \psi^{\beta C b}\left(x_{2}\right)-\sqrt{2} i \bar{\theta}_{2 C \dot{\alpha}}\left|\dot{x}_{2}\right| n^{C D} \tilde{\psi}_{D}^{\dot{\alpha} b}\left(x_{2}\right)\right)\right\rangle \\
&=\operatorname{Tr}\left(T^{a} T^{b}\right)\left\{-\theta_{1}^{A} \dot{x}_{1}^{\alpha}{ }_{\dot{\beta}} \bar{\theta}_{2 C \dot{\alpha}} \dot{x}_{2 \beta}{ }^{\dot{\alpha}}\left\langle\tilde{\psi}_{A}^{\dot{\beta} a}\left(x_{1}\right) \psi^{\beta C b}\left(x_{2}\right)\right\rangle\right. \\
&\left.-2 \theta_{1 \alpha}^{A} \bar{\theta}_{2 C \dot{\alpha}}\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right| \bar{n}_{A B} n^{C D}\left\langle\psi^{\alpha B a}\left(x_{1}\right) \tilde{\psi}_{D}^{\dot{\alpha} b}\left(x_{2}\right)\right\rangle\right\} \\
&=- i g^{2} N^{2} \\
& 4 \pi^{2} \frac{1}{x_{12}^{4}} \theta_{1}^{A} \bar{\alpha}_{2 A \dot{\alpha}}\left(\dot{x}_{1}^{\alpha \dot{\beta}} \dot{x}_{2}^{\beta \dot{\alpha}} x_{12 \beta \dot{\beta}}+\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right| x_{12}^{\alpha \dot{\alpha}}\right) .
\end{aligned}
$$

In the last step, we inserted the gluino propagator (4.25) and made use of the identity (2.32). We now use the Fierz identity (2.20) to rewrite

$$
\theta_{1}^{A}{ }_{\alpha}^{A} \bar{\theta}_{2 A \dot{\alpha}}=-\frac{1}{2}\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{1}\right) \sigma_{\dot{\alpha} \alpha}^{\mu}
$$

Using the identity (2.22) for the trace of four sigma-matrices, we find:

$$
\begin{gathered}
\operatorname{Tr}\left\langle\mathcal{I}_{1,0}\left(\tau_{1}\right) \mathcal{I}_{0,1}\left(\tau_{2}\right)\right\rangle=\frac{i g^{2} N^{2}}{4 \pi^{2}}\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{1}\right)\left\{\dot{x}_{2}^{\mu} \frac{\dot{x}_{1} x_{12}}{x_{12}^{4}}+\dot{x}_{1}^{\mu} \frac{\dot{x}_{2} x_{12}}{x_{12}^{4}}-i \epsilon^{\mu \nu \rho \kappa} \frac{\dot{x}_{2 \nu} x_{12} \dot{x}_{1 \kappa}}{x_{12}^{4}}\right. \\
\left.-\frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{x_{12}^{4}} x_{12}^{\mu}\right\} .
\end{gathered}
$$

We rewrite

$$
\frac{\dot{x}_{1} x_{12}}{x_{12}^{4}}=-\frac{1}{2} \partial_{\tau_{1}} \frac{1}{x_{12}^{2}}, \quad \frac{\dot{x}_{2} x_{12}}{x_{12}^{4}}=\frac{1}{2} \partial_{\tau_{2}} \frac{1}{x_{12}^{2}}
$$

to perform an integration by parts and arrive at the final result:

$$
\left.\begin{array}{l}
\int \mathrm{d} \tau_{1} \mathrm{~d} \tau_{2} \operatorname{Tr}\left\langle\mathcal{I}_{1,0}\left(\tau_{1}\right) \mathcal{I}_{0,1}\left(\tau_{2}\right)\right\rangle=\frac{i g^{2} N^{2}}{8 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left\{\bar{\theta}_{2} \sigma_{\mu} \dot{\theta}_{1} \frac{\dot{x}_{2}^{\mu}}{x_{12}^{2}}-\dot{\bar{\theta}}_{2} \sigma_{\mu} \theta_{1} \dot{x}_{1}^{\mu}\right. \\
x_{12}^{2} \tag{6.85}
\end{array}\right\} .
$$

To calculate the expectation value

$$
\operatorname{Tr}\left\langle\mathcal{I}_{0,0}\left(\tau_{1}\right) \mathcal{I}_{1,1}\left(\tau_{2}\right)\right\rangle
$$

it is convenient to first replace the spinor indices in $\mathcal{I}_{1,1}$ by spacetime indices. Making use of 2.15 and (2.18) we rewrite $F_{\text {lin }}^{\alpha \beta}$ and $F_{\text {lin }}^{\dot{\alpha} \dot{~}}$ in the following way:

$$
\begin{aligned}
F_{\text {lin }}^{\alpha \beta} & =\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right) \sigma^{\mu \nu \alpha \beta}=i\left(\partial_{\mu} A_{\nu}-\partial_{\nu} A_{\mu}\right)\left(\bar{\sigma}^{\mu \alpha \dot{\delta}} \sigma^{\nu \dot{\gamma} \beta}\right) \epsilon_{\dot{\gamma} \dot{\delta}}=2 i \partial^{(\alpha}{ }_{\dot{\gamma}} A^{\beta) \dot{\gamma}}= \\
& =2 i \partial^{\alpha}{ }_{\dot{\gamma}} A^{\beta \dot{\gamma}}-i \epsilon^{\alpha \beta} \partial_{\dot{\gamma}} A^{\dot{\gamma} \dot{\gamma}}, \\
F_{\text {lin }}^{\dot{\alpha} \dot{\beta}} & =-2 i \partial^{\dot{\alpha}}{ }_{\gamma} A^{\dot{\beta} \gamma}-i \epsilon^{\dot{\alpha} \dot{\beta}} \partial_{\gamma \dot{\gamma}} A^{\gamma \dot{\gamma}} .
\end{aligned}
$$

Combining the above, we have:

$$
\begin{aligned}
& \frac{1}{4} \bar{\theta}_{A \dot{\alpha}} \theta_{\alpha}^{A}\left(\dot{x}_{\dot{\beta}}^{\alpha} F_{\operatorname{lin}}^{\dot{\alpha} \dot{\beta}}+\dot{x}_{\beta}^{\dot{\alpha}} F_{\operatorname{lin}}^{\alpha \beta}\right)=\frac{i}{4}\left(\bar{\theta} \sigma_{\mu} \theta\right) \sigma_{\alpha \dot{\alpha}}^{\mu}\left(\dot{x}^{\beta \dot{\alpha}} \partial^{\alpha \dot{\gamma}} A_{\beta \dot{\gamma}}-\dot{x}^{\alpha \dot{\beta}} \partial^{\dot{\alpha} \gamma} A_{\dot{\beta} \gamma}\right)= \\
= & \frac{i}{4}\left(\bar{\theta} \sigma_{\mu} \theta\right)\left(\operatorname{Tr}\left(\bar{\sigma}^{\mu} \sigma^{\nu} \bar{\sigma}^{\lambda} \sigma^{\rho}\right)-\operatorname{Tr}\left(\bar{\sigma}^{\mu} \sigma^{\rho} \bar{\sigma}^{\lambda} \sigma^{\nu}\right)\right) \dot{x}_{\nu} \partial_{\rho} A_{\lambda}=\left(\bar{\theta} \sigma_{\mu} \theta\right) \epsilon^{\mu \nu \lambda \rho} \dot{x}_{\nu} \partial_{\rho} A_{\lambda} .
\end{aligned}
$$

This allows us, to write $\mathcal{I}_{1,1}$ in the following form:

$$
\begin{aligned}
\mathcal{I}_{1,1}= & \left(\bar{\theta} \sigma_{\mu} \theta\right) \epsilon^{\mu \nu \lambda \rho} \dot{x}_{\nu} \partial_{\rho} A_{\lambda}+\frac{i}{2}\left(\bar{\theta} \sigma_{\mu} \theta\right)|\dot{x}| \bar{n}_{B C} \partial^{\mu} \phi^{B C}+2 i\left(\bar{\theta}_{C} \sigma_{\mu} \theta^{A}\right)|\dot{x}| n^{B C} \partial^{\mu} \bar{\phi}_{A B} \\
& +\frac{i}{2}\left(\dot{\bar{\theta}} \sigma_{\mu} \theta-\bar{\theta} \sigma_{\mu} \dot{\theta}\right) \frac{\dot{x}^{\mu}}{|\dot{x}|} \bar{n}_{B C} \phi^{B C} .
\end{aligned}
$$

Hence we have:

$$
\begin{aligned}
\operatorname{Tr} & \left\langle\mathcal{I}_{0,0}\left(\tau_{1}\right) \mathcal{I}_{1,1}\left(\tau_{2}\right)\right\rangle=\operatorname{Tr}\left(T^{a} T^{b}\right)\left\langle\left(\dot{x}_{1}^{\kappa} A_{\kappa}^{a}\left(x_{1}\right)-\frac{1}{2}\left|\dot{x}_{1}\right| \bar{n}_{E F} \phi^{E F a}\left(x_{1}\right)\right)\right. \\
& \left(\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{2}\right) \epsilon^{\mu \nu \lambda \rho} \dot{x}_{2 \nu} \partial_{2 \rho} A_{\lambda}^{b}\left(x_{2}\right)+\frac{i}{2}\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{2}\right)\left|\dot{x}_{2}\right| \bar{n}_{B C} \partial_{2}^{\mu} \phi^{B C b}\left(x_{2}\right)\right. \\
& \left.\left.+2 i\left(\bar{\theta}_{2 C} \sigma_{\mu} \theta_{1}^{A}\right)\left|\dot{x}_{2}\right| n^{B C} \partial_{2}^{\mu} \bar{\phi}_{A B}^{b}\left(x_{2}\right)+\frac{i}{2}\left(\dot{\bar{\theta}}_{2} \sigma_{\mu} \theta_{2}-\bar{\theta}_{2} \sigma_{\mu} \dot{\theta}_{2}\right) \frac{\dot{x}_{2}^{\mu}}{\left|\dot{x}_{2}\right|} \bar{n}_{B C} \phi^{B C b}\left(x_{2}\right)\right)\right\rangle= \\
= & \operatorname{Tr}\left(T^{a} T^{b}\right)\left\{\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{2}\right) \epsilon^{\mu \nu \lambda \rho} \dot{x}_{1}^{\kappa} \dot{x}_{2 \nu} \partial_{2 \rho}\left\langle A_{\kappa}^{a}\left(x_{1}\right) A_{\lambda}^{b}\left(x_{2}\right)\right\rangle\right. \\
& -\frac{i}{4}\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{2}\right)\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right| \bar{n}_{E F} \bar{n}_{B C} \partial_{2}^{\mu}\left\langle\phi^{E F a}\left(x_{1}\right) \phi^{B C b}\left(x_{2}\right)\right\rangle \\
& -i\left(\bar{\theta}_{2 C} \sigma_{\mu} \theta_{2}^{A}\right)\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right| n^{E F} n^{B C} \partial_{2}^{\mu}\left\langle\phi_{E F}^{a}\left(x_{1}\right) \phi_{A B}^{b}\left(x_{2}\right)\right\rangle \\
& \left.-\frac{i}{4}\left(\dot{\bar{\theta}}_{2} \sigma_{\mu} \theta_{2}-\bar{\theta}_{2} \sigma_{\mu} \dot{\theta}_{2}\right) \frac{\left|\dot{x}_{1}\right|}{\left|\dot{x}_{2}\right|} \dot{x}_{2}^{\mu} \bar{n}_{E F} \bar{n}_{B C}\left\langle\phi^{E F a}\left(x_{1}\right) \phi^{B C b}\left(x_{2}\right)\right\rangle\right\} .
\end{aligned}
$$

Inserting the propagators derived in chapter 4.2 and making use of 2.32 we see that the two middle terms cancel each other. Hence we arrive at the following result:

$$
\begin{align*}
\operatorname{Tr}\left\langle\mathcal{I}_{0,0}\left(\tau_{1}\right) \mathcal{I}_{1,1}\left(\tau_{2}\right)\right\rangle= & \frac{g^{2} N^{2}}{4 \pi^{2}}\left\{\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{2}\right) \epsilon^{\mu \nu \lambda \rho} \frac{\dot{x}_{2 \nu} x_{12} \rho \dot{x}_{1 \lambda}}{x_{12}^{4}}\right. \\
& \left.+\frac{i}{2}\left(\dot{\bar{\theta}}_{2} \sigma_{\mu} \theta_{2}-\bar{\theta}_{2} \sigma_{\mu} \dot{\theta}_{2}\right) \frac{\left|\dot{x}_{1}\right|}{\left|\dot{x}_{2}\right|} \frac{\dot{x}_{2}^{\mu}}{x_{12}^{2}}\right\} . \tag{6.86}
\end{align*}
$$

Inserting (6.84), (6.85) and (6.86) into (6.83), we find the $\theta \bar{\theta}$ correction to the vacuum expectation value of the Maldacena-Wilson loop to first order in perturbation theory:

$$
\begin{align*}
\langle\mathcal{W}(C)\rangle_{(1)}= & -\frac{\lambda}{16 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{2}}+\frac{i \lambda}{4 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{1}\right) \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{4}} x_{12}^{\mu} \\
& -\frac{\lambda}{4 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left(\bar{\theta}_{2} \sigma_{\mu} \theta_{1}-\bar{\theta}_{2} \sigma_{\mu} \theta_{2}\right) \epsilon^{\mu \nu \rho \kappa} \frac{\dot{x}_{1 \nu} \dot{x}_{2 \rho} x_{12 \kappa}}{\left(x_{1}-x_{2}\right)^{4}} \\
& -\frac{i \lambda}{8 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left(\bar{\theta}_{2} \sigma_{\mu} \dot{\theta}_{1} \frac{\dot{x}_{2}^{\mu}}{x_{12}^{2}}-\dot{\bar{\theta}}_{2} \sigma_{\mu} \theta_{1} \frac{\dot{x}_{1}^{\mu}}{x_{12}^{2}}\right) \\
& -\frac{i \lambda}{8 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left(\dot{\bar{\theta}}_{2} \sigma_{\mu} \theta_{2}-\bar{\theta}_{2} \sigma_{\mu} \dot{\theta}_{2}\right) \frac{\left|\dot{x}_{1}\right|}{\left|\dot{x}_{2}\right|} \frac{\dot{x}_{2}^{\mu}}{x_{12}^{2}}+\mathcal{O}\left(\theta^{2}\right)+\mathcal{O}\left(\bar{\theta}^{2}\right) . \tag{6.87}
\end{align*}
$$

This result should by construction be invariant under supersymmetry transformations. Hence it is a non-trivial check of our calculation to show that (6.87) is annihilated by
the supersymmetry generators $Q_{A}^{\alpha}$ and $\bar{Q}^{A \dot{\alpha}}$. We will now go through the proof for $\bar{Q}^{A \dot{\alpha}}$, which can easily be transferred to show the invariance under $Q_{A}^{\alpha}$. Note that our result (6.87) only provides the necessary input to check the invariance under $\bar{Q}(Q)$ at order $\theta(\bar{\theta})$. We show that

$$
\begin{equation*}
\int \mathrm{d} s \frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}(s)}\langle\mathcal{W}(C)\rangle_{(1)}-i \sigma_{\mu}^{\dot{\alpha} \alpha} \int \mathrm{d} s \theta_{\alpha}^{A}(s) \frac{\delta}{\delta x_{\mu}(s)}\langle\mathcal{W}(C)\rangle_{(1)} \stackrel{1,0}{=} 0 \tag{6.88}
\end{equation*}
$$

Taking a close look at $\langle\mathcal{W}(C)\rangle_{(1)}$ and noting that

$$
\int \mathrm{d} s \frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}(s)} \dot{\bar{\theta}}_{B \dot{\beta}}(\tau)=0
$$

we read off:

$$
\begin{aligned}
& \int \mathrm{d} s \frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}(s)}\langle\mathcal{W}(C)\rangle_{(1)} \stackrel{1,0}{=} \frac{i \lambda}{4 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \theta_{1}{ }_{\alpha}^{A} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{4}} x_{12}^{\alpha \dot{\alpha}} \\
&-\frac{\lambda}{4 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left(\theta_{1}{ }_{\alpha}^{A}-\theta_{2}{ }_{\alpha}^{A}\right) \sigma_{\mu}^{\alpha \dot{\alpha}} \epsilon^{\mu \nu \rho \kappa} \frac{\dot{x}_{1 \nu} \dot{x}_{2 \rho} x_{12 \kappa}}{\left(x_{1}-x_{2}\right)^{4}} \\
&-\frac{i \lambda}{8 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left(\dot{\theta}_{1}{ }_{\alpha}^{A}-\dot{\theta}_{2}{ }_{\alpha}^{A} \frac{\left|\dot{x}_{1}\right|}{\left|\dot{x}_{2}\right|}\right) \frac{\dot{x}_{2}^{\alpha \dot{\alpha}}}{x_{12}^{2}} \\
&=\frac{i \lambda}{4 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left\{\theta_{1}{ }_{\alpha}^{A} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{4}} x_{12}^{\alpha \dot{\alpha}}-\frac{1}{2}\left(\dot{\theta}_{1}{ }_{\alpha}^{A}-\dot{\theta}_{2}{ }_{\alpha}^{A} \frac{\left|\dot{x}_{1}\right|}{\left|\dot{x}_{2}\right|}\right) \frac{\dot{x}_{2}^{\alpha \dot{\alpha}}}{x_{12}^{2}}\right\} .
\end{aligned}
$$

We used the antisymmetry under ( $\tau_{1} \leftrightarrow \tau_{2}$ ) of the integrand of the second term to reach the last line. Combining the above result with

$$
\begin{gathered}
i \sigma_{\mu}^{\dot{\alpha} \alpha} \int \mathrm{d} s \theta_{\alpha}^{A}(s) \frac{\delta}{\delta x_{\mu}(s)}\langle\mathcal{W}(C)\rangle_{(1)} \stackrel{1,0}{=} \frac{i \lambda}{16 \pi^{2}} \int \mathrm{~d} s \theta_{A}^{\alpha}(s) \sigma_{\mu}^{\dot{\alpha} \alpha} \int \mathrm{d} \tau_{1} \mathrm{~d} \tau_{2}\left\{\frac{1}{x_{12}^{2}}\left(\dot{x}_{1}^{\mu}-\frac{\left|\dot{x}_{1}\right|}{\left|\dot{x}_{2}\right|} \dot{x}_{2}^{\mu}\right)\right. \\
\partial_{\tau_{2}} \delta\left(\tau_{2}-s\right)+\frac{1}{x_{12}^{2}}\left(\dot{x}_{2}^{\mu}-\frac{\left|\dot{x}_{2}\right|}{\left|\dot{x}_{1}\right|} \dot{x}_{1}^{\mu}\right) \partial_{\tau_{1}} \delta\left(\tau_{1}-s\right) \\
\left.-2 \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{4}} x_{12}^{\mu}\left(\delta\left(\tau_{1}-s\right)-\delta\left(\tau_{2}-s\right)\right)\right\} \\
=-\frac{i \lambda}{4 \pi^{2}} \int \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2}\left\{\theta_{1}{ }_{\alpha}^{A} \frac{\dot{x}_{1} \dot{x}_{2}-\left|\dot{x}_{1}\right|\left|\dot{x}_{2}\right|}{\left(x_{1}-x_{2}\right)^{4}} x_{12}^{\alpha \dot{\alpha}}-\frac{1}{2}\left(\dot{\theta}_{1}{ }_{\alpha}^{A}-\dot{\theta}_{2}{ }_{\alpha}^{A} \frac{\left|\dot{x}_{1}\right|}{\left|\dot{x}_{2}\right|}\right) \frac{\dot{x}_{2}^{\alpha \dot{\alpha}}}{x_{12}^{2}}\right\},
\end{gathered}
$$

we indeed find that (6.88) holds.

### 6.3 Yangian Symmetry of the Extended Loop

We now return to the question of Yangian invariance. We have computed the necessary input to calculate the action of the non-local part of the level-1 momentum generator

$$
\begin{aligned}
P_{\mathrm{nl}, \varepsilon}^{(1) \mu} & =P_{\text {bos }, \varepsilon}^{(1) \mu}+P_{\text {ferm }, \varepsilon}^{(1) \mu} \\
P_{\text {bos }, \mathrm{nl}, \varepsilon}^{(1) \mu} & =\int \mathrm{d} s_{1} \mathrm{~d} s_{2}\left\{\left(m^{\mu \nu}\left(s_{1}\right)-d\left(s_{1}\right) \eta^{\mu \nu}\right) p_{\nu}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} \theta\left(s_{2}-s_{1}-d\left(s_{2}, \varepsilon\right)\right), \\
P_{\text {ferm }, \mathrm{nl}, \varepsilon}^{(1) \mu} & =\frac{i}{4} \int \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left\{\bar{q}^{A \dot{\alpha}}\left(s_{1}\right) \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} q_{A}^{\alpha}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right\} \theta\left(s_{2}-s_{1}-d\left(s_{2}, \varepsilon\right)\right),
\end{aligned}
$$

on the expectation value $\langle\mathcal{W}(C)\rangle_{(1)}$ at zero order in the Graßmann variables. The bosonic result 6.37 receives corrections from the action of $P_{\text {ferm }, \mathrm{n} 1, \varepsilon}^{(1) \mu}$ on the terms in (6.87) which are of order $\theta \bar{\theta}$. We rewrite the level- 1 momentum generator to slightly simplify the following calculations,

$$
\begin{aligned}
P_{\text {ferm }, \mathrm{n} 1, \varepsilon}^{(1),} & =\frac{i}{4} \int \mathrm{~d} s_{1} \mathrm{~d} s_{2}\left(\bar{q}^{A \dot{\alpha}}\left(s_{1}\right) \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} q_{A}^{\alpha}\left(s_{2}\right)-\left(s_{1} \leftrightarrow s_{2}\right)\right) \theta\left(s_{2}-s_{1}-\varepsilon\right) \\
& \left.=\frac{i}{4} \int \mathrm{~d} s_{1} \mathrm{~d} s_{2} \bar{q}^{A \dot{\alpha}}\left(s_{1}\right)\right)_{\alpha \dot{\alpha}}^{\mu} q_{A}^{\alpha}\left(s_{2}\right)\left(\theta\left(s_{2}-s_{1}-\varepsilon\right)-\theta\left(s_{1}-s_{2}-\varepsilon\right)\right) .
\end{aligned}
$$

In the above expression the densities of the supersymmetry generators are given by

$$
\begin{aligned}
q_{A}^{\alpha}(s) & =-\frac{\delta}{\delta \theta_{\alpha}^{A}(s)}+i \bar{\theta}_{A \dot{\alpha}}(s) \frac{\delta}{\delta x_{\alpha \dot{\alpha}}(s)}, \\
\bar{q}^{A \dot{\alpha}}(s) & =\frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}(s)}-i \theta_{\alpha}^{A}(s) \frac{\delta}{\delta x_{\alpha \dot{\alpha}}(s)} .
\end{aligned}
$$

In the computation of the zero order of $P_{\text {ferm, } \varepsilon}^{(1) \mu}\langle\mathcal{W}(C)\rangle_{(1)}$, we only need to consider the Graßmann derivatives in the above densities. Hence we consider the generator

$$
\begin{equation*}
\hat{P}_{\text {ferm, nl, } \varepsilon}^{(1) \mu}=-\frac{i}{4} \int \mathrm{~d} s_{1} \mathrm{~d} s_{2} \frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}\left(s_{1}\right)} \bar{\sigma}_{\alpha \dot{\alpha} \dot{\alpha}}^{\mu} \frac{\delta}{\delta \theta_{\alpha}^{A}(s)}\left(\theta\left(s_{2}-s_{1}-\varepsilon\right)-\theta\left(s_{1}-s_{2}-\varepsilon\right)\right) \tag{6.89}
\end{equation*}
$$

and note that

$$
\hat{P}_{\text {ferm }, \mathrm{nl}, \varepsilon}^{(1) \mu}\langle\mathcal{W}(C)\rangle_{(1)} \stackrel{0,0}{=} P_{\text {ferm }, \mathrm{nl}, \varepsilon}^{(1) \mu}\langle\mathcal{W}(C)\rangle_{(1)} .
$$

The bosonic result 6.37 receives corrections from the action of $P_{\text {ferm }, \varepsilon}^{(1) \mu}$ on the terms in (6.87) which are of order $\theta \bar{\theta}$. In order to compute these corrections, we first consider the action of $P_{\text {ferm }, \varepsilon}^{(1) \mu}$ on the relevant $\theta \bar{\theta}$ structures appearing in $\langle\mathcal{W}(C)\rangle_{(1)}$. We have:

$$
\begin{align*}
\hat{P}_{\text {ferm }, \mathrm{nl}, \varepsilon}^{(1) \mu}\left(\bar{\theta}_{2} \sigma^{\nu} \theta_{1}\right)= & -\frac{i}{4} \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} \int \mathrm{d} s_{1} \mathrm{~d} s_{2}\left(\theta\left(s_{2}-s_{1}-\varepsilon\right)-\theta\left(s_{1}-s_{2}-\varepsilon\right)\right) \\
& \frac{\delta}{\delta \bar{\theta}_{A \dot{\alpha}}\left(s_{1}\right)} \frac{\delta}{\delta \theta_{\alpha}^{A}(s)}\left(\bar{\theta}_{B \dot{\beta}}\left(\tau_{2}\right) \sigma^{\nu \dot{\beta} \beta} \theta_{\beta}^{B}\left(\tau_{1}\right)\right) \\
= & -\frac{i}{4} \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} \int \mathrm{d} s_{1} \mathrm{~d} s_{2}\left(\theta\left(s_{2}-s_{1}-\varepsilon\right)-\theta\left(s_{1}-s_{2}-\varepsilon\right)\right) \delta_{B}^{A} \delta_{\dot{\beta}}^{\dot{\alpha}} \delta\left(s_{1}-\tau_{2}\right) \\
& \sigma^{\nu \dot{\beta} \beta}\left(-\delta_{A}^{B} \delta_{\beta}^{\alpha} \delta\left(\tau_{1}-s_{2}\right)\right) \\
= & i \bar{\sigma}_{\alpha \dot{\alpha}}^{\mu} \sigma^{\nu \dot{\alpha} \alpha}\left(\theta\left(\tau_{1}-\tau_{2}-\varepsilon\right)-\theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)\right) \\
= & 2 i \eta^{\mu \nu}\left(\theta\left(\tau_{1}-\tau_{2}-\varepsilon\right)-\theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)\right) . \tag{6.90}
\end{align*}
$$

We made use of the identity (2.9) to reach the last line. In the same way we find:

$$
\begin{align*}
\hat{P}_{\text {ferm }, \mathrm{nl}, \varepsilon}^{(1) \mu}\left(\bar{\theta}_{2} \sigma^{\nu} \theta_{2}\right) & =2 i \eta^{\mu \nu}(\theta(-\varepsilon)-\theta(-\varepsilon))=0,  \tag{6.91}\\
\hat{P}_{\text {ferm }, \mathrm{nl}, \varepsilon}^{(1)}\left(\dot{\bar{\theta}}_{2} \sigma^{\nu} \theta_{1}\right) & =-2 i \eta^{\mu \nu}\left(\delta\left(\tau_{1}-\tau_{2}-\varepsilon\right)+\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)\right),  \tag{6.92}\\
\hat{P}_{\text {ferm, } \mathrm{n}, \varepsilon}^{(1),}\left(\bar{\theta}_{2} \sigma^{\nu} \dot{\theta}_{1}\right) & =2 i \eta^{\mu \nu}\left(\delta\left(\tau_{1}-\tau_{2}-\varepsilon\right)+\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)\right),  \tag{6.93}\\
\hat{P}_{\mathrm{ferm}, \mathrm{nl}, \varepsilon}^{(1) \mu}\left(\dot{\bar{\theta}}_{2} \sigma^{\nu} \theta_{2}-\bar{\theta}_{2} \sigma^{\nu} \dot{\theta}_{2}\right) & =-8 i \eta^{\mu \nu} \delta(\varepsilon) . \tag{6.94}
\end{align*}
$$

Using the above identities we find:

$$
\begin{aligned}
\hat{P}_{\text {ferm, } \mathrm{nl}, \varepsilon}^{(1) \mu}\langle\mathcal{W}(C)\rangle_{(1)} & =\frac{\lambda}{2 \pi^{2}} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}+1}{\left(x_{1}-x_{2}\right)^{2}} x_{12}^{\mu}\left(\theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)-\theta\left(\tau_{1}-\tau_{2}-\varepsilon\right)\right) \\
& +\frac{i \lambda}{2 \pi^{2}} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \epsilon^{\mu \nu \rho \kappa} \frac{\dot{x}_{1 \nu} \dot{x}_{2 \rho} x_{12 \kappa}}{\left(x_{1}-x_{2}\right)^{4}}\left(\theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)-\theta\left(\tau_{1}-\tau_{2}-\varepsilon\right)\right) \\
& +\frac{\lambda}{4 \pi^{2}} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1}^{\mu}+\dot{x}_{2}^{\mu}}{x_{12}^{2}}\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\delta\left(\tau_{1}-\tau_{2}-\varepsilon\right)\right) \\
& -\frac{\lambda}{\pi^{2}} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{2}^{\mu}}{x_{12}^{2}} \delta(\varepsilon) \\
& =\frac{\lambda}{\pi^{2}} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}+1}{\left(x_{1}-x_{2}\right)^{2}} x_{12}^{\mu} \theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)-\frac{\lambda}{\pi^{2}} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{2}^{\mu}}{x_{12}^{2}} \delta(\varepsilon) \\
& +\frac{\lambda}{2 \pi^{2}} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1}^{\mu}}{x_{12}^{2}}\left(\delta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+(\varepsilon \rightarrow-\varepsilon)\right) .
\end{aligned}
$$

We used a change of variables $\left(\tau_{1} \leftrightarrow \tau_{2}\right)$ to reach the last line. All the terms we encounter are already familiar from the discussion of the bosonic result in section 6.1. Dropping the $\delta(\varepsilon)$-term as before and expanding the last term, we arrive at the following result:
$\hat{P}_{\text {ferm, } \mathrm{nl}, \varepsilon}^{(1) \mu}\langle\mathcal{W}(C)\rangle_{(1)}=\frac{\lambda}{\pi^{2}} \int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} \frac{\dot{x}_{1} \dot{x}_{2}+1}{\left(x_{1}-x_{2}\right)^{2}} x_{12}^{\mu} \theta\left(\tau_{2}-\tau_{1}-\varepsilon\right)+\frac{\lambda}{12 \pi^{2}} \int_{0}^{L} \mathrm{~d} \tau \dot{x}^{\mu}(\tau) \ddot{x}^{2}(\tau)$.
Lifting the constraint of arc-length parametrization and combining the above result with the bosonic part (6.37), we reach the final result:

$$
\begin{equation*}
P_{\mathrm{n} 1, \varepsilon}^{(1) \mu}\langle\mathcal{W}(C)\rangle_{(1)} \stackrel{0,0}{=} \frac{7 \lambda}{96 \pi^{2}} \int \mathrm{~d} \tau \dot{x}^{\mu}(\tau)\left(\frac{\ddot{x}^{2}}{\dot{x}^{4}}-\frac{(\dot{x} \cdot \ddot{x})^{2}}{\dot{x}^{6}}\right)+\mathcal{O}(\varepsilon) . \tag{6.95}
\end{equation*}
$$

From this we read of the level- 1 density

$$
\begin{equation*}
p^{(1) \mu}(s)=-\frac{7 \lambda}{96 \pi^{2}} \dot{x}^{\mu}(s)\left(\frac{\ddot{x}^{2}(s)}{\dot{x}^{4}(s)}-\frac{(\dot{x}(s) \cdot \ddot{x}(s))^{2}}{\dot{x}^{6}(s)}\right) . \tag{6.96}
\end{equation*}
$$

The full level-1 momentum generator is thus given by

$$
\begin{equation*}
P_{\varepsilon}^{(1) \mu}=P_{\mathrm{nl}, \varepsilon}^{(1) \mu}-\frac{7 \lambda}{96 \pi^{2}} \int \mathrm{~d} s \dot{x}^{\mu}(s)\left(\frac{\ddot{x}^{2}}{\dot{x}^{4}}-\frac{(\dot{x} \cdot \ddot{x})^{2}}{\dot{x}^{6}}\right) . \tag{6.97}
\end{equation*}
$$

We have shown that this generator annihilates $\langle\mathcal{W}(C)\rangle$ to first order in perturbation theory and zero order in the Graßmann variables,

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} P_{\varepsilon}^{(1) \mu}\langle\mathcal{W}(C)\rangle=0+\mathcal{O}\left(\lambda^{2}\right)+\mathcal{O}(\theta)+\mathcal{O}(\bar{\theta}) \tag{6.98}
\end{equation*}
$$

Having established the invariance under the level-1 momentum generator, one would now like to conclude the invariance under all level-1 generators. This could easily be done by using the commutation relation (3.38),

$$
\left[J_{a}^{(0)}, J_{b}^{(1)}\right]=f_{a b}{ }^{c} J_{c}^{(1)},
$$

of the Yangian, if the invariance under $P^{(1) \mu}$ and the level zero generators were established to all orders in the superspace expansion. However, as this is not the case, we are not yet in a position to do this.

## Chapter 7

## Conclusion and Outlook

Following an idea of Jan Plefka, we have studied a possible Yangian invariance of the Maldacena-Wilson loop in $\mathcal{N}=4$ supersymmetric Yang-Mills theory. For computational simplicity, we restricted our attention to the first-order correction of the expectation value in perturbation theory. Employing a similarity to two-dimensional integrable field theories, the Yangian symmetry generators are given by curve integrals over functional derivatives acting in the loop space. Whilst the non-local structure of the level-1 generators is fixed by the underlying symmetry algebra, they also contain a local path-dependent piece, which can be adjusted in order to reach invariance. By performing an explicit calculation for the non-local part of the level-1 momentum generator $P_{\text {bos, nl }}^{(1) \mu}$ we found that the Maldacena-Wilson loop is not Yangian invariant in this way.

However, we were lead to consider a supersymmetric extension $\mathcal{W}(C)$ of the Malda-cena-Wilson loop which also includes the fermionic fields of $\mathcal{N}=4 \mathrm{SYM}$. In this regard, we also extended the level-1 momentum generator as the inclusion of supersymmetry extends the underlying symmetry algebra from $\mathfrak{s o}(2,4)$ to $\mathfrak{s u}(2,2 \mid 4)$. In the extension, we employed a full non-chiral superspace parametrized by the variables $\{x, \theta, \bar{\theta}, y\}$. The exact form of the extension followed naturally from the requirement that the vacuum expectation value be invariant under the supersymmetry transformations in the given superspace. We have computed the coefficients of the extension up to second order in the anticommuting Graßmann variables $\theta$ and $\bar{\theta}$, setting $y=0$. The reason for this restriction was that we only computed those terms of the expansion, which affect the result for the unextended loop, i.e. at $\theta=\bar{\theta}=y=0$. We were then able to show that the extended loop is indeed invariant under the full level- 1 momentum generator $P^{(1) \mu}$ to first order in perturbation theory and at the lowest order in an extension in the superspace variables. We have thus provided first evidence for a full Yangian symmetry of a supersymmetric extension of the Maldacena-Wilson loop in $\mathcal{N}=4$ supersymmetric Yang-Mills theory.

Another step toward this would be to show the invariance under all level-1 generators at first order in perturbation theory. It would be easy to conclude this from the commutation relations (3.38) of the Yangian algebra, if we had established the invariance under all level- 0 generators and $P^{(1) \mu}$ to all orders in the superspace variables.

Consider for example the following commutator:

$$
\left[K_{\mu}^{(0)}, P_{\nu}^{(1)}\right]=2 M_{\mu \nu}^{(1)}+2 \eta_{\mu \nu} D^{(1)} .
$$

Unfortunately, we are unable to conclude the invariance of $\langle\mathcal{W}(C)\rangle$ under $M_{\mu \nu}^{(1)}$ and $D^{(1)}$ at the leading order as we have not established the conformal invariance of $\langle\mathcal{W}(C)\rangle$ at order $\theta \bar{\theta}$. In order to show the invariance under all level- 1 generators, one would have to derive the $y$-dependence of the extended Maldacena-Wilson loop and check the invariance under the superconformal generators at higher orders in the superspace extension. While this certainly involves several tedious calculations, it seems to be within reach. Of course it would be favorable to derive the full supersymmetric completion of the Maldacena-Wilson loop to all orders in the superspace variables. However, doing this order by order in the way we have constructed the first few orders, seems not to be feasible.

Strikingly, a Yangian invariance has also been found for the dual string description of the Maldacena-Wilson loop in [13]. The non-local parts of the Yangian generators found there are exactly the same as for the weak coupling side and also the functional form of the local part agrees. The difference between the two sides lies in the prefactor $f(\lambda)$ of the local term, for which the following limits are known ${ }^{11}$

$$
f(\lambda \ll 1)=-\frac{7 \lambda}{96 \pi^{2}}, \quad \quad f(\lambda \gg 1)=-\frac{\lambda}{4 \pi^{2}}
$$

A possible explanation for this difference is the existence of a non-trivial interpolating function $f(\lambda)$ which has the asymptotic behaviour given above for small or large $\lambda$. Checking the Yangian invariance also at second order on the weak coupling side should provide further insight into this question.

It was not necessary to consider fermions on the strong coupling side in order to establish the Yangian symmetry. The inclusion of the fermionic degrees of freedom might also change the asymptotic behaviour of $f(\lambda)$ for large $\lambda$. This would require repeating the strong coupling analysis for the Green-Schwarz superstring, which is of course interesting in its own right. It might also yield insights on the form of the supersymmetric extension of the Maldacena-Wilson loop at weak coupling.

Apart from further establishing the Yangian symmetry of the Maldacena-Wilson loop, an interesting question is how the Yangian symmetry might be applied in order to compute new results. To this end, it would be important to understand in which way the Yangian invariance constrains the vacuum expectation value and to classify the Yangian invariants. If not for exact results, one might try to apply the Yangian invariance to so-called wavy Wilson lines [51] to compute further corrections in an expansion in terms of the "waviness" of the loop.

[^15]
## Appendix A

## Killing Metric and Dual Structure Constants for $\mathfrak{s o}(2,4)$

In this appendix we introduce the Killing metric and dual generators for a semi-simple Lie algebra $\mathfrak{g}$ and derive them for the conformal algebra $\mathfrak{s o}(2,4)$.

Consider a semi-simple Lie algebra $\mathfrak{g}=\operatorname{span}\left\{T_{a}\right\}$ over $K=\mathbb{R}$ or $K=\mathbb{C}$ with commutation relations

$$
\begin{equation*}
\left[T_{a}, T_{b}\right]=f_{a b}^{c} T_{c} . \tag{A.1}
\end{equation*}
$$

For every element $x \in \mathfrak{g}$, we define the adjoint map $a d_{x}: \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$
a d_{x}(y):=[x, y] .
$$

This gives rise to a symmetric bilinear form $G: \mathfrak{g} \times \mathfrak{g} \rightarrow K$, which is defined by

$$
\begin{equation*}
G(x, y):=\operatorname{Tr}\left(a d_{x} \circ a d_{y}\right) . \tag{A.2}
\end{equation*}
$$

This bilinear form is called the Killing metric on $\mathfrak{g}$. It can be shown that the Killing metric is non-degenerate, i.e. that the matrix

$$
G_{a b}:=G\left(T_{a}, T_{b}\right)
$$

is invertible. The inverse of $G_{a b}$ is as usual denoted by $G^{a b}$. We then define the dual generators

$$
\begin{equation*}
\hat{T}^{a}:=G^{a b} T_{b} \tag{A.3}
\end{equation*}
$$

By definition, these satisfy the relation

$$
G\left(\hat{T}^{a}, T_{b}\right)=\delta_{b}^{a}
$$

As the metric is non-degenerate this may also be considered as the defining relation of the dual generators. The dual generators satisfy commutation relations with dual structure constants:

$$
\begin{equation*}
\left[\hat{T}^{a}, \hat{T}^{b}\right]=f_{c}^{a b} \hat{T}^{c}, \quad \text { where } \quad f_{c}^{a b}=G^{a d} G^{b e} f_{d e}^{g} G_{g c} . \tag{A.4}
\end{equation*}
$$

We compute the Killing metric for the conformal algebra $\mathfrak{s o}(2,4)$. In the notation of chapter 3.1, the commutation relations read:

$$
\begin{array}{rlrl}
{\left[M_{\mu \nu}, M_{\rho \sigma}\right]} & =\eta_{\mu \sigma} M_{\nu \rho}+\eta_{\nu \rho} M_{\mu \sigma}-\eta_{\mu \rho} M_{\nu \sigma}-\eta_{\nu \sigma} M_{\mu \rho} \\
{\left[P_{\mu}, P_{\nu}\right]} & =0 & {\left[M_{\mu \nu}, P_{\lambda}\right]} & =\eta_{\nu \lambda} P_{\mu}-\eta_{\mu \lambda} P_{\nu} \\
{\left[D, P_{\mu}\right]} & =-P_{\mu} & {\left[D, K_{\mu}\right]} & =K_{\mu}  \tag{A.5}\\
{\left[D, M_{\mu \nu}\right]} & =0 & {\left[M_{\mu \nu}, K_{\rho}\right]} & =\eta_{\nu \rho} K_{\mu}-\eta_{\mu \rho} K_{\nu} \\
{\left[P_{\mu}, K_{\nu}\right]} & =2 M_{\mu \nu}-2 \eta_{\mu \nu} D & {\left[K_{\mu}, K_{\nu}\right]} & =0
\end{array}
$$

These commutation relations can be greatly simplified by the following change of basis:

$$
M_{45}:=D, \quad M_{\mu 5}:=\frac{1}{2}\left(P_{\mu}+K_{\mu}\right), \quad M_{\mu 4}:=\frac{1}{2}\left(P_{\mu}-K_{\mu}\right) .
$$

The commutation relations are then given by

$$
\begin{equation*}
\left[M_{I J}, M_{K L}\right]=\eta_{I L} M_{J K}+\eta_{J K} M_{I L}-\eta_{J L} M_{I K}-\eta_{I K} M_{J L} \tag{A.6}
\end{equation*}
$$

Here, $I, J, K, L$ take values in $\{0,1,2,3,4,5\}$ and the metric of Minkowski space is extended by the diagonal components $\eta_{44}=-1$ and $\eta_{55}=1$ to give the metric of $\mathbb{R}^{2,4}$,

$$
\begin{equation*}
\eta_{I J}=\operatorname{diag}(+,-,-,-,-,+) . \tag{A.7}
\end{equation*}
$$

In order to compute the Killing metric on $\mathfrak{s o ( 2 , 4 )}$, we derive the matrix representation of $a d_{M_{I J}}$ from the commutation relation A.6). Note that a basis of $\mathfrak{s o}(2,4)$ is given by $\left\{M_{I J} \mid I<J\right\}$. Hence the matrix matrix representation of $a d_{M_{I J}}$ must satisfy

$$
\begin{equation*}
\left[M_{I J}, M_{K L}\right]=a d_{M_{I J}}\left(M_{K L}\right)=\sum_{M<N}\left(a d_{M_{I J}}\right)^{M N}{ }_{K L} M_{M N} . \tag{A.8}
\end{equation*}
$$

From A.6), we infer that ${ }^{1}$

$$
\begin{aligned}
{\left[M_{I J}, M_{K L}\right] } & =\left(\eta_{I L} \delta_{J}^{M} \delta_{K}^{N}+\eta_{J K} \delta_{I}^{M} \delta_{L}^{N}-\eta_{J L} \delta_{J}^{M} \delta_{L}^{N}-\eta_{I K} \delta_{I}^{M} \delta_{K}^{N}\right) M_{M N} \\
& =\sum_{M<N} 2\left(\eta_{I L} \delta_{J}^{[M} \delta_{K}^{N]}+\eta_{J K} \delta_{I}^{[M} \delta_{L}^{N]}-\delta_{J}^{[M} \delta_{L}^{N]}-\eta_{I K} \delta_{I}^{[M} \delta_{K}^{N]}\right) M_{M N}
\end{aligned}
$$

Thus, we have:

$$
\begin{aligned}
\left(a d_{M_{I J}}\right)^{M N}{ }_{K L} & =2\left(R_{I J}{ }^{[M N]}{ }_{K L},\right. \\
\left(R_{I J}\right)^{M N}{ }_{K L} & =\eta_{I L} \delta_{J}^{M} \delta_{K}^{N}+\eta_{J K} \delta_{I}^{M} \delta_{L}^{N}-\eta_{J L} \delta_{J}^{M} \delta_{L}^{N}-\eta_{I K} \delta_{I}^{M} \delta_{K}^{N} .
\end{aligned}
$$

This allows us to compute the components of the Killing metric in this basis:

$$
\begin{aligned}
G\left(M_{I J}, M_{K L}\right) & =\operatorname{Tr}\left(a d_{M_{I J}} \circ a d_{M_{K L}}\right)=4 \sum_{X<Y} \sum_{Z<W}\left(R_{I J}\right)^{[X Y]}{ }_{Z W}\left(R_{K L}\right)^{[Z W]}{ }_{X Y} \\
& =\left(R_{I J}\right)^{[X Y]}{ }_{Z W}\left(R_{K L}\right)^{[Z W]}{ }_{X Y}=\left(R_{I J}\right)^{X Y}{ }_{Z W}\left(R_{K L}\right)^{Z W}{ }_{X Y} \\
& =8\left(\eta_{I L} \eta_{J K}-\eta_{I K} \eta_{J L}\right) .
\end{aligned}
$$

[^16]The knowledge of the Killing metric allows us to determine the dual basis of generators $\left\{\hat{T}^{a}\right\}$. We find:

$$
\begin{equation*}
\hat{M}^{\mu \nu}=-\frac{1}{8} \eta^{\mu \rho} \eta^{\nu \sigma} M_{\rho \sigma}, \quad \hat{P}^{\mu}=-\frac{1}{16} \eta^{\mu \nu} K_{\nu}, \quad \hat{K}^{\mu}=-\frac{1}{16} \eta^{\mu \nu} P_{\nu}, \quad \hat{D}=\frac{1}{8} D . \tag{A.9}
\end{equation*}
$$

Although the computation of $G^{a b} T_{b}$ in the given basis can be somewhat tedious, it is easy to check that the given set of dual generators indeed satisfies $G\left(\hat{T}^{a}, T_{b}\right)=\delta_{b}^{a}$. The commutation relations of the dual generators are given by:

$$
\begin{array}{rlrl}
{\left[\hat{M}^{\mu \nu}, \hat{M}^{\rho \sigma}\right]} & =\frac{1}{8}\left(\eta^{\mu \rho} \hat{M}^{\nu \sigma}+\eta^{\nu \sigma} \hat{M}^{\mu \rho}-\eta^{\mu \sigma} \hat{M}^{\nu \rho}-\eta^{\nu \rho} \hat{M}^{\mu \sigma}\right) \\
{\left[\hat{P}^{\mu}, \hat{P}^{\nu}\right]} & =0 & {\left[\hat{M}^{\mu \nu}, \hat{P}^{\lambda}\right]} & =\frac{1}{8}\left(\eta^{\mu \lambda} \hat{P}^{\mu}-\eta^{\mu \lambda} \hat{P}^{\nu}\right) \\
{\left[\hat{D}, \hat{P}^{\mu}\right]} & =\frac{1}{8} \hat{P}^{\mu} & {\left[\hat{D}, \hat{K}^{\mu}\right]} & =-\frac{1}{8} \hat{K}^{\mu}  \tag{A.10}\\
{\left[\hat{D}, \hat{M}^{\mu \nu}\right]} & =0 & {\left[\hat{M}^{\mu \nu}, \hat{K}^{\rho}\right]} & =\frac{1}{8}\left(\eta^{\mu \rho} \hat{K}^{\nu}-\eta^{\nu \rho} \hat{K}^{\mu}\right) \\
{\left[\hat{P}^{\mu}, \hat{K}^{\nu}\right]} & =\frac{1}{16} \eta_{\mu \nu} \hat{D}-\frac{1}{16} \hat{M}^{\mu \nu} & {\left[\hat{K}^{\mu}, \hat{K}^{\nu}\right]} & =0
\end{array}
$$

## Appendix B

## Algebraic Preliminaries

In this appendix, we provide the necessary algebraic preliminaries to follow our discussion of the Yangian algebra in chapter 3.3. The main goals are to explain the universal enveloping algebra of a Lie algebra and to introduce the notion of a Hopf algebra. This appendix is based on [52] as well as [53].

With a view to the definition of Hopf algebras, we define an algebra in a way which may be unfamiliar to the reader.

Definition B.1. An algebra $(A, m, u)$ over a field $K$ consists of a vector space $A$ over $K$ as well as two linear maps, a multiplication $m: A \otimes A \rightarrow A$ and a unit $u: K \rightarrow A$, which satisfy the following properties:

$$
\begin{align*}
m(a \otimes m(b \otimes c)) & =m(m(a \otimes b) \otimes c) & \forall a, b, c \in A,  \tag{B.1}\\
m(u(k) \otimes a) & =k \cdot a=m(a \otimes u(k)) & \forall k \in K, a \in A . \tag{B.2}
\end{align*}
$$

The first property encodes associativity, while the second demands the existence of a unit element $\mathbb{1}=u(1)$. Both properties can be rephrased by demanding that the following diagrams commute:


Having introduced the structure of an algebra, it is natural to define maps which preserve this structure. Such maps are called algebra morphisms.

Definition B.2. Let $\left(A, m_{A}, u_{A}\right)$ and $\left(B, m_{B}, u_{B}\right)$ be algebras over $K$. A linear map $f: A \rightarrow B$ is called an algebra morphism if it satisfies

$$
\begin{equation*}
f\left(m_{A}(a \otimes b)\right)=m_{B}(f(a) \otimes f(b)) \quad \forall a, b \in A, \quad f\left(u_{A}(k)\right)=u_{B}(k) \quad \forall k \in K \tag{B.3}
\end{equation*}
$$

[^17]We express also these properties in the form of commutative diagrams:


For the sake of completeness, we give the definition of a Lie algebra in an analogous way to the definition of algebras.

Definition B.3. A Lie algebra ( $\mathfrak{g}, m_{\mathfrak{g}}$ ) over a field $K$ consists of a vector space $\mathfrak{g}$ over $K$ as well as a (linear) multiplication map $m_{\mathfrak{g}}: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, which satisfies the following properties:

$$
\begin{array}{lr}
0=m_{\mathfrak{g}}(x \otimes y)+m_{\mathfrak{g}}(y \otimes x) & \forall x, y \in L, \\
0=m_{\mathfrak{g}}\left(x \otimes m_{\mathfrak{g}}(y \otimes z)\right)+m_{\mathfrak{g}}\left(y \otimes m_{\mathfrak{g}}(z \otimes x)\right)+m_{\mathfrak{g}}\left(z \otimes m_{\mathfrak{g}}(x \otimes y)\right) & \forall x, y, z \in L .
\end{array}
$$

The first property encodes the antisymmetry of the Lie bracket $[x, y]=m_{\mathfrak{g}}(x \otimes y)$ while the second is the familiar Jacobi identity. Lie algebra morphisms are defined in the same way as algebra morphisms. Note that a general Lie algebra is not associative and does not contain a unit element. From any algebra $A$ one can construct a Lie algebra $L(A)$ by defining the Lie algebra product

$$
m_{L(A)}(x \otimes y):=m_{A}(x \otimes y)-m_{A}(y \otimes x) .
$$

This is by construction antisymmetric, and using the associativity of $m_{A}$ it is easy to show that it satisfies the Jacobi identity.

In order to construct the universal enveloping algebra, we need to understand the construction of a factor algebra. The key to this is the definition of an ideal.

Definition B.4. A left-sided ideal $I$ of an algebra $(A, m, u)$ is a subspace of $A$ which satisfies the following condition:

$$
\begin{equation*}
m_{A}(a \otimes j) \in I \quad \forall j \in I, a \in A \tag{B.6}
\end{equation*}
$$

$I$ is called a right-sided ideal if it instead satisfies

$$
\begin{equation*}
m_{A}(j \otimes a) \in I \quad \forall j \in I, a \in A . \tag{B.7}
\end{equation*}
$$

If $I$ satisfies both $(\overline{\mathrm{B} .6})$ and $(\overline{\mathrm{B} .7})$ it is called two-sided.
Given a two-sided ideal $I$ of an algebra $A$, we can define an equivalence relation on $A$ by

$$
\begin{equation*}
a \sim b \quad \text { if } a-b \in I . \tag{B.8}
\end{equation*}
$$

This defines an equivalence relation as $I$ is a subspace of $A$. The space of equivalence classes $[a]$ is denoted by $A / I$.

Proposition B.1. Let A be an algebra and I a two-sided ideal. The set of equivalence classes A/I becomes an algebra by the following definitions:

$$
\begin{align*}
{[a]+[b]:=[a+b] \quad \forall a, b \in A, \quad k[a]:=[k a] \quad \forall k \in K, a \in A, }  \tag{B.9}\\
u_{A / I}(k):=\left[u_{A}(k)\right] \quad \forall k \in K, \quad m_{A / I}([a] \otimes[b]):=\left[m_{A}(a \otimes b)\right] \quad \forall a, b \in A . \tag{B.10}
\end{align*}
$$

Proof. It is easy to see that (B.9) are well-defined and transfer the vector-space structure of $A$ to $A / I$. It is also simple to check that the product $m_{A / I}$ is well-defined. Take $a^{\prime} \sim a$ and $b^{\prime} \sim b$. Then there are $i, j \in I$ such that $a^{\prime}=a+i$ and $b^{\prime}=b+j$ and we have:

$$
m_{A}\left(a^{\prime} \otimes b^{\prime}\right)=m_{A}((a+i) \otimes(b+j))=m_{A}(a \otimes b)+m_{A}(i \otimes b)+m_{A}(a \otimes j)+m_{A}(i \otimes j)
$$

By the definition of a two-sided ideal, we have

$$
m_{A}(i \otimes b), m_{A}(a \otimes j), m_{A}(i \otimes j) \in I
$$

and hence $m_{A}\left(a^{\prime} \otimes b^{\prime}\right) \sim m_{A}(a \otimes b)$. The properties (B.1) and (B.2) are trivial to check.

## B. 1 The Universal Enveloping Algebra

We are now in a position to define the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. We start by defining enveloping algebras:

Definition B.5. Let $\mathfrak{g}$ be a Lie algebra over $K$. An algebra $(E(\mathfrak{g}), m, u)$ is called an enveloping algebra of $\mathfrak{g}$, if there exists a map $i_{\mathfrak{g}}: \mathfrak{g} \rightarrow E(\mathfrak{g})$ which fulfills

$$
\begin{equation*}
i_{\mathfrak{g}}([x, y])=m\left(i_{\mathfrak{g}}(x) \otimes i_{\mathfrak{g}}(y)-i_{\mathfrak{g}}(y) \otimes i_{\mathfrak{g}}(x)\right) \quad \forall x, y \in \mathfrak{g} . \tag{B.11}
\end{equation*}
$$

Clearly, the map $i_{\mathfrak{g}}$ gives rise to a Lie algebra homomorphism $\mathfrak{g} \rightarrow L(E(\mathfrak{g}))$.
Definition B.6. Let $\mathfrak{g}$ be a Lie algebra over $K$ and $(U(\mathfrak{g}), m, u)$ an enveloping algebra together with the map $i_{\mathfrak{g}}: \mathfrak{g} \rightarrow U(\mathfrak{g}) . \quad(U(\mathfrak{g}), m, u)$ is called a universal enveloping algebra, if it satisfies the following universal property:
For any algebra $A$ and any Lie algebra morphism $f: \mathfrak{g} \rightarrow L(A)$ there exists a unique (up to isomorphism) algebra morphism $\phi: U(\mathfrak{g}) \rightarrow A$ such that $\phi \circ i_{\mathfrak{g}}=f$, i.e. the following diagram commutes:


It can be shown that for any Lie algebra $\mathfrak{g}$ a universal enveloping algebra exists and is unique, see for example [54]. We will now give an explicit construction of the universal enveloping algebra. In order to do this, we consider the tensor algebra $T(\mathfrak{g})$ over the Lie algebra $\mathfrak{g}$. The underlying vector space is given by

$$
\begin{equation*}
T(\mathfrak{g}):=\bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n} \tag{B.12}
\end{equation*}
$$

Here, we used the notation $\mathfrak{g}^{\otimes 0}:=K$ and

$$
\mathfrak{g}^{\otimes n}=\underbrace{\mathfrak{g} \otimes \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{n \text { times }} .
$$

The multiplication and unit are given by $\tilde{m}(x \otimes y)=x \otimes y$ and $\tilde{u}(k)=k$, where $k$ should be viewed as an element of the subspace $K \subset T(\mathfrak{g})$. Furthermore, we define $k \otimes x:=k \cdot x$ for $k \in K \subset T(\mathfrak{g})$ and $x \in T(\mathfrak{g})$. Note also that we have a natural inclusion map $\tilde{i}_{\mathfrak{g}}: \mathfrak{g} \rightarrow T(\mathfrak{g})$, given by $\tilde{i}_{\mathfrak{g}}(x)=x$.

We now consider the two-sided ideal, which is generated by all elements of the form

$$
\begin{equation*}
x \otimes y-y \otimes x-[x, y], \quad x, y \in \mathfrak{g} . \tag{B.13}
\end{equation*}
$$

This is the minimal ideal which contains all these elements, i.e. it is the intersection of all two-sided ideals which contain these elements. By the properties (B.6) and (B.7) it is clear that this ideal must contain all linear combinations of elements of the form

$$
r \otimes(x \otimes y-y \otimes x-[x, y]) \otimes s, \quad r, s \in T(\mathfrak{g}), x, y \in \mathfrak{g}
$$

On the other hand, it is also easy to see that the set

$$
I=\left\{\sum_{i} r_{i} \otimes\left(x_{i} \otimes y_{i}-y_{i} \otimes x_{i}-\left[x_{i}, y_{i}\right]\right) \otimes s_{i} \mid r_{i}, s_{i} \in T(\mathfrak{g}), x_{i}, y_{i} \in \mathfrak{g} \forall i\right\}
$$

forms a two-sided ideal containing all elements of the form (B.13). The universal enveloping algebra $U(\mathfrak{g})$ is then the factor algebra $T(\mathfrak{g}) / I$, with the multiplication and unit transferred from the tensor algebra. Furthermore $i_{\mathfrak{g}}=\pi \circ \tilde{i}_{\mathfrak{g}}$ satisfies B.11. Here, $\pi: x \mapsto[x]$ denotes the projection onto equivalence classes. Note that elements of $K \subset T(\mathfrak{g})$ and $\mathfrak{g} \subset T(\mathfrak{g})$ are never identified with one another by the equivalence relation, such that $i_{\mathfrak{g}}$ is one-to-one. To simplify the notation, we simply write $x$ instead of $i_{\mathfrak{g}}(x)$ for an element $x$ of the Lie algebra $\mathfrak{g}$. Furthermore, we introduce the following definitions:

$$
\begin{equation*}
m(x \otimes y)=: x y, \quad \mathbb{1}=u(1) . \tag{B.14}
\end{equation*}
$$

Considering the structure of the tensor algebra, it is clear that any element of the universal enveloping algebra $U(\mathfrak{g})$ can be written as a linear combination of products of (the image under $i_{\mathfrak{g}}$ of) elements of the Lie algebra.

## B. 2 Hopf algebras

The first step towards the introduction of Hopf algebras is the definition of coalgebras. This definition can be obtained by reversing the arrows in the commutative diagrams defining an algebra.

Definition B.7. A coalgebra $(C, \Delta, \varepsilon)$ over a field $K$ consists of a vector space $C$ over $K$ as well as two linear maps, a coproduct $\Delta: C \rightarrow C \otimes C$ and a counit $\varepsilon: C \rightarrow K$, for which the following diagrams commute:


Here, $1 \otimes$. and.$\otimes 1$ denote the maps $x \mapsto 1 \otimes x$ and $x \mapsto x \otimes 1$ respectively.
Also the definition of coalgebra morphisms follows from reversing the arrows in the respective definition for algebras.

Definition B.8. Let $\left(C, \Delta_{C}, \varepsilon_{C}\right)$ and $\left(D, \Delta_{D}, \varepsilon_{D}\right)$ be coalgebras. A linear map $f: C \rightarrow D$ is called a coalgebra morphism, if the following diagrams commute:


We will now introduce bialgebras, which have both the structure of an algebra and a coalgebra. In order to define a notion of compatibility between these two structures, we need to introduce the algebra and coalgebra structures on tensor products. Consider two algebras $\left(A, m_{A}, u_{A}\right)$ and $\left(B, m_{B}, u_{B}\right)$. The definitions

$$
\begin{align*}
m_{A \otimes B}((a \otimes b) \otimes(c \otimes d)) & :=m_{A}(a \otimes c) \otimes m_{B}(c \otimes d),  \tag{B.15}\\
u_{A \otimes B}(k) & :=u_{A}(k) \otimes u_{B}(k), \tag{B.16}
\end{align*}
$$

make $A \otimes B$ into an algebra. The definition of the multiplication can be rewritten as

$$
\begin{equation*}
m_{A \otimes B}:=\left(m_{A} \otimes m_{B}\right) \circ(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \tag{B.17}
\end{equation*}
$$

where $\tau: A \otimes B \rightarrow B \otimes A$ is the flip, $\tau(a \otimes b)=b \otimes a$.
For coalgebras we can proceed similarly. Consider two coalgebras $\left(C, \Delta_{C}, \varepsilon_{C}\right)$ and ( $D, \Delta_{D}, \varepsilon_{D}$ ). We can define a coalgebra structure on $C \otimes D$ by defining

$$
\begin{align*}
\Delta_{C \otimes D}(c \otimes d): & =(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ\left(\Delta_{C} \otimes \Delta_{D}\right)(c \otimes d)  \tag{B.18}\\
\varepsilon_{C \otimes D}(c \otimes d): & =\varepsilon_{C}(c) \cdot \varepsilon_{D}(d) . \tag{B.19}
\end{align*}
$$

These definitions apply in particular in the cases when $A=B$ or $C=D$.

Definition B.9. Let $A$ be a vector space over $K,(A, m, u)$ an algebra and $(A, \Delta, \varepsilon)$ a coalgebra. The set $(A, m, u, \Delta, \varepsilon)$ is called a bialgebra if one of the following two equivalent conditions holds:

1. The linear maps $\Delta$ and $\varepsilon$ are algebra morphisms.
2. The linear maps $m$ and $u$ are coalgebra morphisms.

The equivalence of the two conditions follows easily from drawing the respective diagrams.

Definition B.10. Let $(A, m, u, \Delta, \varepsilon)$ be a bialgebra. A linear map $S: A \rightarrow A$ is called an antipode, if the following diagram commutes:


A bialgebra with an antipode is called Hopf algebra.

## Appendix C

## Curves

## C. 1 Definitions

In this appendix, we will discuss the basic definitions of the theory of curves in Minkowski space. The aim is to give a characterization of the curves we consider in this thesis and to show that these curves can always be parametrized by arc-length. The discussion is based on 55 but written in physics notation and transferred to Minkowski space.

Intuitively, a curve is a bended line between two points. It may be described by a parametrization, which describes how to run through the curve. Clearly, the choice of a parametrization is not unique for a given curve. However, we use parametrizations to describe what we mean by a curve.

Definition C.1. Let $I \subset \mathbb{R}$ be an interval. A parametrized curve is a smooth map $x: I \rightarrow \mathbb{R}^{(1,3)}$. It is called regular, if the velocity vanishes nowhere along the curve,

$$
\dot{x}(\tau) \neq 0 \quad \forall \tau \in I .
$$

The condition of regularity ensures that the smoothness of the parametrization carries over to the curve. Consider the parametrized curve

$$
x(\tau)=\left(0, \tau^{3}, \tau^{2}, 0\right)
$$



Figure C.1: A parametrized curve which is not regular.

This is clearly a smooth map, however the parametrization is not regular as $\dot{x}(0)=0$. Figure C.1 shows that the respective curve is not smooth although the parametrization is.

As mentioned above the same curve can be described by different parametrizations. The change of parametrization is described by the following definition.

Definition C.2. Let $x: I \rightarrow \mathbb{R}^{(1,3)}$ be a parametrized curve. A parameter transformation of $x$ is a bijective map $\varphi: J \rightarrow I, \dot{\varphi} \neq 0$, where $J \subset \mathbb{R}$ is also an interval. The parametrized curve $\tilde{x}=x \circ \varphi: J \rightarrow \mathbb{R}^{(1,3)}$ is called a reparametrization of $x$. The parameter transformation is called orientation-preserving if $\dot{\varphi}(t)>0 \forall t \in J$ or orientation-reversing if $\dot{\varphi}(t)<0 \forall t \in J$.

Note that, since $\dot{\varphi} \neq 0$, all parameter transformations are either orientation-preserving or orientation-reversing and a reparametrization of a regular, parametrized curve is also regular.

Definition C.3. An oriented curve is an equivalence class of parametrized curves, where two parametrized curves are considered equivalent, if they emerge from each other by an orientation-preserving parameter transformation.

Among all equivalent ways of describing a curve, there is one parametrization that is especially helpful in doing computations:

Definition C.4. A parametrization of a curve is called parametrization by arc-length if $\|\dot{x}(t)\| \equiv 1$.

The following lemma explains when such a parametrization exists:
Lemma C.1. To any regular parametrization $x: I \rightarrow \mathbb{R}^{(1,3)}$ of a curve which satisfies $\|\dot{x}(t)\| \neq 0 \forall t \in I$ there exists a parameter transformation $\varphi: J \rightarrow I$ such that the reparametrization $x \circ \varphi$ is a parametrization by arc-length.

Proof. Let $x: I \rightarrow \mathbb{R}^{(1,3)}$ be as above. Take $t_{0} \in I$ and consider the function

$$
\psi(s):=\int_{t_{0}}^{s}\|\dot{x}(t)\| \mathrm{d} t
$$

Then, $\dot{\psi}(s)=\|\dot{x}(s)\|>0$, and $\psi: I \rightarrow J:=\psi(I)$ is an orientation-preserving parameter transformation. Therefore its inverse, $\varphi:=\psi^{(-1)}: J \rightarrow I$, is also an orientationpreserving parameter transformation and it satisfies

$$
\dot{\varphi}(t)=\frac{1}{\|\dot{x}(\varphi(t))\|} \quad \Rightarrow\left\|\frac{\mathrm{d}}{\mathrm{~d} t}(x \circ \varphi)(t)\right\| \equiv 1 .
$$

One reason why it is helpful to use a parametrization by arc-length in concrete calculations is, that it provides identities for higher parameter derivatives of the parametrization:

$$
\begin{equation*}
\dot{x}^{2} \equiv-1 \Rightarrow \dot{x} \ddot{x} \equiv 0 \Rightarrow \ddot{x}^{2}+\dot{x} x^{(3)} \equiv 0 \Rightarrow 3 \ddot{x} x^{(3)}+\dot{x} x^{(4)} \equiv 0 . \tag{C.1}
\end{equation*}
$$

The condition that $\|\dot{x}(t)\| \neq 0 \forall t \in I$ arises because of the Minkowski signature. In Euclidean space, this condition is always satisfied for a regular parametrization. In Minkowski space, a curve can only be parametrized by arc-length if it is both regular and if its tangent vector is either everywhere time-like or everywhere space-like. In this work, we consider closed curves in Minkowski space. This restricts us to the case of everywhere space-like tangent vectors as the following argument shows: Any light-like tangent vector has a specific time direction. For a closed curve, there must be tangent vectors with positive and tangent vectors with negative time-direction. Hence at some point along the loop the tangent vector must cross from the forward light-cone to the backward light-cone. This is not possible in a continuous way if we demand that $\|\dot{x}(t)\| \neq 0 \forall t \in I$.

In this thesis, we consider closed curves, for which the starting point and the end-point of the parametrization are equal. However, we also need the curves to be smooth in the point where they close. This is essential in the discussion of boundary terms and it seems natural to require it as otherwise the condition of smoothness would depend on the choice of the starting point. The details are captured by the following definition.

Definition C.5. A parametrized curve $x: \mathbb{R} \rightarrow \mathbb{R}^{(1,3)}$ is called periodic with period $l$, if $x(t+l)=x(t) \forall t \in I$. A curve is called closed if it admits a regular, periodic parametrization. It is called simply closed if it admits a regular periodic parametrization $x: \mathbb{R} \rightarrow \mathbb{R}^{(1,3)}$ such that $\left.x\right|_{[0, l)}$ is injective.

The condition that a curve is simply closed encodes that the curve is not selfintersecting. This is important for us in discussing the finiteness of the vacuum expectation value of the Maldacena-Wilson loop operator. There, we also need that any two points $x_{1}$ and $x_{2}$ of the curve are space-like. For a curve with space-like tangent vectors this is automatically satisfied locally, but it need not be satisfied globally as the following example shows: The curve parametrized by

$$
x(\tau):=\left(\cos \frac{\tau}{2},\left(2+\sin \frac{\tau}{2}\right) \cos \tau,\left(2+\sin \frac{\tau}{2}\right) \sin \tau, 0\right)
$$

has everywhere space-like tangent vectors but the points $x(0)$ and $x(2 \pi)$ are time-like separated.

In this thesis, we consider curves described by the following definition:
Definition C.6. We call a parametrized curve space-like if it is regular with everywhere space-like tangent vectors and if the Minkowski distance between any two distinct points is space-like.

Note that any space-like closed curve is also simply closed. For these curves, lemma C. 1 shows that there exists a parametrization by arc-length.

## C. 2 Proof of Reparametrization Invariance

We show that

$$
\int \mathrm{d} \tau \dot{x}^{\mu}(\tau) \frac{\dot{x}(\tau)^{2} \ddot{x}(\tau)^{2}-(\dot{x}(\tau) \cdot \ddot{x}(\tau))^{2}}{\dot{x}(\tau)^{6}}
$$

is a reparametrization invariant curve integral. Consider a reparametrization

$$
\tilde{\tau}:[\tilde{a}, \tilde{b}] \rightarrow[a, b]
$$

and denote by $x:[a, b] \rightarrow \mathbb{R}^{d}$ the old parametrization and by $\tilde{x}:=x \circ \tilde{\tau}:[\tilde{a}, \tilde{b}] \rightarrow \mathbb{R}^{d}$ the new parametrization of $C$. We have

$$
\begin{equation*}
\partial_{s} x^{\mu}(\tilde{\tau}(s))=\dot{x}^{\mu}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s) \Rightarrow \partial_{s}^{2} x^{\mu}(\tilde{\tau}(s))=\ddot{x}^{\mu}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{2}+\dot{x}^{\mu}(\tilde{\tau}(s)) \tilde{\tau}^{\prime \prime}(s) . \tag{C.2}
\end{equation*}
$$

By substitution we have:

$$
\begin{aligned}
& \int_{a}^{b} \mathrm{~d} \tau \dot{x}^{\mu}(\tau) \frac{\dot{x}^{2}(\tau) \ddot{x}^{2}(\tau)-(\dot{x}(\tau) \cdot \ddot{x}(\tau))^{2}}{|\dot{x}(\tau)|^{6}}= \\
= & \int_{\tilde{a}}^{\tilde{b}} \frac{\mathrm{~d} s}{\tilde{\tau}^{\prime}(s)} \dot{x}^{\mu}(\tilde{\tau}(s)) \frac{\dot{x}^{2}(\tilde{\tau}(s)) \ddot{x}^{2}(\tilde{\tau}(s))-(\dot{x}(\tilde{\tau}(s)) \cdot \ddot{x}(\tilde{\tau}(s)))^{2}}{|\dot{x}(\tilde{\tau}(s))|^{6}} .
\end{aligned}
$$

On the other hand, we can use (C.2) to rewrite

$$
\begin{aligned}
& \dot{\dot{x}}^{2}(s) \ddot{\tilde{x}}^{2}(s)-(\dot{\tilde{x}}(s) \cdot \ddot{\ddot{x}}(s))^{2}= \\
= & \dot{x}^{2}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{2}\left(\ddot{x}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{2}+\dot{x}(\tilde{\tau}(s)) \tilde{\tau}^{\prime \prime}(s)\right)^{2} \\
& -\left[\dot{x}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s) \cdot\left(\ddot{x}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{2}+\dot{x}(\tilde{\tau}(s)) \tilde{\tau}^{\prime \prime}(s)\right)\right] \\
= & \dot{x}^{2}(\tilde{\tau}(s)) \ddot{x}^{2}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{6}+2 \dot{x}^{2}(\tilde{\tau}(s)) \dot{x}(\tilde{\tau}(s)) \cdot \ddot{x}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{4} \tilde{\tau}^{\prime \prime}(s)+\dot{x}^{4}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{2} \tilde{\tau}^{\prime \prime}(s)^{2} \\
& -(\dot{x}(\tilde{\tau}(s)) \ddot{x}(\tilde{\tau}(s)))^{2} \tilde{\tau}^{\prime}(s)^{6}-2 \dot{x}^{2}(\tilde{\tau}(s)) \dot{x}(\tilde{\tau}(s)) \cdot \ddot{x}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{4} \tilde{\tau}^{\prime \prime}(s)-\dot{x}^{4}(\tilde{\tau}(s)) \tilde{\tau}^{\prime}(s)^{2} \tilde{\tau}^{\prime \prime}(s)^{2} \\
= & \left(\dot{x}^{2}(\tilde{\tau}(s)) \ddot{x}^{2}(\tilde{\tau}(s))-(\dot{x}(\tilde{\tau}(s)) \ddot{x}(\tilde{\tau}(s)))^{2}\right) \tilde{\tau}^{\prime}(s)^{6} .
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \int_{\tilde{a}}^{\tilde{b}} \mathrm{~d} s \dot{\tilde{x}}^{\mu}(s) \frac{\dot{\tilde{x}}^{2}(s) \ddot{\tilde{x}}^{2}(s)-(\dot{\tilde{x}}(s) \cdot \ddot{\tilde{x}}(s))^{2}}{|\dot{\tilde{x}}(s)|^{6}}= \\
& =\int_{\tilde{a}}^{\tilde{b}} \frac{\mathrm{~d} s}{\tilde{\tau}^{\prime}(s)} \dot{x}^{\mu}(\tilde{\tau}(s)) \frac{\left(\dot{x}^{2}(\tilde{\tau}(s)) \ddot{x}^{2}(\tilde{\tau}(s))-(\dot{x}(\tilde{\tau}(s)) \cdot \ddot{x}(\tilde{\tau}(s)))^{2}\right) \tilde{\tau}^{\prime}(s)^{6}}{|\dot{x}(\tilde{\tau}(s))|^{6} \tilde{\tau}^{\prime}(s)^{6}}
\end{aligned}
$$

proving reparametrization invariance.

## Appendix D

## Proof that the Maldacena-Wilson Loop is Locally $1 / 2$ BPS

In this appendix, we will prove that it is always possible to find 8 linearly independent left-handed Weyl-spinors $\varepsilon$, such that the action of the supersymmetry variation $\delta_{\varepsilon}$ on $W(C)$ vanishes locally. We have already argued in chapter 5.2, that for this to hold true we need to find 8 linearly independent left-handed Weyl spinors which satisfy the condition

$$
\begin{equation*}
A(\tau) \varepsilon=0, \quad A(\tau):=\Gamma_{\mu} \dot{x}^{\mu}(\tau)+\Gamma_{I+3}|\dot{x}(\tau)| n^{I} . \tag{D.1}
\end{equation*}
$$

Whether or not we can also enforce the Majorana condition, depends on the sign of $\dot{x}^{2}$. There exist non-trivial solutions to the condition (D.1) as $A(\tau)$ squares to zero. From this we infer that

$$
\operatorname{rank}\left(\Gamma_{\mu} \dot{x}^{\mu}(\tau)+\Gamma_{I+3}|\dot{x}(\tau)| n^{I}\right) \leq 16 .
$$

To see this, consider a basis $\left\{v_{i}\right\}$ of $\mathbb{C}^{32}$ in which the first $n$ basis vectors form a basis of the kernel of $A(\tau)$. As $A$ squares to zero, we know that

$$
A v_{i} \in \operatorname{ker}(A(\tau)) \forall i>n .
$$

Furthermore, the set of $\left\{A v_{i}, i>n\right\}$ is linearly independent, otherwise there would be a linear combination of the $\left\{v_{i}, i>n\right\}$, which would be mapped to zero, thus leading to a contradiction to our assumption on the form of the basis $\left\{v_{i}\right\}$. Hence we conclude that the kernel of A must at least have dimension 16. Thus there are at least sixteen linearly independent vectors in $\mathbb{C}^{32}$, which are mapped to zero by $A$. To discuss the compatibility of (D.1) with the Majorana-Weyl condition, it is much nicer to consider the matrix

$$
B(\tau):=\Gamma_{0} A(\tau)
$$

As $\Gamma_{0}$ is invertible, the condition $B(\tau) \varepsilon(\tau)=0$ is clearly equivalent to (5.10). The matrix $B(\tau)$ is hermitian and commutes with $\Gamma^{11}$, such that $\left\{B(\tau), P_{L}, P_{R}\right\}$ forms a set
of hermitian operators, which commute with each other. These operators can hence be diagonalized simultaneously. Furthermore, although $B(\tau)$ does not square to zero, we still have $\operatorname{rank}(B(\tau)) \leq 16$. The projection operators $P_{L}$ and $P_{R}$ have the special property that their eigenspaces to the eigenvalue 1 are disjoint and span the whole space,

$$
\mathbb{C}^{32}=\operatorname{Eig}\left(P_{L}, 1\right) \oplus \operatorname{Eig}\left(P_{R}, 1\right)
$$

As $B(\tau)$ commutes with $P_{L}$ and $P_{R}$, we may consider the subspaces

$$
V_{1}:=\left\{v \in P_{L}\left(\mathbb{C}^{32}\right) \mid B(\tau) v=0\right\}, \quad V_{2}:=\left\{v \in P_{R}\left(\mathbb{C}^{32}\right) \mid B(\tau) v=0\right\} .
$$

It is not clear on general grounds how the (at least) sixteen dimensions of $\operatorname{Eig}(B(\tau), 0)$ are distributed among these two. However, considering the matrix $B(\tau)$ explicitly in the notation of chapter 2.33 ,

$$
B=\left(\begin{array}{cccc}
\dot{x}_{\mu} \delta_{B}^{A} \bar{\sigma}_{\alpha \dot{\gamma}}^{0} \sigma^{\mu \dot{\gamma} \beta} & 0 & 0 & -|\dot{x}| n_{I} \Sigma^{I A B} \bar{\sigma}_{\alpha \dot{\beta}}^{0} \\
0 & \dot{x}_{\mu} \delta_{B}^{A} \sigma^{0} \dot{\alpha} \gamma & \bar{\sigma}^{\mu} & |\dot{x}| \Sigma^{I A B} \\
\sigma^{\dot{\alpha} \beta} & 0 \\
0 & -|\dot{x}| n_{I} \bar{\Sigma}^{I A B} \bar{\sigma}_{\alpha \dot{\beta}}^{0} & \dot{x}_{\mu} \delta_{A}^{B} \bar{\sigma}_{\alpha \dot{\gamma}}^{0} \sigma^{\mu \dot{\gamma} \beta} & 0 \\
|\dot{x}| n_{I} \bar{\Sigma}^{I A B} \sigma^{0 \dot{\alpha} \beta} & 0 & 0 & \dot{x}_{\mu} \delta_{A}^{B} \sigma^{0 \dot{\alpha} \gamma} \bar{\sigma}_{\gamma \dot{\beta}}^{\mu}
\end{array}\right)
$$

we find that the restrictions of $B(\tau)$ to the spaces $\operatorname{Eig}\left(P_{L}, 1\right)$ and $\operatorname{Eig}\left(P_{R}, 1\right)$ are similar. Let us be more precise. Consider the restriction of $B(\tau)$ to the space $\operatorname{Eig}\left(P_{R}, 1\right)$ and the transformation

$$
S=\left(\begin{array}{cc}
\delta_{B}^{A} \epsilon_{\alpha \beta} & 0 \\
0 & -\delta_{B}^{A} \epsilon_{\alpha \beta}
\end{array}\right)
$$

We then have, using $\bar{\sigma}^{\mu}=-\epsilon \sigma^{\mu} \epsilon$ and $\epsilon^{2}=-\mathbb{1}$,

$$
\begin{aligned}
\left.S B\right|_{\operatorname{Eig}\left(P_{R}, 1\right)} S^{-1} & =\left(\begin{array}{cc}
\delta_{B}^{A} \epsilon & 0 \\
0 & -\delta_{B}^{A} \epsilon
\end{array}\right)\left(\begin{array}{cc}
\dot{x}_{\mu} \delta_{B}^{A} \bar{\sigma}^{\mu} & |\dot{x}| \Sigma^{I A B} \\
-|\dot{x}| n_{I} \bar{\Sigma}^{I A B} & \dot{x}_{\mu} \delta_{A}^{B} \sigma^{\mu}
\end{array}\right)\left(\begin{array}{cc}
-\delta_{B}^{A} \epsilon & 0 \\
0 & \delta_{B}^{A} \epsilon
\end{array}\right) \\
& =\left.\left(\begin{array}{cc}
\dot{x}_{\mu} \delta_{B}^{A} \sigma^{\mu} & -|\dot{x}| \Sigma^{I A B} \\
|\dot{x}| n_{I} \bar{\Sigma}^{I A B} & \dot{x}_{\mu} \delta_{A}^{B} \bar{\sigma}^{\mu}
\end{array}\right) \hat{=} B\right|_{\operatorname{Eig}\left(P_{L}, 1\right)} .
\end{aligned}
$$

The last equality is of course only meant to hold upon a proper identification of the two subspaces on which the respective matrices act. Note also that the spinor indices are handled somewhat sloppily in the above calculation. Recall that the different indices label, in which representation of $S L(2, \mathbb{C})$ and $S U(4)$ the respective vector transforms. This is however not our concern here, as we are merely discussing an eigenvalue problem in a fixed frame. We conclude that the dimensions of the kernels of the restrictions of $B(\tau)$ to the respective subspaces are equal and thence $V_{1}$ has complex dimension $\geq 8$.

We will now show, that $V_{1}$ contains an 8-dimensional real subspace of left-handed Majorana-Weyl spinors, if $\dot{x}$ is time-like or light-like. To show this, it will be crucial,
that $B(\tau)$ is compatible with the Majorana condition in the following sens $\mathbb{F}^{\text {r }}$

$$
\begin{equation*}
C B^{*}=C \Gamma_{0}^{*}\left(\Gamma_{\mu} \dot{x}^{\mu}(\tau)+\Gamma_{I+3}|\dot{x}(\tau)| n^{I}\right)^{*}=\Gamma_{0}\left(\Gamma_{\mu} \dot{x}^{\mu}(\tau)+\Gamma_{I+3}|\dot{x}(\tau)| n^{I}\right) C=B C . \tag{D.2}
\end{equation*}
$$

Here, we used that $|\dot{x}|$ is real which only holds if $\dot{x}$ is time-like or light-like. We thus have the following implications:

- If $\Psi \in \mathbb{C}^{32}$ satisfies the Majorana condition, then so does $B(\tau) \Psi$ :

$$
C(B \Psi)^{*}=C B^{*} \Psi^{*}=B C \Psi^{*}=B \Psi .
$$

- If $\Psi \in \mathbb{C}^{32}$ is mapped to zero by $B(\tau)$, then so is $\Psi^{(c)}$ :

$$
B \Psi^{(c)}=B C \Psi^{*}=C B^{*} \Psi^{*}=C(B \Psi)^{*}=0 .
$$

In the following, we will use the inner product $(\Psi, \Phi):=\Psi^{\dagger} \Phi$. While this does not give rise to a Lorentz-scalar, it still gives a decent inner product on $\mathbb{C}^{32}$.

We now assume that the maximal number of linearly independent left-handed Majorana-Weyl spinors which are mapped to zero by $B(\tau)$ was $n \leq 7$. Without loss of generality, we choose one of these maximal sets, $\left\{\Psi_{1}, \ldots, \Psi_{n}\right\}$, which satisfies $\left(\Psi_{i}, \Psi_{j}\right)=\delta_{i j}$. As the space of left-handed Weyl-spinors satisfying $B \Psi=0$ has complex dimension eight, we conclude that there must be Weyl-spinors in $V_{1}$, which cannot be written as a complex linear combination of the $\Psi_{i}$. Hence there exists a $\Psi$, which satisfies

$$
\Psi \in V_{1}, \quad\left(\Psi, \Psi_{i}\right)=0 \quad \forall i=1, \ldots, n .
$$

It follows that also $\Psi^{(c)}$ satisfies these conditions:

$$
0=\left(\Psi, \Psi_{i}\right)=\left(\Psi, C \Psi_{i}^{*}\right)=\left(C^{\dagger} \Psi, \Psi_{i}^{*}\right) \Rightarrow 0=\left(C^{\dagger} \Psi, \Psi_{i}^{*}\right)^{*}=\left(C^{T} \Psi^{*}, \Psi_{i}\right)=\left(\Psi^{(c)}, \Psi_{i}\right) .
$$

Here, we used that the charge conjugation matrix is hermitian and has only real entries, which can easily be checked from the explicit form given in (2.34). We can then conclude that $\Psi+\Psi^{(c)}$ satisfies the following properties:

$$
\left(\Psi+\Psi^{(c)}\right)^{(c)}=\Psi+\Psi^{(c)}, \quad B(\tau)\left(\Psi+\Psi^{(c)}\right)=0 \quad\left(\Psi+\Psi^{(c)}, \Psi_{i}\right)=0
$$

As the $\left\{\Psi_{i}\right\}$ were maximal, $\Psi+\Psi^{(c)}$ must be a linear combination of them. However, as it is orthogonal to all of them, we must have $\Psi+\Psi^{(c)}=0$. Then, consider $\varepsilon:=i \Psi$. By construction, this has to satisfy

$$
\varepsilon^{(c)}=\varepsilon, \quad B(\tau) \varepsilon=0, \quad \varepsilon \neq 0, \quad\left(\varepsilon, \Psi_{i}\right)=0 .
$$

This is a contradiction to the assumption that $\left\{\Psi_{i}\right\}$ is maximal. Thus we conclude that there are at least eight linearly independent left-handed Majorana-Weyl spinors which are annihilated by $B(\tau)$.

[^18]Note that in proving the compatibility with the Majorana condition, it was crucial to assume that $\dot{x}$ is not space-like. Indeed, the study of an example vector shows that it is impossible to find a Majorana spinor satisfying $B(\tau) \varepsilon=0$, if $\dot{x}$ is space-like. By performing Lorentz transformations in the ten-dimensional Minkowski space, we are able to transfer the solution for the example vector to any vector of this type. The spinors which solve (D.1) are then rotated accordingly in the spinor space. As the Majorana condition is Lorentz invariant, we conclude that it is also impossible to choose a Majorana spinor among the solutions of (D.1) for other space-like vectors $\dot{x}$.

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## Hilfsmittel

Diese Arbeit wurde in $\mathrm{AT}_{\mathrm{E}} \mathrm{X}$ unter Verwendung von MiKTEX 2.9 gesetzt. Alle Grafiken wurden mithilfe des Open-Source Vektorgrafikedtitors Inkscape erstellt. Numerische Berechnungen wurden in MATLAB durchgeführt. In der Erstellung der Bibliographie wurde BiBTeX mit der von Niklas Beisert verwalteten Stil-Datei "nb.bst" verwendet.

## Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Ort, Datum
Hagen Münkler


[^0]:    ${ }^{1}$ The duality relates scattering amplitudes to light-like loops, for which the Wilson and MaldacenaWilson loop agree.
    ${ }^{2}$ This idea was jointly developed by J. Plefka and N. Drukker.

[^1]:    ${ }^{1}$ Our discussion follows 16 .

[^2]:    ${ }^{2}$ The Weyl and Majorana conditions are known to be compatible only in $d$-dimensional Minkowski space-times for which $d \equiv 2(\bmod 8)$.

[^3]:    ${ }^{1}$ The conventional factor of $2 i$ is chosen to match the commutator of the supersymmetry field transformations, which will be introduced in chapter 4 .

[^4]:    ${ }^{2}$ See appendix A for our conventions on index raising and lowering.

[^5]:    ${ }^{3}$ We follow 28 and set $\alpha=1$. Consider also 31 for an explanation of the structure of these generators.

[^6]:    ${ }^{1}$ In the conventions chosen here, the fields have classical (mass) dimensions $[A]=1$ and $[\Psi]=3 / 2$, while the Yang-Mills coupling constant $g_{10}$ has dimension $\left[g_{10}\right]=-3$.

[^7]:    ${ }^{2}$ See for example 34 for an explanation of this method.

[^8]:    ${ }^{1}$ To avoid confusion, we reserve $W(C)$ for the Maldacena-Wilson loop which we will introduce later.
    ${ }^{2}$ The path-ordering is only necessary if the gauge group is non-Abelian.

[^9]:    ${ }^{3}$ The conformal boundary is the boundary (in the topological sense) of the image of the respective space in its conformal compactification. It should not be mixed up with the usual notion of boundaries of manifolds.

[^10]:    ${ }^{4}$ In the prefactor, we have replaced $\left(N^{2}-1\right) / N$ by $N$ in view of the large $N$ limit. This would be the right result, if we were considering a $\mathfrak{u}(N)$ Lie algebra.

[^11]:    ${ }^{5}$ An easy way to see this is by considering for example the length of curve, $L=\int \mathrm{d} \tau|\dot{x}|$. This scales under dilatations and applying the dilatation operator, one finds $D(L)=i L$. In a parametrization by arc-length, $L=\int_{0}^{L} 1 \mathrm{~d} \tau$ and the calculation of $D(L)$ fails.

[^12]:    ${ }^{6}$ For computational simplicity, the scalar and gauge field contributions are discussed separately here. It should however be pointed out, that only the sum of them is finite. The individual contributions would need to be renormalised which would spoil the conformal invariance.

[^13]:    ${ }^{1}$ In appendix C, we discuss for which curves this is possible. We will restrict ourselves to spacelike curves, for which this can be achieved.

[^14]:    ${ }^{2}$ To reach the third line, we perform an integration by parts without discussing the boundary term, which is given by

    $$
    \begin{aligned}
    B & =-\int_{0}^{L} \mathrm{~d} \tau_{1} \mathrm{~d} \tau_{2} F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right)\left(x_{\nu}(L) \delta\left(\tau_{1}-L-\varepsilon\right) \delta\left(\tau_{1}-L\right)-x_{\nu}(0) \delta\left(\tau_{1}-\varepsilon\right) \delta\left(\tau_{1}\right)\right) \\
    & =-\int_{0}^{L} \mathrm{~d} \tau_{2}\left(F_{\rho \sigma}^{(1)}\left(L, \tau_{2}\right) x_{\nu}(L)-F_{\rho \sigma}^{(1)}\left(0, \tau_{2}\right) x_{\nu}(0)\right) \delta(-\varepsilon)
    \end{aligned}
    $$

    This vanishes because of the periodicity of the parametrization, which carries over to $F_{\rho \sigma}^{(1)}\left(\tau_{1}, \tau_{2}\right)$. The boundary terms appearing in the calculation of the other $C_{i}^{\mu}$ can be discussed similarly.

[^15]:    ${ }^{1}$ The conventions in this thesis differ from 13 by a conventional factor of $(-1)$ in front of all level-1 generators.

[^16]:    ${ }^{1}$ Throughout this thesis, the symmetrization or antisymmetrization of a pair of indices is defined by $F_{[I J]}:=\frac{1}{2}\left(F_{I J}-F_{J I}\right)$ or $F_{(I J)}:=\frac{1}{2}\left(F_{I J}+F_{J I}\right)$.

[^17]:    ${ }^{1}$ We are not interested in discussing general fields here, so the field $K$ is meant to be either $\mathbb{R}$ or $\mathbb{C}$.

[^18]:    ${ }^{1}$ To shorten our notation, we denote the charge conjugation matrix $C_{1,9}$ introduced in chapter 2.3 simply by $C$.

