Minimal surfaces in AdS_n and gluon scattering amplitudes via AdS/CFT

Diplomarbeit

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Berlin, den 15. September 2009

Acknowledgments

First of all I would like to take this opportunity to thank my parents Wilfried and Regina Wuttke for their constant love and support. Furthermore, I want to thank my deceased grandfather Walter Habel for his love and great support, too.

I wish to express my appreciation to Dr. Harald Dorn for supervising my thesis work. During the last year he was always patient and available to answer my questions. His concern for my progress far exceeded what any student could expect from a supervisor.

Prof. Jan Plefka permitted me to graduate in his group and served as the second reviewer of my thesis. I also wish to thank him supporting my work. Thanks also to Dr. George Jorjadze for helpful conversations and comments. The whole group offered a congenial atmosphere for work.

Additionally, I would like to express my gratefulness to colleagues of mine and especially to Markus Hihn and Christian Schön for their moral support, friendship and inspiring conversations about various topics. Thank you to Tina Pietsch for proof reading. Finally, I want to express my appreciation to my dear friend Kati Pietsch for her love, patience and support.

Contents

1 Introduction

This diploma thesis deals with the aspect of gluon scattering amplitudes in the AdS/CFT correspondence. The AdS/CFT correspondence is a conjecture that was established by Juan Maldacena [Mal98]. This conjecture relates four dimensional $\mathcal{N} = 4$ super Yang-Mills theory with ten dimensional type IIb string theory with $AdS_5 \times S^5$ background. The space $AdS_5 \times S^5$ admits a conformal boundary which is $S^1 \times S^3$ with S^1 being a timelike component. But this means that there are closed timelike curves on this conformal boundary which is physically counterintuitive. But the universal covering of this conformal boundary is $\mathbb{R} \times S^3$ which is the so called "static Einstein universe". One half of the universal covering of the conformal boundary of $AdS_5 \times S^5$ is conformal to a Minkowski space which is conjectured to be the physical space for the $\mathcal{N} = 4$ super Yang-Mills theory. The conformal symmetry of the field theory is related to the isometry group of AdS_5. Isometry transformations in AdS_5 act as conformal transformations on the conformal boundary. So the subject of AdS/CFT correspondence is to develop a dictionary between quantities from the string theory and field theoretically quantities. The parameters λ ('t Hooft coupling) and N (dimension of the gauge group) from gauge theory are related with string tension α' and the Radius of S^5 and AdS_5 through

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'} , \qquad \frac{1}{N} \propto g_s$$

where g_s is the string coupling constant. One would also like to have a translation of gluon scattering amplitudes on the gauge theory side into some quantities on the other side of this duality. Scattering amplitudes are very important quantities in field theories. It is conjectured in [AM07] that these gluon amplitudes correspond to certain minimal surfaces in $AdS_5 \times S^5$. To be more precise the minimal surfaces correspond to MHV amplitudes (maximally helicity violating amplitudes). One can perform a color decomposition of gluon scattering amplitudes and factor off the color structure. These amplitudes, that do not carry color indices anymore, are called color ordered amplitudes. Amplitudes corresponding to the case where all gluons have the same helicity alignment or where just one gluon has the opposite helicity alignment are zero. So the maximum helicity violating amplitudes are those where two gluons have the opposite helicity alignment. It is conjectured in [AM07] that the counterparts for these MHV amplitudes are certain minimal surfaces in AdS_5 that we will be dealing with in this diploma thesis.

1.1 The correspondence between gluon scattering amplitudes and spacelike minimal surfaces

We will start with the description of the problem. The MHV amplitudes depend on the momenta of the n gluons. As we are interested in on shell amplitudes these momenta are all lightlike. Because of momentum conservation the momenta form a closed lightlike polygon in the Minkowski space which is a part of the conformal boundary of $AdS_5 \times S^5$. Now we can look for minimal surfaces in $AdS_5 \times S^5$ that reproduce this closed lightlike polygon on the conformal boundary. It is convenient to look for surfaces inside AdS_5 (with a trivial factor in S^5). Then the color ordered planar scattering amplitude for n gluons at strong coupling is of the form

$$\mathcal{A} \sim e^{iS_{cl}} = e^{-\frac{\sqrt{\lambda}}{2}\text{Area}}$$

where S_{cl} is the value of the classical action, i.e. proportional to the area of the solution. Of course this area is divergent so one has to regularize it. So we are interested in the dependence of the regularized area on the kinematical variables of the scattering process. Without regularization the area would be conformally invariant as the isometry group of AdS_n acts as conformal group on the conformal boundary of AdS_n. Some of the dependence on kinematical variables comes from the breaking of conformal symmetry by introducing a regularization. But for configurations with a large number of cusps the conformal group (isometries of AdS_5) is just not big enough thus there are some conformally invariant kinematical parameters.

The problem of finding the surface for a given contour on the conformal boundary is very hard. In the Euclidean case this "Plateau problem" is well understood. For every closed contour in $\mathbb{R}^3$ there is exactly one minimal surface that has the given boundary. Because this correspondence is one to one, we do not explicitely need to calculate the surface in order to calculate its area. There is the Douglas functional which is an integral over the closed contour that allows to calculate the area without finding a suitable minimal surface first. These methods do not work in our case. Additionally there is not much mathematical literature on minimal surfaces in noncompact, curved and Lorentzian spacetimes.

A first solution to the problem appeared in [AM07] by Alday and Maldacena for the tetragon. This solution we will review in chapter 2. In addition we work out an alternative regularization of the tetragon. In chapter 3 we introduce a Pohlmeyer reduction for AdS_n in a similar way we did in [DJW09]. This procedure was first used in [PR79],[Poh76] for an $O(N)$ sigma model.

We will show that there is up to isometries of AdS_5 only one spacelike flat minimal surface and extend this proof for general AdS_n in a following section. This is an interesting point, as the tetragon solution is flat - thus this emphasizes the special role of this solution. We will also examine all lightlike and spacelike minimal surfaces in AdS_5, AdS_4 and AdS_3. In addition to our paper we show that the integrability condition in the Pohlmeyer reduction simply is given by the set of Gauss-, Codazzi-Mainardi- and Ricci-equation for minimal surfaces and thus that the formalism has a clear mathematical interpretation. We also give a characterization for all constantly curved timelike minimal surfaces in AdS_5 which has some similarities with the Weierstrass representation of minimal surfaces in $\mathbb{R}^3$. In our section about invariants we will proof a very interesting formula for all minimal surfaces in AdS_n that relate invariant quantities from outer geometry to the curvature. Chapter 4 is dedicated to the generic n-gon case. We will also summarize some recent results in the AdS_3 case and the octagon that was studied by Alday and Maldacena in [AM09b]. Some explicit calculations can be found in the appendix. In the next section we will start to review some geometric objects and introduce notation. An introduction of geometric quantities and notations will be provided in the next section. The first chapter ends with the introduction of AdS_n and the conformal boundary in more detail.

1.2 Some geometry

Whenever a manifold M is embedded in a manifold (N, g) we can split the tangent space of N in every point of M

$$\mathrm{T}_pN = \mathrm{T}_pM \oplus \mathrm{N}_pM \tag{1}$$

Thus we can introduce a metric on the tangent space of M

$$g_M = g_{|\mathrm{T}M} \tag{2}$$

as the restriction to the tangent bundle of M. On N we have the Levi-Civita connection ∇ that is uniquely determined by the metric via the Koszul formula. The Levi-Civita connection on $\mathbb{R}^n$ is the ordinary derivative and we will denote it by

$$\nabla^{\mathbb{R}^n}_X Y = X \lrcorner dY =: X(Y) \tag{3}$$

where d means the exterior derivative and $\lrcorner$ the inner product. Using this covariant derivative on N, we can write for two vectorfields $X, Y \in \Gamma(\mathrm{T}M)$

$$\nabla_X Y = \underbrace{\mathrm{pr}_{\mathrm{T}M}\left(\nabla_X Y\right)}_{=:\nabla^M_X Y} + \underbrace{\mathrm{pr}_{\mathrm{N}M}\left(\nabla_X Y\right)}_{=:\mathrm{II}(X,Y)} \tag{4}$$

This formula can be seen as a definition for the ("induced") covariant derivative which is the Levi-Civita connection on M and for the second fundamental form. We use the terms "covariant derivative" and "connection" synonymously. Then the second fundamental form applied on two vectorfields is a vectorfield in $\mathrm{N}M$. If we fix a choice of a (orthonormal) local base $B_i \in \mathrm{N}M$ we can project $\mathrm{II}(X, Y)$ onto the normal fields and we will find $\dim(\mathrm{N}M)$ real valued second fundamental forms II_i. A surface is called a "minimal surface" if and only if for all $i \in \{1, \ldots, \dim(\mathrm{N}M)\} \ : \ \mathrm{tr}(\mathrm{II}_i) \equiv 0$. The second fundamental forms II_i are symmetric bilinear forms. Using that ∇ is a metric connection, we have for two tangential vectorfields

$$\begin{aligned} \mathrm{II}_i(V, W) &= \langle \nabla_V W, B_i \rangle = V\left(\langle W, B_i \rangle\right) - \langle W, \nabla_V B_i \rangle \\ &= -\langle W, \nabla_V B_i \rangle \end{aligned} \tag{5}$$

Here we used that the Levi-Civita connection is metric. If we consider a manifold that is embedded in $\mathbb{R}^n$, the last term becomes

$$\langle W, \nabla_V B_i \rangle = \langle W, V(B_i) \rangle = \langle W, V \lrcorner dB_i \rangle \tag{6}$$

which we will use later to show that the embedding $AdS_{n-1} \subset AdS_n$ is geodesic.

Minmal surfaces can be introduced as surfaces whose second fundamental forms are traceless or equivalently as stationary points of the area functional (which is more intuitive in string theory). The variation of the area functional leads to

$$g^{\mu\nu}\left(\nabla_\mu \partial_\nu Y^k(\sigma, \tau) + \partial_\mu Y^j \partial_\nu Y^l \Gamma^k_{jl}(Y(\sigma, \tau))\right) = 0 \tag{7}$$

Here the Christoffel symbol is associated with the covariant derivative in the ambient space. ∇ refers to the induced connection on the surface. $g_{\mu\nu}$ is the induced metric on the surface. It is always possible to choose a conformal parameterization of a surface (sometimes called "isothermal" coordinates) such that the induced metric reads $g_{\mu\nu} = f(\sigma, \tau)\delta_{\mu\nu}$ (or $\eta_{\mu\nu}$ in the timelike case) where $f(\sigma, \tau)$ is strictly positive. In these coordinates the equation of motion reads

$$\Delta Y^k - 2\sqrt{\det g}\, Y^k = 0 \tag{8}$$

where Δ is the flat Laplace operator on $\mathbb{R}^2$. If we introduce the coordinates z and $\bar{z}$ such that $\partial = \partial_z = \partial_\sigma - i\partial_\tau$ and $\bar{\partial} = \partial_{\bar{z}} = \partial_\sigma + i\partial_\tau$ in the spacelike case (or $\partial = \partial_z = \partial_\sigma + \partial_\tau$ and $\bar{\partial} = \partial_{\bar{z}} = \partial_\sigma - \partial_\tau$ in the timelike case) this equation reads

$$\partial\bar{\partial}Y - \langle \bar{\partial}Y, \partial Y \rangle Y = 0 \tag{9}$$

In these coordinates $\langle \bar{\partial} Y, \partial Y \rangle = 2\sqrt{g} = 2f(\sigma,\tau)$. If we calculate the curvature tensor for the metric $g_{\mu,\nu} = f(\sigma,\tau)\delta_{\mu,\nu}$ we find that the only independent entry of the curvature tensor reads

$$R^{\sigma}_{\ \tau,\tau,\sigma} = -\frac{(\partial_\tau f)^2 + \epsilon(\partial_\sigma f)^2 - f(\epsilon\partial_\tau\partial_\tau f + \partial_\sigma\partial_\sigma f)}{2f^2} \tag{10}$$

From here we find that the scalar curvature is given by

$$R = \frac{\epsilon(\partial_\tau f)^2 + (\partial_\sigma f)^2 - f(\epsilon\partial_\tau\partial_\tau f + \partial_\sigma\partial_\sigma f)}{f^3} = -2e^{-\alpha}\partial\bar{\partial}\alpha \tag{11}$$

Here ϵ is one for Euclidean surfaces and minus one for Lorentzian surfaces. We are dealing with conformally parameterized surfaces. Next we show that a holomorphic reparameterization does not disturb conformal gauge. If the surface is given by $Y(\sigma,\tau)$ and we assume $\langle \partial_\sigma Y, \partial_\sigma Y \rangle = \langle \partial_\tau Y, \partial_\tau Y \rangle = f(\sigma,\tau)$ and $\langle \partial_\sigma Y, \partial_\tau Y \rangle = 0$. Now we reparameterize the surface by $\sigma(s,t)$ and $\tau(s,t)$. Calculating the metric we find that two equations have to be fulfilled for the new induced metric to be conformal:

$$(\partial_s\sigma)^2 + (\partial_s\tau)^2 = (\partial_t\sigma)^2 + (\partial_t\tau)^2$$
$$\partial_s\sigma\partial_t\sigma = \partial_s\tau\partial_t\tau$$

which is fulfilled if

$$\partial_s\sigma = \partial_t\tau \qquad \partial_t\sigma = -\partial_s\tau$$

which are the Cauchy-Riemann differential equations. This means, that conformal gauge is preserved for holomorphic reparameterization.

For (nonflat) minimal surfaces in $\mathbb{R}^3$ there is the Weierstrass representation (see for example [AF01] for a proof). If we identify the parameter space (σ,τ) with the complex plane $\mathbb{C}$ via $z = \frac{1}{2}(\tau + i\sigma)$ we can parameterize all nonflat minimal surfaces in $\mathbb{R}^3$ with two holomorphic functions $f(z)$ and $g(z)$. The coordinate representation is given by

$$F = \mathrm{Re}\left(\int \frac{f(z)}{2}(1-g^2(z))dz, \int \frac{f(z)}{2}(1+g^2(z))dz, \int f(z)g(z)dz\right) \tag{12}$$

The flat minimal surfaces in $\mathbb{R}^n$ are planes, i.e. an geodesically embedded $\mathbb{R}^2$.

If we have surface embedded in $\mathbb{R}^n$ (which is always possible due to the embedding theorem of Whitney and Nash), we usually consider an induced metric to be a tensorfield on the parameter space $\mathbb{R}^2$ (for surfaces), because we calculate the pull-back from the tangent space of the surface. Also a second fundamental form is usually given as a tensorfield on $\mathbb{R}^2$. So there is a mathematical question of "integrability". Is it possible to find a map $F: \mathbb{R}^2 \mapsto \mathbb{R}^n$ for given symmetric tensorfields $\tilde{\mathrm{II}}_i$ and a given symmetric positive definite tensorfield $\tilde{g}$ such that the induced second fundamental forms and the induced metric are equal to the given tensorfields, i.e. $F^*g = \tilde{g}$ and $F^*\mathrm{II}_i = \tilde{\mathrm{II}}_i$ on $\mathbb{R}^2$ (at least locally)? This is the case when the tensorfields obey the Gauss-, Codazzi-Mainardi- and Ricci- equation. Then the map F is defined up to isometries of $\mathbb{R}^n$. We will see later that this mathematical integrability condition has a counterpart in our Pohlmeyer reduction in chapter 3. In the next section we review some facts about AdS_n.

1.3 AdS_n and conformal boundary

There are several useful coordinate charts for AdS_n and its conformal boundary, which shall be introduced in this section. AdS_n is given by the set of all points in $\mathbb{R}^{(2,n-1)}$ that satisfy

$$\langle X, X\rangle = -1 \tag{13}$$

where $\mathbb{R}^{(2,n-1)}$ is the $(n+1)$ dimensional real space equipped with the standard scalar product $\langle ., .\rangle$ of index 2. We will index the components of a vector $X \in \mathbb{R}^{(2,n-1)}$ with $i \in \{-1, 0, 1 \ldots n-1\}$. It shall be mentioned that AdS_n is also a homogeneous space. By definition it is obvious that $O(2, n-1)$ acts transitively on AdS_n and that the stabilizer of a point under this action is a conjugacy class of $O(1, n-1)$. The isometry group of AdS_n is $O(2, n-1)$. AdS_n has constant negative scalar curvature $R = -n(n-1)$ and constant sectional curvature -1. We can parameterize AdS_n by

$$\begin{pmatrix} \cosh\alpha\cos\beta \\ \cosh\alpha\sin\beta \\ \sinh\alpha\ \Omega_1 \\ \sinh\alpha\ \Omega_2 \\ \vdots \\ \sinh\alpha\ \Omega_{n-1} \end{pmatrix} = \begin{pmatrix} X_{-1} \\ X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{n-1} \end{pmatrix} \tag{14}$$

where the Ω_i are $(n-1)$ functions of $(n-2)$ parameters that parameterize the $(n-2)$ dimensional unit sphere, $\alpha \in [0, \infty)$ and $\beta \in [0, 2\pi)$. The universal cover of AdS_n is obtained by allowing $\beta \in \mathbb{R}$. Thus the induced metric on AdS_n is

$$G = d\alpha^2 - \cosh^2\alpha d\beta^2 + \sinh^2\alpha \sum_i d\Omega_i^2 \tag{15}$$

Performing a variable transformation yields

$$\begin{aligned}\sinh\alpha &= \tan\theta \\ \cosh^2\alpha &= 1+\tan^2\theta = \frac{1}{\cos^2\theta} \\ d\alpha &= \frac{1}{\cos\theta}d\theta\end{aligned} \tag{16}$$

Then the induced metric reads

$$G = \frac{1}{\cos^2\theta}(d\theta^2 - d\beta^2 + \sin^2\theta\sum_i d\Omega_i^2) \tag{17}$$

The transformation $\sinh\alpha = \tan\theta$ implies that $\theta \in [0, \frac{\pi}{2})$. In the limit $\theta \to \frac{\pi}{2}$ the metric has a divergent conformal factor $\frac{1}{\cos^2\theta}$. If we consider the metric of AdS_n in the same conformal class without this divergent factor, we can introduce the conformal boundary of AdS_n as the limit $\theta \to \frac{\pi}{2}$ and the metric is now well defined on the conformal boundary. Furthermore we observe that the conformal boundary simply is $S^1 \times S^{n-2}$ and its universal covering is $\mathbb{R}^1 \times S^{n-2}$. The conformal group on the conformal boundary of AdS_n is the isometry group of AdS_n which is $O(2, n-1)$. If we consider the universal covering of the conformal boundary, the conformal group is $\widetilde{O(2, n-1)}$, which is a $\mathbb{Z}$ -fibration over $O(2, n-1)$.

However, there is another interesting point of view on the conformal boundary of AdS_n. In Poincaré coordinates we explicitly see that a part of the conformal boundary of AdS_n is conformal to a Minkowski space. The Poincaré coordinate patch is given by

$$\begin{aligned}X^\mu &= \frac{x^\mu}{r} \qquad \mu \in \{0, 1\dots n-2\} \\ X_{-1} + X_{n-1} &= \frac{1}{r} \\ X_{-1} - X_{n-1} &= \frac{r^2 - x_0^2 + x_1^2 + \dots + x_{n-2}^2}{r}\end{aligned} \tag{18}$$

These equations satisfy (13). However, $r \in \mathbb{R}\backslash\{0\}$ and therefore we have a region with positive and negative r that cover both parts of AdS_n space that is cut in two pieces by the $X_{-1} + X_{n-1} = 0$ hypersurface. The induced metric reads

$$G = \frac{1}{r^2}(-dx_0^2 + dx_1^2 + \dots + dx_{n-2}^2 + dr^2) \tag{19}$$

In this coordinate patch we approach the conformal boundary if we take the limit $r \to 0$ and take the metric in the same conformal class without the divergent $\frac{1}{r^2}$ term. Thus we see that a part of the conformal boundary of AdS_n is a $(n-1)$ dimensional Minkowski space.

The canonical embedding $AdS_{n-1} \subset AdS_n \subset \mathbb{R}^{(2,n-1)}$ of AdS_{n-1} in AdS_n is geodesic. This means that every minimal surface in AdS_{n-1} will also be a minimal surface in AdS_n. We will make use of this fact in the following sections. An embedding is geodesic if and only if the second fundamental forms that correspond to this embedding are zero. This implies that they are also traceless which ensures that minimal surfaces in the lower dimensional space are also minimal in the big space. We embed AdS_{n-1} in AdS_n via $AdS_{n-1} = AdS_n \bigcap \{\vec{X} \in \mathbb{R}^{(2,n-1)} \mid X_{n-1} = 0\}$. But obviously the vector $N = (0, 0, \ldots, 1)$ is orthonormal to the hypersurface and therefore also orthonormal to the tangent space of AdS_{n-1}. So we can express the second fundamental form that corresponds to this embedding as

$$\begin{aligned} \mathrm{II}(V, W) = \langle \nabla_V W, N \rangle &= V(\underbrace{\langle W, N \rangle}_{=0}) - \langle W, \nabla_V N \rangle \\ &= -\langle W, V(N) \rangle = -\langle W, V \lrcorner dN \rangle \end{aligned} \tag{20}$$

But we see that

$$dN = d(0, 0, \ldots, 1) = 0 \tag{21}$$

This means that the second fundamental form of this embedding is zero and that the embedding is geodesic. So all minimal surfaces in AdS_m are also minimal in AdS_n for all $m \leq n$.

2 The four point amplitude

In this section we examine the surfaces that correspond to four point amplitudes. In the generic case we have an arbitrary closed lightlike polygon with four cusps given on the conformal boundary of AdS_5 and we are looking for a minimal surface that reproduces the given contour on the conformal boundary of AdS_5. As we have seen in the previous chapter, the embedding of $AdS_3 \subset AdS_5$ is geodesic, so the minimal surfaces that we find in AdS_3 are also minimal in AdS_5 with respect to the canonical embedding $X_5 = X_6 = 0$. So looking for minimal surfaces inside AdS_3 will provide solutions in AdS_5, although this is not the generic configuration. We begin to construct a minimal surface corresponding to a lightlike cusp. These calculations have been done in [AM07].

2.1 The lightlike cusp

We are interested in finding a minimal surface inside AdS_3 that ends on $x_0 = \pm x_1$ on the conformal boundary of AdS_3. The following ansatz has the right boost and scaling symmetry of the problem

$$x_0 = e^\tau \cosh\sigma \quad x_1 = e^\tau \sinh\sigma \quad r = e^\tau w(\tau) \tag{22}$$

The equation for $w(\tau)$ is derived from the variation of the Nambu-Goto

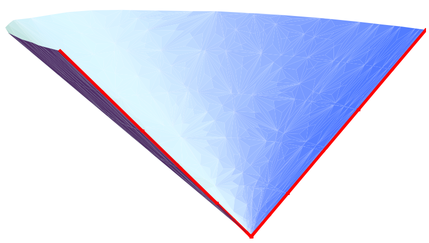

Figure 1: one cusp solution in Poincaré coordinates

action. The Nambu-Goto action simply reads

$$A = \int d\sigma d\tau \sqrt{-\det(\tilde{g})} \tag{23}$$

Here $\tilde{g}$ denotes the induced metric on the surface. Now we start to evaluate this action for the ansatz (22) .

$$\begin{aligned}
A &= \int d\sigma d\tau \sqrt{-\langle\partial_\tau, \partial_\tau\rangle\langle\partial_\sigma, \partial_\sigma\rangle + \langle\partial_\tau, \partial_\sigma\rangle^2} \\
\langle\partial_\tau, \partial_\tau\rangle &= \frac{1}{w(\tau)^2}(-1 + (w(\tau) + \dot{w}(\tau))) \\
\langle\partial_\sigma, \partial_\sigma\rangle &= \frac{1}{w(\tau)^2} \\
\langle\partial_\sigma, \partial_\tau\rangle &= 0 \\
\Rightarrow A &= \int d\sigma d\tau \frac{\sqrt{1 - (w(\tau) + \dot{w}(\tau))^2}}{w(\tau)^2}
\end{aligned}$$

If we vary $w(\tau)$ for this action, we find the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{w + \dot{w}}{w^2\sqrt{1 - (w + \dot{w})^2}} = \frac{w\dot{w} + w^2 + 2(1 - (w + \dot{w})^2)}{w^3\sqrt{1 - (w + \dot{w})^2}} \tag{24}$$

This differential equation is solved by $w(\tau) = \sqrt{2}$. This choice leads to a purely imaginary action. But this is only due to (23), which is the action for lightlike surfaces (string solutions). Our solution here is spacelike. So the right action would be $\int d\sigma d\tau \sqrt{|\ \mathrm{det} g\ |}$. From our calculation which can be found in the appendix 6.1 can be gathered, that this solution really is a solution of the equation of motion for the full Nambu-Goto action. Thus the surface is given by the equation

$$r = \sqrt{2}\sqrt{x_0^2 - x_1^2} \tag{25}$$

Using embedding coordinates of $\mathbb{R}^{(2,4)}$, the surface is given by

$$X_0^2 - X_{-1}^2 = X_1^2 - X_4^2, \quad X_2 = X_3 = 0 \tag{26}$$

2.2 Four light-like segments solution

We now start to consider a surface with four cusps that is a subspace of $AdS_4 \subset AdS_5$ by setting $x_3 = 0$. So the set of coordinates for this AdS_4 is

(r, x_0, x_1, x_2). We assume, that we can use (x_1, x_2) as the parameterization space of the surface. The metric of AdS_4 reads

$$ds^2 = \frac{-dx_0^2 + dx_1^2 + dx_2^2 + dr^2}{r^2} \tag{27}$$

The action is the same as (23) with the induced metric $\tilde{g}$. The components of $\tilde{g}$ are

$$\begin{aligned} \langle\partial_1, \partial_1\rangle &= \frac{1}{r^2}((\partial_1 r)^2 - (\partial_1 x_0)^2 + 1) \\ \langle\partial_2, \partial_2\rangle &= \frac{1}{r^2}((\partial_2 r)^2 - (\partial_2 x_0)^2 + 1) \\ \langle\partial_1, \partial_2\rangle &= \frac{1}{r^2}(\partial_1 r \partial_2 r - \partial_1 x_0 \partial_2 x_0) \end{aligned} \tag{28}$$

This leads to the action

$$iA = \int dx_1 dx_2 \frac{\sqrt{1 + (\partial_1 r)^2 + (\partial_2 r)^2 - (\partial_1 x_0)^2 - (\partial_2 x_0)^2 - (\partial_1 r \partial_2 x_0 - \partial_2 r \partial_1 x_0)^2}}{r^2} \tag{29}$$

We choose the cusps of the square to be at $(x_1, x_2) = (\pm 1, \pm 1)$. Thus the boundary conditions are

$$r(\pm 1, x_2) = r(x_1, \pm 1) = 0, \quad x_0(\pm 1, x_2) = \pm x_2, \quad x_0(x_1, \pm 1) = \pm x_1 \tag{30}$$

Again, we guess a solution that has the right behavior near the cusps.

$$x_0(x_1, x_2) = x_1 x_2, \quad r(x_1, x_2) = \sqrt{(1 - x_1^2)(1 - x_2^2)} \tag{31}$$

In the appendix 6.2 we verify that this is also a solution to the equations of motion. This solution (31) can also be expressed using embedding coordinates.

$$X_3 = X_4 = 0, \quad X_0 X_{-1} = X_1 X_2 \tag{32}$$

Remarkably, (32) and (26) are up to a $SO(2,4)$ transformation the same solution. To get from (26) to (32) we can use the transformation $X_3 \to X_3$, $X_2 \to -X_4$, $X_0 \to \frac{1}{\sqrt{2}}(X_0 + X_{-1})$, $X_{-1} \to \frac{1}{\sqrt{2}}(X_0 - X_{-1})$, $X_1 \to \frac{1}{\sqrt{2}}(X_1 + X_2)$, $X_4 \to \frac{1}{\sqrt{2}}(X_1 - X_2)$. This surface lies in an AdS_3 subspace. However, just one cusp lies at a finite position in a Poincaré patch. So we went one dimension higher and performed an isometry transformation such that all four cusps are now contained in a single Poincaré patch of AdS_4. So far we only discussed the special case $s = t$. To get a solution for general s and t, we perform $SO(2,4)$ transformations on (32). This will be discussed in the following section.

2.3 Boosting the surface and calculation of the area

We start to compute the induced metric on the surface. Starting with (28) we find

$$
\begin{aligned}
\langle \partial_1, \partial_1 \rangle &= \frac{1}{r^2}((\partial_1 r)^2 - (\partial_1 x_0)^2 + 1) = \frac{1}{(1-x_1^2)^2} \\
\langle \partial_2, \partial_2 \rangle &= \frac{1}{r^2}((\partial_2 r)^2 - (\partial_2 x_0)^2 + 1) = \frac{1}{(1-x_2^2)^2} \\
\langle \partial_1, \partial_2 \rangle &= 0
\end{aligned}
$$

This means

$$
ds^2 = \frac{dx_1^2}{(1-x_1^2)^2} + \frac{dx_2^2}{(1-x_2^2)^2} = du_1^2 + du_2^2 \tag{33}
$$

with $x_i = \tanh u_i$. Therefore the worldsheet metric is Euclidean and flat. Using these coordinates we obtain

$$
x_i = \tanh u_i, \quad r = \frac{1}{\cosh u_1 \cosh u_2}, \quad x_0 = \tanh u_1 \tanh u_2 \tag{34}
$$

$$
\begin{pmatrix} X_{-1} \\ X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} \cosh u_1 \cosh u_2 \\ \sinh u_1 \sinh u_2 \\ \sinh u_1 \cosh u_2 \\ \cosh u_1 \sinh u_2 \\ 0 \\ 0 \end{pmatrix}
$$

Via (14) and the described change of coordinates, we can map the whole surface onto a compact space. The plot below shows how the surface is embedded in AdS_3.

In AdS_4 there are isometry transformations such that the surface can be written

$$
\begin{aligned}
r &= \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\
y_0 &= \frac{a\sqrt{1+b^2} \sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\
y_1 &= \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\
y_2 &= \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}
\end{aligned} \tag{35}
$$

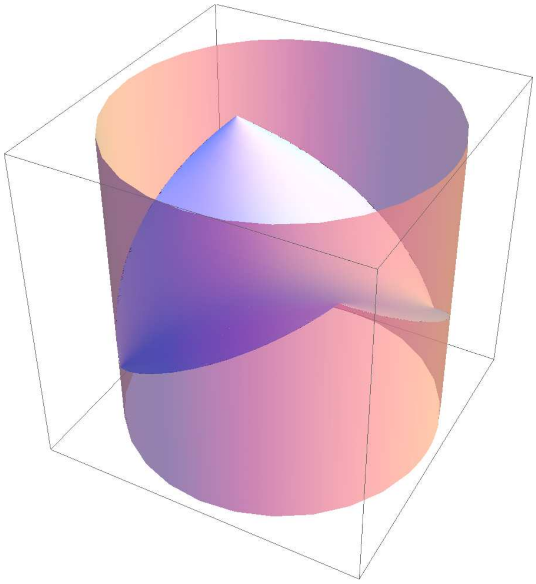

Figure 2: tetragon solution in $\frac{1}{2}S^2 \times \mathbb{R}$

The a and b are parameters that belong to $SO(2,4)$ transformations and will be translated into the kinematical variables s and t. In the appendix 6.3 we prove that this boosted surface can really be obtained using isometry transformations of AdS_4. The a and b can be translated into the kinematical variables s and t via

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2} \qquad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2} \qquad \frac{s}{t} = \frac{(1+b)^2}{(1-b)^2} \tag{36}$$

This will also be shown in the appendix in 6.4. The calculation of the area will of course give some infinite result. So we have to regularize it. This can be done via dimensional regularization or by introducing a cutoff in the radial component. In [Ald08] and [AM07] the authors use both dimensional regularization and regularization via a cutoff at small r in Poincaré coordinates. For the cutoff regularization at constant r_c they give the following result

$$A = \frac{1}{4}\left(\log\left(\frac{r_c^2}{-8\pi^2 s}\right)\right)^2 + \frac{1}{4}\left(\log\left(\frac{r_c^2}{-8\pi^2 t}\right)\right)^2 - \frac{1}{4}\log^2\left(\frac{s}{t}\right) + const. \tag{37}$$

The result is given up to finite pieces that do not depend upon the kinematical variables. This results matches the result they obtained from dimensional regularization. However, the surface is "boosted" with isometry transformations. Thus the area would be independent of the actual isometry transformation and thus independent from the kinematical variables if the area was finite. But because of the area being divergent it has to be regularized. Introducing a cutoff breaks this symmetry and makes the area depending on kinematical variables. So it is a natural question to ask if there is another natural way to do a cutoff. Therefore we will examine the cutoff in the coordinate chart that is given by (14). This map is also a natural choice because it gives a conformal map from the whole conformal boundary onto the static Einstein universe.

2.4 An alternative approach to calculate the area

We start with the generic four cusp case in (35). Applying the relation between the embedding coordinates X^i and the Poincaré coordinate patch we find that the surface is given in terms of the embedding coordinates

$$\begin{pmatrix} \frac{(a^2+1)\cosh u_1 \cosh u_2 - (a^2-1) b \sinh u_1 \sinh u_2}{2a} \\ \sqrt{1+b^2} \sinh u_1 \sinh u_2 \\ \cosh u_2 \sinh u_1 \\ \cosh u_1 \sinh u_2 \\ \frac{-(a^2-1)\cosh u_1 \cosh u_2 + (a^2+1) b \sinh u_1 \sinh u_2}{2a} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \cosh\alpha \cos\beta \\ \cosh\alpha \sin\beta \\ \sinh\alpha \cos\gamma \\ \sinh\alpha \sin\gamma \cos\delta \\ \sinh\alpha \sin\gamma \sin\delta \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix} \tag{38}$$

Here $\{\alpha, \beta, \gamma, \delta\}$ parameterize AdS_4. After applying the coordinate transformation $\sinh\alpha = \tan\theta$ we have a conformal map of AdS_4 to a half Einstein universe. We approach the boundary for $\theta \to \frac{\pi}{2}$.

We want to introduce a cutoff in θ by setting $\theta = \frac{\pi}{2} - \epsilon$. So we have to extract a relation between θ and the surface coordinates $\{u_1, u_2\}$. Therefore we find

$$\gamma = \mathrm{ArcTan}\left(\frac{\sqrt{X_4^2 + X_5^2}}{X_3}\right) \tag{39}$$

and then for

$$\begin{aligned} \tan^2\theta = \sinh^2\alpha = \left(\frac{X_3}{\cos\gamma}\right)^2 = \\ \cosh^2 u_2 \sinh^2 u_1 + \cosh^2 u_1 \sinh^2 u_2 + \\ \frac{((a^2-1)\cosh u_1 \cosh u_2 - (a^2+1) b \sinh u_1 \sinh u_2)^2}{4a^2} \end{aligned} \tag{40}$$

The regularized area is hard to compute exactly. We will approximate the area. In the symmetric case we will be able to show that the error from the approximation tends to a finite value as $\epsilon \to 0$. Then we can approximate the generic case and assume that the error does not diverge for $\epsilon \to 0$.

2.4.1 exact solution in the symmetric case

The symmetric case will appear for $a = 1$ and $b = 0$. Then we have

$$\tan^2\theta = \cosh^2 u_2 \sinh^2 u_1 + \cosh^2 u_1 \sinh^2 u_2 \tag{41}$$

To calculate the area we have to calculate the area of the "round" rectangle below (because the metric determinant in this special parameterization is $\sqrt{\det g} = 1$)

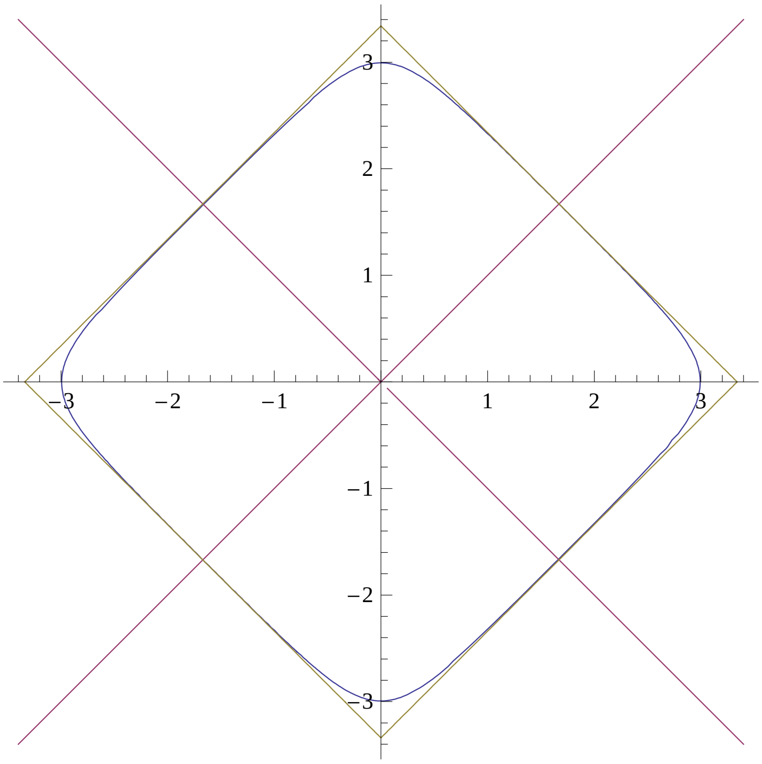

Figure 3: cutoff for $\epsilon = 0.1$ and external rectangle in the u_1u_2 plane

The area of the external rectangle can be computed

$$A = 8\left(\text{ArcCosh}\left(\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{2}{\cos^2(\frac{\pi}{2} - \epsilon)} - 1}}\right)\right)^2 \tag{42}$$

However, there is an error near the cusps of the rectangle. And this error does not tend to 0 as $\epsilon \to 0$. Numerical computations suggest that the error approaches $\frac{\pi^2}{12}$. So if we want to have an exact solution we have to compute

A_{err} and subtract it from the area. The parameterization of the cutoff in the upper right quadrant is

$$u_2(u_1) = \text{ArcSinh}\left(\sqrt{\frac{\frac{1}{\cos^2(\frac{\pi}{2}-\epsilon)} - \cosh^2 u_1}{\cosh^2 u_1 + \sinh^2 u_1}}\right) \tag{43}$$

With $x(\epsilon) = \text{ArcCosh}\left(\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{2}{\cos^2(\frac{\pi}{2}-\epsilon)} - 1}}\right)$ the error can be calculated

$$\frac{A_{err}(\epsilon)}{8} = \int_0^{x(\epsilon)} du_1 \left(-u1 + 2x(\epsilon) - \text{ArcSinh}\left(\sqrt{\frac{\frac{1}{\cos^2(\frac{\pi}{2}-\epsilon)} - \cosh^2 u_1}{\cosh^2 u_1 + \sinh^2 u_1}}\right)\right) \tag{44}$$

$$= \int_0^{x(\epsilon)} du_1 \log\left(\frac{\left(\sqrt{\frac{2}{\cos^2(\frac{\pi}{2}-\epsilon)} - 1} + \sqrt{\frac{2}{\cos^2(\frac{\pi}{2}-\epsilon)} - 2}\right)e^{-u_1}}{\left(\sqrt{\frac{\frac{1}{\cos^2(\frac{\pi}{2}-\epsilon)} - \cosh^2 u_1}{\cosh^2 u_1 + \sinh^2 u_1}} + \sqrt{\frac{\frac{1}{\cos^2(\frac{\pi}{2}-\epsilon)} + \sinh^2 u_1}{\cosh^2 u_1 + \sinh^2 u_1}}\right)}\right) \tag{45}$$

To obtain the latter term, we used the definition of ArcSinh via logarithms and applied logarithm laws. The integral is hard to compute exactly, but we are interested in $\lim_{\epsilon\to 0} A_{err}(\epsilon)$. The term $A_{err}(\epsilon)$ depends on ϵ in two ways. ϵ appears in the upper integration boundary and in the integrand itself. Assuming that the integrand $I(u_1, \epsilon)$ is uniformly convergent in ϵ (which we show in the appendix in 6.5) we can take

$$\begin{aligned} \lim_{\epsilon\to 0} \int_0^{x(\epsilon)} du_1 I(u_1, \epsilon) &= \lim_{\epsilon_1\to 0} \lim_{\epsilon_2\to 0} \int_0^{x(\epsilon_1)} du_1 I(u_1, \epsilon_2) \\ &= \lim_{\epsilon_1\to 0} \int_0^{x(\epsilon_1)} du_1 \lim_{\epsilon_2\to 0} I(u_1, \epsilon_2) \end{aligned} \tag{46}$$

We get

$$\lim_{\epsilon\to 0} \log\left(\frac{\left(\sqrt{\frac{2}{\cos^2(\frac{\pi}{2}-\epsilon)} - 1} + \sqrt{\frac{2}{\cos^2(\frac{\pi}{2}-\epsilon)} - 2}\right)e^{-u_1}}{\left(\sqrt{\frac{\frac{1}{\cos^2(\frac{\pi}{2}-\epsilon)} - \cosh^2 u_1}{\cosh^2 u_1 + \sinh^2 u_1}} + \sqrt{\frac{\frac{1}{\cos^2(\frac{\pi}{2}-\epsilon)} + \sinh^2 u_1}{\cosh^2 u_1 + \sinh^2 u_1}}\right)}\right) = \frac{1}{2}\log\left(1 + e^{-4u_1}\right) \tag{47}$$

So it remains to calculate the integral and we find

$$\lim_{\epsilon\to 0} A_{err}(\epsilon) = 8\int_0^\infty \frac{1}{2}\log\left(1+e^{-4u_1}\right) du_1 = \frac{\pi^2}{12} \tag{48}$$

So we can calculate the expansion in ϵ of the area via (42).

$$A(\epsilon) \approx -\frac{\pi^2}{12} + \frac{9}{2}\log(2)^2 - 6\log(2)\log(\epsilon) + 2\log(\epsilon)^2 + o(1) \tag{49}$$

By $o(1)$ we mean terms that converge to 0 as $\epsilon \to 0$.

2.4.2 An approximation for the generic case

We go back to (40).

$$\begin{aligned}\tan^2\theta = {} & \cosh^2 u_2 \sinh^2 u_1 + \cosh^2 u_1 \sinh^2 u_2 + \\ & \frac{((a^2-1)\cosh u_1 \cosh u_2 - (a^2+1)b\sinh u_1 \sinh u_2)^2}{4a^2}\end{aligned} \tag{50}$$

In this section we assume that the error near the cusps always converges to a fixed number. Then we introduce new variables $u_1 =: x+y$ and $u_2 =: x-y$. By setting $x=0$ and $y=0$ we can calculate the sections with the axes and then calculate area of the rectangle (which is generally not a square) that approximates the cutoff.

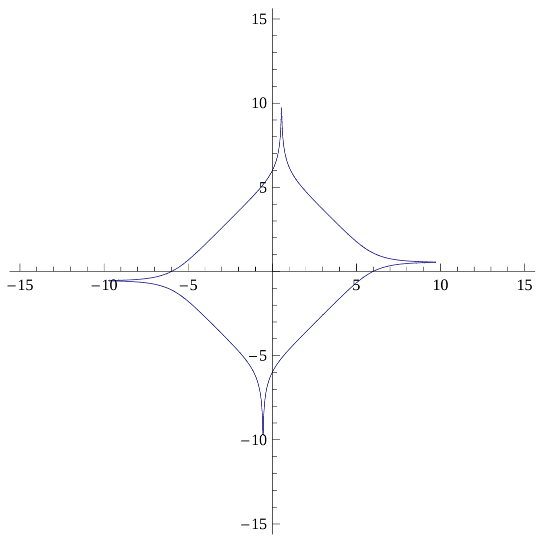

Figure 4: cutoff for $\epsilon = 10^{-5}$, $a = 1000$ and $b = 2$ in u_1u_2-plane

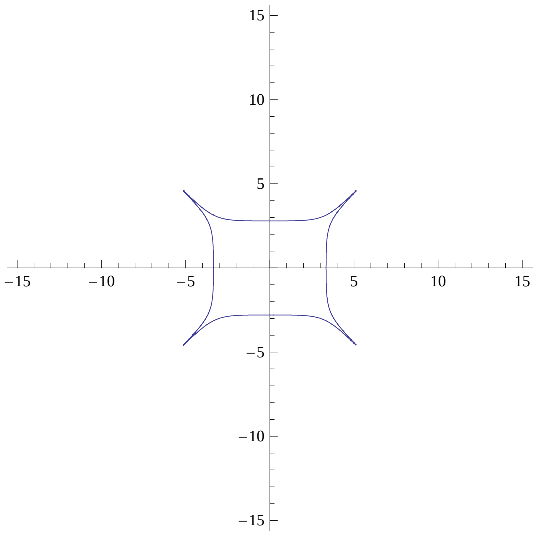

Figure 5: cutoff for $\epsilon = 10^{-5}$, $a = 1000$ and $b = 2$ in xy-plane

By setting $x = 0$ we find for y

$$y_0 = \mathrm{ArcCosh}\left(\sqrt{\frac{A+B}{(b-1)^2 + a^4(1+b)^2 + 2a^2(3+b^2)}}\right) \tag{51}$$

where

$$\begin{aligned} A &= -b + b^2 + a^4 b(1+b) + 2a^2(2+b^2) \\ B &= 2\sqrt{a^2(4a^2 - 2b + 2a^4 b + ((-1+b)^2 + a^4(1+b)^2 + 2a^2(3+b^2))\mathrm{Cot}^2(\epsilon))} \end{aligned} \tag{52}$$

Similarly, we find

$$x_0 = \mathrm{ArcCosh}\left(\sqrt{\frac{C+D}{(b+1)^2 + a^4(b-1)^2 + 2a^2(3+b^2)}}\right) \tag{53}$$

with

$$\begin{aligned} C &= b + b^2 + a^4 b(b-1) + 2a^2(2+b^2) \\ D &= 2\sqrt{a^2(4a^2 + 2b - 2a^4 b + ((1+b)^2 + a^4(b-1)^2 + 2a^2(3+b^2))\mathrm{Cot}^2(\epsilon))} \end{aligned} \tag{54}$$

The regularized surface is then given by the expansion in ϵ of

$$A(\epsilon) = 2 \times 4 x_0 y_0 \tag{55}$$

The extra factor 2 appears because we have calculated the area in the xy Plane. The series expansion in ϵ yields

$$\begin{aligned} A(\epsilon) \approx 2\log\left(\frac{\sqrt{a^4(b-1)^2+(b+1)^2+2a^2(3+b^2)}\epsilon}{8a}\right) \\ \times \log\left(\frac{\sqrt{(b-1)^2+a^4(b+1)^2+2a^2(3+b^2)}\epsilon}{8a}\right) + O(1) \end{aligned} \tag{56}$$

First we observe that if we take the symmetric case $b=0$ and $a=1$ this reproduces (49), of course without the constant error term at the cusps. Resubstituting s and t leads to

$$\begin{aligned} A(\epsilon) \approx 2\log\left(\frac{\sqrt{-\frac{64s}{(\sqrt{s}+\sqrt{t})^4}-\pi^4 t+\frac{16\pi^2(s+t+\sqrt{s}\sqrt{t})}{(\sqrt{s}+\sqrt{t})^2}}\,\epsilon}{8\sqrt{2}\pi}\right) \\ \times \log\left(\frac{\sqrt{-\frac{64t}{(\sqrt{s}+\sqrt{t})^4}-\pi^4 s+\frac{16\pi^2(s+t+\sqrt{s}\sqrt{t})}{(\sqrt{s}+\sqrt{t})^2}}\,\epsilon}{8\sqrt{2}\pi}\right) + O(1) \end{aligned} \tag{57}$$

In the regularization we approximate the cutoff with a rectangle. If we consider large u_1 and u_2 we can approximate the hyperbolic functions with exponential functions. In this approximation we directly see that the sides of the cutoff really become straight lines. The approximation with exponential functions yields

$$\tan^2\theta = \frac{1+6a^2+a^4+(1+a^2)^2b^2-2(a^4-1)b\,\mathrm{Sg}(u_1)\mathrm{Sg}(u_2)}{64a^2}e^{2(|u_1|+|u_2|)} \tag{58}$$

Due to the $\mathrm{Sg}(u_1)\mathrm{Sg}(u_2)$ term this solution is asymmetric and not smooth when we go from one quadrant to another. The plot below shows this approximation But result of this approximation is (up to finite parts) equal to (57). For the discussion we will compare it with a result obtained by Alday in [Ald08]. In order to compare the results we introduce a substitution.

$$V = \left(\frac{\sqrt{a^4(b-1)^2+(b+1)^2+2a^2(3+b^2)}}{8a}\right)^{-1} \tag{59}$$

$$W = \left(\frac{\sqrt{(b-1)^2+a^4(b+1)^2+2a^2(3+b^2)}}{8a}\right)^{-1} \tag{60}$$

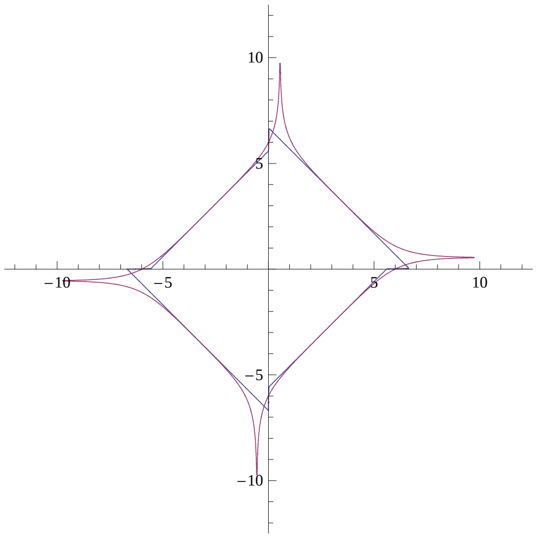

Figure 6: cutoff for $\epsilon = 10^{-5}$, $a = 1000$ and $b = 2$ in u_1u_2-plane and approximation with exponential functions

Then our result can be transformed into

$$\begin{aligned} A(\epsilon) &\approx 2\log(\frac{\epsilon}{V})\log(\frac{\epsilon}{W}) \\ &= (\log(\frac{\epsilon}{\sqrt{s}}\frac{\sqrt{s}}{V})\log(\frac{\epsilon}{\sqrt{s}}\frac{\sqrt{s}}{V}\frac{V}{W}) + \log(\frac{\epsilon}{\sqrt{t}}\frac{\sqrt{t}}{W})\log(\frac{\epsilon}{\sqrt{t}}\frac{\sqrt{t}}{W}\frac{W}{V})) \\ &= (\log\frac{r_1}{\sqrt{s}})^2 + (\log\frac{r_2}{\sqrt{t}})^2 - \log(\frac{V}{W})^2 \end{aligned} \tag{61}$$

with

$$r_1 = \frac{\epsilon\sqrt{s}}{V} \qquad r_2 = \frac{\epsilon\sqrt{t}}{W} \tag{62}$$

In [Ald08] Alday introduces a radial cutoff in a Poincaré patch. It cannot be assumed that our ϵ cutoff leads to a cutoff in Poincaré coordinates with constant cutoff parameter r_c. So r_c is a function of the coordinates of the conformal boundary. But the edges and cusps are located on the conformal boundary and the cutoff r_c is evaluated along the contour of the polygon. So r_c in this sense has a dependence on s and t because they define the contour. In his paper [Ald08] Alday also gives a formula he obtains for cutoff regularization with a cutoff parameter r_c that depends on the coordinates of the conformal boundary and thus also on s and t. Our terms $(\log\frac{r_1}{\sqrt{s}})^2 + (\log\frac{r_2}{\sqrt{t}})^2$ have the same structure. We cannot make a statement about finite terms, as we surely discard some of the s and t dependence of the finite terms.

3 Minimal surfaces in AdS_n

In this chapter we will examine minimal surfaces of AdS_n more closely. The aim is to construct further solutions to the problem. The tetragon solution is however very special. Throughout this section we will prove that this surface is the only flat spacelike minimal surface that exists in AdS_n. It seems that there is a similar picture that is familiar in euclidian geometry. The only flat minimal surface in $\mathbb{R}^n$ is the geodesically embedded $\mathbb{R}^2$. Another very interesting fact is that this is not true for timelike minimal surfaces. There is a big variety of other flat timelike minimal surfaces in AdS_n. Many (timelike) string solutions in AdS_5 are explicitly known. See [FT03] for example for the rigid spinning string, rotating in two different planes. The tetragon solution corresponds via Wick rotation (setting $\tau \to i\tau$ and interpreting the imaginary components as lightlike directions) to a folded rigid spinning string of infinite length with a lightlike trace on the conformal boundary of AdS_n. However, this correspondence cannot be established in the case of other timelike minimal surfaces with lightlike boundary. The folded spinning string can be seen as the 2-spiky string. In [DL08] the authors give a timelike solution for the infinite spiky string with k cusps. In [AM09b] the authors speculate that there might be a correspondence between those infinite spiky spinning strings with k cusps and spacelike minimal surfaces with a lightlike boundary with $2k$ cusps. However, all these infinite spiky string solutions are flat. And we show that there are no further spacelike flat minimal surfaces. So this correspondence would have to be nontrivial. A direct inspection also shows, that there is no simple Wick rotation that gives the correspondence.

The method that we will use in the next section translates the problem of finding a minimal surface that is formulated in embedding coordinates, into a differential equation for an orthogonal frame that moves along the surface. This may seem more difficult but there will be some gauge transformations that simplify the problem. We will treat both timelike and spacelike minimal surfaces simultaneously and specify later. The algorithm that we use was inspired by [dVS93] where the authors examine timelike minimal surfaces in the four dimensional de Sitter space. But the algorithm works in every dimension and on any manifold that can be written as $\{X \in \mathbb{R}^n \mid \langle X, X\rangle = \pm 1\}$ where $\langle ., . \rangle$ indicates a scalar product of arbitrary index. In [JJKV08], [SS09], [AM09a] and [AM09b] the authors use a similar formalism for minimal surfaces in AdS_3.

3.1 A Pohlmeyer reduction in AdS_n

In this section we will present the formalism that we used in [DJW09]. For simplicity we will treat spacelike minimal surfaces here. Nevertheless the formulas translate one to one into the timelike case. We will give the correspondence at the end of this section. Let $Y(s,t)$ be the parameterization of our minimal surface. We choose our coordinate functions such that the induced metric of the surface is conformal to the standard metric of $\mathbb{R}^2$, i.e. $g_s = f(s,t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. This is sometimes called isothermal coordinates. The differential equation for minimal surfaces in conformal gauge reads

$$\Delta Y - 2\sqrt{\det g}Y = 0 \tag{63}$$

Then we introduce a change of variables

$$\begin{aligned} z = \frac{1}{2}(s+it) \quad & \bar{z} = \frac{1}{2}(s-it) \\ \partial = \partial_s - i\partial_t \quad & \bar{\partial} = \partial_s + i\partial_t \end{aligned} \tag{64}$$

In these coordinates the induced metric on the surface reads

$$g_s = f(z,\bar{z})\begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix} \tag{65}$$

Then the equation of motion for the minimal surface is

$$\partial\bar{\partial}Y - \langle\partial Y, \bar{\partial}Y\rangle Y = 0 \tag{66}$$

On the surface we choose a basis of $\mathrm{T}\mathbb{R}^{(2,n-1)}$

$$e = \{Y, \bar{\partial}Y, \partial Y, B_4, \dots, B_{n+1}\} \tag{67}$$

such that $\langle B_i, B_j\rangle = \eta_{ij}$ (with B_4 being timelike) and $\langle B_i, Y\rangle = \langle B_i, \partial Y\rangle = \langle B_i, \bar{\partial}Y\rangle = 0$. Note that here Y is timelike, as it lies in AdS_n and the tangent space of the surface is Euclidean. So the B_i really can be chosen to be an orthonormal set. Then we can define

$$e^{\alpha(z,\bar{z})} := \langle\bar{\partial}Y, \partial Y\rangle \tag{68}$$

Differentiating equation (13) leads to

$$\langle Y, \bar{\partial}Y\rangle = \langle Y, \partial Y\rangle = 0 \tag{69}$$

Differentiating the (1,1) and the (2,2) component of (65) leads to

$$\langle\bar{\partial}Y, \bar{\partial}\bar{\partial}Y\rangle = \langle\partial Y, \partial\partial Y\rangle = 0 \tag{70}$$

Now we express the second derivatives of Y in terms of the basis (67). With the equation of motion (66) it follows

$$\bar{\partial}\partial Y = e^{\alpha(z,\bar{z})}Y \tag{71}$$

For $\partial\partial Y$ and $\bar{\partial}\bar{\partial}Y$ we obtain

$$\partial\partial Y = AY + B\bar{\partial}Y + C\partial Y + \sum_{i=4}^{n+1}\epsilon_i u_i B_i = AY + B\bar{\partial}Y + C\partial Y + u^i B_i \tag{72}$$

$$\bar{\partial}\bar{\partial}Y = DY + E\bar{\partial}Y + F\partial Y + \sum_{i=4}^{n+1}\epsilon_i \bar{u}_i B_i = DY + E\bar{\partial}Y + F\partial Y + u^i B_i \tag{73}$$

Here $\epsilon_i = \eta_{i,i}$. In this formula the coefficients A, B, D, F vanish and $C = \partial\alpha$ and $E = \bar{\partial}\alpha$. (This can easily be verified by rewriting the equations in terms of $\partial_s Y$ and $\partial_t Y$, as this gives an orthonormal base and so it is possible to use the projections.) The $\{u_i\}$ and $\{\bar{u}_i\}$ are the scalar products $u_i = \langle\partial\partial Y, B_i\rangle$ and $\bar{u}_i = \langle\bar{\partial}\bar{\partial}Y, B_i\rangle$, as the basis $\{B_i\}$ is orthonormal. So the second derivatives read

$$\partial\partial Y = \partial\alpha\partial Y + \sum_{i=4}^{n+1}\epsilon_i u_i B_i = \partial\alpha\partial Y + u^i B_i \tag{74}$$

$$\bar{\partial}\bar{\partial}Y = \bar{\partial}\alpha\bar{\partial}Y + \sum_{i=4}^{n+1}\epsilon_i \bar{u}_i B_i = \bar{\partial}\alpha\bar{\partial}Y + \bar{u}^i B_i \tag{75}$$

We need to find the evolution of the basis (67). Thus we express the derivative of the basis in terms of the basis itself which leads to

$$\partial e = A\ e \quad \bar{\partial}e = \bar{A}\ e \tag{76}$$

The upper $3 \times n+1$ block of A is completely determined by the differential equations we found for $\{Y, \bar{\partial}Y, \partial Y\}$. As the B_i are an orthonormal base, we can express their evolution with the scalar products

$$\partial B_i = -e^{-\alpha}u_i\bar{\partial}Y + \sum_{j=4,\, j\neq i}^{n+1}\epsilon_j\langle B_j, \partial B_i\rangle B_j = -e^{-\alpha}u_i\bar{\partial}Y + A_i^{\ j}B_j \tag{77}$$

$$\bar{\partial}B_i = -e^{-\alpha}\bar{u}_i\partial Y + \sum_{j=4,\, j\neq i}^{n+1}\epsilon_j\langle B_j, \bar{\partial}B_i\rangle B_j = -e^{-\alpha}\bar{u}_i\partial Y + \bar{A}_i^{\ j}B_j \tag{78}$$

For $i, j \in \{4, 5, \ldots, n+1\}$ We will also use the notation

$$\begin{aligned} A_i^{\ j} &= \epsilon_j\langle\partial B_i, B_j\rangle \\ \bar{A}_i^{\ j} &= \epsilon_j\langle\bar{\partial}B_i, B_j\rangle \end{aligned} \tag{79}$$

Whenever a quantity has vector- indices these indices belong to the normal space of the surface and indices are raised and lowered with the metric on the normal space. We will use two different conventions. The formulas are shorter if we use Einstein's sum convention and calculate with tensor entries. Then we calculate with upper and lower indices. But we also deal with mathematical coordinate independent tensor calculus. To relate these cases we have to insert a base. Then we end up with sums and coefficients of basis representations that will then be interpreted as tensor entries with upper and lower indices. This means for example $u^i = u_i \epsilon_i$. Whenever the matrix A or $\bar{A}$ carries indices we mean the lower $(n-2) \times (n-2)$ block of A with the above convention.

We can find the expression for A and $\bar{A}$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & \partial\alpha & 0 & \epsilon_4 u_4 & \dots & \epsilon_{n+1} u_{n+1} \\ e^\alpha & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -e^{-\alpha} u_4 & 0 & \dots & \epsilon_{n+1}\langle \partial B_4, B_{n+1}\rangle \\ \vdots & \vdots & \vdots & \vdots & \epsilon_j \langle \partial B_i, B_j \rangle & \vdots \\ 0 & 0 & -e^{-\alpha} u_{n+1} & \epsilon_4 \langle \partial B_{n+1}, B_4 \rangle & \dots & 0 \end{pmatrix} \tag{80}$$

$$\bar{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ e^\alpha & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \bar{\partial}\alpha & \epsilon_4 \bar{u}_4 & \dots & \epsilon_{n+1} \bar{u}_{n+1} \\ 0 & -e^{-\alpha} \bar{u}_4 & 0 & 0 & \dots & \epsilon_{n+1}\langle \bar{\partial} B_4, B_{n+1}\rangle \\ \vdots & \vdots & \vdots & \vdots & \epsilon_j \langle \bar{\partial} B_i, B_j \rangle & \vdots \\ 0 & -e^{-\alpha} \bar{u}_{n+1} & 0 & \epsilon_4 \langle \bar{\partial} B_{n+1}, B_4 \rangle & \dots & 0 \end{pmatrix} \tag{81}$$

$$ \tag{82}$$

Note that the lower $(n-2)\times(n-2)$ blocks in these matrices are antisymmetric (except the fourth line and column) and conjugate to each other. For the system (76) we have to demand

$$\partial\bar{\partial} e_i = \bar{\partial}\partial e_i \tag{83}$$

which leads to the commutation relation

$$\bar{\partial} A - \partial \bar{A} + \left[A, \bar{A}\right] = 0 \tag{84}$$

If we considered timelike minimal surfaces, some adoptions would have to be made.

- The tangent space of the surface is now lorentzian. So we choose a parameterization such that the metric is conformal to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In order to introduce our light cone coordinates z and $\bar{z}$ we define

$$\begin{aligned} z &= \frac{1}{2}(s-t) \quad \bar{z} = \frac{1}{2}(s+t) \\ \partial &= \partial_s + \partial_t \quad \bar{\partial} = \partial_s - \partial_t \end{aligned} \tag{85}$$

  These variables are now two independent real variables.

- The normal space of the surface in AdS_n is now Euclidean, such that

$$\langle B_i, B_j \rangle = \delta_{i,j} \quad \epsilon_i \equiv 1 \forall i \tag{86}$$

- The u_i and $\bar{u}_i$ are again no longer conjugate to each other but are two independent real quantities.

Taking the right metric on the normal space and the proper definition for $\bar{\partial}$ and ∂, we can evaluate the equation (84) for both timelike and spacelike minimal surfaces.

$$\begin{aligned} 0 &= \partial\bar{\partial}\alpha - e^{-\alpha} u^b \bar{u}_b - e^{\alpha} \\ 0 &= \partial \bar{u}_a - A_a^{\ b} \bar{u}_b = \bar{\partial} u_a - \bar{A}_a^{\ b} u_b \\ e^{-\alpha}(\bar{u}_a u^b - u_a \bar{u}^b) &= \partial \bar{A}_a^{\ b} - \bar{\partial} A_a^{\ b} + \bar{A}_a^{\ c} A_c^{\ b} - A_a^{\ c} \bar{A}_c^b \end{aligned} \tag{87}$$

The matrices A_b^a are the lower $(n-2) \times (n-2)$ block of A, i.e. with indices from the normal space. The bar means complex conjugation in the spacelike case. In the timelike case these two are real independent quantities. The metric on the normal space is $\delta_{i,j}$ in the spacelike case and $\eta_{i,j}$ in the timelike case.

3.2 Spacelike minimal surfaces

Next, we examine these equations in the spacelike case. We will start with AdS_5. The known tetragon solution is a flat spacelike minimal surface. So it is a natural question to ask if there are further flat spacelike minimal surfaces that belong to other scattering amplitudes. In this section we will proof that the symmetric tetragon solution is the only (up to isometries of AdS_5) spacelike flat minimal surface in AdS_5. Later we will use similar arguments to show that his result can be extended to AdS_n.

The curvature depends on α (which was shown in the introduction). Assuming that our surface is flat, we will be able to integrate the system of differential equations (76) and prove that the solution is unique up to isometries of AdS_5.

3.2.1 The AdS_5 case

In the AdS_5 case, the $e_i = \{Y, \partial Y, \bar{\partial} Y, B_4, B_5, B_6\}$ are a basis of $\mathrm{T}\mathbb{R}^{(2,4)}$. Here the vectors B_i are chosen to obey $\langle B_i, B_j\rangle = \eta_{i,j}$ with B_4 being timelike. So the matrices A and $\bar{A}$ read

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \partial\alpha & 0 & -u_4 & u_5 & u_6 \\ e^{\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-\alpha}u_4 & 0 & \langle \partial B_4, B_5\rangle & \langle \partial B_4, B_6\rangle \\ 0 & 0 & -e^{-\alpha}u_5 & \langle \partial B_4, B_5\rangle & 0 & \langle \partial B_5, B_6\rangle \\ 0 & 0 & -e^{-\alpha}u_6 & \langle \partial B_4, B_6\rangle & -\langle \partial B_5, B_6\rangle & 0 \end{pmatrix} \tag{88}$$

$$\bar{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ e^{\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\partial}\alpha & -\bar{u}_4 & \bar{u}_5 & \bar{u}_6 \\ 0 & -e^{-\alpha}\bar{u}_4 & 0 & 0 & \langle \bar{\partial} B_4, B_5\rangle & \langle \bar{\partial} B_4, B_6\rangle \\ 0 & -e^{-\alpha}\bar{u}_5 & 0 & \langle \bar{\partial} B_4, B_5\rangle & 0 & \langle \bar{\partial} B_5, B_6\rangle \\ 0 & -e^{-\alpha}\bar{u}_6 & 0 & \langle \bar{\partial} B_4, B_6\rangle & -\langle \bar{\partial} B_5, B_6\rangle & 0 \end{pmatrix} \tag{89}$$

The evaluation of the commutation relation (84) yields some differential equations that have to be fulfilled.

$$\partial\bar{\partial}\alpha + e^{-\alpha}(u_4\bar{u}_4 - u_5\bar{u}_5 - u_6\bar{u}_6) - e^{\alpha} = 0 \tag{90}$$

$$\begin{aligned} \bar{\partial} u_4 - u_5\langle \bar{\partial} B_4, B_5\rangle - u_6\langle \bar{\partial} B_4, B_6\rangle &= 0 \\ \bar{\partial} u_5 - u_4\langle \bar{\partial} B_4, B_5\rangle - u_6\langle \bar{\partial} B_5, B_6\rangle &= 0 \\ \bar{\partial} u_6 - u_4\langle \bar{\partial} B_4, B_6\rangle + u_5\langle \bar{\partial} B_5, B_6\rangle &= 0 \end{aligned} \tag{91}$$

$$\begin{aligned} \partial \bar{u}_4 - \bar{u}_5\langle \partial B_4, B_5\rangle - \bar{u}_6\langle \partial B_4, B_6\rangle &= 0 \\ \partial \bar{u}_5 - \bar{u}_4\langle \partial B_4, B_5\rangle - \bar{u}_6\langle \partial B_5, B_6\rangle &= 0 \\ \partial \bar{u}_6 - \bar{u}_4\langle \partial B_4, B_6\rangle + \bar{u}_5\langle \partial B_5, B_6\rangle &= 0 \end{aligned} \tag{92}$$

So the equations (91) and (92) are conjugate. Based on these equations, we see that $\bar{\partial}(u_4^2 - u_5^2 - u_6^2) = 0$ and $\partial(\bar{u}_4^2 - \bar{u}_5^2 - \bar{u}_6^2) = 0$. So they lie on a hyperboloid whose radius just depends on z (or $\bar{z}$). Whenever $u_4^2 - u_5^2 - u_6^2$ is not constantly zero can locally (near a point where $u_4^2 - u_5^2 - u_6^2$ is nonzero) choose a conformal transformation (that preserves conformal gauge) to choose this radius to be identically 1.

$$\begin{aligned} z \to& h(z) \\ u_i = \langle \partial\partial Y, B_i \rangle =& \langle \frac{\partial^2 Y}{\partial h^2} \left(\frac{\partial h}{\partial z}\right)^2 + \frac{\partial^2 h}{\partial z^2} \frac{\partial h}{\partial z} \frac{\partial Y}{\partial z}, B_i \rangle \\ =& \langle \frac{\partial^2 Y}{\partial h^2} \left(\frac{\partial h}{\partial z}\right)^2, B_i \rangle = \tilde{u}_i (\partial h)^2 \end{aligned} \tag{93}$$

In order to achieve $u_4^2 - u_5^2 - u_6^2 = 1$ we have to integrate

$$\frac{\partial h}{\partial z} = \frac{1}{\sqrt[4]{u_4^2 - u_5^2 - u_6^2}} \tag{94}$$

This can be done locally whenever the denominator is nonzero. Then $h(z)$ is a holomorphic function. Holomorphic functions on the parameter space respect conformal gauge. However, $u_4^2 - u_5^2 - u_6^2$ may have zeros. In this case the transformation is valid at least locally in an open neighborhood of a point where $u_4^2 - u_5^2 - u_6^2 \neq 0$. If $u_4^2 - u_5^2 - u_6^2 \equiv 0$ on an open set then we have an “exceptional” case that we will discuss later in the section about invariants.

The last 3 lines of the integrability condition (84) yield

$$A_{5,6}\bar{A}_{4,6} - \bar{A}_{5,6}A_{4,6} + e^{-\alpha}(u_5\bar{u}_4 - u_4\bar{u}_5) + \bar{\partial}A_{4,5} - \partial\bar{A}_{4,5} = 0 \tag{95}$$
$$A_{4,5}\bar{A}_{5,6} - \bar{A}_{4,5}A_{5,6} + e^{-\alpha}(u_6\bar{u}_4 - u_4\bar{u}_6) + \bar{\partial}A_{4,6} - \partial\bar{A}_{4,6} = 0 \tag{96}$$
$$A_{4,5}\bar{A}_{4,6} - \bar{A}_{4,5}A_{4,6} + e^{-\alpha}(u_6\bar{u}_5 - u_5\bar{u}_6) + \bar{\partial}A_{5,6} - \partial\bar{A}_{5,6} = 0 \tag{97}$$

Now we have to make an explicit choice for the basis B_i to calculate the matrices and differential equations. From now on we will regard the u_i as components of the vector $\sum_i \epsilon_i u_i B_i = u^i B_i$ in the three dimensional complex space that is spanned by $\{B_4, B_5, B_6\}$. As u^i is a complex vector, we can decompose it into two real vectors.

$$u^i = a^i + ib^i. \tag{98}$$

By our choice $1 = u^i u_i = a^i a_i - b^i b_i + 2ia^i b_i$. So these two equations must hold

$$1 = a^i a_i - b^i b_i \tag{99}$$
$$0 = a^i b_i \tag{100}$$

Now we consider three cases where b^i is spacelike, timelike and lightlike. There is a relation between scalar curvature and α (which was shown in the introduction)

$$R = -2e^{-\alpha}\Delta\alpha \tag{101}$$

We will be looking for flat minimal surfaces. This is by (101) equivalent to solutions with harmonic α. The following analysis will be done locally. Nonetheless the solutions of (76) will be defined globally. So the result, that there are no spacelike flat minimal surfaces in AdS_5 will be valid globally.

spacelike case

If $b^ib_i > 0$ we can choose a basis, such that $b^i = (0, \mu, 0)$. But then $b^ia_i = \mu a_5 = 0$. So $a^i = (a_4, 0, a_6)$. But now we can apply a boost in the 4-6 Plane such that we do not change b^i but make $a^i = (0, 0, a_6)$. This is always possible if a^i is not timelike or zero. If a^i is timelike (or zero) we get $a^ia_i - \mu^2 = 1$, which is never true. So let us assume $a^i = (0, 0, a_6)$. But $a^ia_i - b^ib_i = a_6^2 - \mu^2 = 1$. As a_6 and μ are real functions, we can parameterize them with a real parameter $\beta(z, \bar{z})$. So $a^i = (0, 0, \pm\cosh\beta)$ and $b^i = (0, \pm\sinh\beta, 0)$. We start to examine the $+$-case.

+ case

We have

$$u^i = (0, +i\sinh\beta, \cosh\beta) \tag{102}$$
$$\bar{u}^i = (0, -i\sinh\beta, \cosh\beta) \tag{103}$$

The evaluation of the differential equations (91) and (92) with this ansatz yields

$$\langle\bar{\partial}B_5, B_6\rangle = i\bar{\partial}\beta \tag{104}$$
$$\langle\bar{\partial}B_4, B_6\rangle = -i\rho\sinh\beta \tag{105}$$
$$\langle\bar{\partial}B_4, B_5\rangle = \rho\cosh\beta \tag{106}$$

$$\langle\partial B_5, B_6\rangle = -i\partial\beta \tag{107}$$
$$\langle\partial B_4, B_6\rangle = i\bar{\rho}\sinh\beta \tag{108}$$
$$\langle\partial B_4, B_5\rangle = \bar{\rho}\cosh\beta \tag{109}$$

Here ρ is a complex-valued function. With this ansatz the equation (90) becomes

$$\partial\bar{\partial}\alpha - e^{-\alpha}\cosh 2\beta - e^{\alpha} = 0 \tag{110}$$

The equations (95) take the form

$$0 = 2(\bar{\partial}\beta\bar{\rho} - \partial\beta\rho)\sinh\beta + (\bar{\partial}\bar{\rho} - \partial\rho)\cosh\beta \tag{111}$$
$$0 = 2(\bar{\partial}\beta\bar{\rho} + \partial\beta\rho)\cosh\beta + (\bar{\partial}\bar{\rho} + \partial\rho)\sinh\beta \tag{112}$$
$$0 = (\rho\bar{\rho} + e^{-\alpha})\sinh 2\beta + 2\partial\bar{\partial}\beta \tag{113}$$

From (110) it is obvious that there is no solution for beta if we impose the condition $\bar{\partial}\partial\alpha = \Delta\alpha \overset{!}{=} 0$.

- case

We need to consider the minus in the cosh-term only, as sinh is antisymmetric.

$$u^i = (0, +i\sinh\beta, -\cosh\beta) \tag{114}$$
$$\bar{u}^i = (0, -i\sinh\beta, -\cosh\beta) \tag{115}$$

But here we already see that this does not change the equation (110). So again there is no solution if α is harmonic.

lightlike case

If $b^i b_i = 0$ we can perform a transformation such that $b = (1, 1, 0)$. From $a^i b_i = 0$ we know that $a_4 = a_5$. The $a_4 = a_5 = 0$-case will be described below. And again from $a^i a_i - b^i b_i = 1$ we know that $a_6 = \pm 1$. We start to examine the $+$-case.

+ case

We have $b = (1, 1, 0)$ and $a = (\beta, \beta, 1)$. So we can parameterize u and $\bar{u}$ with

$$u^i = (\beta + i, \beta + i, 1) \tag{116}$$
$$\bar{u}^i = (\beta - i, \beta - i, 1) \tag{117}$$

The evaluation of the equations (91) and (92) yields

$$\langle\bar{\partial}B_4, B_6\rangle = \langle\bar{\partial}B_5, B_6\rangle = \rho \tag{118}$$
$$\langle\bar{\partial}B_4, B_5\rangle = \frac{1}{\beta + i}(\bar{\partial}\beta - \rho) \tag{119}$$

$$\langle\partial B_4, B_6\rangle = \langle\partial B_5, B_6\rangle = \bar{\rho} \tag{120}$$
$$\langle\partial B_4, B_5\rangle = \frac{1}{\beta - i}(\partial\beta - \bar{\rho}) \tag{121}$$

The equation (90) for α becomes

$$\partial\bar{\partial}\alpha - e^{-\alpha} - e^{\alpha} = 0 \tag{122}$$

The equations (95) only give 2 independent equations.

$$0 = \bar{\partial}(\frac{1}{\beta - i}(\partial\beta - \bar{\rho})) - \partial(\frac{1}{\beta + i}(\bar{\partial}\beta - \rho)) \tag{123}$$

$$0 = (\beta + i)\partial\beta\rho - (\beta - i)\bar{\partial}\beta\bar{\rho} - 2i\rho\bar{\rho} + (\bar{\partial}\bar{\rho} - \partial\rho - 2ie^{-\alpha})(1 + \beta^2) \tag{124}$$

Again, we look for solutions with harmonic α. From (122) we see that if α is harmonic we have a contradiction.

- case

Again, we have the same result. Taking

$$u^i = (\beta + i, \beta + i, -1) \tag{125}$$

$$\bar{u}^i = (\beta - i, \beta - i, -1) \tag{126}$$

will not change equation (122).

case with $a_4 = a_5 = 0$

If $a_4 = a_5 = 0$ it follows that $a_6 = \pm 1$. But then we evaluate (90) with harmonic α and find

$$e^{-\alpha} + e^{\alpha} = 0 \tag{127}$$

which is a contradiction.

timelike case

If $b^i b_i < 0$ then we can choose a boost such that $b = (\mu, 0, 0)$. Because $0 = a^i b_i = -a_4\mu$ we have $a_4 = 0$. So we can perform a rotation in the $5-6$-plane such that $a = (0, a_5, 0)$. The case $a^i = 0$ will be discussed below. Then we know that $1 = a^i a_i - b^i b_i = a_5^2 + \mu^2$. Thus, we can parameterize u and $\bar{u}$ with

$$u^i = (+i\cos\beta, \sin\beta, 0) \tag{128}$$

$$\bar{u}^i = (-i\cos\beta, \sin\beta, 0) \tag{129}$$

Again, we evaluate the Equations (91) and (92) with this ansatz and we obtain

$$\langle\bar{\partial}B_4, B_5\rangle = -i\bar{\partial}\beta \tag{130}$$

$$\langle\bar{\partial}B_4, B_6\rangle = \rho\sin\beta \tag{131}$$

$$\langle\bar{\partial}B_5, B_6\rangle = i\rho\cos\beta \tag{132}$$

$$\langle \partial B_4, B_5 \rangle = i\partial\beta \tag{133}$$
$$\langle \partial B_4, B_6 \rangle = \bar{\rho}\sin\beta \tag{134}$$
$$\langle \partial B_5, B_6 \rangle = -i\bar{\rho}\cos\beta \tag{135}$$

Thus the equation (90) becomes

$$\partial\bar{\partial}\alpha + e^{-\alpha}\cos 2\beta - e^{\alpha} = 0 \tag{136}$$

The equations (95) read

$$0 = (\rho\bar{\rho} + e^{-\alpha})\sin 2\beta - 2\partial\bar{\partial}\beta \tag{137}$$
$$0 = 2\cos\beta(\bar{\partial}\beta\bar{\rho} - \partial\beta\rho) + \sin\beta(\bar{\partial}\bar{\rho} - \partial\rho) \tag{138}$$
$$0 = 2\sin\beta(\bar{\partial}\beta\bar{\rho} + \partial\beta\rho) - \cos\beta(\bar{\partial}\bar{\rho} + \partial\rho) \tag{139}$$

Here we see that (138) and (139) are not independent. The sum and the difference of (138) and (139) are conjugate to each other. Therefore we only consider the sum.

$$\cos\beta(2\bar{\partial}\beta\bar{\rho} - 2\partial\beta\rho - \bar{\partial}\bar{\rho} - \partial\rho) + \sin\beta(\bar{\partial}\bar{\rho} - \partial\rho - 2\bar{\partial}\beta\bar{\rho} + 2\partial\beta\rho) = 0 \tag{140}$$

From (136) we conclude that if α is harmonic

$$\beta = \frac{1}{2}\mathrm{ArcCos}(e^{2\alpha}) \tag{141}$$
$$\alpha = \frac{1}{2}\log(\cos(2\beta)) \tag{142}$$

Now we assume α to be nonzero and calculate

$$2\partial\bar{\partial}\beta = \partial\bar{\partial}\mathrm{ArcCos}(e^{2\alpha}) = \partial(-\frac{2e^{2\alpha}}{\sqrt{(1-e^{4\alpha})}}\bar{\partial}\alpha) \tag{143}$$
$$= (-\frac{2e^{2\alpha}}{\sqrt{(1-e^{4\alpha})}})\underbrace{\partial\bar{\partial}\alpha}_{=0} - \frac{4e^{2\alpha}}{(1-e^{4\alpha})^{\frac{3}{2}}}\bar{\partial}\alpha\partial\alpha = -\frac{4e^{2\alpha}}{(1-e^{4\alpha})^{\frac{3}{2}}}\bar{\partial}\alpha\partial\alpha \tag{144}$$
$$\sin(2\beta) = \sqrt{1-e^{4\alpha}} \tag{145}$$

and insert into (137)

$$0 = \rho\bar{\rho} + e^{-\alpha} + \frac{4e^{2\alpha}}{(1-e^{4\alpha})^2}\partial\alpha\bar{\partial}\alpha \tag{146}$$

Here we used the fact that $\alpha \neq 0$ is real, so $\bar{\partial}\alpha = \bar{\partial\alpha}$. But in (146) all terms are real and strictly positive. So again, we conclude that there is no solution if $\alpha \neq 0$ is harmonic. If $\alpha \equiv 0$ we know from (136) that $\beta \in \{0, \pi\}$. Equation (137) is fulfilled. For equation (140) to be true

$$\bar{\partial}\bar{\rho} + \partial\rho = 0 \tag{147}$$

So the only case where there exist minimal surfaces, is when $\alpha \equiv 0$ and $\beta \in \{0, \pi\}$.

case where $a^i = 0$

If a^i is zero for all i, we find that $u = (i\mu, 0, 0)$. Inserting into (90) leads to

$$\mu^2 e^{-\alpha} + e^{\alpha} = 0 \tag{148}$$

which cannot be fulfilled.

3.2.2 Integration of the flat case

In general, it is a hard task to integrate this system of differential equations. But in the case $\alpha \equiv 0$ it can be performed quite simply. If $\alpha \equiv 0$ we consider the timelike case from the last section. In this case the matrices A and $\bar{A}$ are

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp i & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mp i\bar{\rho} \\ 0 & 0 & 0 & 0 & \pm i\bar{\rho} & 0 \end{pmatrix} \tag{149}$$

$$\bar{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm i & 0 & 0 \\ 0 & \pm i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pm i\rho \\ 0 & 0 & 0 & 0 & \mp i\rho & 0 \end{pmatrix} \tag{150}$$

Here $\pm$ corresponds to β being $\beta = 0$ or $\beta = \pi$. A and $\bar{A}$ are not constant as they depend on z and $\bar{z}$ via ρ. But as the matrices have a lower 2×2 block, the system of differential equations decouples. We are only interested in the first line of the solution (as it describes the development of Y, which is the surface itself). So we just need to calculate

$$\tilde{e}_i = \exp(A^{[4]}z)\exp(\bar{A}^{[4]}\bar{z}) \tag{151}$$

where $A^{[4]}$ means the upper 4×4 block of A. This leads to

$$\tilde{e}_i = \begin{pmatrix} A & C & B & D \\ B & A & -iD & -iC \\ C & iD & A & iB \\ D & iB & -iC & A \end{pmatrix} \tag{152}$$

with

$$\begin{aligned} A &= \cosh \frac{s}{\sqrt{2}} \cosh \frac{t}{\sqrt{2}} \\ B &= \frac{1}{\sqrt{2}} \left(\frac{1}{2} + \frac{i}{2} \right) \left(\sinh \frac{s-t}{\sqrt{2}} - i \sinh \frac{s+t}{\sqrt{2}} \right) \\ C &= \frac{1}{\sqrt{2}} \left(\frac{1}{2} + \frac{i}{2} \right) \left(-i \sinh \frac{s-t}{\sqrt{2}} + \sinh \frac{s+t}{\sqrt{2}} \right) \\ D &= \sinh \frac{s}{\sqrt{2}} \cosh \frac{t}{\sqrt{2}} \end{aligned} \tag{153}$$

The first line is the coordinate representation of the solution (with the coordinates to B_5 and B_6 being zero) in the basis $\{Y, \partial Y, \bar{\partial} Y, B_4\}$. We have to express the solution in an orthogonal frame. So we calculate back into $\{Y, \partial_s Y, \partial_t Y, B_4\}$, keeping in mind that $\langle \partial_s Y, \partial_s Y \rangle = \langle \partial_t Y, \partial_t Y \rangle = \frac{1}{2}$ if $\alpha = 0$. But now we have the solution given in an orthogonal frame. So we can identify the timelike vectors with the timelike standard vectors of $\mathbb{R}^{(2,2)}$ and the others with the spacelike. Here we see that our minimal surface lies entirely in AdS_3. It has the coordinate representation

$$Y = \begin{pmatrix} \cosh \frac{s}{\sqrt{2}} \cosh \frac{t}{\sqrt{2}} \\ \sinh \frac{s}{\sqrt{2}} \sinh \frac{t}{\sqrt{2}} \\ \sinh \frac{s}{\sqrt{2}} \cosh \frac{t}{\sqrt{2}} \\ \cosh \frac{s}{\sqrt{2}} \sinh \frac{t}{\sqrt{2}} \end{pmatrix} \tag{154}$$

This is the tetragon solution. Here the case $\beta = 0$ was considered. For the case $\beta = \pi$ there appears a minus sign in the first line of (152) in the last place. This sign however does not change the surface. So the conclusion of this calculations is: The Maldacena surface is up to isometries of AdS_5 the only spacelike flat minimal surface. When integrating (76) we can choose a matrix $\mathcal{M} \in SO(2,4)$ as a starting frame in which the solution is given. Here we explicitely see that the solution is given uniquely up to the isometry group of AdS_5. The crucial point why there is a bigger variety of flat minimal surfaces in the timelike case is that the corresponding $\rho\bar{\rho}$ terms in (137) and (241) can have both signs in the timelike case (because ρ and $\bar{\rho}$ are two real independent parameters) while its positive semidefinite for the spacelike case, where ρ and $\bar{\rho}$ are complex conjugate.

3.2.3 Spacelike flat minimal surfaces in AdS_n

In this section we will proof for general n that the symmetric tetragon solution is up to isometries the only flat spacelike minimal surface in AdS_n. We do the proof for the nonexceptional case (the exceptional case will be excluded in the section about invariants 3.6). Again we split $u^i = a^i + ib^i$ and have

$$1 = a^i a_i - b^i b_i \tag{155}$$

$$0 = a^i b_i \tag{156}$$

Similarly to previous sections we assume b^i to be spacelike, lightlike and timelike. Contrary to the AdS_5 section we will not give the whole set of differential equations because we are not so much interested in all the details for generic spacelike minimal surfaces for $n > 5$. We only compute those equations which lead to the conclusion that α has to be zero if α is harmonic. Then we show that the system of differential equations (76) decouples and we have the same upper 4×4 block as in the AdS_5 case. This completes the proof.

spacelike case

Assuming that b^i is spacelike we can perform an transformation such that $b = (0, b, 0, \ldots, 0)$. So we have $1 = a^i a_i - b^2$. This means that a^i has to be spacelike. Because of orthogonality we know that $a_5 = 0$. So we perform another transformation that leaves B_5 invariant to achieve $a = (0, 0, a, 0, \ldots, 0)$. So we have $1 = a^2 - b^2$. We parameterize a and b with $a = \cosh\beta$ and $b = \sinh\beta$. So we have

$$u = (0, i\sinh\beta, \cosh\beta, 0, \ldots, 0) \tag{157}$$

Now we calculate $u^i \bar{u}_i = \sinh^2\beta + \cosh^2\beta > 0$. Assuming that α is harmonic this is a contradiction to the first line in (87).

lightlike case

If b^i is lightlike we know by $1 = a^i a_i - b^i b_i$ that a is spacelike. So there is a base such that $a = (0, 0, \pm 1, 0, \ldots, 0)$ and $b = (1, 1, 0, \ldots, 0)$. That means

$$u = (i, i, \pm 1, 0, \ldots, 0) \tag{158}$$

This means $u^i \bar{u}_i = 1 > 0$. Assuming that α is harmonic this is a contradiction to the first line in (87).

timelike case

When b^i is timelike we chose a transformation such that $b = (b, 0, \ldots, 0)$. From the orthogonality we know that $a_4 = 0$. So a^i is spacelike. So we chose a^i to be $a = (0, a, 0, \ldots, 0)$. We have $1 = a^2 + b^2$. So we parameterize a and b with $a = \sin\beta$ and $b = \cos\beta$. So we find for u^i

$$u = (i\cos\beta, \sin\beta, 0, \ldots, 0) \tag{159}$$

So $u^i \bar{u}_i = -\cos^2\beta + \sin^2\beta = -\cos 2\beta$. Now we use the second line in (87) to parameterize the $A_{i,j}$.

The second line of (87) : $a = 4$ **and** $a = 5$

$$\bar{\partial} u_4 - \bar{A}_4^{\ 5} u_5 = 0 \tag{160}$$

$$\bar{\partial} u_5 - \bar{A}_5^{\ 4} u_4 = 0 \tag{161}$$

From these two equations we see that

$$\bar{A}_4^{\ 5} = \bar{A}_5^{\ 4} = -i\bar{\partial}\beta \tag{162}$$

The second line of (87): $a > 5$

For this case we find

$$0 = \bar{\partial} u_a = \bar{A}_a^{\ 4} u_4 + \bar{A}_a^{\ 5} u_5 \tag{163}$$

which leads to

$$\bar{A}_a^{\ 4} = -\rho_a \sin\beta \tag{164}$$

$$\bar{A}_a^{\ 5} = i\rho_a \cos\beta \tag{165}$$

$$\tag{166}$$

The other $\bar{A}_i^{\ j}$ and $A_i^{\ j}$ are not affected by the choice of u^i and can be regarded as independent complex functions of z. The reason why these “unparameterized” $A_i^{\ j}$ appear in dimensions $n > 5$ is simple. We use $SO(1, n-3)$ transformations on the normal bundle to obtain a specific choice of the vector u^i. In higher dimensions we have several zeros in the vector u^i on which we still can act with orthonormal transformations without changing anything. In higher dimensions we do not fix the gauge anymore. We are left with the following expressions for our matrices $A_i^{\ j}$ and $\bar{A}_i^{\ j}$

$$\bar{A}_i^{\ j} = \begin{pmatrix} 0 & -i\bar{\partial}\beta & -\rho_6 \sin\beta & -\rho_7 \sin\beta & \dots \\ -i\bar{\partial}\beta & 0 & -i\rho_6 \cos\beta & -i\rho_7 \cos\beta & \dots \\ -\rho_6 \sin\beta & i\rho_6 \cos\beta & 0 & \bar{A}_6^7 & \dots \\ -\rho_7 \sin\beta & i\rho_7 \cos\beta & \bar{A}_7^6 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{167}$$

$$A_i^{\ j} = \begin{pmatrix} 0 & i\partial\beta & -\bar{\rho}_6 \sin\beta & -\bar{\rho}_7 \sin\beta & \dots \\ i\partial\beta & 0 & i\bar{\rho}_6 \cos\beta & i\bar{\rho}_7 \cos\beta & \dots \\ -\bar{\rho}_6 \sin\beta & -i\bar{\rho}_6 \cos\beta & 0 & A_6^7 & \dots \\ -\bar{\rho}_7 \sin\beta & -i\bar{\rho}_7 \cos\beta & A_7^6 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{168}$$

Now we need to compute the first line and the first column of the commutator for these two matrices. We find

$$\left[A, \bar{A}\right]_i^{\ j} = \begin{pmatrix} 0 & V & W & \dots \\ V & * & * & \dots \\ W & * & * & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{169}$$

with

$$V = -i \sin 2\beta \sum_a \rho_a \bar{\rho}_a$$
$$W = \cos\beta(\rho_6 \partial\beta - \bar{\rho}_6 \bar{\partial}\beta) + \sin\beta \sum_a (\rho_a A_a^{\ 6} - \bar{\rho}_a \bar{A}_a^{\ 6})$$

Next we evaluate the third line of (87) for $a = 4$ and $b = 5$. This leads to

$$0 = (e^{-\alpha} + \sum_a \rho_a \bar{\rho}_a) \sin 2\beta - 2\partial\bar{\partial}\beta \tag{170}$$

Note that this is the same equation (137) we found in the AdS_5 case with the substitution $\rho\bar{\rho} \longleftrightarrow \sum_a \rho_a \bar{\rho}_a$. Like in this case we conclude that if α is harmonic (and nonzero) we have

$$\beta = \frac{1}{2}\mathrm{ArcCos}(e^{2\alpha}) \tag{171}$$

$$2\partial\bar{\partial}\beta = -\frac{4e^{2\alpha}}{(1 - e^{4\alpha})^{\frac{3}{2}}} \bar{\partial}\alpha\partial\alpha \tag{172}$$

$$\sin 2\beta = \sqrt{1 - e^{4\alpha}} \tag{173}$$

Inserting this into (170) leads to

$$0 = \sum_a \rho_a \bar{\rho}_a + e^{-\alpha} + \frac{4e^{2\alpha}}{(1-e^{4\alpha})^2}\partial\alpha \tag{174}$$

Again this equation has no solution and is a contradiction to the assumption that α is nonzero. This means if α is harmonic it is automatically zero. If alpha is zero it follows by the first line in (87) that $\beta \in \{0, \pi\}$. But if $\beta \in \{0, \pi\}$ the first line and the first column in (167) vanish. So again the system (76) decouples and we can integrate the upper part. That means that we have a proof that the symmetric tetragon solution is the only spacelike flat minimal surface AdS_n.

3.2.4 The AdS_4 and AdS_3 case

The AdS_4 case

In the AdS_4 case we proceed similarly to the AdS_5 case. We just have one independent $A_{4,5}$ left.

$$\begin{aligned} \partial\bar{u}_4 - A_{4,5}\bar{u}_5 &= 0 \\ \partial\bar{u}_5 - A_{4,5}\bar{u}_4 &= 0 \end{aligned} \tag{175}$$

Again, we treat the non-exceptional case ($u^i u_i$ only has discrete zeros) here and perform a holomorphic transformation such that $u^i u_i = \bar{u}^i \bar{u}_i = 1$. We start with splitting u_i into

$$\begin{aligned} u^i u_i = \bar{u}^i \bar{u}_i &= 1 \qquad u^i = a^i + ib^i \\ a^i b_i &= 0 \\ a^i a_i - b^i b_i &= 1 \end{aligned} \tag{176}$$

From these equations it is easy to see that b necessarily has to be timelike. So we can choose a transformation on the normal space such that

$$\vec{b} = (b, 0) \qquad \vec{a} = (0, a) \tag{177}$$

and we are left with

$$1 = a^2 + b^2 \tag{178}$$

So we introduce a parameter $\beta(z, \bar{z})$ such that $a = \cos\frac{\beta}{2}$ and $b = \sin\frac{\beta}{2}$. Because a^i and b^i were real vectors, β should also be real-valued. From (175) we find that $A_{4,5}$ is given by

$$A_{4,5} = -i\frac{\partial\beta}{2} \tag{179}$$

We no longer have the commutator term in the third line of (87). So the third line yields

$$\sin\beta e^{-\alpha} = -\partial\bar{\partial}\beta \tag{180}$$

The first line in (87) yields

$$0 = \partial\bar{\partial}\alpha - e^{-\alpha}\cos\beta - e^{\alpha} \tag{181}$$

The AdS_3 case

In the AdS_3 case we just have one normal direction, so we just have u_4. If u_4 is not identically zero we can locally choose conformal transformations such that $u_4 = 1$. Thus the first line in (87) leads to the sinh Gordon equation (the other lines vanish)

$$\partial\bar{\partial}\alpha - 2\sinh\alpha = 0 \tag{182}$$

Together with the equation for the curvature we find

$$R = -4e^{-\alpha}\sinh\alpha \tag{183}$$

Thus we see that

$$\inf_{\alpha\in\mathbb{R}}(-4e^{-\alpha}\sinh\alpha) = -2 \tag{184}$$

This means that there is no minimal surface in AdS_3 that has a smaller scalar curvature than $R = -2$.

3.3 Geometric interpretation

In this section we demonstrate that the vector u^i encodes the second fundamental forms of the surface in AdS_n. For any codimension the second fundamental form is a $(2,0)$ - tensorfield with values in the normal bundle of the immersion. It is defined by

$$\mathrm{II}(V,W) = (\nabla_V W)^{\perp} \tag{185}$$

where V and W are tangential vectorfields and $()^{\perp}$ means the projection on the normal space. When we choose a basis of the normal space (in our case $\{Y, B_4, B_5, B_6\}$) we can write down several second fundamental forms with values in $\mathbb{R}$ by calculating the projections on every basis vector. We are considering a surface that lies in $\mathbb{R}^{(2,4)}$. Because for $\mathbb{R}^{(2,4)}$ the covariant and ordinary derivative are equivalent, we can write

$$\mathrm{II}(V,W)_i = \langle V(W), B_i\rangle \tag{186}$$

Here $V(W)$ means the derivative of the vectorfield W in the direction of V. To get the matrix S that represents $\mathrm{II}(V,W) = \langle V, S(W)\rangle$ we will evaluate

$\mathrm{II}(V,W)$ on an orthogonal base. The natural choice for an orthogonal base of $\mathrm{T}M^2$ is $\partial_s Y$ and $\partial_t Y$. So the second fundamental forms read

$$S_i = \begin{pmatrix} \langle \partial_s \partial_s Y, B_i \rangle & \langle \partial_t \partial_s Y, B_i \rangle \\ \langle \partial_s \partial_t Y, B_i \rangle & \langle \partial_t \partial_t Y, B_i \rangle \end{pmatrix} =: \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{187}$$

But now we have because of (71)

$$0 = \langle \partial \bar{\partial} Y, B_i \rangle = \langle \partial_s \partial_s Y + \partial_t \partial_t Y, B_i \rangle \tag{188}$$

This means $\gamma = -\alpha$. But we also know that

$$a_i + ib_i = u_i = \langle \partial \partial Y, B_i \rangle = \langle \partial_s \partial_s Y - \partial_t \partial_t Y - i2\partial_s \partial_t Y, B_i \rangle \tag{189}$$

So $\beta = -\frac{b_i}{2}$ and $\alpha = \frac{a_i}{2}$. Finally, we find

$$S_i = \frac{1}{2} \begin{pmatrix} a_i & -b_i \\ -b_i & -a_i \end{pmatrix} \tag{190}$$

Note that this is the formula for the second fundamental forms of the surface inside AdS_5. The second fundamental form for the normal direction of AdS_5 inside $\mathbb{R}^{(2,4)}$ (so the i indicates Y) is given by

$$S_Y = -\frac{e^\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{191}$$

Note here that the trace of S in (190) is zero. This is equivalent to the fact that our surface is minimal inside AdS_5. In equation (191) the trace is nonzero. So this indicates that the surface will not be minimal if regarded as a surface inside $\mathbb{R}^{(2,4)}$. These equations ((190) and (191)) match with those we gave in [DJW09]. However, it is easier for the following calculations to express the second fundamental form in the orthonormal frame $\{\sqrt{2e^{-\alpha}}\partial_s,\ \sqrt{2e^{-\alpha}}\partial_t\}$. This leads to the following expressions

$$S_i = e^{-\alpha} \begin{pmatrix} a_i & -b_i \\ -b_i & -a_i \end{pmatrix} \tag{192}$$

$$S_Y = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{193}$$

As a fundamental theorem of the theory of surfaces states, a (parameterized) surface is uniquely defined (up to isometries of the ambient space) by it is first fundamental form and the second fundamental forms if they obey the Gauss-, the Codazzi-Mainardi- and the Ricci-equation. The first fundamental form is the induced metric on the surface (it is called first fundamental form for historical reasons). So we can evaluate these three equations. These equations can for example be found in [Lan99].

A very great feature of the formalism that we use here is its incapability to distinguish between timelike and spacelike minimal surfaces (the $\rho\bar{\rho}$ terms are an exception). So if we continue to calculate in terms of z and $\bar{z}$ all calculations are valid in both cases. However, our second fundamental forms are given in an unadopted base. So we have to explicitly calculate the second fundamental forms for timelike minimal surfaces. In the base $\{\sqrt{2e^{-\alpha}}\partial_s,\ \sqrt{2e^{-\alpha}}\partial_t\}$ this leads to

$$S_i = e^{-\alpha}\begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix} \tag{194}$$

$$S_Y = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{195}$$

The evaluation on the vectors $\bar{\partial}$ and ∂ is however unaffected if we remember the Lorentzian metric on the tangent space in the timelike case. So the calculation for Ricci and Codazzi-Mainardi equation are automatically valid in both cases.

3.3.1 The Gauss equation

The Gauss equation relates the curvature tensor of the ambient manifold with the curvature tensor of the submanifold.

$$\begin{aligned}&\langle R(X,Y)Z,W\rangle - \langle \tilde{R}(X,Y)Z,W\rangle \\ =&\langle \mathrm{II}(Y,Z),\mathrm{II}(X,W)\rangle - \langle \mathrm{II}(X,Z),\mathrm{II}(Y,W)\rangle\end{aligned} \tag{196}$$

Here $R(X,Y)Z$ is the curvature tensor of $\mathbb{R}^{(2,4)}$ (hence it is vanishing) and $\tilde{R}(X,Y)Z$ the curvature tensor of the surface. $\tilde{R}(X,Y)Z$ is defined

$$\tilde{R}(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{197}$$

We compute the components of the curvature tensor. For a conformally parameterized surface ($g_{i,j} = f(\sigma,\tau)\delta_{i,j}$ or $f(\sigma,\tau) = \frac{e^{\alpha}}{2}$) there is only one independent component. We find

$$\tilde{R}^1_{2,2,1} = -\frac{(\partial_\tau f)^2 + (\partial_\sigma f)^2 - f(\partial_\tau\partial_\tau f + \partial_\sigma\partial_\sigma f)}{2f^2} \tag{198}$$

$$= \frac{1}{2}\bar{\partial}\partial\alpha \tag{199}$$

Hence we find

$$\tilde{R}_{1,2,2,1} = \frac{1}{4}\bar{\partial}\partial\alpha e^{\alpha} \tag{200}$$

The right hand side of (196) becomes

$$\begin{aligned}\tilde{R}_{1,2,2,1} &= \eta_{a,b}(S^a_{1,2}S^b_{2,1} - S^a_{2,2}S^b_{1,1}) &(201)\\ &= \frac{1}{4}(-b_4^2 + b_5^2 + b_6^2 + ... - a_4^2 + a_5^2 + a_6^2 + ... + e^{2\alpha}) &(202)\\ &= \frac{1}{4}(-u_4\bar{u}_4 + u_5\bar{u}_5 + u_6\bar{u}_6 + ... + e^{2\alpha}) &(203)\end{aligned}$$

Here we used $\eta_{a,b}$ as the induced metric on the space that is spanned by $\{Y, B_4, B_5, B_6, \dots\}$ so $\eta_{a,b} = \mathrm{diag}(-1,-1,1,1,\dots)$. In this calculation we used the base $\{\partial_\sigma Y, \partial_\tau Y\}$ and the second fundamental forms from (190). This is possible because there are no covariant derivatives of the second fundamental forms in the Gauss equation. We find that the Gauss equation is in fact one of the differential equations that we derived from the integrability equation

$$0 = \bar{\partial}\partial\alpha - e^{-\alpha}(u^i\bar{u}_i) - e^{\alpha} \tag{204}$$

Here the index i just labels the normal directions inside AdS_n. There are no further independent entries in the curvature tensor of a surface. So the Gauss equation is equivalent to the first line in (87). The calculation is also valid for timelike surfaces.$\eta_{a,b}$ would be $\eta_{a,b} = \mathrm{diag}(-1,1,1,1,\dots)$ then but with the positive definite metric on the normal space and the right second fundamental forms we still arrive at

$$0 = \bar{\partial}\partial\alpha - e^{-\alpha}(u^i\bar{u}_i) - e^{\alpha}$$

3.3.2 The Codazzi- Mainardi equation

The Codazzi-Mainardi equation for a submanifold of any dimension reads

$$(R(X,Y)Z)^{\perp} = (\tilde{\nabla}_X \mathrm{II})(Y,Z) - (\tilde{\nabla}_Y \mathrm{II})(X,Z) \tag{205}$$

Here $(R(X,Y)Z)^{\perp}$ is the normal projection of the curvature transformation of the ambient space. We now consider the ambient space to be $\mathbb{R}^{(2,4)}$. So the curvature tensor vanishes. The vectorfields X, Y, Z are arbitrary tangent vectorfields of the surface. The connection $\tilde{\nabla}$ is defined

$$(\tilde{\nabla}_X \mathrm{II})(Y,Z) = \nabla^{\perp}_X(\mathrm{II}(Y,Z)) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z) \tag{206}$$

Here $\nabla^{\perp}$ is the covariant derivative of the ambient space applied to two vectorfields (one tangent vectorfield and one normal vectorfield) and projected onto the normal bundle. ∇ is the covariant derivative on the surface. On the surface we choose the basis ∂ and $\bar{\partial}$. We want to evaluate the Codazzi Mainardi equation on this base. We calculate (205) for $X = \partial$ and $Y = Z = \bar{\partial}$ (complex conjugation will lead to a conjugate set of equations - for every other choice of basis vectorfields this equation is trivially satisfied). Therefore we need the covariant derivatives, which can be read off from (75), (74) and (71).

$$\begin{aligned} \nabla_{\bar{\partial}}\bar{\partial} &= (\bar{\partial}\bar{\partial}Y)^{\|} = \bar{\partial}\alpha\bar{\partial} \\ \nabla_{\partial}\partial &= (\partial\partial Y)^{\|} = \partial\alpha\partial \\ \nabla_{\bar{\partial}}\partial &= \nabla_{\partial}\bar{\partial} = (\bar{\partial}\partial Y)^{\|} = 0 \end{aligned} \tag{207}$$

Here $()^{\|}$ denotes the projection onto the tangent space of the surface. Further, we have to calculate $S\partial$ and $S\bar{\partial}$.

$$\begin{aligned} S_i\partial &= S_i(\partial_s - i\partial_t) = \sqrt{\frac{e^\alpha}{2}}S_i(e_1 - ie_2) = \frac{u_i}{\sqrt{2e^\alpha}}(e_1 + ie_2) = u_i e^{-\alpha}\bar{\partial} \\ S_i\bar{\partial} &= \bar{u}_i e^{-\alpha}\partial \\ S_Y\partial &= -\partial \\ S_Y\bar{\partial} &= -\bar{\partial} \end{aligned} \tag{208}$$

We are dealing the spacelike and timelike case simultaneously here. All formulas are valid in both cases. In the timelike case the middle of the first line of (208) is different but the left hand side and right hand side are equal in both cases.

Whenever we have a sum over all normal directions (including Y) we use N_i to label the normal fields. If this sum is only over the normal directions of the surface inside AdS_n we use B_i.

The Codazzi Mainardi equation for the ambient space $\mathbb{R}^{(2,4)}$ reads

$$\begin{aligned} 0 = &\overbrace{\nabla^{\perp}_X(\mathrm{II}(Y,Z))}^{T_1} - \overbrace{\mathrm{II}(\nabla_X Y, Z)}^{T_2} - \overbrace{\mathrm{II}(Y, \nabla_X Z)}^{T_3} \\ &- \underbrace{\nabla^{\perp}_Y(\mathrm{II}(X,Z))}_{T_4} + \underbrace{\mathrm{II}(\nabla_Y X, Z)}_{T_5} + \underbrace{\mathrm{II}(X, \nabla_Y Z)}_{T_6} \end{aligned} \tag{209}$$

For our special choice some covariant derivatives vanish. So we have

$$T_2 = T_3 = T_5 = 0 \tag{210}$$

Calculation of T_1:

$$\begin{aligned}\mathrm{II}(\bar\partial,\bar\partial) &= \sum_i \langle S_i\bar\partial,\bar\partial\rangle\epsilon_i N_i = \sum_i \bar u_i\epsilon_i B_i \\ \nabla^\perp_{\bar\partial}\mathrm{II}(\bar\partial,\bar\partial) &= \partial(\sum_i \bar u_i\epsilon_i B_i)^\perp = \sum_i\left(\partial\bar u_i\epsilon_i B_i + \bar u_i\epsilon_i\sum_j \epsilon_j\langle\partial B_i, B_j\rangle B_j\right)\end{aligned} \tag{211}$$

Calculation of T_4:

$$\begin{aligned}\mathrm{II}(\partial,\bar\partial) &= \sum_i \langle S_i\partial,\bar\partial\rangle\epsilon_i N_i = -e^\alpha Y \\ \nabla^\perp_{\bar\partial}\mathrm{II}(\partial,\bar\partial) &= -\bar\partial(e^\alpha Y)^\perp = -\bar\partial\alpha e^\alpha Y\end{aligned} \tag{212}$$

Calculation of T_6:

$$\begin{aligned}\mathrm{II}(\partial,\nabla_{\bar\partial}\bar\partial) &= \bar\partial\alpha\mathrm{II}(\partial,\bar\partial) = \bar\partial\alpha\langle S_Y\partial,\bar\partial\rangle \\ &= -\bar\partial\alpha e^\alpha Y\end{aligned} \tag{213}$$

So we have

$$\begin{aligned}0 = T_1 - T_4 + T_6 &= \sum_i\left(\partial\bar u_i\epsilon_i B_i + \bar u_i\epsilon_i\sum_j \epsilon_j\langle\partial B_i, B_j\rangle B_j\right) \\ &= \partial\bar u^i B_i + \bar u^i A_i^{\ j} B_j\end{aligned} \tag{214}$$

Now we consider each of the $(n-2)$ normal components of the surface inside AdS_n. Then we see that this result (and the complex conjugate of this equation) is equivalent to the equations in the second line of (87). Taking only the B_a component yields

$$\begin{aligned}0 &= \partial\bar u^a + \bar u^b A_b^{\ a} \\ &= \partial\bar u_a + \bar u^b A_{b,a} \\ &= \partial\bar u_a - \bar u^b A_{a,b} \\ &= \partial\bar u_a - A_a^{\ b}\bar u_b\end{aligned} \tag{215}$$

which is precisely the second line in (87).

3.3.3 The Ricci equation

If we have a submanifold M that is embedded in an ambient space N, we can always define a "normal" curvature tensor. Let X, Y be two tangential vectorfields and A, B two normal fields. Then a covariant derivative on the normal space can be defined by

$$\nabla_X A := \hat{\nabla}_X A - (\hat{\nabla}_X A)^{\parallel} \tag{216}$$

Here $\hat{\nabla}$ is the covariant derivative of the ambient space. Then we can build two curvature tensors out of $\hat{\nabla}$ and ∇ and compare them. The curvature tensor corresponding to $\hat{\nabla}$ is the ordinary curvature tensor of the ambient space and we define

$$R^{\perp}(X, Y, A, B) := \langle \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A, B \rangle \tag{217}$$

Then the Ricci equation is

$$R(X, Y, A, B) = R^{\perp}(X, Y, A, B) - \langle [S_A, S_B] X, Y \rangle \tag{218}$$

S_A is the matrix that corresponds to the second fundamental form with respect to A. Now we evaluate this equation on some basis vectorfields. Let us assume $X = \partial$, $Y = \bar{\partial}$, $A = B_i$ and $B = B_k$ (again, this is the only independent possibility). Using $S_i = \frac{1}{2}\begin{pmatrix} 0 & u_i \\ \bar{u}_i & 0 \end{pmatrix}$, we compute

$$[S_i, S_k] = e^{-2\alpha} \begin{pmatrix} \bar{u}_i u_k - \bar{u}_k u_i & 0 \\ 0 & \bar{u}_k u_i - \bar{u}_i u_k \end{pmatrix} \tag{219}$$

So we have to verify

$$R^{\perp}(\partial, \bar{\partial}, B_i, B_k) = \langle [S_i, S_k] \partial, \bar{\partial} \rangle \tag{220}$$

$$\begin{aligned} R^{\perp}(\partial, \bar{\partial}, B_i, B_k) &= \langle \nabla_{\partial} \nabla_{\bar{\partial}} B_i - \nabla_{\bar{\partial}} \nabla_{\partial} B_i, B_k \rangle \\ =& \langle \nabla_{\partial} (\sum_j \epsilon_j \bar{A}_{i,j} B_j) - \nabla_{\bar{\partial}} (\sum_j \epsilon_j A_{i,j} B_j), B_k \rangle \\ =& \langle \sum_j \left(\epsilon_j \partial \bar{A}_{i,j} B_j + \epsilon_j \bar{A}_{i,j} A_j^{\,l} B_l - \epsilon_j \bar{\partial} A_{i,j} B_j - \epsilon_j A_{i,j} \bar{A}_j^{\,l} B_l \right), B_k \rangle \\ =& \partial \bar{A}_{i,k} - \bar{\partial} A_{i,k} + \bar{A}_i^{\,j} A_{j,k} - A_i^{\,j} \bar{A}_{j,k} \end{aligned} \tag{221}$$

The right hand side reads

$$\langle [S_i, S_k] \partial, \bar{\partial} \rangle = e^{-\alpha} (\bar{u}_i u_k - \bar{u}_k u_i) \tag{222}$$

Putting it together we find

$$e^{-\alpha} (\bar{u}_i u_k - \bar{u}_k u_i) = \partial \bar{A}_{i,k} - \bar{\partial} A_{i,k} + \bar{A}_i^{\,j} A_{j,k} - A_i^{\,j} \bar{A}_{j,k} \tag{223}$$

which is precisely the last line of (87). Again the calculation is valid in the timelike and spacelike case.

3.4 Gauge fixing for timelike minimal surfaces in AdS_n

As we are dealing with timelike minimal surfaces, all quantities in (87) are real. At first, we assume that neither $u^i u_i$, nor $\bar{u}^i \bar{u}_i$ are constantly zero (again a "non-exceptional" case). Away from their zeros we can locally perform two conformal transformations to achieve

$$u^i u_i = 1 = \bar{u}^i \bar{u}_i \tag{224}$$

So we can still choose $SO(n-2)$ transformations Ω on the normal space that depend on z and $\bar{z}$. We get the following transformations

$$\begin{aligned} u_i &\longmapsto \Omega_i^{\ j} u_j \\ \bar{u}_i &\longmapsto \Omega_i^{\ j} \bar{u}_j \\ A_i^{\ j} &\longmapsto (\Omega A \Omega^{-1} + \partial\Omega \ \ \Omega^{-1})_i^{\ j} \\ \bar{A}_i^{\ j} &\longmapsto (\Omega \bar{A} \Omega^{-1} + \bar{\partial}\Omega \ \ \Omega^{-1})_i^{\ j} \end{aligned} \tag{225}$$

First, we note that it is possible to choose $\bar{A}_i^{\ j} = 0$. To achieve this we have

$$\bar{\partial}\Omega = -\Omega \bar{A} \tag{226}$$

This differential equation can be solved. So we can still choose a z depending $SO(n-2)$ transformation. With this transformation we transform u^i to be

$$u^i = (0, \ldots, 0, 1) \tag{227}$$

Because $\bar{u}^i \bar{u}_i = 1$ we have for $\bar{u}^i$

$$\bar{u}^i = \left(\chi_4, \chi_5, \ldots, \chi_n, \pm\sqrt{1 - \sum_{a=4}^{n} \chi_a^2} \right) \tag{228}$$

Now the last line in (87) reads

$$\bar{\partial} A = -e^{-\alpha} \begin{pmatrix} 0 & \ldots & 0 & \chi_4 \\ \vdots & & \vdots & \chi_5 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & \chi_n \\ -\chi_4 & \ldots & -\chi_n & 0 \end{pmatrix} \tag{229}$$

We still have the gauge freedom to do z-dependent $SO(n-2)$ transformations that leave B_{n+1} invariant. Using this degree of freedom, we can achieve that all elements of A vanish, except those in the last line and in the last column. Then using the second line in (87) we find

$$\lambda_a = \pm \frac{\partial \chi_a}{\sqrt{1 - \sum_{a=4}^n \chi_a^2}} \quad , \qquad \bar{\partial} \lambda_a = -e^{-\alpha} \chi_a \tag{230}$$

So for A, we have

$$A = \begin{pmatrix} 0 & \dots & 0 & \lambda_4 \\ \vdots & & \vdots & \lambda_5 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \lambda_n \\ -\lambda_4 & \dots & -\lambda_n & 0 \end{pmatrix} \tag{231}$$

Now we completely fixed the gauge. We finally have a non-linear system of second order differential equations for the $(n-2)$ parameters $\alpha, \chi_4, \dots, \chi_n$.

$$\begin{aligned} 0 &= \partial \bar{\partial} \alpha \mp e^{-\alpha} \sqrt{1 - \sum_{a=4}^{n} \chi_n^2} - e^{\alpha} \\ 0 &= \partial \bar{\partial} \chi_b \pm e^{-\alpha} \chi_b \sqrt{1 - \sum_{a=4}^{n} \chi_n^2} + \frac{\sum_{a=4}^n \chi_n \bar{\partial} \chi_n}{1 - \sum_{a=4}^n \chi_n^2} \partial \chi_b \end{aligned} \tag{232}$$

The AdS_3 case

In the AdS_3 case no χ appears in these equations. We find the sinh-Gordon and a "cosh"-Gordon equation

$$\begin{aligned} \partial \bar{\partial} \alpha - 2 \sinh \alpha = 0 \\ \partial \bar{\partial} \alpha - 2 \cosh \alpha = 0 \end{aligned} \tag{233}$$

depending on whether or not the signs of u_4 and $\bar{u}_4$ are equal.

The AdS_4 case

In the AdS_4 case there is only χ_4. If we set $\chi_4 = \sin \beta$ and $\cos \beta = \pm \sqrt{1 - \chi_4^2}$ we find

$$\begin{aligned} \partial \bar{\partial} \alpha - e^{-\alpha} \cos \beta - e^{\alpha} = 0 \\ \partial \bar{\partial} \beta + e^{-\alpha} \sin \beta = 0 \end{aligned} \tag{234}$$

Note that these equations match those we found in the spacelike case.

3.5 Timelike minimal surfaces in AdS_5

From the second equation in (87) we see that $u^i u_i$ is not a function of $\bar{z}$ and $\bar{u}^i \bar{u}_i$ is not a function of z. Assuming that none of them is zero, we can perform a coordinate change locally such that $u^i u_i = \bar{u}^i \bar{u}_i = 1$. Similarly to the spacelike case we set

$$u^i = a^i + b^i \quad \bar{u}^i = a^i - b^i \tag{235}$$

with a and b being two real vectors. Then it follows that

$$\begin{aligned} a^i b_i &= 0 \\ a^i a_i + b^i b_i &= 1 \end{aligned} \tag{236}$$

Now we act with $SO(3)$ transformations on the normal space. If $a^i \neq 0$ and $b^i \neq 0$ we can choose a transformation such that

$$\begin{aligned} a^i &= (a, 0, 0) \\ b^i &= (0, b, 0) \end{aligned} \tag{237}$$

which leads to $a^2 + b^2 = 1$. Thus we parameterize a and b with β

$$\begin{aligned} u &= (\cos\beta, \sin\beta, 0) \\ \bar{u} &= (\cos\beta, -\sin\beta, 0) \end{aligned} \tag{238}$$

With these u^i and $\bar{u}^i$ we find

$$\begin{aligned} A_{4,5} &= -\partial\beta \\ A_{4,6} &= \rho\sin\beta \\ A_{5,6} &= \rho\cos\beta \end{aligned} \tag{239}$$

$$\begin{aligned} \bar{A}_{4,5} &= \bar{\partial}\beta \\ \bar{A}_{4,6} &= \bar{\rho}\sin\beta \\ \bar{A}_{5,6} &= -\bar{\rho}\cos\beta \end{aligned} \tag{240}$$

Inserting this into the third equation of (87) yields

$$0 = (\rho\bar{\rho} + e^{-\alpha})\sin 2\beta + 2\partial\bar{\partial}\beta \tag{241}$$

$$0 = 2\cos\beta(\bar{\rho}\partial\beta - \rho\bar{\partial}\beta) + \sin\beta(\partial\bar{\rho} - \bar{\partial}\rho) \tag{242}$$

$$0 = 2\sin\beta(\rho\bar{\partial}\beta + \bar{\rho}\partial\beta) - \cos\beta(\partial\bar{\rho} + \bar{\partial}\rho) \tag{243}$$

Inserting this into the Gauss equation yields

$$\partial\bar{\partial}\alpha - e^{-\alpha}\cos(2\beta) - e^{\alpha} = 0 \tag{244}$$

3.5.1 A Weierstrass like representation of time like minimal surfaces in AdS_5 with constant curvature

As remarked in previous sections, the curvature of the surface is given by

$$R = -2e^{-\alpha}\bar{\partial}\partial\alpha \tag{245}$$

So if α is a function that fulfills the Liouville equation, the corresponding minimal surfaces will have constant curvature. The generic solution of this differential equation is given by

$$\alpha = \log\left(-\frac{2f'(z)g'(\bar{z})}{\left(1-\frac{R}{2}f(z)g(\bar{z})\right)^2}\right) \tag{246}$$

with two arbitrary free functions $f(z)$ and $g(\bar{z})$ (such that log is defined) that only depend on z and $\bar{z}$. From the Gauss equation for timelike minimal surfaces in AdS_5 (244) we find

$$\cos 2\beta = -e^{2\alpha}\left(\frac{R}{2}+1\right) \tag{247}$$

Thus we can express β and every function of β in terms of $f(z)$ and $g(\bar{z})$. By (241) we can express $\rho\bar{\rho}$ as a function of of $f(z)$ and $g(\bar{z})$.

$$\rho\bar{\rho} = -\frac{\partial\bar{\partial}(2\beta)}{\sin 2\beta} - e^{-\alpha} =: \chi(f,g) \tag{248}$$

Now we multiply (242) with $\sin\beta$ and (243) with $\cos\beta$. Then we calculate (242)+(243) and (242)-(243) and find

$$\bar{\partial}\rho = \sin 2\beta\partial(2\beta)\bar{\rho} - \cos 2\beta\partial\bar{\rho} =: C_1\bar{\rho} - C_2\partial\bar{\rho} \tag{249}$$
$$\partial\bar{\rho} = \sin 2\beta\bar{\partial}(2\beta)\rho - \cos 2\beta\bar{\partial}\rho =: C_3\rho - C_2\bar{\partial}\rho \tag{250}$$

Here $C_1(f,g)$, $C_2(f,g)$ and $C_3(f,g)$ are via β functions of f and g. From (248) we know further that

$$\bar{\rho} = \frac{\chi}{\rho} \quad \rho = \frac{\chi}{\bar{\rho}} \tag{251}$$

This leads to

$$\begin{aligned} \partial\rho &= \left(\frac{C_1C_2}{1-C_2^2} + \frac{\partial\chi}{\chi}\right)\rho - \frac{C_3}{\chi(1-C_2^2)}\rho^3 \\ \bar{\partial}\rho &= \frac{C_1\chi}{1-C_2^2}\frac{1}{\rho} - \frac{C_3C_2}{1-C_2^2}\rho \end{aligned} \tag{252}$$

Similarly one finds for $\bar{\rho}$

$$\begin{aligned}\bar{\partial}\bar{\rho} &= \left(\frac{C_3C_2}{1-C_2^2} + \frac{\bar{\partial}\chi}{\chi}\right)\bar{\rho} - \frac{C_1}{\chi(1-C_2^2)}\bar{\rho}^3 \\ \partial\bar{\rho} &= \frac{C_3\chi}{1-C_2^2}\frac{1}{\bar{\rho}} - \frac{C_1C_2}{1-C_2^2}\bar{\rho}\end{aligned} \tag{253}$$

All coefficients are fully determined by the free functions $f(z)$ and $g(\bar{z})$. Then the partial differential equations (which can be solved separately for ρ and $\bar{\rho}$) have to be solved. After that the differential equation for the orthogonal frame (76) has to be integrated with all entries solely depending on $f(z)$ and $g(\bar{z})$ (and of course on the constant R). Thus the two functions $f(z)$ and $g(\bar{z})$ parameterize all constantly curved timelike minimal surfaces in AdS_5. Here is some analogy to the minimal surfaces in $\mathbb{R}^3$. In Weierstrass representation they are parameterized by two functions, too. To get the coordinate representation a integration has also to be performed.

3.6 Invariants of minimal surfaces in $AdS_n,\ \ n > 3$

In this section we will introduce a torsion quantity T and a further quantity C. T encodes the curvatrue of the normal bundle of the surface. Both quantities will prove to be invariant under both transformations on the normal space and isothermal reparameterization of the surface. With these quantities we will retrieve (together with the Gauss equation) an equation that relates scalar curvature to the torsion T and C.

3.6.1 The torsion quantity

Examining the last equation of (87) we see that the right hand side has the structure of a field strength corresponding to the gauge field $A_\mu^{\ \nu}$. Thus we define

$$T = \frac{1}{8\mid \det g \mid}\epsilon^{\alpha,\beta}\epsilon^{\mu,\nu}\mathrm{tr}(F_{\alpha,\beta}F_{\mu,\nu}) \tag{254}$$

in conformal coordinates we find with (87)

$$T = \frac{1}{2}e^{-2\alpha}\mathrm{tr}(F^2) = e^{-4\alpha}\left((\bar{u}_a u^a)^2 - (\bar{u}_a\bar{u}^a)\left(u_b u^b\right)\right) \tag{255}$$

This quantity is invariant (for a brief proof see appendix 6.6). Together with gauss equation (first line in (87)) and the equation for the scalar curvature (101) we find

$$0 = R + 2 \pm 2e^{-2\alpha}\sqrt{(\bar{u}^a\bar{u}_a)\left(u^b u_b\right) + Te^{4\alpha}} \tag{256}$$

$$C := e^{-4\alpha} \left(\bar{u}^a \bar{u}_a\right) \left(u^b u_b\right) \tag{257}$$

For timelike surfaces $T \leq 0$, while T can have both signs for spacelike surfaces. All cases that we excluded in the last chapters (when at least $\bar{u}_i \bar{u}^i \equiv 0$ or $u_i u^i \equiv 0$) are described by $C = 0$. However, C is also an invariant quantity and positive semidefinite for spacelike minimal surfaces.

It is very interesting to ask, how many gauge invariant scalar parameters are encoded within $F_{i,j}$. The trace of F^2 leads to one. Despite its interpretation as a field strength, F is also just a function on the two vectors u^i and $\bar{u}^i$. So all invariants that come from $F_{i,j}$ have to be functions of these two vectors. The three possible scalar products are the only invariants under transformations on the normal space, that can be constructed. Two of them are conjugate in the spacelike case. But the overall computation leads to three real parameters. Now we have to multiply certain powers of $e^{-\alpha}$ in order to obtain an invariant expression. But only the product $(u^i u_i)(\bar{u}^j \bar{u}_j)$ can lead to an invariant. So the only two invariants are $e^{-2\alpha}(u^i \bar{u}_i)$ and $C = (u^i u_i)(\bar{u}^j \bar{u}_j)e^{-4\alpha}$. Equivalently T and C are the only invariant scalars of outer geometry. Overall we have three independent quantities that characterize a minimal surface in AdS_n. The gaussian curvature $R(z)$ describes the whole inner geometry of the surface at a given point. The outer geometry has two scalar quantities - $C(z)$ and $T(z)$. These three quantities are related with

$$C = \frac{(R+2)^2}{4} - T \tag{258}$$

This equation is very remarkable. Normally quantities of inner and outer geometry are not related in this manner. The relation comes from the fact that we are dealing with minimal surfaces here which gives a relation to the outer geometry, as minimality is an obstruction for the second fundamental forms which is an inherent feature of this formalism.

We also get a necessary condition for two minimal surfaces inside AdS_n being equal. Given two functions of z $R(z, \bar{z})$ and $T(z, \bar{z})$ we can via (258) compute $C(z, \bar{z})$. Then we have a complete set of invariants.

We also remark that these invariants are quite comfortable to calculate if we have an explicitely given minimal surface. To calculate quantities in our formalism you would normally start to calculate the normal vectors $\{B_4, \ldots, B_{n-1}\}$ in every point of the surface which can be quite difficult. This is not necessary if we are only interested in the invariant quantities. At first we need the conformal factor of the metric to calculate R. Then we start to calculate two vectors u and $\bar{u}$ by setting

$$u = \partial\partial Y - \mathrm{pr}_{\mathrm{T}M}(\partial\partial Y) - \mathrm{pr}_Y(\partial\partial Y) \tag{259}$$

$$\bar{u} = \bar{\partial}\bar{\partial} Y - \mathrm{pr}_{\mathrm{T}M}(\bar{\partial}\bar{\partial} Y) - \mathrm{pr}_Y(\bar{\partial}\bar{\partial} Y) \tag{260}$$

Here $\mathrm{pr}_{\mathrm{T}M}$ denotes the projection. Now we construct the scalar products and multiply with $e^{-4\alpha}$. The scalar products are computed with the metric in $\mathbb{R}^{(2,n-1)}$. This is possible because the metric on our normal space is the induced metric from $\mathbb{R}^{(2,n-1)}$. However, these vectors u and $\bar{u}$ cannot be used in any other sense in this formalism as they are not given in an orthonormal base of the normal space.

3.6.2 Exceptional case $C = 0$

For this case we have

$$0 = R + 2 \pm 2\sqrt{T} \tag{261}$$

Let us first consider timelike surfaces. Here we know that $T \leq 0$ and so we have $T \equiv 0$. Thus $R \equiv -2$. From the definition of C we also see that there are two possibilities. If $\bar{u}_i\bar{u}^i \equiv u_i u^i \equiv 0$ the surface is a geodesically embedded $AdS_2 \subset AdS_n$. There is however the possibility that either $\bar{u}_i$ or u_i is not zero. In this case the surface is an $AdS_2 \subset AdS_n$ but not geodesically embedded.

For spacelike surfaces and if $T = 0$, the surface is a hyperbolic plane embedded in AdS_n with $R = -2$. If $T \neq 0$ it follows by (258) that $T > 0$. In this case we can formulate an even stronger equation and have

$$0 = R + 2 + 2\sqrt{T} \tag{262}$$

which will be done in the appendix 6.7. Here we see that in the exceptional case there are no flat minimal spacelike surfaces in AdS_n because setting $R = 0$ and taking $T > 0$ is a contradiction. We can even conclude that for all spacelike exceptional (which both are invariant attributes) the curvature has to be $R \leq -2$.

4 The generic problem

In [AM09b] the authors calculate a regularized area of the eight cusp solution without reconstructing the surface explicitely. However they restrict themselves to a surface that is embedded in an $AdS_3 \subset AdS_5$ subspace. In the four cusp case it was possible to reconstruct the generic four cusp case via certain $SO(2,4)$ transformations. But if we increase the number of cusps we also increase the number of conformally invariant parameters of the configuration on the conformal boundary. Therefore it is not possible to reconstruct the generic eight cusp case in AdS_5 from one given solution. In this section we will first count the number of invariant parameters. Then we will outline the results given in [AM09b].

4.1 Conformal invariants

We want to characterize a scattering process in $\mathbb{R}^{(1,3)}$ with $SO(1,3)$ invariant quantities. To give a n-cusp configuration on the conformal boundary, we have n vectors with 4 components each. Because the whole configuration has to be closed, we are left with $4(n-1)$ parameters. Every vector is lightlike so the number is reduced to $4(n-1)-n$. We are looking for $SO(1,3)$ invariant parameters. So we end up with

$$\#M = 4(n-1) - n - 6 = 3n - 10 \tag{263}$$

lorentz invariant quantities to characterize a generic n cusp configuration on the conformal boundary of AdS_5. For the four cusp case we see that we need 2 quantities to characterize the configuration - s and t.

The conformal group $\widetilde{O(2,d)}$ on the conformal boundary has $\frac{(d+1)(d+2)}{2} = 15$ generators. 6 generators belong to a $SO(1,3)$ subgroup, so they do not change the invariant quantities. The four translations also do not change the configuration. So there remain five generators in the conformal group that change the invariant parameters. So for $n > 5$ we surely cannot reconstruct the generic case from a single solution via conformal transformations. In the $N = 5$ case we have $15 - 10 = 5$ independent scattering parameters. So it might be possible to reconstruct the generic case from a single solution. In [Mey09] the author shows that in the 5 cusp case it is at least locally possible to reconstruct the generic case out of a single solution via conformal transformations, i.e. that the functional determinant of the transformation is nonezero.

4.2 The octagon

Throughout this thesis we dealed with minimal surfaces inside AdS_n using a Pohlmeyer reduction. In [AM09b] and [AM09a] Alday and Maldacena use an analogous formalism for the AdS_3 case. They are able to calculate the area in the octagon case in AdS_3. This is remarkable as they are able to give this result without explicitely solving the problem. As this is a significant progress we will summarize their results in this chapter. They use almost the same variables and vectorfields. For a better comparison we will adopt their notation in this chapter. They introduce $z = \sigma + i\tau$, $\partial = \frac{1}{2}(\partial_\sigma - i\partial_\tau)$ and the following basis frame in $\mathbb{R}^{(2,2)}$

$$\begin{aligned} &q_1 = Y \quad q_2 = e^{-\alpha}\bar{\partial}Y \quad q_3 = e^{-\alpha}\partial Y \quad q_4 = N \\ &q_1^2 = -1 \quad \langle q_2, q_3\rangle = 2 \quad q_2^2 = q_3^2 = 0 \quad q_4^2 = 1 \end{aligned} \tag{264}$$

Their definition of α,N and p is

$$\begin{aligned} \alpha &= \frac{1}{2}\log(\frac{1}{2}\langle\partial Y, \bar{\partial}Y\rangle) \\ N_a &= \frac{e^{-2\alpha}}{2}\epsilon_{a,b,c,d}Y^b\partial Y^c\bar{\partial}Y^d \\ p &= -2\langle N, \partial\partial Y\rangle \ , \quad \bar{p} = 2\langle N, \bar{\partial}\bar{\partial}Y\rangle \end{aligned} \tag{265}$$

Basically, their vectorfield N corresponds to our B_4 in the AdS_3 case and their p corresponds to our u_4 which is a holomorphic function. The area of the surface is then given by

$$A = 4\int e^{2\alpha}d^2z \tag{266}$$

In the AdS_3 case the problem of integration of (76) is in general much more simple because we can use the decomposition of $SO(2,2)$ into $SO(2,2) = SL(2)\times SL(2)$. Furthermore they use this fact to prove the behavior of their solution for large z. They also find the analogon of our Sinh-Gordon equation (182).

$$\partial\bar{\partial}\alpha - e^{2\alpha} + \mid p(z)\mid^2 e^{-2\alpha} = 0 \tag{267}$$

The main difference between their analysis and our calculations is that they do not a priori introduce a conformal transformation on the surface in order to achieve $p^2 = \bar{p}^2 \equiv 1$. They consider $p(z)$ to be an holomorphic function. According to them $p(z)$ is the only function or parameter in the calculation that can carry information about the conformal boundary of the surface inside AdS_3. As we mentioned before, the conformal transformation to achieve $p^2 \equiv 1$ is locally possible away from the zeros of $p(z)$. So they assume that the information about the conformal boundary of the surface is encoded in the zeros of the holomorphic function $p(z)$. Of course it is possible to locally introduce a w-plane by setting

$$dw = \sqrt{p(z)}dz \tag{268}$$

This transformation leads to the real Sinh-Gordon equation.

$$\partial_w \partial_{\bar{w}} \hat{\alpha} - e^{2\hat{\alpha}} + e^{-2\hat{\alpha}}, \qquad \hat{\alpha} := \alpha - \frac{1}{4} \log p\bar{p} \tag{269}$$

However, the information about the zeros of p is not lost because the variable w has branch cuts at the zeros of p. In their paper [AM09b] Alday and Maldacena use this w plane to study the behavior of the surface at large $\mid z \mid$. In the case of the tetragon solution the planes w and z are identified with each other. This solution is given by $\alpha = \hat{\alpha} = 0$ and $p(z) = 1$.

They argue that a polynomial of degree $n-2$ should lead to a solution with $2n$ cusps. In the octagon case they take the polynomial $p(z)$ to be

$$p(z) = z^2 - m \tag{270}$$

They introduce $x^{\pm} = x^0 \pm x^1$ on the conformal boundary in Poincaré coordinates and observe that only one coordinate changes when we go from one cusp to the next. So there are four x_i^+ and four x_i^- that define the cusps. Out of these $x_i^{\pm}$ they construct two conformally independent cross ratios $\chi^{\pm}$. Then they work out a relation between $m \in \mathbb{C}$ and these two cross ratios $\chi^{\pm}$ where $\chi^{\pm}$ only depends on the $x_i^{\pm}$. The computation of the area is done via an asymptotically approximation of the solutions of the Painleve III equation for their $\hat{\alpha}$. The result is an expression of the regularized area of the octagon solution with dependence on m.

5 Conclusions and discussion

In this diploma thesis we examined minimal surfaces in AdS_n inspired by the conjectured correspondence between certain gluon scattering amplitudes and spacelike minimal surfaces with a lightlike polygonal boundary on the conformal boundary of AdS_5 (see [AM07]). We started to review the generic tetragon solution and used an alternative regularization to calculate the area. Our term for the expansion can be interpreted to be similar to an expression that was given in [Ald08] for a cutoff in the radial direction of AdS_5 with a cutoff parameter that is a function of the coordinates on the conformal boundary.

Then we developed a Pohlmeyer reduction for minimal surfaces in AdS_n in a similar way we did in [DJW09] and extended the results. We were able to treat both timelike and spacelike minimal surfaces simultaneously. Using this formalism we showed that the Gauss-, Codazzi-Mainardi- and Ricci- equation for minimal surfaces appear as integrability conditions in the formalism. Further we proved that there is a bigger variety of flat timelike minimal surfaces in AdS_n than in the spacelike case. This is due to the fact that the corresponding $\rho\bar{\rho}$ term can have both signs in the timelike case. Further we showed that every spacelike flat minimal surface in AdS_n is (a part of) a surface that is obtained from the tetragon solution with isometry transformations of AdS_n. So necessarily all further minimal surfaces that correspond to other scattering amplitudes are nonflat. We also derived the differential equations for spacelike and timelike minimal surfaces in AdS_5, AdS_4 and AdS_3. We also noticed that there are no spacelike minimal surfaces inside AdS_3 that have a scalar curvature $R \geq -2$. Further we were able to describe all constantly curved timelike minimal surfaces with two free functions. In the timelike case it was also possible to do a complete gauge fixing for timelike minimal surfaces for every dimension.

For every dimension $n > 3$ we found two invariant scalar quantities C and T that describe the outer geometry of minimal surfaces. They are connected to the scalar curvature with a simple algebraic equation.

$$C = \frac{(R+2)^2}{4} - T \tag{271}$$

This is remarkable because normally invariants of inner and outer geometry are not related in this way. The relation derives from the minimality condition which on one hand is a critical point of the area functional but on the other hand also an obstruction for the second fundamental forms.

We introduced the term "exceptional" surface for the case where $C = 0$ and further showed that every spacelike exceptional minimal surface in AdS_n has a scalar curvature $R \leq -2$. The invariants are also comfortable to use as they can be calculated easily for a given minimal surface.

In the last chapter we added some notes on the conformal invariants of an arbitrary lightlike n-gon in the four dimensional Minkowski space and sketched the new results in [AM09b] where the authors use a similar formalism for AdS_3.

6 Appendix

6.1 Proof that the one cusp solution is a minimal surface

We now explicitely verify that this solution satisfies the equation of motion for the Nambu-Goto action (23).

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^{\mu}}+\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial\mathcal{L}}{\partial x'^{\mu}}-\frac{\partial\mathcal{L}}{\partial x^{\mu}}=0 \tag{272}$$

A dot refers to the derivative with respect to τ and a prime refers to the derivative with respect to σ. The Lagrangian Density $\mathcal{L}$ is simply the integrand from the action

$$\mathcal{L}=\frac{1}{r^2}\sqrt{-(-\dot{x}_0^2+\dot{x}_1^2+\dot{r}^2)(-x_0'^2+x_1'^2+r'^2)+(-x_0'\dot{x}_0+x_1'\dot{x}_1+r'\dot{r})^2} \tag{273}$$

We will evaluate the equation of motion for every $\mu\in\{0,1,2\}$ and we start with $\mu=0$.

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{x}_0}&=\frac{{x_1}'^2\dot{x}_0-{x_0}'{x_1}'\dot{x}_1+r'^2\dot{x}_0-r'{x_0}'\dot{r}}{r^2\sqrt{-(-\dot{x}_0^2+\dot{x}_1^2+\dot{r}^2)(-x_0'^2+x_1'^2+r'^2)+(-x_0'\dot{x}_0+x_1'\dot{x}_1+r'\dot{r})^2}}\\
&=-\frac{\cosh\sigma}{2e^{\tau}}i\\
\frac{\partial\mathcal{L}}{\partial{x_0}'}&=\frac{-{x_1}'\dot{x}_0\dot{x}_1-r'\dot{x}_0\dot{r}+{x_0}'\dot{x}_1^2+{x_0}'\dot{r}^2}{r^2\sqrt{-(-\dot{x}_0^2+\dot{x}_1^2+\dot{r}^2)(-x_0'^2+x_1'^2+r'^2)+(-x_0'\dot{x}_0+x_1'\dot{x}_1+r'\dot{r})^2}}\\
&=-\frac{\sinh\sigma}{2e^{\tau}}i
\end{aligned}$$

Now it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}_0}+\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial\mathcal{L}}{\partial{x_0}'}=\frac{\cosh\sigma}{2e^{\tau}}i-\frac{\cosh\sigma}{2e^{\tau}}i=0$$

For $\mu=1$ we obtain

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot{x}_1}&=\frac{-{x_0}'{x_1}'\dot{x}_0+{x_0}'^2\dot{x}_1-r'^2\dot{x}_1+{x_1}'r'\dot{r}}{r^2\sqrt{-(-\dot{x}_0^2+\dot{x}_1^2+\dot{r}^2)(-x_0'^2+x_1'^2+r'^2)+(-x_0'\dot{x}_0+x_1'\dot{x}_1+r'\dot{r})^2}}\\
&=\frac{\sinh\sigma}{2e^{\tau}}i\\
\frac{\partial\mathcal{L}}{\partial{x_1}'}&=\frac{-{x_0}'\dot{x}_0\dot{x}_1+r'\dot{x}_1\dot{r}+{x_1}'\dot{x}_0^2-{x_1}'\dot{r}^2}{r^2\sqrt{-(-\dot{x}_0^2+\dot{x}_1^2+\dot{r}^2)(-x_0'^2+x_1'^2+r'^2)+(-x_0'\dot{x}_0+x_1'\dot{x}_1+r'\dot{r})^2}}\\
&=\frac{\cosh\sigma}{2e^{\tau}}i
\end{aligned}$$

And again, it turns out that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}_1}+\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial\mathcal{L}}{\partial{x_1}'}=-\frac{\sinh\sigma}{2e^{\tau}}i+\frac{\sinh\sigma}{2e^{\tau}}i=0$$

The last remaining case $\mu=2$ is the only case where the Lagrangian density explicitly depends on x.

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial r}&=-\frac{2\sqrt{-(-\dot{x}_0^2+\dot{x}_1^2+\dot{r}^2)(-x_0'^2+x_1'^2+r'^2)+(-x_0'\dot{x}_0+x_1'\dot{x}_1+r'\dot{r})^2}}{r^3}\\
&=-\frac{1}{\sqrt{2}e^{\tau}}i\\
\frac{\partial\mathcal{L}}{\partial\dot{r}}&=\frac{-{x_0}'r'\dot{x}_0+{x_0}'^2\dot{r}+{x_1}'r'\dot{x}_1-{x_1}'^2\dot{r})}{r^2\sqrt{-(-\dot{x}_0^2+\dot{x}_1^2+\dot{r}^2)(-x_0'^2+x_1'^2+r'^2)+(-x_0'\dot{x}_0+x_1'\dot{x}_1+r'\dot{r})^2}}\\
&=\frac{1}{\sqrt{2}e^{\tau}}i\\
\frac{\partial\mathcal{L}}{\partial r'}&=\frac{r'\dot{x}_0^2-r'\dot{x}_1^2-{x_0}'\dot{x}_0\dot{r}+{x_1}'\dot{x}_1\dot{r}}{r^2\sqrt{-(-\dot{x}_0^2+\dot{x}_1^2+\dot{r}^2)(-x_0'^2+x_1'^2+r'^2)+(-x_0'\dot{x}_0+x_1'\dot{x}_1+r'\dot{r})^2}}\\
&=0
\end{aligned}$$

Finally, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\dot{r}}+\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial\mathcal{L}}{\partial r'}-\frac{\partial\mathcal{L}}{\partial r}=-\frac{1}{\sqrt{2}e^{\tau}}i+0+\frac{1}{\sqrt{2}e^{\tau}}i=0$$

So we see that (22) actually is a solution of the equations of motion.

6.2 Proof that the tetragon solution is a minimal surface

We verify that (31) solves the equations of motion.

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\frac{\partial\mathcal{L}}{\partial(\partial_1x^{\mu})}+\frac{\mathrm{d}}{\mathrm{d}x_2}\frac{\partial\mathcal{L}}{\partial(\partial_2x^{\mu})}-\frac{\partial\mathcal{L}}{\partial x^{\mu}}=0 \tag{274}$$

The action we are using here is

$$\begin{aligned}
&iA=\\
&\int dx_1dx_2\frac{\sqrt{1+(\partial_1r)^2+(\partial_2r)^2-(\partial_1x_0)^2-(\partial_2x_0)^2-(\partial_1r\partial_2x_0-\partial_2r\partial_1x_0)^2}}{r^2}
\end{aligned}$$

For $\mu \in \{1,2\}$ this is automatically true. So let us consider $\mu = 0$.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_1 x_0)} &= \frac{1}{r^2\sqrt{X}}((\partial_1 r \partial_2 x_0 - \partial_2 r \partial_1 x_0)\partial_2 r - \partial_1 x_0) \\
&= -\frac{x_2}{(1-x_2^2)^2} \\
\frac{\partial \mathcal{L}}{\partial(\partial_2 x_0)} &= \frac{1}{r^2\sqrt{X}}(-(\partial_1 r \partial_2 x_0 - \partial_2 r \partial_1 x_0)\partial_1 r - \partial_2 x_0) \\
&= -\frac{x_1}{(1-x_1^2)^2}
\end{aligned}$$

Here

$$X = 1 + (\partial_1 r)^2 + (\partial_2 r)^2 - (\partial_1 x_0)^2 - (\partial_2 x_0)^2 - (\partial_1 r \partial_2 x_0 - \partial_2 r \partial_1 x_0)^2$$

So both total derivatives vanish and the equation of motion holds. The last remaining case is $\mu = 3, \quad x^\mu = r$.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= -\frac{2\sqrt{X}}{r^3} = -\frac{2}{\sqrt{(1-x_1^2)(1-x_2^2)}^3} \\
\frac{\partial \mathcal{L}}{\partial(\partial_1 r)} &= \frac{1}{r^2\sqrt{X}}(-(\partial_1 r \partial_2 x_0 - \partial_2 r \partial_1 x_0)\partial_2 x_0 + \partial_1 r) \\
&= -\frac{x_1}{(1-x_2^2)\sqrt{(1-x_1^2)(1-x_2^2)}} \\
\frac{\partial \mathcal{L}}{\partial(\partial_2 r)} &= \frac{1}{r^2\sqrt{X}}((\partial_1 r \partial_2 x_0 - \partial_2 r \partial_1 x_0)\partial_1 x_0 + \partial_2 r) \\
&= -\frac{x_2}{(1-x_1^2)\sqrt{(1-x_1^2)(1-x_2^2)}}
\end{aligned}$$

So we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x_1}\frac{\partial\mathcal{L}}{\partial(\partial_1 r)}+\frac{\mathrm{d}}{\mathrm{d}x_2}\frac{\partial\mathcal{L}}{\partial(\partial_2 r)}-\frac{\partial\mathcal{L}}{\partial r}\\
=&\frac{-1}{(1-x_2^2)\sqrt{(1-x_1^2)(1-x_2^2)}}-\frac{x_1^2}{\sqrt{(1-x_1^2)(1-x_2^2)}^3}\\
-&\frac{1}{(1-x_1^2)\sqrt{(1-x_1^2)(1-x_2^2)}}-\frac{x_2^2}{\sqrt{(1-x_1^2)(1-x_2^2)}^3}\\
+&\frac{2}{\sqrt{(1-x_1^2)(1-x_2^2)}^3}\\
=&0
\end{aligned}$$

So this is really a minimal surface as stated above.

6.3 Boosted tetragon solution

We proof that (35) is connected with (34) via a $SO(2,3)$ transformation. The (35) solutions depends on a and b. Setting $a=1$ and $b=0$ retrieves the surface (34). First we apply the relation (18) on (34) and find

$$X(u_1,u_2)=\begin{pmatrix}\cosh u_1\cosh u_2\\ \sinh u_1\sinh u_2\\ \sinh u_1\cosh u_2\\ \cosh u_1\sinh u_2\\ 0\end{pmatrix} \tag{275}$$

Further we can show that

$$A\ B\ X(u_1,u_2)=\begin{pmatrix}\frac{(1+a^2)\cosh u_1\cosh u_2-(a^2-1)b\sinh u_1\sinh u_2}{2a}\\ \sqrt{1+b^2}\sinh u_1\sinh u_2\\ \sinh u_1\cosh u_2\\ \cosh u_1\sinh u_2\\ \frac{-(a^2-1)\cosh u_1\cosh u_2+(a^2+1)b\sinh u_1\sinh u_2}{2a}\end{pmatrix} \tag{276}$$

with

$$A=\begin{pmatrix}\frac{1+a^2}{2a}&0&0&0&\frac{1-a^2}{2a}\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ \frac{1-a^2}{2a}&0&0&0&\frac{1+a^2}{2a}\end{pmatrix}\quad B=\begin{pmatrix}1&0&0&0&0\\ 0&\sqrt{1+b^2}&0&0&b\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&b&0&0&\sqrt{1+b^2}\end{pmatrix} \tag{277}$$

But applying the relation (18) we see that (276) is exactly (35). But $A, B \in SO(2,3)$. So (35) is the image under a $SO(2,3)$ transformation from (34), as stated above.

6.4 Dependence of s and t on a and b

We are considering the boosted tetragon solution

$$\begin{aligned} r &= \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\ y_0 &= \frac{a\sqrt{1+b^2} \sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\ y_1 &= \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\ y_2 &= \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \end{aligned} \tag{278}$$

In Poincaré coordinates we approach the conformal boundary for $r \to 0$. So if either u_1 or u_2 goes towards $\pm\infty$ we approach the conformal boundary. Thus the four cusps of the tetragon are given via

1. $u_1 \to \infty$ and $u_2 \to \infty$

In this case we find

$$y_0 = \frac{a\sqrt{1+b^2}}{1+b} \quad y_1 = \frac{a}{1+b} \quad y_2 = \frac{a}{1+b} \tag{279}$$

2. $u_1 \to \infty$ and $u_2 \to -\infty$

This yields

$$y_0 = \frac{a\sqrt{1+b^2}}{b-1} \quad y_1 = \frac{a}{1-b} \quad y_2 = \frac{a}{b-1} \tag{280}$$

3. $u_1 \to -\infty$ and $u_2 \to \infty$

We find

$$y_0 = \frac{a\sqrt{1+b^2}}{b-1} \quad y_1 = \frac{a}{b-1} \quad y_2 = \frac{a}{1-b} \tag{281}$$

4. $u_1 \to -\infty$ and $u_2 \to -\infty$

Finally, we have

$$y_0 = \frac{a\sqrt{1+b^2}}{1+b} \quad y_1 = -\frac{a}{1+b} \quad y_2 = -\frac{a}{1+b} \tag{282}$$

So these four points are the corners on the conformal boundary of AdS_4. Here it is easy to see that the vectors (the four momenta) that join two consecutive corners are lightlike. The four momenta are

$$
\begin{aligned}
P_{12} &= \begin{pmatrix} -\frac{2a\sqrt{1+b^2}}{b^2-1} \\ \frac{2ab}{b^2-1} \\ -\frac{2a}{b^2-1} \end{pmatrix} \qquad P_{13} = \begin{pmatrix} -\frac{2a\sqrt{1+b^2}}{b^2-1} \\ -\frac{2a}{b^2-1} \\ \frac{2ab}{b^2-1} \end{pmatrix} \\
P_{43} &= \begin{pmatrix} -\frac{2a\sqrt{1+b^2}}{b^2-1} \\ -\frac{2ab}{b^2-1} \\ \frac{2a}{b^2-1} \end{pmatrix} \qquad P_{42} = \begin{pmatrix} -\frac{2a\sqrt{1+b^2}}{b^2-1} \\ \frac{2a}{b^2-1} \\ -\frac{2ab}{b^2-1} \end{pmatrix}
\end{aligned} \tag{283}
$$

Here P_{ij} means the coordinates from corner i minus the coordinates from corner j. So the mandelstam variables are

$$
\begin{aligned}
s(2\pi)^2 :&= (P_{12}+P_{14})^2 = -\frac{8a^2}{(b-1)^2} \\
t(2\pi)^2 :&= (P_{43}+P_{14})^2 = -\frac{8a^2}{(b+1)^2} \\
\frac{s}{t} &= \frac{(b+1)^2}{(b-1)^2}
\end{aligned} \tag{284}
$$

which is the result (36).

6.5 Proof of the uniformly convergence

Here we show how to calculate the integral

$$
\frac{A_{err}}{8} = \lim_{\epsilon\to 0}\int_0^{x(\epsilon)} du_1 \log\left(\frac{\left(\sqrt{\frac{2}{\cos^2(\frac{\pi}{2}-\epsilon)}-1}+\sqrt{\frac{2}{\cos^2(\frac{\pi}{2}-\epsilon)}-2}\right)e^{-u_1}}{\left(\sqrt{\frac{\frac{1}{\cos^2(\frac{\pi}{2}-\epsilon)}-\cosh^2 u_1}{\cosh^2 u_1+\sinh^2 u_1}}+\sqrt{\frac{\frac{1}{\cos^2(\frac{\pi}{2}-\epsilon)}+\sinh^2 u_1}{\cosh^2 u_1+\sinh^2 u_1}}\right)}\right) \tag{285}
$$

with $x(\epsilon) = \mathrm{ArcCosh}\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{2}{\cos^2(\frac{\pi}{2}-\epsilon)}-1}}\right)$. The correct way would be to calculate the integral for all ϵ and then take the limit $\epsilon \to 0$. However this turns out to be difficult. So we label the ϵ in the upper integration boundary with ϵ_1 and the ϵ in the integrand with ϵ_2. If we call the integrand $I(u_1,\epsilon_2)$ we have

$$
\frac{A_{err}}{8} = \lim_{\epsilon_1\to 0}\lim_{\epsilon_2\to 0}\int_0^{x(\epsilon_1)} du_1 I(u_1,\epsilon_2) \tag{286}
$$

Now we would like to take the limit $\epsilon_2 \to 0$, integrate then and finally take $\epsilon_1 \to 0$. For this procedure to be correct we have to verify that the integrand converges uniformly.

$$\lim_{\epsilon_2 \to 0} I(u_1, \epsilon_2) = \frac{1}{2} \log\left(1 + e^{-4u_1}\right) =: I(u_1) \tag{287}$$

Now we show that this convergence is uniformly. Therefore we have to show that

$$\lim_{\epsilon_2 \to 0} \sup_{u_1 \in (0, x(\epsilon_1))} \mid (I(u_1, \epsilon_2) - I(u_1)) \mid = 0 \tag{288}$$

By direct inspection we see that $\mid I(u_1, \epsilon_2) - I(u_1) \mid = I(u_1) - I(u_1, \epsilon_2)$. Further this term is monotonely falling and therefore

$$\sup_{u_1 \in (0, x(\epsilon_1))} \mid (I(u_1, \epsilon_2) - I(u_1)) \mid = I(0) - I(0, \epsilon_2) \tag{289}$$

But the limit

$$\lim_{\epsilon_2 \to 0} (I(0) - I(0, \epsilon_2)) \tag{290}$$

is 0. That means that we can first take the limit $\epsilon_2 \to 0$ and integrate then.

6.6 Proof of the invariance of T and C

First, let us consider transformations on the normal space. A matrix $A \in SO(1, n-3)$ or $A \in SO(n-2)$ acts on the normal space

$$u_a \to A_a^{\ b} u_b \tag{291}$$

So every scalar product on the normal space is invariant under this action. However, it is not invariant under conformal reparameterization of the surface. We already mentioned that if we reparameterize the parameter space with a holomorphic function $z \to h(z)$ we find

$$\begin{aligned} u_i &\to u_i (\partial h)^2 \\ \bar{u}_i &\to \bar{u}_i (\bar{\partial} \bar{h})^2 \end{aligned} \tag{292}$$

Under this reparameterization the metric transforms

$$e^{\alpha} \to e^{\alpha} \mid \partial h \mid^2 \tag{293}$$

So the following combinations are invariant : $(u^i \bar{u}_i) e^{-2\alpha}$ and $(u^i u_i)(\bar{u}^j \bar{u}_j) e^{-4\alpha}$. But the right hand side of (255) is constructed out of invariant quantities thus T and C are invariant.

6.7 Proof of $0 = R+2+2\sqrt{T}$ in the exceptional spacelike case for $T \neq 0$

Because we are examining the exceptional case $C = 0$ here. We start with the equation for the curvature together with the Gauss equation.

$$R = -2e^{-\alpha}\partial\bar{\partial}\alpha = -2(u^a\bar{u}_a e^{-2\alpha} + 1) \tag{294}$$

So we have for $C = 0$

$$R + 2 + 2u^a\bar{u}_a e^{-2\alpha} = R + 2 + 2\sqrt{T} = 0 \tag{295}$$

as stated above.

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Hilfsmittel

Diese Diplomarbei wurde mit $\LaTeX\,2_\varepsilon$ gesetzt. Die Grafiken wurden mit Hilfe von Mathematica 6 (Wolfram Research) erstellt. Die in dieser Arbeit enthaltenen Rechnungen wurden unter Einbeziehung von Mathematica 6 erstellt.

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Diplomarbeit selbständig sowie ohne unerlaubte fremde Hilfe verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

Mit der Auslage meiner Diplomarbeit in den Bibliotheken der Humboldt-Universität zu Berlin bin ich einverstanden.

Berlin, den 15.09.2009

Sebastian Johannes Wuttke