# The evaluation of loop integrals via differential equations

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## Abstract

Multi-loop Feynman integrals occurring in $\mathcal{N} = 4$ super Yang-Mills theory are being calculated, namely planar and non-planar two-loop three-point functions with massive propagators. By means of integration-by-parts identities the corresponding topologies of scalar integrals are reduced to master integrals. The evaluation of the master integrals is done, using differential equations in the masses. The results of the planar integrals are presented as Laurent series in $\epsilon$ in terms of harmonic polylogarithms, whereas the non-planar functions are found to contain elliptic integrals. In addition we discuss the reduction of massive two-loop four-point functions to master integrals.

# Acknowledgements

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# Contents

# Chapter 1

# Introduction

In the era of the LHC experiments of increasing accuracy become possible. Hence it is necessary to achieve more accurate results for measurable quantities at the theoretical level. According to perturbation theory, higher order corrections to amplitudes have to be considered. To evaluate such corrections in quantum field theory, it is necessary to compute multi loop Feynman diagrams. While for instance in quantum-chromodynamics higher order computations are available for various processes, in other cases it is more difficult because of the presence of large, non-negligible masses of particles like the Higgs boson or the top quark. The evaluation of such tensor integrals, coming directly from the application of Feynman rules, can be reduced to that of scalar integrals.

The scalar integrals are closely related to the original Feynman diagram. The denominator of their integrand is formed by the denominators of propagators present in the diagram and their numerator contains scalar products of external momenta and loop variables. In general the tensor decomposition leads to a large number of linear dependent scalar integrals. The first step to reduce this number, is to classify them into smaller sets of independent scalar integrals, also called topologies. The next step uses the integration-by-parts identities [1]. These identities allow to reduce integrals of a single topology to a much smaller subset of master integrals. One of the strategies, how to use the identities, is the Laporta method [2][3]. Several tools have been developed, which base on this method [4][5][6].

The method of differential equations is one of the techniques for the computation of the remaining master integrals. This method is based on the reduction with the integration-by-parts identities. The use of differential equations in the masses was first proposed by Kotikov [7]. Remiddi extended it to more general differential equations in the Mandelstam variables [8]. Gehrmann and Remiddi fully developed the method by showing its effectiveness through the application to a non-trivial class of functions [9]. More calculations with this method can be found in [10][11][12][13]. There are also reviews on a pedagogical level in [14][15].

Instead of performing loop integrations, differential equations have to be solved. Within the dimensional regularization one uses the Laurent-expansion in $\epsilon$ to obtain simplified coupled differential equations. They are solved using a proper basis of special functions: the harmonic polylogarithms [16]. Of course boundary conditions are needed to get the complete expressions of the master integrals. The general strategy is to take a look on special kinematical points. For example the integrals can be analysed in the limits of small or large masses.

In this thesis we use the method of differential equations to calculate previously unknown planar and non-planar two-loop three-point functions with massive propagators. In order to obtain boundary conditions we use the Mellin-Barnes representation (see for example [17]) of the integrals to expand them in the limit of small external momenta. Two of the external legs are on-shell and the internal masses are uniform. So they refer for example to the production of a Higgs boson via the fusion of two massless fermions with a top quark in the loop. It is also of interest to compare our results with the corresponding massless amplitudes to analyse the mass corrections.

These integrals are of particular interest for $\mathcal{N} = 4$ super Yang-Mills theory. They contribute to a form factor within massive regularization [18]. One of the integrals calculated in this paper is relevant for the computation of a massive four-point topology [19]. We discuss the reduction of these more complicated functions. We propose a suitable choice for the master integrals. They are also part of a massive form factor in $\mathcal{N} = 4$ super Yang-Mills theory.

The thesis is organised as follows. In chapter 2 we explain the general form of the scalar integrals, which result from the tensor decomposition. We describe the so called auxiliary diagram scheme, which is used to classify these integrals into topologies. Chapter 3 is about the IBP identities, how to derive them and how to use them to reduce the number of independent scalar integrals. We also explain the Laporta method, as one of the strategies for this reduction. Chapter 4 discusses the method of differential equation. We describe how to generate these equations and how to solve them. For the latter the basic analysis is explained as well as specific procedures, such as the expansion in $\epsilon$ and the definition of the harmonic polylogarithms. We give also a short discussion on boundary conditions in this chapter. In chapter 5 we discuss the calculation of the above mentioned integrals. The results are summarized in the second section 5.2. The reduction of the massive four-point functions to master integrals is the topic of Chapter 6. The evaluation of the boundary conditions uses the Mellin-Barnes representation and is summarized in the appendix A. In appendix B we give an example for the in- and output of the tools used for the calculations in this thesis.

# Chapter 2

# Scalar Feynman integrals

We start with the description of the scalar integrals, resulting from the tensor decomposition. One way how to classify them into topologies, is explained by means of an example.

A scalar integral of a corresponding diagram with $l$ loops, $g$ independent external momenta and $f$ internal lines has the form

$$\int \frac{d^d k_1}{i\pi^{d/2}} \cdots \int \frac{d^d k_l}{i\pi^{d/2}} \frac{\prod_{j=1}^{b} S_j^{n_j}}{\prod_{i=1}^{f} D_i}, \tag{2.1}$$

where

- $S_j \quad (j = 1, 2, ..., b)$ are the possible scalar products of either one independent external momentum and one internal loop momentum, or of two loop momenta. There are

  $$b = l \cdot g + \frac{l(l+1)}{2} \tag{2.2}$$

  different scalar products. They appear with arbitrary powers $n_j$ ( $n_j \geq 0$).

- $D_i = q_i^2 + m_i^2 \quad (i = 1, 2, ..., f)$ is the denominator of the propagator with momentum $q_i$ and mass $m_i$. From now, we call $D_i$ propagator.

To minimize the number of integrals to be computed, they can be classified into sets of independent integrals, also called topologies. One way to do this, is to express every integral in terms of propagators. Since there are $b$ independent scalar products, we need the same number of different propagators for this representation. The $f$ propagators of (2.1) appear with arbitrary powers $a_i \quad (i = 1, 2, ..., f)$, also called indices, while the additional $(b - f)$ propagators are restricted to non-positive indices $a_i \leq 0$ $(i = f + 1, ..., b)$:

$$\int \frac{d^d k_1}{i\pi^{d/2}} \cdots \int \frac{d^d k_l}{i\pi^{d/2}} \frac{1}{\prod_{i=1}^{b} D_i^{a_i}} \tag{2.3}$$

Integrals of a given topology are conveniently written in this short form:

$$I\{a_1, a_2, ..., a_b\} \tag{2.4}$$

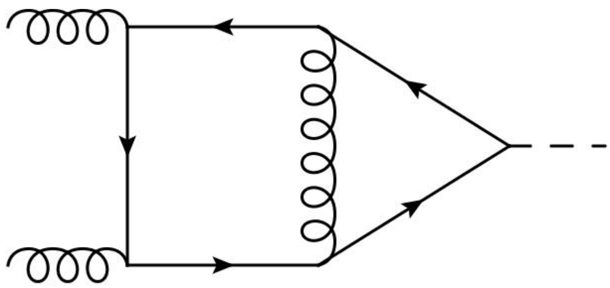


Figure 2.1: Relevant Feynman diagram for Higgs production via gluon fusion with a quark in the loop.

Let us consider for instance the two loop correction to Higgs production via gluon fusion, i.e. the process

$$g + g \to H. \tag{2.5}$$

A contributing diagram would be that in fig. 2.1 with a quark running in the loop. Tensor decomposition leads to scalar integrals of this form

$$\int \frac{d^d k_1}{i\pi^{d/2}} \int \frac{d^d k_2}{i\pi^{d/2}} \frac{(k_1^2)^{n_1}(k_2^2)^{n_2}(k_1\cdot k_2)^{n_3}(k_1\cdot p_1)^{n_4}(k_1\cdot p_2)^{n_5}(k_2\cdot p_1)^{n_6}(k_2\cdot p_2)^{n_7}}{D_1D_2D_3D_4D_5D_6}, \tag{2.6}$$

where

- $k_1$ and $k_2$ are the loop momenta,
- $p_1$ and $p_2$ are the external momenta, with the conditions $p_1^2 = p_2^2 = 0$ and $2p_1 \cdot p_2 = s$,
- $D_i \quad (i = 1, 2, ..., 6)$ are the denominators of the propagators of the corresponding internal lines present in fig. 2.1.

The numerator contains the scalar products with one or two loop momenta. So one additional propagator is necessary to write the scalar integrals in the form

$$\int \frac{d^d k_1}{i\pi^{d/2}} \int \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{D_1^{a_1}D_2^{a_2}D_3^{a_3}D_4^{a_4}D_5^{a_5}D_6^{a_6}D_7^{a_7}} = I\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}, \tag{2.7}$$

with arbitrary integers $a_i \quad (i = 1, 2, ..., 6)$ and $a_7 \leq 0$. A possible choice for the propagators in (2.7) is

$$\begin{aligned}
D_1 &= k_1^2 + m^2 \\
D_2 &= (k_1 + p_1 + p_2)^2 + m^2 \\
D_3 &= k_2^2 + m^2 \\
D_4 &= (k_2 + p_1)^2 + m^2 \\
D_5 &= (k_2 + p_1 + p_2)^2 + m^2 \\
D_6 &= (k_1 - k_2)^2 \\
D_7 &= (k_1 + p_1)^2 + m^2.
\end{aligned} \tag{2.8}$$

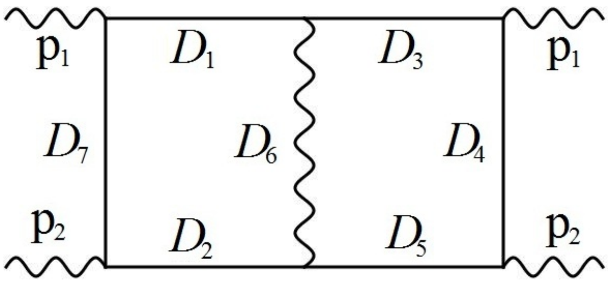



Figure 2.2: Diagram for the topology (2.7). Wavy lines denote massless particles both external and internal. Internal massive lines are denoted by straight lines. This notation is the same for the rest of this thesis.

Indeed every scalar product in the numerator of (2.6) can be expressed in terms of these propagators and kinematical invariants:

$$\begin{aligned} k_1^2 &= D_1 - m^2 \\ k_2^2 &= D_3 - m^2 \\ k_1 \cdot k_2 &= \frac{1}{2}\left(D_1 + D_3 - D_6\right) - m^2 \\ k_1 \cdot p_1 &= \frac{1}{2}\left(D_7 - D_1\right) \\ k_1 \cdot p_2 &= \frac{1}{2}\left(D_2 - D_7 - s\right) \\ k_2 \cdot p_1 &= \frac{1}{2}\left(D_4 - D_3\right) \\ k_2 \cdot p_2 &= \frac{1}{2}\left(D_5 - D_4 - s\right) \end{aligned} \tag{2.9}$$

A topology is determined by its propagators, like (2.8) for the above example. Since the integrals are invariant with respect to shifts in the loop momenta, there is some freedom in choosing the propagators. A clear definition of a topology is given by the diagram containing all the propagators. For (2.7) it is the diagram in fig. 2.2, with the remark, that $D_7$ is restricted to non-positive indices.

There is an alternative to this auxiliary-denominator scheme. It is called shift scheme and uses scalar products of external and loop momenta in the numerator instead of auxiliary propagators. For detailed descriptions see for example [9][11].

In both ways, all the scalar integrals of the form (2.1) can be reduced to a minimal set of linear independent ones. Another advantage of this classification becomes apparent in the next section. There are identities for every topology. These so-called integration-by-parts identities allow a further reduction.

# Chapter 3

# Integration-by-parts identities

In general the tensor reduction leads to a large number of scalar integrals with different sets of indices. It is however possible to reduce the number of integrals to be computed by using special identities. These are the integration-by-parts (IBP) identities. IBP identities relate integrals of a single topology. For each integral of the form (2.3) we can write

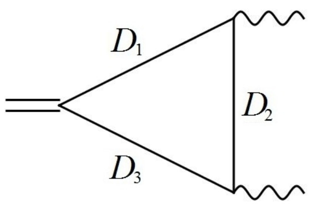
$$\int \frac{d^d k_1}{i\pi^{d/2}} \cdots \int \frac{d^d k_l}{i\pi^{d/2}} \frac{\partial}{\partial k_{j,\mu}} \left( \frac{v_\mu}{\prod_{i=1}^{b} D_i^{a_i}} \right) = 0. \tag{3.1}$$

The vanishing of this integrals can be proved with an extension to $d$-dimensional spaces of the divergence theorem [1]. The vector $k_{i,\mu}$ is one of the loop momenta ($i = 1, 2, ..., l$), whereas $v_\mu$ is an arbitrary momentum, loop or external. With $g$ independent external momenta, there are $l(l+g)$ IBP identities for each set of indices.

It is most suitable to explain the explicit calculation of the identities (3.1) and how to use them for the reduction by means of a simple example. This is done in section 3.1. The Laporta method, which will be explained in section 3.2, is a useful technique to reduce more complicated topologies. The idea is that of replacing explicit values for the indices in the IBP identities. This way a system of linear equations is generated, whose unknowns are the integrals themselves.

## 3.1 IBP reduction

An easy example is the one-loop three-point function with uniform internal masses:




$$= \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3}} = I\{a_1, a_2, a_3\}$$

With two external on-shell momenta $p_1$ and $p_2$ ($p_1^2 = p_2^2 = 0$; $2p_1 \cdot p_2 = s$), loop momentum $k$ and mass $m$, a possible choice for the propagators is

$$\begin{aligned} D_1 =& k^2 + m^2 \\ D_2 =& (k + p_1)^2 + m^2 \\ D_3 =& (k + p_1 + p_2)^2 + m^2. \end{aligned} \tag{3.2}$$

There are three independent identities for each set of indices, since $v_\mu$ can be $p_{1,\mu}$, $p_{2,\mu}$ or $k_\mu$. The identity with $v_\mu = p_{1,\mu}$ reads

$$0 = \int \frac{d^dk}{i\pi^{d/2}} \frac{\partial}{\partial k_\mu} \frac{p_{1,\mu}}{D_1^{a_1} D_2^{a_2} D_3^{a_3}}. \tag{3.3}$$

The derivations of the propagators are easily obtained:

$$\begin{aligned} p_{1,\mu} \frac{\partial}{\partial k_\mu} D_1 =& 2p_1 \cdot k = D_2 - D_1 \\ p_{1,\mu} \frac{\partial}{\partial k_\mu} D_2 =& 2p_1 \cdot k + 2p_1^2 = D_2 - D_1 \\ p_{1,\mu} \frac{\partial}{\partial k_\mu} D_3 =& 2p_1 \cdot k + 2p_1^2 + 2p_1 \cdot p_2 = D_2 - D_1 + s \end{aligned} \tag{3.4}$$

In the last steps the derivatives have been expressed in terms of propagators. Substituting these expressions, we get

$$\begin{aligned} & \int \frac{d^dk}{i\pi^{d/2}} (D_1^{a_1} D_2^{a_2} D_3^{a_3})^{-1} \left( a_1 - a_2 - a_1 \frac{D_2}{D_1} + a_2 \frac{D_1}{D_2} + a_3 \frac{D_1}{D_3} - a_3 \frac{D_2}{D_3} - a_3 \frac{s}{D_3} \right) \\ =& a_1 - a_2 - a_1 1^+ 2^- + a_2 2^+ 1^- + a_3 3^+ 1^- - a_3 3^+ 2^- - a_3 s 3^+ = 0. \end{aligned} \tag{3.5}$$

In the second step the operators $i^+$ and $i^-$ are introduced. Applied on the generic integral $I\{a_1, a_2, a_3\}$, which is omitted as an overall factor, they raise or lower the associated index, e.g.

$$1^+ 2^- I\{a_1, a_2, a_3\} = I\{a_1 + 1, a_2 - 1, a_3\}. \tag{3.6}$$

The calculation of the IBP identities for $v_\mu = p_{2,\mu}$ and $v_\mu = k_\mu$ is done in the same way:

$$p_{2,\mu}: \quad a_2 - a_3 + a_1 1^+ 2^- - a_1 1^+ 3^- - a_2 2^+ 3^- + a_3 3^+ 2^- + a_1 s 1^+ = 0 \tag{3.7}$$

$$\begin{aligned} k_\mu: \quad & d - 2a_1 - a_2 - a_3 - a_2 2^+ 1^- - a_3 3^+ 1^- \\ & + 2m^2 (a_1 1^+ + a_2 2^+ + a_3 3^+) + a_3 s 3^+ = 0 \end{aligned} \tag{3.8}$$

These identities relate integrals not only with different indices, but also with different sums of indices. The most important terms are the ones with $1^+$, $2^+$ and $3^+$: the sum of indices is one greater than for all the other integrals. So it is possible to express more complicated integrals of a topology step by step in terms of simpler ones with a smaller sum of indices.

As an example we reduce the integral $I\{2, 1, 1\}$ in the following. With $\{a_1, a_2, a_3\} = \{1, 1, 1\}$ eq. (3.7) reads

$$I\{2, 1, 1\} = \frac{1}{s} \left[ I\{2, 1, 0\} + I\{1, 2, 0\} - I\{2, 0, 1\} - I\{1, 0, 2\} \right]. \tag{3.9}$$

Due to the symmetric limits of integration and the invariance in shifting the loop momentum $k$, we easily proof that $I\{a_1, a_2, 0\} = I\{a_2, a_1, 0\}$:

$$D_1^{a_1} D_2^{a_2} = \left[k^2 + m^2\right]^{a_1} \left[(k + p_1)^2 + m^2\right]^{a_2}$$
$$\xrightarrow{k \to -k - p_1} \left[(-k - p_1)^2 + m^2\right]^{a_1} \left[(-k)^2 + m^2\right]^{a_2} = D_2^{a_1} D_1^{a_2} \tag{3.10}$$

In the same way we can show that $I\{a_1, 0, a_3\} = I\{a_3, 0, a_1\}$. So we arrive at

$$I\{2, 1, 1\} = \frac{2}{s} \left[I\{2, 1, 0\} - I\{2, 0, 1\}\right]. \tag{3.11}$$

Choosing $\{a_1, a_2, a_3\} = \{2, 1, 0\}$ in (3.5) and $\{a_1, a_2, a_3\} = \{1, 0, 1\}$ in (3.8) gives us the identities

$$I\{2, 1, 0\} = I\{3, 0, 0\} \tag{3.12}$$

and

$$-I\{2, 0, 1\} = \frac{1}{4m^2 + s} \left[(d - 3)I\{1, 0, 1\} - I\{2, 0, 0\}\right]. \tag{3.13}$$

When we substitute both in (3.11), the integral $I\{2, 1, 1\}$ is expressed in terms of bubble and tadpole integrals (fig. 3.1). With the indices $\{a_1, a_2, a_3\} = \{a_1, 0, 0\}$ in (3.8) we obtain the identity

$$I\{a_1 + 1, 0, 0\} = \frac{2a_1 - d}{2m^2 a_1} I\{a_1, 0, 0\}, \tag{3.14}$$

which can be used to reduce the tadpole integrals:

$$I\{2, 0, 0\} = \frac{2 - d}{2m^2} I\{1, 0, 0\} \tag{3.15}$$

$$I\{3, 0, 0\} = \frac{4 - d}{4m^2} I\{2, 0, 0\} = \frac{(4 - d)(2 - d)}{8m^4} I\{1, 0, 0\} \tag{3.16}$$

The final reduction then is

$$I\{2, 1, 1\} = \frac{(d - 2)\left(4(d - 3)m^2 + (d - 4)s\right)}{4m^4 s(4m^2 + s)} I\{1, 0, 0\}$$
$$+ \frac{2(d - 3)}{s(4m^2 + s)} I\{1, 0, 1\}. \tag{3.17}$$

The remaining integrals $I\{1, 0, 1\}$ and $I\{1, 0, 0\}$ are irreducible. They are called master integrals (MI). This topology has a third MI: $I\{1, 1, 1\}$. Every integral of this topology can be reduced to these three MIs. This simple example shows, that the IBP identities are a useful tool to reduce the number of integrals to be computed.

There is some freedom in choosing the basis of MIs. For example (3.15) allows to use $I\{2, 0, 0\}$ instead of $I\{1, 0, 0\}$ as a MI. Certainly one chooses simpler integrals, which generally implies a smaller sum of indices or less propagators. Further criteria for more complicated topologies are a better convergence or a higher symmetry.

There are additional identities, such as Lorentz invariance identities or symmetry relations (see for example [10]). But it is proven that these are included by the IBP identities [20].

Obtaining a complete IBP reduction for one-loop integrals is much simpler than for multi-loop ones. The reason is the following: The aim of a reduction is to eliminate

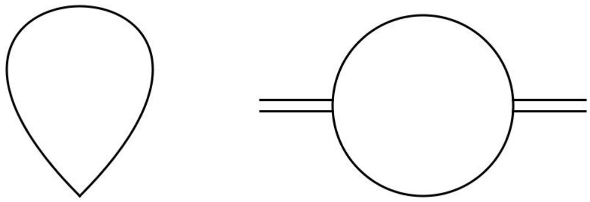


Figure 3.1: Tadpole integral $I_T$ and bubble integral $I_B$. The double straight lines denote external lines with mass square s.

propagators, meaning that their indices should become zero. When this succeeds, the corresponding internal line contracts to a point. Since in the one-loop case every endpoint of an internal line is connected to a an external line, the contraction allows to combine the associated external momenta. So the elimination of a propagator reduces the number of independent external momenta, i.e. also the number of differing scalar products. Hence the integral can be classified into a topology with a smaller set of propagators. Of course the IBP identities of the so-called subtopology do not contain operators or the index of the eliminated propagator. This is not the case for multi-loop integrals. There are propagators with endpoints, which are not connected to external lines, e.g. $D_i \ \ (i = 1, 2, 3, 5)$ in fig. 2.2. To further simplify integrals with one of these propagators eliminated, one has to use identities, which still contain the associated operators. Therefore one has to deal with integrals with negative indices of already eliminated propagators. This makes the reduction of multi-loop topologies a difficult task. A good strategy is necessary to prevent the reduction from going around in circles. One of these is the Laporta method, which is explained in the next section.

The simplicity of the reduction of one-loop topologies can also be explained with (2.2). The number of differing scalar products for one-loop topologies ($l = 1$) is given by $g + 1$. In general this is the number of external lines and in the case of one-loop integrals this equals the number of internals lines. Thus no auxiliary propagators with non-positive indices are needed for one-loop topologies.

The definition of subtopologies will also be important for chapter 4. Let us consider for instance the topology of the functions $I\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ in fig. 2.2 again. A subtopology is given by $I\{a_1, a_2, a_3, 0, a_5, a_6, 0\}$, because $D_4$ connects the external on-shell lines.

## 3.2 Laporta method

The Laporta method, originally introduced in [2][3], is a useful strategy for the IBP reduction. The idea is to build IBP identities with explicit values for the indices. In this way a system of linear equations is generated, whose unknowns are the integrals themselves. The system can be solved with the method of elimination of variables, where we have to decide, which integrals have to be solved first. Of course we choose the more complicated amplitudes, so that the remaining ones, which are the MIs, are as simple as possible. As already mentioned in the last section, the basis of MIs depends on the definition of “simple”. Possible criteria are:

- **The number of different propagators**: This is usually the primary criterion, since tadpole integrals are simpler than bubbles, while bubbles are less complicated then vertices and so on. Of course there are many integrals with the same number of propagators, so additional criteria are needed.

- **The sum of positive indices**: In general an integral with an quadratic propagator ($a_i = 2$) is more complicated than the one with the according index $a_i = 1$. So this is a good secondary condition.
- Since there may still be no clear order, further criteria could be imposed. For instance, they can include the number of negative indices. At some point the choice can also be random.

With only a few identities the system is badly under-determined. For example the identity (3.5) had already six unknowns for $a_1, a_2, a_3 \neq 0$ and $a_1 \neq a_2$. But by enlarging the size of the system, the number of equations grows faster than the number of unknowns [2]. Thus a smaller or equal set of MIs is obtained. There is a "critical mass" of equations, above which the number of MIs stays the same. Solving this system leads to a minimal basis of MIs.

The main advantage of the Laporta method is, that it can be automated in a rather simple way. There are some public implementations based on this method [4][5][6]. For the calculations of this thesis, the `Mathematica` implementation `FIRE` has been used in combination with the `IBP` tool [21], which allows to generate IBP identities. The reduction with `AIR` was used for checking purposes.

The basic input for `FIRE` consists of the momenta, internal an external, and the propagators of a topology. When the code is ready to work, any integral of the given topology can be reduced. The output is a linear combination of MIs. An example for the use of `FIRE` and `IBP` is in appendix B. There we take the topology from section 3.1 and reduce the integral $I\{2, 1, 1\}$. The result is identical to (3.17).

It is worth mentioning, that there is no strict mathematical proof, that the number of MIs obtained with the Laporta method is minimal, i.e. that they are independent from each other. In any case the final number of the MIs is small, compared to the several hundred integrals occurring in a typical calculation. So this reduction is after all a great progress.

# Chapter 4

# The method of differential equations

After classifying the scalar integrals into topologies and carrying out the IBP reduction we are left with the MIs. The method of differential equations allows to compute the MIs with the help of the IBP identities. Instead of performing loop integrations, differential equations have to be solved to get the MIs. How to generate these equations is explained in section 4.1 with two one-loop examples. In 4.2 we discuss how boundary conditions are obtained. The rest of this chapter discusses how to solve the differential equations. We explain the basic analysis, such as separation of variables, as well as more specific techniques, e.g. the $\epsilon$-expansion.

## 4.1 Generating the equations

Differential operators of the masses or the external momenta applied on scalar integrals raise or lower the indices of specific propagators. The resulting integrals can then be reduced to the MIs using the IBP identities. In building these equations for every MI, a system of differential equations is obtained.

We define the differential operator

$$p_{x,\mu}\frac{\partial}{\partial p_{y,\mu}}, \tag{4.1}$$

where $p_x$ and $p_y$ $(x, y = 1, 2, ..., g)$ are independent external momenta of an integral

$$I(m_j^2, p_v \cdot p_w) = \int \frac{d^d k_1}{i\pi^{d/2}} \cdots \frac{d^d k_l}{i\pi^{d/2}} \frac{1}{\prod_{i=1}^{b} D_i^{a_i}}. \tag{4.2}$$

Since it is a scalar integral, it can only depend on scalar quantities, like the mass squares $m_j^2$ $(j = 1, 2, ..., b)$ and the scalar products of the external momenta $p_v \cdot p_w$ $(v, w = 1, 2, ..., g)$. Applying (4.1) to a propagator of (4.2), we get

$$\begin{aligned} p_{x,\mu}\frac{\partial}{\partial p_{y,\mu}} D_i =& p_{x,\mu}\frac{\partial}{\partial p_{y,\mu}}(q_i^2 + m_i^2) \\ =& p_{x,\mu}\frac{\partial}{\partial p_{y,\mu}}((\hat{q}_i + p_y)^2 + m_i^2) \\ =& 2p_x \cdot (\hat{q}_i + p_y), \end{aligned} \tag{4.3}$$

assuming that $D_i$ contains $p_y$, since otherwise the derivative would vanish. The scalar products in the last line can then be expressed in terms of propagators, as shown in section 2. Hence it is possible to express the quantity

$$p_{x,\mu} \frac{\partial}{\partial p_{y,\mu}} I(m_j^2, p_v \cdot p_w) \tag{4.4}$$

in terms of integrals of the same topology as $I(m_j^2, p_v \cdot p_w)$, because we can take the partial derivative inside the integral. Now the IBP reduction can be used to write (4.4) in terms of MIs. And that is the main idea, how to get a system of differential equations, with one equation for every MI. Fortunately great parts of the system can be separated, because in the differential equations of MIs from subtopologies will not appear more complicated ones. That should be clear, if we recall the considerations at the end of section 3.1. As a result we need to begin with the calculation of the simplest MI.

As already mentioned, a scalar integral can not depend on the momenta $p_y$ themselves, bot only on scalar products $s_{xy} = 2p_x \cdot p_y$ of external momenta. The differential operator for $s_{xy}$ is a linear combination of the operators (4.1), which can be obtained using the chain rule.

Alternatively we can set up differential equations in the masses. Because of

$$\frac{\partial}{\partial m_j^2} D_i = \frac{\partial}{\partial m_j^2}(q_i^2 + m_i^2) = \begin{cases} 1 & \text{for } m_j = m_i \\ 0 & \text{else,} \end{cases} \tag{4.5}$$

the quantity

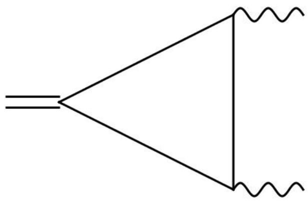
$$\frac{\partial}{\partial m_j^2} I(m_j^2, p_v \cdot p_w) \tag{4.6}$$

can also be expressed in terms of integrals of the same topology.

A scalar integral $I$ can be factorized into a dimensionless function $F$ and a factor, which carries the mass dimension. It is then possible to set up the differential equations of $F$ in the dimensionless quantities the function depends on. This is useful, because scalar integrals usually contain logarithms and polylogarithms.

In the following are two examples.

- **The massive three-point function** from section 3.1 for $d = 4$. We consider one of its MIs:



$$= \int \frac{d^4k}{i\pi^2} \frac{1}{D_1 D_2 D_3} = I\{1,1,1\} =: I_V(s, m^2)$$

The other MIs are the tadpole integral $I\{1,0,0\} =: I_T$ and the bubble integral $I\{1,0,1\} =: I_B$. $I_V$ can only depend on the kinematical invariant $s$ and the mass square $m^2$. The mass dimension of the differential $d^4k$ is 4, while every propagator of the integrand reduces it by 2. So a proper ansatz is

$$I_V(s, m^2) = \frac{1}{s} F_V(\tau_s) \qquad \text{with} \quad \tau_s = \frac{m^2}{s}. \tag{4.7}$$

The function $F_V$ can only depend on one independent dimensionless quantity, which we choose to be the quotient $\tau_s$. The derivative of $I_V$ with respect to $m^2$ is given by

$$\frac{\partial}{\partial m^2} I\{1,1,1\} = -I\{2,1,1\} - I\{1,2,1\} - I\{1,1,2\}, \tag{4.8}$$

since the derivative of a massive propagator is

$$\frac{\partial}{\partial m^2}\frac{1}{D_i} = -\frac{1}{D_i^2}. \tag{4.9}$$

The reduction of $I\{2,1,1\}$ is explained in detail in section 3.1. We do the same for the other two integrals on the r.h.s. This leads to

$$\frac{\partial}{\partial m^2} I_V = \frac{I_T + m^2 I_B}{4m^6 + m^4 s}, \tag{4.10}$$

where the r.h.s does not contain the integral $I_V$ itself. That is a feature of this simple example. In general the differential equations do not take this trivial form. The inhomogeneous term contains the known MIs of subtopologies and rational factors. Inserting $I_T$ and $I_B$, which can be calculated with Feynman parametrization (see for example [17]), and using (4.7), we arrive at

$$\frac{\partial}{\partial \tau_s} F(\tau_s) = \frac{1}{\tau_s\sqrt{1+4\tau_s}} \log\left(\frac{\sqrt{1+4\tau_s}-1}{\sqrt{1+4\tau_s}+1}\right). \tag{4.11}$$

The integrals $I_T$ and $I_B$ are divergent in 4 dimensions, but their poles cancel. With the substitution

$$x_s = \frac{\sqrt{1+4\tau_s}-1}{\sqrt{1+4\tau_s}+1} \tag{4.12}$$

the differential equation simplifies to

$$\frac{\partial}{\partial x_s} F(x_s) = \frac{1}{x_s}\log(x_s). \tag{4.13}$$

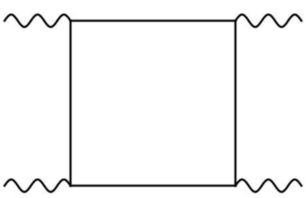
Integration on both sides yields the solution

$$F(x_s) = \frac{1}{2}\log^2(x_s) + C. \tag{4.14}$$

We need a boundary condition to determine the constant $C$. This topic is explained in the next section.

We could also start with the differential equation in $s = 2p_1 \cdot p_2$. But since $F_V$ only depends on one variable, this differential equation is linear dependent to the one in $m^2$ and leads to the same in $\tau_s$.

- **The massive four-point function** with all external momenta on-shell in $d = 4$:



$$= \int \frac{d^4k}{i\pi^2}\frac{1}{D_1D_2D_3D_4} := I_4(s,t,m^2)$$

This function is a MI of the according topology with arbitrary powers of the propagators $D_i$ $(i = 1, 2, 3, 4)$. Besides $m^2$, this integral depends on two kinematical invariant $s = 2p_1 \cdot p_2$ and $t = 2p_2 \cdot p_3$, because $p_1 \cdot p_3$ can be expressed in terms of $s$ and $t$. With the on-shell conditions and the conservation of momentum

$$p_i^2 = 0 \quad (i = 1, 2, 3) \tag{4.15}$$
$$p_4^2 = (-p_1 - p_2 - p_3)^2 = 0, \tag{4.16}$$

it is easy to prove that

$$2p_1 \cdot p_3 = -s - t. \tag{4.17}$$

A suitable factor to isolate a dimensionless function is $(st)^{-1}$, because $I_4(s, t, m^2)$ is symmetric in $s$ and $t$. There are two independent quantities, the dimensionless function $F_4$ can depend on. A convenient choice is $m^2/s = \tau_s$ and $m^2/t = \tau_t$. So the integral can be written as

$$I_4(s, t, m^2) = \frac{1}{st} F_4(\tau_s, \tau_t). \tag{4.18}$$

To obtain the complete informations on $F(\tau_s, \tau_t)$ we have to consider a system of partial differential equations, i.e. one equation for each of the arguments $\tau_s$ and $\tau_t$.

The differential operators must not violate the conditions (4.15) and (4.16). The first one is preserved by every linear combination of the operators $p_1 \cdot \partial_{p_1}$, $p_2 \cdot \partial_{p_2}$ and $p_3 \cdot \partial_{p_3}$. With the ansatz

$$(\alpha p_1 \partial_{p_1} + \beta p_2 \partial_{p_2} + \gamma p_3 \partial_{p_3})(p_1 + p_2 + p_3)^2 = 0 \tag{4.19}$$

we find the possible differential operators (besides the trivial solution $\alpha = \beta = \gamma$, which only gives a dimensional information)

$$D(\alpha, \gamma) = \alpha p_1 \partial_{p_1} + \frac{\alpha t + \gamma s}{t + s} p_2 \partial_{p_2} + \gamma p_3 \partial_{p_3}. \tag{4.20}$$

Using the chain rule we obtain expressions for $\partial_{\tau_s}$ and $\partial_{\tau_t}$ in terms of two independent operators of the form (4.20), e.g.

$$\partial_{\tau_s} = -\frac{1}{\tau_s} D(1, 0) + \frac{1}{2(\tau_s + \tau_t)} \left[ D(1, 0) + D(0, 1) \right]. \tag{4.21}$$

Of course differential operators of $m^2$ automatically satisfies conditions of the momenta.

In general there is a partial differential equation for every independent dimensionless quantity, which can be formed with the masses and the kinematical invariants the integral depends on. But there has to be at least one such quantity. The method only works for integrals with more than one scale. Differential equations of integrals like $I_T$, depending only on $m^2$, give only trivial dimensional informations. Especially for multi-loop topologies, systems of differential equations of two or more MIs have to be solved, since they can not always be separated.

## 4.2 Boundary conditions

In order to obtain the complete expression for a MI, a boundary condition has to be imposed to the general solution of the differential equation. That means we have to know the MI in a given kinematical point. A convenient strategy is to analyse the integrals in special limits, like the massless limit, i.e. $m^2 \to 0$, or for small external momenta. In this thesis we use the latter one. For the integrals calculated in chapter 5, this refers to the limit $s \to 0$. We use the Mellin-Barnes (MB) representation (see for example [17]) together with several tools [22] to obtain the expansions for this limit. The MB representation allows to write loop integrals as integrals over contours in a complex plane along the imaginary axis of products and ratios of gamma functions. The calculations are summarized in appendix A.

## 4.3 Recursive solution in $\epsilon$

To handle the divergences of integrals in four dimensions, one uses dimensional regularization. This means to evaluate the integrals for generic dimensions $d$. In general the differential equations are too complicated to be solved exactly in $d$ dimensions. An efficient way to handle this problem involves the Laurent expansion in $\epsilon$ $(d = 4 - 2\epsilon)$ of the equation itself.

First order differential equations are of the general form

$$\frac{\partial}{\partial x} F(x;\epsilon) = A(x;\epsilon)F(x;\epsilon) + B(x;\epsilon). \tag{4.22}$$

Every quantity of this equation can be expanded in $\epsilon$:

- The coefficient $A(x;\epsilon)$ of the unknown function $F(x;\epsilon)$ is a rational function of the dimensionless variable $x$. It has the form

  $$A(x;\epsilon) = \sum_{i=0}^{\infty} \epsilon^i A^{(i)}(x), \tag{4.23}$$

  since in general it does not contain poles.

- The inhomogeneous term $B(x;\epsilon)$ is related to the subtopologies (cf. (4.10)). Its expansion

  $$B(x;\epsilon) = \sum_{i=-p}^{\infty} \epsilon^i B^{(i)}(x) \tag{4.24}$$

  starts with the pole of highest order $(\mathcal{O}(\epsilon^{-p}))$.

- We expects the same to be true for the unknown MI and the according function $F(x;\epsilon)$:

  $$F(x;\epsilon) = \sum_{i=-p}^{\infty} \epsilon^i F^{(i)}(x) \tag{4.25}$$

Inserting (4.23), (4.24) and (4.25) in (4.22) allows to write down a series of coupled differential equations, e.g. for $p = 2$:

$$
\begin{aligned}
\frac{\partial}{\partial x} F^{(-2)}(x) &= A^{(0)}(x)F^{(-2)}(x) + B^{(-2)}(x) \\
\frac{\partial}{\partial x} F^{(-1)}(x) &= A^{(0)}(x)F^{(-1)}(x) + A^{(1)}(x)F^{(-2)}(x) + B^{(-1)}(x) \\
\frac{\partial}{\partial x} F^{(0)}(x) &= A^{(0)}(x)F^{(0)}(x) + A^{(1)}(x)F^{(-1)}(x) + A^{(2)}(x)F^{(-2)}(x) + B^{(0)}(x) \\
&\vdots
\end{aligned}
\tag{4.26}
$$

The first equation, for the coefficient $F^{(-2)}(x)$ of the highest pole, is the first one to be solved. The solution can than be inserted in the second equation for $F^{(-1)}(x)$ and solved for the latter function, and so on. We usually stop with the order of $\epsilon^0$, because it refers to the value for $d = 4$. It is sometimes necessary to compute integrals for higher orders in $\epsilon$, since these terms appear in differential equations of more complicated MIs.

The individual equations are solved with the method of variation of constants. Let us consider, for instance, the equation for the coefficient $F^{(-2)}(x)$. Let $\hat{F}(x)$ be the solution of the associated homogeneous equation, i.e. the equation with the non-homogeneous part, in this case $B^{(-2)}(x)$, set to zero:

$$
\frac{\partial}{\partial x} \hat{F}(x) = A^{(0)}(x)\hat{F}(x) \tag{4.27}
$$

It is important to notice, that the above equation is the same for every order in $\epsilon$. It is solved with separation of variables:

$$
\hat{F}(x) = \exp\left[\int^x dx' A^{(0)}(x')\right] \tag{4.28}
$$

The solution of the non-homogeneous equation is then given by

$$
F^{(-2)}(x) = \hat{F}(x)\left[\int^x dx' \hat{F}^{-1}(x')B^{(-2)}(x') + C\right]. \tag{4.29}
$$

The integration constant of (4.28) can be neglected, since it is included in (4.29).

Often differential equations of some MI can not be separated. One way to solve the resulting systems of differential equations is to start from (4.22) and treat $F(x;\epsilon)$ and $B(x;\epsilon)$ as $l$-component vectors and $A(x;\epsilon)$ as a $l \times l$ matrix ($l$ is the number of differential equations). The steps leading to (4.29) can be reinterpreted as well. But in practice this method turns out to be inappropriate. The problem is, that due to the matrix exponential in (4.28) the matrix $\hat{F}(x)$ or rather its inverse is to complicated to be integrated in (4.29).

Another way is to build the associated differential equation of order $l$. The eq. (4.29) can be generalized for this case. The solution of a differential equation of order $l$

$$
\frac{\partial^l}{\partial x^l} F(x) = \sum_{i=0}^{l-1} A_i(x) \frac{\partial^i}{\partial x^i} F(x) + B(x) \tag{4.30}
$$

is then given by

$$
F(x) = \sum_{i=1}^{l} \hat{F}_i(x)\left[\int^x dx' W^{-1}(x')M_i(x')B(x') + C_i\right], \tag{4.31}
$$

where

- $\hat{F}_i$ are independent particular solutions of the homogeneous equation, i.e.

$$\sum_{i=1}^{l} \hat{C}_i \hat{F}_i \tag{4.32}$$

  is the general solution with constants $\hat{C}_i$.

- $C_i$ are the constants of integration.

- $W(x)$ is the Wronskian

$$W(x) = \begin{vmatrix} \hat{F}_1(x) & \hat{F}_2(x) & \cdots & \hat{F}_l(x) \\ \hat{F}_1'(x) & \hat{F}_2'(x) & \cdots & \hat{F}_l'(x) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{F}_1^{(l-1)}(x) & \hat{F}_2^{(l-1)}(x) & \cdots & \hat{F}_l^{(l-1)}(x) \end{vmatrix} \tag{4.33}$$

- $M_i(x)$ are the minors of the $\hat{F}_i^{(l-1)}(x)$ in (4.33).

The main problem of this way is, that there is no universal procedure for finding the homogeneous solutions $\hat{F}_i$. This has to be done by trial and error. To carry out the integrations in (4.28) a special type of functions is necessary: the harmonic polylogarithms. Their definitions and characteristics are summarized in the next section.

## 4.4 Harmonic polylogarithms

With the method of variation of constant (4.29) the general solutions of the coupled differential equations (4.26) take the form

$$\begin{aligned} F^{(-2)}(x) =& \hat{F}(x) \int^x dx' \hat{F}^{-1}(x') B^{(-2)}(x') \\ F^{(-1)}(x) =& \hat{F}(x) \int^x dx' \hat{F}^{-1}(x') B^{(-1)}(x') \\ &+ \hat{F}(x) \int^x dx'(x') A^{(1)}(x') \int^{x'} dx'' \hat{F}^{-1}(x'') B^{(-2)}(x'') \\ F^{(0)}(x) =& \hat{F}(x) \int^x dx' \hat{F}^{-1}(x') B^{(0)}(x') \\ &+ \hat{F}(x) \int^x dx'(x') A^{(2)}(x') \int^{x'} dx'' \hat{F}^{-1}(x'') B^{(-2)}(x'') \\ &+ \hat{F}(x) \int^x dx'(x') A^{(1)}(x') \int^{x'} dx'' \hat{F}^{-1}(x'') B^{(-1)}(x'') \\ &+ \hat{F}(x) \int^x dx'(x') A^{(1)}(x') \int^{x'} dx'' A^{(1)}(x'') \int^{x''} dx''' \hat{F}^{-1}(x''') B^{(-2)}(x'''), \end{aligned} \tag{4.34}$$

where we omitted the constants. We note, that these expressions require repeated integrations of products of the function $\hat{F}^{-1}(x)$ and the coefficients of the non-homogeneous terms $B^{(i)}(x)$. The idea is to express these integrations in terms of functions with a recursive definition. These are the harmonic polylogarithms (HPL) [16].

The integrands in (4.34) consist of rational factors and logarithms or polylogarithms. The rational part can be expressed as a linear combination of a minimal set of fractions. For most of the integrals, used in this thesis as MIs, the fractions

$$\begin{aligned} f(-1;x) =& \frac{1}{1+x} \\ f(0;x) =& \frac{1}{x} \\ f(1;x) =& \frac{1}{1-x} \end{aligned} \tag{4.35}$$

are sufficient. The simplest HPLs, the ones of weight one, are defined as integrals of these factors:

$$\begin{aligned} H[-1;x] &= \int_0^x \frac{dx'}{1+x'} = \log(1+x) \\ H[0;x] &= \int_1^x \frac{dx'}{x'} = \log(x) \\ H[1;x] &= \int_0^x \frac{dx'}{1-x'} = -\log(1-x) \end{aligned} \tag{4.36}$$

This seems to be needless, because these are simple logarithms. But since there are repeated integrations in (4.34), it may be necessary to perform one of these:

$$\begin{aligned} \int_1^x \frac{dx'}{x'} \log(x') &= \frac{1}{2} \log^2(x) \\ \int_1^x \frac{dx'}{x'} \log(1+x') &= \mathrm{Li}_2(x), \end{aligned} \tag{4.37}$$

where $\mathrm{Li}_2(x)$ is the dilogarithm. Those two examples are HPLs of weight two. So for higher orders in $\epsilon$, polylogarithms of increasing order appear beside the rational factors. This is also true for more complicated integrals, because the non-homogeneous parts of their differential equations, which are part of the integrands in (4.34), contain simpler MIs. The integration of such terms is nontrivial. This is where the recursive definition of the HPLs becomes useful:

$$H[a,\vec{w};x] = \int_0^x f(a;x') H[\vec{w};x'] dx' \tag{4.38}$$

for $[a,\vec{w}] \neq [0,\vec{0}_w]$ and

$$H[0,\vec{0}_w;x] = \frac{1}{w!} \log^w(x), \tag{4.39}$$

where $a$ is a index, which takes the values $-1$, $0$ or $+1$. The vector $\vec{w}$ is list of $w$ indices, each one taking the values $-1$, $0$ or $+1$ and $\vec{0}_w$ is a list of $w$ zeros.

We need additional fractions for the calculations in this thesis:

$$\begin{aligned} f(-r;x) =& \frac{1}{\sqrt{x(4+x)}} \\ f(r;x) =& \frac{1}{\sqrt{x(4-x)}} \end{aligned} \tag{4.40}$$

The definition of the related HPLs stays the same. We just need to extend the list of possible indices. These functions are related to integrals with thresholds or pseudothresholds at $s = \pm 4m^2$ [12].

A lot of progress has been made in analysing the characteristics of these functions, see e.g. [23][24]. For the analysis of the HPL in chapter 5, we used the package `HPL` [25] for `Mathematica`.

# Chapter 5

# Computation of previously unknown three-point integrals

## 5.1 Detailed calculations

We want to calculate two versions of two-loop three-point integrals, a planar and a non-planar one, shown in fig. 5.1. The masses of massive internal propagators are uniform, two external lines are on-shell ($p_1^2 = p_2^2 = 0$) and the third obtains $(p_1 + p_2)^2 = s$. We use the same approach for the integrals as in [13]:

$$I_x = \left( \frac{\Gamma(1+\epsilon)}{1-\epsilon} \right)^l (m^2)^{m_{dim}/2} F_x \tag{5.1}$$

where

- $l$ is the number of loops.
- $m_{dim}$ is the mass dimension of $I_x$. For an integral with $b$ propagators of powers $a_i \ \ (i = 1, 2, ..., b)$ it is given by

$$m_{dim} = l \cdot d - 2 \sum_{i=1}^{b} a_i. \tag{5.2}$$

- $F_x$ is the dimensionless function.

The factor with the gamma function has the advantage, that the dimensionless function of the tadpole integral takes the simple form $F_T = -\epsilon^{-1}$.

The results of the integrals calculated in this chapter are summarized in the last section.

### 5.1.1 Planar topology

The first topology we want to consider is the planar one (fig. 5.1 (a)):

$$\int \frac{d^d k_1}{i\pi^{d/2}} \int \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7}} = I_p\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\} \tag{5.3}$$

A possible choice for the propagators is given in (5.46). The MIs of this topology are shown in fig. 5.2. We chose this set to use as many already known integrals as possible.

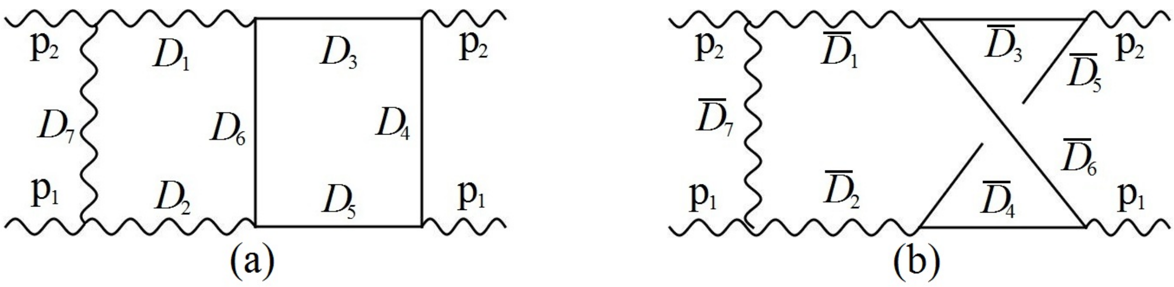
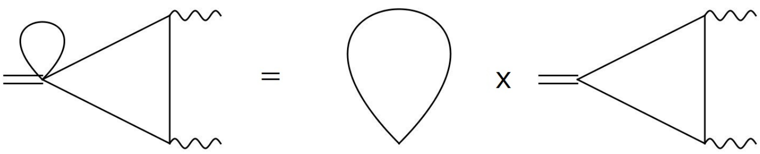
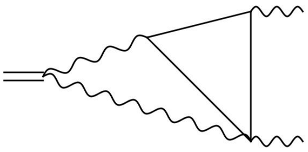



Figure 5.1: Planar (a) and non-planar topologies (b) with $D_7$ and $\overline{D}_7$ as auxiliary propagators.

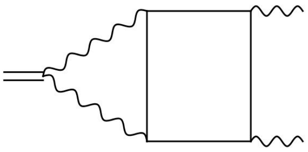
The integrals in the first two lines are factorizable. There is no need to calculate them by means of their differential equations, because they can be written as products of simple one-loop integrals shown in fig. 5.3 (for the calculation of these integrals see for example [17]), e.g.



That is where the simple form of $F_T$ is useful. The integrals in lines three and four have been calculated in [13] and [12]. The computation of the remaining two integrals



$$= I_p\{1,1,1,1,0,1,0\} =: I_\alpha(s,m^2)$$



$$= I_p\{1,1,1,1,1,1,0\} =: I_\beta(s,m^2)$$

is explained in the following.

### Calculation of $I_\alpha$

Let us start with the integral $I_\alpha(s,m^2)$, because it appears in the differential equation of $I_\beta(s,m^2)$ but not vice versa. Taking the derivative with respect to $m^2$ and performing the IBP reduction on the r.h.s., we get

$$\begin{aligned}\frac{\partial}{\partial m^2} I_\alpha =& \frac{d-4}{2m^2} I_\alpha + \frac{(d-2)^2(30+(d-12)d)}{8(d-5)(3-d)(3d-10)m^8} I_{TT}\\ &+ \frac{4(d-4)m^2+(18-5d)s}{4(30-19d+d^2)m^4} I_{31} + \frac{(d-4)(4m^2+s)}{(30-19d+3d^2)m^4} I_{32}\\ &- \frac{d-2}{4m^4} I_{41} + \frac{3d-10}{4m^4} I_{42} + \frac{s}{2m^4} I_{43},\end{aligned} \tag{5.4}$$

where the arguments of the integrals are omitted from now on. The MIs on the r.h.s. are defined in fig. 5.2. Using (5.1), we obtain the differential equation for $F_\alpha$ with the

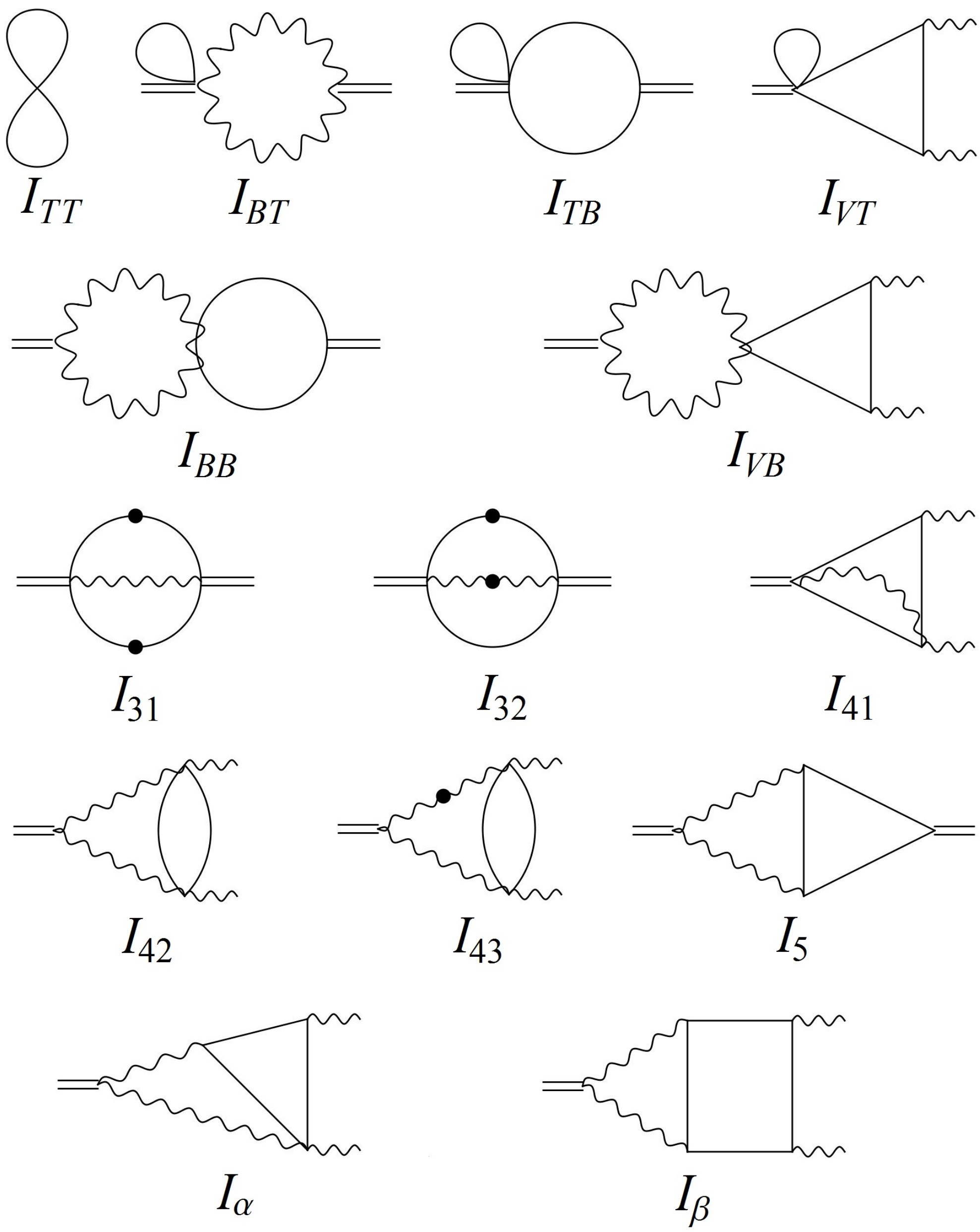



Figure 5.2: Master integrals of the planar topology. We use the same notation as in [13]: Wavy lines denote massless particles both external and internal. Internal massive lines are denoted by single straight lines. Double straight lines denote external lines with mass square s. Each dot on a propagator line denotes an additional power of the propagator.

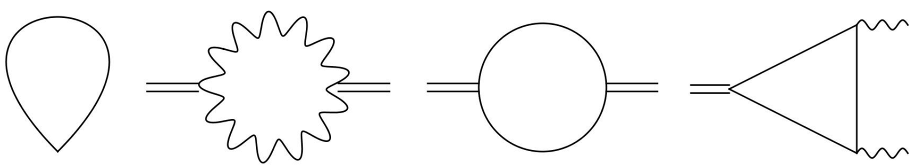


Figure 5.3: One loop integrals: the massive tadpole, the massless bubble, the massive bubble and the massive vertex integral

product rule:

$$\frac{\partial}{\partial m^2} F_\alpha = \left(\frac{1-\epsilon}{\Gamma(1+\epsilon)}\right)^2 (m^2)^{1+2\epsilon} \frac{\partial}{\partial m^2} I_\alpha + \frac{1+2\epsilon}{m^2} F_\alpha \tag{5.5}$$

Then we substitute $m^2$ for a dimensionless quantity, where we choose $x_s$ from (4.12). The $\epsilon$-expansion (4.26) leads to the differential equations for the coefficients of $F_\alpha$. Since the non-homogeneous part is finite for $d = 4$, we can start with the order $\mathcal{O}(\epsilon^0)$:

$$\begin{aligned} \frac{\partial}{\partial x_s} F_\alpha^{(0)} =& \frac{1+x_s}{x_s(1-x_s)} F_\alpha^{(0)} + \frac{1+x_s}{(1-x_s)^3} \{2\zeta(3) - H[0,0,0;x_s] \\ & -2H[1,0,0;x_s] + H[r,r,0;\hat{\tau}_s]\} \end{aligned} \tag{5.6}$$

The HPL with the argument $\hat{\tau}_s = 1/\tau_s$ comes from the MIs $I_{42}$ and $I_{43}$. This function can not be expressed in terms of $x_s$. We use the method of variation of the constants (4.29) to solve this equation. So we need to start with separation of variables (4.28) to obtain the solution of the corresponding homogeneous equation. Partial fraction decomposition is useful to perform the integration:

$$\begin{aligned} \hat{F}_\alpha =& \exp\left[\int^{x_s} dx'_s \frac{1+x'_s}{x'_s(1-x'_s)}\right] \\ =& \exp\left[\int^{x_s} dx'_s \left(\frac{1}{x'_s} + \frac{2}{1-x_s}\right)\right] \\ =& \exp\left[\log(x_s) - 2\log(1-x_s)\right] \\ =& \frac{x_s}{(1-x_s)^2} \end{aligned} \tag{5.7}$$

We use a hat to denote homogeneous solutions, i.e. $\hat{F}_\alpha$ is the solution of the differential equation of $F_\alpha^{(0)}$ with the non-homogeneous terms set to zero. We omit the index (0), since the homogeneous equations are the same for every order in $\epsilon$.

The next task is to perform the integration in (4.29). The integrand is the product of the non-homogeneous part in (5.6) and the inverse of the homogeneous solution (5.7). We obtain

$$\begin{aligned} & \frac{1+x_s}{x_s(1-x_s)} \left(2\zeta(3) - H[0,0,0;x_s] - 2H[1,0,0;x_s] + H[r,r,0;\hat{\tau}_s]\right) \\ =& \left(\frac{4}{1-x_s} + \frac{2}{x_s}\right)\zeta(3) + \left(-\frac{2}{1-x_s} - \frac{1}{x_s}\right) H[0,0,0;x_s] \\ & + \left(-\frac{4}{1-x_s} - \frac{2}{x_s}\right) H[1,0,0;x_s] + \frac{1+x_s}{x_s(1-x_s)} H[r,r,0;\hat{\tau}_s], \end{aligned} \tag{5.8}$$

where we used partial fraction decomposition to bring it in a form, which allows to use the definitions of the HPLs (4.38) and (4.39). The integration of the first three terms is then straightforward. To integrate the last term we change the variable of integration:

$$\begin{aligned} \int^{x_s} dx'_s \frac{1+x'_s}{x'_s(1-x'_s)} H[r,r,0;\hat{\tau}'_s] =& -\int^{\hat{\tau}_s} \frac{d\hat{\tau}'_s}{\hat{\tau}'_s} H[r,r,0;\hat{\tau}'_s] \\ =& -H[0,r,r,0;\hat{\tau}_s] \end{aligned} \tag{5.9}$$

Thus the general solution of (5.6) is

$$\begin{aligned}F_\alpha^{(0)} =&\frac{x_s}{(1-x_s)^2}\Big\{C + 2\zeta(3)H[0;x_s] + 4\zeta(3)H[1;x_s] - H[0,0,0,0;x_s]\\&- 2H[1,0,0,0;x_s] - 2H[0,1,0,0;x_s] - 4H[1,1,0,0;x_s]\\&- H[0,r,r,0;\hat{\tau}_s]\Big\}.\end{aligned} \tag{5.10}$$

To determine the constant of integration $C$, we analyse the behaviour of $F_\alpha^{(0)}$ for small external momenta, i.e. for $\hat{\tau}_s = s/m^2 \to 0$. To obtain the expansion of $F_\alpha^{(0)}$ in this limit, we calculate the MB representation of $I_\alpha$ for $d = 4$ (see (A.17)). With the tools, also listed in the appendix, we obtain the following expansion for small $\hat{\tau}_s$:

$$m^2 I_\alpha = \frac{3}{2} - \frac{1}{2}\log(\hat{\tau}_s) + \mathcal{O}(\hat{\tau}_s) \tag{5.11}$$

Now we need the same for the general solution (5.10). The part with the HPLs with indices 0 and 1 can be expanded, using the `HPL` package:

$$\begin{aligned}&\frac{x_s}{(1-x_s)^2}\Big\{C + 2\zeta(3)H[0;x_s] + 4\zeta(3)H[1;x_s] - H[0,0,0,0;x_s]\\&- 2H[1,0,0,0;x_s] - 2H[0,1,0,0;x_s] - 4H[1,1,0,0;x_s]\Big\}\\=&\left(C - \frac{\pi^4}{30}\right)\hat{\tau}_s^{-1} + \mathcal{O}\left(\hat{\tau}_s^0\right)\end{aligned} \tag{5.12}$$

For the HPL with indices $r$ and 0 we use the approximation

$$\int_0^{\hat{\tau}_s} \frac{dx}{\sqrt{x(4\pm x)}} \xrightarrow{\hat{\tau}_s \to 0} \int_0^{\hat{\tau}_s} \frac{dx}{\sqrt{4x}}. \tag{5.13}$$

Thus we can write

$$\begin{aligned}H[0,r,r,0;\hat{\tau}_s] &= \int_0^{\hat{\tau}_s} \frac{da}{a} \int_0^a \frac{db}{\sqrt{b(4-b)}} \int_0^b \frac{dc}{\sqrt{c(4-c)}} \log(c)\\&\xrightarrow{\hat{\tau}_s \to 0} \frac{1}{4} \int_0^{\hat{\tau}_s} \frac{da}{a} \int_0^a \frac{db}{\sqrt{b}} \int_0^b \frac{dc}{\sqrt{c}} \log(c)\\&= \left(-2 + \frac{1}{2}\log(\hat{\tau}_s)\right)\hat{\tau}_s.\end{aligned} \tag{5.14}$$

This yields the expansion

$$-\frac{x_s}{(1-x_s)^2} H[0,r,r,0;\hat{\tau}_s] = \left(2 - \frac{1}{2}\log(\hat{\tau}_s)\right) + \mathcal{O}(\hat{\tau}_s). \tag{5.15}$$

Now we can compare the sum of (5.12) and (5.15) with (5.11). Because there is neither a term of order $\mathcal{O}(\hat{\tau}_s^{-1})$ in (5.11) nor in (5.15) the coefficient in (5.12) must vanish. This is the case for

$$C = \frac{\pi^4}{30}. \tag{5.16}$$

**Calculation of $I_\beta$**

The steps for the calculation of $I_\beta$ are essentially the same as those for $I_\alpha$. The differential equation in $m^2$ after the IBP reduction is

$$\begin{aligned}\frac{\partial}{\partial m^2} I_\beta = &\left(\frac{d-4}{m^2} + \frac{6-2d}{4m^2+s}\right) I_\beta + \frac{(d-2)^2(210+d(9d-91))}{8(d-5)(d-3)(3d-10)m^8(4m^2+s)} I_{TT} \\ &+ \frac{d-3}{m^2(4m^2+s)} I_{BV} + \frac{d-2}{2m^4(4m^2+s)} I_{VT} \\ &+ \frac{3(4(d-4)m^2+(18-5d)s)}{2(30-19d+3d^2)m^4(4m^2+s)} I_{31} + \frac{6(d-4)}{(30-19d+3d^2)m^4} I_{32} \\ &+ \frac{6-3d}{8m^6+2m^4 s} I_{41} + \frac{3d-10}{m^4(4m^2+s)} I_{42} + \frac{2s}{4m^6+m^4 s} I_{43} \\ &+ \frac{d-3}{m^2(4m^2+s)} I_{51} + \frac{4-d}{4m^4+m^2 s} I_\alpha. \end{aligned} \tag{5.17}$$

The non-homogeneous part of this equation is finite for $d = 4$. With the general approach (5.1) the $\epsilon$-expansion leads to the following differential equation for the coefficient of $F_\beta$ of order $\mathcal{O}(\epsilon^0)$:

$$\begin{aligned}\frac{\partial}{\partial x_s} F_\beta^{(0)} = &\frac{2(1+x_s+x_s^2)}{x_s(1-x_s^2)} F_\beta^{(0)} + \frac{x_s}{(1+x_s)(1-x_s)^3} \{18\zeta(3) - 3H[0,0,0;x_s] \\ &+ 4H[0,0,1;x_s] + 4H[0,1,0;x_s] - 14H[1,0,0;x_s] \\ &+4H[r,r,0;\hat{\tau}_s]\} \end{aligned} \tag{5.18}$$

Using separation of variables for the associated homogeneous equation we get

$$\hat{F}_\beta = -\frac{x_s^2}{(1+x_s)(1-x_s)^3}. \tag{5.19}$$

The product of the inverse of (5.19) and the non-homogeneous part of (5.18) can then be written in the following form:

$$\begin{aligned}-\frac{18}{x_s}\zeta(3) + \frac{3}{x_s} H[0,0,0;x_s] - \frac{4}{x_s} H[0,0,1;x_s] - \frac{4}{x_s} H[0,1,0;x_s] \\ +\frac{14}{x_s} H[1,0,0;x_s] - \frac{4}{x_s} H[r,r,0;\hat{\tau}_s] \end{aligned} \tag{5.20}$$

The integration is easily done, using the definition of the HPL. We need to change the variable of integration for the last term again:

$$\begin{aligned}\int^{x_s} dx_s' \frac{1}{x_s'} H[r,r,0;\hat{\tau}_s'] = &- \int^{\hat{\tau}_s} d\hat{\tau}_s' \frac{1}{\sqrt{\hat{\tau}_s'(4+\hat{\tau}_s')}} H[r,r,0;\hat{\tau}_s'] \\ = &- H[-r,r,r,0;\hat{\tau}_s] \end{aligned} \tag{5.21}$$

Thus the general solution of (5.18) is

$$\begin{aligned}F_\beta^{(0)} = &\frac{x_s^2}{(1+x_s)(1-x_s)^3} \Bigg\{ - C + 18\zeta(3) H[0;x_s] - 3H[0,0,0,0;x_s] \\ &+ 4H[0,0,0,1;x_s] + 4H[0,0,1,0;x_s] - 14H[0,1,0,0;x_s] \\ &- 4H[-r,r,r,0;\hat{\tau}_s] \Bigg\}. \end{aligned} \tag{5.22}$$

Now we determine the constant $C$. The MB representation of $I_\beta$ for $d = 4$ is given in (A.16). The expansion for small $\hat{\tau}_s$ is

$$(m^2)^2 I_\beta = -\frac{1}{6}\log(\hat{\tau}_s) + \frac{13}{36} + \mathcal{O}(\hat{\tau}_s). \tag{5.23}$$

We expand the part of the solution with the ordinary HPLs:

$$\begin{aligned}
&\frac{x_s^2}{(1+x_s)(1-x_s)^3}\Big\{ -C + 18\zeta(3)H[0;x_s] - 3H[0,0,0,0;x_s] \\
&+ 4H[0,0,0,1;x_s] + 4H[0,0,1,0;x_s] - 14H[0,1,0,0;x_s]\Big\} \\
&= \left(C + \frac{5\pi^4}{9}\right)\left(-\frac{1}{2\hat{\tau}_s^{3/2}} + \frac{1}{16\hat{\tau}_s^{1/2}}\right) + \frac{1}{6}\log(\hat{\tau}_s) - \frac{31}{36} + \mathcal{O}(\hat{\tau}_s)
\end{aligned} \tag{5.24}$$

For the HPL with indices $\pm r$ we use the approximation (5.13). This results in the following expansion for the remaining term:

$$-\frac{x_s^2}{(1+x_s)(1-x_s)^3} 4H[-r,r,r,0;\hat{\tau}_s] = -\frac{1}{3}\log(\hat{\tau}_s) + \frac{11}{9} + \mathcal{O}(\hat{\tau}_s) \tag{5.25}$$

The sum of (5.24) and (5.25) must equal (5.23). This only holds for

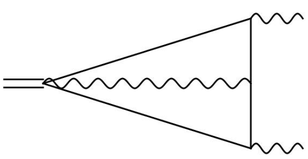
$$C = -\frac{5\pi^4}{9}. \tag{5.26}$$

### 5.1.2 Non-planar topology

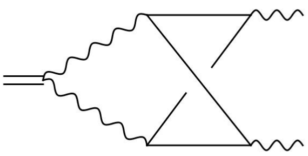
With the propagators (5.49) the non-planar topology of fig. 5.1 (b) is given by

$$\int \frac{d^d k_1}{i\pi^{d/2}} \int \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{\overline{D}_1^{a_1}\overline{D}_2^{a_2}\overline{D}_3^{a_3}\overline{D}_4^{a_4}\overline{D}_5^{a_5}\overline{D}_6^{a_6}\overline{D}_7^{a_7}} = I_{np}\{a_1,a_2,a_3,a_4,a_5,a_6,a_7\}. \tag{5.27}$$

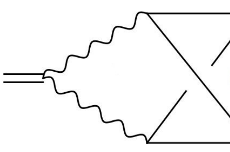
The MIs of this topology are shown in fig. 5.4. Most of the MIs are identical to those of the planar topology, e.g. $I_\alpha$ from the last section. The others are factorizable or have been calculated in [13] and [12]. There are three integrals remaining:



$= I_{np}\{0,1,1,1,1,1,0\} =: I_\gamma(s,m^2)$



$= I_{np}\{1,1,1,1,1,1,0\} =: I_\delta(s,m^2)$



$\times\, \overline{D}_7^2 = I_{np}\{1,1,1,1,1,1,-2\} =: I_\eta(s,m^2)$

Their calculation is explained in the following.

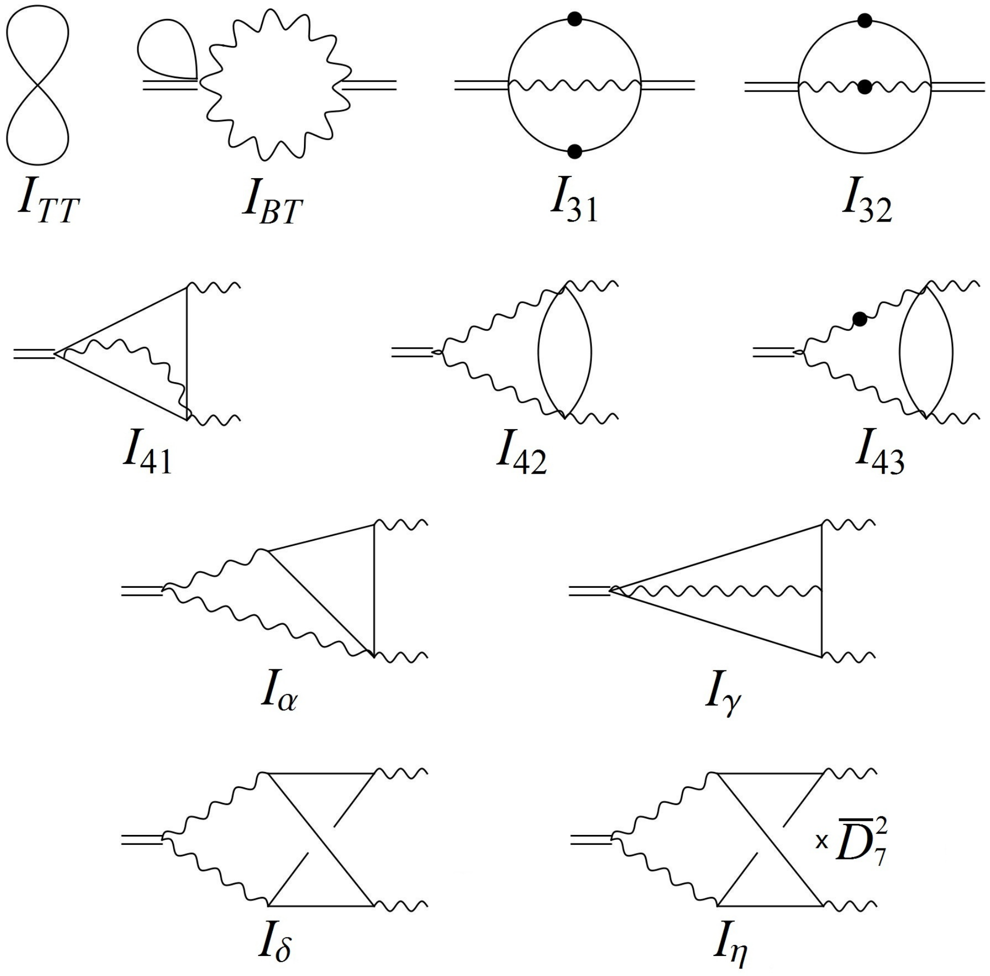



Figure 5.4: Master integrals of the non-planar topology

**Calculation of $I_\gamma$**

The differential equation in $m^2$ for the integral $I_\gamma$ after performing the IBP reduction on the r.h.s. is

$$\begin{aligned}\frac{\partial}{\partial m^2} I_\gamma =& \frac{d-4}{m^2} I_\gamma - \frac{(d-2)^2(5d-18)}{8(30-19d+3d^2)m^8} I_{TT} + \frac{(5d-18)s-4(d-4)m^2}{2(30-19d+3d^2)m^4} I_{31} \\ &- \frac{2(d-4)(4m^2+s)}{(30-19d+3d^2)m^4} I_{32} + \frac{d-2}{2m^4} I_{41}. \end{aligned} \tag{5.28}$$

The non-homogeneous term is finite in 4 dimensions. The differential equation for $F_\gamma^{(0)}$ is

$$\begin{aligned}\frac{\partial}{\partial x_s} F_\gamma^{(0)} =& \frac{1+x_s}{x_s(1-x_s)} F_\gamma^{(0)} + \frac{2(1+x_s)}{(1-x_s)^3} \{-2\zeta(3) + H[0,0,0;x_s] \\ &+2H[1,0,0;x_s]\}. \end{aligned} \tag{5.29}$$

With separation of variables we obtain the solution of the homogeneous part:

$$\hat{F}_\beta = \frac{x_s}{(1-x_s)^2}. \tag{5.30}$$

The term that has to be integrated in order to get the general solution is

$$\left(-\frac{4}{x_s} - \frac{8}{1-x_s}\right)\zeta(3) + \left(\frac{2}{x_s} + \frac{4}{1-x_s}\right) H[0,0,0;x_s] \\ + \left(\frac{4}{x_s} + \frac{8}{1-x_s}\right) H[1,0,0;x_s]. \tag{5.31}$$

Thus we arrive at

$$F_\gamma^{(0)} = \frac{x_s}{(1-x_s)^2}\Big\{C - 4\zeta(3)H[0;x_s] - 8\zeta(3)H[1;x_s] + 2H[0,0,0,0;x_s] \\ + 4H[1,0,0,0;x_s] + 4H[0,1,0,0;x_s] + 8H[1,1,0,0;x_s]\Big\}. \tag{5.32}$$

The expansion of the MB representation (A.19) for small $\hat{\tau}_s$ is

$$m^2 I_\gamma = 1 + \mathcal{O}\left(\hat{\tau}_s\right). \tag{5.33}$$

There are no HPLs with arguments $\pm r$ in this general solution. The expansion of the complete expression can be computed with the help of the `HPL` package. We get

$$F_\gamma^{(0)} = \left(C + \frac{\pi^4}{15}\right)\hat{\tau}_s^{-1} + \mathcal{O}\left(\hat{\tau}_s^0\right). \tag{5.34}$$

There is no term of this order in (5.33), so we have

$$C = -\frac{\pi^4}{15}. \tag{5.35}$$

### Calculation of $I_\delta$ and $I_\eta$

The differential equations of $I_\delta$ and $I_\eta$ are coupled. We concentrate on the homogeneous parts, with the solutions $\hat{I}_\delta$ and $\hat{I}_\eta$:

$$\frac{\partial}{\partial m^2}\hat{I}_\delta = \frac{8(d-5)m^2 + (d-3)s}{16m^4 - m^2 s}\hat{I}_\delta + \frac{14-4d}{16m^4 s - m^2 s^2}\hat{I}_\eta \tag{5.36}$$

$$\frac{\partial}{\partial m^2}\hat{I}_\eta = \frac{(2d-7)(8m^2 - s)}{16m^4 - m^2 s}\hat{I}_\eta + \frac{s^2(-4(d-1)m^2 + (d-3)s)}{32m^4 - 2m^2 s}\hat{I}_\delta \tag{5.37}$$

We build the differential equations for the corresponding dimensionless functions in the quantity $\hat{\tau}_s$. Since they are the same for every order in $\epsilon$, we can write

$$\frac{\partial}{\partial\hat{\tau}_s}\hat{F}_\delta = -\frac{24-\hat{\tau}_s}{\hat{\tau}_s(16-\hat{\tau}_s)}\hat{F}_\delta + \frac{2}{\hat{\tau}_s^2(16-\hat{\tau}_s)}\hat{F}_\eta, \tag{5.38}$$

$$\frac{\partial}{\partial\hat{\tau}_s}\hat{F}_\eta = -\frac{8-\hat{\tau}_s}{\hat{\tau}_s(16-\hat{\tau}_s)}\hat{F}_\eta + \frac{\hat{\tau}_s(12-\hat{\tau}_s)}{2(16-\hat{\tau}_s)}\hat{F}_\delta. \tag{5.39}$$

The associated second order differential equation for $\hat{F}_\delta$ is

$$\partial^2_{\hat{\tau}_s}\hat{F}_\delta = -\frac{64-5\hat{\tau}_s}{\hat{\tau}_s(16-\hat{\tau}_s)}\partial_{\hat{\tau}_s}\hat{F}_\delta - \frac{4(9-\hat{\tau}_s)}{\hat{\tau}_s^2(16-\hat{\tau}_s)}\hat{F}_\delta. \tag{5.40}$$

With the ansatz

$$\hat{F}_\delta = \frac{\bar{F}_\delta}{\hat{\tau}_s^{\frac{3}{2}}}, \tag{5.41}$$

we obtain

$$\partial^2_{\hat{\tau}_s} \bar{F}_\delta = -\frac{2(8-\hat{\tau}_s)}{\hat{\tau}_s(16-\hat{\tau}_s)} \partial_{\hat{\tau}_s} \bar{F}_\delta + \frac{1}{4\hat{\tau}_s(16-\hat{\tau}_s)} \bar{F}_\delta. \tag{5.42}$$

If we substitute $\hat{\tau}_s \to \bar{\tau}_s = 16\hat{\tau}_s^2$ we get a differential equation, which can be written in the following form:

$$\partial_{\bar{\tau}_s} \left[ \bar{\tau}_s (1-\bar{\tau}_s^2) \partial_{\bar{\tau}_s} \bar{F}_\delta \right] = \bar{\tau}_s \bar{F}_\delta \tag{5.43}$$

The solution of this differential equation is the complete elliptic integral of the first kind $\mathrm{K}(\bar{\tau}_s)$. Elliptic integrals are integrals of square roots of polynomials of degree three or four. This special type is defined as

$$\mathrm{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}. \tag{5.44}$$

A second solution to equation (5.43) is $\mathrm{K}\left(\sqrt{1-\bar{\tau}_s^2}\right)$. Thus the general solution of (5.40) is

$$\hat{F}_\delta = \frac{C_1}{\hat{\tau}_s^{\frac{3}{2}}} \mathrm{K}\left(\frac{\sqrt{\hat{\tau}_s}}{4}\right) + \frac{C_2}{\hat{\tau}_s^{\frac{3}{2}}} \mathrm{K}\left(\sqrt{1-\frac{\hat{\tau}_s}{16}}\right). \tag{5.45}$$

It is not possible to represent any master integral in terms of harmonic polylogarithms, when elliptic integrals are involved (for a general mathematical introduction, see for example [26]). Indeed finding any closed analytical representation is an unsolved problem (see for example [27][28][29][30]). In [29] similar non-planar MIs (with different mass configurations) were also found to contain elliptic integrals.

## 5.2 Results of the calculated integrals

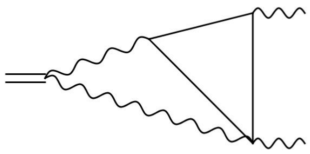
In this section we summarize the results from the last section. We note, that all calculated integrals are finite for $d = 4$. The ordinary HPLs with indices 0 and 1 are real valued in the region $s > 0$, whereas the generalized HPLs with a index $r$ are real valued in the region $0 < s < 4m^2$. To express the following integrals in other regions, one has to use analytic continuation (see for example [13]). The following MIs have also been computed numerically with `FIESTA` [31]. The analytic expressions agree fully with the numerical evaluation.

### 5.2.1 Planar topology

Propagators:

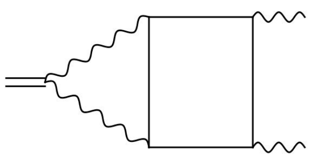
$$
\begin{aligned}
D_1 &= (k_1 + p_1 + p_2)^2 \\
D_2 &= k_1^2 \\
D_3 &= (k_2 + p_1 + p_2)^2 + m^2 \\
D_4 &= (k_2 + p_1)^2 + m^2 \\
D_5 &= k_2^2 + m^2 \\
D_6 &= (k_1 - k_2)^2 + m^2 \\
D_7 &= (k_1 + p_1)^2
\end{aligned}
\tag{5.46}
$$

Calculated MIs:



$$
\begin{aligned}
&= \int \frac{d^d k_1}{i\pi^{d/2}} \int \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{D_1 D_2 D_3 D_4 D_6} \\
&= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m^2)^{-2\epsilon-1} F_\alpha^0 + \mathcal{O}(\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
F_\alpha^{(0)} =& \frac{x_s}{(1-x_s)^2} \bigg\{ \frac{\pi^4}{30} + 2\zeta(3) H[0;x_s] + 4\zeta(3) H[1;x_s] - H[0,0,0,0;x_s] \\
&- 2H[1,0,0,0;x_s] - 2H[0,1,0,0;x_s] - 4H[1,1,0,0;x_s] \\
&- H[0,r,r,0;\hat{\tau}_s] \bigg\}
\end{aligned}
\tag{5.47}
$$



$$
\begin{aligned}
&= \int \frac{d^d k_1}{i\pi^{d/2}} \int \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6} \\
&= \left(\frac{\Gamma(1+\epsilon)}{1-\epsilon}\right)^2 (m^2)^{-2\epsilon-2} F_\beta^0 + \mathcal{O}(\epsilon)
\end{aligned}
$$

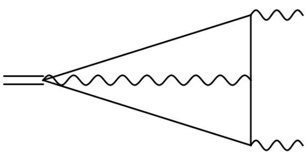
$$
\begin{aligned}
F_\beta^{(0)} =& \frac{x_s^2}{(1+x_s)(1-x_s)^3} \bigg\{ \frac{5\pi^4}{9} + 18\zeta(3) H[0;x_s] - 3H[0,0,0,0;x_s] \\
&+ 4H[0,0,0,1;x_s] + 4H[0,0,1,0;x_s] - 14H[0,1,0,0;x_s] \\
&- 4H[-r,r,r,0;\hat{\tau}_s] \bigg\}
\end{aligned}
\tag{5.48}
$$

### 5.2.2 Non-planar topology

Propagators:

$$
\begin{aligned}
\overline{D}_1 &= (k_2 - k_1 + p_2)^2 \\
\overline{D}_2 &= (k_1 - k_2 + p_1)^2 \\
\overline{D}_3 &= (k_2 + p_2)^2 + m^2 \\
\overline{D}_4 &= (k_1 + p_1)^2 + m^2 \\
\overline{D}_5 &= k_2^2 + m^2 \\
\overline{D}_6 &= k_1^2 + m^2 \\
\overline{D}_7 &= (k_1 - k_2)^2
\end{aligned}
\tag{5.49}
$$

Calculated MI:



$$
\begin{aligned}
&= \int \frac{d^d k_1}{i\pi^{d/2}} \int \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{\overline{D}_2 \overline{D}_3 \overline{D}_4 \overline{D}_5 \overline{D}_6} \\
&= \left( \frac{\Gamma(1+\epsilon)}{1-\epsilon} \right)^2 (m^2)^{-2\epsilon-1} F_\gamma^0 + \mathcal{O}(\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
F_\gamma^{(0)} = & \frac{x_s}{(1-x_s)^2} \Bigg\{ -\frac{\pi^4}{15} - 4\zeta(3) H[0; x_s] - 8\zeta(3) H[1; x_s] + 2H[0,0,0,0; x_s] \\
& + 4H[1,0,0,0; x_s] + 4H[0,1,0,0; x_s] + 8H[1,1,0,0; x_s] \Bigg\}
\end{aligned}
\tag{5.50}
$$

# Chapter 6

# Reduction of the four-point topology

In this chapter we want to consider four-point functions of the type shown in fig. 6.1. All external momenta are on-shell:

$$p_i^2 = 0 \qquad (i = 1, 2, 3) \tag{6.1}$$
$$p_4^2 = (-p_1 - p_2 - p_3)^2 = 0 \tag{6.2}$$

These integrals depend on the invariants $m^2$, $s = 2p_1 \cdot p_2$ and $t = 2p_2 \cdot p_3$. Since there are three independent external momenta and two loops, we need, according to (2.2), nine independent propagators to construct the corresponding topology. We write the topology as

$$\int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{\tilde{D}_1^{a_1} \tilde{D}_2^{a_2} \tilde{D}_3^{a_3} \tilde{D}_4^{a_4} \tilde{D}_5^{a_5} \tilde{D}_6^{a_6} \tilde{D}_7^{a_7} \tilde{D}_8^{a_8} \tilde{D}_9^{a_9}} = I_{box}\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}, \tag{6.3}$$

where $\tilde{D}_8$ and $\tilde{D}_9$ are auxiliary propagators. The diagram of this topology is in fig. 6.2. A possible choice for the propagators is

$$\begin{aligned}
\tilde{D}_1 &= k_1^2 + m^2 \\
\tilde{D}_2 &= (k_1 + p_1)^2 + m^2 \\
\tilde{D}_3 &= (k_1 + p_1 + p_2)^2 + m^2 \\
\tilde{D}_4 &= (k_2 + p_1 + p_2)^2 + m^2 \\
\tilde{D}_5 &= (k_2 + p_1 + p_2 + p_3) + m^2 \\
\tilde{D}_6 &= k_2^2 + m^2 \\
\tilde{D}_7 &= (k_1 - k_2)^2 \\
\tilde{D}_8 &= (k_1 + p_1 + p_2 + p_3)^2 + m^2 \\
\tilde{D}_9 &= (k_2 + p_1)^2 + m^2.
\end{aligned} \tag{6.4}$$

The reduction with `FIRE` gives 27 MIs. It is important to identify all symmetries of this topology. This is explained in appendix B. In fig. 6.3 are the six factorizable MIs. They can be factorized into the massive one-loop tadpole, bubble and vertex integrals, which were also of interest for the planar three-point topology (fig. 5.3). In fig. 6.4 are

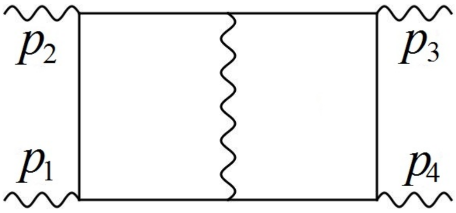



Figure 6.1: Two-loop four-point function

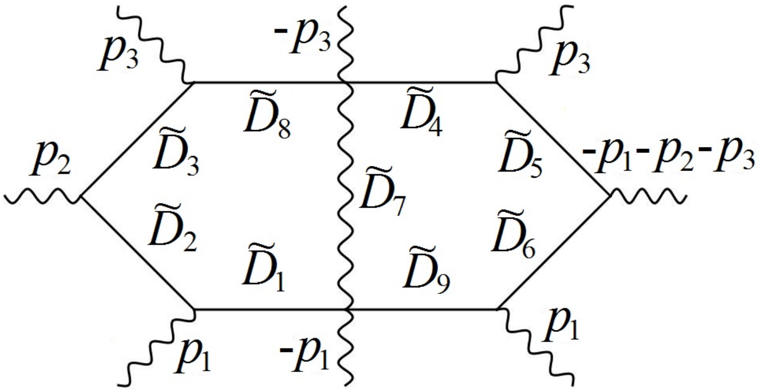



Figure 6.2: Diagram for the topology of (6.3). All external momenta are incoming. The auxiliary propagators are $\tilde{D}_8$ and $\tilde{D}_9$.

the two- and three-point MIs. Some of them occur with different external momenta, e.g. the two-point functions can have $p_1 + p_2$ or $p_2 + p_3$ as external momenta. We choose the basis to match the set of MIs in [13]. Indeed all the integral except $I_\gamma$ were calculated in this reference. Thus all two- and three-point MIs are known.

Let us concentrate on the remaining 12 four-point functions. They can be classified into the four topologies in fig. 6.5, where we omitted the auxiliary propagators. There are

- 4 MIs of the type $T_1$,
- 3 of $T_2$,
- 2 of $T_3$
- and 3 of the standard type $T_4$.

Both $T_1$ and $T_2$ can be computed separately, e.g. the IBP reduction of every integral of type $T_1$ leads to the 4 MIs of this topology and to the MIs in fig. 6.3 and fig. 6.4. The MIs of $T_1$ and of $T_2$ are necessary to calculate the MIs of $T_3$ and all MIs these three topologies are needed for the computation of the MIs of $T_4$:

$$\begin{tikzcd} T_1 \arrow[dr] & & \\ & T_3 \arrow[r] & T_4 \\ T_2 \arrow[ur] & & \end{tikzcd}$$

The considerations for the one-loop four-point function in section 4.1 also applies to the four-point MIs of this topology: The corresponding dimensionless functions depend on two independent dimensionless quantities and we need to consider one partial differential equation in each of them. For example we need to solve two systems of four

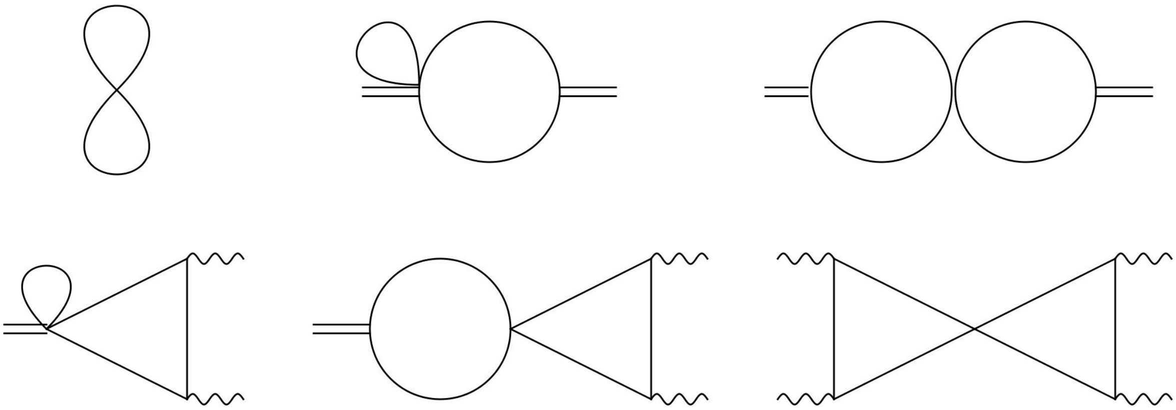
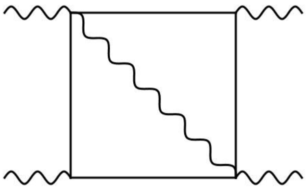


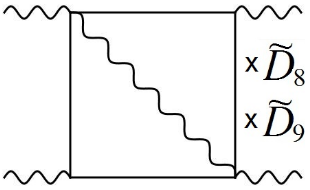
Figure 6.3: Factorizable MIs of the four-point topology

differential equations, e.g. one in $m^2/s = \tau_s$ and one in $m^2/t = \tau_t$, to obtain the MIs of $T_1$. It is possible to simplify this task by using the $\epsilon$-expansion and symmetries in $s$ and $t$.

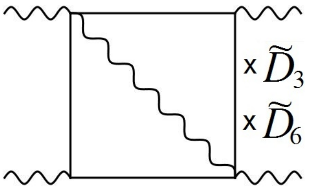
Let us start with $T_1$. A suitable choice for the MIs is



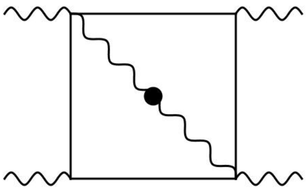
$$= I_{box}\{1,1,0,1,1,0,1,0,0\} =: I_{T_1;1}(s,t,m^2)$$




$$= I_{box}\{1,1,0,1,1,0,1,-1,-1\} =: I_{T_1;2}(s,t,m^2)$$




$$= I_{box}\{1,1,-1,1,1,-1,1,0,0\} =: I_{T_1;3}(s,t,m^2)$$



$$= I_{box}\{1,1,0,1,1,0,2,0,0\} =: I_{T_1;4}(s,t,m^2).$$

The integral $I_{T_1;1}$ is obviously symmetric with respect to the exchange $s \leftrightarrow t$. If we solve the differential equation in $\tau_s$, we automatically know the solution for the equation in $\tau_t$. The same is true for $I_{T_1;4}$. Exchanging $s$ and $t$ in $I_{T_1;2}$ yields the integral $I_{T_1;3}$ and vice versa, i.e. the solution of $I_{T_1;2}$ in $\tau_s$ with $s \leftrightarrow t$ solves the differential equation for $I_{T_1;3}$ in $\tau_t$ and so on.

Another advantage of this set of MIs is, that the differential equation of $I_{T_1;4}$ decouples within the $\epsilon$-expansion from the equations of $I_{T_1;1}$, $I_{T_1;2}$ and $I_{T_1;3}$. Hence we

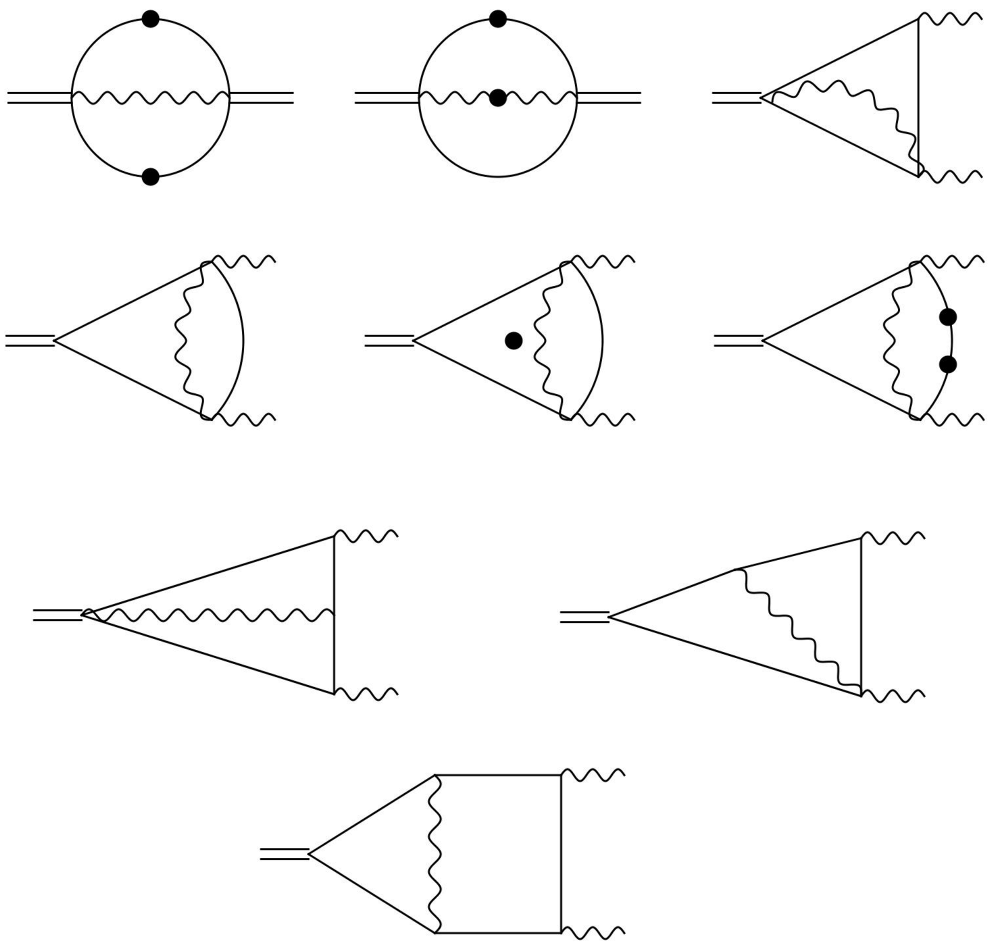
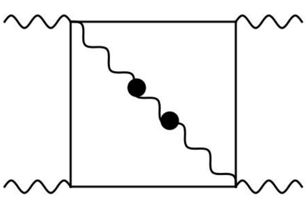


Figure 6.4: Two- and three-loop MIs of the four-point topology. The diagram with a dot in the middle contains a numerator and is defined in [13]. The double straight lines denote external legs with masses $s = 2p_1 \cdot p_2$ or $t = 2p_2 \cdot p_3$.

need to solve a third order differential equation and a single one instead of a fourth order equation.

Decoupling within the $\epsilon$-expansion means the following: Supposed we have two MIs $I_1$ and $I_2$ with coupled differential equations, where $I_2$ appears in the equation of $I_1$ with a factor of $(d-4) = 2\epsilon$. Than the equations for the coefficients of the $\epsilon$-expansion of $I_1$ only contain coefficients of lower order of $I_2$. For example the coefficient $I_1^{(p)}$ appears in the equation for $I_2^{(p)}$, but not vice versa. We need to start with the lowest order in $\epsilon$ and calculate higher orders recursively as usual. But in every order $\mathcal{O}(\epsilon^p)$ we start with the differential equation of $I_1^{(p)}$ and substitute the result in the equation of $I_2^{(p)}$.

Alternatively to $I_{T_1;2}$ or $I_{T_1;3}$ we can use the integral



$$= I_{box}\{1,1,0,1,1,0,3,0,0\} =: I_{T_1;5}(s,t,m^2)$$

as a MI, since it also decouples from $I_{T_1;1}$, $I_{T_1;2}$ and $I_{T_1;3}$. Thus we need to handle

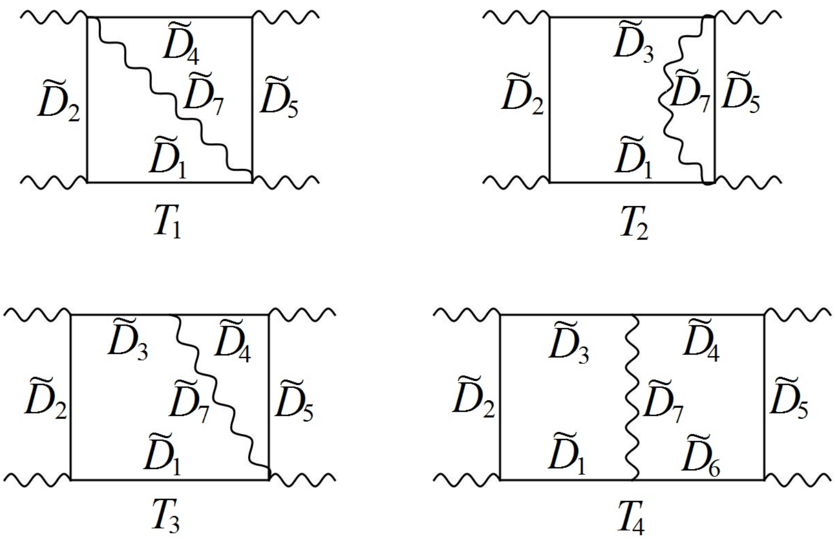
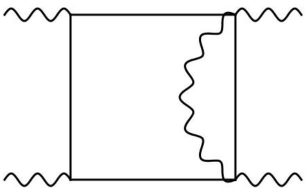


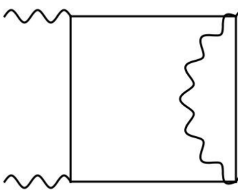

Figure 6.5: Types of four-point MIs. The external legs are according to fig. 6.2 with the missing propagators contracted to points, i.e. $p_1$, $p_2$, $p_3$ and $p_4 = -p_1 - p_2 - p_3$ beginning from the lower left leg and going clockwise (like in fig. 6.1).

two systems of two differential equations. In general this is easier to handle than a system of three equations, but of course it depends on the coefficients of the equations. It turns out that they are very extensive for the system of $I_{T_1;4}$ and $I_{T_1;5}$.

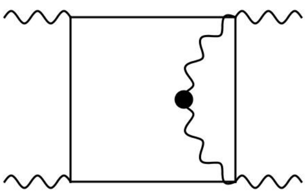
It is not possible to use any $s$-$t$-symmetries for the other topologies in fig. 6.5. But there are also integrals, whose differential equations decouple due to the $\epsilon$-expansion. A proper choice for the basis of $T_2$ is



$$= I_{box}\{1,1,1,0,1,0,1,0,0\} =: I_{T_2;1}(s,t,m^2)$$



$$\times \widetilde{D}_4 \; = I_{box}\{1,1,1,-1,1,0,1,0,0\} =: I_{T_2;2}(s,t,m^2)$$



$$= I_{box}\{1,1,1,0,1,0,2,0,0\} =: I_{T_2;3}(s,t,m^2),$$

where the differential equation of $I_{T_2;3}$ is decoupled from the system of $I_{T_2;1}$ and $I_{T_2;2}$. For the topologies in fig. 6.5 this is probably a feature of integrals with higher powers of the massless propagator and can also be used for $T_3$ and $T_4$.

The calculation of the four-point topology is much more complicated than it was for the three-point integrals in the last chapter. The number of MIs is greater and their differential equations can not always be separated. Thus systems of differential equations or higher order differential equations respectively have to be solved. In addition we need to handle partial differential equations. This results in finding a proper basis of two-dimensional HPLs.

# Chapter 7

# Conclusion

We have calculated three previously unknown three-point scalar integrals with massive propagators, which are of particular interest for $\mathcal{N} = 4$ super Yang-Mills theory. Two of them are MIs of a planar topology. Since all the other MIs are known, we can find an analytical expression for every integral of this topology. The results are presented as Laurent series in $\epsilon$ in terms of HPL and are finite in four dimensions. This is also true for the third integral, which is a MI of a non-planar topology. It is not possible to obtain an expression for the two remaining non-planar MIs in terms of HPLs, since we have shown, that they contain elliptic integrals. It is an open problem, to find a closed analytical representation for such integrals.

The method of differential equations has proven to be a useful procedure for calculating massive multi-loop integrals not only in this thesis. It is remarkable, that the computation is done without performing any loop integrations.

The solutions of simpler integrals are also of interest for more complicated ones. Especially one of the planar integrals evaluated in this thesis is necessary to compute four-point functions, which are the subject of [19]. We have performed the IBP reduction of this topology and proposed a suitable choice for the MIs.

This method also has its limits. Instead of solving loop integrations, differential equations have to be evaluated. This is not always straightforward. Systems of differential equations or higher order differential equations respectively lack in universal solution procedures. The main problem is the evaluation of the corresponding homogeneous equations. The more complex the topology, the greater the number of MIs with coupled differential equations. But we have seen, especially for the four-point integrals, that in some cases this number can be reduced by using the freedom of choosing the basis of MIs. Due to the $\epsilon$-expansion differential equations of some MIs can be separated. So it is a good criterion for the IBP reduction, to use MIs, whose differential equations decouple within this expansion.

Integrals which depend on more kinematical invariants, e.g. integrals with more external legs or different internal or external masses, require the analysis of higher dimensional systems of partial differential equations. This is the case for the four-point integrals considered in this thesis. Partial differential equations in two independent variables have to be solved. We have seen, that this task can be simplified by taking advantage of symmetries in these quantities. Furthermore two-dimensional HPLs are needed to find analytical expressions for these integrals.

# Appendix A

# Mellin-Barnes representations of the integrals

In this section we give the Mellin-Barnes (MB) representations of the integrals computed in chapter 5. A detailed description of this method can be found in [17]. We use the same strategy as in [19]. Since we need these representations to get boundary conditions for the differential equations, we use the tools `MB`, `MBasymptotics` and `barnesroutines` [22] to expand the integrals for small $s$. We describe the calculation of the integral $I_\beta$ in detail. The MB representations of $I_\alpha$ and $I_\gamma$ are achieved with the same steps. The representations are derived for $d = 4$, because we only need boundary conditions for the coefficients of the $\epsilon$-expansion of order $\mathcal{O}(\epsilon^0)$.

The required formulae are the following:

$$\begin{aligned}(a_1 + a_2 + \ldots + a_n)^{-\lambda} =& \frac{1}{\Gamma(\lambda)} \int \frac{dz_1}{2\pi i} \cdots \int \frac{dz_{n-1}}{2\pi i} \Gamma[-z_1] \ldots \Gamma[-z_{n-1}] \\ & \cdot \Gamma[z_1 + \ldots + z_{n-1} + \lambda] a_1^{z_1} \ldots a_{n-1}^{z_{n-1}} a_n^{-z_1 - \ldots - z_{n-1} - \lambda}\end{aligned} \tag{A.1}$$

$$\int_0^1 \prod_{j=1}^{n} d\alpha_j \alpha_j^{q_j - 1} \delta\left(\sum_{i=1}^{n} \alpha_i - 1\right) = \frac{\Gamma(q_1) \ldots \Gamma(q_n)}{\Gamma(q_1 + \ldots + q_n)} \tag{A.2}$$

## MB representation of $I_\beta$

The integral can be written in terms of dual coordinates:

$$I_\beta = \int \frac{d^4 x_i}{i\pi^2} \int \frac{d^4 x_j}{i\pi^2} \frac{1}{P_{1i} P_{3i} P_{1j,m} P_{2j,m} P_{3j,m} P_{ij,m}}, \tag{A.3}$$

where $P_{lk,m} = x_{lk}^2 + m^2$ and $P_{lk} = x_{lk}^2$ with $x_{lk} = x_l - x_k$. Detailed informations on this notation can be found in [19]. We start with the $x_j$ subintegral

$$I_{\beta,j} = \int \frac{d^4 x_j}{i\pi^2} \frac{1}{P_{1j,m} P_{2j,m} P_{3j,m} P_{ij,m}}, \tag{A.4}$$

so that

$$I_\beta = \int \frac{d^4 x_i}{i\pi^2} \frac{I_{\beta,j}}{P_{1i} P_{3i}}. \tag{A.5}$$

We introduce $\alpha$ parameters (see e.g. [17]) to transform the loop integrations into one-dimensional integrals:

$$I_{\beta,j} = \int_0^\infty \frac{d\alpha_{1,2,3,4}\delta(\sum_i \alpha_i - 1)}{\left[\alpha_1\alpha_3 s + \alpha_4(\alpha_1 P_{1i} + \alpha_2 P_{2i} + \alpha_3 P_{3i}) + (\alpha_1+\alpha_2+\alpha_3+\alpha_4)^2 m^2\right]^2} \tag{A.6}$$

The range of the sum in the delta function is arbitrary. We choose $\sum_{i=1}^4$ to simplify the $m^2$ term. Thus we get

$$I_{\beta,j} = \int_0^1 \frac{d\alpha_{1,2,3,4}\delta(\sum_{i=1}^4 \alpha_i - 1)}{\left[\alpha_1\alpha_3 s + \alpha_4(\alpha_1 P_{1i} + \alpha_2 P_{2i} + \alpha_3 P_{3i}) + m^2\right]^2}. \tag{A.7}$$

Now we can use (A.1) to transform the denominator:

$$\begin{aligned} I_{\beta,j} = &\int \frac{dz_{1,2,3,4}}{(2\pi i)^4}\Gamma(2+z)\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)s^{z_1}(m^2)^{-2-z} \\ &\cdot P_{1i}^{z_2}P_{2i}^{z_3}P_{3i}^{z_4}\int_0^1 d\alpha_{1,2,3,4}\delta(\sum_{i=1}^4\alpha_i - 1)\alpha_1^{z_1+z_2}\alpha_2^{z_3}\alpha_3^{z_1+z_4}\alpha_4^{z_2+z_3+z_4}, \end{aligned} \tag{A.8}$$

with $z = z_1 + z_2 + z_3 + z_4$. The $\alpha$-integrals can then be evaluated with (A.2):

$$I_{\beta,j} = \int \frac{dz_{1,2,3,4}}{(2\pi i)^4} f^{(j)}(z_{1,2,3,4}) s^{z_1}(m^2)^{-2-z} P_{1i}^{z_2}P_{2i}^{z_3}P_{3i}^{z_4}, \tag{A.9}$$

where

$$\begin{aligned} f^{(i)}(z_{1,2,3,4}) = &\frac{\Gamma(2+z)}{\Gamma(4+2z)}\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(1+z_3) \\ &\cdot\Gamma(1+z_1+z_2)\Gamma(1+z_1+z_4)\Gamma(1+z_2+z_3+z_4). \end{aligned} \tag{A.10}$$

Substituting the above integral in (A.5), we obtain

$$I_\beta = \int \frac{dz_{1,2,3,4}}{(2\pi i)^4} f^{(j)}(z_{1,2,3,4}) s^{z_1}(m^2)^{-2-z} I_{\beta,i}, \tag{A.11}$$

with the subintegral

$$I_{\beta,i} = \int \frac{d^4x_i}{i\pi^2}\frac{1}{P_{1i}^{1-z_2}P_{2i}^{-z_3}P_{3i}^{1-z_4}}. \tag{A.12}$$

With the $\alpha$-representation this can be written as

$$\begin{aligned} I_{\beta,i} = &\frac{\Gamma(-z_2-z_3-z_4)}{\Gamma(1-z_2)\Gamma(-z_3)\Gamma(1-z_4)}\int_0^\infty d\alpha_{1,2,3}\delta(\sum_i\alpha_i - 1)s^{z_2+z_3+z_4} \\ &\cdot\alpha_1^{z_3+z_4}\alpha_2^{-1-z_3}\alpha_3^{z_2+z_3}(\alpha_1+\alpha_2+\alpha_3)^{-2-z_1-z_2-z_3}. \end{aligned} \tag{A.13}$$

We choose $\sum_{i=1}^3$ as the range of the sum. As a result we do not have to introduce additional MB parameters. We can directly perform the $\alpha$-integrations by using (A.2):

$$I_{\beta,i} = s^{z_2+z_3+z_4} f^{(i)}(z_{2,3,4}), \tag{A.14}$$

where

$$f^{(i)}(z_{1,2,3,4}) = \frac{\Gamma(-z_2-z_3-z_4)\Gamma(1+z_3+z_4)\Gamma(1+z_2+z_3)}{\Gamma(1-z_2)\Gamma(1-z_4)\Gamma(2+z_2+z_3+z_4)}. \tag{A.15}$$

Plugging (A.14) into (A.11), we arrive at the result

$$I_\beta = \int \frac{dz_{1,2,3,4}}{(2\pi i)^4}\frac{1}{(m^2)^2}\left(\frac{s}{m^2}\right)^{2+z} f^{(j)}(z_{1,2,3,4}) f^{(i)}(z_{2,3,4}). \tag{A.16}$$

## MB representation of $I_\alpha$

$$I_\alpha = \int \frac{dz_{1,2}}{(2i\pi)^2} \frac{1}{m^2} \left(\frac{s}{m^2}\right)^z f(z_{1,2}) \tag{A.17}$$

with

$$f(z_{1,2}) = \frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma^3(1+z)\Gamma(1+z_1)\Gamma^2(1+z_2)\Gamma(-z)}{\Gamma(3+2z)\Gamma(1-z_1)\Gamma(2+z)} \tag{A.18}$$

and $z = z_1 + z_2$.

## MB representation of $I_\gamma$

$$I_\gamma = \int \frac{dz_{1,2,3,4}}{(2i\pi)^4} \frac{1}{m^2} \left(\frac{s}{m^2}\right)^{z_4} f(z_{1,2,3,4}) \tag{A.19}$$

with

$$\begin{aligned} f(z_{1,2,3,4}) =& \frac{\Gamma(-z_1)\Gamma^2(-z_2)\Gamma(-z_3)\Gamma(-z_4)\Gamma(1+z_4)\Gamma(1+z_2+z_4)\Gamma(z_3-2z_2)}{\Gamma(3+z_2+2z_4)\Gamma(-2z_2)} \\ & \cdot \Gamma(1+z_1+z_3)\Gamma(1+z_1+z_2+z_4)\Gamma(-z_1+z_2-z_3). \end{aligned} \tag{A.20}$$

# Appendix B

# Algorithm FIRE

## B.1 Input and Output

As an example we give the input for the IBP reduction of the massive three-point topology of section 3.1 with `IBP` and `FIRE`. The algorithm is explained in detail in [5]. Most of the variables are self-explanatory.
Input:

```
Get["ibp.m"];
Get["FIRE_3.0.0.m"];
Internal = {k}; External = {p1, p2};
Propagators = {k^2 + m^2, (k + p1)^2 + m^2,
  (k + p1 + p2)^2 + m^2};
PrepareIBP[]
reps = {p1^2 -> 0, p2^2 -> 0, p1*p2 -> s/2};
startinglist = {IBP[k, k], IBP[k, p1], IBP[k, p2]} /. reps;
SYMMETRIES = {{3, 2, 1}};
Prepare[]
Burn[]
```

Output:

```
FIRE, version 3.0.0
UsingIBP: True
UsingFermat: False
Prepared
Dimension set to 3
```

The reduction of any integral of this topology can then be started with a command like `F[2,1,1]`. The answer is a linear combination of terms looking the same way, but with `G` instead of `F`.
Input:

```
F[{2, 1, 1}]
```

Output:

```
((−2 + d) G[{0, 0, 1}])/( m^2 s (4 m^2 + s))
 + ((−4 + d) (−2 + d) G[{0, 1, 0}])/(4 m^4 s)
 + ( 2 (−3 + d) G[{1, 0, 1}])/(s (4 m^2 + s))
```

This reduction matches with (3.17), since `G[0,0,1]` and `G[0,1,0]` are identical.

## B.2 Symmetries

It is important to specify all symmetries of a topology in the variable `SYMMETRIES`. This reduces both the times for the calculations and the number of MIs. The symmetry in the example of the last section is related to the invariance with respect to the exchange of the on-shell momenta $p_1$ and $p_2$. Since this integral depends only on $s = 2p_1 \cdot p_2$ and $m^2$, this symmetry is quite obvious. One has to provide a list of possible permutations of indices for the variable `SYMMETRIES`. In this example, the replacement $p_1 \leftrightarrow p_2$ is apparently realized through exchanging the propagators $D_1$ and $D_3$ (cf. (3.2)). So the list has just one entry: `{3,2,1}`.

This symmetry also applies for the three-point topologies in fig. 5.1. From these diagrams, the corresponding permutations should also be obvious: `{2,1,5,4,3,6,7}` for the planar and `{2,1,4,3,6,5,7}` for the non-planar topology. This is the only symmetry for these three-point functions.

It is worth to take a closer look on the four-point topology in fig. 6.2. There are three independent symmetries. These integrals depend on $s = 2p_1 \cdot p_2$, $t = 2p_2 \cdot p_3$ and $m^2$, hence there is one symmetry related to

$$\begin{aligned} &p_1 \leftrightarrow p_2 \\ \text{and}\quad &p_3 \leftrightarrow p_4 = -p_1 - p_2 - p_3 \end{aligned} \tag{B.1}$$

and one related to

$$\begin{aligned} &p_1 \leftrightarrow p_4 = -p_1 - p_2 - p_3 \\ \text{and}\quad &p_2 \leftrightarrow p_3. \end{aligned} \tag{B.2}$$

In addition we have a symmetry referring to the exchange of the two loop momenta. It is not always straightforward, to obtain the corresponding permutations with the help of the diagram, like it was for the three-point functions. So we take the explicit propagators (6.4), apply the symmetric exchanges and try to transform the resulting propagators into the standard ones, e.g. for (B.1):

| | propagators in (6.4) | transformations (B.1) | $k_{1,2} \to -k_{1,2} - p_1 - p_2$ | |
|---|---|---|---|---|
| $\tilde{D}_1$ | $k_1^2 + m^2$ | $k_1^2 + m^2$ | $(-k_1 - p_{12})^2 + m^2$ | $\tilde{D}_3$ |
| $\tilde{D}_2$ | $(k_1 + p_1)^2 + m^2$ | $(k_1 + p_2)^2 + m^2$ | $(-k_1 - p_1)^2 + m^2$ | $\tilde{D}_2$ |
| $\tilde{D}_3$ | $(k_1 + p_{12})^2 + m^2$ | $(k_1 + p_{12})^2 + m^2$ | $(-k_1)^2 + m^2$ | $\tilde{D}_1$ |
| $\tilde{D}_4$ | $(k_2 + p_{12})^2 + m^2$ | $(k_2 + p_{12})^2 + m^2$ | $(-k_2)^2 + m^2$ | $\tilde{D}_6$ |
| $\tilde{D}_5$ | $(k_2 + p_{123}) + m^2$ | $(k_2 - p_3) + m^2$ | $(-k_2 - p_{123}) + m^2$ | $\tilde{D}_5$ |
| $\tilde{D}_6$ | $k_2^2 + m^2$ | $k_2^2 + m^2$ | $(-k_2 - p_{12})^2 + m^2$ | $\tilde{D}_4$ |
| $\tilde{D}_7$ | $(k_1 - k_2)^2$ | $(k_1 - k_2)^2$ | $(-k_1 + k_2)^2$ | $\tilde{D}_7$ |
| $\tilde{D}_8$ | $(k_1 + p_{123})^2 + m^2$ | $(k_1 - p_3)^2 + m^2$ | $(-k_1 - p_{123})^2 + m^2$ | $\tilde{D}_8$ |
| $\tilde{D}_9$ | $(k_2 + p_1)^2 + m^2$ | $(k_2 + p_2)^2 + m^2$ | $(-k_2 - p_1)^2 + m^2$ | $\tilde{D}_9$ |

with $p_{ij} = p_i + p_j$. The transformations in the column to the right are allowed, due to the symmetric limits of integration and the invariance in shifting the loop momentum. So we can directly read the permutation, which is related to (B.1): `{3,2,1,6,5,4,7,8,9}`. This also works for the other symmetries. Since the variable `SYMMETRIES` has to contain all symmetries, not only the independent ones, the complete input for the four-point topology should be

```
{{3, 2, 1, 6, 5, 4, 7, 8, 9}, {6, 9, 4, 3, 8, 1, 7, 5, 2},
 {1, 8, 3, 4, 9, 6, 7, 2, 5}, {4, 9, 6, 1, 8, 3, 7, 5, 2},
 {3, 8, 1, 6, 9, 4, 7, 2, 5}, {6, 5, 4, 3, 2, 1, 7, 9, 8},
 {4, 5, 6, 1, 2, 3, 7, 9, 8}}
```

We note, that the symmetry, which is related to the exchange of the loop momenta, applies only if one chooses the propagators $\tilde{D}_8$ and $\tilde{D}_9$ to be massive. It can also be seen in fig. 6.2, that the diagram is less symmetric for massless $\tilde{D}_8$ and $\tilde{D}_9$. Therefore it is useful to choose a mass configuration for the auxiliary propagators, which yields a higher symmetry of the topology.

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# Hilfsmittel

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# Eigenständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Berlin, den 06.03.2012