Hopf Algebras in the AdS/CFT Correspondence

Diplomarbeit

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Chapter 1

Summary

In this thesis, we want to explore applications of Hopf algebras to integrable systems arising in the AdS/CFT correspondence. To do so, we will first provide some details of the AdS/CFT correspondence in chapter 3, introducing briefly both theories of the correspondence, namely type II.B superstring theory on $AdS_5 \times S^5$ background, and $\mathscr{N} = 4$ super Yang Mills theory. We will go on stating the correspondence, and conclude by giving some details on the BMN limit.

In chapter 4 we will give some self contained introduction to Hopf algebras, starting with some elementary facts about Lie algebras. Then we will give the definition of Hopf algebras, and finally quasitriangular Hopf algebras, with a brief comment on the general construction method of the quantum double, and the q-deformation of $sl(2)$ as an example. We will hardly make any reference to physical systems in this chapter, which is also reflected in the mathematical style.

Chapter 5 starts by motivating how spin chains arise in planar super Yang Mills theory, and by studying the $su(2)$ subsector it is shown how one can use the Bethe ansatz to calculate energies of the spin chain, which turn out to be the anomalous dimensions of the corresponding Yang Mills operators. We will then study the spin chain whose S matrix is symmetric under $su(2|2)$, because it is of major interest to the AdS/CFT correspondence. The chapter ends with some brief comparison to string theory.

In the last chapter we will give some applications of Hopf algebras to the $su(2|2)$ spin chain. Hopf algebras arise for this spin chain because some of the symmetry generators change the length of the chain, and the length changing operator is precisely what makes the Hopf algebra of this chain distinct from ordinary Lie algebra symmetry. We will also present a universal R matrix for this Hopf algebra, even though it does not reproduce the S matrix found on the fundamental representation of $su(2|2)$. We will also include a brief discussion on the Zamolodchikov-Faddeev algebra, and on the crossing relation.

Zusammenfassung

In dieser Diplomarbeit untersuchen wir Anwendungen von Hopf Algebren auf integrable Systeme, die in der AdS/CFT Korrespondenz auftauchen. Wir werden zuerst die nötigen Details der AdS/CFT Korrespondenz vorstellen, II.B Superstring Theorie und Super-Yang-Mills-Theorie. Dann werden wir werden die Korrespondenz selbst beschreiben, und das Kapitel mit dem BMN Limes abschliessen.

In Kapitel 4 werden wir dann Hopf Algebren definieren, insbesondere auch quasitriangulare Hopf Algebren. Wir werden kurz dass Quanten Doppel besprechen, mit einer Anwendung auf die q-deformierte $sl(2)$. Dieses Kapitel ist sehr mathematisch gehalten, ohne besondere physikalische Anwendungen.

In Kapitel 5 werden wir motivieren, wie Spinketten in planarer super Yang Mills Theorie auftauchen. Wir werden dies anhand des Beispiels des $su(2)$ Untersektors tun, und zeigen, wie man den Bethe Ansatz benutzen kann, um die Energien der Ketten zu berechnen, die den anomalen Dimensionen der Eichtheorieoperatoren entsprechen. Wir werden anschliessend Spinketten studieren, deren S Matrizen invariant unter $su(2|2)$ sind, da diese besonders wichtig für die AdS/CFT Korrespondenz sind. Wir werden das Kapitel mit einem Vergleich zur Stringtheorie beenden.

Im letzten Kapitel werden wir Anwendungen von Hopf Algebren auf die $su(2|2)$ Spinkette besprechen. Die Hopf Symmetrie dieser Kette kommt daher, da gewisse Symmetrieoperatoren von $su(2|2)$ die Länge der Kette ändern. Es ist genau der Längenänderungsoperator, in dem sich die Hopf Symmetrie von der normalen Lie algebra Symmetrie unterscheidet. Wir werden das Kapitel mit einer kurzen Besprechung der Zamolodchikov-Faddeev Algebra und der Crossing Symmetrie abschliessen.

Chapter 2

Introduction

String theory has long been the main candidate for the theory of quantum gravity, or even the theory of everything. However, so far it has failed to produce any observable data. Nevertheless, it has shown many interesting and surprising relations to different branches of mathematics, in a depth not achieved by any other physical theory.
Originally, string theory was introduced in the late 1960's to explain certain effects of the strong interaction. However, with the advent of quantum chromodynamics (QCD) in the early 70's string theory became less popular within the community, even though at about the same time it was proposed by Scherk and Schwarz [1] that string theory might in fact describe gravitation, and not only aspects of the strong force. An interesting relation between string and gauge theory was noted by 't Hooft [2], who showed that gauge theories have string-like behaviour if one allows the rank N of the gauge group to be an extra free parameter, and studies the large N behaviour of the gauge theory. However, the focus of the community shifted rather into the direction of regarding string theory as a theory of gravity. Indeed, in 1984 Green and Schwarz [3] showed that certain previously encountered anomalies appearing when quantising the string cancelled for supersymmetric string theory in ten dimensions. This showed that superstring theory can be regarded as a consistent theory of quantum gravity, and helped string theory to become one of the major branches of theoretical high energy physics and gravitational physics. The problem of needing ten dimensions for consistency of the theory was also turned into an advantage. It was shown that when compactifying the six not observed dimensions one can interpret this compactified space as some space of internal degrees of freedom, as needed for gauge theories. However, there is a huge number of consistent compactifications, which makes it difficult to find the correct one which should give in some limit the established standard model of particle physics. Today, studies of this so called landscape, referring to the huge number of metastable vacua or possible compactifications, is a very active research area within string theory. Another important development we should mention is M theory, which was proposed in the early 90's as an eleven dimensional theory containing membranes which unifies the five previously known consistent string theories. However, for the moment, it is far from being a completed theory.

Let us come to the field of string theory on which this thesis tries to give a humble contribution, the AdS/CFT correspondence. As we noted above already in the 70's it was known that large N gauge theories can exhibit string like behaviour, but the strings at that time were not thought of describing gravity, but only effects of the strong interaction, i.e. their scale was taken to be in the range of the QCD scale. In late 1997, Maldacena [4] proposed that type II.B string theory on an $AdS_{d+1} \times K$ background, should be dual to a conformal field theory on the boundary of AdS_{d+1}, which is d dimensional conformally flat space. K denotes a compact space. This means that on the superstring side we have a theory of gravity, which is linked to a dual field theory without gravity. The best understood case of the AdS/CFT correspondence is between strings on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang Mills theory on the boundary. Soon after Maldacenas proposal, Gubser, Klebanov and Polyakov [5] and Witten [6] gave a more precise formulation of the correspondence.

Since then, this field of string and gauge dualities has become one of the most active research areas of string theory. Even though in a certain sense string theory on $AdS_5 \times S^5$ is the next simplest thing after flat space and plane wave string theory, in the sense that the isometry algebra $psu(2,2|4)$ is as large as possible, and $\mathcal{N} = 4$ super Yang Mills theory in four dimensions is the most restrictive Yang Mills theory, being the maximally supersymmetric theory in four dimensions, with the largest possible symmetry algebra, which is not by chance also $psu(2,2|4)$, the superconformal algebra. Here, the conformal symmetry holds, unlike for QCD, also at the quantum level. Despite the large and restrictive symmetry, both theories are extremely rich, and far from being fully understood, or even solved.

One important aspect of the correspondence is that it is a strong/weak duality. Taking the rank of the gauge group N large, we get a new effective coupling constant λ [1], such that gauge theory behaves perturbatively for small λ, whereas perturbative string theory requires λ to be large. Even though this is a problem for precise tests of the correspondence, it can be turned into an advantage, since it allows to study usually inaccessible strong coupling regimes on both sides of the correspondence via the dual perturbative theory. In particular, one is interested in the strong coupling behaviour of the strong interaction, which is confining, and far from being completely understood. Hence, there is lots of activity in using the AdS/CFT correspondence[2] to describe features of strongly interacting particles such as mesons, or even heavy ions.

An open problem is, even almost ten years after its discovery, a proof of the AdS/CFT correspondence even in the best understood case of $AdS_5 \times S^5$. In fact, in the first years one had to restrict oneself to only a small class of operators for tests of the correspondence. The situation dramatically improved early this century with a series of remarkable discoveries. First of all, the plane wave limit for $AdS_5 \times S^5$ was derived and shown to be another

[1] We will define the correspondence in some more detail in chapter 3.

[2] We should mention that $\mathcal{N} = 4$ super Yang Mills theory itself is not confining, and also, unlike QCD, supersymmetric, so one either restricts to deformations of $AdS_5 \times S^5$, or tries to work with other versions of gauge/string duality, or one studies only qualitative features.

maximally supersymmetric background [7], [8]. Then it was realised by Metsaev [9] that II.B string theory on this background plus Ramond-Ramond fields is exactly quantisable in the light cone gauge, and detailed solutions were presented by Metsaev and Tseytlin in [10]. Berenstein, Maldacena and Nastase [11] went on and showed that one can directly map string states on plane waves to a certain class of gauge theory operators with large R charge in the large N limit. Minahan and Zarembo [12] made the next big step showing that one can think of those gauge theory operators, at least in the subsector of scalar fields, as some integrable spin chains, and that one can use the Bethe ansatz to compute the energies of those spin chains, which correspond to the one-loop anomalous dimensions of the gauge theory operators. Beisert and Staudacher [13] could soon thereafter show that all one-loop anomalous dimensions could be computed using the Bethe ansatz for a $psu(2,2|4)$ symmetric spin chain. Since then integrability has played an important role for the AdS/CFT correspondence, as it promises precise spectroscopic tests, or even a proof of the correspondence, at least in the large N limit. Important and interesting progress we want to mention is the investigation of the higher conserved charges, which one needs for integrability, on the string [14] and gauge side [15], [16]; the development of novel long range spin chains [17], [18] or investigations of relations to other condensed matter systems such as the Hubbard model [19], [20]. An important direction was given by Staudacher in [21] where it was argued that it is the S matrix which one should look for as the important ingredient for finding the whole spectrum of the theory. For the spin chain symmetric under the full superconformal algebra, the S matrix was found by Beisert [22] up to a prefactor, which could later be fixed [23], [24] using constraints coming from crossing symmetry [25].

Integrable systems have long been of major interest to mathematicians. Advanced Bethe ansatz techniques have been developed, especially by the Leningrad school, see e.g. [26] for a review. In the late 1980's, certain Hopf algebras with an interesting element, the universal R matrix, have been investigated [27], [28]. These Hopf algebras are called quantum groups[3]. Main classes of quantum groups one gets through certain deformations of universal enveloping algebras of Lie algebras, namely the q-deformations of affine Lie algebras, Yangians and elliptic quantum groups. On representations, their respective R matrices lead to trigonometric, rational and elliptic solutions of the Yang Baxter equation, which is automatically satisfied for the R matrix of a quantum group. Mathematically, one can use such R matrix to show that the tensor product of two representation spaces V, W is almost commutative in the sense that $V \otimes W \cong W \otimes V$, where the isomorphism should respect the algebra. This is not trivial, since, unlike for ordinary Lie algebras, a generic Hopf algebra does not act on the tensor product simply as the sum of the actions on each individual tensor product factor. The problem of having non-commutative tensor products in ultimately connected to two dimensional systems, such as spin chains and string worldsheets. Indeed, Hopf algebras have already been investigated in connection with conformal field theories in $1+1$ dimensions, see e.g. [29], [30] [31]. Strangely, even

[3]We warn the reader that the term quantum group is not consistently defined in the literature. Often, it is only used to denote certain q-deformations of universal enveloping algebras, which we will study later.

though it is now widely believed that the AdS/CFT correspondence in the planar limit becomes an integrable systems, and various methods from integrability, especially the Bethe ansatz, have successfully been applied, abstract methods from the theory of Hopf algebras have not played a major role so far. Albeit, the higher conserved charges explored in [15], [16] have been shown to be related to a Yangian, but it is probably fair to say that the Hopf algebraic aspects of Yangian symmetry have not at all been fully explored.

Another application of Hopf algebras was described in [32], [33], where it was shown that some of the symmetry generators transforming the excitations of an $su(2|3)$ symmetric spin chain [34], [22] do not act with the usual trivial coproduct[4] on tensor products, but with a slightly deformed one, where a length changing operator appears in the coproduct of those generators. One might speculate whether the S matrices derived in certain subsectors, or even for the full model, can be related to a universal R matrix of some Hopf algebra. Indeed, in [35] it was noted that the S matrix of the $su(1|1)$ sector can be related to the universal R matrix of a quantum affine algebra. Is this purely coincidental, because the Lie algebra symmetry is already highly constraining, and any possible Hopf algebra should naturally be build on the Lie algebra? In this thesis, we cannot give an answer to this question, but at least we want to speculate on a possible Hopf symmetry, which would beautifully allow to derive the full S matrix including the prefactor which would then automatically satisfy the important Yang Baxter and crossing equations. Also, the Hopf algebra would possibly shed some new light on the higher conserved charges.

[4]By a trivial coproduct we mean that the symmetry generators act on each tensor product individually, as one expects this for ordinary Lie algebras. We will explore this further in chapter 4.

Chapter 3

The AdS/CFT correspondence

The AdS/CFT correspondence [4] is one of the most fascinating discoveries in string and gauge theories of the last decade. In fact, it links two seemingly different subjects, gauge theory in four dimensional flat space, and string theory on a curved ten dimensional space, in a holographic way. Before investigating this correspondence further, we want to summarise some facts from string and gauge theory. For a general introduction to string theory, we refer the reader to [36], [37], [38], [39], [40]. Reviews of the AdS/CFT correspondence can be found in [41], [42].

3.1 The $AdS_5 \times S^5$ superstring

On the string side of the duality, we deal with type II.B superstring theory on a supersymmetric $AdS_5 \times S^5$ background. Hereby, we make use of the fact that we can write AdS_n and S^n as homogeneous spaces

$$AdS_n = \frac{SO(n-1,2)}{SO(n-1,1)}, \qquad S^n = \frac{SO(n+1)}{SO(n)}, \tag{3.1}$$

and that we have $SO(4,2) \times SO(6) \simeq SU(2,2) \times SU(4)$. We can combine those two bosonic isometry groups into the supergroup $PSU(2,2|4)$, giving the supersymmetric target space[1]

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)}. \tag{3.2}$$

Sometimes it is useful to think of AdS_5 and S^5 embedded into flat space. We have $S^n = \{\sum_{k=1}^{n+1} X_k^2 = R^2\}$ and $AdS_n = \{Y_0^2 + Y_n^2 - \sum_{k=1}^{n-1} Y_k^2 = L^2\}$, where X_k, Y_k are the standard cartesian coordinates in flat space. More precisely, whereas for S^n we have the standard $\mathbb{R}^{n+1}$ with Euclidean signature as the embedding space, for AdS_n we should choose $\mathbb{R}^{2,n-1}$ with signature $(-,+,+,\ldots,+,-)$, then the isometry groups are easily seen to be $SO(n+1)$ or $SO(2,n-1)$ respectively. One sees that AdS_n describes a hyperboloid in flat space. Both spaces are homogeneous and isotropic, and have constant curvature

[1] Which we will, for simplicity, also call $AdS_5 \times S^5$.

$R_{AdS_n} = -\frac{n(n-1)}{L^2}$, $R_{S^n} = \frac{n(n-1)}{R^2}$. For the combination of both AdS_5 and S^5 in one super coset space the radii have to be the same, that is $L = R$. The isometry group of this super coset space is $PSU(2,2|4)$.
Metsaev and Tseytlin [43] wrote down the action for the supersymmetric $AdS_5 \times S^5$ coset model, however, for simplicity we will only deal with the bosonic part, and mainly follow the convention of [44]. With the metric given in global coordinates by

$$\begin{aligned} ds^2 &= ds^2_{AdS_5} + ds^2_{S^5} \\ ds^2_{AdS_5} &= d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho d\Omega_3^2 \\ ds^2_{S^5} &= d\gamma^2 + \cos^2\gamma d\phi_3^2 + sin^2\gamma(d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2), \end{aligned} \tag{3.3}$$

we get the bosonic part of the string action

$$I = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \gamma_{ab}(G^{AdS_5}_{mn}\partial^a y^m \partial^b y^n + G^{S^5}_{mn}\partial^a x^m \partial^b x^n). \tag{3.4}$$

We have used $\gamma_{ab} = \sqrt{h}h_{ab}$, and $\sqrt{\lambda}$ in the prefactor instead of the standard string tension, because it will have a natural counterpart on the dual gauge side, namely, we will have

$$\lambda = g^2_{YM}N, \tag{3.5}$$

the usual 't Hooft coupling. In the AdS/CFT correspondence it will be related to the inverse string tension via

$$\lambda = \frac{R^4}{\alpha'^2}, \tag{3.6}$$

with the joint radius R of both AdS_5 and S^5 We choose the conformal gauge with the worldsheet metric fixed to $\gamma_{ab} = \mathrm{diag}(-1,1)$.
We can also write the action in the $(6+6)$ dimensional embedding space with coordinates X_M, $M = 1,\ldots,6$ and $Y_P, P = 0,\ldots,5$ respectively, when we include the constraints for the submanifolds via Lagrange multipliers $\Lambda(\tau,\sigma), \tilde{\Lambda}(\tau,\sigma)$. The Lagrangian reads

$$\begin{aligned} L_{AdS_5} &= -\frac{1}{2}\eta^{(-1,+1,\cdots+1,-1)}_{PQ}\partial^a Y^P \partial_a Y^Q + \frac{1}{2}\tilde{\Lambda}(\eta^{PQ}Y_P Y_Q + 1) \\ L_{S^5} &= -\frac{1}{2}\eta^{(euclid)}_{MN}\partial^a X^M \partial_a X^N + \frac{1}{2}\Lambda(\eta^{MN}X_M X_N - 1). \end{aligned} \tag{3.7}$$

We have $\eta^{(euclid)}_{MN} = \delta_{MN}$, and $\eta^{(-1,+1,\cdots+1,-1)}_{PQ} \equiv \eta_{PQ} = diag(-1,+1,\cdots+1,-1)$. Note that the fields are rescaled in such a way that the dependence on the radius R appears only in the overall factor $\sqrt{\lambda}$ which we have in front of the action

$$I = \sqrt{\lambda}\int d\tau d\sigma \frac{L_{AdS_5} + L_{S^5}}{2\pi}, \tag{3.8}$$

so we work with an effective radius 1. We also want to remind the reader that we deal with closed string and $\sigma \in [0, 2\pi]$.
Working with the gauge fixed action we should not forget the Virasoro constraints

$$\dot{X}_M \dot{X}^M + X'_M X'^M + \dot{Y}_P \dot{Y}^P + Y'_P Y'^P = 0, \tag{3.9}$$

$$\dot{Y}_P Y'^P + Y_M X'^M = 0. \tag{3.10}$$

The Lagrange equations simply give the constraint for the embedding space

$$Y_P Y^P = -1 \tag{3.11}$$

$$X_M X^M = 1. \tag{3.12}$$

The conserved charges corresponding to the global symmetries $SO(4,2)$ and $SO(6)$ are given by

$$S_{PQ} = \frac{\sqrt{\lambda}}{2\pi} \int (Y_P \dot{Y}_Q - Y_Q \dot{Y}_P) d\sigma \tag{3.13}$$

$$J_{MN} = \frac{\sqrt{\lambda}}{2\pi} \int (X_M \dot{X}_N - X_N \dot{X}_M) d\sigma. \tag{3.14}$$

3.2 Super Yang Mills theory

In this section, we want to introduce the most important facts about the other side of the AdS/CFT duality, namely $\mathcal{N} = 4$ super Yang Mills (SYM) theory in four dimensions with $SU(N)$ gauge group. This is the maximally supersymmetric four dimensional Yang Mills theory [45], [46], [47]. Its Lagrangian is uniquely given by

$$L = -\frac{2}{g_{YM}^2} tr \left(\frac{1}{4} F^2 + \frac{1}{2} (D_\mu \phi_i)^2 - \frac{1}{4} [\phi_i, \phi_j] [\phi_i, \phi_j] + \frac{1}{2} \bar{\chi} \Gamma_\mu D_\mu \chi - \frac{i}{2} \bar{\chi} \Gamma_i [\phi_i, \chi] \right). \tag{3.15}$$

The field content consists of six scalars ϕ_i, one gauge field A_μ and four Majorana gluinos. The only free parameters are the coupling constant g_{YM} and the rank of the gauge group $SU(N)$. One can construct this Lagrangian via dimensional reduction from ten dimensional $\mathcal{N} = 1$ Super Yang Mills theory [45]

$$L = -\frac{1}{2g} tr (F_{mn} F^{mn} - 2i \bar{\chi} \Gamma^m D_m \chi) \tag{3.16}$$

after compactifying on a six dimensional torus. That is why we write the gluinos in one ten dimensional Majorana Weyl spinor field χ_α, with $\alpha = 1, \dots, 16$. In this theory one has one gauge field A_m, $m = 1, \dots, 10$, which splits up after the reduction to four dimensions to A_μ and ϕ_i, and the ten dimensional Lorentz algebra $so(1,9)$ splits up into the four

dimensional Lorentz algebra $so(1,3)$ and the R symmetry algebra $so(6)$ transforming the scalars. For the four dimensional Yang Mills theory the bosonic space-time symmetry is enhanced to the conformal symmetry $so(2,4)$, and using $so(6) \simeq su(4)$, $so(2,4) \simeq su(2,2)$ and taking the supersymmetries into account, we arrive at the full symmetry algebra of this theory, the superconformal algebra $psu(2,2|4)$. One notes that this global symmetry algebra is identical to the isometry algebra of $AdS_5 \times S^5$. Furthermore, we should note that all fields transform in the adjoint representation of the gauge group.
An important property of $\mathcal{N} = 4$ SYM theory is that the superconformal symmetry survives at the quantum level. There occur no ultraviolet divergences in correlation functions of the fundamental fields in the perturbative quantisation, the β function vanishes to all orders in perturbation theory [48], [49]. That implies that the fundamental fields ϕ_i, A_μ, χ have nonrenormalised mass dimensions

$$[D, \phi_i] = 1\phi_i \tag{3.17}$$

$$[D, \chi] = \frac{3}{2}\chi \tag{3.18}$$

$$[D, A_\mu] = 1A_\mu. \tag{3.19}$$

Here, D is the dilatation operator, see [50] for an extensive review. One can split it up into $D = D_0 + \delta D$, where D_0 measures the classical dimension, and δD the anomalous dimension, i.e. the quantum correction. D_0 is part of the superconformal algebra, whereas δD is an external generator commuting with the full superconformal algebra.

3.2.1 The superconformal algebra

Let us briefly describe the full superconformal algebra $psu(2,2|4)$. It consists of the spacetime rotations $\mathfrak{L}^\alpha_\beta$, $\dot{\mathfrak{L}}^{\dot\alpha}_{\dot\beta}$, $\alpha, \beta, \dot\alpha, \dot\beta = 1,2$ forming two $su(2)$'s as the Lorentz part of the conformal algebra $su(2,2) \simeq so(2,4)$, which is completed by the translations $\mathfrak{P}_{\dot\alpha\beta}$, the conformal boosts $\mathfrak{K}^\beta\dot\alpha$ and the dilatation generator D. Furthermore, one has $\mathfrak{R}^a_b$, $a, b = 1, \ldots, 4$ forming the $su(4) \simeq so(6)$ R symmetry algebra. Additionally, one has the supersymmetry generators $\mathfrak{Q}^a_\alpha, \dot{\mathfrak{Q}}_{\dot\alpha a}, \mathfrak{S}^\alpha_a, \dot{\mathfrak{S}}^{\dot\alpha a}$. For the full commutation relations we refer the reader to the appendix of [50], but the structure is clear: The odd generators form a representation of the even Lie algebras, and the indices are chosen in such a way that one can think of the even generators as acting on the appropriate indices of the odd generators.
One can extend $psu(2,2|4)$ by a one dimensional nontrivial central charge $\mathfrak{C}$ and gets $su(2,2|4)$. Additionally, one can add an external automorphism j to the algebra, which we will call the hypercharge[2]. Then one gets $u(2,2|4)$.

[2] $psl(n|n)$ is the only series of classical Lie superalgebras which allows for a nontrivial central extension and an external automorphism. We will elaborate a bit more on this fact later. We should also warn the reader that j, $\mathfrak{C}$ introduced here are not the same j,$\mathfrak{C}$ we will use later in chapter 5 as central charge or automorphisms of $psu(2|2)$, even if $psu(2|2)$ is regarded as a subalgebra of $psu(2,2|4)$.

3.2.2 Operators

We are interested in getting all local gauge invariant operators of super Yang Mills theory. They are composed of the fundamental operators ϕ_i, A_μ, χ, or rather, we should take the gauge covariant objects $F_{\mu\nu}, D_\mu$ instead of the gauge field. Let $\mathcal{X} = D^k\phi_i, D^k\chi, D^kF$, then a generic operator is a linear combination of multi trace operators

$$\mathcal{O} = tr(\mathcal{X}_1 \dots \mathcal{X}_n)tr(\mathcal{X}_{n+1} \dots \mathcal{X}_{n+m})tr(\dots)\dots \quad . \tag{3.20}$$

Such operator is required to be local, that is, all fields are taken at the same space time point x. Making use that the theory is conformal, two point functions of two quasi-primary operators[3] $\mathcal{O}_1, \mathcal{O}_2$ have only one free parameter (see [51], [52] for general informations on conformal field theory), the scaling dimension, which has to be the same for both operators, and we simply have

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y)\rangle \propto \frac{\delta_{\Delta_1,\Delta_2}}{|x-y|^{2\Delta_1}}. \tag{3.21}$$

This is why the scaling dimensions play a key role in conformal field theory.
In general, for generic operators, the scaling dimension receives quantum corrections, which can be calculated perturbatively as a series in the free parameters of the theory, i.e. in g_{YM}^2 and $\frac{1}{N}$. However, we want to write the expansion in terms of the 't Hooft coupling $\lambda = g_{YM}^2 N$, getting

$$\Delta = \Delta_0 + \sum_{k=1}\sum_{l=0}\frac{\lambda^k}{N^{2l}}\Delta_{k,l} \tag{3.22}$$

Now a fundamental task is to diagonalise the dilatation operator order by order in perturbation theory. This is generically a difficult problem. We will see later how one can use methods of integrability to dramatically simplify this in the planar limit, i.e. $N \to \infty$.

3.3 The AdS/CFT conjecture

The so far unproven AdS/CFT correspondence simply states that the two theories we discussed in this chapter, ten dimensional type II B superstring theory on an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ Super Yang Mills theory in four dimensions with $SU(N)$ gauge group, are equivalent.
In fact, the correspondence is a duality. It links weakly coupled gauge theory to strongly coupled string theory, and vice versa. On the one hand this fact makes it hard to test the conjecture, on the other hand, it opens up the possibility to investigate nonperturbative effects on the one side of the correspondence by ordinary perturbative methods on the other side.
The full conjecture is supposed to hold for all values of g_{YM} and N. But neither is $\mathcal{N} = 4$ super Yang Mills theory solved, nor is it known how to fully quantise string theory

[3] These operators scale like $\mathcal{O}(x) \to \left|\frac{\partial x'}{\partial x}\right|^{\Delta/4} \mathcal{O}(x')$ under global conformal transformations.

on a curved background, not even on this highly (in fact maximally) supersymmetric background $AdS_5 \times S^5$. This is why one usually restricts oneself to certain limits. The easiest one is as follows: On the gauge theory side, it has long been known [2] that the theory can dramatically simplify in the so called 't Hooft limit: One sends the gauge coupling g_{YM} to zero and the number of colours N to infinity, but in a controlled way such that $\lambda = g_{YM}^2 N =$ fixed. In this limit of large N, nonplanar diagrams are suppressed by powers of $\frac{1}{N}$. On the string side, due to the identification $g_{YM}^2 = g_s$, [4] the string coupling also goes to zero, hence we deal with classical string theory. The word classical is somehow misleading, it simply means that there are no string loops, but we can still have a quantum theory on the worldsheet, which is then restricted to an ordinary cylinder. A further simplifying limit is sending additionally $\lambda \to \infty$, this gives II.B supergravity on the AdS side.

A further remarkable feature of the AdS/CFT correspondence is that it is holographic. This is not merely related to the fact that one theory lives in four and the other in ten dimensions. One can even think of the four dimensional gauge theory as living on the boundary of AdS_5. This can be seen as follows: If one considers II.B string theory with N coincident D3 branes, the low energy excitations of open strings ending on the branes correspond precisely to $\mathscr{N} = 4$ SYM with $SU(N)$ gauge group. On the other hand, we have closed strings in the theory, whose low energy behaviour is described by supergravity. We can also look at the theory from a different point of view. Instead of treating the D3 branes simply as Dirichlet boundary conditions for the open strings, we consider them as massive objects which curve spacetime. In particular, a solution for the metric with N D3 branes of II.B supergravity is given by

$$ds^2 = (1 + \frac{R^4}{y^4})^{-\frac{1}{2}} \eta_{ij} dx^i dx^j + (1 + \frac{R^4}{y^4})^{\frac{1}{2}} (dy^2 + y^2 d\Omega_5^2). \tag{3.23}$$

The constants are linked via $R^4 = 4\pi g_s \alpha'^2 N$. In the near horizon $y \ll R$ the metric reduces to

$$ds^2 = R^2 (\frac{du^2 + \eta_{ij} dx^i dx^j}{u^2} + d\Omega_5^2), \quad u = \frac{R^2}{y} \tag{3.24}$$

which is precisely the $AdS_5 \times S^5$ metric.

Having stated the correspondence as an equivalence of two theories, and having motivated a bit how it was originally derived, the main question is how one can test the correspondence, i.e. which quantities on both sides one has to calculate and compare. We have seen that the symmetry groups $PSU(2,2|4)$ are the same, physical fields are given in terms of the representations of the group, so we require a matching of the representations. In particular, for each operator on the gauge side with a definite scaling dimension Δ, there should be a corresponding string state with definite energy E, with the matching

$$\Delta = E. \tag{3.25}$$

[4] Sometimes, one puts a factor 4π between both couplings.

As we mentioned above, we have a strong/weak duality, which means that the natural expansion for Δ is in terms of small λ, but for E we need large λ, making it insufficient to apply standard perturbative methods on both sides. Hence, in the first years after the discovery of the AdS/CFT duality one had to restrict oneself to comparatively simple limits, i.e. supergravity, and operators such as BPS operators on the gauge side, which are protected from quantum corrections. The situation improved dramatically with the discovery of an interesting limit, the plane wave and BMN limit. There, one has a string and gauge duality where, at least for certain parts, one can compare perturbative results on both sides.

3.3.1 Plane wave strings and BMN limit

Besides the flat space and $AdS_5 \times S^5$, or rather, its analogue superspaces, there is one other maximally supersymmetric background, the plane wave background, which is also a solution to type IIB supergravity [7]. Later, in [8] it was shown that this plane wave background can arise as the Penrose limit from supersymmetric $AdS_5 \times S^5$. In fact, the existence of such background was already shown for any classical space-time manifold by Penrose. Here we have the case that the dimension of the isometry algebra is not changed when going from $AdS_5 \times S^5$ to the plane wave, but merely contracted [7]. On the string side type II.B string theory on the plane wave Ramond-Ramond background gives a quadratic Lagrangian in the light-cone gauge [9], [10], which allows for an exact quantisation. We will see soon that the plane wave geometry can be seen as a kind of deformation of flat space. Taking certain strings with large angular momentum J on the S^5, one can compare them with simple single trace operators with J scalar fields in the trace [11]. This opened up completely new opportunities of testing and using the correspondence, see [53] for an extensive review on this subject.
Let us first describe the geometric picture. Starting with the $AdS_5 \times S^5$ metric (3.3), and a light-like geodesic, e.g. along the ϕ_3 direction, with $\rho = 0$, $\theta = 0$, we want to look at the geometry close to this trajectory. We will choose suitable light cone coordinates

$$x^+ = \frac{t+\phi_3}{2\mu} \qquad\qquad x^- = R^2\mu\frac{t-\phi_3}{2} \tag{3.26}$$

$$r = R\rho \qquad\qquad y = R\gamma, \tag{3.27}$$

where we have performed an additional rescaling with the radius R and a new parameter μ having energy dimension one. Recalling that in front of the action (3.4) we had a factor $\sqrt{\lambda} \sim R^2$ we include the R^2 in the metric and perform the limit $R \to \infty$, getting the simplified metric

$$ds^2 = -4dx^+dx^- - \mu^2(x^I)^2(dx^+)^2 + (dx^I)2. \tag{3.28}$$

Here, we introduced new transverse coordinates x^I, $\quad i = 1, \ldots, 8$, of which half come from AdS_5, the others from S^5. We also see that μ controls the correction to flat space, which we get for $\mu \to 0$. Later we want to study the light cone energy of string states in this limit. The canonical momenta corresponding to the light cone variables $x^{\pm}$ are given by

$$p_+ = -i\partial_{x^+} = -i\mu(\partial_t + \partial_{\phi_3}) \tag{3.29}$$

$$p_- = -i\partial_{x^-} = -\frac{i}{\mu R^2}(\partial_t - \partial_{\phi_3}). \tag{3.30}$$

We will often use p^+, p^-, where the raising with the metric (3.28) in light cone coordinates changes plus and minus and one picks up a factor -2. In the old coordinates the time derivative corresponds to the energy, $E = i\partial_t$, whereas the derivative with respect to the angle ϕ_3 corresponds to the angular momentum $J = -i\partial_{\phi_3}$ in S^5.
Let us go on and consider string theory on this simplified plane-wave background. We will work in the light cone gauge, setting

$$X^+ = p^+\tau, \tag{3.31}$$

where τ is the standard worldsheet time, and get

$$I = \frac{1}{2\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left(\frac{1}{2}(\partial X^I)^2 - \frac{1}{2}\mu^2(X^I)^2 + \text{fermions}\right). \tag{3.32}$$

This is simply a theory with eight massive bosons satisfying the Klein Gordon equation

$$(\partial_\tau^2 - \partial_\sigma^2 + \mu^2)X^I = 0. \tag{3.33}$$

We can do the standard mode expansion subject to the periodic boundary conditions on the world sheet cylinder, which gives

$$\begin{aligned} X^I &= x_0^I\frac{\cos(\mu\tau)}{\mu} + p_0^I\frac{\sin(\mu\tau)}{\mu} + \sum_{n\neq 0}\frac{i}{\sqrt{2\omega_n}}\left(\alpha_n^I e^{-i(\omega_n\tau - k_n\sigma)} + \tilde{\alpha}_n^I e^{-i(\omega_n\tau + k_n\sigma)}\right) \\ P^I &= \dot{X}^I \end{aligned} \tag{3.34}$$

with the dispersion relation $\omega_n = \text{sign}(n)\sqrt{k_n^2 + \mu^2}$, $k_n = \frac{n}{\alpha' p^+}$. Then the canonical quantisation of the oscillator modes reads

$$\begin{aligned} \left[\alpha_n^I, \alpha_m^J\right] &= \delta_{n,-m}\delta^{IJ} & \left[\tilde{\alpha}_n^I, \tilde{\alpha}_m^J\right] &= \delta_{n,-m}\delta^{IJ} \\ \left[p_0^I, x_0^J\right] &= -i\delta^{IJ} & \left[\alpha_n^I, \tilde{\alpha}_m^J\right] &= 0, \end{aligned} \tag{3.35}$$

hence for $n \geq 1$ the oscillators are interpreted as annihilation operators, and for $n \leq -1$ as creation operators. The Hamiltonian is given by

$$\begin{aligned} H_{lc} \equiv 2p^- &= \frac{1}{p^+}\int d\sigma((P^I)^2 + (\partial_\sigma X^I)^2 + \mu^2(X^I)^2 + \text{fermions}) \\ &= \mu\alpha_0^{\dagger I}\alpha_0^I + \sum_{n\geq 1}\sqrt{k_n^2 + \mu^2}\left(\alpha_{-n}^{\dagger I}\alpha_n^I + \tilde{\alpha}_{-n}^{\dagger I}\tilde{\alpha}_n^I\right) + \text{fermions}, \end{aligned} \tag{3.36}$$

where we defined the zero modes $\alpha_0^I = \frac{1}{\sqrt{2\pi}}(p_0^I + i\mu x_0^I)$ and its conjugates, which also appears in harmonic oscillator form in the Hamiltonian. For $\mu \to 0$ we recover the flat space results. We get the eigenvalues

$$E_{lc} = \mu N_0 + \sum_{n\geq 1}^{\infty}(N_n + \tilde{N}_n)\sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}}, \tag{3.37}$$

where $N_n, \tilde{N}_n, N_0$ are the occupation numbers of the n-th left or rightmoving mode and the zero mode, respectively. We want to skip further details and refer the reader to [10], [11] or the review [53]. We will also ignore the scale μ in what follows. Instead, we want to investigate how we can compare string states to gauge theory operators. Let us start with a light cone string ground state, i.e. $E_{lc} = 0$. It is additionally characterised by p^+. In the rescaled coordinates we had $2p^+ = \frac{E+J}{R^2}$. Furthermore, we have argued above that in the AdS/CFT correspondence we should identify the energy E with the scaling dimension Δ. We have angular momentum J in one direction on the S^5, and the single J is a $u(1) = so(2) \subset so(6)$ generator as part of the isometry algebra of S^5, which is on the gauge side identified with the R symmetry algebra, so we expect an $so(2)$ invariant operator in the scalar sector to correspond to this string state. The one to identify with the string state $E_{lc} = 0$, i.e. $\Delta = J$, is uniquely given by

$$\frac{1}{\sqrt{JN^J}} tr(Z^J), \tag{3.38}$$

with $Z = \phi^5 + i\phi^6$ written in complex language. Such operator is protected from quantum corrections.

We want to take the Penrose limit, so to keep p^+ finite for $R \to \infty$ we see that we should have $J^2 \sim N$. The complete, so called BMN limit, is given by

$$N \to \infty, \qquad J \to \infty \tag{3.39}$$

$$\frac{J^2}{N}, g_{YM} \quad \text{fixed}, \qquad g_{YM} \ll 1. \tag{3.40}$$

On the string side this limit makes no trouble, as we argued above, it corresponds to the Penrose plane wave limit, but on the gauge side we recall that $\lambda = g_{YM}^2 N$, hence $\lambda \to \infty$ in this limit, so in principle we cannot do perturbation theory. The new feature of this limit is that one can still deal with a certain class of operators, which are somehow close to those operators protected from quantum corrections. Let us see how this works. We start with the $E_{lc} = 0$ operator as given above, and insert a small number of the other fields into the trace. We want to deal with operators which are eigenvectors of $\Delta - J$, and starting with the eigenvalue $\Delta - J = 1$ we get the eight bosonic operators $\frac{1}{N^J} Tr(\phi_i Z^J)$, $\frac{1}{N^J} Tr(D_\mu Z Z^{J-1})$ and the eight fermionic operators[5] $\frac{1}{N^J} Tr(\psi_A Z^J)$. These operators are

[5]These are the eight fermions with $J = \frac{1}{2}$, hence $\Delta - J = 1$.

again protected, and have the following corresponding supergravity modes on the string side:

$$\alpha_0^{\dagger i}|0,p^+\rangle \equiv \frac{1}{\sqrt{N^J}} tr(\phi_i Z^J) \tag{3.41}$$

$$\alpha_0^{\dagger \mu}|0,p^+\rangle \equiv \frac{1}{\sqrt{N^J}} tr(D_\mu Z Z^{J-1}) \tag{3.42}$$

$$\theta_{0A}^{\dagger}|0,p^+\rangle \equiv \frac{1}{\sqrt{N^J}} tr(\psi_A Z^J) \tag{3.43}$$

Here we split up the index $I = 1,\ldots,8$ of the transverse coordinates in two parts $\mu = 1,\ldots,4$ living on AdS and $i = 1,\ldots 4$ living on S, and θ_{0A} denotes the previously dropped fermionic zero mode. If one excites more than one other supergravity mode, one has to symmetrise the corresponding insertion of fields on the gauge side, e.g. one has

$$\alpha_0^{\dagger j}\alpha_0^{\dagger i}|0,p^+\rangle \equiv \frac{1}{\sqrt{JN^J}} \sum_{l=0}^{J} tr(\phi_i Z^l \phi_j Z^{J-l}), \tag{3.44}$$

which has eigenvalue $\Delta - J = 2$. So far, the BMN limit gave nothing new, we dealt with supergravity states corresponding to some protected operators. The great virtue of this limit is that one can go on and get operators corresponding to massive string excitations like

$$\alpha_{-n}^{j}|0,p^+\rangle, \tag{3.45}$$

with the corresponding operator [11] proportional to

$$\sum_{l=1}^{J} tr(\phi_j Z^l Z^{J-l}) e^{\frac{2\pi i n l}{J}}, \tag{3.46}$$

but such operator vanishes by the cyclicity of the trace, which is not surprising, since the corresponding string state doesn't satisfy the level matching condition. Let us instead look for the corresponding operator of the string state

$$\alpha_{-n}^{j}\tilde{\alpha}_{-n}^{i}|0,p^+\rangle, \tag{3.47}$$

which turns out to have a similar form as the supergravity ones, but will not be protected anymore. One gets

$$\frac{1}{\sqrt{JN^J}} \sum_{l=0}^{J} tr(\phi_i Z^l \phi_j Z^{J-l}) e^{2\pi i n l/J}. \tag{3.48}$$

Indeed, one can calculate that the scaling dimension of this operators gives perturbatively

$$\Delta - J = 2 + \frac{g_{YM}^2 N}{J^2} n^2 + ..., \tag{3.49}$$

which is written as an expansion in the new effective coupling

$$\lambda' := \frac{g_{YM}^2 N}{J^2}. \tag{3.50}$$

In the stringy variables this coupling is given by $\lambda' = \frac{1}{(\alpha' p^+ \mu)^2}$, and one can expand the square root of the string energy (3.37) and gets the same answer as $\Delta - J$. So one can go on and one obtains a one-to-one correspondence between all type II.B string states on the plane wave background and the gauge theory operators described above in this BMN limit.

Another feature of the BMN limit we should emphasis is that it allows to go beyond the planar limit, since $\frac{J^2}{N}$ appears as a new genus counting parameter, and is finite. Now a natural question is what happens if one take instead of the large angular momentum J on S^5 a large spin S on AdS_5. This has been investigated in [5]. The next step, allowing for large momenta on both S^5 and on AdS_5, was done in [54], [55], see [44] for a review of these so called spinning string solutions, and [56] for a review which also includes the comparison to the gauge side using methods of integrability, which we partially want to introduce in chapter 5.

Chapter 4

Hopf Algebras

In this chapter we want to give a short introduction to Hopf algebras, and follow the main references [57], [58], [59], [60], [61]. We will start with some basic definitions about Lie algebras and superalgebras, and go on with the definition of Hopf algebras. For simplicity we will mainly not explicitly work with Hopf superalgebras, even though we will need them in chapter 6, because the generalisation is straightforward. The last section will be on quasitriangular Hopf algebras, which form an interesting class of Hopf algebras because they have a universal R matrix, an object which can serve as an intertwiner of modules, and which automatically satisfies the Yang Baxter equation. We will also briefly introduce the quantum double. In the case of q-deformed universal enveloping algebras, the double has an interesting additional element, which will be interpreted as a length-changing operator in chapter 6.

4.1 Lie algebras and Lie superalgebras

Let us briefly recall some well known facts about Lie algebras and Lie superalgebras which we will need later. We mainly used the standard references [62], [63], [64], [65] [66], [67]. Explicit informations about superalgebras can be found in [68], [69].

Definition 1 (Lie algebra). *A Lie algebra $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$* [1]*, equipped with a bilinear mapping $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, which satisfies*

$$[A, B] = -[B, A] \tag{4.1}$$

$$[A, [B, C]] + [B, [C, A,]] + [C, [A, B]] = 0. \tag{4.2}$$

The last equation is called the Jacobi identity.

Definition 2 (Lie superalgebra). *A Lie superalgebra $\mathfrak{g}$ is a vector space over a field $\mathbb{K}$ equipped with a bilinear mapping*[2] *$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, and a $\mathbb{Z}_2$-grading such that $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$,*

[1] We will only use $\mathbb{K} = \mathbb{R}, \mathbb{C}$

[2] We will usually just use one symbol $[,]$ for the generalised Lie bracket. When dealing exclusively with the symmetric or antisymmetric bracket we will add a subscript $\pm$.

and $[\mathfrak{g}_\mathfrak{i}, \mathfrak{g}_\mathfrak{j}] \subset \mathfrak{g}_{(i+j)mod2}$. *We will say* $A \in \mathfrak{g}$ *is even (or of degree* 0 *or bosonic) if* $A \in \mathfrak{g}_0$, *and* A *is called odd (or of degree* 1 *or fermionic) if* $A \in \mathfrak{g}_1$. *We denote the degree of* A *by* $degA$ *or* $|A|$. *The generalised Lie bracket satisfies*

$$[A, B] = (-1)^{|A||B|+1}[B, A] \tag{4.3}$$

$$(-1)^{|A||C|}\,[A\,[B, C]] + (-1)^{|A||B|}[B, [C, A,]] + (-1)^{|C||B|}[C, [A, B]] = 0 \tag{4.4}$$

It is obvious from the definition that the even part of a Lie superalgebra forms a Lie algebra. Now let us introduce some notions and results from the classification theory of simple Lie algebras and superalgebras.

Definition 3 (Basic classical Lie superalgebra). *A basic classical Lie superalgebra is a Lie superalgebra with a non-degenerate, supersymmetric, consistent invariant bilinear form.*

Supersymmetric simply means $(a, b) = (-1)^{|a||b|}(b, a)$, and invariant means $(a, [b, c]) = ([a, b], c)$. Consistent means that $(a, b) = 0$ when a is odd and b is even.
For semisimple Lie algebras, such form is given by the Killing form. For superalgebras, some subtleties can arise, and the Killing form can be zero even for simple Lie superalgebras. We will encounter such cases in chapter 5, namely, for superalgebras of type $A(n|n)$. There, the bilinear form will be provided via the supertrace of generators in the fundamental representation.

Definition 4 (Serre-Chevalley basis). [3] *Let* $\mathfrak{g}$ *be a Lie (super)algebra. Then the algebra is described in terms of its Cartan elements* $\mathfrak{h} := \{h_1, ..., h_r\}$, *a simple root system* $\Delta^0 = \{\alpha_1, ...\alpha_r\}$ *and its corresponding simple root generators* $e_i^\pm$ $(i = 1, ...r)$, *the (symmetric)*[4] *Cartan matrix* A_{ij}, *an index set* $\tau \subseteq \{1, ..., r\}$ *determining the odd generators (i.e.* $dege_i^\pm = 1$ *if* $i \in \tau$*)and the following relations:*

$$[h_i, h_j] = 0 \tag{4.5}$$
$$[h_i, e_j^\pm] = \pm A_{ij} e_j^\pm \tag{4.6}$$
$$[e_i^+, e_j^-] = \delta_{ij} h_i \tag{4.7}$$
$$(ade_i^\pm)^{N_{ij}} e_j^\pm = 0 \quad i \neq j \tag{4.8}$$

The last equation is called the Serre relation. One gets the Matrix N_{ij} *from the Cartan matrix in the following way:*

$$N_{ij} = \begin{cases} 1 & if \quad A_{ij} = A_{ii} = 0 \\ 2 & if \quad A_{ii} = 0, A_{ij} \neq 0 \\ 1 - 2\frac{A_{ij}}{A_{ii}} & if \quad A_{ii} \neq 0 \end{cases} \tag{4.9}$$

For a basic Lie superalgebra there are some mathematical subtleties, one needs supplementary conditions for odd roots of zero length, see e.g. [66]. We will ignore those subtleties in what follows.

[3] Here we use the notation of [70] because we will follow their notations later for quantised enveloping algebras.

[4] It can be obtained from the usual Cartan matrix by multiplication with a diagonal matrix D

4.2 Basics of Hopf algebras

4.2.1 Algebras

In this section we will briefly review some basic facts about associative algebras[5]. We will formulate the standard multiplication and the unit as maps, and properties like associativity in terms of compositions of those maps. This might look awkward at first, but having established this language it is straightforward to formulate the axioms for coalgebras, and finally, Hopf algebras.

Definition 5 (Associative Algebra). *An associative algebra $\mathfrak{a}$ over a field $\mathbb{K}$ with identity is a linear vector space over $\mathbb{K}$ together with linear maps $\mu : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a}$, the multiplication, and $\eta : \mathbb{K} \to \mathfrak{a}$, the identity, such that the following diagrams commute:*

$$\begin{array}{ccc} \mathfrak{a}\otimes\mathfrak{a}\otimes\mathfrak{a} & \xrightarrow{id\otimes\mu} & \mathfrak{a}\otimes\mathfrak{a} \\ {\scriptstyle \mu\otimes id}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathfrak{a}\otimes\mathfrak{a} & \xrightarrow{\mu} & \mathfrak{a} \end{array}$$

$$\begin{array}{ccc} \mathfrak{a}\otimes\mathbb{K} & \xrightarrow{id\otimes\eta} & \mathfrak{a}\otimes\mathfrak{a} \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathfrak{a} & \xrightarrow{id} & \mathfrak{a} \end{array}$$

$$\begin{array}{ccc} \mathbb{K}\otimes\mathfrak{a} & \xrightarrow{\eta\otimes id} & \mathfrak{a}\otimes\mathfrak{a} \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathfrak{a} & \xrightarrow{id} & \mathfrak{a} \end{array}$$

The $\cong$ denotes the canonical identification $\lambda \otimes a = \lambda a, \quad \lambda \in \mathbb{K}, a \in \mathfrak{a}$.
It is easily seen that the first diagram expresses the well known associativity law: Let $a, b, c \in \mathfrak{a}$, then we get[6] $\mu(\mu(a, b), c) = (\mu(a, \mu(b, c))$, or, writing simply $ab := \mu(a, b)$, $(ab)c = a(bc)$. The diagrams involving the identity express nothing else than $\mu(\eta \otimes id(\lambda \otimes a)) = \lambda 1a = \lambda a = a\lambda = \mu(id \otimes \eta(a \otimes \lambda)), \lambda \in \mathbb{K}$. Commutativity, which we do not require in general, can be expressed with the help of the permutation map $\sigma : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$ defined by $\sigma(a \otimes b) = b \otimes a$. Then commutativity is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathfrak{a}\otimes\mathfrak{a} & \xrightarrow{\mu} & \mathfrak{a} \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle id} \\ \mathfrak{a}\otimes\mathfrak{a} & \xrightarrow{\mu} & \mathfrak{a} \end{array}$$

[5]When we just speak about an algebra, we usually assume associativity, unless otherwise stated.

[6]Writing $\mu(a, b)$ might seem to be not completely correct, since this would usually mean that $(a, b) \in \mathfrak{a} \times \mathfrak{a}$, and not in $\mathfrak{a} \otimes \mathfrak{a}$, as we required in the definition. Nevertheless, defining μ just on the cartesian product is still sufficient, since it would then uniquely extend to the tensor product. Indeed, this is the way one usually proceeds.

This simply gives $\mu(a,b) = \mu(b,a)$.

4.2.2 Coalgebras

With the definition of an algebra given in the way above, it is easy to give the definition of a coalgebra, whose structure can be understood as dual to the structure of an algebra. One simply reverses the arrows in the defining diagrams for algebras.

Definition 6. *A coalgebra* $\mathfrak{a}$ *over a field* $\mathbb{K}$ *is a linear vector space with linear maps* $\Delta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$, $\epsilon : \mathfrak{a} \to \mathbb{K}$, *such that the following diagrams commute:*

$$\begin{array}{ccc} \mathfrak{a}\otimes\mathfrak{a}\otimes\mathfrak{a} & \xleftarrow{id\otimes\Delta} & \mathfrak{a}\otimes\mathfrak{a} \\ {\scriptstyle \Delta\otimes id}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathfrak{a}\otimes\mathfrak{a} & \xleftarrow{\Delta} & \mathfrak{a} \end{array}$$

$$\begin{array}{ccc} \mathfrak{a}\otimes\mathbb{K} & \xleftarrow{id\otimes\epsilon} & \mathfrak{a}\otimes\mathfrak{a} \\ {\scriptstyle \cong}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathfrak{a} & \xleftarrow{id} & \mathfrak{a} \end{array}$$

$$\begin{array}{ccc} \mathbb{K}\otimes\mathfrak{a} & \xleftarrow{\epsilon\otimes id} & \mathfrak{a}\otimes\mathfrak{a} \\ {\scriptstyle \cong}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathfrak{a} & \xleftarrow{id} & \mathfrak{a} \end{array}$$

We call Δ the comultiplication, or coproduct, and ϵ the counit. Unlike in the case for multiplication in an algebra, there might, at first, not be an intuitive way of writing the coproduct, like $\mu(a,b) = ab$. It is conventional to use Sweedlers notation, $\Delta a = \sum a_{(1)} \otimes a_{(2)}$. The sum goes over some elements in $\mathfrak{a}\otimes\mathfrak{a}$, to be specified for a particular coproduct. Then, coassociativity is written as[7]

$$\begin{aligned} (\Delta\otimes id)\Delta a &= (\Delta\otimes id)a_{(1)}\otimes a_{(2)} = a_{(1)(1)}\otimes a_{(1)(2)}\otimes a_{(2)} \\ &\equiv a_{(1)}\otimes a_{(2)(1)}\otimes a_{(2)(2)} = (id\otimes\Delta)\Delta a \end{aligned}$$

We ask the reader to wait for the examples to get a more intuitive understanding. Similarly to the notion of commutativity of an algebra, one calls a coalgebra cocommutative if the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{a}\otimes\mathfrak{a} & \xleftarrow{\Delta} & \mathfrak{a} \\ {\scriptstyle \sigma}\uparrow & & \uparrow{\scriptstyle id} \\ \mathfrak{a}\otimes\mathfrak{a} & \xleftarrow{\Delta} & \mathfrak{a} \end{array}$$

[7] As usual, we drop the sum sign.

Later, it will become a task of major importance to investigate coalgebras which are not cocommutative, but almost in a well defined sense. That is why we introduce an extra symbol for the opposite comultiplication, $\Delta^{op} := \sigma\Delta$. Then cocommutativity simply means $\Delta a = \Delta^{op} a, \quad \forall a \in \mathfrak{a}$.

4.2.3 Hopf algebras

In this section we want to merge the two structures of the previous sections, algebras and coalgebras, into one object called bialgebra. This is to be done in a compatible way. If such a bialgebra allows for an additional map called antipode, we get a Hopf algebra.

Definition 7 (Hopf algebra). *A bialgebra $\mathfrak{a}$ over a field $\mathbb{K}$ is a bialgebra, i.e. an algebra and a coalgebra s.t. the comultiplication and counit are algebra homomorphisms, i.e.*

$$\Delta(ab) = \Delta(a)\Delta(b) \qquad \Delta(1) = 1 \otimes 1 \tag{4.10}$$
$$\epsilon(ab) = \epsilon(a)\epsilon(b) \qquad \epsilon(1) = 1, \tag{4.11}$$

and the multiplication and unit are coalgebra homomorphisms. This means that the first three diagrams below commute. If there exist an additional antihomomorphism $S : A \to A$, i.e. $S(ab) = S(b)S(a)$ such that the last two diagrams below commute, we call the bialgebra a Hopf algebra.

$$\begin{array}{ccc} \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{\Delta\circ\mu} & \mathfrak{a} \otimes \mathfrak{a} \\ {\scriptstyle \Delta\otimes\Delta}\downarrow & & \uparrow{\scriptstyle \mu\otimes\mu} \\ \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{id\otimes\sigma\otimes id} & \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a} \end{array}$$

$$\begin{array}{ccc} \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{\epsilon\otimes\epsilon} & \mathbb{K} \otimes \mathbb{K} \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathfrak{a} & \xrightarrow{\epsilon} & \mathbb{K} \end{array}$$

$$\begin{array}{ccc} \mathfrak{a} \otimes \mathfrak{a} & \xleftarrow{\eta\otimes\eta} & \mathbb{K} \otimes \mathbb{K} \\ {\scriptstyle \Delta}\uparrow & & \uparrow{\scriptstyle \cong} \\ \mathfrak{a} & \xleftarrow{\eta} & \mathbb{K} \end{array}$$

$$\begin{array}{ccc} \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{S\otimes id} & \mathfrak{a} \otimes \mathfrak{a} \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle \mu} \\ \mathfrak{a} & \xrightarrow{\eta\circ\epsilon} & \mathfrak{a} \end{array}$$

$$\begin{array}{ccc} \mathfrak{a} \otimes \mathfrak{a} & \xrightarrow{id\otimes S} & \mathfrak{a} \otimes \mathfrak{a} \\ {\scriptstyle \Delta}\uparrow & & \downarrow{\scriptstyle \mu} \\ \mathfrak{a} & \xrightarrow{\eta\circ\epsilon} & \mathfrak{a} \end{array}$$

The first three diagrams express the homomorphism properties of the appropriate maps, and the last two are part of the definition for the antipode. The appearance of the permutation map in the first diagram can be understood as follows: In general, if $\mathfrak{a}, \mathfrak{b}$ are two coalgebras, then their tensor product $\mathfrak{a} \otimes \mathfrak{b}$ can naturally be equipped with a coalgebra structure with the counit $\epsilon^{\mathfrak{a}\otimes\mathfrak{b}}(a \otimes b) = \epsilon^{\mathfrak{a}}(a)\epsilon^{\mathfrak{b}}(b)$ and coproduct $\Delta^{\mathfrak{a}\otimes\mathfrak{b}}(a \otimes b) = a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}$, where we used Sweedlers notation $\Delta^{\mathfrak{c}}(c) = c_{(1)} \otimes c_{(2)}$, $c = a, b$. The permutation of the second and third factor is necessary, since the coproduct should live in $\mathfrak{a} \otimes \mathfrak{b} \otimes \mathfrak{a} \otimes \mathfrak{b}$. If $\mathfrak{a} = \mathfrak{b}$, the permutation of course remains.

Before introducing some examples, let us emphasise that the definitions of this section can be easily generalised to superalgebras. Basically, one can use physicists intuition, i.e. one should always pick up a minus sign whenever interchanging two odd elements. For instance, the permutation map should be substituted by the graded permutation $\sigma(a \otimes b) = (-1)^{|a||b|}(b \otimes a)$.

4.2.4 Examples

Universal enveloping algebra

Let us come to a very easy example of a Hopf algebra, which we will also generalise later. We will show that, in a certain sense, all Lie algebras can be equipped with a Hopf structure. However, Lie algebras themselves are not even associative algebras, so they cannot be directly turned into Hopf algebras. Instead, we will introduce the universal enveloping algebra of a Lie algebra, and show that it is indeed a Hopf algebra.

As a physicists, one usually thinks of Lie algebras as some matrix algebras. For such, it is no problem to perform operations like the ordinary product of two or more generators, or compute things like Casimirs. However, from the purely mathematical side, a Lie algebra $\mathfrak{g}$ is only defined as a vector space with an antisymmetric Lie bracket which satisfies the Jacobi identity. Only on representation spaces, this Lie bracket becomes the ordinary commutator, i.e. $[A, B] = AB - BA$, in general, $[A, B]$ simply denotes another element in $\mathfrak{g}$. Additionally, as mentioned above, a Lie algebra is not associative (except for few, simple cases). Mathematically, one would like to have associativity, to carry over general results from the theory of associative algebras to Lie algebras. And for physics one would like to have a mathematically rigorous defined object which somehow confirms the physicists intuition. For all this, one introduces the universal enveloping algebra of a Lie algebra. One starts with the tensor algebra $T(\mathfrak{g})$, which consists formally of all powers of elements of $\mathfrak{g}$, but one would like to identify an element like $AB - BA$ with the Lie bracket $[A, B]$. For this, one divides out the ideal I generated by elements of the form $AB - BA - [A, B]$. This gives the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Besides the standard literature cited at the beginning of this chapter, some explicit informations on universal enveloping algebras can be found in [71].

Definition 8 (Universal enveloping algebra). *The tensor algebra $T(\mathfrak{g})$ of a Lie alge-*

bra[8] *is defined as*

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}, \tag{4.12}$$

where $\mathfrak{g}^0 \equiv \mathbb{K}$. *It has the tensor product as its natural product, i.e.*

$$(a_1 \otimes \cdots \otimes a_n)(b_1 \otimes \cdots \otimes b_m) := (a_1 \otimes \dots a_n \otimes b_1 \cdots \otimes b_m). \tag{4.13}$$

The universal enveloping algebra is defined by

$$U(\mathfrak{g}) = T(\mathfrak{g})/I, \tag{4.14}$$

where I *is the ideal generated by elements* $AB - BA - [A, B], A, B \in \mathfrak{g}$.

An element like the quadratic Casimir $C = \kappa_{ab} T^a T^b$ of a semi simple Lie Algebra lives in the universal enveloping algebra. Whereas it is easily seen that $U(\mathfrak{g})$ forms an algebra, with the product inherited from the tensor product of the tensor algebra, it turns out that it can also be equipped with a coproduct and an antipode, making $U(\mathfrak{g})$ a Hopf algebra:

$$\begin{aligned} \Delta(J) &= J \otimes 1 + 1 \otimes J & (4.15)\\ \epsilon(J) &= 0 & (4.16)\\ S(J) &= -J \qquad \forall J \in \mathfrak{g} & (4.17) \end{aligned}$$

$$\begin{aligned} \Delta(1) &= 1 \otimes 1 & (4.18)\\ \epsilon(1) &= 1 & (4.19)\\ S(1) &= 1 & (4.20) \end{aligned}$$

For all other elements of U, one gets Δ, ϵ, S via the homomorphism property. What one needs to show is that these definitions are compatible with the Lie algebra structure, e.g. that $\Delta(AB - BA) = \Delta([A, B])$. This is easy to show using the homomorphism property of Δ. One notices that, independently of whether we started with an abelian Lie algebra, $U(\mathfrak{g})$ is always cocommutative. From the construction it is clear that $U(\mathfrak{g})$ is infinite dimensional. By the Poincare-Birkhoff-Witt theorem, a basis is given by ordered monomials in the basis elements T_i of the Lie algebra, i.e. elements $T_{i1} T_{i2} ... T_{1k}$, with $i_1 \leq i_2 \leq ... \leq i_k$ form a basis. However, the representation theory of $U(\mathfrak{g})$ is basically the same as the one of $\mathfrak{g}$.
Let us show by an example that the introduced structures appear naturally in physics. To calculate the quantum numbers of a tensor product, that is, a multi particle state, which are associated to certain continuous symmetries, one expects additivity. Take for simplicity an $su(2)$ symmetry, and two spin s particles, with z-components s_1, s_2. Then

[8] This definition works for any vector space.

the tensor product state $|s_1 s_2\rangle = |s_1\rangle \otimes |s_2\rangle$ is expected to have total spin $s_1 + s_2$, that is, the eigenvalue of the S_z acting on the tensor product should be $s_1 + s_2$. Mathematically, the generators of a Lie algebra act on double tensor products via the coproduct. We get

$$\pi \otimes \pi(\Delta S_z)|s_1 s_2\rangle = (\pi(S_z) \otimes 1 + 1 \otimes \pi(S_z))|s_1 s_2\rangle = (s_1 + s_2)|s_1 s_2\rangle, \tag{4.21}$$

[9]

which is exactly what we want. The action of the other generators is given in the same fashion.

Now there is the obvious question of how to act with the Lie algebra on higher tensor products. Let us see what happens for the triple tensor product. We have two possibilities to get an action on it: either via $(\Delta \otimes 1)\Delta$, or via $(1 \otimes \Delta)\Delta$. But those two are the same by the coassociativity property of the Hopf algebra. Let us see what happens for $U(\mathfrak{g})$. We simply get

$$\begin{aligned} \Delta^{(2)}(J) &:= (1 \otimes \Delta)\Delta(J) \equiv (\Delta \otimes 1)\Delta && (4.22) \\ &= 1 \otimes 1 \otimes J + 1 \otimes J \otimes 1 + J \otimes 1 \otimes 1. && (4.23) \end{aligned}$$

Inductively, we define

$$\Delta^{(n)} := (1 \otimes ... \otimes 1 \otimes \Delta)\Delta^{(n-1)}. \tag{4.24}$$

[10]

We see that the coproduct gives us precisely what we want, namely, that a symmetry algebra acts on a tensor product as a sum of the actions on the individual factors of the tensor product.

Group algebras

If we have a (finite) group G, we can also embed it into a larger, associative structure which can be equipped with a Hopf structure. If $\{g_1, \dots g_n\}$ are the elements of the group, we can consider the space $\mathbb{K}G$ of formal linear combinations $a = \sum a_k g_k, \quad a_k \in \mathbb{K}, g_k \in G$ of those elements, so we get a vector space. Then the product is naturally given by the group multiplication, the unit by the unit of the group, and the additional Hopf algebra structures are defined by

$$\begin{aligned} \Delta(g) &= g \otimes g \\ \epsilon(g) &= 1 \\ S(g) &= g^{-1} \qquad \forall g \in G. \end{aligned} \tag{4.25}$$

Again, by linearity of the maps, those definitions extend to the whole group algebra $\mathbb{K}G$. The Hopf algebra axioms are easily verified. One notices that the antipode plays the role of the inverse. We will not deal with pure group algebras in what follows. However, we will encounter certain important elements in our Hopf algebras of interest which satisfy equation (4.25), they will be called grouplike.

[9] π denotes the representation. We will often be sloppy and drop π, using the same symbol for a generator and its matrix representation.

[10] This definition remains valid for any coalgebra.

4.3 Quasitriangular Hopf algebras

4.3.1 Definitions

In the examples we encountered so far, the additional Hopf algebraic structures, that is Δ, S, ϵ, were rather trivial. They have simply cast some expected things, e.g. the additivity of quantum numbers of continuous symmetries, into a new language. The Hopf algebras have also been cocommutative, i.e. $\Delta = \Delta^{op}$. In this section we want to drop this property of cocommutativity, though not completely, but in a controlled way. We want to consider Hopf algebras where the opposite coproduct is given via conjugation of the standard coproduct. This leads us to the notion of quasi cocommutative Hopf algebras[11].

Definition 9 (quasi cocommutative Hopf algebras). *A Hopf algebra is called quasi cocommutative if there exists $R \in \mathfrak{a} \otimes \mathfrak{a}$, which is invertible and satisfies*

$$\Delta^{op}(a) = R\Delta(a)R^{-1} \quad \forall a \in \mathfrak{a}. \tag{4.26}$$

For reasons which will become clear later, we are especially interested in quasi cocommutative Hopf algebras where R satisfies two additional conditions. This leads us to the notion of quasitriangularity. Before giving the definition, let us fix some notation. Let us write $R = \sum r_i \otimes r_j$. This sum is not necessarily understood as a base expansion, but a formal sum over any elements $r_i, r_j \in \mathfrak{a}$. We will also allow for infinite series, ignoring mathematical subtleties arising in such case. Furthermore, we will use the handy expressions $R_{12} = R \otimes 1, R_{13} = \sum r_i \otimes 1 \otimes r_j, R_{23} = 1 \otimes R$ when working in the triple tensor product.

Definition 10 (quasitriangular Hopf algebra). *A Hopf algebra is called quasitriangular, if it is quasi cocommutative and the following additional identities hold:*

$$(\Delta \otimes 1)(R) = R_{13}R_{23} \tag{4.27}$$

$$(1 \otimes \Delta)(R) = R_{13}R_{12} \tag{4.28}$$

The element R of a quasitriangular Hopf algebra is called the universal R matrix. The importance of this abstract definition becomes clear with the following theorem:

Theorem 1. *The universal R matrix of a quasitriangular Hopf algebra satisfies the following relations:*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{4.29}$$

$$(S \otimes 1)R = (1 \otimes S^{-1})R = R^{-1} \tag{4.30}$$

These equations are called the Yang Baxter equation (YBE) and the crossing equation. They play an important role in the theory of integrable systems, as we will see later.

[11] also called almost cocommutative

4.3.2 Quantised universal enveloping algebras

Let us come to a first example of a Hopf algebra which is not cocommutative anymore, but still quasi triangular. We start with a semisimple Lie algebra or basic Lie superalgebra $\mathfrak{g}$, or more precisely, its universal enveloping algebra $U(\mathfrak{g})$. It turns out that $U(\mathfrak{g})$ allows for a deformation $U_q(\mathfrak{g})$ with a complex parameter $q \neq 0$, such that the resulting structure is a quasitriangular Hopf algebra. Unlike $U(\mathfrak{g})$, it is not cocommutative anymore. However, one recovers $U(\mathfrak{g})$ in the limit $q \to 1$. We will first give the definition in terms of the deformed Serre-Chevalley generators. Later, we will briefly discuss the general construction method of the quantum double.

Definition 11 (Quantised (super)algebras). *The quantised (super)algebra $U_q(\mathfrak{g})$ of a Lie (super)algebra $\mathfrak{g}$ is defined by the generators $h_i, e_i^{\pm}, i = 1, ...r$, a set $\tau \subseteq \{1, \ldots, r\}$ denoting the odd generators and relations*

$$[h_i, h_j] = 0 \tag{4.31}$$
$$[h_i, e_j^{\pm}] = \pm A_{ij} e_j^{\pm} \tag{4.32}$$
$$[e_i^+, e_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \tag{4.33}$$
$$(ad_{q^{\pm}} e_i^{\pm})^{N_{ij}} e_j^{\pm} = 0 \quad i \neq j \tag{4.34}$$

We used[12] $k_i := q^{h_i}$, and the notations of definition 4. Furthermore, we introduced the q-commutator or q-deformed adjoint

$$(ad_q e)e' = [e, e']_q := ee' - (-1)^{(|e||e'|)} q^{(e,e')} e'e, \tag{4.35}$$

with the scalar product of the corresponding roots (e, e') in the root space.
$U_q(\mathfrak{g})$ can be equipped with a coproduct, counit and antipode, making it a Hopf algebra:

$$\begin{aligned}
\Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i && (4.36)\\
\Delta(e_i^+) &= e_i^+ \otimes 1 + k_i^{-1} \otimes e_i^+ && (4.37)\\
\Delta(e_i^-) &= e_i^- \otimes k_i + 1 \otimes e_i^- && (4.38)\\
S(h_i) &= -h_i && (4.39)\\
S(e_i^+) &= -k_i e_i^+ && (4.40)\\
S(e_i^-) &= = -e_i^- k_i^{-1} && (4.41)\\
\epsilon(h_i) &= \epsilon(e_i^{\pm}) = 0 \qquad \epsilon(1) = 1 && (4.42)
\end{aligned}$$

This Hopf algebra is quasitriangular with the following universal R matrix (see [70]):

[12] Rigorously, one should consider k_i, k_i^{-1} as two independent generators with $k_i k_i^{-1} = 1$, and not as an infinite series. One could write the commutation relations without explicitly referring to h_i. However, to write down the universal R matrix one needs h_i anyway, so we will ignore those mathematical subtleties and allow those infinite power series to live in the algebra.

$$
\begin{aligned}
R &= \left(\prod_{\gamma \in \text{positive roots}} R_\gamma\right) K && (4.43)\\
R_\gamma &:= exp_{q_\gamma}((-1)^{|e_\gamma|} a_\gamma^{-1}(q-q^{-1}) e_\gamma^+ \otimes e_\gamma^-) && (4.44)\\
K &:= q^{(\sum_{ij}(A^{-1})_{ij} h_i \otimes h_j)} && (4.45)
\end{aligned}
$$

Here we introduced the q-exponential function $exp_q(x) := \sum \frac{x^n}{(n)_q!}$, where $(n)_q := \frac{1-q^n}{1-q}$, $(n)_q! := (1)_q(2)_q...(n)_q$ and $q_\gamma := (-1)^{|e_\gamma|} q^{-(\gamma,\gamma)}$. [13] The number a_i one gets for the non simple roots γ fixing appropriately the relation $[e_\gamma, e_{-\gamma}] = a_\gamma \frac{k_\gamma - k_\gamma^{-1}}{q-q^{-1}}$.

The quantum double

After giving these direct definitions of a quantised universal enveloping algebra $U_q(\mathfrak{g})$, we want to sketch how one can obtain it via the quantum double. The quantum double is a much more general method to construct quasitriangular Hopf algebras, which is one of the reasons we want to discuss it here. The other reason is that the quantum double for the q deformed enveloping algebra contains an extra central grouplike element, which we will relate in chapter 6 to a length changing operator of a certain spin chain.
The quantum double can be constructed for any two Hopf algebras $\mathfrak{a}, \mathfrak{b}$, provided there exists a non-degenerate bilinear pairing $\langle , \rangle : \mathfrak{a} \times \mathfrak{b} \to \mathbb{K}$ which makes the Hopf structures of $\mathfrak{a}, \mathfrak{b}$ dual to each other, that is

$$
\begin{aligned}
\langle a, b_1 b_2 \rangle &= \langle \Delta a, b_1 \otimes b_2 \rangle\\
\langle a_1 a_2, b \rangle &= \langle a_2 \otimes a_1, \Delta b \rangle\\
\langle 1, b \rangle &= \epsilon(b)\\
\langle a, 1 \rangle &= \epsilon(a)\\
\langle S(a), S(b) \rangle &= \langle a, b \rangle\,. && (4.46)
\end{aligned}
$$

We see that this bracket makes the product and unit of the one Hopf algebra dual to the coproduct and counit of the other Hopf algebra.
The quantum double $\mathfrak{D}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is the unique Hopf algebra which contains $\mathfrak{a}, \mathfrak{b}$ as Hopf subalgebras and satisfies the additional requirements

$$
\begin{aligned}
&\mathfrak{a} \otimes \mathfrak{b} \ni a \otimes b \mapsto ab \in \mathfrak{D} \quad \text{is a vector space isomorphism}\\
&ba = \sum \langle a_1, S(b_1) \rangle \langle a_3, b_3 \rangle\, a_2 b_2. && (4.47)
\end{aligned}
$$

We do not want to dwell too long on those details, we simply want to mention that the first requirement means that $\mathfrak{a} \otimes \mathfrak{b}$ is canonically and isomorphically embedded into $\mathfrak{D}$,

[13] There are different conventions in the literature concerning the definition of the q-exponential. A one often used is $exp_{q^2}(x) = \sum \frac{q^{-m(m-1)/2}}{[m]!} x^m$, with $[n] = \frac{q^n - q^{-n}}{q-q^{-1}}$.

i.e. if one has bases $a_i \in \mathfrak{a}, b_j \in \mathfrak{b}$, then $a_i b_j$ is a basis in $\mathfrak{D}$. The second line tells us how to express elements $ba \in \mathfrak{D}$ in terms of elements of the form ab, hence gives us the opportunity to express those elements in terms of the above mentioned basis $a_i b_j$. Instead we want to emphasise that the quantum double is a quasitriangular Hopf algebra, with the simple looking universal R matrix

$$R = \sum a_i \otimes b_i, \tag{4.48}$$

where a_i is, as before, a basis of $\mathfrak{a}$, but b_i is now the corresponding dual basis of $\mathfrak{b}$, identified with the dual bracket.

Let us sketch as an example how one can use this construction for our deformed universal enveloping algebras. We can take $\mathfrak{a} = b^+, \mathfrak{b} = b^-$, where $b^\pm$ are the q deformed Borel subalgebras containing the positive and negative roots, respectively. Both also contain the Cartan subalgebra $\mathfrak{h}$. Indeed, we can find a non degenerate dual pairing between those two. Let us restrict to $U_q(sl(2))$, with the Borel subalgebras $\mathfrak{a} = \tilde{e}, k^\pm = q^{\pm h}$ and $\mathfrak{b} = \tilde{f}, \bar{k}^\pm = q^{\pm \bar{h}}$. Here we doubled the Cartan subalgebra, getting $k^\pm, \bar{k}^\pm$ as independent elements. Roughly speaking, k and $\bar{k}$ are dual via the dual bracket, as are $\tilde{e}$ and $\tilde{f}$. Interestingly, one can do the variable transformation $e = \tilde{e}$, $f = B\tilde{f}$, $k = Bt$, $\bar{k} = B^{-1}t$, where it turns out that B is grouplike and central. Let us state the complete relations for the quantum double in this form[14]:

$$tet^{-1} = qe, \qquad tft^{-1} = q^{-1}f,$$

$$[e, f] = \frac{t^2 - t^{-2}}{q - q^{-1}} \tag{4.49}$$

$$\tag{4.50}$$

$$\begin{aligned} \Delta(e) &= e \otimes t + (Bt)^{-1} \otimes e, & \Delta(f) &= f \otimes t + Bt^{-1} \otimes f \\ \Delta t^\pm &= t^\pm \otimes t^\pm, & \Delta B^\pm &= B^\pm \otimes B^\pm \\ \epsilon(e) &= \epsilon(f) = 0, & \epsilon(t^\pm) &= \epsilon(B^\pm) = 1 \\ S(e) &= -B^{-1}e, & S(f) &= -Bf, \\ S(t^\pm) &= t^\mp, & S(B^\pm) &= B^\mp \end{aligned} \tag{4.51}$$

We see that B decouples from the commutation relations, e.g. it is not only central, but also doesn't appear on the right hand side of the products of the other elements. Hence one has the algebra isomorphism $\mathfrak{D} \cong U_q(sl(2)) \otimes span(B^n, n \in \mathbb{Z})$. However, the comultiplication part is twisted. One can use a so called twist element to untwist the coproduct, i.e. one can recover the usual $U_q(sl(2))$ coproduct and antipode. However, usually in the literature one obtains $U_q(sl(2))$ directly by setting $B \to 1$. We presented

[14]Note the slight change of definition of the Hopf structure of the q-deformed part, compared to definition 11. Both definitions are equivalent. One can use different conventions by rescaling the roots with some powers of q^h.

the quantum double in this form due to the similarity of this element B with a length changing operator appearing at a spin chain, which we will discuss in section 6
Let us finally present the universal R matrix (4.48) in terms of the generators used before. We need to set $t = q^{\frac{h}{2}}$, $B =: e^w$ with w as a new central generator, and for convenience we will also rescale $v := \frac{w}{\ln(q)}$, $t = q^h$ and get [58]

$$R = q^{\frac{-1}{2}h\otimes v}(q^{-\frac{1}{2}v\otimes v}q^{\frac{1}{2}h\otimes h}e_{q^{-1}}^{(1-q^{-2})q^{\frac{h}{2}}e\otimes q^{-\frac{h}{2}}f})q^{\frac{1}{2}v\otimes h}. \tag{4.52}$$

As we said, in the literature one usually takes the limit $B \to 1$, i.e. $w, v \to 0$. Then the pieces to the left and right of the bracket would vanish, and one would get the universal R matrix of $U_q(sl(2))$. Here, we want to take $q \to 1$ instead[15]. Then the part in the middle would vanish, except for the piece $q^{-\frac{1}{2}v\otimes v} = e^{-\frac{1}{2\ln(q)}w\otimes w}$, which would diverge, but it is central and quasitriangular on its own, so discarding it leaves us with a universal R matrix

$$R = q^{\frac{-1}{2}h\otimes v}q^{\frac{1}{2}v\otimes h} = e^{\frac{-1}{2}h\otimes w}e^{\frac{1}{2}w\otimes h} \tag{4.53}$$

which is still quasitriangular, and intertwines (4.51) for $q \to 1$, i.e. $t \to 1$. In this limit the ordinary $sl(2)$ commutation relations are restored. The remaining deformation of the coproduct can be undeformed with a twist[16] $F = e^{\frac{-1}{2}h\otimes w}$, which then automatically gives the universal R matrix.

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[15]I thank A. Torrielli for showing the consistent interchange of the two limits.

[16]I thank P.Schupp for showing me this twist deformation.

Chapter 5

Spin chains and the AdS/CFT correspondence

In 2002, Minahan and Zarembo [12] made the exciting discovery that there is a correspondence between operators in planar SYM theory in the sector of the scalar fields, which transform under the $so(6)$ R symmetry algebra, and $so(6)$ symmetric spin chains. The one-loop dilatation operator corresponds to the Hamiltonian of this spin chain, i.e. the energy eigenvalues correspond to the scaling dimensions. This correspondence was soon generalised to all sectors at one-loop in [13]. The strength of this correspondence lies in the fact that the spin chain is integrable, allowing the use of the Bethe ansatz to diagonalise the Hamiltonian. This method was originally introduced by Bethe [72] to solve the Heisenberg XXX spin chain, with much more advanced mathematical techniques being developed later, see the review [26] for algebraic Bethe ansätze. We will only briefly sketch how one can use Bethe ansätze to obtain scaling dimensions of gauge theory operators, starting with the simplest example of operators in the $su(2)$ subsector. For more details concerning the applications of spin chains, Bethe ansätze and the role of the dilatation operator in the AdS/CFT correspondence we refer the reader to the reviews [56], [50], [73], [74]. We then want to shift our interest towards the S matrix of the spin chain with the full superconformal symmetry, because the S matrix, as argued in [21], is the object of greatest importance to uncover the spectrum in this, as widely believed, integrable model. This S matrix is invariant under a residual $u(1) \ltimes (psu(2|2))^2 \ltimes \mathbb{R}^3$ as part of the full superconformal algebra, and has been, up to a prefactor, derived in [22]. In fact, the invariance under the additional $u(1)$ symmetry could not been shown directly for the spin chain S matrix, since it could not be represented on the standard representation space. We will study the fundamental representation of $psu(2|2) \ltimes \mathbb{R}^3$ in some detail, and present a novel infinite dimensional representation space on which on can also represent the $u(1)$. We will then argue that the S matrix is also invariant under this $u(1)$. We will conclude the chapter comparing the results from the spin chain with results from string theory.

5.1 $su(2)$ chain

Let us consider the subsector $su(2)$ of the $so(6)$ R-symmetry algebra, and let Z, W be two complex scalars transforming as an $su(2)$ doublet. They are given in terms of the real scalars from section 3.2 via $Z = \phi^5 + i\phi_6$ and $W = \phi^3 + i\phi^4$. We consider gauge theory in the planar limit, and are interested in gauge invariant single trace local operators [1] transforming under this symmetry. They looks like

$$\mathfrak{O}^{J_1 J_2} := tr(ZWWZ\ldots W) + \ldots \quad , \tag{5.1}$$

where J_1, J_2 denote the number of fields Z, W inside the trace. The dots behind the trace should indicate some linear combinations of traces with permutations of the fields Z, W, which we will later choose such that the resulting operators are diagonal with respect to the dilatation operator. Setting $|\uparrow\rangle := Z$, $|\downarrow\rangle := W$, we can write the operator $TrZWWZ\ldots W$ as a spin chain

$$\mathfrak{O}^{J_1 J_2} = |\uparrow\downarrow\downarrow\uparrow \ldots \downarrow\rangle \tag{5.2}$$

which transforms in the spin $\frac{1}{2}$ representation of $su(2)$. One thing to note is that the trace is cyclic, hence, we should implement periodic boundary conditions for the spin chain. Now as usual in conformal field theories we are interested in the scaling dimensions Δ of the operators. The scaling dimensions are the eigenvalues of the dilatation operator, which we write in planar perturbation theory as

$$\mathfrak{D} = \sum_{n=0} g_{YM}^{2n} \mathfrak{D}^{2n}. \tag{5.3}$$

The classical dimension is simply the length of the spin chain $\Delta^0 \equiv L = J_1 + J_2$, because the dimension of a scalar field is one. As was first observed by Minahan and Zarembo in [12] for the full $so(6)$ sector, it turns out that the one-loop correction to the dilatation operator is the Hamiltonian of the spin chain, which is in the case of $su(2)$ in spin $\frac{1}{2}$ representation the Heisenberg $XXX_{\frac{1}{2}}$ spin chain:

$$\begin{aligned}
\mathfrak{D}^2 &= g^2 H && (5.4)\\
H &= \sum_{k=1}^{L} H_{k,k+1}, \quad \text{where} && (5.5)\\
H_{k,k+1} &= I_{k,k+1} - P_{k,k+1} = \frac{1}{2}(1 - \vec{\sigma}_k \vec{\sigma}_{k+1}) && (5.6)
\end{aligned}$$

This operator acts on nearest neighbours, $I_{k,k+1}$ is simply the identity, and $P_{k,k+1}$ permutes the two sites k and $k+1$. Hence, diagonalising the dilatation operator (to one loop order) is equivalent to finding the spectrum of this Hamiltonian. One could do so by ordinary methods of linear algebra, but for longer chains the Hilbert space is too large to do this

[1] Multi trace operators need not be considered in the planar limit, since the different traces of a multi trace operator do not interact.

efficiently. However, the eigenvalue problem can also be solved by the much more efficient Bethe ansatz techniques, which were first developed by Bethe [72].
Let us start with the vacuum state of the XXX spin chain. Due to the positive sign in front of the Hamiltonian, the vacuum is ferromagnetic. We choose

$$|0\rangle = |\downarrow \ldots \downarrow\rangle. \tag{5.7}$$

Then obviously $H|0\rangle = 0$. Let us find the first excited states, the magnons. We denote by

$$|n\rangle := |\downarrow ... \downarrow\uparrow\downarrow ... \downarrow\rangle \tag{5.8}$$

the state where an up spin sits at the n-th site, and all others are spin down.
Due to the homogeneity of the Hamiltonian we expect a plane wave as the energy eigenstate. We find

$$|p\rangle = \sum_{n=1}^{L} e^{ipn}|n\rangle, \tag{5.9}$$

where we regard p as the magnon momentum. Such state is an eigenvector of the Hamiltonian with eigenvalue

$$H|p\rangle = 4\sin^2\frac{p}{2}|p\rangle. \tag{5.10}$$

Lets go on and find two particle eigenvectors. Again, we first define the position eigenstate as

$$|n_1 n_2\rangle = |\ldots \downarrow\uparrow\downarrow \ldots \downarrow\uparrow\downarrow \ldots\rangle, \tag{5.11}$$

where the upspins sit at the n_1-th and n_2-th site, and $n_1 < n_2$. To find the general two magnon energy eigenstate we make the general ansatz

$$|\Psi\rangle = \sum_{n_1<n_2} \Psi(n_1, n_2)|n_1 n_2\rangle, \tag{5.12}$$

which we plug into the Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$. Now we have to distinguish two cases, whether or not the two magnons are adjacent:

$$\begin{aligned} n1 < n_2 - 1 : E\Psi(n_1, n_2) &= 2\Psi(n_1, n_2) - \Psi(n_1 + 1, n_2) - \Psi(n_1, n_2 + 1)+ \qquad (5.13)\\ &\quad +2\Psi(n_1, n_2) - \Psi(n_1 - 1, n_2) - \Psi(n_1, n_2 - 1) \\ n1 = n_2 - 1 : E\Psi(n_1, n_2) &= 2\Psi(n_1, n_2) - \Psi(n_1 - 1, n_2) - \Psi(n_1, n_2 + 1) \qquad (5.14) \end{aligned}$$

The solution was found in [72] to be a linear combination of an incoming plane wave, and and outgoing plane wave where the particles have exchanged their momenta, namely

$$\Psi(n_1, n_2) = e^{ip_1 n_1 + ip_2 n_2} + S(p_2, p_1)e^{ip_2 n_1 + ip_1 n_2}. \tag{5.15}$$

Here we introduced the two particle scattering matrix $S(p_1, p_2)$. We can also easily see that the energy of the two magnon state is simply the sum of the energies of both magnons, i.e.

$$E = 4\sin^2\frac{p_1}{2} + 4\sin^2\frac{p_2}{2}. \tag{5.16}$$

One can also read off the two particle S matrix to be

$$S(p_1, p_2) = -\frac{e^{i(p_1+p_2)} - 2e^{ip_1} + 1}{e^{i(p_1+p_2)} - 2e^{ip_2} + 1}. \tag{5.17}$$

In particular, no particle production or annihilation takes place, and the magnons can only exchange their momenta.
We should also implement appropriate boundary conditions for a finite system of length L, i.e. the wave function should satisfy

$$\Psi(n_1, n_2) = \Psi(n_2, n_1 + L). \tag{5.18}$$

Note that we have interchanged the arguments due to the convention that the first argument is always smaller than the second. From this we get the Bethe equations

$$e^{ip_1 L} = S(p_1, p_2) \tag{5.19}$$
$$e^{ip_2 L} = S(p_2, p_1). \tag{5.20}$$

Together with $S(p_2, p_1) = S(p_1, p_2)^{-1}$, this gives the solutions $p_1 + p_2 \in 2\pi\mathbb{Z}$ for the two magnon problem. Implementing also the total momentum constraint $p_1 = -p_2$ one can solve the two magnon problem and gets $p_1 = \frac{2\pi n}{L-1}$.

As we mentioned before, the XXX spin chain is an integrable model, meaning that we have infinitely many conserved charges. This implies that not only the total momentum is conserved, but the momenta of each magnon are conserved individually. Scattering can at most exchange the momenta of the particles, and no particle production or annihilation can appear. Additionally, the process of M magnon scattering factorises into several two magnon scatterings, and the order of those two particle scattering processes does not matter. We only need to show this for the three magnon scattering process, as in figure 5.1.
This leads to the Yang Baxter equation

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}. \tag{5.21}$$

For the $su(2)$ chain this means when scattering one magnon with the $M-1$ others we just pick up one phase for each individual two particle scattering process, i.e. instead of the Bethe equation above we simply have the M magnon Bethe equations

$$e^{ip_k L} = \prod_{i\neq k}^{M} S(p_k, p_i). \tag{5.22}$$

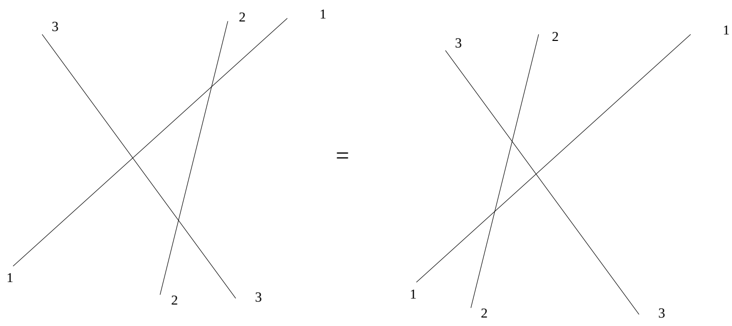

Figure 5.1: Yang Baxter equation

The Yang Baxter equation is trivial because the S matrix has no real matrix structure, it is only a one by one matrix. Let us end the discussion of this sector by writing the Bethe equation in an algebraic form, via the transformation $u_k = \frac{1}{2}\cot\frac{p_k}{2}$:

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{i\neq k}^{M} \frac{u_k - u_i + i}{u_k - u_i - i} \tag{5.23}$$

The total momentum constraint $\sum p_k = 0$ can be expressed as

$$\prod_{k}^{M} \frac{2u_k + i}{2u_k - i} = 1, \tag{5.24}$$

using $\cot^{-1} z = \frac{i}{2}\ln\frac{z+i}{z-i}$.

The other two rank one sectors of super Yang Mills theory, $su(1|1)$ and $sl(2,\mathbb{R})$ are given by operators of the form $Tr\psi^M \mathcal{Z}^{L-M}$ and $Tr(D\mathcal{Z})^M \mathcal{Z}^{L-M}$, respectively, where ψ denotes an adjoint fermion of super Yang Mills theory and D is the covariant derivative. One can write down appropriate Bethe ansätze for those sectors, see, e.g. [21] for some discussions. The results so far have been simple. We were only dealing with a rank one subsector, and the nearest neighbour Hamiltonian (5.4) only gives the dilatation operator to one loop. So one might wonder whether one can use similarly nice and, compared to direct diagonalisation of the dilatation operator, simple Bethe ansätze for the other sectors of higher rank and/or for higher loops. This is indeed true. In [13] it was shown that the complete one loop dilatation operator for the full superconformal algebra can be regarded as a spin chain Hamiltonian, and the appropriate Bethe ansätze were generalised. However, the more fundamental object suitable to proceed to higher loops and larger sectors turned out to be the S matrix [21]. Conjectures for all order S matrices and its associated Bethe ansätze have been proposed in [18], and the S matrix for the $su(2|1)$ subsector has been derived using symmetry in [35]. The object of main interest, the S matrix for the spin chain transforming under the full superconformal symmetry, which itself transforms under a residual $u(1) \ltimes psu(2|2) \ltimes psu(2|2) \ltimes u(1)$ symmetry [50] after choosing an appropriate vacuum, has been derived in [22]. We will investigate this model in the next section.
Before doing so we should address some of the questions arising from the discussions in

this section. One of them is that we argued that for integrability we need infinitely many conserved charges [2]. These charges arise as part of a Yangian symmetry and have been investigated to leading order in gauge theory in [15], [16]. Recent progress on Yangian symmetry at higher orders in certain subsectors has been done in [75], [76].

Let us finish this section with some thoughts about the meaning of higher orders, and the dual string theory. We were starting with $\mathcal{N} = 4$ super Yang Mills theory in the planar limit, i.e. $N \to \infty$, and started as usual perturbatively, i.e. looked at the first corrections to the scaling dimensions in λ. Indeed, it seems integrability holds beyond one loop, so we do not need to calculate higher loop Feynman diagrams, as would be the standard case in perturbative quantum field theory, but can use the Bethe ansatz instead to calculate energies of spin chains. As we learned in chapter 3, this particular gauge theory is dual to a string theory on $AdS_5 \times S^5$, so we might wonder if integrability also shows up there. This is indeed the case. In [14] infinitely many nonlocal charges of the classical superstring on $AdS_5 \times S^5$ have been shown to exist, and it was possible to construct Bethe ansätze for the quantum string, see [77], [78]. So we have an integrable model both at large λ, the classical superstring, and at small λ, a spin chain which describes Yang Mills operators, and on both sides quantum corrections were calculated and integrability persisted. This lets one hope that actually the complete model at large N is integrable! Above, we argued that the S matrix plays a key role for integrability. Hence, naively one would expect that the S matrices derived on the string and gauge side in some corresponding subsector, or for the full model, should agree. Indeed, the matrix structure is the same, but from perturbative calculations on both sides the matrices differ by a dressing factor [79], which is an overall scalar factor depending only on the spectral parameter and the coupling $g \propto \sqrt{\lambda}$. This is precisely caused by the fact that string and gauge theory are perturbatively treatable only in exactly opposite limits. In fact, getting the correct dressing factor is crucial for the AdS/CFT correspondence itself, since, after fixing the matrix structure of the S matrix, this factor is the object which interpolates between string and gauge theory. Hence, we should have only one S matrix with one dressing phase, whose large g expansion should reproduce the perturbative results from string theory, and the small g expansion should give the same answer as perturbative gauge theory. We will address the question of how to constrain the dressing factor later in section 6.3. A complete dressing factor based on this constraint has been proposed in [23], [24].

5.2 $su(2|2)$ chain

In this section we want to investigate a spin chain whose S matrix has $su(2|2)$ symmetry [22][3]. Let us start with an $su(2|3)$ symmetric chain, but if we choose in the $3|2$ dimen-

[2]This holds, of course, for theories with infinitely many degrees of freedom, such as field theories. Classically, in ordinary symplectic geometry, the theorem of Liouville-Arnold states that one needs n independent integrals of motion on a $2n$ dimensional symplectic manifold for integrability to hold.

[3]Here, as in large parts of the literature, we sometimes speak of a $\mathfrak{g}$ spin chain if the sites itself transform covariant under $\mathfrak{g}$, and sometimes when the S matrix transforming excitations is invariant under $\mathfrak{g}$. This is somewhat bad language, we hope we can make clear what is meant with the introduction to this section.

sional representation a bosonic vacuum, the excitations transform under $su(2|2)$. However, the $2|2$ dimensional fundamental representation requires a central charge $C = \pm\frac{1}{2}$, see [80], [81], [82] and [20]. But the eigenvalue of the central charge should give, in our model, the energy of the spin chain, which is in general not a multiple of $\frac{1}{2}$. Luckily, this algebra is very special: It allows for two other non-trivial central charges $\mathfrak{P}, \mathfrak{K}$. Adding them to the algebra, only the combination $\mathfrak{C}^2 - \mathfrak{P}\mathfrak{K}$ is fixed to the value $\frac{1}{4}$ in the fundamental representation [20], and $\mathfrak{C}$ can have continuous values.
Let us briefly explain why this particular symmetry algebra is important. Of course, the most important symmetry algebra in the AdS/CFT correspondence is the full superconformal algebra $psu(2,2|4)$, and we are interested in a spin chain with this symmetry. The S matrix is only invariant under a residual symmetry of $psu(2,2|4)$, which is $psu(2|2)^2 \ltimes \mathbb{R}^3$, see [50]. Both copies of $psu(2|2)$ share the same central charges. Hence the full S matrix scattering those $psu(2|2)^2 \ltimes \mathbb{R}$ excitations factorises into $\mathcal{S}_{psu(2|2)^2} = S_0(S_{psu(2|2)} \otimes S_{psu(2|2)})$, and we only need to determine the S matrix in one $psu(2|2)$ sector, which needs to be centrally extended, as argued above. This was done by Beisert in [22] using only the invariance of the S matrix under symmetry algebra, where he fixed the S matrix up to the scalar dressing factor S_0, which remained unconstrained by the symmetry. A constraining equation coming from crossing symmetry was derived by Janik in [25], which we will briefly discuss later. Solutions for the dressing factor were finally presented in [23], [24].

5.2.1 The centrally extended $psu(2|2)$ algebra

Let us first give the definition of the Lie superalgebra under investigation. The simple superalgebra $psu(2|2)$ consists of the following parts: We have two bosonic $su(2)$'s, denoted by $\{\mathfrak{R}, \mathfrak{R}^+, \mathfrak{R}^-\}$, $\{\mathfrak{L}, \mathfrak{L}^+, \mathfrak{L}^-\}$ with the usual commutation relations

$$\begin{aligned} \left[\mathfrak{R}, \mathfrak{R}^\pm\right] &= \pm 2\mathfrak{R}^\pm && (5.25)\\ \left[\mathfrak{R}^+, \mathfrak{R}^-\right] &= \mathfrak{R} && (5.26)\end{aligned}$$

$$\begin{aligned} \left[\mathfrak{L}, \mathfrak{L}^\pm\right] &= \pm 2\mathfrak{L}^\pm && (5.27)\\ \left[\mathfrak{L}^+, \mathfrak{L}^-\right] &= \mathfrak{L} && (5.28)\end{aligned}$$

Furthermore, we have 8 fermionic elements, labled by $\mathfrak{Q}_a^\alpha, \mathfrak{S}_\beta^b, \quad a, b, \alpha, \beta = 1, 2$, where the Latin index corresponds to the representation of the $R - su(2)$, whereas the Greek index corresponds to the $L - su(2)$. To make this more precise we set

$$\begin{aligned} \mathfrak{R} &= 2\mathfrak{R}_1^1 = -2\mathfrak{R}_2^2 & \mathfrak{L} &= 2\mathfrak{L}_1^1 \\ \mathfrak{R}^+ &= \mathfrak{R}_2^1 & \mathfrak{L}^+ &= \mathfrak{L}_2^1 \\ \mathfrak{R}^- &= \mathfrak{R}_1^2 & \mathfrak{L}^- &= \mathfrak{L}_1^2, \end{aligned} \qquad (5.29)$$

and write the commutators simply as

$$\begin{aligned}
[\mathfrak{R}^a_b, \mathfrak{S}^c_\alpha]_- &= \delta^c_b \mathfrak{S}^a_\alpha - \frac{1}{2}\delta^a_b \mathfrak{S}^c_\alpha, \\
[\mathfrak{R}^a_b, \mathfrak{Q}^\alpha_c]_- &= -\delta^a_c \mathfrak{Q}^\alpha_b + \frac{1}{2}\delta^a_b \mathfrak{Q}^\alpha_c, \\
\left[\mathfrak{L}^\alpha_\beta, \mathfrak{Q}^\gamma_a\right]_- &= \delta^\gamma_\beta \mathfrak{Q}^\alpha_a - \frac{1}{2}\delta^\alpha_\beta \mathfrak{Q}^\gamma_a, \\
\left[\mathfrak{L}^\alpha_\beta, \mathfrak{S}^a_\gamma\right]_- &= -\delta^\alpha_\gamma \mathfrak{S}^a_\beta + \frac{1}{2}\delta^\alpha_\beta \mathfrak{S}^a_\gamma, \\
\left[\mathfrak{Q}^\alpha_a, \mathfrak{S}^b_\beta\right]_+ &= \delta^b_a \mathfrak{L}^\alpha_\beta + \delta^\alpha_\beta \mathfrak{R}^b_a \\
\left[\mathfrak{Q}^\alpha_a, \mathfrak{Q}^\beta_b\right]_+ &= 0, \\
\left[\mathfrak{S}^a_\alpha, \mathfrak{S}^b_\beta\right]_+ &= 0\,.
\end{aligned} \tag{5.30}$$

The algebra allows for a universal central extension (see [83]) with three central charges $\mathfrak{C}, \mathfrak{P}, \mathfrak{K}$, such that the following commutators get modified:

$$\begin{aligned}
\left[\mathfrak{Q}^\alpha_a, \mathfrak{S}^b_\beta\right]_+ &= \delta^b_a \mathfrak{L}^\alpha_\beta + \delta^\alpha_\beta \mathfrak{R}^b_a + \delta^b_a \delta^\alpha_\beta \mathfrak{C} \\
\left[\mathfrak{Q}^\alpha_a, \mathfrak{Q}^\beta_b\right]_+ &= \epsilon^{\alpha\beta}\epsilon_{ab}\mathfrak{P}, \\
\left[\mathfrak{S}^a_\alpha, \mathfrak{S}^b_\beta\right]_+ &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathfrak{K}
\end{aligned} \tag{5.31}$$

Additionally, the algebra has three outer automorphisms [84] $\{j, j^+, j^-\}$, which form an $sl(2)$ algebra and act like [4]

$$\begin{aligned}
[j, \mathfrak{P}] &= 2\mathfrak{P} & [j, \mathfrak{Q}] &= \mathfrak{Q} \\
[j, \mathfrak{K}] &= -2\mathfrak{K} & [j, \mathfrak{S}] &= -\mathfrak{S} \\
[j^-, \mathfrak{Q}^1_1] &= \mathfrak{S}^2_2 & [j^+, \mathfrak{S}^2_2] &= \mathfrak{Q}^1_1 \\
[j^-, \mathfrak{Q}^1_2] &= -\mathfrak{S}^1_2 & [j^+, \mathfrak{S}^1_2] &= -\mathfrak{Q}^1_2 \\
[j^-, \mathfrak{Q}^2_1] &= -\mathfrak{S}^2_1 & [j^+, \mathfrak{S}^2_1] &= -\mathfrak{Q}^2_1 \\
[j^-, \mathfrak{Q}^2_2] &= \mathfrak{S}^1_1 & [j^+, \mathfrak{S}^1_1] &= \mathfrak{Q}^2_2 \\
[j^-, \mathfrak{P}] &= 2\mathfrak{C} & [j^+, \mathfrak{K}] &= 2\mathfrak{C} \\
[j^-, \mathfrak{C}] &= \mathfrak{K} & [j^+, \mathfrak{C}] &= \mathfrak{P}.
\end{aligned} \tag{5.32}$$

The other commutators are zero.
Let us hold on for some moment and comment on the meaning of the central extension and the automorphisms. For ordinary bosonic Lie algebras $\mathfrak{g}$, it is known (see e.g. [65]) that any

[4] Originally, we introduced j with the action given, and checked the consistence with the algebra. This was done to get an operator which measures the number of inserted $\mathcal{Z}$ fields, see the discussion in section 6. The action of j^+, j^- can then be derived using the Jacobi identity, and that the automorphisms together form an $sl(2)$.

central extension by n central elements is trivial, that is, the central extension is the direct sum $\mathfrak{g} \oplus (u(1))^n$. In fact, this result almost carries over to basic classical Lie superalgebra. It holds for almost all of them, except for the series $A(n-1, n-1) = psl(n|n), n \geq 2$, which is of main interest here[5]. Let us see how those algebras look in their fundamental representation. Generically for $sl(n|m)$, we have $(n+m) \times (n+m)$ matrices of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.33}$$

with the bosonic parts A, D and the fermionic part C, D. The supertrace, defined by $strM = trA - trB$, is per definition zero for $sl(n|m)$. One can choose a basis such that for the bosonic part one almost has $A \in su(n)$, $B \in su(m)$, but for $n \neq m$ the element

$$\begin{pmatrix} mId_{n\times n} & 0 \\ 0 & nId_{m\times m} \end{pmatrix} \tag{5.34}$$

should also be included in the bosonic part. For $n = m$ this element becomes a multiple of the identity, hence we get the problem that the identity matrix also satisfies $\mathrm{str}(Id) = 0$, and multiples of the identity will form an abelian ideal, so $sl(n|n)$ cannot be simple. To obtain a simple Lie superalgebra out of $sl(n|n)$ we have to factor out this ideal, getting $psl(n|n) = sl(n|n)/\mathrm{span}(Id)$. Generically, it is nothing unusual that one can add the identity matrix to a matrix algebra, the same happens, e.g., for $sl(n)$, where we get $gl(n)$ after adding the identity, or $sl(n|m), n \neq m$, where we get $gl(n|m)$. But in those cases we have, as argued above, a direct sum of the old, simple Lie algebra and the centre, i.e. $gl(n) = sl(n) \oplus \mathrm{span}(Id)$. The novel feature of $sl(n|n)$ is that its centre is tightly fixed to the rest, we only have a semidirect product $sl(n|n) = psl(n|n) \ltimes \mathrm{span}(Id)$. Additionally, those algebras allow for a continuous outer (or external) automorphism j, which is also unprecedented for basic, classical Lie superalgebras, where one usually just has discrete outer automorphisms. In terms of the above fundamental matrix representation it is natural to have this additional automorphism j, it would simply have the block diagonal form

$$j = \begin{pmatrix} Id_{n\times n} & 0 \\ 0 & -Id_{n\times n} \end{pmatrix}. \tag{5.35}$$

Hence adjoining j to $sl(n|n)$ gives $gl(n|n)$, which consists of all $2n \times 2n$ matrices with the appropriate grading.
This situation reminds a bit of loop algebras $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ of semisimple Lie algebras $\mathfrak{g}$. They also allow for one and only one nontrivial central extension, and one external automorphism. This is exactly the same for $psl(n|n), n \geq 3$. For $n = 2$, the algebra

[5] Here, and in fact in large parts of this thesis, we are interested only in algebraic details, so we will work with complex algebras. Taking appropriate real forms gives the real algebras of interest, such as $psu(2|2)$.

which we are studying in this section, the situation is even more weird, and, as far as we know, without any precedence. If we would adjoin only $\mathfrak{C}$ and j we would be in the same situation as for $psl(n|n), n \geq 3$, and we could work with the ordinary $2|2$ dimensional matrix representation with $\mathfrak{C}$ and j represented as before. But it turns out that the additional central elements $\mathfrak{P}, \mathfrak{K}$ introduced in (5.31) are also nontrivial, and the centrally extended algebra $psl(2|2) \ltimes \mathrm{span}(\mathfrak{C}, \mathfrak{P}, \mathfrak{K})$ is now 17 dimensional. If we still want to have a $2|2$ dimensional representation, $\mathfrak{P}$ and $\mathfrak{K}$ should, by Schur's Lemma, also be mapped to a multiple of the identity, so the representation cannot be faithful anymore. Recalling the nontrivial commutation relations of the automorphism j with $\mathfrak{P}$ and $\mathfrak{K}$, equation (5.32), we see that j can neither be represented by (5.35), nor can it be represented by any 4×4 matrix, since it would then necessarily commute with the identity. Similarly, neither j^+ or j^- can be represented by 4×4 matrices. This situation seems mathematically strange, and one might wonder if one should worry too much about this, in particular, if it can effect physics. For this we should note that $\mathfrak{P}$ and $\mathfrak{K}$ are not part of the full superconformal symmetry algebra $psu(2,2|4)$, and neither are j^+, j^-. However, to construct an S matrix of this model, it seems we are forced to do this central extension [22]. For this construction we still do not need any of the automorphisms, so it might not be that serious to leave them out completely. However, there are mathematical as well as physical reasons why we might need at least the Cartan part j of the automorphisms. We will deal with the explicit representation in the next section, and show a way how one can finally represent the automorphisms in a novel representation. In section 5.2.6 we will argue that the S matrix of the system is invariant under j. For more discussions on automorphisms of Lie superalgebras we refer the reader to [84].

We want to close this section with some observations about the curious relation of the algebra $psl(2|2)$ to the exceptional Lie superalgebra $D(2,1;\alpha)$. $D(2,1;\alpha)$ is the only family of basic, classical Lie superalgebras with a continuous parameter α. They are all 17 dimensional, and for different $\alpha' s$ they are non-isomorphic, except for $\alpha^{\pm}, -(1+\alpha)^{\pm}, \left(\frac{-\alpha}{1+\alpha}\right)^{\pm}$. For $\alpha = 1$ one recovers the standard Lie superalgebra $D(2,1)$. Interestingly, for $\alpha \to 0$, several thing can happen. Let us first describe the full algebra $D(2,1;\alpha)$: The bosonic part consists of three $sl(2)$'s which we denote by $\mathfrak{R}^a_b$, $\mathfrak{L}^\alpha_\beta$ and $\mathfrak{M}^\mathfrak{a}_\mathfrak{b}$ similarly as in (5.29). The eight dimensional fermionic part forms a representation of the three $sl(2)' s$, hence we write them as $\mathfrak{Q}^{a\alpha\mathfrak{a}}$ with the three different indices indicating which $sl(2)$ acts on them, i.e.

$$
\begin{aligned}
[\mathfrak{R}^a_b, \mathfrak{Q}^{c\alpha\mathfrak{a}}] &= \delta^c_b \mathfrak{Q}^{a\alpha\mathfrak{a}} - \frac{1}{2}\delta^a_b \mathfrak{Q}^{c\alpha\mathfrak{a}} \\
\left[\mathfrak{L}^\alpha_\beta, \mathfrak{Q}^{a\gamma\mathfrak{a}}\right] &= \delta^\gamma_\beta \mathfrak{Q}^{a\alpha\mathfrak{a}} - \frac{1}{2}\delta^\alpha_\beta \mathfrak{Q}^{a\gamma\mathfrak{a}} \\
[\mathfrak{M}^\mathfrak{a}_\mathfrak{b}, \mathfrak{Q}^{a\alpha\mathfrak{c}}] &= \delta^\mathfrak{c}_\mathfrak{b} \mathfrak{Q}^{a\alpha\mathfrak{a}} - \frac{1}{2}\delta^\mathfrak{a}_\mathfrak{b} \mathfrak{Q}^{a\alpha\mathfrak{c}}.
\end{aligned}
\tag{5.36}
$$

The commutator of two fermions gives

$$
\left[\mathfrak{Q}^{a\alpha\mathfrak{a}}, \mathfrak{Q}^{b\beta\mathfrak{b}}\right] = \sigma_1 \epsilon^{ac}\epsilon^{\alpha\beta}\epsilon^{\mathfrak{a}\mathfrak{b}}\mathfrak{R}^b_c + \sigma_2 \epsilon^{ab}\epsilon^{\alpha\gamma}\epsilon^{\mathfrak{a}\mathfrak{b}}\mathfrak{L}^\beta_\gamma + \sigma_3 \epsilon^{ab}\epsilon^{\alpha\beta}\epsilon^{\mathfrak{a}\mathfrak{c}}\mathfrak{M}^\mathfrak{b}_\mathfrak{c}. \tag{5.37}
$$

Those three extra constants $\sigma_1, \sigma_2, \sigma_3$ we introduced here effectively reduce to the one parameter α, after imposing the constraint $\sigma_1 + \sigma_2 + \sigma_3 = 0$ and rescaling some of the generators, in particular one gets α as a ratio of the $\sigma's$. Let us choose $\sigma_1 = -1-\alpha, \sigma_2 = 1, \sigma_3 = \alpha$. We want to investigate now what happens in the limit $\alpha \to 0$, or, equivalently $\alpha \to -1, \infty$ [84]. In the first step we simply set $\alpha = 0$ and see that, using the conventions above, the third $sl(2)$ denoted by $\mathfrak{M}^{\mathfrak{a}}_{\mathfrak{b}}$ does not appear on the right hand side of any commutator $[A, B]$ provided that at least one of the elements A, B is not in this $sl(2)$. The other elements which are not in this $sl(2)$ form a $psl(2|2)$ algebra. Hence, $psl(2|2)$ is an ideal in $D(2,1;\alpha)$, for $\alpha = 0$. The generators $\mathfrak{M}^{\mathfrak{a}}_{\mathfrak{b}}$ act as outer automorphisms on $psl(2|2)$, and give exactly the same as the generators j, j^+, j^- we discussed before.

Let us now follow [22], [83] and show how one can also obtain the centrally extended $psl(2|2) \ltimes \mathbb{C}^3$ from the exceptional $D(2,1;\alpha)$ algebra. Interestingly, we will get it for the same value $\alpha \to 0$, but only after rescaling the third $sl(2)$ as

$$\begin{aligned} \mathfrak{C} &= -\alpha \mathfrak{M}^1_1 \\ \mathfrak{P} &= \alpha \mathfrak{M}^1_2 \\ \mathfrak{K} &= -\alpha \mathfrak{M}^2_1. \end{aligned} \tag{5.38}$$

We take the limit $\alpha \to 0$ and obtain the centrally extended $psl(2|2) \ltimes \mathbb{C}^3$ as a contracted $D(2,1;\alpha)$ for $\alpha = 0$. To get the same convention as before we set

$$\begin{aligned} \epsilon^{ac}\mathfrak{Q}^{\alpha}_{c} &= \mathfrak{Q}^{a\alpha 1} \\ \epsilon^{\alpha\gamma}\mathfrak{S}^{a}_{\gamma} &= \mathfrak{Q}^{a\alpha 2}. \end{aligned} \tag{5.39}$$

It is very interesting that we can obtain both the centrally extended $psl(2|2) \ltimes \mathbb{C}^3$ and the algebra with automorphisms, $sl(2) \ltimes psl(2|2)$, from $D(2,1;\alpha)$. Of course, we cannot get both automorphisms and central elements from $D(2,1;\alpha)$ at the same time. In case we study a model with $psl(2|2) \ltimes \mathbb{C}^3$, as in this section, it will probably not be possible to make use of this exceptional algebra as long as we also need one or more automorphisms. However, we should stress that the full model S matrix we are interested in is invariant under $u(1) \ltimes psl(2|2)^2 \ltimes u(1)$, so one could speculate whether two copies of $D(2,1;\alpha)$ could provide the required symmetry.

5.2.2 Fundamental representation

We will now give the $2|2$ dimensional fundamental representation of the centrally extended $psu(2|2)$ algebra [22], [20]. Denote by V the $2|2$ dimensional vector space where the first two entries in a column vector are even, the other two are odd. The representation is labled by the 3 eigenvalues of the central charges $\mathfrak{C}, \mathfrak{P}, \mathfrak{K}$, which we write in the form $\mathfrak{P} = ab, \mathfrak{K} = cd, \mathfrak{C} = \frac{1}{2}(ad + bc)$. Then those four parameters a,b,c,d have to satisfy the constraint ad-bc=1, by consistency with the Lie superalgebra axioms. In terms of the eigenvalues of the central charges this would read

$$\mathfrak{C}^2 - \mathfrak{P}\mathfrak{K} = \frac{1}{4}. \tag{5.40}$$

This equation is the shortening condition for the fundamental representation, see [20] for more group theoretical details. The parameters a, b, c, d allows for a handy representation of the fermionic generators, which read [6]:

$$\mathfrak{Q}^1_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathfrak{S}^1_1 = \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix} \tag{5.41}$$

$$\mathfrak{Q}^1_2 = \begin{pmatrix} 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathfrak{S}^1_2 = \begin{pmatrix} 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.42}$$

$$\mathfrak{Q}^2_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix} \qquad \mathfrak{S}^2_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 \end{pmatrix} \tag{5.43}$$

$$\mathfrak{Q}^2_2 = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix} \qquad \mathfrak{S}^2_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.44}$$

$$\tag{5.45}$$

For the even generators we get

$$\mathfrak{R}^1_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathfrak{L}^1_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{-1}{2} \end{pmatrix} \tag{5.46}$$

$$\mathfrak{R}^1_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathfrak{L}^1_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.47}$$

$$\mathfrak{R}^2_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathfrak{L}^2_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5.48}$$

$$\tag{5.49}$$

[6] As usual, we will use the same symbol for the abstract generator and its matrix representation.

As argued in the previous section, the automorphisms cannot act on this four dimensional representation, since they commute nontrivially with the central charges, which are by Schur's lemma given by multiples of the identity, hence they commute with any 4×4 matrix. However, we can get a representation of the automorphisms by extending the representation in the following way: Instead of dealing solely with the four dimensional vector space V and constant numbers a, b, c, d, we deal with the space $\tilde{V} := V \otimes \mathbb{C}[a, b, c, d]$[7]. Now a, b, c, d are regarded as formal parameters. Then the automorphisms act on this space as the following derivations:

$$j = a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} - d\frac{\partial}{\partial d} \quad (5.50)$$

$$j^{+} = a\frac{\partial}{\partial c} + b\frac{\partial}{\partial d} \quad (5.51)$$

$$j^{-} = c\frac{\partial}{\partial a} + d\frac{\partial}{\partial b} \quad (5.52)$$

Since they are first order differential operators, the commutation relations (5.32) are easily verified.

For convenience with results to be discussed later we want to perform the following change of variables, from a, b, c, d to $x^{+}, x^{-}, \alpha, \gamma, g$:

$$a = \sqrt{g}\gamma, \qquad b = \sqrt{g}\frac{\alpha}{\gamma}(1 - \frac{x^{+}}{x^{-}}) \quad (5.53)$$

$$c = \sqrt{g}\frac{i\gamma}{\alpha x^{+}}, \qquad d = \sqrt{g}\frac{1}{i\gamma}(x^{+} - x^{-}) \quad (5.54)$$

The number g is related to the 't Hooft coupling, $g \propto \sqrt{\lambda}$.
In those new variables the constraint $ad - bc = 1$ translates to

$$x^{+} + \frac{1}{x^{+}} - x^{-} - \frac{1}{x^{-}} = \frac{i}{g}, \quad (5.55)$$

whereas the central charges look like

$$\mathfrak{C} = ig(x^{-} - x^{+}) - \frac{1}{2}$$
$$\mathfrak{P} = g\alpha\left(1 - \frac{x^{+}}{x^{-}}\right)$$
$$\mathfrak{K} = \frac{g}{\alpha}\left(1 - \frac{x^{-}}{x^{+}}\right). \quad (5.56)$$

[7] $\mathbb{C}[a, b, c, d]$ denotes polynomials in a,b,c,d over the complex numbers. However, in principle there should be no problem to allow for power series or even for Laurent series. For unitary representations, which we are ultimately interested in, we should also restrict to real numbers.

We can write the automorphisms as

$$j = \gamma\partial_\gamma + 2\alpha\partial_\alpha \tag{5.57}$$

$$j^+ = i\alpha\frac{(x^+)^2((x^-)^2-1)}{x^-(x^-+x^+)}\partial_{x^+} + \frac{i\alpha}{x^++x^-}x^-((x^+)^2-1)\partial_{x^-} + \frac{i\alpha^2}{x^++x^-}\frac{x^+}{x^-}(1+x^+x^-)\partial_\alpha \tag{5.58}$$

$$j^- = -\frac{i}{\alpha}\frac{x^+((x^-)^2-1)}{x^-+x^+}\partial_{x^+} + \frac{i}{\alpha}\frac{(x^-)^2(1/x^+-x^+)}{x^-+x^+}\partial_{x^-} + \frac{i\gamma}{\alpha x^+}\partial_\gamma + i\frac{x^-}{x^+}\frac{1+x^-x^+}{x^-+x^+}\partial_\alpha, \tag{5.59}$$

keeping g fixed. The action on tensor products is not clear for the moment. The problem is that α, g should be global parameters, whereas $x^\pm, \gamma$ take different values on different tensor products [20]. Hence, it makes a difference for the automorphisms if one works with the variables a_i, b_i, c_i, d_i or $x_i^\pm, \gamma_i, \alpha, g$.

5.2.3 Bilinear Form and Casimir

In this subsection we want to calculate a non degenerate, supersymmetric bilinear form for our Lie superalgebra, and use it to calculate the second order Casimir in the canonical fashion. On $psu(2|2) \ltimes \mathbb{R}^3$ we take the bilinear form given by $(A, B) = str(A, B)$, where A, B live in the fundamental representation given above, and str denotes the supertrace defined in the last section. On $psu(2|2)$ this form is non degenerate, but including the central charges gives a degenerate form on $psu(2|2) \ltimes \mathbb{R}^3$, because the supertrace of a central element with any other element gives zero. By taking the full algebra $sl(2) \ltimes psu(2|2) \ltimes \mathbb{R}^3$ we again get a non degenerate bilinear form[8], where the coefficients on the $sl(2) \ltimes \mathbb{R}^3$ part are calculated by the invariance requirement $([A, B], C) = (A, [B, C])$.
We get for the nonzero elements of $psu(2|2)$

$$(\mathfrak{R},\mathfrak{R}) = 2 \qquad (\mathfrak{L},\mathfrak{L}) = -2 \tag{5.60}$$
$$(\mathfrak{R}^+,\mathfrak{R}^-) = 1 \qquad (\mathfrak{L}^+,\mathfrak{L}^-) = -1 \tag{5.61}$$
$$(\mathfrak{Q}^1_1,\mathfrak{S}^1_1) = -1 \qquad (\mathfrak{Q}^1_2,\mathfrak{S}^2_1) = -1 \tag{5.62}$$
$$(\mathfrak{Q}^2_1,\mathfrak{S}^1_2) = -1 \qquad (\mathfrak{Q}^2_2,\mathfrak{S}^2_2) = -1. \tag{5.63}$$

For $sl(2)$ we have the usual relations[9]

$$(j, j) = 2, \qquad (j^+, j^-) = 1. \tag{5.64}$$

[8] This situation again reminds of loop algebras, whose central extensions have a degenerate bilinear form, and one need to extend the algebra by an automorphism, getting an affine algebra with non-degenerate form.

[9] More general, the scalar product of the $sl(2)$ automorphism is not completely fixed, one only has the relation $(j, j) = 2(j^+, j^-)$. This is the reason that the central charge part of the Casimir decouples from the rest.

Furthermore, we get

$$(j, \mathfrak{C}) = (j^+, \mathfrak{K}) = -(j^-, \mathfrak{P}) = (\mathfrak{Q}_2^2, \mathfrak{S}_2^2) \equiv -1. \tag{5.65}$$

This allows us to write down the quadratic Casimir $J^2 = J_1^2 + J_2^2 + J_3^2$, where J_1^2 is the $psu(2|2)$ part, J_2^2 is the mixed part containing the automorphisms and central charges, and J_3^2 is the contribution from only the central charges. Since J_3^2 commutes separately with everything, it can be regarded as an independent Casimir. The usual form is

$$J^2 = \sum \kappa_{ab} T^a T^b, \tag{5.66}$$

with κ_{ab} being the inverse of the Bilinearform $\kappa^{ab} = (T^a, T^b)$.
We get

$$\begin{aligned} J_1^2 &= \frac{1}{2}\mathfrak{R}^2 + [\mathfrak{R}^+, \mathfrak{R}^-]_+ - (\frac{1}{2}\mathfrak{L}^2 + [\mathfrak{L}^+, \mathfrak{L}^-]_+) + [\mathfrak{Q}_a^\alpha, \mathfrak{S}_\alpha^a]_- \\ &\to \frac{1}{2}diag(-1,-1,1,1). \end{aligned} \tag{5.67}$$

The last line denotes the value of the Casimir in the fundamental representation $\tilde{V}$. Interestingly, unlike one would expect for a Casimir, it is not proportional to the identity on this representation. It needs the other differential operator part to commute with all generator:

$$\begin{aligned} J_2^2 &= (-[j, \mathfrak{C}]_+ - [j^+, \mathfrak{K}]_+ + [j^-, \mathfrak{P}]_+) \\ &= -2j\mathfrak{C} - 2j^+\mathfrak{K} + 2j^-\mathfrak{P} \\ &= -2\mathfrak{C}j - 2\mathfrak{K}j^+ + 2\mathfrak{P}j^- \\ &\to (-a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} + d\frac{\partial}{\partial d}) \end{aligned} \tag{5.68}$$

The third part is only composed of central charges , we get

$$J_3^2 = 2(-\mathfrak{C}^2 + \mathfrak{P}\mathfrak{K}). \tag{5.69}$$

On the fundamental representation it takes the value $-\frac{1}{4}$ by the shortening condition. Obviously, $[J_3^2, psu(2|2) \ltimes \mathbb{C}^3] = 0$. So we only need to check the commutation relations with the automorphisms,

$$\begin{aligned} \left[j, -\mathfrak{C}^2 + \mathfrak{P}\mathfrak{K}\right] &= (\mathfrak{P}j + 2\mathfrak{P})\mathfrak{K} - \mathfrak{P}(j\mathfrak{K} + 2\mathfrak{K}) = 0 \\ \left[j^+, -\mathfrak{C}^2 + \mathfrak{P}\mathfrak{K}\right] &= \mathfrak{C}(j^+\mathfrak{C} - \mathfrak{P}) - (\mathfrak{C}j^+ + \mathfrak{P})\mathfrak{C} + 2\mathfrak{C}\mathfrak{P} = 0 && (5.70) \\ \left[j^-, -\mathfrak{C}^2 + \mathfrak{P}\mathfrak{K}\right] &= \mathfrak{C}(j^-\mathfrak{C} - \mathfrak{K}) - (\mathfrak{C}j^- + \mathfrak{K})\mathfrak{C} + 2\mathfrak{C}\mathfrak{K} = 0. && (5.71) \end{aligned}$$

We shall drop J_3^2 in what follows, since it commutes with everything separately, as shown above. Let us do some checks on the more interesting part of the Casimir, and start by checking the commutation relations with the central charges, which obviously commute with J_1^2, so we need to show that they commute also with J_2^2. We will work on the basis of the representation. We get

$$\begin{aligned}
\left[J_2^2, \mathfrak{P}\right] &= \frac{1}{2}(ab - ab) = 0 \\
\left[J_2^2, \mathfrak{K}\right] &= \frac{1}{2}(cd - cd) = 0 \\
\left[J_2^2, \mathfrak{C}\right] &= \frac{1}{4}(ad - bc + bc - ad) = 0.
\end{aligned} \tag{5.72}$$

The most interesting part are the fermionic generators, since here $[J_{1,2}^2, \mathfrak{Q}] \neq 0$, $[J_{1,2}^2, \mathfrak{S}] \neq 0$. Lets take, as an example, $\mathfrak{Q}_2^1$:

$$\begin{aligned}
\left[J_1^2, \mathfrak{Q}_2^1\right] &= \left[\frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{5.73}$$

On the other hand,

$$\left[J_2^2, \mathfrak{Q}_2^1\right] = \begin{pmatrix} 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.74}$$

Hence, the contributions from J_1^2 and J_2^2 exactly compensate. Similarly, one can show that this works for all other generators.
A strange thing is that this Casimir cannot be truncated consistently to the matrix part. Working only with $psu(2|2) \ltimes \mathbb{R}^3$ the formally truncated Casimir will not commute, so we need the extension by the automorphisms, and work with the representation space $\tilde{V}$. One can now use the full Casimir $J^2 = J_1^2 + J_2^2$ to determine irreducible representations, by solving the combined matrix and differential equation.

5.2.4 Asymptotic states

Lets take a step back to the 3|2 dimensional representation of $su(2|3)$ consisting of the fields $\{\mathcal{Z}, \phi^1, \phi^2, \psi^1, \psi^2\}$, where the first three are bosonic and the latter two are fermionic. We consider infinitely long spin chains, and choose as the ground state the chain with all elements being $\mathcal{Z}$:

$$|0\rangle = |\ldots \mathcal{Z}\mathcal{Z}\mathcal{Z}\ldots\rangle \tag{5.75}$$

An elementary excitation, a magnon with definite momentum p is given by

$$|\mathcal{X}\rangle = \sum_n e^{ipn}|...\mathcal{Z}\mathcal{X}\mathcal{Z}...\rangle, \tag{5.76}$$

where the $\mathcal{X}$ is inserted at the n-th site. This is similar to the situation for the $su(2)$ case section 5.1, just that the excitations transform now under the residual symmetry algebra $su(2|2)$, i.e. $\mathcal{X}$ is an element of the vector space V given in section 5.2.2, spanned by $\{\phi^1, \phi^2, \psi^1, \psi^2\}$. We also encounter a unique feature of this spin chain with this symmetry algebra [22]: The momentum p of an excitation is linked with the labels a, b, c, d of the representation, in the following fashion:

$$\begin{aligned} e^{-ip} &= 1 + abg\alpha \equiv 1 + g\alpha\mathfrak{P} \\ e^{ip} &= 1 + cd\frac{g}{\alpha} \equiv 1 + \frac{g}{\alpha}\mathfrak{K}. \end{aligned} \tag{5.77}$$

Here g is the coupling constant, and α some extra parameter we already encountered in 5.53, where we would get the connection $e^{ip} = \frac{x^+}{x^-}$. Plugging this into the shortening condition (5.40), we get for the energy $\mathfrak{C}$ the dispersion relation

$$\mathfrak{C}^2 = \frac{1}{4} - 4g^2 \sin^2\frac{p}{2}. \tag{5.78}$$

We see from (5.76) that when we insert or remove another ground state field $\mathcal{Z}$ to the left of an excitation, we can exchange it with the excitation and pick up a phase:

$$|\mathcal{Z}^{\pm}\mathcal{X}\rangle = e^{\mp ip}|\mathcal{X}\mathcal{Z}^{\pm}\rangle \tag{5.79}$$

Formally, we could introduce operators $\tilde{B}, \tilde{B}^{-1}$, which create or destroy vacuum sites:

$$\tilde{B}^{\pm}|\mathcal{X}\rangle = |\mathcal{Z}^{\pm}\mathcal{X}\rangle = e^{\mp ip}|\mathcal{X}\mathcal{Z}^{\pm}\rangle \tag{5.80}$$

We will make use of those, or similar, operators in chapter 6.
For states with single excitations, those inserted $\mathcal{Z}$ markers result in an absolute phase. We should not forget that we have, as for the $su(2)$ chain, an overall momentum constraint $\sum p_n = 0$, so for one magnon this phase factor even equals one. However, this phase will become important for multi magnon states since then it will be a relative phase. Lets first consider a two magnon state

$$|\mathcal{X}\mathcal{Y}\rangle = \sum_{n_1 \ll n_2} e^{ip_1 n_1 + ip_2 n_2}|...\mathcal{Z}\mathcal{X}\mathcal{Z}...\mathcal{Z}\mathcal{Y}\mathcal{Z}...\rangle, \tag{5.81}$$

or more generally, an m magnon state

$$|\mathcal{X}_1 \ldots \mathcal{X}_m\rangle = \sum_{n_1 \ll \cdots \ll n_m} e^{ip_1 n_1 + \cdots + ip_m n_m}|...\mathcal{Z}\mathcal{X}_1\mathcal{Z}\ldots\mathcal{Z}\mathcal{X}_m\mathcal{Z}\ldots\rangle. \tag{5.82}$$

$\mathcal{X}_k$ is inserted at the m_k-th site. We consider asymptotic states with $n_k \ll n_{k+1}$ such that the distance between two excitations is much larger than the range of interaction, which grows with the order in the coupling g. Then in principle we could think of the symmetry algebra $su(2|2)$ acting on each excitation individually. In the language of universal enveloping algebras, as studied in chapter 4.2.4, this is the same as acting with the trivial coproduct of the universal enveloping algebra on the excitations, i.e. $su(2|2)$ acts like

$$\Delta_0 J = 1 \otimes J + J \otimes 1 \quad \forall J \in su(2|2) \tag{5.83}$$

on two excitations. However, for this spin chain we encounter a new phenomenon: From the $su(2|3)$ symmetric point of view, the two fermion state $\psi_{[1}\psi_{2]}$ and the three boson state $\phi_{[1}\phi_2\phi_{3]}$ have the same quantum numbers [34]. Here, the bracket $[,]$ means total antisymmetrisation, and $\phi_3 \equiv \mathcal{Z}$. In particular, both states have classical energy dimension three since scalars have dimension one and spinors have dimension $\frac{3}{2}$ in four dimensional space time[10], and their $su(2)$ and $su(3)$ charges are zero. Having the same quantum numbers makes them effectively indistinguishable, hence those two states are expected to mix. But those two states have different length. This implies that the length is not conserved in this picture, which is not the standard case for spin chains studied by condensed matter physicists.

As a consequence, at least some of the generators of the residual symmetry algebra $su(2|2)$ have a length changing effect. This could not be seen from the matrix representation of section 5.2.2, since, as argued above, for single magnons the length changing has no observable effect. We will make the length changing visible with the following presentation as given in [22]:

$$\begin{aligned}
\mathfrak{R}^a{}_b|\phi^c\rangle &= \delta^c_b|\phi^a\rangle - \frac{1}{2}\delta^a_b|\phi^c\rangle, \\
\mathfrak{L}^\alpha{}_\beta|\psi^\gamma\rangle &= \delta^\gamma_\beta|\psi^\alpha\rangle - \frac{1}{2}\delta^\alpha_\beta|\psi^\gamma\rangle, \\
\mathfrak{Q}^\alpha{}_a|\phi^b\rangle &= a\,\delta^b_a|\psi^\alpha\rangle, \\
\mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle &= b\,\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^b\mathcal{Z}^+\rangle, \\
\mathfrak{S}^a{}_\alpha|\phi^b\rangle &= c\,\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^\beta\mathcal{Z}^-\rangle, \\
\mathfrak{S}^a{}_\alpha|\psi^\beta\rangle &= d\,\delta^\beta_\alpha|\phi^a\rangle \qquad (5.84) \\
\mathfrak{P}|\mathcal{X}\rangle &= ab|\mathcal{X}\mathcal{Z}\rangle \\
\mathfrak{K}|\mathcal{X}\rangle &= cd|\mathcal{X}\mathcal{Z}^{-1}\rangle \quad \forall \mathcal{X} \in span\{\phi^1, \phi^2, \psi^1, \psi^2\}. \qquad (5.85)
\end{aligned}$$

Indeed, those extra $\mathcal{Z}$ markers have no effect when restricting to one magnon, and the equations above tell us the same as those in section 5.2.2, setting

[10] In general, scalars have energy dimension $\frac{d}{2} - 1$ and spinors have dimension $\frac{d-1}{2}$ in d space-time dimensions. For $d = 4$ we already discussed this in chapter 3.2.

$$\phi^1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \phi^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$
$$\psi^1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad \psi^2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{5.86}$$

However, this presentation will make it easier to see the effect of the length changing for multi magnon states, which we will discuss in the next section. Before doing so, let us change the basis of the representation space a bit. Let us forget that $\mathcal{Z}$ stands for an individual vacuum field, and formally rescale the bosonic states $\phi \to \phi\sqrt{\mathcal{Z}}$. Then the odd generators act like

$$\begin{aligned}
\mathfrak{Q}^\alpha{}_a|\phi^b\rangle &= a\delta^b_a|\psi^\alpha\sqrt{\mathcal{Z}}\rangle, \\
\mathfrak{Q}^\alpha{}_a|\psi^\beta\rangle &= b\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^b\sqrt{\mathcal{Z}}\rangle, \\
\mathfrak{S}^a{}_\alpha|\phi^b\rangle &= c\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^\beta\mathcal{Z}^{-\frac{1}{2}}\rangle, \\
\mathfrak{S}^a{}_\alpha|\psi^\beta\rangle &= d\delta^\beta_\alpha|\phi^a\mathcal{Z}^{-\frac{1}{2}}\rangle \\
\mathfrak{P}|\mathcal{X}\rangle &= ab|\mathcal{X}\mathcal{Z}\rangle \\
\mathfrak{K}|\mathcal{X}\rangle &= cd|\mathcal{X}\mathcal{Z}^{-1}\rangle,
\end{aligned} \tag{5.87}$$

whereas the action of the even generators remains unchanged.
We do this strange rescaling, because this new basis will be suitable to introduce a Hopf algebra interpretation of the length changing in chapter 6. One can avoid this rescaling with a square root, as its done in [32], and still get a Hopf algebra. However, on the string side a structure where only the fermionic generators change the length has arisen [85], which we will briefly discuss in section 5.3.1. To get the same for the spin chain we need this strange rescaling. Mathematically this rescaling with square roots works fine, since it just results in the appearance of another phase, but the physical meaning remains obscure.

5.2.5 Multi magnon states

Above, we argued that for single magnons an extra $\mathcal{Z}$ or $\mathcal{Z}^{\frac{1}{2}}$ inserted or removed does not matter. We are dealing with infinitely long chains, and we were, as seen above, allowed to shift such marker across an excitation picking up only a phase. Lets use the convention of (5.87), and act in the usual way with the trivial coproduct Δ_0 of the universal enveloping algebra on multiply excited states. For the fermionic generators as well as $\mathfrak{P}, \mathfrak{K}$, we then encounter some new phenomenon: Some $\mathcal{Z}^{\frac{1}{2}}$ markers will appear between two excitations. Shifting the markers to the right we pick up a relative phase. Let us illustrate this with a particular action of some generator on a two magnon state:

$$
\begin{aligned}
\Delta_0 \mathfrak{Q}^1_1 |\phi^1\phi^1\rangle &\equiv (1\otimes \mathfrak{Q}^1_1 + \mathfrak{Q}^1_1 \otimes 1)|\phi^1\phi^1\rangle \\
&= a_1 |\psi^1\sqrt{\mathcal{Z}}\phi^1\rangle + a_2|\phi^1\psi^1\sqrt{\mathcal{Z}}\rangle \\
&= a_1 e^{-i\frac{p_2}{2}}|\psi^1\phi^1\sqrt{\mathcal{Z}}\rangle + a_2|\phi^1\psi^1\sqrt{\mathcal{Z}}\rangle \\
&\equiv a_1 e^{-i\frac{p_2}{2}}|\psi^1\phi^1\rangle + a_2|\phi^1\psi^1\rangle
\end{aligned} \tag{5.88}
$$

Here, a_i means the value of a in the $i-th$ tensor product, i.e. the i-th excitation. As argued above, we drop the markers once we shifted them to the right. As we will see in the next section, one can get the same physical effects by trading those length changing effects of the generators themselves for a deformed coproduct.
Let us briefly comment on the meaning of the strange central extension with the elements $\mathfrak{P}, \mathfrak{K}$. We were coming from an $su(2|3)$ symmetric spin chain, which does not include $\mathfrak{P}, \mathfrak{K}$. In the whole picture, we have a $psu(2,2|4)$ symmetry, and we argued before that there are no $\mathfrak{P}, \mathfrak{K}$ either. On this spin chain picture, we can think of those generators as generating some kind of gauge transformation [22]. They also arise on the string side, on which we will briefly comment in sections 5.3.1,5.3.2.
Furthermore, we said that due to the cyclicity of the trace all physical excitations on the spin chain together should have vanishing momentum, i.e. $e^{i(p_1+\ldots p_n)} = 1$. Due to the linking (5.77) of the representation labels with the momentum p, this is equivalent of having vanishing total $\mathfrak{P}, \mathfrak{K}$ on any tensor product:

$$
\begin{aligned}
\Delta^{(n-1)}\mathfrak{P}|\mathcal{X}_1\ldots\mathcal{X}_n\rangle &= \sum a_k b_k |\mathcal{X}_1\ldots\mathcal{X}_k\mathcal{Z}\ldots\mathcal{X}_n\rangle \\
&= \sum a_k b_k e^{-i(p_{k+1}+\ldots p_n)} \\
&= \alpha\sum (e^{-ip_k}-1)e^{-i(p_{k+1}+\ldots p_n)} \\
&= \alpha\sum (e^{-i(p_k+\ldots p_n)} - e^{-i(p_{k+1}+\ldots p_n)}) \\
&= \alpha(e^{-i(p_1+\ldots p_n)}-1)
\end{aligned} \tag{5.89}
$$

$$
\begin{aligned}
\Delta^{(n-1)}\mathfrak{K}|\mathcal{X}_1\ldots\mathcal{X}_n\rangle &= \sum c_k d_k |\mathcal{X}_1\ldots\mathcal{X}_k\mathcal{Z}^{-1}\ldots\mathcal{X}_n\rangle \\
&= \sum c_k d_k e^{i(p_{k+1}+\ldots p_n)} \\
&= \beta\sum (e^{ip_k}-1)e^{i(p_{k+1}+\ldots p_n)} \\
&= \beta\sum (e^{i(p_k+\ldots p_n)} - e^{i(p_{k+1}+\ldots p_n)}) \\
&= \beta(e^{i(p_1+\ldots p_n)}-1)
\end{aligned} \tag{5.90}
$$

In particular, on single magnon states the central charges vanish, so we still have $su(2|2)$ as the symmetry algebra for physical states satisfying the momentum constraint. We conclude that enhancing the symmetry algebra from $su(2|2)$ to $psu(2|2) \ltimes \mathbb{R}^3$ is only seen on the tensor product.

5.2.6 The $su(2|2)$ S matrix

We now want to discuss the two particle scattering matrix of the $su(2|3)$ symmetric spin chain we discussed above, which is invariant under the residual $su(2|2)$ symmetry. The S matrix is a map

$$S : V_1 \otimes V_2 \to V_2' \otimes V_1', \tag{5.91}$$

where both V_i, V_2' are supposed to be fundamental representations[11], but the label i denotes different sets of labels a_i, b_i, c_i, d_i or central charges $\mathfrak{C}_i, \mathfrak{P}_i, \mathfrak{K}_i$, respectively. We have argued that we have an integrable model, so the S matrix should only permute the momentum p. Since $\mathfrak{C}$ is uniquely fixed by p, $\mathfrak{C}$ should also only be interchanged by the scattering matrix. However, $\mathfrak{P}$ and $\mathfrak{K}$ can change [20], but are restricted by momentum conservation

$$\begin{aligned} \mathfrak{P}_1 + \mathfrak{P}_2 &= \mathfrak{P}_1' + \mathfrak{P}_2' && (5.92)\\ \mathfrak{K}_1 + \mathfrak{K}_2 &= \mathfrak{K}_1' + \mathfrak{K}_2', && (5.93)\end{aligned}$$

whereas the shortening condition (5.40) yields

$$\mathfrak{P}_i\mathfrak{K}_i = \mathfrak{P}_i'\mathfrak{K}_i'. \tag{5.94}$$

Besides the trivial solution $\mathfrak{P}_i = \mathfrak{P}_i'$, $\mathfrak{K}_i = \mathfrak{K}_i'$, we have the nontrivial one

$$\begin{aligned} \mathfrak{P}_i' &= \mathfrak{K}_i \frac{\mathfrak{P}_1 + \mathfrak{P}_2}{\mathfrak{K}_1 + \mathfrak{K}_2} && (5.95)\\ \mathfrak{K}_i' &= \mathfrak{P}_i \frac{\mathfrak{K}_1 + \mathfrak{K}_2}{\mathfrak{P}_1 + \mathfrak{P}_2}. && (5.96)\end{aligned}$$

The trivial solution leads obviously to a trivial S matrix, i.e. simply a permutation operator. The nontrivial one is found up to an overall factor by solving the invariance equation

$$\Delta^{21} J S_{12} = S_{12} \Delta^{12} J \quad \forall J \in psu(2|2) \ltimes \mathbb{R}^3. \tag{5.97}$$

Δ^{ij} simply means that the first tensor product factor lives in representation i, and the second in representation j, because S was intertwining the representations. The result of [22], which we present here, was derived using $\Delta = \Delta^{op}$ keeping track of the inserted marker $\mathcal{Z}^{\pm}$, or the resulting phase factors, and with the length-changing convention (5.84).

[11] Here, we can work with the ordinary four dimensional representation. We can switch to the representations $\tilde{V}$ without problems.

$$\begin{aligned}
S_{12}|\phi^a\phi^b\rangle &= A_{12}|\phi^{\{a}\phi^{b\}}\rangle + B_{12}|\phi^{[a}\phi^{b]}\rangle + \frac{1}{2}C_{12}\epsilon^{ab}\epsilon_{\alpha\beta}|\psi^\alpha\psi^\beta\mathcal{Z}^{-1}\rangle \\
S_{12}|\psi^\alpha\psi^\beta\rangle &= D_{12}|\psi^{\{\alpha}\psi^{\beta\}}\rangle + E_{12}|\psi^{[\alpha}\psi^{\beta]}\rangle + \frac{1}{2}F_{12}\epsilon_{ab}\epsilon^{\alpha\beta}|\phi^a\phi^b\mathcal{Z}^{+1}\rangle \\
S_{12}|\psi^\alpha\phi^b\rangle &= K_{12}|\psi^\alpha\phi^b\rangle + L_{12}|\phi^b\psi^\alpha\rangle \\
S_{12}|\phi^a\psi^\beta\rangle &= G_{12}|\psi^\beta\phi^a\rangle + H_{12}|\phi^a\psi^\beta\rangle
\end{aligned} \tag{5.98}$$

$$\begin{aligned}
A_{12} &= S^0_{12}\frac{x_2^+ - x_1^-}{x_2^- - x_1^+}, \\
B_{12} &= S^0_{12}\left(-1 + \frac{x_1^+x_2^+ - 2x_1^-x_2^+ + x_1^-x_2^-}{2x_1^+x_1^-x_2^+x_2^-(1-1/x_1^-x_2^-)}\frac{x_1^+ - x_2^+}{x_2^- - x_1^+}\right), \\
C_{12} &= S^0_{12}\frac{2\gamma_1\gamma_2}{\alpha x_1^-x_2^-}\frac{1}{1-1/x_1^-x_2^-}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+} \\
D_{12} &= -S^0_{12}, \quad E_{12} = S^0_{12}\left(\frac{x_2^+ - x_1^-}{x_2^- - x_1^+} - \frac{x_1^+x_2^+ - 2x_1^+x_2^- + x_1^-x_2^-}{2x_1^+x_1^-x_2^+x_2^-(1-2/x_1^-x_2^-)}\frac{x_1^+ - x_2^+}{x_2^- - x_2^+}\right), \\
F_{12} &= -S^0_{12}\frac{2\alpha(x_1^+ - x_1^-)(x_2^+ - x_2^-)}{\gamma_1\gamma_2x_1^+x_2^+}\frac{1}{1-1/x_1^-x_2^-}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}, \\
G_{12} &= S^0_{12}\frac{x_2^+ - x_1^+}{x_2^- - x_1^+}, \quad H_{12} = S^0_{12}\frac{\gamma_1}{\gamma_2}\frac{x_2^+ - x_2^-}{x_2^- - x_1^+}, \\
K_{12} &= S^0_{12}\frac{\gamma_2}{\gamma_1}\frac{x_1^+ - x_1^-}{x_2^- - x_1^+}, \quad L_{12} = S^0_{12}\frac{x_2^- - x_1^-}{x_2^- - x_1^+}.
\end{aligned} \tag{5.99}$$

Here we made use of the variables (5.53), which defer slightly from the ones used in [22]. When we change the length-changing convention, as in (5.87), we pick up extra factors of the form $e^{ip/2}$ in some elements of the S matrix. The overall factor S^0_{12} could not be determined by the action of the symmetry generators, it just drops out of the equation $[\Delta J, S_{12}] = 0$. One needs an additional constraint, the crossing equation, which for our algebra was first derived by Janik [25]. We will investigate this later.
An important property of this S matrix is that it satisfies the Yang Baxter equation $S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}$. This was calculated perturbatively, see [22], and could later also be proved using group theoretic methods [20]. It also allows for a classical limit, i.e. an expansion in $\frac{1}{\sqrt{\lambda}}$. This was studied in [86], and the part in first order in $\frac{1}{\sqrt{\lambda}}$ was shown to satisfy the classical Yang Baxter equation, hence it is the classical r matrix. Investigating the classical r matrix can be useful to see the structure of the full quantum R matrix more easily.

Let us close with some comments about the invariance of the S matrix under the automorphism j, which looked in these variables like

$$j = \gamma\partial_\gamma + 2\alpha\partial_\alpha \tag{5.100}$$

on one representation. As we will discuss in chapter 6 in more detail, j will not have any length changing effect, so its action on a two magnon state is simply given by

$$\Delta j = \gamma_1 \partial_{\gamma_1} + 2\alpha_1 \partial_{\alpha_1} + \gamma_2 \partial_{\gamma_2} + 2\alpha_2 \partial_{\alpha_2}. \tag{5.101}$$

The problem is that α is usually regarded as a global parameter, i.e. $\alpha_1 = \alpha_2$. Allowing α_1, α_2, formally to be independent, then we expect that in the S matrix the currently appearing α should be either α_1 or α_2[12]. In this case it is easily seen that the S matrix is invariant under Δj, since only combinations $\frac{\gamma}{\gamma}$ and $\frac{\alpha}{\gamma\gamma}$ appear in the S matrix coefficients, and j does not act on the matrix structure.

5.3 Comparison to the string side

5.3.1 Off-shell symmetry algebra in light cone gauge

We argued above that the symmetry algebra $psu(2|2) \times psu(2|2) \ltimes \mathbb{R}^3$ plays a special role for the spin chain picture, namely, it is the residual algebra which acts on the excitations and leaves the vacuum invariant. In particular, we encountered the need for a nontrivial central extension of $psu(2|2)$, where only one generator $\mathfrak{C}$ has a natural interpretation as a Hamiltonian, or energy eigenvalue, whereas the other central elements $\mathfrak{P}, \mathfrak{K}$ can be thought of as gauge transformations [22]. One might wonder if this algebra, and in particular its central extension, also arises on the dual string side. This is indeed the case, and was investigated in [85], so we briefly want to sketch this approach. We start with the standard sigma model string action [43] with the generalised uniform light cone gauge [79], [87] and kappa-symmetry fixed, in the phase space form obtained in [78], and have

$$S = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \int_{-r}^{r} d\sigma d\tau (p_I \dot{x}_I + str(\chi^\dagger \dot{\chi}) - \mathcal{H}), \tag{5.102}$$

where x_I, p_I are the transverse coordinates and its conjugate momenta, χ are the 16 remaining fermions, and $\mathcal{H}$ is the light cone Hamiltonian density, which is the same as $-p_-$. One can go on and expand the Hamiltonian in the inverse string tension, or number of fields, after rescaling the fields with the fourth root of the inverse tension. The world sheet variable σ should also be rescaled such that

$$r = \frac{\pi P_+}{\sqrt{\lambda}}, \tag{5.103}$$

gives the string length. P_+ is the light cone momentum. This formula one gets from choosing the light cone gauge such that $p_+ = 1$, i.e. $P_+ = \frac{\sqrt{\lambda}}{2\pi} \int_{-r}^{r} = \frac{\lambda}{\pi} r$. Then one can quantise the string and calculate the energy order by order in perturbation theory. The residual unphysical degree of freedom which is left in the gauge fixed form is the level matching condition, which enforces the vanishing of the total world-sheet momentum

[12] We should confirm this with a detailed calculation, but have not done this so far.

$$p_{ws} = \int d\sigma x'_-, \qquad x'_- = -\frac{2\pi}{\sqrt{\lambda}}(p_I x'_I - \frac{i}{2} str(\Sigma_+ \chi \chi')). \tag{5.104}$$

Here $\Sigma_\pm = diag(\pm 1, \pm 1, \mp 1, \mp 1, 1, 1, -1, -1)$ are Cartan generators of the full $psu(2,2|4)$, and Σ_+ corresponds to the Hamiltonian, whereas Σ_- corresponds to P_+. The full algebra which leaves the Hamiltonian invariant is given by

$$\Sigma_- \times psu(2|2) \times psu(2|2) \times \Sigma_+, \tag{5.105}$$

whereas Σ_- decouples in the limit $P_+ \to \infty$ which we will consider here. Note that a difference to the plane wave limit is that the $AdS_5 \times S^5$ radius can still be finite in the $P_+ \to \infty$ limit. This is why we get no contraction of the algebra, as in the plane wave limit [7], [8]. Σ_- is still a symmetry of the system. It corresponds to the generator j which we introduced as an external automorphism in the spin chain picture. In the quantum theory it is conjugate to the zero mode $\hat{x}_-^{(0)}$ of the light cone variable, i.e. $\left[\hat{P}_+, \hat{x}_-^{(0)}\right] = -i$. This results in the interpretation of $e^{i\alpha\hat{x}_-}$ as a length changing operator, since the action on a state with definite P_+ gives

$$\hat{P}_+ e^{i\alpha\hat{x}_-}|P_+\rangle = (\alpha + P_+) e^{i\alpha\hat{x}_-}|P_+\rangle, \tag{5.106}$$

hence we still have an eigenstate of $\hat{P}_+$ with changed eigenvalue. As argued before, P_+ is basically the string length, so we have found an appropriate analogon of the length changing operators $B^\pm$ on the spin chain. Indeed, some of the symmetry generators indeed contain this length changing operator. In [85] it was found that these are precisely the fermionic generators and the two extra central charges $\mathfrak{P}, \mathfrak{K}$. These generators have the form

$$\mathfrak{Q}^\alpha_b \propto \int d\sigma e^{-\frac{1}{2}x_-} f(\text{transverse fields}) \tag{5.107}$$

$$\mathfrak{S}^b_\alpha \propto \int d\sigma e^{+\frac{1}{2}x_-} g(\text{transverse fields}), \tag{5.108}$$

where we want to draw the attention to the appearance of the length changing operator, and not the dependence on the field content, which we just wrote as some functions f, g of the fields. As expected, the central charge $\mathfrak{P} = [\mathfrak{Q}, \mathfrak{Q}]_+$ then contains the factor e^{-x_-}. We should mention that this form of the length changing generators is only valid in the $P_+ \to \infty$ limit, so one might ask the question how serious one should take this length changing from a physical point of view, when it is only valid for infinite lengths. On the other hand, it will be an interesting question to study what will happen to the central charges for finite string length. Let us finally comment that the scattering matrix for this sigma model has been derived perturbatively in [88], where the authors made explicit use of a Hopf algebra, which we will introduce in chapter 6.

5.3.2 Giant magnons

In the above section we have identified the symmetry algebra of interest on the string side, whereas in this section, we want to study which appropriate string states can be identified with the spin chains or gauge theory operators studied in chapter 5.
On the gauge side we were dealing with operators which can be considered as magnons, i.e. excitations of a spin chain. On the string side these magnons correspond to so called giant magnons, which have been introduced by Hofman and Maldacena in [89]. One gets them as follows: We start with the 't Hooft limit, and a free string with angular momentum J on S^5. Then we take

$$\begin{aligned} J &\to \infty, \\ \lambda, p, (E-J) &= fixed. \end{aligned} \tag{5.109}$$

p is the magnon momentum, which also occurred in the BMN operators (3.48) as $p = \frac{n}{J}$. The advantage over the BMN limit is that one can work with finite λ, p. This leads to a decoupling of quantum and finite volume corrections, i.e. corrections in λ and J, which we both sent to infinity in the BMN limit. Let us sketch the geometric picture, suppressing quantum corrections, i.e. taking λ large. Let ϕ be the angle on S^5 corresponding to the angular momentum J, and let θ the axial coordinate, and the string ground state shall be sitting at $\theta = \frac{\pi}{2}$ and a fixed point of AdS_5. Hence the motion of the string takes place on $\mathbb{R} \times S^2$. The energy of this string configuration is found to be

$$E - J = \frac{\sqrt{\lambda}}{\pi} \sin \frac{\Delta\phi}{2}, \tag{5.110}$$

where $\Delta\phi$ is the difference of angle between the two endpoints of the string at a fixed time. When one remembers that this result was derived for large λ, and that the energy of corresponding gauge theory operators was given by

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}}, \tag{5.111}$$

the energies agree upon identifying

$$p = \Delta\phi. \tag{5.112}$$

For more detailed discussions the reader is referred to [89], here we want to continue to briefly discuss the role of the extra central charges $\mathfrak{P}, \mathfrak{K}$, which we encountered in the spin chain picture. One can choose coordinates [13] on $AdS_5 \times S^5$ such that two S^3's and the time are fibred over a space with coordinates x_1, x_2, y, and for $y = 0$ one of the spheres shrinks smoothly to a point. The remaining space spanned by x_1, x_2 is flat, and a giant magnon looks like a straight stretched string in this plane.

[13] We will be very sketchy, because we won't use those coordinates further. However, they give some nice picture for the giant magnons. Details on these coordinates can be found in [90].

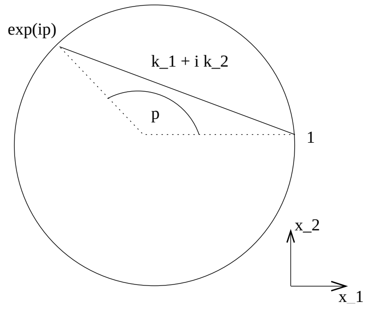

Figure 5.2: giant magnons

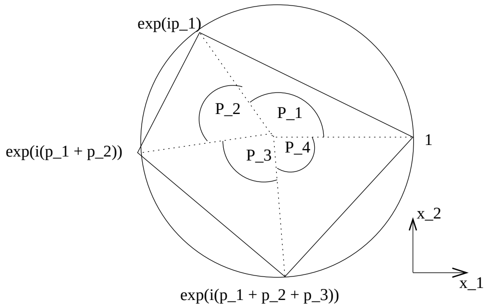

Figure 5.3:

Let k_1, k_2 be the projections of this string on the directions x_1, x_2, then the angle p is given by the phase $arg(k_1 + ik_2)$. It turns out that that those two projections correspond to the two central elements [14] of the extended supersymmetry algebra $su(2|2) \ltimes \mathbb{R}^2$. In fact, we get

$$k_l^1 + ik_l^2 \propto i\sqrt{\lambda}e^{i\sum_{j<l} p_j}(e^{ip_l} - 1), \tag{5.113}$$

so $k_l^1 + ik_l^2$ depends on the sum of all momenta to the left of the considered excitation. This is the same phenomenon we encountered on the gauge theory side (5.89), the central charges on multi magnon states depend nonlocally on other excitations[15]. The condition of having total vanishing momentum here arises because we should have a closed string as in figure 5.3.2.

From this figure it is also clear why the value of a particular k_i depends on the values of $k_j, j < i$, we simply have $k_l^1 + ik_l \propto e^{i(p_i + p_{i-1} + \dots p_1)} - e^{i(p_{i-1} + \dots p_1)}$ We can think of k^1, k^2 as momenta in $2+1$ dimensions, with the energy corresponding to the central charge of $su(2|2)$. We can extend these three-momentum generators by appropriate Lorentz generators, which correspond to the generators j, j^+, j^- we identified for the $psu(2|2) \ltimes \mathbb{R}^3$ spin chain. Again, it is clear that the rotation of k^1, k^2, which is generated by j, is a

[14] We use the convention of [89] using k_1, k_2 instead of $\mathfrak{P}, \mathfrak{K}$ to make clear that they are part of a momentum three vector, together with the energy k_0. Using a unitary representation in the convention we had before, i.e. with central charges $\mathfrak{P}, \mathfrak{K}$, we have $\mathfrak{K} = \mathfrak{P}^\dagger \propto k_1 + ik_2$ and $\mathfrak{C} = k_0$.

[15] On the gauge side, we used the "right-shifting" convention as in [22], but this plays no decisive role. We will investigate the algebraic origin of this nonlocality further in chapter 6.

symmetry of this system, since it merely rotates figure 5.3.2. However, in [89] it was argued that the boosts j^+, j^- are no symmetry of this system, since we can see from (5.113) that the absolute value is bounded.
From the point of view of the full superalgebra $psu(2|2) \ltimes \mathbb{R}^3$, we have the strange situation that it is not an ordinary super Poincare algebra, since both the three momenta and the $su(2)$'s appear on the right hand side of the supercharges. Anyway, we can effectively consider scattering in $2+1$ dimensions, since the states we are scattering are labled by the central charge three vector. Since we assume that we have an integrable system [14], the scattering not only factorises, but also the momenta are only exchanged, as seen before. The momenta in question are the worldsheet momenta p, from the $2+1$ dimensional point of view it means that $k^0 \equiv \mathfrak{C}$ is unchanged, since it only depends on the momentum p of the excitation we are scattering. Similarily, the magnitude of $k^1 + ik^2$ is also fixed, but not its phase, i.e. the orientation. The matrix structure of the scattering matrix is completely fixed by the symmetry, so we get the same answer as for the spin chain with the same symmetry algebra. The only difference is the phase factor, which interpolates between strong and weak coupling, i.e. between perturbative string and perturbative gauge theory.
We want to end this chapter mentioning that giant magnons have become of major interest within the community, see e.g. [91], [92], [93], [94], [95], [96], [97], [98].
In [99] some limit has been introduced which interpolates between the giant magnon regime and the plane wave regime.

Chapter 6

Algebraic aspects of AdS/CFT

In this chapter, we want to apply at least some of the mathematics developed in chapter 4 to the $su(2|2)$ symmetric spin chain. In the first section we will use a length changing operator introduced in [32], [33] to deform the coproduct for those symmetry generators which change the length. We also want to present a novel universal R matrix, which intertwines the nontrivial coproduct. Furthermore, we will briefly discuss the crossing equation found in [25], and close with a short sector on the Zamolodchikov-Faddeev algebra.

6.1 The Hopf algebra of the $su(2|2)$ spin chain

Some of the generators of the algebra we described in the sections 5.2.5, 5.3.1 had an unusual feature: They changed the length of the spin chain, or world sheet, respectively. On multi magnon states this resulted in the appearance of a phase. The action of the algebra on multi particle states was still given by the trivial coproduct. In this chapter we want to give an alternative action of the symmetry generators on multi particle states via a non trivial coproduct [32], [33]. We will use the operators $B^{\pm}$ defined by

$$B^{\pm}|X\rangle = |\mathcal{Z}^{\pm\frac{1}{2}}\mathcal{X}\rangle = e^{\frac{\pm ip}{2}}|\mathcal{X}\rangle, \tag{6.1}$$

and work in the setting were the fermionic generators as well as $\mathfrak{P}, \mathfrak{K}$ change the length. To be more precise, we do not want the generators themselves to change the length, but the new operator $B^{\pm}$. This means our generators act now simply as

$$\begin{aligned}
\mathfrak{Q}^{\alpha}{}_{a}|\phi^{b}\rangle &= a\delta_{a}^{b}|\psi^{\alpha}\rangle, \\
\mathfrak{Q}^{\alpha}{}_{a}|\psi^{\beta}\rangle &= b\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^{b}\rangle, \\
\mathfrak{S}^{a}{}_{\alpha}|\phi^{b}\rangle &= c\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^{\beta}\rangle, \\
\mathfrak{S}^{a}{}_{\alpha}|\psi^{\beta}\rangle &= d\delta_{\alpha}^{\beta}|\phi^{a}\rangle \\
\mathfrak{P}|\mathcal{X}\rangle &= ab|\mathcal{X}\rangle \\
\mathfrak{K}|\mathcal{X}\rangle &= cd|\mathcal{X}\rangle
\end{aligned} \tag{6.2}$$

on one magnon states, but on two magnon states we define the action to be

$$\begin{aligned}
\Delta\mathfrak{Q} &= \mathfrak{Q}\otimes B + 1\otimes\mathfrak{Q} \\
\Delta\mathfrak{S} &= \mathfrak{S}\otimes B^{-1} + 1\otimes\mathfrak{S} \\
\Delta\mathfrak{P} &= \mathfrak{P}\otimes B^{2} + 1\otimes\mathfrak{P} \\
\Delta\mathfrak{K} &= \mathfrak{K}\otimes B^{-2} + 1\otimes\mathfrak{K} \\
\Delta J &= J\otimes 1 + 1\otimes J \quad \forall J\in su(2)\oplus su(2)\ltimes span\{\mathfrak{C}\}
\end{aligned} \tag{6.3}$$

Thinking of these equations as living in the 2|2 representation, with $B|X_i\rangle = e^{-\frac{1}{2}p_i}|X_i\rangle$, one easily sees that one gets the same result as in (5.88). However, one can think of these equations as abstract equations in an algebra which is almost the universal enveloping algebra of $psu(2|2)\ltimes\mathbb{R}^3$, but with the two additional central bosonic generators[1] $B^{\pm}$ and powers of them. Additionally, the coproduct is deformed as above. In the $B\to 1$ limit, one can think of recovering the usual universal enveloping algebra.
In what follows we want to show that the structure we have introduced with equation (6.3) indeed leads to a consistent Hopf algebra. The antipode is shown to follow uniquely. The first thing one needs to check is the compatibility of the coproduct with the commutation relations, i.e. $\Delta[J_1,J_2] = [\Delta J_1,\Delta J_2]$. But this immediately follows from the following observations:

$$[\mathfrak{Q},\mathfrak{S}]\subset su(2)\oplus su(2)\ltimes span\{\mathfrak{C}\} \tag{6.4}$$

and [2]

$$\begin{aligned}
[\Delta\mathfrak{Q},\Delta\mathfrak{S}] &= [\mathfrak{Q},\mathfrak{S}]\otimes BB^{-1} + \mathfrak{Q}\otimes[B,\mathfrak{S}] + \mathfrak{S}\otimes[B,\mathfrak{Q}] + 1\otimes[\mathfrak{Q},\mathfrak{S}] \\
&= [\mathfrak{Q},\mathfrak{S}]\otimes 1 + 1\otimes[\mathfrak{Q},\mathfrak{S}] \\
&\subset (su(2)\oplus su(2)\ltimes span\{\mathfrak{C}\})\otimes 1 + \\
&+ \quad 1\otimes(su(2)\oplus su(2)\ltimes span\{\mathfrak{C}\}).
\end{aligned} \tag{6.5}$$

Further,

$$[\mathfrak{Q},\mathfrak{Q}]\subset span\{\mathfrak{P}\}, \tag{6.6}$$

so

$$[\Delta\mathfrak{Q},\Delta\mathfrak{Q}] = [\mathfrak{Q},\mathfrak{Q}]\otimes B^2 + 1\otimes[\mathfrak{Q},\mathfrak{Q}] \tag{6.7}$$

$$\subset span\{\mathfrak{P}\}\otimes B^2 + 1\otimes span\{\mathfrak{P}\}. \tag{6.8}$$

[1] Even if B^{-1} is the inverse of B, they are linearly independent in this language. Unlike in some footnote in [32], we also do not want to demand $B = 1+\frac{\mathfrak{P}}{\alpha}$, $B^{-1} = 1+\frac{\mathfrak{K}}{\beta}$ abstractly, but only for the eigenvalues on the representation we are interested in. One can still have a universal R matrix with the help of the automorphisms, as we will see later. The argument in [32], that one needs a symmetric coproduct on the centre, is still correct. But upon adjoining j $\mathfrak{C},\mathfrak{P},\mathfrak{K}$ are not central anymore.

[2] We remind the reader that [,] denotes the general supercommutator, and in the tensor product $(a\otimes b)(c\otimes d) = (-1)^{|b||c|}ac\otimes bd$

A similar relation holds, as expected, for $\mathfrak{S}$.
Since here $B^{\pm}$ are considered to be part of the Hopf algebra, we need the action of the coproduct on it. We can uniquely determine it via coassociativity, e.g. from

$$\begin{aligned}(\Delta\otimes 1)\Delta\mathfrak{Q} &= (\Delta\otimes 1)(\mathfrak{Q}\otimes B+1\otimes\mathfrak{Q}) \\ &= \mathfrak{Q}\otimes B\otimes B+1\otimes\mathfrak{Q}\otimes B+1\otimes 1\otimes\mathfrak{Q} \\ &\equiv (1\otimes\Delta)(\mathfrak{Q}\otimes B+1\otimes\mathfrak{Q}) \\ &= \mathfrak{Q}\otimes\Delta B+1\otimes\mathfrak{Q}\otimes B+1\otimes 1\otimes\mathfrak{Q} \end{aligned} \tag{6.9}$$

we can read off

$$\Delta B = B\otimes B. \tag{6.10}$$

The other constraining equations give exactly the same for B, and for B^{-1} they give

$$\Delta B^{-1} = B^{-1}\otimes B^{-1}. \tag{6.11}$$

Hence, $B^{\pm}$ are grouplike central elements.

The counit is unchanged, compared to the universal enveloping algebra, i.e.

$$\begin{aligned}\epsilon(J) &= 0 \quad \forall J\in psu(2|2)\ltimes\mathbb{R}^3 \\ \epsilon(B^{\pm}) &= 1. \end{aligned} \tag{6.12}$$

We can also derive the unique antipode from the equation

$$\mu(S\otimes 1)\Delta(J) = \mu(1\otimes S)\Delta(J) = \eta\circ\epsilon(J). \tag{6.13}$$

Of course, (6.13) gives us for the non-deformed generators the same answer as the universal enveloping algebra, i.e.

$$S(J) = -J \quad \forall J\in su(2)\oplus su(2)\ltimes \mathrm{span}\{\mathfrak{C}\}. \tag{6.14}$$

For the deformed generators we get

$$S(\mathfrak{Q}) = -B^{-1}\mathfrak{Q} \tag{6.15}$$
$$S(\mathfrak{S}) = -B\mathfrak{S} \tag{6.16}$$
$$S(\mathfrak{P}) = -B^{-2}\mathfrak{P} \tag{6.17}$$
$$S(\mathfrak{K}) = -B^{2}\mathfrak{K}. \tag{6.18}$$

The action of the antipode on $B^{\pm}$ is the standard one for grouplike elements:

$$S(B^{\pm}) = B^{\mp} \tag{6.19}$$

When we enlarge the algebra by the outer automorphisms j, j^+, j^-, we can still have a consistent Hopf algebra with the following coproduct:

$$\begin{aligned} \Delta j &= j \otimes 1 + 1 \otimes j \\ \Delta j^+ &= j^+ \otimes B^2 + 1 \otimes j^+ \\ \Delta j^- &= j^- \otimes B^{-2} + 1 \otimes j^- \end{aligned} \tag{6.20}$$

and the following antipode

$$\begin{aligned} S(j) &= -j \\ S(j^+) &= -j^+ B^{-2} \\ S(j^-) &= -j^- B^2. \end{aligned} \tag{6.21}$$

Before going on with introducing the universal R matrix, we briefly want to compare the Hopf algebra introduced here with the one in [32]. There, an $su(1|2)$ symmetric language was used in the sense that generators belonging to an $su(1|2)$ subalgebra of $psu(2|2) \ltimes \mathbb{R}^3$ have no length changing effect, but all other generator. One can obtain a basis where the generators behave this way from the starting point (5.84) and rescaling only one of the bosons by $\mathcal{Z}$. Hence both Hopf algebras can be related on the representation via rescaling of some fields.

6.1.1 The universal R matrix

In this section we will discuss a universal R matrix for the coproduct above. Since this R matrix cannot give the S matrix of [22] on the four dimensional representation, these results have so far been unpublished [3]. We hope that, albeit it cannot give the full answer, it might play some part in a bigger picture, which is why we want to present it here.
If we simply work with the algebra $psu(2|2) \ltimes \mathbb{R}^3$, we were so far unable to write down an R matrix which intertwines the coproduct introduced in the previous section. It might be that the structure found so far has to be embedded in a larger algebraic structure, like a Yangian, and possibly additional deformations. When one looks at the quantum double as introduced for $su(2)$ in chapter 4.3.2, one can see that one can obtain the same deformed coproduct from the double when one takes the limit $q \to 1$, instead of taking $B \to 1$ as it is usually done in the literature. For $psu(2|2)$ or its central extension there is one obstruction for constructing the double: The Cartan matrix is degenerate. Again, this seems unusual for basic classical Lie superalgebras. Indeed, the series $A(n|n)$ is the only series were this happens. The q deformation can nevertheless be done, upon adjoining j. The problems for constructing an intertwiner for $psu(2|2) \ltimes \mathbb{R}^3$ as for $su(2)$ is simply that there is no element in the Cartan subalgebra which "measures" how many B's appear in the coproduct. This is of course related to the fact that the Cartan matrix of $psu(2|2)$ is

[3] I want to thank Jan Plefka, Peter Schupp and Alessandro Torrielli for the collaboration in which the results of this section have been derived

degenerate. We can cure this by enlarging the Cartan subalgebra with the generator j of the $sl(2)$ automorphism algebra. We can also add the generators j^+, j^-, the R matrix we are going to write down will also intertwine their coproduct.
As in chapter 4.3.2 we need to write $B = e^w$, with a new abstract generator w satisfying $\Delta w = w \otimes 1 + 1 \otimes w$. w will also be central, so one could simply think of w as being more fundamental than B[4]. Having this, the universal R matrix is given by

$$R = e^{w \otimes j - j \otimes w}. \tag{6.22}$$

One can easily check the intertwining condition with the Baker-Campell-Hausdorff formula. Let $[j, J] = \lambda J$ be any generator with a definite charge λ under j. The crucial point is that this means precisely that $\Delta J = J \otimes B^\lambda + 1 \otimes J$. In detail, we get

$$\begin{aligned} R\Delta(J)R^{-1} &= e^{w\otimes j - j \otimes w}(J \otimes B^\lambda + 1 \otimes J)e^{-w\otimes j + j \otimes w} \\ &= \sum \frac{1}{n!}([w \otimes j - j \otimes w, \ldots [w \otimes j - j \otimes w, J \otimes B^\lambda + 1 \otimes J], \ldots]) \\ &= \sum \frac{1}{n!}(w^n \otimes \lambda^n J + (-1)^n \lambda^n J \otimes B^\lambda w^n) \\ &= B^\lambda \otimes J + J \otimes 1 \equiv \Delta^{op}(J). \end{aligned} \tag{6.23}$$

Furthermore, simple calculation gives

$$\begin{aligned} (\Delta \otimes id)R &= e^{w\otimes 1 \otimes j + 1 \otimes w \otimes j - 1 \otimes j \otimes w - j \otimes 1 \otimes w} \\ &= R_{13}R_{23} \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} (id \otimes \Delta)R &= e^{w\otimes 1 \otimes j + w \otimes j \otimes 1 - j \otimes w \otimes 1 - j \otimes 1 \otimes w} \\ &= R_{13}R_{12}. \end{aligned} \tag{6.25}$$

Hence, our Hopf algebra is quasitriangular. Due to the appearance of j, we cannot work on the standard four dimensional representation space V, but need $\tilde{V} := V \otimes \mathbb{C}(a, b, c, d)$. Anyway, since $w = -i\frac{p_i}{2}$ on the representation, the matrix structure is diagonal, so it cannot reproduce Beiserts S matrix [22]. This leads to the conclusion that the discovered Hopf algebra structure is either coincidental, just part of a bigger Hopf algebra, or the way to intertwine the coproduct with the help of j is just another solution, whereas there might be one without referring to the automorphisms. The solution on the four dimensional representation of Beisert [22] also does not depend on any automorphisms.

[4]This situation reminds of the construction of the universal R matrix for q deformed universal enveloping algebras. The Hopf algebra works fine with generators $k^\pm$, but for the R matrix one needs to work with Cartan generators h and $k = q^h$. This mathematical subtlety arises because when one takes h as fundamental, q^h is an infinite linear combination. Hence, effectively we work in some completion of the Hopf algebra, and will not worry about such mathematical subtleties.

However, there are reasons to believe that at least the j, or all automorphisms, are in fact needed. Indeed, the standard Hopf algebras such as q-deformed enveloping algebras or Yangian, which we plan to investigate in the future, need a non-degenerate Cartan matrix, which one can get by adjoining j.

6.2 The Zamolodchikov-Faddeev algebra

The Zamolodchikov-Faddeev algebra is a natural framework for many of the things we encountered so far. For our system, it has recently been derived by Arutyunov, Frolov and Zamaklar [100].The Zamolodchikov-Faddeev algebra is generated by operators $A_i^\dagger(p), A(p)_i$, which create or annihilate asymptotic states of fields $\mathcal{X}_i$, i.e.

$$|\mathcal{X}_{i_1}, \ldots, \mathcal{X}_{i_n}\rangle^{(in)} = A_{i_1}^\dagger(p_1)\ldots A_{i_n}^\dagger(p_n)|0\rangle, \quad p_1 > \cdots > p_n \tag{6.26}$$

$$|\mathcal{X}_{i_1}, \ldots, \mathcal{X}_{i_n}\rangle^{(out)} = A_{i_n}^\dagger(p_n)\ldots A_{i_1}^\dagger(p_1)|0\rangle, \quad p_1 > \cdots > p_n. \tag{6.27}$$

The momenta are ordered in the way above because otherwise no scattering would occur for our $1+1$ dimensional system. It is the S matrix which links in and out states, so we expect the S matrix to appear in the algebraic relation of the creation operators as follows:

$$\begin{aligned} |\mathcal{X}_{i_1}\mathcal{X}_{i_2}\rangle^{(in)} &\equiv A_{i_1}^\dagger(p_1)A_{i_2}^\dagger(p_2)|0\rangle = S_{i_1 i_2}^{k_1 k_2}(p_1,p_2)|\mathcal{X}_{k_1}\mathcal{X}_{k_2}\rangle^{(out)} \\ &= S_{i_1 i_2}^{k_2 k_1}(p_1,p_2)A_{k_1}^\dagger(k_1)A_{i_2}^\dagger(p_2)|0\rangle \end{aligned} \tag{6.28}$$

Hence, from the abstract point of view we have the algebraic relation

$$A_{i_1}^\dagger(p_1)A_{i_2}^\dagger(p_2) = A_{k_1}^\dagger(k_1)A_{i_2}^\dagger(p_2)S_{i_1 i_2}^{k_2 k_1}(p_1,p_2), \tag{6.29}$$

or, without matrix indices,

$$A_{i_1}^\dagger A_{i_2}^\dagger = A_{i_2}^\dagger A_{i_1}^\dagger S. \tag{6.30}$$

The corresponding relation for the annihilation operators is

$$A_{i_1}A_{i_2} = SA_{i_2}A_{i_1}, \tag{6.31}$$

and the mixed relation are

$$A_{i_1}A_{i_2}^\dagger = A_{i_2}^\dagger S_{21}A_{i_1} + \delta_{p_1-p_2}. \tag{6.32}$$

Consistency of the algebra is now equivalent to having the Yang Baxter equation and unitarity, additionally for our system we should demand crossing symmetry. Unitarity simply states that

$$\mathcal{S}_{12}(p_1,p_2)\mathcal{S}_{21}(p_2,p_1) = 1, \tag{6.33}$$

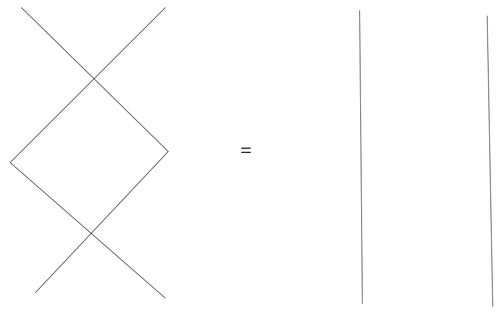

Figure 6.1: Unitarity

where $\mathcal{S}_{12} = S_0 S_{12}$ is the S matrix including the scalar prefactor, which played no role before, but is constraint by unitarity.
Crossing we want to investigate in the next chapter. The specific form of the S matrix is again constraint by the symmetry algebra, so in this framework one gets the same answer we previously encountered in section 5.2.6. What one needs to specify first is how the symmetry generators commute with $A_i^\dagger$. One puts

$$J^a A_i^\dagger(p) = J^{ak}{}_l(p) A_k^\dagger(p) \Theta_{bi}^{al} + A_m^\dagger(p) \tilde{\Theta}_{bi}^{am}(p, \mathfrak{p}) J^b \tag{6.34}$$

to get the right form of the structure constants of the generators J^a on multi particle states, which is here again given by the trivial coproduct. The role of the length changing operators $B^\pm$ or the markers $\mathcal{Z}$ is taken by the braidings Θ, $\tilde{\Theta}$, which depend on the world sheet or magnon momentum operator $\mathfrak{p}$ which satisfies

$$\mathfrak{p} A_i^\dagger(p) = A_i^\dagger(p)(\mathfrak{p} + p). \tag{6.35}$$

This is reminiscent of the length changing encountered at the spin chain and the world sheet.
A particularly nice thing about the Zamolodchikov-Faddeev is that one can easily identify an infinite set of commuting charges, which we need for integrability. In fact, it turns out that operators of the form

$$I_\omega = \int dp \omega(p) A_i^\dagger(p) A^i(p) \tag{6.36}$$

for any function $\omega(p)$ form an abelian subalgebra of the full Zamolodchikov-Faddeev . In particular, one gets

$$\mathfrak{p} = \int dp p A_i^\dagger(p) A^i(p) \tag{6.37}$$

$$\mathfrak{C} = \int dp C(p) A_i^\dagger(p) A^i(p). \tag{6.38}$$

6.3 The crossing equation

In this section we want to discuss how one can constrain the prefactor S_{12}^0 of the S matrix $\mathcal{S}_{12} = S_0 S_{12}$, where S_{12} was given in equation (5.98). As we argued above, this constraint cannot come from the invariance under the symmetry generators. Demanding additional crossing symmetry a constraining equation on the prefactor can be derived, and this was done by Janik [25] for the $psu(2|2) \ltimes \mathbb{R}^3$ spin chain, or string theory worldsheet S matrix. In standard affine $1+1$ dimensional integrable quantum field theory, one has additional requirements which fix, or at least constrain the prefactor. The first is unitarity, which we introduced in the last chapter. The second is the crossing symmetry, which allows one to express the S matrix for the scattering of two particles through the S matrix describing the scattering where one of the particles is substituted by its antiparticle.
Let us work with the R matrix $R = PS$, where P is the graded permutation operator, i.e. we have $R : V_1 \otimes V_2 \to V_1 \otimes V_2$. We then have to solve [5]

$$\begin{aligned}(S \otimes 1)\mathscr{R} &= \mathscr{R}^{-1} \\ (1 \otimes S^{-1})\mathscr{R} &= \mathscr{R}^{-1},\end{aligned} \tag{6.39}$$

with the total R matrix $\mathscr{R} = S_0 R$. These are the same relations we encountered in chapter 4, there they followed from requiring quasitriangularity. Particle and antiparticle representation are related by a linear bosonic transformation $\mathscr{C}$, whose coefficients one can get via the antipode. The antiparticle representation is parametrised by new labels $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, which one can relate to the original parameters a, b, c, d from the equation

$$\pi(S(J)) = \mathscr{C}^{-1} \bar{\pi}(J)^{st} \mathscr{C}, \tag{6.40}$$

where st denotes the supertranspose. For the manifestly $su(1|2)$ symmetric case this was done in [32]. The first crossing equation becomes

$$(\mathscr{C}^{-1} \otimes id)\mathscr{R}(-p_1, p_2)^{st_1}(\mathscr{C} \otimes 1)\mathscr{R}(p_1, p_2) = 1, \tag{6.41}$$

which constrains the scalar prefactor to

$$S_0(-p_1, p_2) S_0(p_1, p_2) = f(p_1, p_2). \tag{6.42}$$

In terms of the variables x^+, x^- one gets the solution

$$f(p_1, p_2) = \frac{(\frac{1}{x_1^+} - x_2^-)(x_1^+ - x_2^+)}{(\frac{1}{x_1^-} - x_2^-)(x_1^- - x_2^+)}. \tag{6.43}$$

However, it is not easy to extract the prefactor from this constraint. The parameters $x^\pm$ do not live in the ordinary complex plane, but on a generalised rapidity plane [25]. Crossing symmetric solutions for S_0 were only found recently in [23], [24].

[5] Here one should not confuse the antipode with the S matrix.

Chapter 7

Conclusions and outlook

In this thesis, we gave some comparatively simple applications of Hopf algebras to integrable systems arising in the AdS/CFT correspondence. We started by introducing the necessary details both from the AdS/CFT correspondence and Hopf algebras, went on studying the $su(2|2)$ spin chain and finally applied Hopf algebras to this spin chain. In particular, we used a length changing operator to deform the coproduct of the universal enveloping algebra. We should emphasise that to derive the S matrix the Hopf algebra was not needed, provided that one includes instead some $\mathcal{Z}$ markers, which stand for an inserted vacuum field [22]. However, one can also employ this Hopf algebra showing that the S matrix is indeed intertwining the nontrivial coproduct. This was done, at least for the perturbatively derived S matrix of the string sigma model, in [88]. One might wonder if the Hopf algebraic interpretation of length changing is merely an equivalent mathematical possibility, or if it has some deep meaning. We cannot give a final answer here. In this context, we should mention that spin chains with fluctuating length have not been extensively studied in the literature. Hence, it might well be that it is precisely this central, grouplike operator B, which should be studied when dealing with length changing spin chains. From the mathematical point of view, we have argued that B can be seen as an element arising in the quantum double construction. In particular, we showed that B can arise in the quantum double of q-deformed enveloping algebras. This does not necessarily mean that the q-deformed enveloping algebra is the wanted Hopf algebra, which we hope would give us the S matrix from 5.2.6 including the prefactor. We hope that B can also be obtained for other double constructions, e.g. Yangian doubles.

One difficulty in identifying the correct Hopf algebra for the $su(2|2)$ symmetric spin chain is that it seems its S matrix does not fall into one of the standard mathematical classification schemes, see [86] for some recent discussions. In particular, the S matrix is not seen to be a rational, trigonometric or elliptic solution of the Yang Baxter equation, hence one might wonder if it can be at all related to the standard Hopf algebras, i.e. Yangians, q-deformed affine algebras or elliptic quantum groups. We know that there are some Yangian charges in the system [15], [16], and should note that the $su(2|2)$ symmetric S matrix is special in the sense that its central charge eigenvalues are linked to the momenta of the scattered magnons, and it depends on the momenta of both scattered magnons separately. This is not the standard case, at least for a fully relativistic theory, where the S matrix can by Lorentz invariance only depend on the difference of the rapidities. For string theory, Lorentz invariance is broken since one usually works in the light cone gauge. Can we possibly still work with one of the standard Hopf algebras, and get the nontrivial momentum dependence automatically via the representation labels?

One question one should definitely answer, whichever construction will work at the end, is which role is played by the automorphisms. It seems that the q-deformation of $psu(2|2) \ltimes \mathbb{R}^3$, the Yangian and also the R matrix which we presented in chapter 6 needs at least the automorphism j. As we have seen in chapter 5, to represent j we have to work with an infinite dimensional representation. In case j is indispensable, we should study this infinite dimensional representation further. Another option might be that one can remove j from the universal R matrix, instead picking up some twist, which would probably also modify the Yang Baxter equation. A similar thing is done for quantum affine algebras, where one often wants to remove the derivation from the universal R matrix. The derivation of an affine Lie algebra indeed plays a similar role as j does for $su(2|2)$, with two main differences: The algebra $sl(2) \ltimes psu(2|2) \ltimes \mathbb{R}^3$ with all automorphisms and central elements is still finite dimensional, whereas the loop algebra of a Lie algebra including the automorphism and the central charge is infinite dimensional. Furthermore, the centre of $psu(2|2) \ltimes \mathbb{R}^3$ is three dimensional and not only one dimensional, as for the affine algebra. In fact, affine algebras can in a certain sense be used to introduce a spectral parameter into ordinary Lie algebra symmetry. Here, we already have spectral parameters for the finite dimensional $psu(2|2) \ltimes \mathbb{R}^3$. Do we need an affinisation anyway? We want to continue to investigate those questions of the algebraic origin of integrable structures in the AdS/CFT correspondence further in the future. Even if it might be difficult to apply standard classification theorems, we think it will be worth the effort. At the end one might get out only the known S matrix with the prefactor, which was written down in [23], [24], but is in no way proved to be the only correct one, hence deriving the prefactor rigorously would still be an important thing.

Chapter 8

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Diplomarbeit selbständig sowie ohne unerlaubte fremde Hilfe verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

Mit der Auslage meiner Diplomarbeit in den Bibliotheken der Humboldt Universität zu Berlin bin ich einverstanden.

Berlin, 15. Februar 2007 Fabian Spill

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