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Wilson Loops and the static Quark Antiquark Potential in the AdS/CFT Correspondence

Diplomarbeit
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Abstract

In this diploma thesis the proposal for evaluating the expectation value of Wilson loops in the AdS/CFT correspondence will be considered. Many physical observables related to the quarks depend on how the strings, whose endpoints on some probe branes represent the quarks, are embedded in certain backgrounds. Using this prescription, quantities like static interquark potential, screening length, drag force and jet quenching parameter can be determined. Two backgrounds are of special interest in this work, one background dual to $\mathcal{N} = 2$ SYM and the Lorentz-boosted AdS black hole background dual to $\mathcal{N} = 4$ SYM at finite temperatures. The latter background is used as an approximation to explore some interesting properties of the Quark-Gluon-Plasma.

This thesis primarily concerns the possible dependence of the above mentioned observables on various quark orientations inside the internal space (S^5). It was found that the relative S^5-angle of a quark and an antiquark has strong influence on their static potential and the related screening length but it does not change the confinement behavior, and the drag force related to a moving single heavy quark in a QGP is independent from its various internal orientations. For comparison with QCD a method to average over all relative S^5-angles is proposed.

Keywords: AdS/CFT correspondence, Wilson loops, Quark-Gluon-Plasma

Inhaltsangabe

Diese Diplomarbeit untersucht das Konzept zur Bestimmung der Erwartungswerte von Wilson-Schleifen in der AdS/CFT Korrespondenz. In dieser Beschreibung werden Quarks durch die Endpunkte des Strings auf Testbranen dargestellt, so dass viele Quarks-spezifische Messgrössen davon abhängen, wie der String in verschiedenen Raumzeiten eingebettet ist. Von besonderem Interesse für die vorliegende Arbeit sind dabei ein zu einer $\mathcal{N} = 2$ SYM Theorie dualer Hintergrund, sowie der als AdS Schwarzes Loch genannte Hintergrund. Letztere Raumzeit wurde als Approximation gewählt um einige sehr interessante Eigenschaften des Quark-Gluon-Plasmas studieren zu können. So ist es möglich, mittels der beschriebenen Methode physikalische Grössen wie statisches Quarkspotential, Screening-Länge, die sogenannte Drag-Force und Jet-Quenching-Parameter zu bestimmen.

Der Schwerpunkt dieser Diplomarbeit liegt dabei auf der Untersuchung einer möglichen Abhängigkeit der obengenannten Messgrössen von verschiedenen Quarkorientierungen im S^5-Raum. Es wurde festgestellt, dass der relative Winkel zwischen einem Quark und einem Antiquark grossen Einfluss auf das statische Interquarkpotential und die dazugehörende Screening-Länge hat. Für das eventuell vorhandene Confinement-Verhalten sowie die Drag-Force bezüglich eines einzelnen bewegten schweren Quarks konnte jedoch keine Abhängigkeit von der S^5-Orientierung nachgewiesen werden. Um Kontakt mit QCD knüpfen zu können, wurde eine Mittelungsmethode über allen relativen S^5-Winkeln vorgeschlagen.

Schlüsselwörter: AdS/CFT Korrespondenz, Wilson-Schleifen, Quark-Gluon-Plasma

Contents

1. Motivation and Introduction

The strong interaction between the quarks described by exchanging of gluons is governed best by quantum chromodynamics (QCD). The well-known running coupling has the profound consequence that the interactions become weak at short distances, which is known as *asymptotic freedom*, and strong at large distances, which leads to *confinement* of charges. QCD theory has been tested and confirmed with success in high energy experiments and at low energies, where the theory is non-perturbative, lattice calculations provide strong evidences for the correctness of QCD, but there still seems to lack a satisfactory description for confinement. Currently there are many indications that string theory might be useful to describe the strong interactions in the non-perturbative regimes of QCD.

String theory arose in the late sixties in an attempt to describe the strong nuclear force which binds the quarks together. This was motivated from experimental data relevant to hadronic scattering which provided an apparently infinite tower of resonances with mass and angular momenta related by

$$J \approx \alpha' m_J^2 \ ,$$

where $\alpha' \approx 1GeV^{-2}$ is the Regge slope. This relation emerges naturally from considering a rigid rotating string with constant string tension proportional to $1/\alpha'$ [1]. But this bosonic string theory suffered from several unphysical features, the absence of fermions, the presence of a tachyon and spin-2 particles, and it is only consistent in 26-dimensional spacetime. Due to these difficulties and the incorrect prediction of the theory at high energies, the theory was abandoned and taken over by QCD in early seventies. During the seventies supersymmetry has been built into string theory to form the superstring theory, which is free of tachyons and consistent in ten-dimensional spacetime [2]. The theory contains massless spin-2 excitations and has been regarded as a consistent theory of quantum gravity. Due to the discovered connection between superstring and super-Yang Mills theory in the late nineties [6] called the AdS/CFT correspondence, the relation between string theory and strongly interacting world of hadrons has been reconstructed.

The AdS/CFT correspondence [6], for reviews see e.g. [7][8], relates type IIB string theory on $AdS_5 \times S^5$ space-time (AdS) and $\mathcal{N} = 4$ superconformal Yang-Mills (SYM) theory living on the conformal boundary of AdS which is the 4-dimensional Minkowski space. Taking the *'t Hooft limit* [5] first and then the *Maldacena limit* (decoupling or low-energy limit) [6], the correspondence describes a *weak/strong* coupling duality. In these limits, type IIB string theory is approximated to type IIB *supergravity* (low energy) and $\mathcal{N} = 4$ SYM describes the strongly coupled $\mathcal{N} = 4$ SYM theory in the large N limit where N denotes the number of colors. The correspondence conjectures the existence of maps that identify parameters and correlation functions of both theories, thus making the two theories as two different descriptions of the same theory. Hence, if a physical observable is hard to compute in one theory, the problem might be solved more easily using the dual description.

In field theory the static quark antiquark potential can be obtained from the expectation value of a Wilson loop having a rectangular contour $\mathcal{C}$ with the temporal sides T much larger than the spatial sides L representing the separation between *heavy* quark and antiquark [12]. Confinement occurs if the so-called *area law* holds, which basically states that the expectation value giving the *static* interquark potential is proportional to the area of the loop. A prescription how to evaluate $\langle W[\mathcal{C}]\rangle$ using the dual string picture was proposed in references [13][14]. There, the quarks are represented by fundamental strings ending on the boundary of AdS. The quark mass is associated to the string length and depending on the string orientation, the string endpoint at the boundary can be seen as quark or antiquark. Hence, a quark pair representing a meson is thought to be the string with both endpoints on the boundary. The authors of [13][14] proposed the Wilson loop to be defined by this open string and its expectation value is given by the string minimal world-sheet ending on the loop $\mathcal{C}$ at the boundary of $AdS_5 \times S^5$. In leading order, the potential can be extracted from the minimal surface found by extremizing the Nambu-Goto action. The result describes a potential of coulombic type [13][14], since the dual field theory is a conformal one and hence does not show confinement.

After the proposal of the correspondence, an impressive amount of work has been taken to understand and extend these ideas with the ultimate aim to find a gravity dual background to QCD. The technique presented in [13][14] can serve as a first check to explore the confinement property of the background. As a first step to this direction, the temperature was introduced into the background to break the conformal and supersymmetry of the theory. This is realized by compactifying the Euclidean time on a circle with periodic boundary conditions [16]. Utilizing this idea, explicit calculations in [17][18] showed that at finite temperature the quarks potential still has the Coulomb-behavior, however, only up to some critical separation before it vanishes. The calculation of Wilson loop along two space directions at finite temperature can be found in [19], the result exhibits confinement and was interpreted as the potential of the 3-dimensional pure Yang-Mills theory which is the limit of the discussed cases in [17][18] at infinite temperature. A general setup for determining the potential was constructed in [20] and was applied for several models. A theorem that determines the leading behavior of the classical potential for a generic metric was proven, in particular a corollary of this theorem states the sufficient conditions for the potential to have a confining nature. Some corrections to the leading order of the potential in certain backgrounds can be found in [21][22][23].

Along the line of development, the technique to obtain the potential has been applied for most known dual backgrounds whenever the metric can be given in explicit form. Some of the backgrounds which have dual non-conformal field theory with less symmetry have been found, e.g.: the background in [24] describes a supersymmetric solution dual to $\mathcal{N}=2$ SYM and the potential exhibits confinement at large quarks separation distances; confining potential has also been found for some backgrounds dual to $\mathcal{N}=1$ field theories [25]; and the background described in [26] shows confinement and string breaking at some critical distance.

Recently, experimental results from the Relativistic Heavy Ion Collider (RHIC) indicated the existence of a new state of strongly interacting matter at extremely high temperatures and densities called Quark Gluon Plasma (QGP) [33] where the quarks and gluons are no longer confined in hadrons. This medium is created in Au-Au collision at velocity close to c. Some observed phenomena like suppression of heavy mesons and energy loss of a single quark are related to the expectation value of the Wilson loop. Using lattice QCD to study QGP, one encounters the difficulty that quarks and mesons are produced with some initial *velocity* relative to QGP. The dual gravity description can solve that problem in a natural way, namely by boosting the background, thus this strongly coupled medium might serve as an ideal testing ground for the AdS/CFT correspondence. Since there exists no known gravity dual background to QCD, the AdS black hole metric dual to $\mathcal{N}=4$ at finite temperature was used as an approximation, since there are good reasons that the two theories might share common properties under these extreme conditions [38] .

Theoretical analyzes making use of the AdS/CFT correspondence for evaluating the shear viscosity of the $\mathcal{N}=4$ plasma [34, 35, 36] have confirmed the conjecture that this hot nuclear medium behave like a nearly perfect fluid, an effect which was concluded from measurements of flow parameters [33]. The resulting value for the viscosity seems to obey a universal law [36] and is well compatible with the values for QGP found at RHIC, thus this success motivates the use of AdS/CFT correspondence to study further properties of the plasma.

Since then there have been many AdS/CFT calculations, e.g. to determine the *velocity-induced* suppression of the screening length beyond which quarkonium bound states dissociate [37][39]. The result is obtained from evaluating the static quarks potential using gravity dual description in a boosted *AdS* black hole background. The investigation on the dependence of the screening length on the relative plasma wind direction and the chemical potential have been carried out in [40, 41, 42]. Another phenomena observed at RHIC are mono-jets and the suppression of back-to-back jets compared to the case of proton-proton scattering which can be explained by the energy loss of heavy quarks moving in the $\mathcal{N}=4$ SYM plasma [43]. The jet quenching parameter describing the energy loss can be obtained from the expectation value in the *short distance limit* of the closed light-like Wilson loop in the *adjoint* representation [44]. Another approach to determine the jet quenching parameter comes from considering the motion of a single heavy quark moving in the $\mathcal{N} = 4$ SYM plasma [46, 47, 48] which also underlies the concept of the AdS/CFT correspondence.

This diploma thesis mainly concerns the computations of the Wilson loops via string world-sheets in 10-dimensional backgrounds where an additional degree of freedom is switched on. This allows the string to move in the *internal space* which for the to be considered backgrounds is given by the S^5-part of the metric. The string endpoints representing quark and antiquark at the boundary can have different positions on the S^5, so the string connecting the quarks describes a curve inside the internal space. There seems to exist only a few papers discussing on the potentials [13][22][28][29] where the internal orientation of the Wilson loop is *not* kept fixed along the rectangular contour. The reason not to turn on the relative orientation of the quark and antiquark is the absence of any interpretation of such degree of freedom in QCD. However, when studying supersymmetric Wilson loops [13][31][32], the correlation between the loop contour in Minkowski space and the contour on S^5 is not trivial.

In this thesis the calculation of Wilson loops will include the relative S^5-orientation between the quark and antiquark for the $\mathcal{N} = 2$ dual background [24] and the Lorentz-boosted AdS black hole background [16][37]. Possible dependence of the potential, confinement behavior, screening length, drag force and jet quenching parameter on the relative S^5-angle will be considered.

The thesis is organized as follows:

In chapter two there will be a brief review of some main ideas of the AdS/CFT correspondence [7][8], the Wilson loops in field theory will be introduced and the technique how to evaluate the expectation value of Wilson loops using AdS/CFT [13][14] will be presented.

In chapter three the technique applied to the $AdS_5 \times S^5$ metric [13] will be reviewed, the result for the potential extracted from the background dual to $\mathcal{N} = 2$ [24] will be examined and extended by switching on the relative orientation of the quarks. A proposal on averaging over all relative angles [60] will be introduced, a short summary and discussions on confinement behavior and concavity [28] of the potential can be found at the last section.

In chapter four some properties like the screening length [37], drag force [47] and jet quenching parameter [38] [44] related to a massive single quark and heavy mesons moving in QGP will be considered. In each case the internal orientation of quarks will be switched on. Discussions on the results are attached at the end of each section.

A short summary of the results and the outlook are given in the last chapter.

2. AdS/CFT Correspondence and the Wilson Loops

2.1. The Correspondence

In this section the correspondence between a gauge theory that arises at low energies on a stack of N coincident D3-branes and a type IIB superstring theory in a certain spacetime background will be described. Here we follow closely the references [6][7][8].

2.1.1. The Conjecture

The AdS/CFT duality was conjectured by Maldacena [6] and originally states that $\mathcal{N} = 4$ U(N) SYM theory in 3+1 dimensions describes the same physics as type IIB string theory on an $AdS_5 \times S^5$ background, where AdS_5 is five-dimensional anti-de Sitter space and S^5 a five-sphere.
The $\mathcal{N} = 4$ SYM theory is a gauge theory with one gauge field A_μ, four Weyl fermions χ_a and six real scalars ϕ_i, all in the adjoint representation of the color group. Its Lagrangian can be described by [8]

$$S = \frac{1}{g_{YM}^2} \int d^4x \; Tr\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi_i)^2 + \frac{1}{2}\bar{\chi}_a \not{D}\chi_a - \frac{1}{2}\bar{\chi}_a[\phi_i, \chi_a] - \frac{1}{4}[\phi_i, \phi_j][\phi_i, \phi_j]\Big) \; . \tag{2.1}$$

It is a conformal field theory (CFT) with vanishing beta function. The theory has two parameters, the number of colors N and the gauge coupling g_{YM}. When the number of colors is large, the perturbation theory is controlled by the so called *'t Hooft coupling*

$$\lambda \equiv g_{YM}^2 N \; .$$

The type IIB string theory is a superstring theory, which contains a *finite* number of massless fields, including the graviton, the dilaton and the Kalb-Ramond antisymmetric 2-form. Furthermore it comprises of the fermionic superpartners and an infinite number of massive string excitations. This theory has two parameters, the string coupling g_s and the string length $l_s = \sqrt{\alpha'}$ where α' is the slope parameter. In the low energy limit (long-wavelength limit), when all fields vary over length scales much larger than l_s, the massive modes decouple and one is left with type IIB *supergravity* in 10 dimensions. The Lagrangian for this theory can be described by [8]

$$S_{SUGRA} = \frac{1}{16\pi G^{(10)}} \int d^{10}x \sqrt{-g} \, e^{-2\Phi_D} \left(\mathcal{R} + 4\,\partial^\mu\Phi_D \; \partial_\mu\Phi_D + \cdots\right) , \tag{2.2}$$

where $G^{(10)}$ is the ten-dimensional Newton's gravitational constant, g the determinant of the metric, $\mathcal{R}$ the Ricci scalar, Φ_D the dilation field and the dots denotes contributions from some other fields.

To motivate the duality, let us first consider the excitations around the ground state of type IIB string theory in the presence of N coincident D3-branes in flat, ten dimensional Minkowski space. Dp-branes are massive, charged objects extending in p-spatial directions which are defined by the property that strings can start and end on them. The D stands for Dirichlet, there are $(10-p-1)$ Dirichlet boundary conditions for the string endpoint coordinates transverse to the brane and $(p+1)$ Neumann boundary conditions for string endpoint coordinates parallel to the brane. The D3-branes are extended along a $(3+1)$ dimensional plane in ten dimensional space time. The excitations of the system consist of open and closed strings, as displayed in Fig.2.1(a), in interaction with each other. Quantization of the strings leads to a spectrum containing a massless $\mathcal{N}=4$ vector multiplet plus a tower of massive string excitations. Since the open string endpoints are attached on the D3-branes, all these modes propagate in the four dimensional worldvolume of these branes. Similarly, quantization of closed strings provides a massless graviton supermultiplet plus a tower of massive string modes, all propagating in flat, ten dimensional spacetime.

At energies smaller than the string scale $1/l_s$, where l_s denotes the string length, only massless string states can be excited. Interactions in gravity are determined by the value of Newton's gravitational constant which in ten dimensions is given by [1]

$$G^{(10)} \sim g_s^2 l_s^8 = g_s^2 (\alpha')^4. \tag{2.3}$$

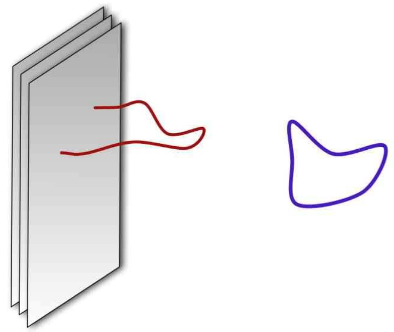
Taking the decoupling limit, i.e. keeping the energy and all the dimensionless parameters like the string coupling constant g_s and the number of colors N fixed while sending $l_s \to 0$ ($\alpha' \to 0$), the closed strings become non-interacting. The interactions between open strings are controlled by the $\mathcal{N}=4$ SYM coupling constant $g_{YM}^2 \sim g_s$, so the open string massless states are governed by the low-energy effective Lagrangian of $\mathcal{N}=4$ U(N) SYM theory [11] and the closed string massless states by the low-energy effective Lagrangian which turns out to be that of type IIB supergravity. [2]

The complete effective action of the massless modes has the following form

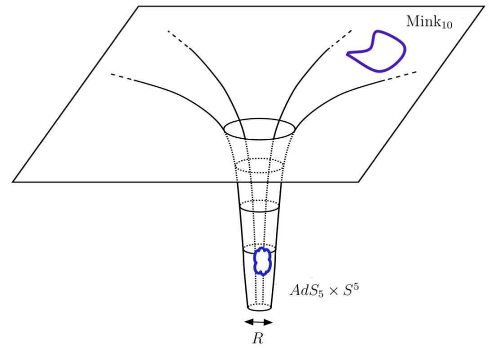
$$S_{total} = S_{bulk} + S_{brane} + S_{int} + \textit{some corrections}\ , \tag{2.4}$$

where S_{bulk} describes the ten dimensional supergravity, S_{brane} the physics on the worldvolume of D3-branes containing the $\mathcal{N}=4$ SYM and S_{int} describes the interactions between the brane modes and the bulk modes. In the low energy limit S_{int} relating the bulk and the branes vanishes and there is no interactions between closed and massless "open" strings. In this limit S_{total} is left with two decoupled systems: the free gravity in the bulk; and a conformal field theory which is known to be the pure $\mathcal{N}=4$ U(N) gauge theory in four dimensions.

Next, let us examine the same limit in the second description where the low-energy limit consists of focusing on excitations that have arbitrarily low energy with respect to an *observer in the asymptotically flat Minkowski region* (very far away from the branes).



(a)



(b)

Figure 2.1.: *(a) Excitations in the presence of D-branes (b) Excitations in the bulk and near-horizon region of D3-branes*

Since D-branes are massive charged objects, they deform their embedding space and a D3 brane solution [7][9] of supergravity is given by

$$\begin{aligned} ds^2 &= f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \ , \\ F_5 &= (1 + *) \ dt dx_1 dx_2 dx_3 df^{-1} \ , \\ f &= 1 + \frac{R^4}{r^4} \ , \qquad R^4 \equiv 4\pi g_s \alpha'^2 N \ . \end{aligned} \tag{2.5}$$

The D3- brane solution above is described by a ten-dimensional metric, a self-dual five-form field strength F indicating a four-form potential, since on general grounds a p-brane will be a source of a $p+1$-gauge potential. Because g_{tt} is non-constant, the energy E_r of an object as measured by an observer at a constant position r and the energy E measured by an observer at infinity are related by the *redshift* factor

$$E = f^{-1/4} E_r \ . \tag{2.6}$$

From the point of view of an observer at $r \to \infty$, there are two kinds of excitations at the *low-energy limit*: the massless graviton supermultiplet propagating at the ten-dimensional Minkowski region (bulk region); and the whole tower of massive string excitations which are sent closer and closer to $r = 0$ (near-horizon or throat region) and due to the redshift appear to have arbitrarily low energy as seen by observer at infinity, see Fig.2.1(b).

As argued before, by taking the $\alpha' \to 0$ limit while keeping the energy fixed, the massless particles propagating in the bulk become non-interacting providing free gravity. Moreover, these modes also decouple from the modes in the throat regions, since at low energies the wave length of these modes becomes much larger than the size of the throat which is of order R and therefore cannot enter this region. Similarly, the modes living near the D-branes cannot escape to infinity, since otherwise they have to climb up a gravitational potential. As a result, the configuration is again well approximated by two decoupled systems: free gravity in flat ten-dimensional spacetime and interacting closed strings in the near-horizon region, whose geometry for $r \ll R$ turns out[1] to be that of $AdS_5 \times S^5$

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \; . \tag{2.7}$$

Both descriptions contain a subsystem of free closed strings in flat spacetime, so it is tempting to conjecture that *$\mathcal{N} = 4$ U(N) SYM theory in 3+1 dimensions is the same as (or dual to) type IIB superstring theory on $AdS_5 \times S^5$* [6].

2.1.2. Matching of Symmetries and Parameters

Matching of symmetries In the last subsection a heuristic argument based on a low energy limit for the AdS/CFT correspondence has been introduced. Let us now consider the symmetry argument. The isometry group of AdS_5 is $SO(2,4)$, and this is also the conformal group in 3+1 dimensions [8]. The isometry group of AdS_5 can be understood by the fact that the five-dimensional anti-de Sitter spacetime can be obtained by taking the hyperboloid

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2 \tag{2.8}$$

embedded in a flat six-dimensional spacetime with the metric $\eta = diag(-1,-1,1,1,1,1)$. R is called the constant anti-de Sitter radius or sometimes the radius of the spacetime. By a suitable change of variables

$$\begin{aligned} r &= X_{-1} + X_4 \\ v &= X_{-1} - X_4 = \frac{R^2}{r} + \frac{x^2 r}{R^2} \\ x_\mu &= \frac{X_\mu R}{r} \; , \qquad \mu = 0,1,2,3 \end{aligned} \tag{2.9}$$

and using the relations

$$dv = \left(-\frac{R^2}{r^2} + \frac{x^2}{R^2}\right) dr + 2\frac{xr}{R^2}\, dx \; , \tag{2.10}$$

the induced metric on the hyperboloid takes exactly the form of the AdS_5-part of (2.7)

$$ds^2_{AdS_5} = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}dr^2 \; . \tag{2.11}$$

Thus this geometry has the symmetry group $SO(2,4)$. The coordinates x_μ may be thought of as the coordinates along the worldvolume of the D3-branes and may be

[1]The 1 in $f = 1 + \frac{R^4}{r^4}$ can be neglected.

identified with the gauge theory coordinates. At this point it is interesting to note that the metric is invariant under the action of the dilatation operator $D : x^\mu \to \Lambda x^\mu$ provided that is accompanied by the rescaling $r \to r/\Lambda$, where Λ is a constant. Since $\mathcal{N} = 4$ U(N) SYM describes a conformal theory, it is also invariant under the action of D. Hence, if the correspondence hold, that would mean that short-distance physics in the gauge theory is associated to physics near the AdS boundary ($r \to \infty$) and the long-distance physics to the physics near the horizon ($r \to 0$).
In addition, the isometries of S^5 form the group $SO(6) \sim SU(4)$ which can be identified with the R-symmetry of the $\mathcal{N} = 4$ SYM theory. To be more precise, after including the fermionic generators required by supersymmetry, the full isometry supergroup of the $AdS_5 \times S^5$ background is $SU(2,2|4)$, which is identical to the $\mathcal{N} = 4$ superconformal symmetry group [8].

Matching of parameters In this paragraph the parameters entering the definition of each theory and the map between them will be considered. The gauge theory can be specified by the number of colors N and the 't Hooft coupling constant $\lambda = g_{YM}^2 N$. The string theory is determined by the string coupling constant g_s and the size of the AdS_5 and S^5 spaces, which can be expressed by the same radius of curvature R. There is a precise relation between the 't Hooft coupling and the parameters on the string theory side, namely

$$\lambda = \frac{R^4}{\alpha'^2} = 4\pi g_s N. \tag{2.12}$$

The number of colors N on field theory side appears on the other side as the flux of the five-form Ramond-Ramond field strength through the S^5, $\int_{S^5} F_5 = N$.

In its strongest form, the correspondence is supposed to hold for arbitrarily values of N and λ. Since superstring theory on $AdS_5 \times S^5$ is still not completely understood, it proves more convenient to work with its low energy-limit, the supergravity description.

At first the so-called *'t Hooft limit* is taken which consists in keeping $\lambda = g_{YM}^2 N$ fixed while sending N to infinity. In this 't Hooft limit, field theory reorganizes itself in a topological expansion [5]. On the other side, due to $g_s \sim \lambda/N$, this limit corresponds to the classical type IIB string theory on $AdS_5 \times S^5$. In the second limit $\alpha' \to 0$, the curvature radius R is assumed to be large compared to the string length $l_s = \sqrt{\alpha'}$, thus this corresponds to the low-energy limit where supergravity becomes an effective description. On the field theory side, due to $\lambda = R^4/\alpha'^2$, this implies large 't Hooft coupling indicating a strongly coupled theory. Hence in the large N-, large λ-limit, the AdS/CFT correspondence describes a *weak/strong* coupling duality.

2.1.3. The Field/Operator Correspondence

The correspondence states the duality between type IIB string theory describing on $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ SYM field theory living on the conformal boundary of AdS. Both theories have the same global symmetries. Then, if these two theories are indeed equivalent, it must be possible to specify for each operator $\mathcal{O}(\vec{x})$ of the field theory living on the boundary at $r \to \infty$, the corresponding field $\phi(\vec{x}, r)$ of the bulk string theory and to show that the computations of the corresponding correlators in these two theories should provide the same result. The relation between correlation functions on the two sides was first proposed in [6] which is expressed by the generating functional on the field theory side and the string partion function on the other side. The connection is written as

$$\left\langle e^{\int d^4x \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \right\rangle_{CFT} = \mathcal{Z}_{string} \left[\phi\left(\vec{x}, r\right) |_{r \to \infty} = \phi_0\left(\vec{x}\right) \right] \ , \tag{2.13}$$

where $\mathcal{O}(\vec{x})$ describes any gauge-invariant local[2] operator and $\phi_0\left(\vec{x}\right)$ denotes an arbitrary function specifying the boundary values of the bulk field $\phi\left(\vec{x}, r\right)$.

The left-hand side of (2.13) encodes all the physical information in the gauge theory, since it allows the calculation of correlation function of arbitrary gauge-invariant local operators by taking the derivatives with respect to $\phi_0\left(\vec{x}\right)$ and taking it to zero afterwards. The string partition function on the right-hand side is in general not easy to compute, but in the large N-, large λ-limit, it dramatically reduces to

$$\mathcal{Z}_{string} \approx e^{i \cdot S_{sugra}} \ , \tag{2.14}$$

where S_{sugra} is the on-shell supergravity action.

2.1.4. Some Comments about D-Branes

D-branes in String Theory A Dp-brane is an extended object with p *spatial* dimensions where open strings can end, hence p imposes the number of Neumann conditions on the motion of the open string endpoints [10]. With the time dimension the world-volume of a Dp brane is $(p+1)$-dimensional. Not all extended objects in string theory are D-branes, for example, strings are 1-branes but not D1-branes.

By quantizing open strings ending on a Dp-brane, one has to introduce $(p+1)$ Neumann conditions for the string endpoints moving freely on the brane and $(D-p-1)$ Dirichlet conditions fixing the Dp-brane position in space. Here, D denotes the spacetime dimension of the embedding space. The massless excitations of open strings coupling to the Dp-brane describe the low energy dynamics of the brane. Using the light-cone gauge and taking care all the boundary conditions, the open string wave function satisfies the equation

$$(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0, \tag{2.15}$$

where τ and σ are the parameters in the light-cone gauge and X^μ the string coordinates [1]. The solution of the wave equation can be written as a Fourier expansion. The creation $a_n^{i\dagger}$ and annihilation a_n^i operators can be constructed from the coefficients in the

[2]Wilson loops are non-local operators, so this prescription must be extended, see next section.

Fourier expansion and the mass operator of the excited states of the quantized system can be given as[3]

$$M^2 = \frac{1}{\alpha'} \left(-1 + \sum_{n=1}^{\infty} \sum_{i=2}^{p} n \; a_n^{i\dagger} a_n^i + \sum_{m=1}^{\infty} \sum_{j=p+1}^{D-1} m \; a_m^{j\dagger} a_m^j \right) . \tag{2.16}$$

The operator $a_n^\dagger a_n$ just counts the number of particles excited in the n-mode. For oscillators arising from coordinates tangential to the brane, the massless states can be constructed by applying the creation operator $a_n^{i\dagger}$ on the vacuum state only for $n = 1$. Here, i runs from 2 to p, so there are $(p+1) - 2$ massless states. Since they carry Lorentz index for the brane and hence transform as a Lorentz vector, and moreover, the number of states equals the spacetime dimensionality of the brane minus two, so they are identified to be photon states whose associated field is a Maxwell gauge field living on the brane. The other massless states can be realized by acting $a_m^{j\dagger}$ on the vacuum for $m = 1$. This set of $(D - p - 1)$ massless states arise from coordinates normal to the brane. They do not carry a Lorenz index and can be interpreted as *scalar* fields parameterizing the position of the D-brane in the *transverse* directions.

D-branes and Gauge Fields We have seen that quantization of open strings in the presence of D-branes gives the rise for gauge fields living on the brane. A single D-brane supports on its world-volume a single $U(1)$ multiplet whose massless vector arises from zero length strings starting and ending at the same point on this brane. In the presence of more than one brane, let's say N branes, open strings can be labeled by $[ij]$ meaning strings extending from brane i to brane j, where i and j are integers running from 1 to N. These strings are also said to be in $[ij]$ sector. The discrete labels i, j used to label the branes and the various open string sectors are called *Chan-Paton* indices [1]. Each of the ends of the strings carries a Chan-Paton label of the gauge group, that is an index in the fundamental representation, so the vector multiplet is then left with a fundamental and an antifundamental index (ingoing and outcoming) and hence can be described by adjoint fields. When no D-branes coincide, there is just one massless vector each, or $U(1)^N$ in all. Now, if N D-branes coincide, there are new massless states because strings which are stretched between these branes can achieve vanishing length. In total, there will be N^2 massless vectors which form the adjoint of a $U(N)$ gauge group and it was found that the effective action is dimensionally reduced from ten-dimensional $U(N)$ supersymmetric gauge theory down to $(p+1)$-dimensional world-volume of the D-brane [10][11][4].

[3]In order to use the light-cone gauge, one of the spatial NN string coordinates X^i (tangential to Dp) will be used together with X^0 to define the coordinate $X^\pm$. So this construction is only valid for $p \geq 1$.

[4]The gauge group $U(N)$ can be written as $SU(N) \times U(1)$ and the center group $U(1)$ is identified to describe the center of mass of motion of the stack of coincident N D-branes.

Branes Higgsing Now let us consider the situation when two coincident D-branes get separated. Strings of zero length starting from one brane and ending on the other become massive whose mass is determined by the distance between the branes times the string tension. These strings carry two Chan-Paton indices, each one from the $U(1)$ of each brane, and are naturally identified as W-bosons of a broken $U(2)$ gauge group to $U(1) \times U(1)$. This is the stringy version of the Higgs effect which is sometimes called *branes Higgsing*. The distance between the branes determines the Higgs expectation value. For the case of a stack of coincident N D3-branes, the near horizon geometry will be that of $AdS_5 \times S^5$ and the $r \to \infty$ region describes the flat 10-dim Minkowski space. The separation of the branes will be realized along the radial coordinate r fixing the mass of the string and inside the internal space[5] S^5. The AdS/CFT correspondence states a duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory in 4 dimensions. This gauge theory does not contain quarks in the fundamental representation. Usually, massive charged particles can be introduced via symmetry breaking the gauge group by giving expectation values to the scalars of the theory, e.g. $U(N+1) \to U(N) \times U(1)$. On the string theory side, that can be realized via separating a single D-brane far away[6] from the stack of coincident $(N+1)$-branes. The ground states of open string stretching between the single brane and the remaining N-branes are identified with the W-bosons transforming in the fundamental representation of the $U(N)$ [13, 14].

Branes as Probes Another possibility to have particles in the fundamental representation is introducing *probe branes* into the dual geometry. Having a probe brane in the background means introducing an additional brane which is assumed not cause any kind of deformations to the geometry and neglecting all the back reactions of this brane to the original branes. The open strings between the probe brane and the original branes are used to represent additional particles in the theory. Probing the background with a D-brane is one natural way to learn something about string theory from Yang-Mills theory and vice versa. As described in the previous paragraph, a gauge theory lives on the world-volume of the D-brane. The background geometry will be encoded in this gauge theory via the matter content, the amount of unbroken supersymmetry and the interaction potentials, so solving problems in gauge theory provides information about the background and vice versa.

[5]It was found that the stationary solution for a single straight string has constant position in the internal space of AdS, hence the quark mass does not depend on the S^5-position of quark in this background [13, 14].

[6]The meaning of "far away" will be talked in the next chapter.

2.2. Wilson Loops and the static Quark Antiquark Potential

The expectation value of the Wilson loop in the form of an infinite rectangular with infinite time-like sides T and transverse side L denoting the distance between very massive quark and antiquark gives the static interquark potential [4][12]. In QCD this quantity is used for characterizing the confinement behavior [12]. Confinement occurs if the expectation value of an rectangular Wilson loop gives the famous area law. In this section, some properties of the Wilson loop operator in QCD and $\mathcal{N} = 4$ SYM will be reviewed. At the end of the section, the construction and calculation of the Wilson loop expectation value on the string theory side will be presented.

2.2.1. Wilson Loops in Field Theory

The Wilson line has its origin from an operator $\hat{U}(\mathbf{y}, \mathbf{z})$ called parallel transporter of complex vector fields $\Phi = (\Phi_1, ..., \Phi_N)$, which transform in some representation of the gauge group, along some curve $\mathcal{C}$ connecting two spacetime points $\mathbf{y}$ and $\mathbf{z}$. This operator can be expressed by

$$\hat{U}\left(\mathcal{C}_{\mathbf{yz}}\right)\Phi(\mathbf{z}) = \Phi(\mathbf{y}) \ . \tag{2.17}$$

In order to have a local gauge-invariant theory, one needs to introduce a gauge potential $A_\mu(\mathbf{x})$ with its specific transformation law for constructing the covariant derivative. The parallel transporter has the transformation law

$$\hat{U}\left(\mathbf{y}, \mathbf{z}\right) \to \Sigma^{-1}(\mathbf{y})\hat{U}\left(\mathbf{y}, \mathbf{z}\right)\Sigma(\mathbf{z}) \ , \tag{2.18}$$

where Σ is an element of some gauge group. It was found that the expression

$$\hat{U}\left(\mathcal{C}_{\mathbf{yz}}\right) = exp\left(ig\int_{\mathcal{C}} A_\mu dx^\mu\right) \ , \tag{2.19}$$

respects the gauge transformation laws. Here, $\mathcal{C}$ represents a curve connecting $\mathbf{y}$ and $\mathbf{z}$ and g the coupling constant of the theory. This object is called the *Wilson line*. To construct the *Wilson loop* one sets $\mathbf{y} = \mathbf{z}$ and takes trace. These objects can be generalized to the non-abelian case, where a *path-ordering* operator $\mathcal{P}$ will be introduced in order to take care the non-commuting property of the theory.
In QCD, the expectation of a Wilson loop has the form

$$\langle W(\mathcal{C})\rangle = \frac{1}{N}\left\langle Tr\mathcal{P}exp\left(ig\oint_{\mathcal{C}} A_\mu dx^\mu\right)\right\rangle \ , \tag{2.20}$$

where A_μ stands for the vector-potential[7] of the gluonic field, g the QCD coupling constant, $\mathcal{C}$ some closed contour, $\mathcal{P}$ the path-ordering operator, $\frac{1}{N}$ the averaging over all

[7] For the non-Abelian case, $A_\mu \equiv A^a_\mu t^a$ with $a = 1, ..., N^2-1$ and $t^a = (t^a)^{ij}$ is the Hermitean generator of the color group $SU(N)$ in the fundamental representation

colors, and Tr the trace in the fundamental representation. Since a trace is taken over this operator, and a gauge transformation for $\hat{U}$ is given by

$$\frac{1}{N}\mathcal{P}exp\left(ig\oint_{\mathcal{C}} A_\mu dx^\mu\right) \to \Sigma^{-1}(\mathbf{x})\frac{1}{N}\mathcal{P}exp\left(ig\oint_{\mathcal{C}} A_\mu dx^\mu\right)\Sigma(\mathbf{x}) \ , \qquad (2.21)$$

the invariance of the trace under cyclic permutation guarantees the gauge invariance of the Wilson loop. Another crucial property of the Wilson loop (or Wilson line in general) is that it depends on the path $\mathcal{C}$. By construction, this quantity is a nontrivial function of A_μ and is gauge invariant. In fact, all gauge-invariant functions of A_μ can be recovered from Wilson loops for various choices of the path $\mathcal{C}$ [3].

2.2.2. $q\bar{q}$-Potential and the Wilson's Criterion of Confinement

In order to get the static quark antiquark potential, one considers the Wilson loop with closed contour $\mathcal{C}$ as a rectangular [4][12] lying in the plane (x_1, t) with the size along the t-axis denoted by T and the size L along the x_1-axis. The expectation value of such a Wilson loop for $T \gg L$ gives the static potential of heavy quark and antiquark separated by the distance L which is given by[8]

$$V_{q\hat{q}}(L) = -\lim_{T\to\infty}\frac{1}{T}\log\langle W(\mathcal{C})\rangle \, .$$

Quantizing a gauge field theory on a discrete lattice in *Euclidean* space-time [12], Wilson showed that, for sufficiently strong coupling, QCD exhibits *confinement of colors*. The so-called *Wilson's criterion of confinement* says that the confinement of quarks holds if the Wilson loops obey the area law and the associated string tension is not zero. That can be written as

$$\langle W_{\mathcal{C}}\rangle \sim e^{-\alpha\cdot Area_{min}(\mathcal{C})} = e^{-\alpha\cdot L\cdot T} = e^{-V_{q\bar{q}}(L)\cdot T}, \qquad (2.22)$$

where α is the *QCD string tension* and $V_{q\bar{q}}(L)$ corresponds to the linearly rising potential between a quark and an antiquark. From this we learn that the force between the quarks is constant at every distance. The picture of this *QCD string* is seen as the contraction of the gluonic field between the quarks into a color tube (string), whose energy is proportional to its length.

2.2.3. Heavy Quark and Wilson Loops in $\mathcal{N} = 4$ SYM

On the gauge theory side a heavy particle is realized by breaking the gauge group, e.g. $U(N+1)$ gauge theory to $U(N) \times U(1)$, through the Higgs mechanism. The resulting W boson transforms in the fundamental of the gauge group $U(N)$ and its

[8]The Euclidean version is used in this subsection.

mass is proportional to the scalar expectation value. In the dual picture when one D-brane is separated from a stack of original coincident $(N+1)$ D-branes, the W boson is represented by an open string connecting the single brane and the remaining branes. Its mass is given by the string length times the string tension.

In QCD, an infinitely massive quark in the fundamental representation moving along the loop $\mathcal{C}$ will be described by the phase factor in (2.20) [12]. In the dual description of $\mathcal{N}=4$, the "quarks" are thought to be W-bosons representing the ground states of open strings as described above. In order to make the quarks very massive, the length of the open strings are sent to infinity. Usually, these strings will pull the N branes and will cause deformations to the branes, which are described by scalar fields. One can see these couplings explicitly by writing the full $U(N+1)$ Lagrangian, putting in the Higgs expectation value and calculating the equation of motion for the massive fields [15]. Hence, in the presence of scalar fields, in the $\mathcal{N}=4$ super Yang-Mills dual to type IIB superstring theory, the phase factor in the path-integral representation of the W-boson propagator involves not only the gauge field $\mathbf{A}$, but also scalars Φ_I [13, 15].

$$W(\mathcal{C}) = \frac{1}{N} Tr\mathcal{P}\ exp\left(i \oint \left(A_\mu(\mathbf{x}(s))\dot{x}^\mu(s) + |\dot{x}|\Phi_I(\mathbf{x}(s))\theta^I(s)\right) ds\right) . \tag{2.23}$$

This expression is given in the Lorentzian[9] signature metric, where s is the curve parameter, Φ_I $(I=4,..,9)$ denote the scalar fields, θ^I angular coordinates of magnitude 1 and can be regarded as coordinates on S^5, so $\vec{\theta}(s)$ can be seen as a function mapping each point of the loop to a point on the five-sphere. Eq. (2.23) restricts the coupling of the Wilson loop only to the bosonic fields, a detail derivation of this formula and the additional coupling to fermionic fields leading to the complete loop equation for $\mathcal{N}=4$ SYM can be found at [15].

2.2.4. The static Quark Antiquark Potential in the dual Picture

As in QCD, where the string is represented by a color tube connecting the quark and antiquark, a similar situation in string theory will be considered. Let us come back to the situation where a D3-brane is separated far away from the N D3-branes. A heavy quark in this picture is represented by an open string stretching between the far away single D3-brane and the stack coincident N branes, see Fig.2.2(a) [13]. For the field theory living on the world-volume of the D3-brane, the quarks are referred to the string endpoints attached to the brane. The string orientation (incoming or outgoing) determines whether it represents quark or antiquark. Hence, a string starting at some point on the brane, penetrating along the radial coordinate r of the $AdS_5 \times S^5$ metric down to some distance and turning back to the single brane can be seen as a "meson". This situation is sketched in Fig.2.2(b).

[9] In [13] the Euclidean signature metric is used, but with Wick-rotated θ^I which is not the case here.

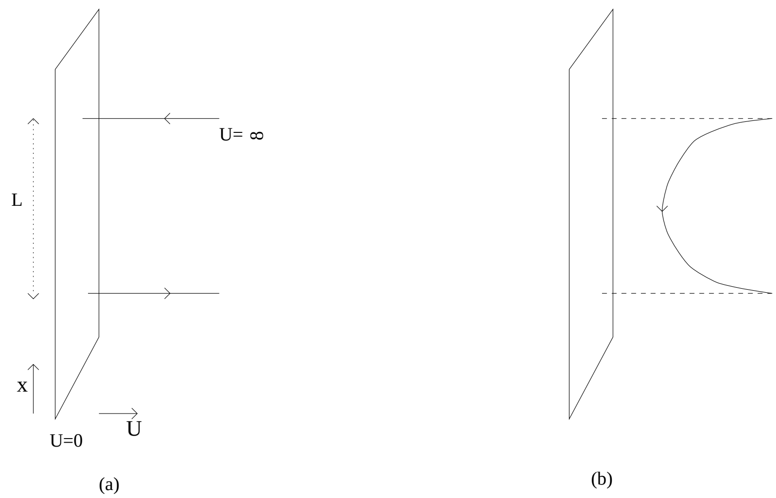


Figure 2.2.: *(a) Representation of massive W-bosons (quarks) as strings coming from the far away D-brane at $U = \frac{r}{\alpha'} \to \infty$ down to the stack of N coincident D3-branes situated at $U = 0$. (b) A string connecting quark and antiquark (depending on whether it is a starting or an ending endpoint) at the boundary. The quark antiquark potential is given by the difference of the total length of the strings in (a) and (b).*

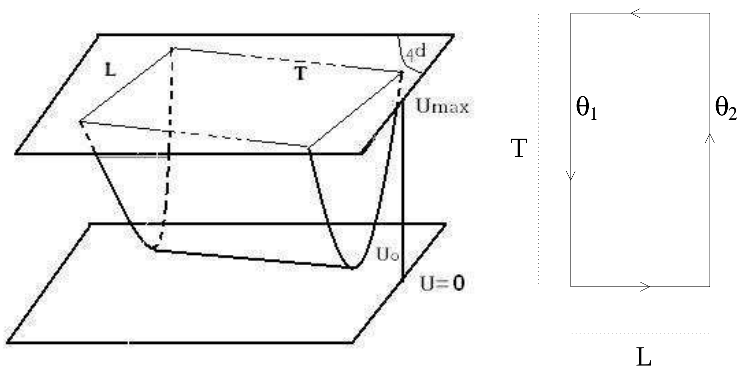
In order to compute the expectation value of the non-local operator (2.23), and hence the quark antiquark potential, the field/operator correspondence (2.13) has to be generalized by the condition that the string world-sheet has to *end on the loop* $\mathcal{C}$ at the boundary. This mapping between the non-local Wilson loop operator $W(\mathcal{C})$ in the gauge theory and the string partition function $\mathcal{Z}(\mathcal{C})$ with the above-mentioned boundary condition was first proposed in [13, 14] through the relation

$$\langle W(\mathcal{C}) \rangle = \mathcal{Z}(\mathcal{C}) = e^{i \cdot S(\mathcal{C})} \ , \tag{2.24}$$

where $S(\mathcal{C})$ is the classical action of the string. This proposal comes from the agreement in studying the dynamics of the type IIB open string using supergravity limit at large N and large 't Hooft coupling constant λ and of the same string realized as world-volume soliton on D3-brane using Born-Infeld world volume action [14]. For a rectangular contour $\mathcal{C}$ of the form $L \times T$, the static potential $V_{q\bar{q}}(L)$ between the heavy quark and antiquark [12] in the large N-, large λ-limit can be extracted from the following relation

$$e^{iV_{q\bar{q}}(L)T} = \langle W(\mathcal{C}) \rangle = e^{i \cdot S(\mathcal{C})} \tag{2.25}$$

In these limits when all quantum fluctuations can be neglected, the string action $S(\mathcal{C})$ describing the proper area of the string world-sheet is calculated by the Nambu-Goto



(a) (b)

Figure 2.3.: *(a) The basic setup of the Wilson loop with $U = r/\alpha'$ denoting the radial coordinate. (b) The rectangular shape of the loop at the boundary with the quarks separation distance L, large time T and the S^5 positions of quark θ_1 and antiquark θ_2.*

action

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det_{\alpha,\beta}\left[G_{MN}\partial_\alpha X^M \partial_\beta X^N\right]} \ . \tag{2.26}$$

In the above equation, G_{MN} gives the ten-dimensional background metric, X^M the string coordinates in ten-dimensional spacetime, τ and σ parameterize the string world-sheet and α, β can take the value of the set $\{\tau,\ \sigma\}$.

For evaluating the *static* interquark potential we need to find the string configuration yielding the minimal surface S_{NG}^{min} which satisfies condition that the surface ends on the rectangular loop at the boundary. The problem of determining the expectation value of the Wilson loop is mapped to the problem of *solving the classical equations of motion for extremizing the Nambu-Goto action.*

Usually, the minimal Nambu-Goto action S_{NG}^{min} diverges since the surface has to go all the way to the boundary at infinity, thus an appropriate method is needed for regularizing this action. In field theory, due to the interaction of the quarks with the gluonic field, the self-energy of the quarks need to be subtracted from the total energy to obtain the length-dependent interaction potential. In the present case, the infiniteness of the minimal surface is interpreted as if it contains terms for the infinitely massive W-bosons, which need to be subtracted from the action in order to get the potential. The mass of the W-boson, which from now on will always be called "quark", is proportional to the length of the string stretching along the radial coordinate r from the far away D3-brane down to $r = 0$.

Consider a metric of the generic form

$$ds^2 = G_{MN}dX^M dX^N = -G_{00}(r)dt^2 + G_{x_{||}x_{||}}(r)dx_{||}^2 + G_{rr}(r)dr^2 + G_{x_T x_T}(r)dx_T^2 \ , \tag{2.27}$$

where r describes the radial coordinate (the fifth coordinate of AdS_5), $x_{||} = \{x_1, x_2, x_3\}$ and t are the coordinates of the usual 4-dim Minkowski space M_4, and x_T the coordinates transverse coordinates.

Let us consider the computation of a rectangular Wilson loop as described in Fig.2.3. Because of the rotational invariance in the four-dimensional Minkowski space M_4, the quark and antiquark position can be set along the $x_1 \equiv x$-coordinate, let us say at $-L/2$ and $L/2$, respectively. Here, the string only stretches along the radial coordinate and within the (x,t)-plain in M_4 (as in Fig.2.3(a)). For the moment, all the quark positions in the transverse coordinates are assumed to be constant. For the static case and the limit $T \to \infty$, the translation invariance along t can be assumed. Choosing the world-sheet coordinates $\sigma = x$ and $\tau = t$, the Nambu-Goto string action (2.26) takes the form

$$S_{NG} = \frac{T}{2\pi\alpha'} \cdot \int_{-L/2}^{L/2} d\sigma \sqrt{\mathcal{F}^2\left(r\left(\sigma\right)\right) + \mathcal{G}^2\left(r\left(\sigma\right)\right)\left(\partial_\sigma r\right)} \ , \tag{2.28}$$

with

$$\mathcal{F}^2\left(r\left(\sigma\right)\right) \equiv G_{00}(r(\sigma))G_{x_{||}x_{||}}(r(\sigma)) \tag{2.29}$$

$$\mathcal{G}^2\left(r\left(\sigma\right)\right) \equiv G_{00}(r(\sigma))G_{rr}(r(\sigma)) \ . \tag{2.30}$$

In this setting, the mass of the quark can be read off from a straight string with a constant value of x stretching from $r = 0$ to $r = r_{max}$, the position on the radial direction where the single D-brane is placed. Each quark has then the mass

$$m_q = \frac{1}{2\pi\alpha'} \int_0^{r_{max}} \mathcal{G}(r) dr \ . \tag{2.31}$$

For the $AdS_5 \times S^5$ metric the solution for the minimal surface traced out by the single string implies constant position in the transverse space [14]. For other backgrounds considered in this thesis, the same result is found and will be discussed later. Then, the quark antiquark potential can be calculated by extremizing the Nambu-Goto action and afterwards subtracting the quark masses $2m_q$ from it.

Assuming the string not to move in the transverse coordinates, the analysis in [20] for the generic metric (2.27) showed that if there exists a minimum surface satisfying the boundary condition, the potential can be brought to the general form

$$V_{q\bar{q}} = \mathcal{F}(r_0) \cdot L + K(r_0) \ , \tag{2.32}$$

where L denotes the separation between the quark and antiquark, r_0 the fathermost position of the string along the radial coordinate r from the boundary which due to the symmetry occurs at $\sigma = 0$ and K is some function of the coefficients of the generic metric.

It has been shown in [20] that *(assuming without loss of generality that $\mathcal{F}(r)$ has a minimum or $\mathcal{G}(r)$ diverges at $r = 0$) confinement occurs if and only if $\mathcal{F}(0) > 0$ and the* *corresponding non-vanishing string tension is* $\mathcal{F}(0)$. This observation comes from the fact that if a minimal surface can still be found for very large L, (2.32) describes an area law, and since r_0 will be then very close to zero, $\mathcal{F}(0)$ can be interpreted as the *effective* string tension.

In the frame of this diploma thesis, the string can also stretch in the internal space, which is described by the x_T- coordinates in (2.27). In order to have different internal positions of quark and antiquark, it is necessary to break the $U(N+2)$ gauge group into $U(N) \times U(1)_1 \times U(1)_2$ by giving two expectation values to the two $U(1)$ factors [13]. Picturally, this can be realized by moving two D3-branes far away along the radial coordinate r from the stack of original $(N+2)$ D3-branes with the additional requiring that the two far away D3-branes can have different positions in the transverse space. Although this degree of freedom is absent in QCD, it is given in superstring theory where strings are allowed to move in ten-dimensional space-time, and hence have some interpretation in $\mathcal{N} = 4$ SYM (coupling to the scalar fields). In the next chapters, it will be shown how quantities like quark antiquark potential, screening length, drag force, jet quenching parameter and confinement behavior depend on the relative internal orientations of the quark and antiquark.

3. Conformal and non-conformal Backgrounds

In the following chapter, the method for calculating the quark antiquark potential presented in the last chapter will be applied for two different backgrounds. The first one will be that of $AdS_5 \times S^5$. Since the string theory of type IIB described in this background is conjectured to be equivalent to the conformal $\mathcal{N} = 4$ SYM theory, a non-confinement behavior of $V_{q\bar{q}}(L)$ is expected. The second background is claimed to be dual to $\mathcal{N} = 2$ SYM [24], and hence non-confinement behavior of $V_{q\bar{q}}(L)$ might be changed. In each case, the additional dependence of $V_{q\bar{q}}(L)$ on the relative internal orientations of quark and antiquark will be considered.

3.1. The Prototype of $AdS_5 \times S^5$

3.1.1. The near-horizon Region

The near horizon geometry of a stack of coincident D3-branes is described by the $AdS_5 \times S^5$ metric (2.7). In order to introduce the quarks into the theory, one or more D3-branes need to be moved "far away" from a stack of coincident D-branes. In the following, the meaning of "far away" will be explained. The $AdS_5 \times S^5$ metric has the form

$$ds^2_{AdS} = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \quad , \; R^4 = 4\pi g_s \alpha'^2 N \tag{3.1}$$

Define a new variable $U \equiv r/\alpha'$, the metric becomes

$$ds^2_{AdS} = \alpha' \left\{ \frac{U^2}{R'^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R'^2}{U^2}dU^2 + R'^2 d\Omega_5^2 \right\} \; , \tag{3.2}$$

where $R'^4 = 4\pi g_s N$. In most calculations, the prime in R' will be omitted. We should keep in mind the explicit form of R while using r or U as the radial coordinate[1]. We are interested in the static potential between the quarks, so the quark and antiquark mass should be sent to infinity. Note, U has the dimension of energy, so the mass of the quark is proportional to the distance along the U-direction between the single and the stack of the remaining D-branes. By taking the decoupling limit $\alpha' \to 0$, the only requiring condition is keeping fixed the energy measured from infinity, which is expressed by r/α' [7]. Hence, it is possible to have very massive quarks (large separation in U) while keeping r small (staying in the near horizon region). Effectively, introducing the

[1] Thoughout this thesis, R is dimensionful in a metric with r representing the radial coordinate and is dimensionless if the coordinate $U = r/\alpha'$ is used instead of r.

very massive quarks can be realized by moving one single brane very far away along the U-coordinate from the remaining branes, then taking the decoupling limit $\alpha' \to 0$ while keeping the energy proportional to the distance along $U = r/\alpha'$ fixed, r becomes very small and the geometry (2.5) can be seen as $AdS_5 \times S^5$ everywhere.

3.1.2. Case of constant Angle

The computations of the potential in this and the next subsection follow closely the calculations in [13]. The quark antiquark potential can be extracted from the expectation value of the Wilson loop in the form of an rectangular as in Fig. 2.3(b). Following the prescription described in subsection 2.2.4, the potential will be calculated. For $T \to \infty$, the problem is translational invariant along the time direction. The quark and antiquark are set at $-L/2$ and $L/2$ along the x direction, respectively, and for the case of constant S^5-position, $\vec{\theta}(x) = \text{const}$ for all values of x, so $\theta_1 = \vec{\theta}(-L/2) = \vec{\theta}(L/2) = \theta_2$. Because of the static configuration it is natural to take $t = \tau$ and $x = \sigma$ to parameterize the string world-sheet. The boundary conditions are set on the single D-brane to be

$$U(\pm L/2) = U_{max} \to \infty$$

in order to make the quarks very massive. Applying the Nambu-Goto action (2.26) on the AdS metric (3.2), we get

$$S = \frac{T}{2\pi} \int_{-L/2}^{L/2} d\sigma \sqrt{(\partial_\sigma U)^2 + U^4/R^4} \ . \tag{3.3}$$

The Lagrangian density does not depend explicitly on σ, thus from equation

$$\frac{d}{d\sigma} \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial U'} U' \right) = 0, \quad U' \equiv \partial_\sigma U \tag{3.4}$$

one gets a conserved quantity ϵ_M along σ

$$\frac{U^4/R^4}{\sqrt{U'^2 + U^4/R^4}} = \epsilon_M = \frac{U_0^2}{R^2}, \tag{3.5}$$

where U_0 is defined to be the minimum value of U, which by symmetry occurs at $\sigma = 0$. From this expression it is possible to express $\partial U/\partial\sigma$ as a function of U alone, and so σ can be given in the following form

$$\sigma = \frac{R^2}{U_0} \int_1^{U_{max}/U_0} \frac{dy}{y^2 \sqrt{(y^4 - 1)}} \ , \tag{3.6}$$

where $y \equiv \frac{U}{U_0}$ and U_{max} gives the distance of the single D-brane to the remaining D-branes along the radial coordinate. Taking the limit $U_{max} \to \infty$ for infinite massive quarks, the position U_0 can be uniquely determined for a given separation distance L

$$L = 2\frac{R^2}{U_0} \int_1^{\infty} \frac{dy}{y^2 \sqrt{(y^4 - 1)}} = 2\frac{R^2}{U_0} \frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2} \ . \tag{3.7}$$

Using the relation (3.5) to rewrite the square root term in the action (3.3) and replacing U by y and by doing so $d\sigma$ by dy, the equation (3.3) takes the form

$$S = \frac{TU_0}{\pi} \int_1^{U_{max}/U_0} \frac{y^2}{\sqrt{(y^4-1)}}\, dy \ . \tag{3.8}$$

This expression diverges for $U_{max} \to \infty$, since it contains the quark masses (2.31), which need to be subtracted. The quark and antiquark mass together is given by

$$m_q + m_{\bar{q}} = 2m_q = \frac{TU_0}{\pi} \int_0^{U_{max}/U_0} dy \ . \tag{3.9}$$

The regularized potential $V_{q\bar{q}}(L)$ (2.25) becomes

$$V = \frac{U_0}{\pi} \left[\int_1^{\infty} dy \left(\frac{y^2}{\sqrt{y^4-1}} - 1 \right) - 1 \right] \ . \tag{3.10}$$

The integration gives a finite result

$$\lim_{\epsilon \to 0} \int_1^{\infty} dy\ y^{\epsilon} \left(\frac{y^2}{\sqrt{y^4-1}} - 1 \right) - 1 = -\frac{\sqrt{2}\ \pi^{3/2}}{\Gamma(1/4)^2} \ , \tag{3.11}$$

then using relation $R^2 = \sqrt{4\pi g_s N} = \sqrt{g_{YM}^2 N}$ and (3.7) to express U_0 in term of L, the final expression for $V_{q\bar{q}}(L)$ reads

$$V_{q\bar{q}}(L) = -\frac{4\pi^2 \sqrt{g_{YM}^2 N}}{\Gamma(1/4)^4 L} \ . \tag{3.12}$$

Due to the conformal invariance, the $1/L$ dependence of the coulombic potential occurs as expected, and that can serve as a first test for the correctness of the proposal (2.24), since due to the conjecture the $AdS_5 \times S^5$ geometry is dual to a *conformal* field theory. Furthermore, in the large λ-limit the $\sqrt{g_{YM}^2 N} = \sqrt{\lambda}$ dependence in the leading order is a non-trivial fact of strong coupling which differs from weak coupling where the result usually depends on $g_{YM}^2 N$ at the leading order. This information confirms the before mentioned *weak/strong* duality of the AdS/CFT at large N-, large λ-limit.

3.1.3. Case of non-constant Angle

In this subsection the S^5-dependence of the quark and antiquark will be turned on. This means that $\theta_1 = \vec{\theta}(-L/2)$ on one vertical line of Fig.2.3(b) will differ from $\theta_2 = \vec{\theta}(L/2)$ on the other vertical line [13]. The configuration can be realized by separating two D3-branes from the original $(N+2)$ coincident D3-branes. The positions of these two D3-branes in the internal space S^5 are not necessary identical, hence the S^5 position of the quark and antiquark sitting on each brane can be different. The considered world-sheet traced out by the string connecting the quarks goes at $\sigma = -L/2$ to $U = \infty$ and to the point θ_1 of the five-sphere and at $\sigma = L/2$ to $U = \infty$ and to the point θ_2. The time-translational invariance is still assumed for large T, hence to any time, this string describes a U-shape in the AdS_5 space with the tip going down into the radial direction and on the S^5-sphere it connects the two points $\vec{\theta}(-L/2)$ and $\vec{\theta}(L/2)$.

Due the symmetry of the problem this string will lie along a great circle of the sphere. By an appropriate parameterization of the sphere the relative positions of the quarks on the five-sphere can be completely determined by an angle[2]

$$\theta = |\vartheta(-L/2) - \vartheta(L/2)| \ , \tag{3.13}$$

where $\vartheta(\sigma)$ is one of the five angles parameterizing the S^5. Due to the symmetry around $\sigma = 0$ one can set the boundary conditions as

$$\vartheta(\pm\frac{L}{2}) = \pm\frac{\theta}{2} \ , \qquad U(\pm\frac{L}{2}) = \infty \ . \tag{3.14}$$

The Nambu-Goto action (2.26) for this case contains an additional term describing the string stretching inside the internal space and reads

$$S = \frac{T}{2\pi}\int_{-L/2}^{L/2} d\sigma \sqrt{(\partial_\sigma U)^2 + U^4/R^4 + U^2(\partial_\sigma \vartheta)^2} \ . \tag{3.15}$$

The Lagrangian in (3.15) does not explicitly depend on σ and ϑ, thus one obtains two conserved quantities ε_m and j_m. From the equation (3.4) one has

$$\frac{U^4}{\sqrt{U'^2 + U^4/R^4 + U^2\vartheta'^2}} = \varepsilon_m = \frac{U_0^4}{\sqrt{U_0^4/R^4 + U_0^2\vartheta_0'^2}} \ , \tag{3.16}$$

where U_0 again gives the radial position of the tip of the U-shape string occurring at $\sigma = 0$, and ϑ_0' the slope of the angle $\vartheta(\sigma)$ at $\sigma = 0$. The second conserved quantity arises, when we demand

$$\frac{d}{d\sigma}\frac{\partial \mathcal{L}}{\partial \vartheta'} - \frac{\partial \mathcal{L}}{\partial \vartheta} = 0 \ , \tag{3.17}$$

namely

$$\frac{U^2\vartheta'}{\sqrt{U'^2 + U^4/R^4 + U^2\vartheta'^2}} = j_m = \varepsilon_m \cdot \frac{\vartheta_0'}{U_0^2} \ . \tag{3.18}$$

Giving the quark antiquark separation L on the boundary and the angle difference θ in the internal space, it is necessary to solve the equations (3.16) and (3.18) in order

[2]Note, θ without indices gives the angle difference between quark and antiquark or in other words a segment of the great circle, while θ_i (with indices) gives the position on S^5

to determine the string configuration with the minimal world-sheet. After doing some basic algebra and using the boundary conditions (3.14), one get the following relations

$$\sigma = \frac{R^2}{U_0}\sqrt{1-\zeta^2}\int_1^{U/U_0}\frac{dy}{y^2\sqrt{(y^2-1)(y^2+1-\zeta^2)}}\ , \tag{3.19}$$

$$\vartheta = \zeta\int_1^{U/U_0}\frac{dy}{\sqrt{(y^2-1)(y^2+1-\zeta^2)}}\ , \tag{3.20}$$

where $y \equiv U/U_0$ and ζ is identified by

$$\zeta^2 \equiv \frac{R^4\vartheta_0'^2}{U_0^2 + R^4\vartheta_0'^2}\ . \tag{3.21}$$

By this definition ζ is restricted on the interval $(-1,1)$, but due to the symmetry it is sufficient to take values of $\zeta \in [0,1)$. Using the boundary conditions (3.14), one finds the characteristics of the stationary string's shape U_0 and ϑ_0' expressed by $\{U_0, \zeta\}$ for given values of L and θ, namely

$$L = 2\,\frac{R^2}{U_0}\sqrt{1-\zeta^2}\ I_1(\zeta) \tag{3.22}$$

$$\theta = 2\,\zeta\ I_2(\zeta)\ , \tag{3.23}$$

where

$$I_1(\zeta) = \frac{1}{(1-\zeta^2)\sqrt{2-\zeta^2}}\left[\left(2-\zeta^2\right)E\left(\sqrt{\frac{1-\zeta^2}{2-\zeta^2}}\right) - F\left(\sqrt{\frac{1-\zeta^2}{2-\zeta^2}}\right)\right] \tag{3.24}$$

$$I_2(\zeta) = \frac{1}{\sqrt{2-\zeta^2}}\ F\left(\sqrt{\frac{1-\zeta^2}{2-\zeta^2}}\right) \tag{3.25}$$

with F, E are complete elliptic integrals[3] of the first and second kind, respectively. Using relation (3.16) to replace the term under the square root of the action (3.15) by a function depending on U, U_0 and ϑ_0', and carrying out a change of variable $y \equiv U/U_0$ by making used of the relation (3.19), one ends up with the following expression for the action

$$S = \frac{T}{\pi}\int_1^{U_{max}/U_0} dy\ \frac{U_0 y^2}{\sqrt{(y^2-1)(y^2+1-\zeta^2)}}\ . \tag{3.26}$$

[3] $F(m) = \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-m^2 sin^2\theta}} = \int_0^1\frac{dy}{\sqrt{(1-y^2)(1-m^2y^2)}}$
$E(m) = \int_0^{\frac{\pi}{2}} d\theta\sqrt{1-m^2 sin^2\theta} = \int_0^1\frac{\left(1-m^2y^2\right)}{\sqrt{(1-y^2)}}dy$

Subtracting the infinite quark masses (3.9) ($U_{max} \to \infty$) from the last equation, the renormalized quarks potential can be determined

$$V(U_0, \zeta) = \frac{U_0}{\pi}\left[\int_1^\infty dy \left(\frac{y^2}{\sqrt{(y^2-1)(y^2+1-\zeta^2)}} - 1\right) - 1\right] \tag{3.27}$$

$$V(L, \zeta) = -\frac{2}{\pi}\frac{\sqrt{g_{YM}^2 N}}{L}(1-\zeta^2)^{3/2} I_1^2(\zeta) \, . \tag{3.28}$$

Equation (3.23) gives an one-to-one relation between θ and ζ with $\theta(0) = 0$ and $\theta(1) = \pi$. For vanishing angle θ, the potential above reduces to the case of constant angle (3.12) as expected. It is interesting to note that for $\theta = \pi$ (this special case can only achieved for $\zeta = 1$ demanding $R^2\vartheta_0' \gg U_0 \to 0$ which follows from the definition (3.21)) the potential goes to zero. This behavior is expected since it is a BPS construction [14]. The picture of this 1/4 BPS "loop" is also in agreement with the Zarembo's construction [31], in the sense that the tangent vectors of the two large sides of the rectangle point to two opposite poles on the S^5.

3.2. $\mathcal{N} = 2$ Dual

We have seen from the last section that the quark antiquark potential exhibits only a Coulomb branch and does not show confinement behavior. That is a good match to the dual picture since the $\mathcal{N} = 4$ SYM theory describes a conformal field theory and hence does not have running coupling. In order to make contact with QCD, it is necessary to investigate dual backgrounds to SYM theories with less supersymmetry and those which break conformal invariance. In [24] the form of F_5 as in the case of $AdS_5 \times S^5$ remains unchanged, the $SO(6)$ symmetry of the S^5 is preserved, and since it is of interest to have gauge theory defined on four dimensional Minkowski space-time, the Poincaré invariance $ISO(1,3)$ for the remaining metric without the radial part will be assumed. The deformation of the $AdS_5 \times S^5$ metric occurs when the couplings to the dilaton and axion field are turned on. Solutions with non-constant dilaton field will be crucial to the confinement behavior, since the expectation value of the dilaton is directly related to the coupling "constant" of the theory. A supersymmetric solution for this specific problem has been found in [24] which breaks half of the supersymmetries. The boundary gauge theory is identified to be $\mathcal{N} = 2$ SYM which, due to the non-vanishing beta function, turns out to have running coupling. Using the method presented in the last chapter, the quark antiquark potential will be evaluated.

3.2.1. The metric dual to $\mathcal{N} = 2$ SYM

Now we are going to consider above mentioned deformed version of the AdS background which is claimed to be dual to the $\mathcal{N} = 2$ SYM, the metric for the supersymmetric solution in the string frame in that case reads

$$ds^2 = \alpha' \left(\frac{R^4}{\rho^4} + \frac{\chi_0 R}{4g_s^{1/4}}\right)^{1/2} \left(d\rho^2 + dx_\mu dx^\mu + \rho^2 d\Omega_5^2\right) \tag{3.29}$$

and by changing the variable $\rho = 1/U$ and interpreting U as the energy of the boundary field theory [24], the metric is

$$ds^2 = \alpha' \left(R^4 + \frac{\chi_0}{4g_s^{1/4}}\frac{R}{U^4}\right)^{1/2} \left(\frac{dU^2}{U^2} + U^2 dx_\mu dx^\mu + d\Omega_5^2\right) . \tag{3.30}$$

where U denotes the radial coordinate, $R = (4\pi g_s N)^{1/4}$ the curvature radius of the AdS_5, g_s the string coupling constant, and χ_0 is a dimensionless positive integration constant.

3.2.2. Quark Antiquark Potential with constant Angle on S^5

Following the computational techniques described in subsection 2.2.4, the result for the potential in [24] will be examined. The expectation value of a rectangular Wilson loop, from which the quark antiquark potential can be extracted, is determined from the minimal surface described by a dual string lying in space with endpoints on the boundary representing the quark and antiquark positions. We take the ansatz for the background string as

$$x_0 = \tau, \quad x_1 = \sigma, \quad U = U(\sigma)$$

and rest of the string positions remains constant in τ and σ. The quarks separation distance will be further denoted by L and the boundary conditions are set at $U \to \infty$, where the probe brane is placed in order to make the quark masses very massive,

$$U(\pm\frac{L}{2}) = \infty, \quad \vartheta(\sigma) = \text{const.} \tag{3.31}$$

The Nambu-Goto action for the string world-sheet is

$$S = \frac{1}{2\pi}\int d\sigma d\tau \left[\left(R^4 + \frac{\chi_0}{4g_s^{1/4}}\frac{R}{U^4}\right)\left(U^4 + (\partial_\sigma U)^2\right)\right]^{1/2} . \tag{3.32}$$

The Euler-Lagrange equation can be solved using the fact that the Lagrangian density does not explicitly depend on σ, so we demand $\mathcal{L} - \frac{\partial \mathcal{L}}{\partial U'}U' = const$, where the prime denotes differentiating with respect to σ, and get

$$\left(R^4 + \frac{\chi_0}{4g_s^{1/4}}\frac{R}{U^4}\right)^{1/2} \frac{U^4}{\sqrt{U^4 + (\partial_\sigma U)^2}} = \varepsilon_K . \tag{3.33}$$

Again, defining U_0 to be the minimum value of U, which by symmetry occurs at $\sigma = 0$, and taking care the vanishing of $(\partial_\sigma U)\,|_{\sigma=0}$, the constant ε_K along σ-axes can be given in terms of U_0

$$\begin{aligned} \varepsilon_K &= \left(R^4 U_0^4 + A\right)^{1/2} \ , && (3.34) \\ A &\equiv \frac{\chi_0 R}{4 g_s^{1/4}} \ . && (3.35) \end{aligned}$$

The relation (3.33) can be brought to the following form by writing $U = U_0 y$

$$\sigma = \frac{\sqrt{U_0^4 R^4 + A}}{U_0^3 R^2} \int_1^{U_{max}/U_0} \frac{1}{\sqrt{y^4\left(y^4 - 1\right)}}\, dy \ , \tag{3.36}$$

and using the boundary conditions (3.31), a relation between U_0 and L is found to be

$$\frac{L}{2} = \frac{\sqrt{U_0^4 R^4 + A}}{U_0^3 R^2}\, \eta \quad , \qquad \eta = \frac{\sqrt{\pi}\ \Gamma(3/4)}{\Gamma(1/4)} \ . \tag{3.37}$$

Since at the end it is of interest to have the potential as a function of the separation distance, we need to express U_0 in terms of L. Noting by definition the positivity of the turning point of the trajectory U_0, (3.37) is equivalent to

$$\begin{aligned} f(z) &= z^3 - P^2 z^2 - P^2 B = 0 \ , && (3.38) \\ z &\equiv U_0^2 \ , \qquad P \equiv \frac{2\eta}{L} \ , \qquad B \equiv \frac{A}{R^4} = \frac{\chi_0}{R^3 4 g_s^{1/4}} \ . && (3.39) \end{aligned}$$

Since $f(z_i) < 0$ for $\forall z_i$ with $(\partial_z f(z))|_{z=z_i} = 0$, $f(z)$ has only one real solution and U_0 is given by

$$\begin{aligned} U_0 &= \left(\frac{1}{6}\Delta^{1/3} + \frac{2}{3}\frac{P^4}{\Delta^{1/3}} + \frac{P^2}{3}\right)^{1/2} \ , \\ \Delta &\equiv 108 P^2 B + 8 P^6 + 12\sqrt{12 P^8 B + 81 P^4 B^2} \ . && (3.40) \end{aligned}$$

In order to calculate the action, (3.33) will be inserted in (3.32). Carrying out the integration $d\tau$ we have

$$S = \frac{T}{\pi} \int_{U_0}^{U_{max}} \left(\frac{R^2 U^2}{\sqrt{U^4 - U_0^4}} + \frac{A}{R^2 U^2 \sqrt{U^4 - U_0^4}} \right) dU \ . \tag{3.41}$$

The action consists of two parts, while the second term in the integral converges the first one diverges. This infiniteness of the action is due to the contribution from the masses of the quark and antiquark, which in the dual picture are represented by open strings connecting the N branes with the far away "probe" brane. This quantity is related to the length of the fundamental string stretching between those two regions, and in this case (3.30) it is

$$2\ m_q = \frac{1}{\pi}\int_0^{U_{max}} \sqrt{R^4 + \frac{A}{U^4}}\ dU\ . \tag{3.42}$$

Subtracting this contribution from the total energy extracted from (3.41), the potential becomes finite [4] can be found. For more details about this subtraction procedure, see appendix A.5.

$$V_{q\bar{q}} = \frac{1}{\pi}\frac{\sqrt{\pi}\ \Gamma(3/4)}{\Gamma(1/4)}\left(-R^2U_0 + \frac{A}{R^2U_0^3}\right)\ . \tag{3.43}$$

Here we have introduced an cutoff not only at some U_{max} but also at some U_{IR}, since the integral diverges as U approaches to zero. We are only interested in the behavior of the quark antiquark potential depending on $U_0(L)$, so basically the terms containing U_{IR} can be left out and serve as a shift of the potential. Hence, this finite result is exact up to a constant depending on where the cutoff U_{IR} is taken. From (3.40) and (3.43) it is possible to examine the behavior of $V_{q\bar{q}}$ depending on the quark antiquark separation distance L.

It is of interest to have some analytic expressions showing how the potential depends on the quarks separation distance for small and large values of L compared to the constant $B^{1/4} = A^{1/4}/R$. From (3.40) one has for $L \ll B^{-1/4}$,

$$\Delta \simeq 8P^6, \qquad U_0 \simeq P = \frac{2\eta}{L}$$

$$V_{q\bar{q}} \simeq -\frac{2\eta^2}{\pi}\frac{R^2}{L}\ . \tag{3.44}$$

This usual Coulombic law behavior as in [13, 14] is expected, since for large U the metric (3.30) turns to that of $AdS_5 \times S^5$ by a reparametrization $U \to U'/R^2$. However, in the opposite limit where $L \gg B^{-1/4}$, the second term in (3.43) dominates, giving

$$\Delta \simeq 216P^2B, \qquad U_0 \simeq \left(P^2B\right)^{1/6} = \left(\frac{2\eta}{L}\right)^{1/3}\left(\frac{A}{R^4}\right)^{1/6}$$

$$V_{q\bar{q}} \simeq \frac{1}{2\pi}\sqrt{A}L\ . \tag{3.45}$$

This confining behavior is new compared to the case of AdS where we have only the Coulombic term in the potential. The confinement is a consequence of the running coupling in the dual field theory. From another point of view, this result once more suggests the correctness of the conjectured mapping (2.24).

[4] This result is identical with that of [24] by a rescaling $U_0 \to \frac{U_0}{R^2}$. In [24] the change of variable $\rho = R^2/U$ was taken, instead of $\rho = 1/U$ as in (3.30).

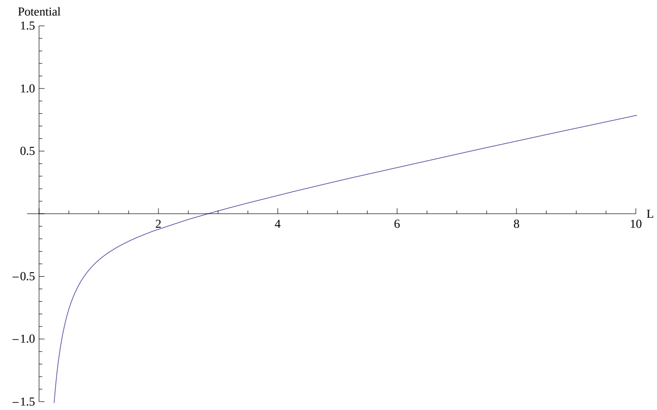


Figure 3.1.: *Plot of the quark antiquark potential, in units of $RA^{1/4}$, as a function of separation distance L, in units of $RA^{-1/4} = B^{-1/4}$.*

To have a whole picture how the quark antiquark potential depends on the distance L between the quarks, a plot of $V_{q\bar{q}}$ is given in Fig.3.1. Here we rescaled U_0 by $U_0 = \frac{A^{1/4}}{R} U_r$ and the potential (3.43) becomes

$$V_{q\bar{q}} = \frac{1}{\pi} \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)} RA^{1/4} \left(-U_r + \frac{1}{U_r^3} \right) \quad , \tag{3.46}$$

and from (3.37) the relation between L and U_r is given by

$$L = 2 \ RA^{-1/4} \frac{\sqrt{U_r^4 + 1}}{U_r^3} \ . \tag{3.47}$$

From the last equation it is possible to give a relation between U_r and L (in units of $RA^{-1/4}$), then replacing $U_r(L)$ in the potential (3.46), a plot of the general course of $V_{q\bar{q}}(L)$ can be generated, see Fig. 3.1.

3.2.3. Quark Antiquark Potential with non-constant Angle on S^5

In this subsection we are going to consider the case where the quarks have different positions on the S^5 of the deformed AdS metric (3.30). Because of the symmetry, as described in chapter 1, the relative quarks position on the S^5 can be fully determined by an angle $\vartheta(s)$ where s parameterizes the loop variable. Taking the configuration where the Wilson loop contour $\mathcal{C}$ forms an rectangular as usual and letting the angle ϑ be time-independent and vary only along the spatial coordinate σ, we can write the angle as $\vartheta(\sigma)$.

In the following, we want to investigate how the coulombic and confinement behavior of the quark antiquark potential are influenced by the relative angle θ, we also want to check whether a BPS configuration as considered in [13, 14] exists. At the end, the results will be compared to those of the case of the vanishing relative angle.

Considering the string stretching on the (x_1, t)-plain, along the radial coordinate and on the S^5, we take the following boundary conditions

$$U(\pm\frac{L}{2}) \to \infty, \quad \vartheta(\pm\frac{L}{2}) = \pm\frac{\theta}{2} \ . \tag{3.48}$$

Taking the usual parameterization of the world-sheet by $x_1 = \sigma$ and $t = \tau$, and applying the Nambu-Goto action

$$S = \frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\det\left(G_{MN}\partial_\alpha X^M \partial_\beta X^N\right)} \tag{3.49}$$

on (3.30), we have

$$S = \frac{T}{2\pi}\int_{-L/2}^{L/2} d\sigma \left[\left(R^4 + \frac{A}{U^4}\right)\left(U^4 + U'^2 + U^2\vartheta'^2\right)\right]^{1/2} \ , \tag{3.50}$$

where the prime denotes the partial differentiation with respect to σ. The Lagrangian does not depend explicitly on σ and is independent of ϑ, thus we have two conserved quantities.

$$\begin{aligned} \sqrt{R^4 + \frac{A}{U^4}}\frac{U^4}{\sqrt{U^4 + U'^2 + U^2\vartheta'^2}} &= \varepsilon_k \ , \\ \varepsilon_k \ \frac{\vartheta'}{U^2} &= j_k \ . \end{aligned} \tag{3.51}$$

The quantities ε_k and j_k are conserved quantities along σ. Due to the symmetry, it is then convenient to determine their values at $\sigma = 0$ by expressing (3.51) as functions of the turning point U_0 and the slope of the angle in the internal space ϑ'_0.

$$\varepsilon_k = \sqrt{R^4 + \frac{A}{U_0^4}}\frac{U_0^4}{\sqrt{U_0^4 + U_0^2\vartheta_0'^2}} \ , \tag{3.52}$$

$$j_k = \sqrt{R^4 + \frac{A}{U_0^4}}\frac{U_0^2\vartheta'_0}{\sqrt{U_0^4 + U_0^2\vartheta_0'^2}} \ . \tag{3.53}$$

Dividing (3.53) by (3.52), one finds

$$\frac{\vartheta'}{U^2} = \frac{\vartheta'_0}{U_0^2} \quad \Leftrightarrow \quad \vartheta' = \frac{\vartheta'_0}{U_0^2}U^2 \ . \tag{3.54}$$

Inserting the above relation for ϑ' in (3.52) and after some calculations one gets

$$U' = \sqrt{U^4\left[\frac{\left(U^4R^4 + A\right)\left(U_0^4 + U_0^2\vartheta_0'^2\right)}{U_0^8R^4 + AU_0^4} - \left(1 + \frac{U^2}{U_0^4}\vartheta_0'^2\right)\right]} \ . \tag{3.55}$$

From this it is possible to determine the U-coordinate of the string depending on σ when U_0 and ϑ'_0 are known. After setting y as $U = yU_0$ and introducing a new parameter l being identified with

$$l^2 = \frac{\vartheta_0'^2 \left(U_0^4 R^4 + A\right)}{U_0^4 R^4 \left(U_0^2 + \vartheta_0'^2\right)} \ , \tag{3.56}$$

we end up with

$$\sigma = \frac{l}{\theta_0} \int_1^{U/U_0} \frac{dy}{y^2 \sqrt{\left(y^2-1\right)\left(y^2+1-l^2\right)}} \ . \tag{3.57}$$

Using the relation (3.54), $\vartheta(\sigma)$ can be given by

$$\begin{aligned} \vartheta\left(\sigma\right) &= \int \frac{\vartheta_0}{U_0^2} U^2\left(\sigma\right) d\sigma = \vartheta_0 \int y^2 d\sigma \\ &= l \int_1^{U/U_0} \frac{dy}{\sqrt{\left(y^2-1\right)\left(y^2+1-l^2\right)}} \ . \end{aligned} \tag{3.58}$$

Using the boundary conditions (3.48), where U in the integration boundary goes to infinity, relations between the quarks separation distance L, the relative angle θ, the turning point along the radial coordinate U_0 and the slope ϑ'_0 at $\sigma = 0$ are given by

$$\begin{aligned} L &= 2\,\frac{l}{\vartheta'_0} I_1\left(l\right) \\ \theta &= 2\, l I_2\left(l\right) \ , \end{aligned} \tag{3.59}$$

where

$$I_1(l) = \frac{1}{\left(1-l^2\right)\sqrt{2-l^2}} \left[\left(2-l^2\right) E\left(\sqrt{\frac{1-l^2}{2-l^2}}\right) - F\left(\sqrt{\frac{1-l^2}{2-l^2}}\right)\right] \tag{3.60}$$

$$I_2(l) = \frac{1}{\sqrt{2-l^2}}\, F\left(\sqrt{\frac{1-l^2}{2-l^2}}\right) \tag{3.61}$$

with F, E are complete elliptic integrals [5] of the first and second kind, respectively.

From equation (3.59) one has an one-to-one relation between θ and l . Since l is restricted to the interval $(-1, 1)$, the relative angle θ can take values between $-\pi$ and π. Because of the symmetry, it is sufficient to let l having values in $[0, 1)$. A plot of equation (3.59) is given in Fig.3.2.

[5] $F(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m^2 sin^2\theta}} = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-m^2y^2)}}$

$E(m) = \int_0^{\frac{\pi}{2}} d\theta\sqrt{1-m^2 sin^2\theta} = \int_0^1 \frac{\left(1-m^2y^2\right)}{\sqrt{(1-y^2)}} dy$

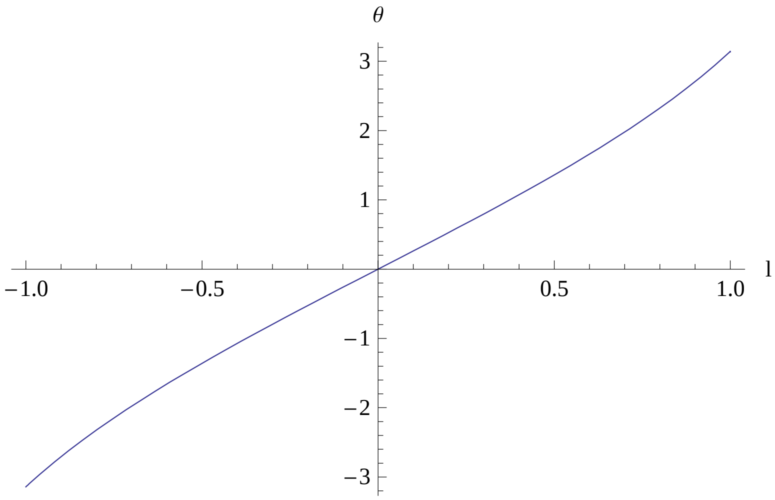


Figure 3.2.: Plot of the relative angle θ and the parameter l.

The quark antiquark potential The next step is to calculate the extremal Nambu-Goto action as a function of L and θ which is expressed through l in this case. Because of the symmetry of the problem around $\sigma=0$ in the (σ,U)-plain, the action (3.50) now reads

$$S=\frac{2T}{2\pi}\int_0^{L/2}dx\sqrt{\left(R^4+\frac{A}{U^4}\right)(U^4+U'^2+U^2\vartheta'^2)}\ . \tag{3.62}$$

From (3.52) we have

$$\sqrt{(U^4+U'^2+U^2\theta'^2)}=\sqrt{\frac{R^4+\frac{A}{U^4}}{R^4+\frac{A}{U_0^4}}}\cdot\frac{U^4}{U_0^4}\cdot\sqrt{(U_0^4+U_0^2\theta_0^2)}\ , \tag{3.63}$$

then using (3.55) for replacing $d\sigma$ by dU and afterwards introducing y as usual $U=U_0y$, the action takes the following form

$$S=\frac{T}{\pi}\int_1^{U/U_0}dy\left[\frac{A}{U_0^3R^2}\frac{1}{y^2\sqrt{(y^2-1)\,(y^2+1-l^2)}}+R^2U_0\frac{y^2}{\sqrt{(y^2-1)\,(y^2+1-l^2)}}\right]. \tag{3.64}$$

The action diverges because of the second term. After regularizing this expression by subtracting the contribution from the mass of the quark and antiquark (3.42) (for more details see appendix A for the case of non-constant angle), the action becomes finite

$$\frac{S}{T}=\frac{1}{\pi}\left[\frac{A}{U_0^3R^2}I_1(l)-R^2U_0I_3(l)\right]=V_{q\bar{q}}\ , \tag{3.65}$$

where

$$I_3(l) = E\left(\frac{\pi}{2}, \sqrt{l^2-1}\right) - F\left(\frac{\pi}{2}, \sqrt{l^2-1}\right) \ . \tag{3.66}$$

Similarly to the case of vanishing relative angle, this expression for the potential is exact up to a constant shift, since in order to get a finite result, an cutoff $U_{IR} \to 0$ has been taken. From the equations (3.59), for given values of L and θ, U_0 and ϑ_0' can be fully determined.

Let us fix the relative angle between quark and antiquark at the boundary by choosing a constant value for $l \in [0,1)$. Taking care the positivity of U_0 and A, the following expression

$$l^2 = \frac{\vartheta_0'^2\left(U_0^4 R^4 + A\right)}{U_0^4 R^4 \left(U_0^2 + \vartheta_0'^2\right)} \ , \qquad \vartheta_0' = \frac{2 l I_1(l)}{L} \tag{3.67}$$

is equivalent to

$$f(z) = z^3 - \frac{M}{L^2} z^2 - \frac{NB}{L^2} = 0 \ , \tag{3.68}$$

where

$$z = U_0^2, \qquad M = 4 I_1^2(l)\left(1 - l^2\right), \qquad N = 4 I_1^2(l), \qquad B = \frac{A}{R^4} \ . \tag{3.69}$$

Since $f(z_i) < 0$ for $\forall z_i$ with $\left(\partial_z f(z)\right)|_{z=z_i} = 0$, equation (3.68) has only one real solution and which is given by

$$U_0 = \sqrt{\frac{1}{6}\frac{Z^{1/3}}{L^2} + \frac{2}{3}\frac{M^2}{L^2 Z^{1/3}} + \frac{1}{3}\frac{M}{L^2}} \tag{3.70}$$

with

$$Z = 108 N B L^4 + 8 M^3 + 12\sqrt{3}\sqrt{NB\left(27 N B L^4 + 4 M^3\right)}\ L^2 \tag{3.71}$$

With these results we are able to consider the cases where L becomes large or small compared[6] to $\frac{\left(1-l^2\right)^{3/4}}{B^{1/4}}\ I_1(l)$.

For $L \gg \frac{\left(1-l^2\right)^{3/4}}{B^{1/4}}\ I_1(l)$, the terms proportional to M^3 will be neglected, and we have

$$Z \simeq 216 N B L^4, \qquad U_0 \simeq \left(\sqrt{NB}\frac{1}{L}\right)^{1/3}$$

$$V_{q\bar{q}} \simeq \frac{1}{\pi}\left[\frac{\sqrt{A}}{2} L - R^2 \left(NB\right)^{1/6} I_3(l) \frac{1}{L^{1/3}} + \mathcal{O}\left(\frac{1}{L^{5/2}}\right)\right] . \tag{3.72}$$

[6] The reason why L should be compared with this quantity comes from considering (3.71) in order to see which terms dominate for various values of L while keeping l (θ) fixed.

Compared with the constant angle case (3.45), there exist additional terms in (3.72), but for very "large" L the potential shows the same confinement behavior in both cases expressing by the same slope parameter denoting the constant force between the quarks for all distances.

In the opposite case, where $L \ll \frac{\left(1-l^2\right)^{3/4}}{B^{1/4}} I_1(l)$, we have

$$Z \simeq 8M^3, \qquad U_0 \simeq \frac{\sqrt{M}}{L}$$

$$V_{q\bar{q}} \simeq \frac{1}{\pi}\left[\frac{A}{R^2 M^{3/2}} L^3 - \frac{R^2\sqrt{M}}{L} I_3(l)\right]. \tag{3.73}$$

The second term in the above equation dominates for "small" L showing a Coulombic behavior. This behavior differs from the case of constant angle (3.44) by the dependence on $l(\theta)$. As a mild check of consistency l is set to be 0 and the second term in the above expression is

$$\frac{-2I_1(0)I_3(0)}{\pi}\frac{R^2}{L} = \frac{-2\eta^2}{\pi}\frac{R^2}{L} \ .$$

Hence, for $l = 0$ ($\theta = 0$) equation (3.73) reduces to the case of constant angle (3.44) as it should.

Plotting the Potential

Recall the quark antiquark potential

$$V_{q\bar{q}} = \frac{1}{\pi}\left[\frac{A}{U_0^3 R^2} I_1(l) - R^2 U_0 I_3(l)\right] \ . \tag{3.74}$$

We can rescale U_0 by $U_0 = \frac{A^{1/4}}{R} U_{0r}$ and the potential becomes

$$V_{q\bar{q}} = A^{1/4} R \cdot \frac{1}{\pi}\left[\frac{1}{U_{0r}^3} I_1(l) - U_{0r} I_3(l)\right] \ . \tag{3.75}$$

Since we want to investigate the dependence of the potential on L, we need to know the relation between L and U_{0r} and that can be done by solving the equation (3.68) after rescaling U_0 as above and L by $L_r = L A^{1/4} R^{-1}$. Equation (3.68) is the equivalent to

$$\frac{A^{3/2}}{R^6}\left(z_r^3 - \frac{M}{L_r^2} z_r^2 - \frac{N}{L_r^2}\right) = 0 \ , \qquad z_r = U_{0r}^2 \ . \tag{3.76}$$

Solving this equation gives a relation between U_{0r} and L_r, which is L in units of $A^{1/4}R^{-1}$. Making use of this relation, the potential V in units of $A^{1/4}R$ can be completely expressed by terms depending on L_r and $l(\theta)$. The plots are given in Fig.3.3.

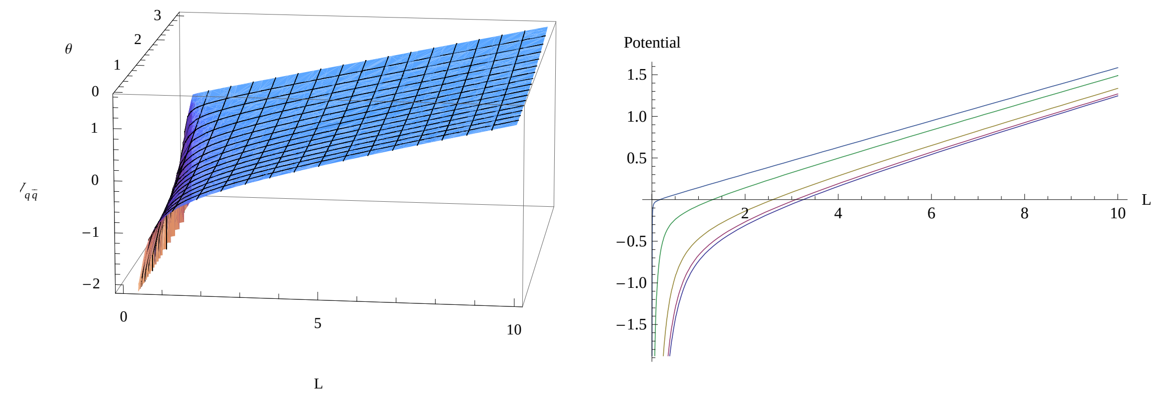


Figure 3.3.: *(lhs)The potential, in units of $A^{1/4}R$, as a function of separation distance L, in units of $A^{1/4}R^{-1}$, and θ. (rhs) $V_{q\bar{q}}(L)$ with $\theta = (.99\pi, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}, 0)$ from the top to bottom ,respectively*

3.3. Summary and Discussions

Summary We have applied the proposal (2.24) for evaluating the quark antiquark potential in two different backgrounds. The $AdS_5 \times S^5$ background is dual to a conformal field theory which does not have running coupling. As expected, the calculated static potential shows only a Coulombic branch, which states that the quarks behave like two static charged points at all separation distances. However, when one considers the background proposed in [24], which is claimed to be dual to $\mathcal{N} = 2$ SYM with running coupling, the potential determined via string dual picture indeed shows confinement behavior at large quarks separation distances. Both results can be interpreted as positive tests for the proposal (2.24). Furthermore, the weak/strong coupling duality in large N, large λ-limit is expressed by the $\sqrt{g_{YM}^2 N} = \sqrt{\lambda}$ dependence in the leading order which is a non-trivial fact of strong coupling which differs from the perturbative YM results which usually depend on $g_{YM}^2 N$ in the leading order.

We are also interested in the case where an additional degree of freedom is turned on which allows the movement of the open string inside the internal space S^5. In this picture, the quarks are represented by the endpoints of open strings on some probe brane. This degree of freedom can be expressed through the relative angle θ between the quarks on the five-sphere.

For the case of $AdS_5 \times S^5$ the Coulombic behavior of the potential does not change by switching on the S^5-orientation, but compared to the case of constant angle ($\theta = 0$) there exists a prefactor $X(\theta)$ which scales the $V(\theta = 0, L)$-potential. The prefactor X depends only on θ which can be expressed by the parameter ζ

$$X(\zeta) = \frac{\Gamma^4(1/4)}{2\pi^3}(1-\zeta^2)^{3/2} I_1^2(\zeta) \ . \tag{3.77}$$

If the relative angle approaches to π, $X(\zeta \to 1)$ goes to zero and the potential vanishes independently from L. This situation was mentioned before as a BPS state which can be realized only for $U_0 \to 0$. The solution then looks like two strings starting from infinity going straight all the way down to $U = 0$. For the second background (3.30) we do not observe an BPS state (non-vanishing potential for any L at $\theta = \pi$). That can be explained by the breaking of conformal and supersymmetry, also there is a shift in the potential which we do not consider.

Averaging the Results It is clear that SYM theory differs from QCD in many ways, and there is no degree of freedom in QCD which can be interpreted as the internal degree of freedom described above. However, if one carries out calculations in the string dual picture and wants to get rid of the S^5 degree of freedom, one could average over all possible θ-angles [60]. That is one possibility to reduce the degrees of freedom in going towards QCD. Our ansatz comes from letting the quarks and antiquarks having arbitrary positions on the five-sphere and average over all possible configurations. The weight to each configuration of relative angle θ will be $\omega(\theta)$ given by the volume per θ on S^5 divided by the total S^5 volume[7]

$$\omega(\theta) = \frac{volS^4}{volS^5}\ \sin^4\theta\ =\ \frac{8}{3\pi}\ \sin^4\theta\ . \tag{3.78}$$

The integration of the weight over all angles $\theta \in [0, \pi]$ gives unity as it should. To have a picture of what is meant above let us consider the simplified case where the quark and antiquark are assumed to have arbitrary positions on the S^2-sphere. For the static configuration the string connecting them is a segment of the great circle. Due to the symmetry the relative position between the quarks can be parameterized by an angle[8] θ. The weight is then given by the circumference of the circle at the polar angle θ divided by the surface area of S^2. By doing this, $\theta = 0$ and $\theta = \pi$ have the minimal weight while $\theta = \pi/2$ has the maximal one.

Applying this method on (3.77), then the average

$$\overline{X(\theta)}\ =\ \int_0^\pi X(\theta)\ \omega(\theta)\ d\theta \tag{3.79}$$

can be evaluated numerically by

$$\overline{X(\theta)}\ =\ \int_0^1 X(l)\ \omega(l)\frac{d\theta}{dl}dl\ \simeq\ 0.623\ . \tag{3.80}$$

[7] $volS^n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma[\frac{n+1}{2}]}$

[8] We are just interested in the relative positions between the quarks, not in the precise positions of them on the sphere

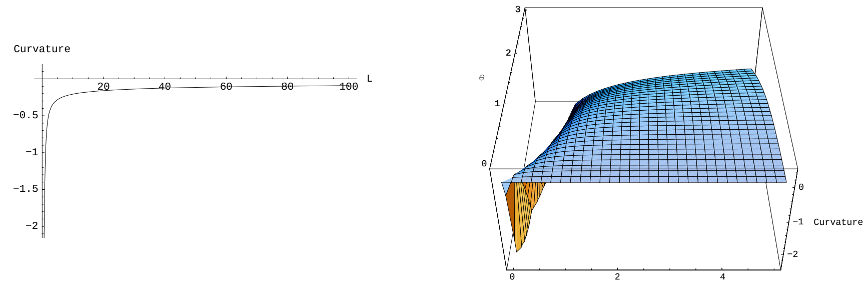


Figure 3.4.: *(lhs) Curvature for $\theta = 0$. (rhs) Plot of* (3.85) *for the whole(L,θ)-plane*

Due to our proposed method, after averaging over all angles θ, this factor of 0.623 will stand infront of (3.12) giving

$$V(L, \overline{\theta}) = -0.623 \, \frac{4\pi^2 \sqrt{g_{YM}^2 N}}{\Gamma(1/4)^4 L} \; . \tag{3.81}$$

For the second discussed background (3.30) dual to $\mathcal{N} = 2$ SYM, the S^5-angle dependence does not cause any change to the confinement behavior (the same slope parameter for $V(L, \theta)$ at any θ), since the attractive force between the quarks for large enough L remains the same for both cases, namely for the case of vanishing and non-vanishing relative angle. At small L compared to some constant given by the geometry, the potential is Coulombic and at large L it shows confinement. Switching on the relative angle θ, the definition of "small" and "large" L changes, it depends strongly on θ. As an extremal case for $\theta = \pi$, the sufficient condition for L to be considered as "large" holds, if it has a non-vanishing value.

From the analysis in the last chapter for the non-constant angle case, L is considered as large (small) when it is much greater (smaller) than $\frac{R\left(1-l^2\right)^{3/4}}{A^{1/4}} I_1(l)$. We can consider this quantity as the separation distance, proportional to which confinement behavior sets in, then using the averaging method presented before, a suppression factor to the case of constant angle can be found by averaging over all angles θ

$$\int_0^1 \; (1-l^2)^{3/4} \frac{I_1(l)}{I_1(0)} \omega(l) \frac{d\theta}{dl} dl \; \simeq .7799 \; . \tag{3.82}$$

Check of Concavity In the last chapter, the quark antiquark potential have been determined using string dual picture for two different backgrounds, one dual to a conformal and the other to a non-conformal field theory. To check whether the proposal (2.24) is a good one, it needs to pass certain consistency checks. One of which is the stability condition under small fluctuations. For the case of $AdS_5 \times S^5$, equations of motion for the fluctuations have been obtained in [27] which provide positive results.

In this paragraph we follow the analysis in [28] to check the concavity of the potential obtained via string dual picture in all directions of the (L, θ) plane. From the field theoretical point of view the potential has to be concave and monotonically growing with L [30],

$$\frac{\partial V}{\partial L} > 0, \qquad \frac{\partial^2 V}{\partial L^2} \leqslant 0 \ , \tag{3.83}$$

which states that the force between the quarks is always attractive and its magnitude is a never increasing function of separation distance. In [28] a general expression for those conditions has been found if the string does not extend in the S^5-part of a general 10-dimensional metric (2.27)

$$\frac{1}{2\pi}\sqrt{b_0} > 0, \qquad \frac{db_0(U_0)}{dU_0}\frac{\partial U_0}{\partial L} \leqslant 0 \ , \tag{3.84}$$

with $b_0 \equiv b(U_0)$ for $b = -G_{tt}G_{||}$. There is no known dual geometry violating the above condition. If the S^5 degree of freedom is turned on, it is necessary to check the concavity condition case by case. Concavity for the whole (L, θ)-plane holds if the following inequation is valid

$$\left(L^2\frac{\partial^2}{\partial L^2} + 2L\theta\frac{\partial^2}{\partial L \partial \theta} + \theta^2\frac{\partial^2}{\partial \theta^2}\right) V(L, \theta) \leq 0 \ . \tag{3.85}$$

In large T-limit it reduces to a milder condition, namely standard concavity at $\theta = 0$ and $V(L, 0) \leq V(L, \theta)$. Due to the complexity of the problem, the validity of (3.85) for quarks potential (3.65) could not be solved analytically. In Fig.3.4 we present two graphical representations indicating the concavity of the potential for the non-conformal case.

4. Quark-Gluon-Plasma

Hadronic matter undergoes a phase transition to a deconfined phase above some critical temperature $T_c \sim 200MeV$, where a gas of hadrons turns into a gas of quarks and gluons. This critical temperature can be obtained from models in lattice QCD and there are many hints from experiments at RHIC[33] indicating that this QCD state of matter has been realized by Au-Au collision at velocity close to c. Due to the strong coupling of the medium, conventional theoretical methods for investigating the properties of QGP encounter many difficulties. The gauge/gravity correspondence provides a stimulating framework to explore the strong coupling regime of the gauge theories using the dual string description. Although QCD and $\mathcal{N} = 4$ SYM are very different theories at zero temperature, they might share some common properties above the critical temperature [38]. Considering the QGP as a $\mathcal{N} = 4$ SYM plasma, physical quantities like viscosity, drag force, jet quenching parameter, interquark potential of heavy meson *moving* inside the plasma and its screening length can be explored. In the following chapter, phenomena registered by experimental data like suppression of charmonium states and back-to-back jets compared to the case of proton-proton collision will be analyzed via string dual description.

4.1. The AdS Black Hole background

According to the AdS/CFT, raising the temperature of the gauge theory corresponds to introducing a black brane (black hole) in to the center of AdS_5 [16], and the Hawking temperature of the black hole related to the horizon of the geometry is identified with the temperature of the gauge theory living on the boundary. In thermal field theory, expectation values of physical observables in thermal ensemble can be calculated using partition function $Tr\ e^{-\beta H}$ which is written in the form of Euclidean functional integral. In order to have Euclidean time one needs to carry out a Wick rotation $t \to i\tau$ and the trace may be implemented in that way that the partion function is (anti)-periodical with respect to $\tau \sim \tau + \beta$, where the inverse of β gives the temperature of the system.

From the other side, the metric dual to $\mathcal{N} = 4$ SYM theory at non-zero temperature is the black three-brane metric[16],

$$ds^2 = -f dt^2 + \frac{r^2}{R^2}(dx_1^2 + dx_2^2 + dx_3^2) + \frac{1}{f} dr^2 + R^2 d\Omega_5^2 \ , \quad f \equiv \frac{r^2}{R^2}\left(1 - \frac{r_H^4}{r^4}\right) \ . \quad (4.1)$$

The event horizon is located at $r = r_H$, where $G_{tt} = 0$. The horizon is the three dimensional flat space (in x_1, x_2, x_3 directions), that is the reason why this metric is called black three-brane. The Hawking temperature is determined completely by the behavior of the metric close to the horizon $r \to r_H$. The interesting part in this region are the terms containing G_{tt} and G_{rr}, so let us concentrate only on the (r, t) part of the metric. After Taylor expanding $f(r)$ in the neighborhood of r_H,

$$f(r) = \frac{4r_H}{R^2}(r - r_H) + \mathcal{O}\left((r - r_H)^2\right) \tag{4.2}$$

the metric takes the form

$$ds'^2 = -\frac{4r_H}{R^2}(r - r_H)dt^2 + \frac{R^2}{4r_H\,(r - r_H)}dr^2. \tag{4.3}$$

Changing the radial variable from r to ς by $r = r_H + \varsigma^2/r_H$ and Wick-rotating $t \to i\tau$, the metric has the form of the flat metric in cylindrical coordinates

$$ds'^2 = \frac{R^2}{r_H^2}(d\varsigma^2 + \frac{4r_H^2}{R^4}\varsigma^2 d\tau^2). \tag{4.4}$$

By doing so, the Euclidean time coordinate τ just gets compactified since τ plays the role of the angular coordinate in this parameterization. Identifying the periodicity mentioned above and noting that the Hawking temperature gives the value where the conical singularity ($\varsigma = 0$) disappears, the Hawking temperature reads

$$\frac{1}{\beta} = T_H = \frac{r_H}{\pi R^2}\ . \tag{4.5}$$

This so-called Hawking temperature will be identified with the temperature of the field theory living at the boundary of the spacetime [1].

Since we are still working with geometry dual to $\mathcal{N} = 4$ SYM, and the dual field theory contains only fields in the adjoint representation of the gauge group, we have to add a probe brane into the geometry in order to have particles in the fundamental representation. A D3-brane will be introduced at the boundary of AdS_5 and lying along $\vec{n}$ on S^5 where $\vec{n}$ is a unit vector in $\mathbf{R}^6$. The D-brane is situated at some large, fixed value of the radial coordinate. The reason is twofold, first the quark masses proportional to the string length have to be large since we are interested in the static potential between them, and as a second reason the contour of the Wilson loop should be in a 4-dimensional Minkowski spacetime. If the quark and antiquark can have different positions on the five-sphere, two probe D3-branes sitting at the same radial coordinate r but different positions in the transverse space (having different $\vec{n}$) need to be introduced, hence in this picture the quark and antiquark are placed on different D3-branes.

[1] To be more precise, there is another geometry competing with this geometry in the Euclidean partion sum $Z = \sum exp[-\beta H]$. The temperature is introduced by replacing the conformal boundary geometry $\mathbf{R^4}$ of AdS by $S^3 \times S^1$, where S^1 represents the periodic imaginary time. There are two different manifolds with this conformal boundary, the finite temperature version of AdS and the AdS black hole metric. The dominant one is the one with smaller Euclidean action, which at sufficient high temperature turns out to be the AdS black hole metric [16].

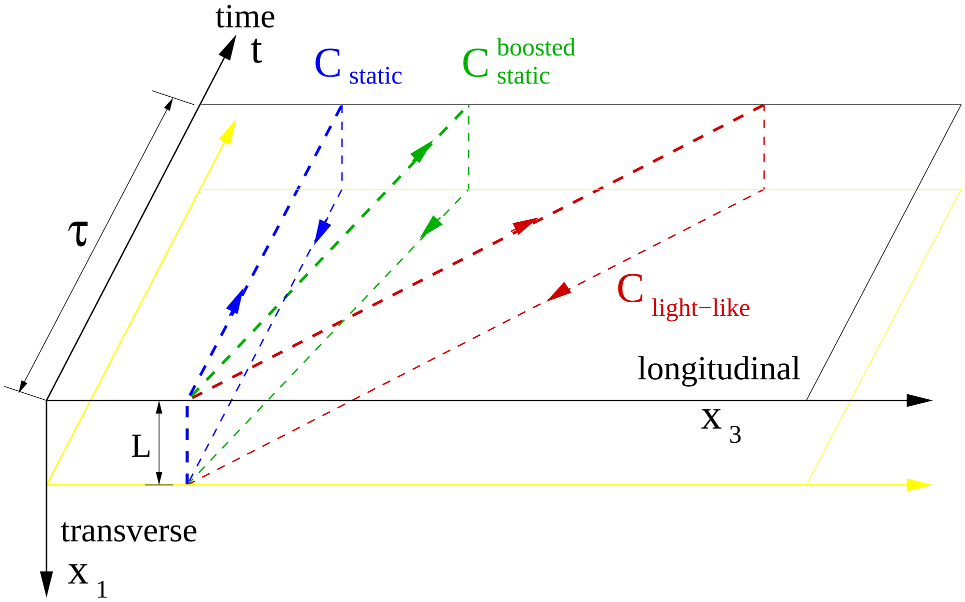


Figure 4.1.: *Schematic illustration of the shape of Wilson loop at different velocities with respect to the medium.* $\mathcal{C}_{static}$ *corresponds to* $v = 0$, $\mathcal{C}_{boosted}$ *to* $0 < v < c$ *and* $\mathcal{C}_{light-like}$ *to* $v = c$.

In heavy ion collisions quark antiquark pairs are produced moving with some velocity v with respect to the medium. This causes great challenges when the conventional methods are applied for studying the physics of the moving quarks. Using the string dual description, this problem can be solved in a very natural way. It proves very convenient to work in the rest frame of the pair, so the geometry will be Lorentz-boosted. Let the $q\bar{q}$ pair with constant length extending along the x_1 direction move with constant velocity v in x_3 direction, i.e. perpendicular to its separation, we boost the system to the rest frame (t', x_3') of the $q\bar{q}$ pair using the Lorentz transformation

$$\begin{aligned} dt &= dt' \cosh\eta - dx_3' \sinh\eta, \\ dx_3 &= -dt' \sinh\eta + dx_3' \cosh\eta, \end{aligned} \tag{4.6}$$

where the rapidity is given by $\tanh\eta = v$ or $\gamma = 1/\sqrt{1-v^2} = \cosh\eta$. Then the $q\bar{q}$ pair can be seen as at rest in a moving quark-gluon plasma wind and after dropping the prime, the boosted AdS black hole metric reads

$$ds^2 = -Adt^2 - 2Bdtdx_3 + Cdx_3^2 + \frac{r^2}{R^2}(dx_1^2 + dx_2^2) + \frac{1}{f}dr^2 + R^2 d\Omega_5^2 \tag{4.7}$$

with

$$A = \frac{r^2}{R^2} - \frac{r_1^4}{r^2R^2}\,, \quad B = \frac{r_1^2 r_2^2}{r^2R^2}\,, \quad C = \frac{r^2}{R^2} + \frac{r_2^4}{r^2R^2}\,, \tag{4.8}$$

and

$$r_1^4 = r_H^4 \cosh^2 \eta = r_H^4 \gamma^2 , \quad r_2^4 = r_H^4 \sinh^2 \eta = r_H^4 \gamma^2 v^2 \ .$$

Working in this frame, the Wilson loop is static and the QGP is moving with some velocity v in the negative x_3-direction. This Wilson loop can be used to calculate the quark antiquark potential in a moving quark gluon plasma in the next section. In Fig. 4.1, one finds an illustration of the Wilson loops at different velocities [38].

4.2. Quark-Antiquark System at finite Temperature

The quark antiquark potential in a background atfinite temperatures has been calculated in [17, 18] using the proposal in [13, 14]. It was found that there is a maximal separation length, above which no stationary solution exists. This phenomenon is interpreted as string breaking and the critical length is called the *screening length*. Below the screening length, the turning point along the radial coordinate is a *double-valued* function of the separation distance L. Accordingly, the potential $V_{q\bar{q}}(L)$ is a double-valued function describing two classical string configurations satisfying the same boundary conditions.

In this section there will be a review of ref.[37, 38] showing how the screening length and the potential are effected when the quarks are not at rest, but moving though the QGP with some velocity v. Furthermore we will extend the problem by switching on the relative orientation θ on the five-sphere between quark and antiquark and examine how the screening length and potential depend on the additional parameter θ. At the end there is a discussion from the field theoretical point of view showing that one of the two classical string configurations mentioned above is unstable.

4.2.1. Screening Length and $q\bar{q}$-Potential at $\theta = 0$

First we need to set up the shape of the rectangular Wilson loop in the Minkowski space at the boundary. The short side L of $\mathcal{C}$ is chosen to lie along the x_1 direction and long side along the time direction t. By doing so the problem is restricted to the case where the plasma wind is blowing perpendicular to the quark antiquark pair[2].

The ansatz for the background string can be taken as

$$t = \tau , \quad x_1 = \tilde{\sigma} \in [-L/2, L/2] \, , \quad r = r(\tilde{\sigma})$$

[2]It is possible to chose the short side L lying in the (x_1, x_3)-plane by introducing an angle ϕ between the extention of L and the x_3-coordinate. Then the problem will be generalized to any direction with respect to plasma wind. The problem can be solved numerically and it was found that the wind direction has very mild influence on the screening length and potential (ϕ only causes a shift, the shape of of the curve $L_{max}(\phi)$ and $V_{q\hat{q}}(\phi)$ does not change). It was found in ref. [41] that the screening length has the minimum when $\phi = \pi/2$ and the maximum when $\phi = 0$, while $L_{max}(\pi/2)/L_{max}(0) \simeq 0.9$. For the sake of simplicity and clarity, $\phi = \pi/2$ is assumed thoughout the thesis

and the rest of the string positions remains constant in τ and $\tilde{\sigma}$. The quark and antiquark are set at $\tilde{\sigma} = -L/2$ and $\tilde{\sigma} = L/2$, respectively. The probe D3-brane at the boundary of AdS black hole is located at some radial coordinate which will be taken to infinity in order to make the quarks infinitely massive. Both, the quark and antiquark are situated on this brane, so they are sitting at the same position in the internal space S^5. Introducing following dimensionless variables

$$y \equiv r/r_H \ , \quad \ell \equiv Lr_H/R^2 = L\pi T_H \ , \tag{4.9}$$

and rescaling $\sigma = \tilde{\sigma} r_H/R^2$, the boundary conditions become

$$y\left(\pm\frac{\ell}{2}\right) = \Lambda \to \infty \ . \tag{4.10}$$

Applying the Nambu-Goto action for the string world-sheet on the background (4.7)

$$S = \frac{1}{2\pi\alpha'}\int_{-L/2}^{L/2} d\tilde{\sigma}d\tau\sqrt{A\left(\frac{r^2}{R^2}+\frac{(\partial_{\tilde{\sigma}}r)^2}{f}\right)} \ , \tag{4.11}$$

transforming all the dimensionful parameters above into dimensionless variables defined in (4.9) and after intergrating over τ, one gets

$$\begin{aligned} S &= \frac{T}{2\pi\alpha'}\int_{-\ell/2}^{\ell/2} d\sigma\sqrt{(y^4-\cosh^2\eta)(1+\frac{y'^2}{y^4-1})r_H^2} \\ &= T\ T_H\sqrt{\lambda}\int_0^{\ell/2} d\sigma\sqrt{(y^4-\cosh^2\eta)(1+\frac{y'^2}{y^4-1})} \ . \end{aligned} \tag{4.12}$$

Here, T denotes the proper time, which should be assumed to be large, $\lambda = R^4/\alpha'^2$ the 't Hooft coupling, T_H the Hawking temperature and the prime in y' the differentiation with respect to σ. At this place, it is interesting to note that the action can be real or imaginary depending on the sign of $(y^4 - \cosh^2\eta)$ inside the square root. Actually, the reality or imaginarity of the action for a given velocity will depends on where the probe brane is placed at some the radial coordinate Λ, where the boundary conditions are set. The action above has real value if $\sqrt{\Lambda} > \cosh\eta$ and the string will start from the boundary at Λ going down to some radial coordinate $y_0 \geq \sqrt{\cosh\eta}$, make a turn there and come back to the boundary. From the Lagrangian above, a conserved quantity along σ can be constructed, namely

$$\mathcal{L} - y'\frac{\mathcal{L}}{\partial y'} = \frac{y^4-\cosh^2\eta}{\sqrt{(y^4-\cosh^2\eta)(1+\frac{y'^2}{y^4-1})}} = \sqrt{y_0^4-\cosh^2\eta} \ , \tag{4.13}$$

where y_0 gives the turning point of the string world-sheet along the radial coordinate [3]. One can rewrite the above equation as

[3] y_0 plays the role of U_0, but is dimensionless. Note our earlier convention, in a metric with radial coordinate denoted by U, R is dimensionless and is defined by $R^4 \equiv R'^4 = 4\pi g_s N$, while in metric with r as radial coordinate, $R^4 = 4\pi g_s N\alpha'^2$ is dimensionful.

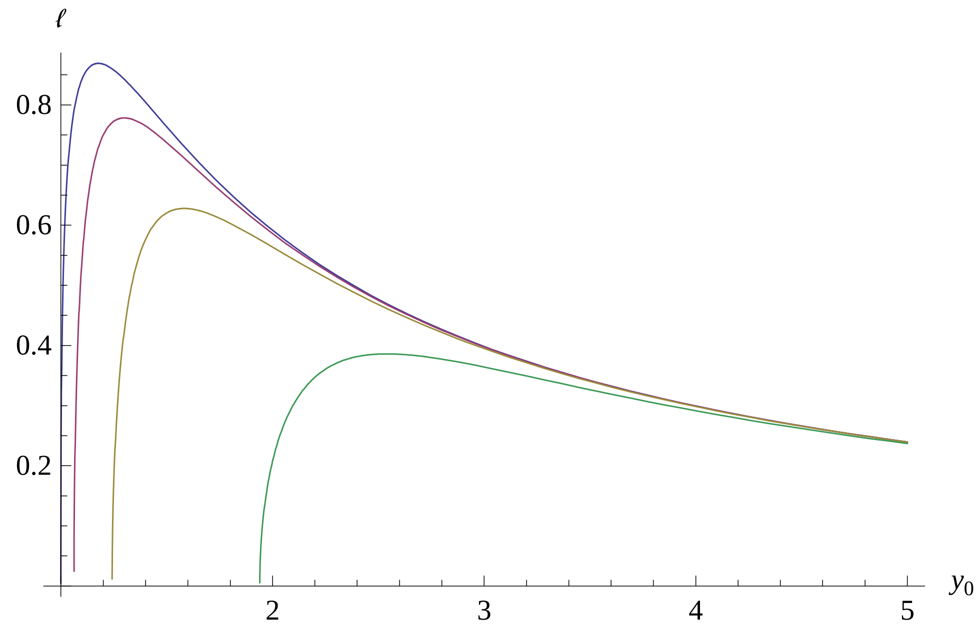


Figure 4.2.: *Separation ℓ length as a function of the turning point y_0 for $\eta = \{0, .5, 1, 2\}$ ($\gamma = \cosh\eta$) from the top to bottom, respectively.*

$$\frac{\partial y}{\partial \sigma} = y' = \frac{1}{\sqrt{y_0^4 - \gamma^2}} \sqrt{(y^4 - 1)(y^4 - y_0^4)} \ , \tag{4.14}$$

where $\cosh\eta$ has been replaced by $\gamma = 1/\sqrt{1 - v^2}$. From the boundary conditions (4.10), using $y(0) = y_0$ and $\frac{\ell}{2} = \int_0^\ell d\sigma$, we have

$$\ell = 2 \int_{y_0}^{\Lambda} \frac{\sqrt{y_0^4 - \gamma^2}}{\sqrt{(y^4 - 1)(y^4 - y_0^4)}} \ dy \ . \tag{4.15}$$

Substituting y by y_0/w and taking the limit where Λ goes to infinity, the following expression can be found for the $q\bar{q}$ separation length

$$\ell = \frac{2\sqrt{y_0^4 - \gamma^2}}{y_0} \int_0^1 \frac{w^2 dw}{\sqrt{(1 - w^4)(y_0^4 - w^4)}} \tag{4.16}$$

$$= \frac{2\sqrt{\pi}}{3} \frac{\Gamma(7/4)}{\Gamma(5/4)} \frac{\sqrt{y_0^4 - \gamma^2}}{y_0^3} \ {}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \frac{1}{y_0^4}\right) \ , \tag{4.17}$$

where ${}_2F_1(a, b, c; z)$ is the standard hypergeometric function. This equation states that for any given velocity and separation length, y_0 determined by (4.17) would describe the characteristic value of the stationary U-shape string which minimizes the string world-sheet. One can then ask the question whether it is always possible to find a stationary string shape with some y_0 satisfying the minimizing problem for given values of L and v. Let us take a look at Fig. 4.2 showing the above relation between ℓ and y_0 for various velocities.

The screening length The plot shows that a solution for y_0 exists only for values of ℓ up to a critical ℓ_{max}. From now on this maximal value ℓ_{max} will be called *screening length*, since above this value no stationary solution for the minimal world-sheet bounded by the Wilson loop $\mathcal{C}$ can be found. Later we will see this behavior again when the potential is evaluated. It is interesting to note that for every distance $\ell < \ell_{max}$ there are two solutions for y_0 satisfying the same boundary conditions. This signifies the existence of two classical string configurations, the one with larger value of y_0 is called the "short" string and the other one with smaller value of y_0 the "long" string.

In QGP the screening length gives the maximal interquark distance above which heavy mesons (like charmonium bound states) dissociate. This quantity depends strongly on the relative velocity between the quark antiquark pair and the medium. Unfortunately, an analytical expression for ℓ_{max} can not be found from (4.17). However, for the high velocity limit, which is assumed for particles produced inside the QGP at RHIC, γ is considered to be large. Due to the reality condition for ℓ, $y_0 \geq \sqrt{\gamma}$ is large, thus we can take the first derivative of ℓ with respect to y_0, and after neglecting all the terms proportional to $1/\gamma^4$, an approximated expression for ℓ_{max} depending on the velocity can be written as

$$\ell_{max}(\gamma) \approx \ell\left(3^{1/4}\sqrt{\gamma}\right) = 0.7433\frac{1}{\sqrt{\gamma}}\ {}_2F_1\left(\frac{1}{2},\frac{3}{4},\frac{5}{4},\frac{1}{3\gamma^2}\right) . \tag{4.18}$$

Numerical analysis of equation (4.17) shows that $\ell_{max} = 0.869$ for $\gamma = 1$ which does not differ much from the value of 0.837 obtained by interpolating to $\gamma = 1$ from the above approximation for large γ, and this motivates writing the screening length as [37]

$$L_{max} = \frac{g(v)}{\pi T_H}(1-v^2)^{1/4}. \tag{4.19}$$

The introduced function of velocity $g(v)$ can be plotted for velocity between [0,1] and it was found in [37] that $g(v)$ has a very mild influence[4] on L_{max} and the dominant v-dependence[5] of L_{max} is the factor $(1-v^2)^{1/4}$. At this place a first statement about the velocity-dependence of physical observables can be made, namely that the screening length decreases with increasing velocity which as an approximation can be expressed by

$$L_{max}(v) \simeq L_{max}(0)(1-v^2)^{1/4} . \tag{4.20}$$

[4]Numerical analysis shows $g(0) = 0.869$ and $g(1) = 0.7433$.

[5]Another analysis of screening length [39] shows that the L_s behavior for some region of v not close to c is actually closer to $(1-v^2)^{1/3}$ than $(1-v^2)^{1/4}$. Throughout this thesis we work with the large γ-limit.

It was argued in ref.[44] that if the discovered velocity-scaling of L_{max} holds for QCD, it will have qualitative consequences for quarkonium suppression. For example, lattice calculations of quark antiquark potential indicate that J/Ψ state dissociates at temperature around $2.1T_c$ where as excited states of $c\bar{c}$ like χ_c and Ψ' cannot survive temperatures above $1.2T_c$ [44]. If collisions at RHIC reach temperature between $1.2T_c$ and $2.1T_c$, one would expect the suppression of the excited charmonium states, but not the suppression of the J/Ψ. However, taking the equation (4.20), the temperature[6] $T_{diss} \sim 1/L_{max}$ needed to dissociate J/Ψ decreases as $(1-v^2)^{1/4}$ and that might be one of the reasons for the observed suppression of the production of charmonium states.

The Potential For evaluating the minimal surface traced out by the stationary string, relations (4.13) and (4.14) will be substituted into (4.12),

$$S = T\; T_H\sqrt{\lambda}\int_{y_0}^{\Lambda} dy \frac{y^4-\gamma^2}{\sqrt{(y^4-1)(y^4-y_0^4)}} \; . \tag{4.21}$$

This action diverges when the limit $\Lambda \to \infty$ is taken. In order to have a finite result, a subtraction for removing the "self-energies" of the quark and antiquark propagating independently along the long sides of the Wilson loop has to be carried out. The quark masses are proportional to the minimal surface of two straight string stretching along the radial coordinate from the boundary to the horizon of this background. Since we are working in the rest frame of the quark antiquark pair in a flowing medium (boosted background), we need to check whether this straight string configuration still describes the stationary solution.

In the following a stationary solution for a *single* quark in a moving plasma, let us say in x_3-direction, will be considered. The string world-sheet of the single quark stretches in a three dimensional subspace spanned by t-, r- and x_3-coordinate. This world-sheet can be parameterized by two variables τ and σ as

$$\tau = t, \qquad \sigma = r, \qquad x_3 = x_3(\sigma) \; . \tag{4.22}$$

The quark mass $m_q = S/T$ can be obtained after extremizing the Nambu-Goto action applied on the boosted AdS black hole metric (4.7)

$$S_q = \frac{T}{2\pi\alpha'}\int_{r_H}^{r_H\Lambda} dr\sqrt{\frac{A}{f} + (AC+4B^2)(\partial_r x_3)^2} \; , \tag{4.23}$$

where Λ denotes the radial position of the probe brane in the usual $y = \frac{r}{r_H}$ coordinate. Introducing the dimensionless variable $z = \frac{r_H x_3}{R^2}$, the action S_q becomes

$$S_q = \frac{T\; T_H\sqrt{\lambda}}{2}\int_1^{\Lambda} dy\sqrt{\frac{y^4-\gamma^2}{y^4-1} + (y^4-1)z'^2} \; . \tag{4.24}$$

[6] It was found from lattice QCD that the critical temperature needed to dissociate a quark antiquark pair is proportional to $1/L_{max}$ [38].

This action does not depend explicitly on z, thus the Euler-Lagrange equations of motion implies a conserved quantity

$$c_q = \left[\sqrt{\frac{y^4-\gamma^2}{y^4-1} + (y^4-1)z'^2}\right]^{-1} z'(y^4-1) \ , \tag{4.25}$$

which leads to

$$z'^2 = c_q^2 \frac{1}{(y^4-1)^2} \frac{y^4-\gamma^2}{y^4-1-c_q^2} \ . \tag{4.26}$$

We are still interested in the case where $\Lambda > \sqrt{\gamma}$ which, as discussed at the beginning of the last subsection, is necessary in order to avoid imaginary action. The string related to the single quark stretches between the horizon at $y=1$ and the boundary at $y=\Lambda$. The only possible solution for the above equation is called drag solution [46][47] which arises from the observation that the denominator and the numerator of the last factor have to change sign at the same radial coordinate. Otherwise z'^2 would be negative when the string passes some critical point on the way from the boundary at $\sqrt{\gamma}$ to the horizon at $y=1$. This condition restricts the value of c_q to be

$$c_q^2 = \gamma^2 - 1 \ . \tag{4.27}$$

Integrating z' above for $c_q^2 = \gamma^2-1$ gives the drag solution describing the shape of the open string stretching from the boundary down to the horizon

$$z(y) = \sqrt{\gamma^2-1}\left(\frac{1}{4}\log(y-1) - \frac{1}{4}\log(y+1) - \frac{\arctan(y)}{2}\right) + const \ . \tag{4.28}$$

After inserting the relation $z'(c_q)$ into (4.24), the quark mass reads

$$m_q = \frac{T_H\sqrt{\lambda}}{2}\int_1^\Lambda dy \ . \tag{4.29}$$

Subtracting quark masses $2m_q$ from the action (4.21), substituting y by $w=\frac{y_0}{y}$ and sending Λ to infinity, one obtains the interquark potential from the regularized action

$$\begin{aligned}\frac{V_{q\bar{q}}(y_0,\gamma)}{\sqrt{\lambda}T_H} &= \int_0^1 \frac{y_0}{w^2}\left(\frac{y_0^4-w^4\gamma^2}{y_0^2\sqrt{(1-w^4)(y_0^4-w^4)}}-1\right)dw-(y_0-1) \qquad (4.30)\\ &= 1-\frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2}\left[\frac{5(\gamma^2+y_0^4+1)\,{}_2F_1\left(\frac{1}{2},\frac{3}{4},\frac{5}{4},\frac{1}{y_0^4}\right)-9{}_2F_1\left(\frac{1}{2},\frac{7}{4},\frac{9}{4},\frac{1}{y_0^4}\right)}{5y_0^3}\right] \ .\end{aligned}$$

Both, the separation length ℓ from (4.17) and the potential are functions of velocity v and the turning point y_0. There is no analytical expression giving the explicit relation

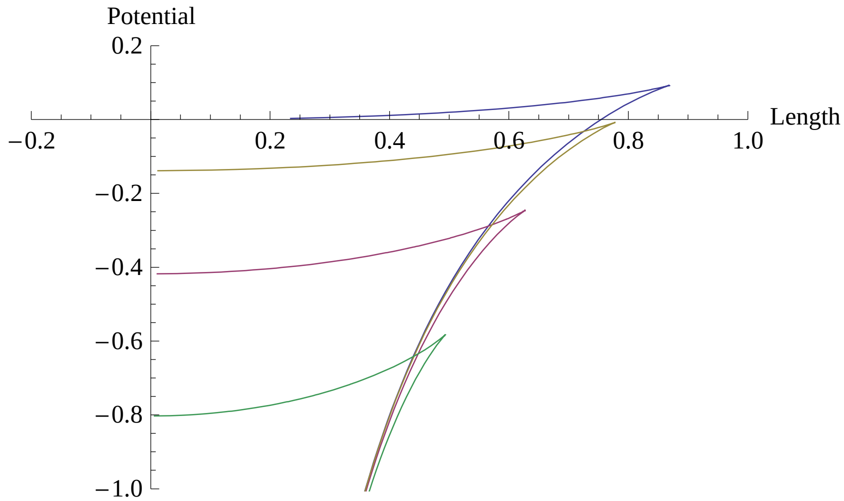


Figure 4.3.: *Plot of the potential, in units of $\sqrt{\lambda}T_H$, as a function of the separation ℓ, in units of $\frac{1}{\pi T_H}$, for $\eta = \{0, .5, 1, 2\}$ ($\gamma = \cosh\eta$) from the top to bottom, respectively.*

between ℓ and $V_{q\hat{q}}$, but it is possible to show the dependence of $V_{q\hat{q}}$ on ℓ graphically via parametric plotting using y_0 as curve parameter for certain values of γ. A plot of $V_{q\bar{q}}(L)$ for various γ is given in Fig.4.3. From the plot we see again the presence of a maximal separation distance above which only trivial solution for two disconnected strings exist. Below the critical distance there are two branches of the potential. As seen in Fig.4.2 and keeping in mind the meaning of y_0 as the furthermost radial point from the boundary, one expects that the configuration with larger value of y_0 has shorter length and so energetically more favorable configuration. Then the short string configuration with larger value of y_0 corresponds to the lower branch of the potential. That can also be conceived by the fact that the world-sheet traced out by the short string is smaller compared to the one traced out by the longer string. Later we will show in the discussion that due to concavity conditions the longer string with smaller value of y_0 represents an unstable string configuration.

Another characteristic of the quarks potential evaluated via $\mathcal{N} = 4$ dual background at finite temperature can be seen from the above figure. For small values of γ and ℓ close to ℓ_{max} the potential can cross zero which means that this configuration has higher total energy than the “straight”[7] parallel string pair. This configuration is sometimes called metastable and will be suppressed by higher velocities as seen from the above plot.

[7] Due to the drag force the shape of the single string (4.28) is not straight.

4.2.2. Screening Length and $q\bar{q}$-Potential at $\theta \neq 0$

The content of this subsection is closely related to [60]. Allowing the background string to stretch additionally in the internal space S^5 and basically taking the same ansatz for the remaining directions and using all the notations mentioned in subsection, a string world-sheet obeying the following boundary conditions will be considered:

$$y\left(\pm\frac{\ell}{2}\right) = \Lambda \to \infty \ , \quad \vartheta\left(\pm\frac{\ell}{2}\right) = \pm\frac{\theta}{2} \ , \tag{4.31}$$

where ϑ is the σ-dependent angle on the great circle connecting the S^5 position of q and $\bar{q}$. Then the Nambu-Goto action with the induced metric for a string world-sheet, approaching on the boundary of AdS black hole ($r \to \infty$) the just discussed rectangle, turns out to be (as usual using translation invariance in time for large T)

$$\begin{aligned} S(\mathcal{C}) &= \frac{1}{2\pi\alpha'}\int_{-\frac{L}{2}}^{-\frac{L}{2}} d\tilde{\sigma}d\tau\sqrt{A\left(\frac{r^2}{R^2}+\frac{1}{f}\left(\partial_{\tilde{\sigma}}r\right)^2+R^2\left(\partial_{\tilde{\sigma}}\vartheta\right)^2\right)} & (4.32)\\ &= \sqrt{\lambda}TT_H\int_0^{\frac{\ell}{2}} d\sigma\sqrt{(y^4-\gamma^2)\left(1+\frac{y'^2}{y^4-1}+\frac{\vartheta'^2}{y^2}\right)}\ , & (4.33)\end{aligned}$$

with the prime denoting the differentiation with respect to σ. Since the Lagrangian does not depend explicitly on σ and is independent of ϑ, there are two conserved quantities ϵ and j

$$\begin{aligned} \epsilon &= \sqrt{\frac{y^4-\gamma^2}{1+\frac{y'^2}{y^4-1}+\frac{\vartheta'^2}{y^2}}} &= \sqrt{\frac{y_0^2\left(y_0^4-\gamma^2\right)}{y_0^2+{\vartheta'_0}^2}}\ ,\\ j &= \epsilon\cdot\frac{\vartheta'}{y^2} &= \epsilon\frac{\vartheta'_0}{y_0}\ . \end{aligned} \tag{4.34}$$

The rightmost sides express these conserved quantities in terms of geometric characteristics of the string world-sheet: y_0 the minimal y value realized for symmetry reasons at $\sigma = 0$ where $y' = 0$, and $\vartheta'_0 = \vartheta'(0)$ giving the slope of the angle at $\sigma = 0$. Replacing $\vartheta' = \vartheta'_0 y^2/y_0^2$ in equation (4.34), we find

$$y' = \sqrt{\frac{(y^4-1)(y^2-y_0^2)(y^2y_0^4+y^2y_0^2\vartheta'^2_0+y_0^6+\gamma^2\vartheta'^2_0)}{y_0^4(y_0^4-\gamma^2)}}\ . \tag{4.35}$$

Screening length for $\theta \neq 0$ Using the above equation, the boundary conditions (4.31) and after changing the coordinate y to w as $w = y_0/y$, relations between ℓ, θ and y_0, ϑ'_0

can be found

$$\theta = 2\sqrt{1-h}\int_0^1 \frac{dw}{\sqrt{(1-kw^4/\gamma^2)(1-w^2)(1+hw^2)}}\ ,$$
$$\ell = \frac{2k^{1/4}}{\gamma^{1/2}}\sqrt{h-k}\int_0^1 \frac{w^2\ dw}{\sqrt{(1-kw^4/\gamma^2)(1-w^2)(1+hw^2)}}\ , \tag{4.36}$$

where we switched from $y_0,\ \vartheta_0'$ parameterizing the conserved quantities in (4.34) to h and k defined by

$$h = \frac{y_0^6+\gamma^2\vartheta_0'^2}{y_0^6+y_0^4\vartheta_0'^2}\ , \qquad k = \frac{\gamma^2}{y_0^4}\ . \tag{4.37}$$

Real ϵ and j in (4.34) require $y_0^4 \geq \gamma^2$, i.e. $k \in (0,1]$, which together with reality for θ and ℓ in (4.36) constrains the parameters h and k to

$$0 < k \leq h \leq 1\ . \tag{4.38}$$

Unfortunately we can not find some exact analytical expressions for these two integrals, but in the large velocity limit expanding the first factor under the square root in the integrals for large γ, one gets a representation in terms of elliptic integrals

$$\theta = 2\sqrt{1-h}\left\{K(-h)+\frac{k}{6h^2\gamma^2}\Big(2(h-1)E(-h)+(2-h)K(-h)\Big)+\mathcal{O}(\gamma^{-4})\right\}\ , \tag{4.39}$$

$$\begin{aligned}\ell = \frac{2k^{1/4}\sqrt{h-k}}{\gamma^{1/2}}\bigg\{\frac{E(-h)-K(-h)}{h} &+ \frac{k}{30h^3\gamma^2}\Big((8+h(8h-7))E(-h) \\ &+ (h(3-4h)-8)K(-h)\Big)+\mathcal{O}(\gamma^{-4})\bigg\}\ ,\end{aligned} \tag{4.40}$$

where K and E denote the standard complete elliptic integrals of the first and second kind. In leading large γ approximation the relative S^5-angle θ depends on h only and, in addition, the $\theta \leftrightarrow h$ relation is one to one with $\theta(0)=\pi$ and $\theta(1)=0$. This simplifies the further analysis considerably, and we will restrict ourselves to this level of approximation throughout this section.

For fixed θ, i.e. fixed h, the rescaled dimensionless q-$\bar{q}$ separation ℓ depends on the remaining conserved quantity k via the trivial $k^{1/4}\sqrt{h-k}$ factor taking into account the constraint (4.38). This defines a maximal value $\ell_{max}=\ell(k=h/3)$ which plays the role of a screening length, since for larger q-$\bar{q}$ separation one obviously finds no solution of the stationarity condition for the Nambu-Goto action (4.33)

$$\ell_{max} = Z(\theta)\cdot\gamma^{-1/2} + \mathcal{O}(\gamma^{-5/2})\ ,$$
$$Z(\theta) = \frac{2\sqrt{2}}{3^{3/4}}\,\frac{E(-h)-K(-h)}{(h(\theta))^{1/4}}\ . \tag{4.41}$$

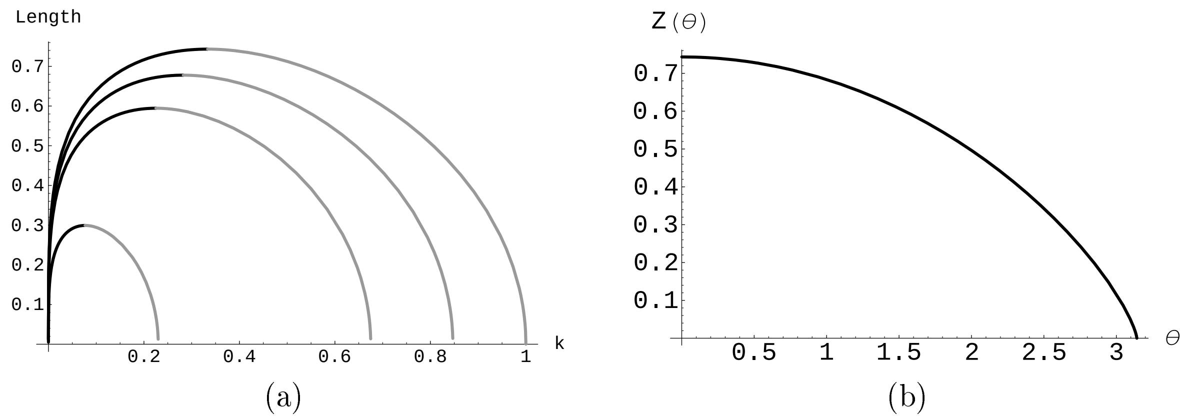


Figure 4.4.: *(a) The quark antiquark separation ℓ, in units of $(\pi T_H)^{-1}$, as a function of $k = \frac{\gamma^2}{y_0^4}$ with $\theta = \left\{0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$ from the top to bottom ,respectively. The dark lines refer to the "short strings" while the gray lines to the "long strings". (b) The prefactor Z as a function of the angle θ.*

The large velocity scaling $\ell_{max} \propto \gamma^{-1/2}$ holds for all $\theta \in [0, \pi)$. The prefactor $Z(\theta)$ monotonously decreases from $Z(0) = 0.7433$, known already from [37], to $Z(\pi) = 0$, see also Fig.4.4. For $\ell < \ell_{max}$ there are two solutions to the stationarity condition, one for $k < h/3$, the "short string" and one for $k > h/3$, the "long string".

We have seen that the screening length depends strongly on the relative S^5 orientation of quark and antiquark. This maximal separation even shrinks to zero if the quarks were situated at antipodal points on the five-sphere. To get out of this analysis something which, assuming some kind of universality, should be compared to QCD, we apply the in chapter 2 presented method to average $Z(\theta)$ over all θ with a weight $\omega(\theta)$ given by the volume per θ on Ω_5 divided by the total Ω_5 volume. Then the average

$$\overline{Z(\theta)} \;=\; \int_0^{\pi} Z(\theta)\; \omega(\theta)\; d\theta \tag{4.42}$$

using (4.39) up to the leading order can be expressed as an integral over h and evaluated numerically. The result is $\overline{Z} \;=\; 0.5797$ and gives a suppression factor $\overline{Z}/Z(0) \;=\; 0.7797$ relative to ref. [37].

The $q\bar{q}$ potential for $\theta \neq 0$ The next step is to examine the dependence of $q\bar{q}$ potential (4.33) on ℓ and θ. Using relations (4.34) and (4.35), one gets

$$V \;=\; \frac{T_H\sqrt{\lambda}}{\epsilon} \int_0^{\ell/2} (y^4 - \gamma^2)\; dx \;= T_H\sqrt{\lambda} \int_{y_0}^{\infty} \frac{(y^4 - \gamma^2)\; dy}{\sqrt{(y^4-1)(y^4-\gamma^2-\epsilon^2-j^2y^2)}}\;. \tag{4.43}$$

The integral is divergent at $y \to \infty$. We adopt the usual procedure to subtract the minimal action for a string world-sheets described by two independently straight string hanging down from the boundary to the horizon. In this case we allow quark and antiquark to have different positions on the five-sphere, so one might wonder whether the quark masses depend on this additional degree of freedom. Intuitively, one might expect an increasing quark mass due to the fact that the single string must stretch also on the S^5, and by doing so it will gain additional length.

The analysis is similar to the one carried out explicitly for the constant angle case. The only difference is that we have to consider the string moving in ϑ direction along the way from the boundary to the horizon. Taking the ansatz

$$\tau = t, \qquad \sigma = r, \qquad x_3 = x_3(\sigma), \qquad \vartheta = \vartheta(\sigma) \ , \tag{4.44}$$

the action for the single string reads

$$\begin{aligned} S_q &= \frac{T}{2\pi\alpha'}\int_{r_H}^{r_H\Lambda} dr\sqrt{\frac{A}{f} + (AC+4B^2)(\partial_r x_3)^2 + AR^2\vartheta'^2} \ , \\ &= \frac{T\ T_H\sqrt{\lambda}}{2}\int_1^{\Lambda} dy\sqrt{\frac{y^4-\gamma^2}{y^4-1} + (y^4-1)z'^2 + (\frac{y^4-\gamma^2}{y^2}\vartheta'^2)} \ , \end{aligned} \tag{4.45}$$

where the prime in the last line denotes differentiation with respect to y. The above action contains an additional term for the S^5 part and does not depend on z and ϑ. Solving the equation of motions and identifying two conserved quantities c_z and c_ϑ along z and ϑ, respectively, one finds

$$z'^2 = \frac{c_z^2}{y^4-1}\left(\frac{1}{y^4-1} + \frac{\vartheta'^2}{y^2}\right)\frac{y^4-\gamma^2}{y^4-1-c_z^2} \ , \tag{4.46}$$

$$\vartheta'^2 = \frac{c_\vartheta^2 y^4}{y^4-1}\frac{(y^4-\gamma^2)+(y^4-1)^2 z'^2}{(y^4-\gamma^2)(y^4-\gamma^2-c_\vartheta^2 y^2)} \ . \tag{4.47}$$

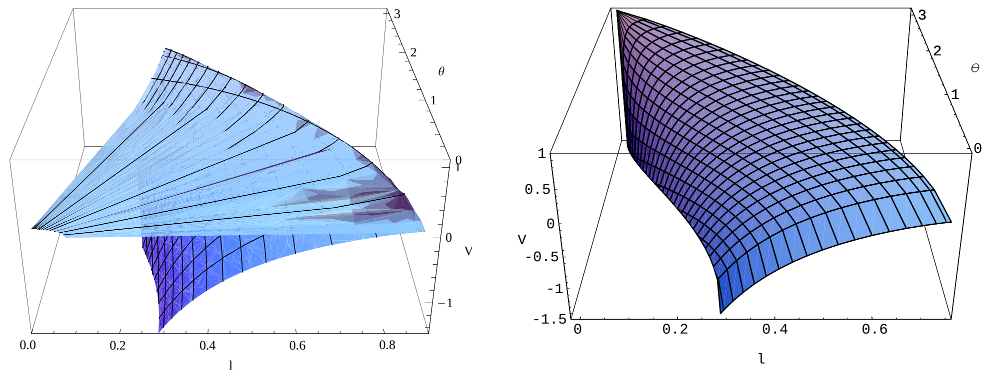
The expressions on the right hand sides of the two equations above have to be non-negative, and since we are interested in a string configuration stretching from the $\Lambda \geq \sqrt{\gamma}$ down to the horizon at $y = 1$, the only solution satisfying the stationary conditions of such a string world-sheet turns out to be

$$c_\vartheta = 0, \quad c_z^2 = \gamma^2 - 1 \ , \tag{4.48}$$

which is exactly the same solution obtained for the vanishing relative angle case[8]. Thus the quark mass does not depend on its various orientations on the five-sphere. Subtracting the quark and antiquark mass from the minimal Nambu-Goto action, expressing in addition ϵ and γ via (4.34),(4.37) by h and k and transforming the integration variable by $y = y_0/w$ we get

$$\frac{V_{q\bar{q}}}{\lambda^{1/2}T_H} = \left\{1 + \frac{\gamma^{\frac{1}{2}}}{k^{\frac{1}{4}}}\left(-1 + \int_0^1 \Big(\frac{1-kw^4}{\sqrt{(1-\frac{kw^4}{\gamma^2})(1-w^2)(1+hw^2)}} - 1\Big)\frac{dw}{w^2}\right)\right\} . \tag{4.49}$$

[8]Later, this solution will be used for studying the possible S^5-dependence of the drag force.



(a) (b)

Figure 4.5.: *(a)A numerical plot of potential V, in units of $\sqrt{\lambda}T_H$, as a function of the separation ℓ, in units of $\frac{1}{\pi T_H}$, and the angle θ for $\gamma = 1$ using (4.36) and (4.49). The upper "plane" corresponds to the long strings while the lower one to the short ones. (b)The leading order of potential V for the short string branch, in units of $\sqrt{\lambda}T_H$, as a function of the separation ℓ, in units of $\frac{1}{\pi T_H}$, and the angle θ for $\gamma = 1$.*

As above in the integrals for ℓ and θ, the expansion for large γ allows a representation in terms of elliptic integrals

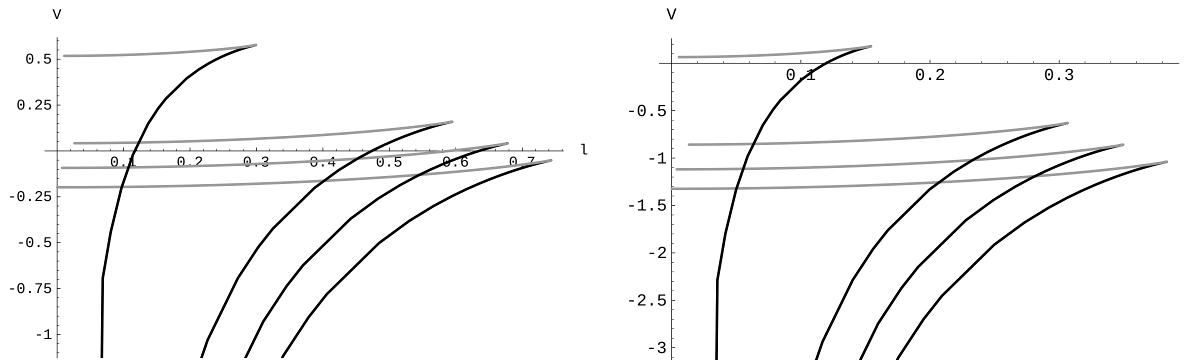
$$V_{q\bar{q}}(L,\theta,v) \;=\; T_H\lambda^{1/2}\left(1+\frac{\gamma^{1/2}(h+k)}{hk^{1/4}}\Big(K(-h)-E(-h)\Big)+\mathcal{O}(\gamma^{-3/2})\right)\;. \qquad (4.50)$$

A graphical representation of the potential as a function of the q-$\bar{q}$ separation ℓ for various values of the velocity v and the S^5 angle θ can be generated by fixing $h(\theta)$ and using (4.40) and (4.50) for parametric plots with k as parameter.

4.2.3. Summary and Discussions

We have evaluated the screening length and the potential of the quark antiquark pair produced with high transverse momentum in a QGP. As an approximation to QCD at high temperature, a boosted background dual to $\mathcal{N} = 4$ SYM has been used. Turning on the relative S^5 angle, we still found two possible string configurations satisfying the same stationary boundary conditions. The large velocity scaling law (4.19) for the screening length holds for any angle θ, but the screening length strongly depends on an additional prefactor which is a function of θ. Averaging over all possible relative S^5 orientations gives a suppression factor of 0.7797 compared to the case of the quark and antiquark having the same position on the five-sphere.

In the following we will discuss the instability of the long string configuration from the field theoretical point of view and will give an argument why the averaging method makes sense for all θ ranging between zero and π at large velocity limit.



(a) (b)

Figure 4.6.: *(a) Potential $V(\ell)$ for $\gamma = 1$ and $\theta = \left\{\frac{5\pi}{6}, \frac{\pi}{2}, \frac{\pi}{3}, 0\right\}$ from the top to bottom, respectively. The dark lines refer to the "short strings" while the gray lines to the "long strings". (b) Basically the same representation in(a) but for $\gamma = cosh(2)$.*

In ref.[50] a stability analysis via small fluctuations in various directions to the quark antiquark axis have been discussed with the result that the long string configurations are unstable and hence physically irrelevant. Using the generalization of concavity conditions for the whole (L, θ)-plane given in [28], we will examine the stability of the solution (4.50). Using (4.40), (4.50) one gets

$$\left(\frac{\partial V_{q\bar{q}}}{\partial \ell}\right)_\theta = \frac{\gamma\sqrt{h-k}}{2\sqrt{k}} \, , \tag{4.51}$$

$$\left(\frac{\partial^2 V_{q\bar{q}}}{\partial \ell^2}\right)_\theta = \frac{h^2\gamma^{3/2}}{2(h-3k)k^{3/4}(K(-h)-E(-h))} \, . \tag{4.52}$$

With (4.38) and $K(-h) - E(-h) < 0$, $\forall h \in (0, 1]$ we see that, while monotony is always realized, concavity holds on the short string branch ($k < h/3$) only. This perfectly fits with the stability analysis on the string side, where it has been shown that the short (long) string branch is stable (unstable) with respect to small fluctuations. We did not analytically check the extended concavity in the (L, θ)-plane, but the graph in Fig.4.5(b) suggests its validity.

As seen from figure above, for large enough θ, part of the short string branch reaches positive values for the potential. Due to our renormalization prescription this implies a metastable situation [50]. In spite of the stability with respect to small fluctuations, the configuration of two separated world-sheets located at fixed σ and stretching along y from the horizon to infinity would be favored.

At first sight this metastability in the neighborhood of the screening length could obstruct our averaging proposal for the screening length advocated above. However,

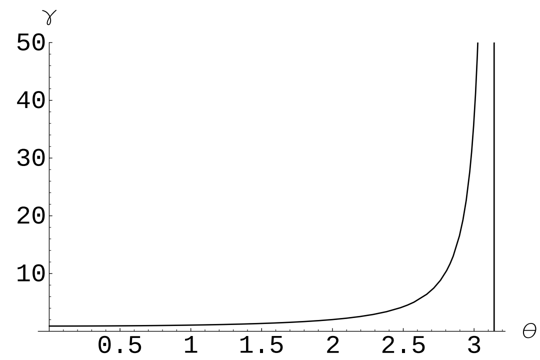


Figure 4.7.: *Stability analysis represented on the γ-θ-plane, the short string configuration is stable (metastable) in the regions above (below) the curve.*

another look at Fig.4.6(b) indicates a weakening of this effect for increasing γ such that the critical value for θ is driven to π for $\gamma \to \infty$. That can be seen at Fig.4.7. where the curve describes the zero value of $V(\ell_{max})$ in the (γ, θ)-plane. Since after all our discussion of the screening length concerns its leading behavior for large γ, no objection to an averaging over the whole S^5 remains.

4.3. Comment on Drag Force

In this section we make a comment about the S^5-dependence of the drag force [46][47] acting on a moving heavy quark in a thermal medium described by the AdS black hole metric. In ref.[47] it was argued that due to the infinite mass of the quark, after some initial fluctuations its velocity *should* be constant[9] relative to the plasma. The below derivation of drag force follows closely the description in [47]. Working in the restframe of the plasma, the motion of the string along the x_3 coordinate can be described by

$$x_3(r,t) = vt + \xi(r), \tag{4.53}$$

where $\xi(r)$ describes the trajectory of the dual string along the radial coordinate. Recall the metric for AdS-Black hole

$$ds^2 = -f dt^2 + \frac{r^2}{R^2}(dx_1^2 + dx_2^2 + dx_3^2) + \frac{1}{f} dr^2 + R^2 d\Omega_5^2 \ , \quad f \equiv \frac{r^2}{R^2}\left(1 - \frac{r_H^4}{r^4}\right) \ , \tag{4.54}$$

[9] This ansatz can also be understood as assuming the constant velocity of the quark, then it is of interest to find out how much force is needed to maintain the constant motion.

and taking the usual parameterization $t = \tau$, $r = \sigma$ while all other coordinates are independent of τ and σ, the Nambu-Goto action (2.26) reads

$$S = \frac{1}{2\pi\alpha'}\int d\tau dr \sqrt{1+\frac{r^2 f}{R^2}\xi'^2 - \frac{r^2}{R^2 f}v^2} \tag{4.55}$$

$$= \frac{1}{2\pi\alpha'}\int d\tau dr \sqrt{1+\frac{n}{H}\xi'^2 - \frac{v^2}{n}} \quad ; \; H \equiv \frac{R^4}{r^4}, \quad n \equiv f\sqrt{H} \; . \tag{4.56}$$

A conserved quantity $\pi_\xi = \frac{\partial\mathcal{L}}{\partial\xi'}$ can be obtained from the above equation since the Lagrangian does not depend on ξ, and from this relation we get

$$\xi'^2 = \pi_\xi^2 \frac{H^2}{n^2}\frac{n-v^2}{n-\pi_\xi^2 H} \tag{4.57}$$

$\xi(r)$ describes the string shape dangling from the boundary at infinity to the horizon at r_H, so ξ'^2 has to be non-negative in this interval[10]. Since n takes the values in [0,1), $(n-v^2)$ will switch its sign at some radial position for $v>0$. To avoid a negative left hand side of the above equation, $n-\pi_\xi^2 H$ has to change its sign at the same radial position as its numerator. Hence, this condition leads to

$$\pi_\xi^2 = \frac{v^2}{1-v^2}\frac{r_H^4}{R^4} \; . \tag{4.58}$$

Plugging π_ξ into (4.57), we get

$$\xi' = v\frac{r_H^2 R^2}{r^4 - r_H^4} \tag{4.59}$$

which after integrating ξ' gives the same string shape along the radial coordinate compared to the result[11] of (4.28). This solution describes a bending string trailing out behind the quark, arcing downwards into the horizon. This string exerts a drag force on the external quark which will be determined next.

The equations of motion for the Nambu-Goto action may be expressed as

$$\frac{\partial\mathcal{P}_\mu^\tau}{\partial\tau} + \frac{\partial\mathcal{P}_\mu^\sigma}{\partial\sigma} = 0 \quad ; \qquad \mathcal{P}_\mu^\kappa \equiv \frac{\partial\mathcal{L}}{\partial(\frac{\partial X^\mu}{\partial\kappa})} \; . \tag{4.60}$$

$\mathcal{P}_\mu^\tau$ and $\mathcal{P}_\mu^\sigma$ are the current densities in the τ and σ directions of the p_μ component of the spacetime momentum. The drag force is then described by the time derivative of the momentum $\frac{dp_{x_3}}{dt}$ where

$$p_{x_3}(\tau) = \int \mathcal{P}_{x_3}^\tau(\tau,\sigma)d\sigma \; . \tag{4.61}$$

[10] For zero velocity $\xi'(r)$ vanishes giving the usual straight string hanging from the boundary down to the horizon.

[11] The same result is obtained after boosting back to the plasma restframe. In (4.28) $z = \frac{r_H x_3}{R^2}$ and $y = r/r_H$.

The momentum p_{x_3} above is computed by integrating the flux $\mathcal{P}_{x_3}$ over a constant τ-path on the string world-sheet and since the $\mathcal{P}^\sigma_{x_3}$ is parallel to any constant τ-path it does not contribute to the flux. From another point of view p_{x_3} can be seen as the conserved charge obtain by integrating the zeroth component $\mathcal{P}^\tau_{x_3}$ of the current over space.
Using (4.60) the drag force is

$$\frac{dp_{x_3}}{dt} = \int \frac{\partial \mathcal{P}^\tau_{x_3}}{\partial \tau} d\sigma = -\int \frac{\partial \mathcal{P}^\sigma_{x_3}}{\partial \sigma} d\sigma = -\mathcal{P}^\sigma_{x_3} \; . \tag{4.62}$$

We have calculated $\mathcal{P}^\sigma_{x_3} = \frac{\partial \mathcal{L}}{\partial x_3'} = \frac{\partial \mathcal{L}}{\partial \xi'} = \pi_\xi$ before in (4.59), using this result the final expression for the drag force reads

$$\frac{dp_{x_3}}{dt} = -\frac{1}{2\pi\alpha'} \frac{v}{\sqrt{1-v^2}} \frac{r_H^2}{R^2} = -\frac{\pi\sqrt{\lambda}\, T_H^2}{2} \frac{v}{\sqrt{1-v^2}}. \tag{4.63}$$

There is a world-sheet momentum $\mathcal{P}^\sigma_{x_3}$ flowing into the horizon. In order to maintain the constant motion, a force acting on the quark has to be added which depends on the velocity of the quark and the temperature of the medium[12]

We have seen that in the absence of an additional force, the quark will slow down while transferring its momentum into the medium. However, this solution might still be in agreement with the ansatz (4.53) if there were a force acting on the string endpoint and feeding momentum into the string. In ref.[46] it was argued that a constant electric field on the probe brane would provide precisely such a force so that the quark will approach an equilibrium value v at which the rate of momentum loss to the plasma is balanced by the driving force exerted by the electric field. This was proposed to be another[13] back-to-back jet solution with external forcing in which quark and antiquark move apart at constant velocity after dissociation.

At this point we want to make a comment about the possible dependence of the drag force on the S^5 position of the single quark. This situation was analyzed in the last section in order to find out whether the quark mass depends on its internal orientation, see (4.46) and (4.47). It was found that there exists no minimal surface traced out by a single string which changes its S^5 position along the radial coordinate. For any S^5 orientation of the string endpoint the shape of the string is described by (4.28) and $\vartheta'(\sigma) = 0$. Hence, $\mathcal{P}^\sigma_{x_3}$ keeps the same form for all quark's internal orientations and we conclude that the drag force does not depend on the S^5 position of the quark.

[12]For a string with both endpoints on the probe brane there is no drag force [57] acting on its endpoints, which can be interpreted as that a color singlet state does not interact with the thermal medium.
[13]The other "solution" is the trivial one where the dissociation process occurs near the boundary transverse to the plasma where the quarks can escape the medium without suffering significant loss of energy.

4.4. Jet Quenching Parameter

When a quark antiquark pair is produced inside a particle collider, the quarks will typically end up hadronizing into back-to-back jets. If the pair is produced in a QGP and gets dissociated close to the boundary of the plasma ball, the situation could be different. One of the quarks might be able to escape the plasma without significant loss of energy and form the jet as usual, while the other one has to travel through the plasma and will lose energy to the medium. This event will be observed as a single jet and the phenomenon is known as *jet quenching* telling us something about the interaction of the quarks with the QGP. The plasma is a strongly interacting medium and the string/gauge duality might be appropriate to compute physical quantities describing this process of energy loss. The result for the drag force discussed in the last section can be used to compute the *jet quenching parameter* [46] which describes the energy loss of a quark moving in the QGP, since the quantity

$$\frac{dp_{x_3}}{dt} = \frac{1}{v}\frac{dE}{dt} = \frac{dE}{dx_3} \tag{4.64}$$

can be interpreted as energy loss per distance traveled. The analysis in [46] gives the result

$$\frac{d}{dt}\langle(\vec{p}_\perp)^2\rangle = 2\pi\sqrt{\lambda}\, T_H^3 \ , \tag{4.65}$$

which is the rate of change of the mean square transverse momentum of a quark and is sometimes called the *jet quenching parameter*.

In this section, however, we are going to consider another method arising from investigating radiative loss of a light-like projectile in a strongly interacting thermal medium [44], since it is believed that the dominant energy loss is due to gluons radiation.

4.4.1. Field Theoretical Background

The authors of ref.[44] proposed a non-perturbative definition of jet quenching parameter $\hat{q}$ which can be obtained by computing the thermal expectation value of a Wilson loop in the *adjoint representation* whose contour has the form of a rectangular loop $\mathcal{C}$ with large parallel light-like edges[14] L^- separated by a small extension L,

$$\langle W^A(\mathcal{C})\rangle \approx exp[-\frac{1}{4}\hat{q}\frac{L^-}{\sqrt{2}}L^2] \ . \tag{4.66}$$

There exists a relation[15] between the expectation value of the Wilson loop in the adjoint and fundamental representation, which in the large N is approximated as

$$\langle W^A(\mathcal{C})\rangle \approx \langle W^F(\mathcal{C})\rangle^2 \ . \tag{4.67}$$

[14] L^- is in light cone coordinates.

[15] For $SU(N)$, there is an identity for Wilson lines $TrW = trW \cdot trW^\dagger - 1$, where Tr and tr denote traces in the adjoint and fundamental representations, respectively.

According to the AdS/CFT correspondence, the calculation of the expectation value of the Wilson loop in the fundamental representation is equivalent to the probem of finding minimal surface presented before. Until now, the expectation of the Wilson loop is evaluated as the exponential $exp(iS)$ of the Nambu-Goto action for a string with boundary conditions corresponding to the Wilson loop $\mathcal{C}$ on the probe brane. To have the correct behavior compared to (4.66), the exponential suppression requires an imaginary action. In [38][44] this requiring is fulfilled by taking the light-like limit of the spacelike strings. Memorizing the discussion at the beginning of this chapter, the spacelike string configuration is realized if we place the probe brane at some radial coordinate smaller than $\sqrt{\gamma}$ yielding imaginary action. We will come back to more details in the next subsection. Using this method and assuming the physics of QGP can be described by $\mathcal{N}=4$ SYM, a theoretical prediction for $\hat{q}$ can be compared to the results at RHIC.

The complete argumentation why the jet quenching parameter $\hat{q}$ obtained by (4.66) should give information about the medium-dependent energy loss of a single quark moving in a strongly coupled medium is beyond the scope of this diploma thesis, more details about this proposal can be found at [38][51][52][53] and references therein. We follow the description in [38] will try to give a sketch how the Wilson loop in the adjoint representation arises in the calculation of interest.

We start from the *eikonal formalism* in which S-matrix amplitudes are determined in terms of *eikonal Wilson lines* in the target field. This formalism comes from the idea that at high energy, the propagation time through the target is short (the target gets Lorentz-contracted), then the partons propagate independently of each other and do not change their transverse potisions during the propagation. The gluons can only be produced *before* or *after* interacting with the target. An incoming hadronic projectile can be described as a superposition of the partonic states[16]

$$\Psi^{\alpha}_{\text{in}} = |\alpha(\mathbf{0})\rangle + \int d\mathbf{x}\, f(\mathbf{x})\, T^{b}_{\alpha\beta}|\beta(\mathbf{0}); b(\mathbf{x})\rangle\ . \tag{4.68}$$

In the above equation $|\alpha(\mathbf{0})\rangle$ describes a quark with color α at transverse position $\mathbf{0}$, the ket $|\beta(\mathbf{0}); b(\mathbf{x})\rangle$ describes the two-parton state, consisting of a quark with color β at transverse position $\mathbf{0}$ and a gluon of color b at transverse position $\mathbf{x}$. The Weizsäcker-Williams field $f(\mathbf{x}) \propto g\frac{\mathbf{x}}{\mathbf{x}^2}$ with g describing the strong coupling constant is built up by the coherent state of quasi-real gluons. Subscripts in greek letters denote the fundamental while latin ones the adjoint indices.

In the eikonal approximation, the process of scattering of the projectile with the target at high energy is described by adding an eikonal phase to each projectile component denoting its color rotation, then the outgoing wave can be written as

$$\Psi^{\alpha}_{\text{out}} \;=\; W^{F}_{\alpha\,\gamma}(\mathbf{0})\,|\gamma\rangle + \int d\mathbf{x}\, f(\mathbf{x})\, T^{b}_{\alpha\,\beta} W^{F}_{\beta\,\gamma}(\mathbf{0})\, W^{A}_{b\,c}(\mathbf{x})\,|\gamma\,; c(\mathbf{x})\rangle\ , \tag{4.69}$$

[16] Lorenz, spin indices and integration over the distribution of radiated gluon are suppressed in the following expression.

where $W^r(\mathbf{x}) = \mathcal{P}\exp\{i\int dz^- T^r A_r^+(\mathbf{x}, z^-)\}$ is a straight light-like Wilson line in lightcone coordinates where the lightcone gauge $A^- = 0$ was used, z gives the moving direction of the projectile, $\mathbf{A}$ is the gauge field in the target[17] and T^r are the generators of the gauge group in a r-representation corresponding to a given parton.

The interaction in the target field changes the relative phases between the components of the wave function and thus *decoheres* the initial state. As a result, the outgoing state differs from the initial one and one can interpreted the finale state as if it contains emitted gluons. The next step is to calculate the number spectrum of the produced gluons. The difference between the incoming and outgoing state is the subspace of Ψ_{out} which is orthogonal to Ψ_{in} and can be expressed by

$$
\begin{aligned}
|\delta\Psi^\alpha\rangle &= \left[1 - \sum_\gamma |\Psi_{in}^\gamma\rangle\langle\Psi_{in}^\gamma|\right] |\Psi_{out}^\alpha\rangle &\quad (4.70)\\
&= \int d\mathbf{x}\, f(\mathbf{x}) \left[T^b_{\alpha\beta} W^F_{\beta\gamma}(\mathbf{0})\, W^A_{bc}(\mathbf{x}) \;-\, T^c_{\beta\gamma} W^F_{\alpha\beta}(\mathbf{0})\right] |\gamma\,; c(\mathbf{x})\rangle \;. &\quad (4.71)
\end{aligned}
$$

The number spectrum of the *produced* gluons with momentum $\mathbf{k}$ is obtained by calculating the expectation value of the number operator in the state $|\delta\Psi_\alpha\rangle$ averaged over the incoming color index α

$$
N_{\text{prod}}(\mathbf{k}) = \frac{1}{N}\sum_{\alpha,d} \left\langle \delta\Psi^\alpha | a_d^\dagger(\mathbf{k})\, a_d(\mathbf{k}) |\, \delta\Psi^\alpha \right\rangle \;. \quad (4.72)
$$

Noting the annihilation and creation operator acting on $|\delta\Psi^\alpha\rangle$ as

$$
\begin{aligned}
a_d(\mathbf{x})|\delta\Psi^\alpha\rangle &= \int d\mathbf{x}\, f(\mathbf{x}) \left[T^b_{\alpha\beta} W^F_{\beta\gamma}(\mathbf{0})\, W^A_{bd}(\mathbf{x}) \;-\, T^d_{\beta\gamma} W^F_{\alpha\beta}(\mathbf{0})\right] |\gamma\rangle \;, &\quad (4.73)\\
\langle\delta\Psi^\alpha| a_d^\dagger(\mathbf{y}) &= \int d\mathbf{y}\, f(\mathbf{y}) \langle\gamma| \left[W^{A\dagger}_{d\bar{b}}(\mathbf{y}) \left(W^{F\dagger}(\mathbf{0}) T^{\bar{b}}\right)_{\gamma\alpha} - \left(T^d W^{F\dagger}(\mathbf{0})\right)_{\gamma\alpha}\right],
\end{aligned}
$$

and using the relations

$$
Tr\left[T^b T^c\right] = \frac{\delta^{bc}}{2} \;, \quad W^A_{bc}(\mathbf{x}) = Tr\left[T^b\, W^F(\mathbf{x})\, T^c\, W^{F\dagger}(\mathbf{x})\right] \;, \quad (4.74)
$$

then number spectrum of the produced gluons reads

$$
\begin{aligned}
N_{\text{prod}}(\mathbf{k}) = \int d\mathbf{x}\, d\mathbf{y}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} f(\mathbf{x}) f(\mathbf{y}) \frac{C_F}{N^2-1} \Big[&\left\langle \mathrm{Tr}\left[W^{A\dagger}(\mathbf{0})\, W^A(\mathbf{0})\right]\right\rangle \quad (4.75)\\
- \left\langle \mathrm{Tr}\left[W^{A\dagger}(\mathbf{x})\, W^A(\mathbf{0})\right]\right\rangle - \left\langle \mathrm{Tr}\left[W^{A\dagger}(\mathbf{y})\, W^A(\mathbf{0})\right]\right\rangle &+ \left\langle \mathrm{Tr}\left[W^{A\dagger}(\mathbf{y})\, W^A(\mathbf{x})\right]\right\rangle \Big] .
\end{aligned}
$$

[17] The gauge fields in the target can be seen as static classical sources and the field distribution is taken care by a method called *target averaging* [53]

The $\mathbf{x}$ and $\mathbf{y}$ denote the transverse positions of the *gluon* in the amplitude and complex conjugate amplitude. The only information about the target which enters in the above is that encoded in the transverse size dependence of the expectation value of two light-like adjoint Wilson lines.

We have seen that due to the color algebraic identities (4.74) all the eikonal phases in the fundamental representation have been replaced by adjoint ones. This allows us to write the gluon spectrum in terms of expectation values of products of adjoint Wilson lines. Determining the expectation values of the above expressions means averaging over the gluon field of the target. There are many procedures which describing how the *target averaging* is taken, depending on the what type of scattering process we want to consider [53]. Using the *dipole approximation* which is valid for small transverse distance $L = |\mathbf{x}-\mathbf{y}|$, and is taken in the limit where the projectile undergoes many scatterings. In this approximation the vector potentials of different scattering centers are uncorrelated in color space and the result for averaging is [38][53]

$$\frac{1}{N^2-1}\left\langle \mathrm{Tr}\left[W^{A\dagger}(\mathbf{y})\, W^A(\mathbf{x})\right]\right\rangle \longrightarrow \left\langle W^A(\mathcal{C}_{\text{light-like}})\right\rangle = \exp\left[-\hat{q}\frac{1}{4}\frac{L^-}{\sqrt{2}}\,L^2\right] + \mathcal{O}\left(\frac{1}{N^2}\right) \,, \tag{4.76}$$

where the light-cone distance $L^-/\sqrt{2} = \Delta z$ is identified with the in-medium path length and $\hat{q}$ characterizes the average transverse momentum squared transferred from the target to the projectile per unit path length. By the "arrow" it is meant that in order to obtain a gauge-invariant formulation, the two long light-like Wilson lines will be connected by two short transverse segments of length L yielding the closed rectangular loop $\mathcal{C}_{\text{light-like}}$, that is how the *adjoint light-like Wilson loop* arises in the calculation[18].

4.4.2. The Calculation in the dual Picture

This computation of Wilson loop at small ℓ-limit follows closely the calculation in [38]. Until now, in order to calculate the screening length and $q\bar{q}$-potential in large velocity limit, the probe brane radial position $\Lambda > \sqrt{\gamma}$ is taken to infinity first and the limit $\gamma \to \infty$ afterwards. For calculating the jet quenching parameter the order of taking large limit of Λ and γ changes. Since we are interested in constructing the light-like Wilson loop, the $\gamma \to \infty$ limit with large and fixed Λ is taken first, so that $\sqrt{\gamma} > \Lambda$. In this case the action becomes imaginary which yields a real quantity in the exponent of e^{iS}.

Assuming the string to extend in the $x_1 = \sigma$ and the radial direction y, introducing the boundary conditions $y \pm (\ell/2) = \Lambda$, using the result in (4.12) and noting the condition

[18] To describe the energy loss of a quark moving through a QGP, one has to go beyond the eikonal approximation, since gluons are produced within the target. However, this refined kinematical description [38][51] does not involve additional information about the medium beyond that already encoded in the jet quenching parameter $\hat{q}$ that has been introduced above.

$\sqrt{\gamma} > \Lambda$, the action becomes

$$\begin{aligned} S &= T\, T_H \sqrt{\lambda} \int_0^{\ell/2} d\sigma \sqrt{(y^4-\gamma^2)(1+\frac{y'^2}{y^4-1})} \\ &= i\, T\, T_H \sqrt{\lambda} \int_0^{\ell/2} d\sigma \sqrt{(\gamma^2-y^4)(1+\frac{y'^2}{y^4-1})}\ . \end{aligned} \tag{4.77}$$

This action does not depend explicitly on σ, thus we have a conserved quantity q

$$\begin{aligned} q &= \sqrt{(\gamma^2-y^4)\left(1+\frac{y'^2}{y^4-1}\right)^{-1}} \\ \hookrightarrow y' &= \frac{1}{q}\sqrt{(y^4-1)\,(\gamma^2-q^2-y^4)}\ . \end{aligned} \tag{4.78}$$

From this equation we see, that the string connecting $-\ell/2$ and $\ell/2$ has two possible turning points along the radial coordinate, namely at the horizon $y=1$ and at the position where $y=(\gamma^2-q^2)^{1/4}$. Since y runs from the horizon at $y=1$ to the boundary at $y=\Lambda$ and the right hand side of the last equation should be real, it is necessary to demand $(\gamma^2-q^2)\geq\Lambda^4$. By doing so[19] the latter turning point can only be realized at $y=\Lambda$ which indicates the trivial solution $y(\sigma)=\Lambda=const.$

However, this trivial solution does not fulfill the Euler-Lagrange equation derived from (4.77), so we are left with the only string configuration which starts from Λ at $\sigma=-\ell/2$, goes all the way down to the horizon at $\sigma=0$, due to symmetry makes a turn there and comes back to the boundary Λ at $\sigma=\ell/2$. Note, the string always touches the horizon for any value of ℓ. From the boundary conditions, using $\frac{\ell}{2}=\int_0^{\frac{\ell}{2}} d\sigma$ and (4.78), we have

$$\ell = 2q\int_1^{\Lambda} dy\,\frac{1}{\sqrt{(\gamma^2-q^2-y^4)(y^4-1)}}\ , \tag{4.79}$$

$$S = i\, T\, T_H\sqrt{\lambda}\int_1^{\Lambda} dy\, \frac{\gamma^2-y^4}{\sqrt{(y^4-1)(\gamma^2-q^2-y^4)}}\ . \tag{4.80}$$

In order to have analytical expressions for the equations above and according to the discussion at the beginning, the $\gamma\to\infty$-limit will be taken first followed by the $\Lambda\to\infty$-limit

$$\begin{aligned} \ell &= \frac{2q}{\gamma}\int_1^{\Lambda} dy \frac{1}{\sqrt{y^4-1}} + \mathcal{O}\left(\frac{1}{\gamma^3}\right) \\ \ell &\approx \lim_{\Lambda\to\infty}\frac{2q}{\gamma}\int_1^{\Lambda} dy\frac{1}{\sqrt{y^4-1}} = \frac{2q}{\gamma}\sqrt{\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\ . \end{aligned} \tag{4.81}$$

[19] That means the *necessary* condition for the existing of the purely imaginary extremal surface for certain ℓ requires the placing of the probe brane not to close to $\sqrt{\gamma}$, so that $\gamma^2-\Lambda^4 \geqslant q^2(\ell,\gamma)$.

$q(\gamma)$ was assumed to be small when the above limits was taken. The reason is that we are interested in evaluating the expectation value of the Wilson loop at small ℓ-limit, and that indicates small q. This comes from considering the conserved quantity q in (4.78) along σ. Let us determine at q at $\sigma = \ell/2$ indicating $y = \Lambda$

$$q = \sqrt{(\gamma^2 - \Lambda^4)\left(1 + \frac{y_\Lambda'^2}{\Lambda^4 - 1}\right)^{-1}}\,, \tag{4.82}$$

where y'_Λ denotes the slope of $y(\sigma)$ at $\sigma = \ell/2$. For $\ell \to 0$ at given values of γ and Λ, y'_Λ goes to infinity since the string goes straight down to the horizon, thus making q small.

In the following the action will be Taylor-expanded in small q-limit, i.e. small ℓ

$$S(\ell) = S^{(0)} + q^2 S^{(1)} + O(q^4)\,, \tag{4.83}$$

yielding

$$S^{(0)} = i\,T\,T_H\sqrt{\lambda}\int_1^\Lambda dy\,\sqrt{\frac{\gamma^2 - y^4}{y^4 - 1}}\,, \tag{4.84}$$

$$\begin{aligned} q^2 S^{(1)}(\ell) &= \frac{i\,T\,T_H\sqrt{\lambda}}{2} q^2 \int_1^\Lambda dy\,\frac{1}{\sqrt{(\gamma^2 - y^4)(y^4 - 1)}} \\ &\approx \frac{i\,T\,T_H\sqrt{\lambda}\,q^2}{2\gamma}\sqrt{\pi}\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = i\frac{\sqrt{\lambda}\pi^{3/2}\,\Gamma\left(\frac{3}{4}\right)T_H^3}{8\,\Gamma\left(\frac{5}{4}\right)}(T\gamma)\,L^2 \end{aligned} \tag{4.85}$$

where in the last equation (4.81) and $\ell = \pi\,L\,T_H$ have been used and the integral was carried out in leading order of γ. The equation (4.84) describes the infinite part of the action and should cancel out with the minimal surfaces traced out by two straight strings dangling from the boundary at $y = \Lambda$ to the horizon.
Recall the results in (4.24) and (4.26)

$$\begin{aligned} S_q &= \frac{T\,T_H\sqrt{\lambda}}{2}\int_1^\Lambda dy\sqrt{\frac{y^4 - \gamma^2}{y^4 - 1} + (y^4 - 1)z'^2}\,, \\ c_q &= \left[\sqrt{\frac{y^4 - \gamma^2}{y^4 - 1} + (y^4 - 1)z'^2}\right]^{-1} z'(y^4 - 1)\,, \\ z'^2 &= c_q^2\frac{1}{(y^4 - 1)^2}\frac{y^4 - \gamma^2}{y^4 - 1 - c_q^2}\,. \end{aligned}$$

We are looking for solutions with imaginary action for $\sqrt{\gamma} > \Lambda$, so inserting z' into the action and demanding the expression under the square root to be negative for all values of $y \in [1, \Lambda]$, we need to set condition for c_q so that

$$\frac{y^4 - \gamma^2}{y^4 - 1 - c_q^2} < 0\,. \tag{4.86}$$

Since $\sqrt{\gamma} > \Lambda$ the numerator is always negative, then the sufficient condition for having imaginary action is $y^4 - 1 - c_q^2 > 0$ which is trivial due to the definition of purely imaginary c_q above. [20] Note, z'^2 is well-defined (non-negative) for any value of imaginary value of c_q. The action for a single space-like string is

$$S_q = \frac{i\, T\, T_H \sqrt{\lambda}}{2} \int_1^\Lambda dy \sqrt{\frac{\gamma^2 - y^4}{y^4 - 1 - c_q^2}}\ . \tag{4.87}$$

Motivating from the physical expectation that

$$\lim_{\ell \to 0} [S(\ell) - 2S_q] = S^{(0)} - 2S_q = 0 \tag{4.88}$$

since, as argued in [38], the expression inside the square bracket of (4.75) giving the probability amplitude for the scattering process should vanish at this limit. On the mathematical side, the solution in this case would look like a string going straight from the boundary touching the horizon and coming back at almost the same σ-position, hence subtracting two times the quark mass should yield zero. From (4.84) and (4.87) the only reasonable solution for the subtracting procedure is (4.87) with $c_q = 0$.

The jet quenching parameter $\hat{q}$ in (4.76) can now be read off from (4.85). Identifying $(T\gamma) = L^-/\sqrt{2}$, where L^- is the extension in light-cone coordinates of the Wilson loop in the light-like direction, and noting the expectation value of the adjoint Wilson loop differs by a factor of 2 in the exponent S from the expectation value of the fundamental Wilson loop, the jet quenching parameter[21] reads

$$\hat{q} = \frac{\pi^{3/2}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} \sqrt{\lambda}\, T_H^3\, . \tag{4.89}$$

Once again, we see the non-trivial $\sqrt{\lambda}$-dependence which is a consequence of strong coupling. In order to compare this result with experimental data at RHIC and thinking $\alpha_{QCD} = 1/2$ being reasonable for temperature not far above the QCD phase transition, the authors of [44] have set $N = 3$ and $\alpha_{SYM} = 1/2$. From (4.89), one finds for the temperature of 300 MeV, which is an estimated value the average the temperature at RHIC, the jet quenching parameter

$$\hat{q} = 4.5\ \mathrm{GeV}^2/\mathrm{fm} \tag{4.90}$$

compared to the RHIC data with values for the time-averaged[22] $\hat{q}$ around 5-15 $\mathrm{GeV}^2/\mathrm{fm}$ [54].

[20] c_q is purely imaginary in this case, since the action is imaginary.

[21] The jet quenching parameter can be obtained directly using light-cone coordinates [44] where the extremal surfaces are finite yielding the same result.

[22] $\hat{q}$ decreases with time as the QGP expands and cools.

4.4.3. Discussions

The result for jet quenching parameter evaluated by the proposed method described above is close to the experimental estimate. Anyway, one should have in mind that the calculation was done for a $\mathcal{N} = 4$ dual background which is not a dual background to QCD. Clearly, QCD is very different from $\mathcal{N} = 4$ SYM which is a conformal, supersymmetric theory with no fundamental quarks, no running coupling and hence no confinement. Compared to QCD $\mathcal{N} = 4$ SYM has additional scalars, fermionic fields in the adjoint representation and it contains additional global symmetry. However, as argued in reference [38], these two theories might share common properties at temperature higher than T_c where supersymmetry is badly broken and in QCD, there is no confinement. In order to make a meaningful statement for QCD, the authors of [38] propose that the ratio of the QCD jet quenching parameter to that of $\mathcal{N} = 4$ SYM is given by the square root of the ratio of the number of degrees of freedom between the two theories yielding a suppression factor of 0.63 for (4.89). This suppression would enlarge the discrepancy between theoretical prediction and experimental measurement which could be interpreted as signal indicating that additional energy loss sources besides gluons radiation might be important.

The jet quenching parameter has been calculated also for non-conformal background [55], deformed background [56], background dual to $\mathcal{N} = 4$ with non-zero chemical potential [57], or background with no asymptotical AdS_5 component [59] with the main lesson that the proposed definition of jet quenching parameter [44] is gauge theory specific and therefore not universal. It was found that the jet quenching parameter increases as one goes from a confining gauge theory to a conformal theory.

The result for the jet quenching parameter (4.65) evaluated via drag force is 16.5% smaller than the one calculated from the expectation value of the light-like Wilson loop (4.89). One argument for this difference is that the describing of the energy loss of a quark moving in a QGP in these two methods arises from different perspectives, one coming from the dragging reason and the another from the gluon radiation reason.

In this diploma thesis we are interested in the possible dependence of physical observables on the internal degree of freedom (S^5). It has been found that the drag force does not depend on the S^5-orientation of the string endpoint on the probe brane, thus the jet quenching parameter obtained directly via drag force [46] is independent of various internal quark orientations. The jet quenching parameter as proposed in [38][44] is obtained from the expectation value of a light-like Wilson loop with small transverse distance in the adjoint representation. However, due to the relation (4.67) the expectation value of the Wilson loop in the *fundamental representation* is evaluated. Hence, this loop relates to the quarks and it is tempting to raise the question what happens if the long light-like edges of the loop are *assumed* to have different positions in the internal space.

Taking the boundary as usual for a string connecting two points on the probe brane and switching on the internal dependence, we are going to extremize the action

$$S = i\ T\ T_H \sqrt{\lambda} \int_0^{\ell/2} d\sigma \sqrt{(\gamma^2 - y^4)(1 + \frac{y'^2}{y^4 - 1} + \frac{\vartheta'^2}{y^2})}\ . \tag{4.91}$$

Since the action does not depend explicitly on σ and is independent of ϑ, there exist two conserved quantities

$$\mathcal{L} - \frac{\partial \mathcal{L}}{\partial y'} y' \equiv q_j\ , \quad \frac{\partial \mathcal{L}}{\partial \vartheta'} \equiv p_j \quad \text{with} \quad \frac{p_j}{q_j} = \frac{\vartheta'}{y^2}.$$

Using the relation between q_j and p_j, one finds

$$y' = \frac{1}{q_j} \sqrt{(y^4 - 1)(\gamma^2 - y^4 - q_j^2 - p_j^2 y^2)}\ . \tag{4.92}$$

Reality condition for the above expression is necessary for the extremal surface to exist and this imposes a maximal value for y given in terms of γ, q_j and p_j. This critical value gives the largest possible radial coordinate where a probe brane can be placed[23]

$$\Lambda_m = \left[\frac{1}{2}\left(\sqrt{4\gamma^2 - 4q_j^2 + p_j^4} - p_j^2\right)\right]^{1/2}\ . \tag{4.93}$$

For the light-like Wilson loop the large γ-limit at fixed Λ will be taken first. Then using the boundary condition and $\frac{\ell}{2} = \int_0^{\ell/2} d\sigma$, we get for the small ℓ-limit

$$\ell = \frac{2\pi\ \Gamma(5/4)}{\Gamma(3/4)} \frac{q_j}{\gamma} + \mathcal{O}(\frac{1}{\gamma^3})\ , \tag{4.94}$$

where the large γ-limit was taken first, followed from the large Λ-limit. The relative angle θ can be evaluated using the boundary conditions, $\frac{\theta}{2} = \int_0^{\theta/2} d\vartheta = \int_0^{\ell/2} \frac{p_j}{q_j} y^2 d\sigma$ and after transforming $y = \sqrt{w}$, we have

$$\theta = \int_1^{\Lambda^2} dw \frac{p_j \cdot \sqrt{w}}{\sqrt{(w^2 - 1)\left(w + \Lambda_m^2 + p_j^2\right)(\Lambda_m^2 - w)}}\ . \tag{4.95}$$

Unfortunately, we do not know how to solve this integral explicitly. To have an idea of what is represented by (4.95), let us look at a short numerical analysis of this integral. Noting the condition (4.93), taking large γ and then large Λ limit would mean that the Λ in the integration boundary approaches Λ_m. Numerical analysis shows that the relative angle can take values between zero and π for various combinations of Λ_m and p_j.

[23]Placing the probe brane between Λ_m and $\sqrt{\gamma}$ does not give purely imaginary action.

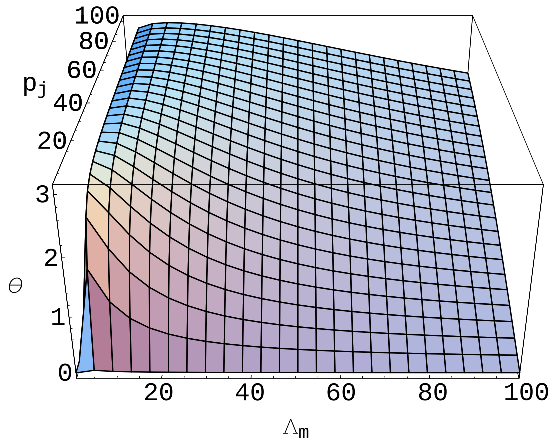


Figure 4.8.: *Numerical plot of* (4.95)

The action (4.91) in the small q_j, i.e. small ℓ limit[24], can be brought to the following form

$$
\begin{aligned}
S(\ell) &= S^{(0)} + q_j^2 S^{(1)} + O(q_j^4) \\
S^{(0)} &= i\, T\, T_H \sqrt{\lambda} \int_1^{\Lambda} dy \, \frac{\gamma^2 - y^4}{\sqrt{(y^4-1)\left(\gamma^2 - y^4 - y^2 p_j^2\right)}}\,, \qquad (4.96) \\
q_j^2 S^{(1)}(\ell) &= \frac{i\, T\, T_H \sqrt{\lambda}}{2} q_j^2 \int_1^{\Lambda} dy . \frac{\gamma^2 - y^4}{\left(\gamma^2 - y^4 - y^2 p_j^2\right)^{3/2} \sqrt{y^4-1}} \qquad (4.97)
\end{aligned}
$$

Compared that to the case of vanishing relative angle, there is an additional term depending on p_j. This remnant can be explained by the enlargement of the extremal surface for $\theta \neq 0$ since it has to stretch additionally inside the internal space.

At small ℓ and in large γ-, large Λ-limit, the leading order the conserved quantity q_j is completely determined by ℓ (4.94), then the jet quenching parameter $\hat{q}$ can be extracted from the coefficient of $\ell^2 \sim q_j^2$.

At this place I encounter many difficulties, since I do not know how to solve the above integrals explicitly. The large γ-limit cannot be taken properly, since this would change the original geometry and hence p_j. From Fig.4.8. one recognizes that angle θ close to π demands $p_j \gg \Lambda_m$, hence from (4.93) that would mean $p_j^4 \gg 4\gamma^2 - 4q_j^2$. When the angle θ is close to zero, it just describes the opposite case. Because I have not found any sufficient approximation to the solution of the problem, the answer to the question, whether the jet quenching parameter obtained via light-like Wilson loop in the adjoint

[24] Compared to q in (4.78), (4.82), the denominator of q_j gets an additional positive term arising from the slope of the angle, thus the small-ℓ small q_j-relation still holds.

representation at small distance ℓ depends on the relative S^5-angle, will be left for future work.

5. Summary and Outlook

Summary We have applied the method proposed in [13][14] to calculate the quark antiquark potential using the dual description. The quarks and antiquarks in this description are represented by string endpoints on the probe brane, thus a meson can be seen as a string with its both endpoints on this brane. It is known from field theory that the static quark antiquark potential can be extracted from the expectation value of a Wilson loop, whose shape has the form of a rectangular with the sides along the time direction much larger than the spatial sides representing the distance between the quarks [12]. Using the AdS/CFT correspondence, the expectation value of a Wilson loop in the fundamental representation of $\mathcal{N} = 4$ SYM can be evaluated by solving the problem of finding minimal surface in the dual background. Since the Wilson loop is a non-local operator, the field/operator correspondence (2.13), which is only valid for gauge invariant local operator, has been extended. The proposal basically said that the minimal surface described by the extremal Nambu-Goto action has to end on the Wilson loop lying on the probe brane at the boundary. The extremal surface is infinite since the integration is taken up to the boundary at infinity. In order to have a finite expression for the potential, a subtraction procedure needs to be introduced for removing the infinite part of the action.

Motivated from the fact that the potential depends only on the separation[1] between the quarks, the infinite part, which does not depend on variables parameterizing the separation, is found to be the mass of the string with one endpoint on the brane and the other entpoint at the horizon of the geometry, which sometimes is interpreted as the quark mass. After subtracting twice this mass, the potential becomes finite.

The superstring connecting the quarks lives in ten-dimensional space-time, hence the problem can be extended by letting these endpoints having different positions in the internal space. In all the discussed backgrounds the internal space is identified with the five-sphere. The situation of a meson with a quark and an antiquark having different positions on the five-sphere is realized by introducing two probe D3 branes into the geometry which have coincident world-volume and position along the radial coordinate, but different S^5 orientations. Then the relative angle can be introduced by setting the quark and antiquark on different probe brane.

The main motivation of this diploma thesis is confronted with the problem how some physical quantities related to the quarks depend on the additional degree of freedom, the S^5-orientation of the quarks. We found:

[1] Separation between the quarks at the boundary includes also the θ-depending part for the case of non-vanishing relative angle on S^5.

(i) For the metric dual to $\mathcal{N} = 2$ SYM [24], the confinement behavior does not depend on the relative angle θ between the quark and antiquark, which is expressed by the same force strength for large quarks separation L in four-dimensional Minkowski space. However, the meaning of large separation L depends strongly on θ, see discussion in 3.3.

(ii) The screening length ℓ_{max} of heavy meson produced with some velocity $\gamma = 1/\sqrt{1-v^2}$ relative to the QGP still scales with $1/\sqrt{\gamma}$ [37], but there exists a prefactor which gives the dependence of the screening length on θ, see eq. (4.41).

(iii) The drag force of a single quark moving relative to the QGP does not depend on its position on S^5, thus the jet quenching parameter obtained via drag force is independent of θ.

(iv) Since all the results were obtained from dual backgrounds to SYM theories, in order to make contact to QCD, where the discussed internal degree of freedom is absent, we proposed a method to average the results over all possible relative S^5-orientations of the quarks [60].

Outlook We have investigated the two backgrounds whose internal space is the five-sphere. There are many other dual backgrounds, for example those given in [25][26], where the internal space is a deformed version of S^5. It would be interesting to consider the internal degree of freedom of such metrics. In contrast to the five-sphere, where the geodesics are the great circles and the relative position between the quarks can be parameterized by one single angle, it is not easy to find the geodesic line connecting two arbitrary points inside the deformed S^5. However, the deformed internal space can always be parameterized by five angles, so one could let the string be constant along some internal coordinates and investigate the dependence of the expectation value of the Wilson loop on the remaining angles of interest.

Recently, a prescription for computing planar gluon scattering amplitudes at strong coupling using the AdS/CFT correspondence was proposed in [64][65]. In the classical approximation the calculation of gluon scattering amplitudes is formally the same to the calculation of the expectation value of a Wilson loop by finding the minimal surface which ends on a sequence of *lightlike* segments at the boundary[2] which are specified by the momenta of the gluons. In [65] the configuration of the gluons is approximated as a rectangular Wilson loop with no couplings to the scalar. The computation was done only for the AdS_5-part of the AdS metric, so it raises the question whether it is physical to turn on the S^5-part of the metric.

[2] In this case the boundary conditions are set at $r \to 0$!

A. Appendix: Regularizing the Action

The method in general

We have to deal with some expressions of the form

$$\underbrace{\int f\left(U\right)dU}_{\to\infty} - \underbrace{\int h\left(U\right)dU}_{\to\infty} \tag{A.1}$$

with the hope that the difference provides a finite value. Fortunately, all the integrals have the form

$$\int_a^\infty g\left(U\right)dU \quad with \quad g\left(U\to\infty\right)\to finite\ . \tag{A.2}$$

and after a transformation of parameters as $U=\frac{1}{x}$, the integral takes the form

$$\int_a^\infty g\left(U\right)dU = \int_{1/a}^{\epsilon} -\frac{g(1/x)}{x^2}dx = \int_\epsilon^{1/a}\frac{\tilde{g}(x)}{x^2}dx \equiv G\left(\epsilon\right)\ , \tag{A.3}$$

with $\epsilon \equiv \lim_{U\to\infty}\frac{1}{U}$. Because of (A.2) $\tilde{g}(x\to 0)\to finite$. If a is zero, it is necessary to introduce an cutoff U_{IR} which can take values arbitrarily close to zero. The term $G\left(\epsilon\right)$ can be seen as

$$\begin{aligned} G\left(\epsilon\right) &= \int_\epsilon^{1/a}\left(\frac{\tilde{g}(x)-\tilde{g}(0)-x\tilde{g}'(0)}{x^2}+\frac{\tilde{g}(0)+x\tilde{g}'(0)}{x^2}\right)dx \\ &= \int_\epsilon^{1/a}\frac{\tilde{g}(x)-\tilde{g}(0)-x\tilde{g}'(0)}{x^2}dx - \tilde{g}(0)\cdot\frac{1}{x}\Big|_\epsilon^{1/a} + \tilde{g}'(0)\log x\Big|_\epsilon^{1/a}\ , \end{aligned} \tag{A.4}$$

where the prime denotes partial differentiation on x. Giving explicit expressions for $f(U)$ and $h(U)$, we can find $\tilde{f}(x)$ and $\tilde{h}(x)$ like above and then determine $F(\epsilon)$ and $H(\epsilon)$ in the form of (A.4). In some cases the difference $F(\epsilon)-H(\epsilon)$ might provide finite result.

$\mathcal{N}=2$ background: The case of constant angle

From (3.41) and (3.42) we have

$$\begin{aligned} V_{q\bar{q}}T &= S-2m_q \\ &= \frac{T}{\pi}\int_{U_0}^{U_{max}}\left(\frac{R^2U^2}{\sqrt{U^4-U_0^4}}+\frac{A}{R^2U^2\sqrt{U^4-U_0^4}}\right)dU - \frac{T}{\pi}\int_0^{U_{max}}\sqrt{R^4+\frac{A}{U^4}}\,dU \end{aligned} \tag{A.5}$$

The second part in the first integral is finite and after changing of variables as $U = U_0 y$, one finds

$$\int_{U_0}^{U_{max}} \frac{A}{R^2 U^2 \sqrt{U^4 - U_0^4}} dU = \frac{A}{R^2 U_0^3} \int_1^{U_{max}/U_0} \frac{dy}{\sqrt{y^4\left(y^4-1\right)}} = \frac{A}{R^2 U_0^3} \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)} \ . \tag{A.6}$$

Identifying $\int f(U)dU$ with the remaining part of the first integral and $\int h(U)dU$ with the second one, we have

$$f(x) = \frac{R^2}{\sqrt{1-U_0^4 x^4}} \ , \qquad h(x) = \sqrt{R^4 + A x^4} \ . \tag{A.7}$$

Since $h(U)$ is integrated over the interval $[0,U_{max}]$ and will diverge as U gets closed to zero, a cutoff at U_{IR} is introduced. Using (A.4) and since $f'(0) = h'(0) = 0$, $f(0) = h(0) = R^2$ the difference $F(\epsilon) - H(\epsilon)$ reads

$$F - H = \int_\epsilon^{\frac{1}{U_0}} \frac{R^2}{x^2}\left(\frac{1}{\sqrt{1-U_0^4x^4}} - 1\right) dx - \int_\epsilon^{\frac{1}{U_{IR}}} \frac{\sqrt{R^4+Ax^4}-R^2}{x^2}\, dx - R^2\left(U_0 - U_{IR}\right) \ . \tag{A.8}$$

We are only interested in the behavior of the quark antiquark potential depending on L and will consider such term like

$$-\int_\epsilon^{\frac{1}{U_{IR}}} \frac{\sqrt{R^4+Ax^4}-R^2}{x^2}\, dx + R^2 U_{IR} \tag{A.9}$$

as a constant value, which shifts the curve $V_{q\bar{q}}(L)$ along the energy axis. Thus, we will not consider this term in further calculations. Setting $t = U_0 x$ and taking the limes $\epsilon \longrightarrow 0$, the rest of (A.8) becomes [1]

$$\begin{aligned}
\lim_{\epsilon\to 0}\left(F(\epsilon) - H(\epsilon)\right) &= R^2 U_0 \left[\int_0^1 \frac{1}{t^2}\left(\frac{1}{\sqrt{1-t^4}} - 1\right) dt - 1\right] \\
&= R^2 U_0 \left[F\left(\frac{\pi}{2}, \sqrt{-1}\right) - E\left(\frac{\pi}{2}, \sqrt{-1}\right) + \frac{1-\sqrt{1-t^4}}{t}\Big|_0^1 - 1\right] \\
&= -R^2 U_0 \frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)} \ .
\end{aligned} \tag{A.10}$$

With (A.6) and (A.10)we have found a finite result for the potential

$$V_{q\bar{q}} = \frac{1}{\pi}\frac{\sqrt{\pi}\,\Gamma(3/4)}{\Gamma(1/4)}\left(-R^2 U_0 + \frac{A}{R^2 U_0^3}\right) \ . \tag{A.11}$$

[1] $F(\frac{\pi}{2}, m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m^2 sin^2\theta}} = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-m^2y^2)}}$
$E(\frac{\pi}{2}, m) = \int_0^{\frac{\pi}{2}} d\theta\sqrt{1-m^2 sin^2\theta} = \int_0^1 \frac{\left(1-m^2y^2\right)}{\sqrt{(1-y^2)}} dy$

$\mathcal{N}=2$ background: The case of non-constant angle

Recall the equation (3.64)

$$S=\frac{T}{\pi}\int_1^{U_{max}/U_0}dy\left[\frac{A}{U_0^3R^2}\frac{1}{y^2\sqrt{(y^2-1)\,(y^2+1-l^2)}}+R^2U_0\frac{y^2}{\sqrt{(y^2-1)\,(y^2+1-l^2)}}\right].$$

The first part of the action is finite, namely

$$\int_1^{U_{max}/U_0}\frac{A}{U_0^3R^2}\frac{dy}{y^2\sqrt{(y^2-1)\,(y^2+1-l^2)}}=\frac{A}{U_0^3R^2}I_1(l)\ , \tag{A.12}$$

with

$$I_1(l)=\frac{1}{(1-l^2)\sqrt{2-l^2}}\left[\left(2-l^2\right)E\left(\frac{\pi}{2},\sqrt{\frac{1-l^2}{2-l^2}}\right)-F\left(\frac{\pi}{2},\sqrt{\frac{1-l^2}{2-l^2}}\right)\right]$$

The second part of the action diverges, so we are going to subtract the quark masses $2m_q$ from this term. While $h(U)$ in (A.1) keeps the same form, $f(U)$ is identified with the second part of the action which in U coordinate looks like

$$\begin{aligned}\int_1^{U_{max}/U_0}R^2U_0\frac{y^2dy}{\sqrt{(y^2-1)\,(y^2+1-l^2)}}&\overset{y=U/U_0}{\longrightarrow}\int_{U_0}^{U_{max}}\frac{R^2U^2dU}{\sqrt{(U^2-U_0^2)\,(U^2+U_0^2-l^2U_0^2)}}\\&\overset{U=1/x}{\longrightarrow}\int_\epsilon^{1/U_0}\frac{R^2}{x^2}\frac{dx}{\sqrt{(1-x^2U_0^2)\,(1+x^2\,(U_0^2-l^2U_0^2))}}\\&=\int_\epsilon^{1/U_0}\frac{dx}{x^2}f(x)\ .\end{aligned} \tag{A.13}$$

Taking the difference $F(\epsilon)-H(\epsilon)$ in the form of (A.4) with $f(0)=R^2$ and $f'(0)=0$ we have

$$\begin{aligned}F(\epsilon)-H(\epsilon)\quad&=\quad\int_\epsilon^{1/U_0}\frac{R^2}{x^2}\left(\frac{1}{\sqrt{(1-x^2U_0^2)\,(1+x^2U_0^2\,(1-l^2))}}-1\right)dx-R^2U_0\\&\overset{\epsilon\to0,\ t=xU_0}{\longrightarrow}\quad R^2U_0\left[\int_0^{1/U_0}\left(\frac{1}{t^2\sqrt{(1-t^2)\,(1+t^2\,(1-l^2))}}-\frac{1}{t^2}\right)dt-1\right]\ ,\end{aligned}$$

here we left out the "constant" term (depending on U_{IR})

$$-\int_\epsilon^{\frac{1}{U_{IR}}}\frac{\sqrt{R^4+Ax^4}-R^2}{x^2}\,dx+R^2U_{IR}\ .$$

$$\begin{aligned}\lim_{\epsilon\to 0}\left(F(\epsilon)-H(\epsilon)\right) &= R^2U_0\left[-E\left(\frac{\pi}{2}\right),\sqrt{l^2-1}+F\left(\frac{\pi}{2},\sqrt{l^2-1}\right)\right]- \\ &\quad R^2U_0\left[\left(\frac{-1}{t}|_0^1+\frac{\sqrt{(1-t^2)\left(1+t^2\left(1-l^2\right)\right)}}{t}|_0^1\right)+1\right] \\ &= R^2U_0\left[-E\left(\frac{\pi}{2},\sqrt{l^2-1}\right)+F\left(\frac{\pi}{2},\sqrt{l^2-1}\right)\right]=-R^2U_0I_3(l)\end{aligned} \tag{A.14}$$

Adding (A.12)and (A.14) together the potential becomes

$$V_{q\bar{q}}=\frac{1}{\pi}\left[\frac{A}{U_0^3R^2}I_1(l)-R^2U_0I_3(l)\right]\ . \tag{A.15}$$

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