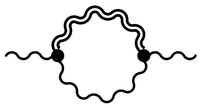

One-Loop Divergences of the Yang-Mills Theory Coupled to Gravitation

(revised version of August 20, 2008)

Diplomarbeit
zur Erlangung des akademischen Grades
des Diplom-Physikers (Dipl.-Phys.)

eingereicht von
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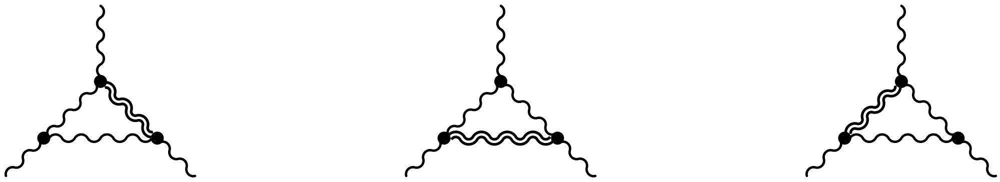
in Berlin-Lichtenberg



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Berlin, den 28. September 2007

Zusammenfassung

Diese Arbeit untersucht den Beitrag der Quantengravitation zu den Ein-Schleifen-Divergenzen abelscher und nicht-abelscher Eichtheorien. Ein besonderes Augenmerk der Arbeit liegt hierbei auf dem Einfluss der Gravitation auf das Laufen der Kopplungskonstante. Anders als vorangegangene Arbeiten wird statt der Hintergrundfeld-Methode ein diagrammatischer Ansatz verwendet. Zur Bestimmung der Divergenzen werden sowohl die Cut-Off- als auch die dimensionale Regularisierung angewandt, was eine zusätzliche Überprüfung der Ergebnisse ermöglicht.

Die Rechnungen zeigen, dass die Kopplungskonstante der Eichtheorie *keine* gravitativen Korrekturen erfährt und alle gravitativen Divergenzen durch einen einzigen Konterterm kompensiert werden können, welcher auf der Massenschale verschwindet und durch Feldredefinition des Vektorpotentials entfernt werden kann.

Abstract

This thesis examines the quantum gravity contributions to the one-loop divergences of Abelian and non-Abelian gauge theories. A special focus is on the gravitational influence on the running of the coupling constant. In contrast to preceding works we use a diagrammatical approach and not the background field method. The divergences are calculated in both cut-off and dimensional regularization, which allows for an additional verification of the results.

The calculations show that the coupling constant receives *no* gravitational corrections and all gravitational divergences are compensable by one single counter-term, which vanishes on-shell and can be removed by a field redefinition of the gauge potential.

Hilfsmittel

Diese Diplomarbeit wurde mit $\LaTeX\,2_\varepsilon$ gesetzt. Die Grafiken wurden mit Hilfe von `feynMP` und METAPOST erstellt. Die in dieser Arbeit enthaltenen Rechnungen wurden unter Einbeziehung von FORM 3.2 (Jos Vermaseren) und MATHEMATICA 5.2 (Wolfram Research) erstellt.

Selbstständigkeitserklärkung

Hiermit erkläre ich, die vorliegende Diplomarbeit selbständig sowie ohne unerlaubte fremde Hilfe verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

Mit der Auslage meiner Diplomarbeit in den Bibliotheken der Humboldt-Universität zu Berlin bin ich einverstanden.

Berlin, den 28. September 2007

Andreas Rodigast

Acknowledgments

First of all, at this opportunity I would like to thank my family, namely my parents Marlies and Rainer, my grandmother Edeltraut and my sister Sabine for their constant support throughout my life and helping me to keep up my studies.

I wish to express my gratitude to my supervisor Prof. Jan Plefka for his time, helpful advises and encouragement.

Also, I wish to express my appreciation to Prof. Dietmar Ebert for dedicating plenty of time to support this work and for being devotedly active in teaching and reminding students to keep in mind to regard the overall picture in physics and life.

I would like to thank all former and present members of the Quantum Field Theory and String Theory group at the Humboldt University Berlin for the friendly and personal atmosphere, especially Sylvia Richter for her administrative assistance and her dedication for social concerns and Dr. Hans-Jörg Otto for maintaining our computers.

Last but no least I would like to express my cordial thanks to my office colleagues and my fellow graduands Nikolai Beck, Volker Branding, Danilo Diaz, Max Dohse, Alexander Hentschel, Ngo Than Hai, Ralf Sattler, Fabian Spill, Per Sundin, Johannes Vetter, Aiko Voigt and Konstantin Wiegandt for many fruitful discussions, some about physical and lots about other topics in life.

This diploma thesis is available at
`http://qft.physik.hu-berlin.de`.

Contents

Conventions and Symbols

We use lowercase greek letters to denote Lorentz indices and lowercase roman letters for indices of the gauge group. The background metric will always be the Minkowski metric with the siganture

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in the computations involving gravitons, two forth level tensor will be needed frequently:

$$\begin{aligned} I^{\mu\nu,\alpha\beta} &\equiv \tfrac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) \\ P^{\mu\nu,\alpha\beta} &\equiv \tfrac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta}) \end{aligned}$$

We work in natural units, i. e.

$$c = \hbar = 1$$

So derivative operator gets

$$\begin{aligned} \partial_\mu &= (\partial_0, \nabla) \\ \partial^\mu &= (\partial_0, -\nabla) \end{aligned}$$

Derivation with respect to coordinates become in momentum space

$$p_\mu = i\partial_\mu \, .$$

To indicate symmetrized and anti-symmetrized indices we will use parathese and brackets respectively:

$$T^{(\mu\nu)} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}) \qquad \text{symmetrized indices} \tag{1}$$

$$T^{[\mu\nu]} = \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}) \qquad \text{anti-symmetrized indices} \tag{2}$$

The momenta in all Feynman graphs are counted as ingoing.

I
Introduction

Modern physics describes nature by relativistic field theories. At high energy scales which corresponds to small distances the laws of quantum mechanics apply. The quantization of interacting field theories always induces divergences due to the (self-)interaction of the fields. To cancel these divergences, which are physically not present, counter-terms are introduced. In general new divergences occur at each multi-loop order, thus new counter-terms are needed. For the Lagrangian to have a finite number of different terms the counter-terms have to be proportional to terms already present in the classical Lagrangian. A theory bearing this feature is called (perturbatively) renormalizable. For a quantum field theory to be regarded as fundamental its renormalizability is essential. Non-renormalizable field theories are known and find also applications, e. g. the four-fermi of weak decays or chiral perturbation theory, but they are all effective low energy descriptions of renormalizable theories. Beyond a specific threshold energy scale the effective theories have to be substitute by these fundamental theories, in the mentioned cases these are the electron-weak theory by Glashow, Salam and Weinberg or quantum chromo dynamics.

Three of the four *fundamental forces* were formulated as renormalizable quantum field theories since the second half of the last century. But general relativity, which is found to be the accurate description of gravity at all acessable scales, still resists its quantization. The occurrence of quantum effects is expected near the Planck scale corresponding to energies $\sim 10^{19}$GeV or lengths $\sim 10^{-33}cm$.

The underlying fundamental theory for general relativity – quantum gravity – has still not been found. Great advances in this direction were achieved in supersymmetric string theory.

The non-renormalizability of pure quantized general relativity was pointed out first by 't Hooft and Veltman [1]. The inclusion of matter fields, thoroughly examined by Deser, van Nieuwenhuizen et. al. [3, 4, 5, 6], also does not improve the situation.

Despite its non-renormalizability the results of a nonperturbative ansatz by Reuter[7, 8, 9, 10] indicate that quantized general relativity is asymptotically save.

Quantum general relativity as an effective field theory was advocated by Donoghue [11, 12, 13]. The energies of physics accessable by present day's experiments are well below the Planck scale, thus the effective theory should be applicable.

In the last year Robinson and Wilczek [14] initiated a discussion on effective quantum gravity coupled to gauge theories. They utilized the formulation of quantum gravity as an effective field theory to compute the running of the Yang-Mills coupling g in the Einstein-Yang-Mills system. Their background field calculation of the Calan-Symanzik β-function yields the result

$$\beta_g(g,E) = -\frac{b_0}{16\pi^2}g^3 + \frac{a_0}{16\pi^2}g\kappa^2E^2 \tag{1.1}$$

with a non-vanishing $a_0 = -3/2$. This would render any theories – including Abelian gauge theories – asymptotically free, when the energy E approaches the Planck mass. Their a_0 originates from quadratic divergences at one-loop order. This result appears to be a wonderful example of a successfull application of the effective quantum gravity, however Pietrykowski [16] and recently Toms [17] doubted this result. Pietrykowski reconsidered the calculations of [14], but in an alternative gauge of the graviton field. Toms utilized a sightly different background-field method developed by DeWitt, which ensures the independence of the results from gauge conditions. Deser, Tsao and van Nieuwenhuizen already studied the Einstein-Yang-Mills system in 1974 [6] using dimensional regularization. The only gravitational contribution to the pure Yang-Mills sector they found was a dimension-six counter-term $\sim (DF)^2$.

In 1975, Berends and Gastmans [18] examined QED coupled to gravity. They found no gravitational influence on the vertex function, too.

Outline

This thesis is organized as follows: First in chapter 2 we will introduce the Lagrangian of the Einstein-Yang-Mills theory and derive the linearized gravity. We will describe how to derive the Feynman rules for the gluon–graviton interaction. These rules will be needed to compute the one-loop pertubations in chapter 4.

Before the divergent diagrams are calculated, we will introduce the new dimesion-six terms entering the Einstein-Yang-Mills Lagrangian due to the quantum corrections in chapter 3. The new terms will be necessary due to the existence of a coupling with negative mass dimension.

In chapter 4 we will present the divergent as well as the finite parts of the one-loop graphs at order g^2 and κ^2 for two external gluons and g^3 and $g\kappa^2$ for three external gluons. The diagrams consisting solely of Yang-Mills entities are used to check the applied methods by comparing their results with the known literature values [19]. The regularization of the diagrams, which will be necessary because of the diverging integrals of the loop momenta, will be done in two schemes: in cut-off and in dimensional regularization. The first one will reveal all divergences including the quadratic ones, but its results can violate the Slavnov-Taylor-Ward identity required by the gauge symmetry. So parts of its outcome have to be considered as artefacts of its deficiency and have to be dropped. The later scheme is more elegant and respects the gauge symmetry, but it will not unveil the quadratic divergences, a major shortcoming, especially because the found quadratic divergences are the main result of Robinson and Wilczek [14]. In our results no quadratical divergences will remain, in accordance with [6, 16, 17]. The logarithmical divergences in both regularization schemes will be found to be identical.

In chapter 5 the counter-terms to cancel the obtained one-loop divergences are determined. To cancel the found pure Yang-Mills divergences we will need the known Yang-Mills counter-terms. The gravitational divergence will correspond to dimension-six terms as introduces in chapter 3. Our result will agree with the result of [6], now obtained in a diagrammatical approach.

Finally chapter 6 is devoted to the β-functions of the Yang-Mills coupling and the new dimension-six couplings.

II

Derivation of the Feynman Rules

Although the Einstein-Yang-Mills system is, as our calculations will support, not renormalizable, quantum gravity is a workable and powerful effective field theory[13] for energies well below the Planck scale $\sim 10^{19}$GeV. We will compute the first perturbative corrections of two- and three-gluon amplitude. This is done diagrammatically using Feynman graphs of the effective quantum gravity coupled to the fields of the considered gauge theory. To write down the graphs we first have to derive the Feynman rules for the linearized Einstein-Yang-Mills theory. These will result from the classical Einstein-Yang-Mills Lagrangian. The quantum corrections will lead to new terms of higher mass dimension, which will be discussed in chapter 3.

We start with the Yang-Mills Lagrangian in an arbitrary metric $\mathbf{g}_{\mu\nu}$

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{2}\sqrt{-\mathbf{g}}\,\mathbf{g}^{\mu\rho}\mathbf{g}^{\nu\sigma}\,\mathrm{tr}\,[F_{\mu\nu}F_{\rho\sigma}] \tag{2.1}$$

with the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$

which will provide the coupling terms of gravitons and gauge bosons.

Analogous, we can consider the Einstein-Maxwell theory with the Maxwell Lagrangian

$$\mathcal{L}_{\mathrm{Maxwell}} = -\tfrac{1}{4}\sqrt{-\mathbf{g}}\,\mathbf{g}^{\mu\rho}\mathbf{g}^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}$$

The field strength tensor in this case is simply $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, thus the photon field is free and no analogon to the three-gluon vertex exists. Yet, the two-gluon results in order κ^2 can be adopted without modification.

Furthermore the graviton propagator is derived from the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\mathrm{EH}} = \frac{2}{\kappa^2}\sqrt{-\mathbf{g}}\,\mathbf{R} \tag{2.2}$$

of the complete metric $\mathbf{g}_{\mu\nu}$ with the Ricci scalar $\mathbf{R}$.

The sum of both

$$\mathcal{L}_{\mathrm{EYM}} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{EH}}$$

forms the Einstein-Yang-Mills Lagrangian which is the classical foundation of our quantum calculations.

All additional terms of the Langragian due to the quantization of the gauge field – gauge fixing and ghost field terms – will not be introduced before the linearization of gravity, otherwise the gravitational coupling would become gauge dependent. Separating the generators t^a of the gauge group: $F_{\mu\nu} = F^a_{\mu\nu}t^a$, one can use the commutator relation and write

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

using the structure constants f^{abc}. The trace over the generators in fundamental representation yields $\operatorname{tr}\left[t^a t^b\right] = \frac{1}{2}\delta^{ab}$ for all $SU(N)$ gauge groups. Hence the trace in the Lagrangian can be written as

$$-\frac{1}{2}\operatorname{tr}\left[F_{\mu\nu}F_{\rho\sigma}\right] = -\frac{1}{4}F^a_{\mu\nu}F^a_{\rho\sigma}$$

The metric tensor $\mathbf{g}_{\mu\nu}$ is split in a fix part $g_{\mu\nu}$ and a dynamical part $h_{\mu\nu}$:

$$\mathbf{g}_{\mu\nu} \equiv g_{\mu\nu} + \kappa h_{\mu\nu}\,. \tag{2.3}$$

For our computations we will restrict ourselves to an background which is the flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$. The dynamic part $h_{\mu\nu}$ is our graviton field. The coefficient κ is the gravitational coupling constant which is proportional to the squareroot of the Newton constant $\kappa = \sqrt{32\pi G}$ and is in natural units basically the inverse Planck mass $\kappa = \sqrt{32\pi}/M_{\mathrm{Pl}}$.

Now all metric dependent quantities can be expanded in κ. Here we will only need the expansions up to quadratic order because at one-loop level and without external graviton lines no higher order interactions will occur. For (2.1) we need the expansions of the square root of the metric's determinant and of the inverse metric which are

$$\begin{aligned}\sqrt{-\mathbf{g}} &= 1 + \tfrac{\kappa}{2}h + \tfrac{\kappa^2}{8}\left(h^2 - 2h^{\alpha\beta}h_{\alpha\beta}\right) + \mathcal{O}\left(\kappa^3\right)\\ \mathbf{g}^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha}{h_\alpha}^{\nu} + \mathcal{O}\left(\kappa^3\right)\end{aligned}$$

where $h = h^\alpha_\alpha$ and indices are raised and lowered by the background metric $\eta_{\mu\nu}$.

The zeroth order of (2.1) reproduces the Yang Mills Langrangian in flat space and therefore the pure Yang Mills theory. This part can be quantized in the usual way[20, 21], which will lead to the common Feynman rules, compiled in Appendix C. The higher orders correspond to interaction terms between gauge bosons and one or more gravitons. The expansion of (2.2) will yields at zeroth order the graviton propagator and in higher orders its self interactions which are not needed for our considerations.

2.1 Technique to acquire the Rules

The Feynman rules for the vertices are all derived by using the computer algebra system Form[22] in version 3.2 by Jos Vermaseren. This system allows to manipulate complex tensorlike mathematical expressions. Therefore it can handle the expressions with comparatively small effort but without loss of accuracy. Form e. g. automatically evaluates sums over doubled indices and allows to factor out and substitute vector-, tensor-like and scalar entities.

The input for the scripts are the contributing terms of the Lagrangian, viz. the term with two or three gluon fields. It is written in momentum space substituting the spatial derivations by the momentum of the corresponding field:

$$\partial_\mu \to -ip_\mu\,.$$

The appearing gluon fields are differently tagged as A_1, A_2, A_3, so that the expressions have to be symmetrized afterwards. This is necessary to distinguish the momenta and to account for the group indices which are not treated in the FORM script but only separately by hand. So for example the two-gluon vertices are derived from

$$\sqrt{-\mathbf{g}}\,\mathbf{g}^{\mu\rho}\mathbf{g}^{\nu\sigma}(-ip_\mu A_{1\,\nu} + ip_\nu A_{1\,\mu})(-iq_\rho A_{2\,\sigma} + iq_\sigma A_{2\,\rho})$$

Now we substitute the expansions in $\kappa h_{\mu\nu}$ for $\sqrt{-\mathbf{g}}$ etc. and extract the right order in the fields. For the graviton fields no distinguishing labels are needed because they carry only Lorentz indices, which are part of the FORM objects, and no derivatives of h appear in the interaction terms. Hence the vertex expressions depend on the gravitons only by their Lorentz indices. Since the vertices result from expansion of the action exponential $e^{iS} \simeq 1 + iS$, the corresponding terms of the Lagrangian are all multiplied by i to obtain the vertex expressions.

The expressions obtained this way are symmetrized in the two Lorentz indices of each involved graviton field. Now all permutations of the equivalent fields of the vertex are added, viz. symmetrized in the fields without weighting factor. The chosen separate treatment of the group structure has no consequences for the two-gluon terms because the group indices of the fields are carried by an symmetric tensor, which can be and is chosen to be a Kronecker delta δ^{ab}. The three-gluon terms on the other hand are always multiplied by the the antisymmetric structure constant f^{abc}. To allow for this, we antisymmetrize the group index free FORM expression whereby the complete vertex will be symmetric in the gluon fields.

To verify the procedure, the more simple rules for the vertices with two vector boson were also calculated by hand. Additionly the program delivered the three-gluon vertex without any coupling to gravity. The identity of this result to the well known rule Appendix C also confirms our algorithm.

The graviton propagator was acquired completely by hand from the Einstein-Hilbert Lagrangian (2.2).

The FORM expressions for the vertex rules are subsequently passed to the scripts which will calculate the one-loop diagrams, see 4.2.

2.2 Vertices with two Vector Bosons

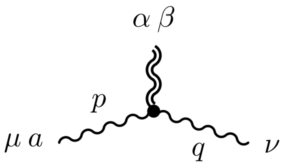
The contributing part of the Lagrangian for the vertices with two gauge bosons is

$$-\sqrt{-\mathbf{g}}\,\mathbf{g}^{\mu\rho}\mathbf{g}^{\nu\sigma}\partial_\mu A^a_\nu\,\partial_{[\rho}A^a_{\sigma]}\,.$$

The summand of it's expansion at order κ

$$\kappa\left(h^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\rho}h^{\nu\sigma} - \tfrac{1}{2}h\,\eta^{\mu\rho}\eta^{\nu\sigma}\right)\partial_\mu A^a_\nu\,\partial_{[\rho}A^a_{\sigma]}$$

corresponds to the two bosons one graviton vertex. The method described above leads to the Feynman rule



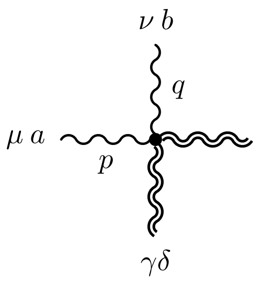
$$= -i\kappa\delta^{ab}\big[P^{\mu\nu,\alpha\beta}p\cdot q+\eta^{\mu\nu}p^{(\alpha}q^{\beta)}+\tfrac{1}{2}\eta^{\alpha\beta}p^{\nu}q^{\mu} \\ -p^{\nu}\eta^{\mu(\alpha}q^{\beta)}-q^{\mu}\eta^{\nu(\alpha}p^{\beta)}\big]\,. \tag{2.4}$$

Here $P^{\mu\nu,\alpha\beta}\equiv\frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta})$ is a four tensor which will frequently appear in all calculations involving the graviton.

Analogously the κ^2 term

$$-\kappa^2\big(h^{\mu\rho}h^{\nu\sigma}+h^{\mu\alpha}h_{\alpha}{}^{\rho}\eta^{\nu\sigma}+\eta^{\mu\rho}h^{\nu\alpha}h_{\alpha}{}^{\sigma}-\tfrac{1}{2}h\,(h^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\rho}h^{\nu\sigma}) \\ +\tfrac{1}{8}(h^2-2h^{\alpha\beta}h_{\alpha\beta})\eta^{\mu\rho}\eta^{\nu\sigma}\big)\partial_{\mu}A^a_{\nu}\,\partial_{[\rho}A^a_{\sigma]}$$

leads us to the two bosons two gravitons vertex:



$$= \frac{i}{2}\kappa^2\delta^{ab}\big[(p^{\nu}q^{\mu}-p\cdot q\,\eta^{\mu\nu})P^{\alpha\beta,\gamma\delta} \\ +p\cdot q(I^{\mu\nu,\alpha\gamma}\eta^{\beta\delta}+I^{\mu\nu,\alpha\delta}\eta^{\beta\gamma} \\ +I^{\mu\nu,\beta\gamma}\eta^{\alpha\delta}+I^{\mu\nu,\beta\delta}\eta^{\alpha\gamma} \\ -I^{\mu\nu,\alpha\beta}\eta^{\gamma\delta}-I^{\mu\nu,\gamma\delta}\eta^{\alpha\beta}) \\ +2p^{(\alpha}q^{\beta)}P^{\mu\nu,\gamma\delta}+2p^{(\gamma}q^{\delta)}P^{\mu\nu,\alpha\beta} \\ -p^{\nu}(q^{\alpha}P^{\mu\beta,\gamma\delta}+q^{\beta}P^{\alpha\mu,\gamma\delta} \\ +q^{\gamma}P^{\alpha\beta,\mu\delta}+q^{\delta}P^{\alpha\beta,\gamma\mu}) \\ -q^{\mu}(p^{\alpha}P^{\nu\beta,\gamma\delta}+p^{\beta}P^{\alpha\nu,\gamma\delta} \\ +p^{\gamma}P^{\alpha\beta,\nu\delta}+p^{\delta}P^{\alpha\beta,\gamma\nu}) \\ +p^{\alpha}q^{\gamma}\eta^{\mu[\nu}\eta^{\delta]\beta}+p^{\gamma}q^{\alpha}\eta^{\mu[\nu}\eta^{\beta]\delta} \\ +p^{\alpha}q^{\delta}\eta^{\mu[\nu}\eta^{\gamma]\beta}+p^{\delta}q^{\alpha}\eta^{\mu[\nu}\eta^{\beta]\gamma} \\ +p^{\beta}q^{\gamma}\eta^{\mu[\nu}\eta^{\delta]\alpha}+p^{\gamma}q^{\beta}\eta^{\mu[\nu}\eta^{\alpha]\delta} \\ +p^{\beta}q^{\delta}\eta^{\mu[\nu}\eta^{\gamma]\alpha}+p^{\delta}q^{\beta}\eta^{\mu[\nu}\eta^{\alpha]\gamma}\big]\,. \tag{2.5}$$

Additional to $P^{\mu\nu,\alpha\beta}$ here $I^{\mu\nu,\alpha\beta}\equiv\frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha})$ is introduced.

2.3 Vertices with three Vector Bosons

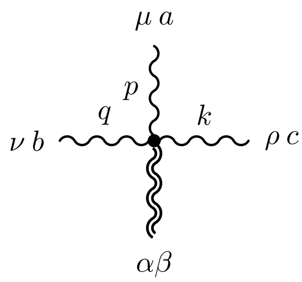
The three gauge boson term

$$-\tfrac{1}{2}g\sqrt{-\mathbf{g}}\,\mathbf{g}^{\mu\rho}\mathbf{g}^{\nu\sigma}f^{abc}\partial_{\mu}A^a_{\nu}\,A^b_{\rho}A^c_{\sigma}$$

is at order κ

$$\tfrac{1}{2}g\kappa\left(h^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\rho}h^{\nu\sigma}-\tfrac{1}{2}h\,\eta^{\mu\rho}\eta^{\nu\sigma}\right)f^{abc}\partial_{\mu}A^a_{\nu}\,A^b_{\rho}A^c_{\sigma}\,,$$

so the vertex with one graviton is

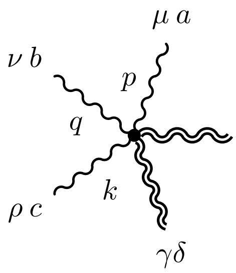


$$
\begin{aligned}
= -g\kappa f^{abc}\Big[&P^{\alpha\beta,\mu\nu}(p-q)^\rho \\
&+P^{\alpha\beta,\nu\rho}(q-k)^\mu \\
&+P^{\alpha\beta,\rho\mu}(k-p)^\nu \\
&+\eta^{\mu\nu}\eta^{\rho(\alpha}(p-q)^{\beta)} \\
&+\eta^{\nu\rho}\eta^{\mu(\alpha}(q-k)^{\beta)} \\
&+\eta^{\rho\mu}\eta^{\nu(\alpha}(k-p)^{\beta)}\Big]\,.
\end{aligned}
\tag{2.6}
$$

At order κ^2 we have

$$
\begin{aligned}
-\tfrac{1}{2}g\kappa^2\big(&h^{\mu\rho}h^{\nu\sigma} + h^{\mu\alpha}h_\alpha{}^\rho\eta^{\nu\sigma} + \eta^{\mu\rho}h^{\nu\alpha}h_\alpha{}^\sigma - \tfrac{1}{2}h\,(h^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\rho}h^{\nu\sigma}) \\
&+ \tfrac{1}{8}(h^2 - 2h^{\alpha\beta}h_{\alpha\beta})\eta^{\mu\rho}\eta^{\nu\sigma}\big)f^{abc}\partial_\mu A^a_\nu\, A^b_\rho A^c_\sigma
\end{aligned}
$$

and the vertex rule becomes



$$
\begin{aligned}
\alpha\beta = \frac{1}{2}g\kappa^2 f^{abc}\big[&(p-q)^\alpha(\eta^{\mu\nu}P^{\rho\beta,\gamma\delta} + \eta^{\rho\beta}I^{\mu\nu,\gamma\delta}) \\
&+(p-q)^\beta(\eta^{\mu\nu}P^{\alpha\rho,\gamma\delta} + \eta^{\alpha\rho}I^{\mu\nu,\gamma\delta}) \\
&+(p-q)^\gamma(\eta^{\mu\nu}P^{\alpha\beta,\rho\delta} + \eta^{\rho\delta}I^{\mu\nu,\alpha\beta}) \\
&+(p-q)^\delta(\eta^{\mu\nu}P^{\alpha\beta,\gamma\rho} + \eta^{\gamma\rho}I^{\mu\nu,\alpha\beta}) \\
&+(p-q)^\rho(I^{\mu\nu,\alpha\gamma}\eta^{\beta\delta} + I^{\mu\nu,\alpha\delta}\eta^{\beta\gamma} \\
&\qquad +I^{\mu\nu,\beta\gamma}\eta^{\alpha\delta} + I^{\mu\nu,\beta\delta}\eta^{\alpha\gamma} \\
&\qquad -I^{\mu\nu,\alpha\beta}\eta^{\gamma\delta} - I^{\mu\nu,\gamma\delta}\eta^{\alpha\beta} \\
&\qquad -\eta^{\mu\nu}P^{\alpha\beta,\gamma\delta}) \\
&+(q-k)^\alpha(\eta^{\nu\rho}P^{\mu\beta,\gamma\delta} + \eta^{\mu\beta}I^{\nu\rho,\gamma\delta}) \\
&+(q-k)^\beta(\eta^{\nu\rho}P^{\alpha\mu,\gamma\delta} + \eta^{\alpha\mu}I^{\nu\rho,\gamma\delta}) \\
&+(q-k)^\gamma(\eta^{\nu\rho}P^{\alpha\beta,\mu\delta} + \eta^{\mu\delta}I^{\nu\rho,\alpha\beta}) \\
&+(q-k)^\delta(\eta^{\nu\rho}P^{\alpha\beta,\gamma\mu} + \eta^{\gamma\mu}I^{\nu\rho,\alpha\beta}) \\
&+(q-k)^\mu(I^{\nu\rho,\alpha\gamma}\eta^{\beta\delta} + I^{\nu\rho,\alpha\delta}\eta^{\beta\gamma} \\
&\qquad +I^{\nu\rho,\beta\gamma}\eta^{\alpha\delta} + I^{\nu\rho,\beta\delta}\eta^{\alpha\gamma} \\
&\qquad -I^{\nu\rho,\alpha\beta}\eta^{\gamma\delta} - I^{\nu\rho,\gamma\delta}\eta^{\alpha\beta} \\
&\qquad -\eta^{\nu\rho}P^{\alpha\beta,\gamma\delta}) \\
&+(k-p)^\alpha(\eta^{\rho\mu}P^{\nu\beta,\gamma\delta} + \eta^{\nu\beta}I^{\nu\rho,\gamma\delta}) \\
&+(k-p)^\beta(\eta^{\rho\mu}P^{\alpha\nu,\gamma\delta} + \eta^{\alpha\nu}I^{\nu\rho,\gamma\delta}) \\
&+(k-p)^\gamma(\eta^{\rho\mu}P^{\alpha\beta,\nu\delta} + \eta^{\nu\delta}I^{\nu\rho,\alpha\beta}) \\
&+(k-p)^\delta(\eta^{\rho\mu}P^{\alpha\beta,\gamma\nu} + \eta^{\gamma\nu}I^{\nu\rho,\alpha\beta}) \\
&+(k-p)^\nu(I^{\rho\mu,\alpha\gamma}\eta^{\beta\delta} + I^{\rho\mu,\alpha\delta}\eta^{\beta\gamma} \\
&\qquad +I^{\rho\mu,\beta\gamma}\eta^{\alpha\delta} + I^{\rho\mu,\beta\delta}\eta^{\alpha\gamma} \\
&\qquad -I^{\rho\mu,\alpha\beta}\eta^{\gamma\delta} - I^{\rho\mu,\gamma\delta}\eta^{\alpha\beta} \\
&\qquad -\eta^{\rho\mu}P^{\alpha\beta,\gamma\delta})\big]\,.
\end{aligned}
\tag{2.7}
$$

2.4 The Graviton Propagator

Now the propagator of the graviton $h_{\mu\nu}$ has to be constructed. Its derivation follows [13].

The graviton propagator arises from the Einstein-Hilbert Lagrangian (2.2). For its quadratic expansion in κ we need the expansion of the Ricci scalar which is

$$\begin{aligned}\mathbf{R} = \kappa\left(\Box h - \partial_\mu\partial_\nu h^{\mu\nu}\right) \\ + \kappa^2\big(\tfrac{1}{4}\partial_\mu h\partial^\mu h - \partial_\mu h^{\mu\nu}\partial_\nu h + \partial_\mu h^{\mu\nu}\partial^\rho h_{\nu\rho} - \tfrac{3}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} \\ + \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + 2h^{\mu\nu}\partial_\mu\partial^\rho h_{\rho\nu} - h^{\mu\nu}\Box h_{\mu\nu}\big) + \mathcal{O}\left(\kappa^3\right) .\end{aligned}$$

Insertion in (2.2) and partial integration yields

$$\begin{aligned}\mathcal{L}_{\mathrm{EH}} =& \frac{2}{\kappa^2}\sqrt{-\mathbf{g}}\,\mathbf{R} \\ =& \tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \tfrac{1}{2}\partial_\mu h\partial^\mu h + \partial_\mu h^{\mu\nu}\partial_\nu h - \partial_\mu h^{\mu\nu}\partial^\rho h_{\rho\nu} + \mathcal{O}\left(\kappa\right) .\end{aligned}$$

To gain a quantum theory of the linearized gravity we use the Faddeev-Popov quantization scheme[20]. The chosen gauge for the propagator is the harmonic (de Donder) gauge which takes the form

$$\begin{aligned} 0 =& G_\mu \\ \text{with} \quad G_\mu =& \partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h\end{aligned}$$

in flat Minkowski background. So we need to add the gauge fixing term[1]

$$\begin{aligned}\mathcal{L}_{\text{g.f.}} =& G_\mu G^\mu \\ =& (\partial^\nu h_{\mu\nu} - \tfrac{1}{2}\partial_\mu h)(\partial_\rho h^{\mu\rho} - \tfrac{1}{2}\partial^\mu h)\end{aligned}$$

to the Lagrangian. We also have to introduce a gravitational ghost field, which is a vector like field with fermionic statistics. b^μ:

$$\mathcal{L}_{\mathrm{gh}} = b^{*\mu}\left(\kappa\frac{\delta G_\mu}{\delta\varepsilon^\nu}\right)b^\nu .$$

Under general coordniate transformation $x^\mu \to x^\mu - \varepsilon^\mu(x)$ the *complete* metric transforms as a tensor:

$$\mathbf{g}'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x) .$$

Therefore the graviton field must transform according

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\alpha h_{\mu\nu}\varepsilon^\alpha + (\tfrac{1}{\kappa}\eta_{\alpha\nu} + h_{\alpha\nu})\partial_\mu\varepsilon^\alpha + (\tfrac{1}{\kappa}\eta_{\mu\alpha} + h_{\mu\alpha})\partial_\nu\varepsilon^\alpha$$

in order to keep the background metric fixed. The resulting behavior of the gauge fixing expression G_μ yields the gravitational ghost Lagrangian

$$\mathcal{L}_{\mathrm{gh}} = b^{*\mu}\left(\eta_{\mu\nu}\Box + \kappa\left(h_{\mu\nu}\Box + \partial_\nu G_\mu + G_\nu\partial_\mu + (\partial_\nu h_{\alpha\mu} - \partial_\mu h_{\alpha\nu} + \partial_\alpha h_{\mu\nu})\partial^\alpha\right)\right)b^\nu .$$

The interaction terms do not interfere with our consideration because we are not interested in processes with outer gravitons, therefore graviton–ghost vertices will first appear at two-loop order, but only the one-loop contributions to Yang-Mills fields are examined.

Integration by parts and collection of the terms leads to the final very simple form of the kinetic part of the gravitational action:

$$S_{\text{grav}} = \int \mathrm{d}^d x \tfrac{1}{2}\partial^\alpha h_{\mu\nu} \left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\right) \partial_\alpha h_{\rho\sigma} + b^{\mu *}\Box b_\mu + \mathcal{L}_{\text{grav}}^{\text{int}} \tag{2.8}$$

which agrees with Donoghue's action[13] in the considered limit of flat background metric $\eta_{\mu\nu}$. Now we easily derive the graviton propagator

$$\alpha\beta \;\underset{p}{\sim\!\sim\!\sim}\; \gamma\delta \;=\; \frac{i\left(\frac{1}{2}(\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\beta\gamma}) - \frac{1}{d-2}\eta^{\alpha\beta}\eta^{\gamma\delta}\right)}{p^2 + i\varepsilon} \tag{2.9}$$

which is also in agreement with [13]. We keep the spacetime dimension d in the formula which allows to include it in dimensional regularization. The ε is introduced to avoid the branch cut in expressions containing momenta and must not be mistaken for the ϵ which will appear in the dimensional renormalization scheme.

Together with the well known rules for the pure Yang Mills theory – collected in Appendix C – we now have all rules we need to do the computation of the one-loop perturbations up to order κ^2.

III

New Terms of the Einstein-Yang-Mills Theory

The renormalization of the one-loop divergences in the Einstein-Yang-Mills system will necessitate counter-terms of order κ^2. Due to the mass dimension of κ which is not zero but minus one in four dimensional space, the logarithmic divergences cannot be canceled by a $\mathrm{tr}\, F^2$ term. Such a term already has the mass dimension four, thus its coupling must be a dimensionless parameter. Quadratic divergences of the form $\kappa^2\Lambda^2$ however would be canceled by such terms as described by Robinson and Wilczek [14].

To cancel the logarithmic divergences we need new dimension-six terms with couplings of the same mass dimension as κ^2 or $g\kappa^2$. These terms will enter the Lagrangian additionally to $\mathcal{L}_{\mathrm{YM}}$ and $\mathcal{L}_{\mathrm{EH}}$. The requirement of gauge invariance leaves four possible terms:

$$\begin{aligned}
\mathcal{O}_1 &= \mathrm{tr}\,[(D_\mu F_{\nu\rho})(D^\mu F^{\nu\rho})] \\
\mathcal{O}_1' &= \mathrm{tr}\,[(D_\mu F_{\nu\rho})(D^\nu F^{\mu\rho})] \\
\mathcal{O}_2 &= \mathrm{tr}\,[(D_\mu F^{\mu\rho})(D^\nu F_{\nu\rho})] \\
\mathcal{O}_3 &= i\,\mathrm{tr}\left[F_\alpha{}^\beta F_\beta{}^\gamma F_\gamma{}^\alpha\right] .
\end{aligned} \tag{3.1}$$

The second term $\mathcal{O}_1'$ is proportional to $\mathcal{O}_1$. It can be transformed to

$$\mathrm{tr}\,[(D_\mu F_{\nu\rho})(D^\nu F^{\mu\rho})] = \frac{1}{2}\,\mathrm{tr}\,[(D_\mu F_{\nu\rho})(D^\mu F^{\nu\rho})] . \tag{3.2}$$

using the Bianchi identity $D_{[\alpha}F_{\beta\gamma]} = 0$. Thus we will not denote in separately.

Also the three remaining terms are not independent in the action integral. One term can be eliminated and expressed by the others plus total derivatives. The latter are purely topological, thus uninteresting in the examined flat space time. We can for example express $\mathcal{O}_2$ by $\mathcal{O}_1$ and $\mathcal{O}_3$:

$$\begin{aligned}
\mathcal{O}_2 &= \mathrm{tr}\left[D_\alpha F^{\alpha\gamma} D_\beta F^\beta{}_\gamma\right] \\
&= \mathrm{tr}\big[\partial_\alpha F^{\alpha\gamma}\partial_\beta F^\beta{}_\gamma \\
&\qquad -ig[A_\alpha, F^{\alpha\gamma}]\partial_\beta F^\beta{}_\gamma - ig\partial_\alpha F^{\alpha\gamma}[A_\beta, F^\beta{}_\gamma] \\
&\qquad -g^2[A_\alpha, F^{\alpha\gamma}][A_\beta, F^\beta{}_\gamma]\big] \\
&= \mathrm{tr}\big[\partial_\beta F^{\alpha\gamma}\partial_\alpha F^\beta{}_\gamma \\
&\qquad +ig\partial_\beta[A_\alpha, F^{\alpha\gamma}]\partial_\beta F^\beta{}_\gamma + igF^{\alpha\gamma}\partial_\alpha[A_\beta, F^\beta{}_\gamma] \\
&\qquad -g^2[A_\alpha, F^{\alpha\gamma}][A_\beta, F^\beta{}_\gamma]\big] + \text{total derivatives}
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{tr}\big[\partial^\beta F^{\alpha\gamma}\partial_\alpha F_{\beta\gamma} \\
&\quad -ig[A^\beta, F^{\alpha\gamma}]\partial_\alpha F_{\beta\gamma} - ig\partial^\beta F^{\alpha\gamma}[A_\alpha, F_{\beta\gamma}] \\
&\quad +ig(\partial^\beta A_\alpha - \partial_\alpha A^\beta)[F^{\alpha\gamma}, F_{\beta\gamma}] \\
&\quad -g^2([A^\beta, F^{\alpha\gamma}][A_\alpha, F_{\beta\gamma}] + [A^\beta, A_\alpha][F^{\alpha\gamma}, F_{\beta\gamma}])\big] + \text{t. d.} \\
&= \operatorname{tr}\big[(\partial^\beta F^{\alpha\gamma} - ig[A^\beta, F^{\alpha\gamma}])(\partial_\alpha F_{\beta\gamma} - ig[A_\alpha, F_{\beta\gamma}]) \\
&\quad +ig(\partial^\beta A_\alpha - \partial_\alpha A^\beta - ig[A^\beta, A_\alpha])[F^{\alpha\gamma}, F_{\beta\gamma}]\big] + \text{t. d.} \\
&= \operatorname{tr}\big[D^\beta F^{\alpha\gamma} D_\alpha F_{\beta\gamma}\big] + ig \operatorname{tr}\big[F^\beta{}_\alpha [F^{\alpha\gamma}, F_{\beta\gamma}]\big] + \text{t. d.}
\end{aligned}
$$

The first summand is $\mathcal{O}_1' = \frac{1}{2}\mathcal{O}_1$. Hence in the action integral, where total derivatives can be neglected, the identification

$$\mathcal{O}_2 = \frac{1}{2}\mathcal{O}_1 - 2g\mathcal{O}_3 \tag{3.3}$$

is possible. We refrain from choosing the terms used in the Langrangian until we have determined the counter-term values in chapter 5. Then it will be possible to pick the most simple combination of terms, i. e. the combination with the most simple structure in the counter-terms.

In the choice of dimension-six terms one should also take into account the special nature of the second term. $\mathcal{O}_2$ is proportional to the Yang-Mills equation of motion – $D^\mu F_{\mu\nu} = 0$, hence vanishes on-shell. Also, the non-linear field redefinition

$$A_\mu \to A_\mu' = A_\mu + \frac{d_2}{2} D^\nu F_{\mu\nu} \tag{3.4}$$

removes $\mathcal{O}_2$:

$$
\begin{aligned}
-\tfrac{1}{2}\operatorname{tr}\big[(F_{\mu\nu}')^2\big] &= -\tfrac{1}{2}\operatorname{tr}\Big[(2\partial_{[\mu}A_{\nu]}' - ig[A_\mu', A_\nu'])^2\Big] \\
&= -\tfrac{1}{2}\operatorname{tr}\big[(2\partial_{[\mu}A_{\nu]} - ig[A_\mu, A_\nu] + \tfrac{d_2}{2}(2\partial_{[\mu}D^\alpha F_{\nu]\alpha} \\
&\quad - ig([A_\mu, D^\alpha F_{\nu\alpha}] - [A_\nu, D^\alpha F_{\mu\alpha}])) + \mathcal{O}(d_2^2))^2\big] \\
&= -\tfrac{1}{2}\operatorname{tr}\big[(2\partial_{[\mu}A_{\nu]} - ig[A_\mu, A_\nu])^2 \\
&\quad + 2d_2 D_{[\mu}D^\alpha F_{\nu]\alpha}(2\partial^{[\mu}A^{\nu]} - ig[A^\mu, A^\nu]) + \mathcal{O}(d_2^2)\big] \\
&= -\tfrac{1}{2}\operatorname{tr}\big[(F_{\mu\nu})^2 + 2d_2 D_\mu D^\alpha F_{\nu\alpha}F^{\mu\nu}\big] + \mathcal{O}(d_2^2) \\
&= -\tfrac{1}{2}\operatorname{tr}\big[(F_{\mu\nu})^2\big] - d_2 \operatorname{tr}[F^{\mu\nu}\partial_\mu(D^\alpha F_{\nu\alpha}) - F^{\mu\nu} ig[A_\mu, D^\alpha F_{\nu\alpha}]] \\
&\quad + \mathcal{O}(d_2^2) \\
&= -\tfrac{1}{2}\operatorname{tr}\big[(F_{\mu\nu})^2\big] - d_2 \operatorname{tr}\big[-\partial_\mu(F^{\mu\nu})D^\alpha F_{\nu\alpha} \\
&\quad - ig(F^{\mu\nu}A_\mu D^\alpha F_{\nu\alpha} - F^{\mu\nu}D^\alpha F_{\nu\alpha}A_\mu)\big] + \mathcal{O}(d_2^2) + \text{t. d.} \\
&= -\tfrac{1}{2}\operatorname{tr}\big[(F_{\mu\nu})^2\big] + d_2 \operatorname{tr}[(\partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}])D^\alpha F_{\nu\alpha}] \\
&\quad + \mathcal{O}(d_2^2) + \text{t. d.} \\
&= -\tfrac{1}{2}\operatorname{tr}\big[(F_{\mu\nu})^2\big] - d_2\mathcal{O}_2 + \mathcal{O}(d_2^2) + \text{t. d.}
\end{aligned}
$$

In the Abelian theory no traces are present and symmetry prohibits a $\mathcal{O}_3$ term. For the sake of uniformity of the results the dimension-six terms are defined as

$$\begin{aligned}\mathcal{O}_1^{\text{Maxwell}} =& \frac{1}{2}(\partial_\mu F_{\nu\rho})(\partial^\mu F^{\nu\rho}) \\ \mathcal{O}_2^{\text{Maxwell}} =& \frac{1}{2}(\partial_\mu F^{\mu\rho})(\partial^\nu F_{\nu\rho}) \\ \mathcal{O}_3^{\text{Maxwell}} =& 0\end{aligned} \tag{3.5}$$

in the Maxwell case. Consequently the identity (3.3) reduces to

$$\mathcal{O}_2^{\text{Maxwell}} = \frac{1}{2}\mathcal{O}_1^{\text{Maxwell}} + \text{t. d.} \tag{3.6}$$

These terms form the Yang-Mills part of the dimension-six Lagrangian[1], which can be written as:

$$\mathcal{L}_{\text{dim}6} = d_1\mathcal{O}_1 + d_2\mathcal{O}_2 + d_3\mathcal{O}_2 \tag{3.7}$$

The mass dimensions of the new introduced couplings d_1, d_2 and d_3 are determined from the requirement $[d_i] + [\mathcal{O}_i] = d$. Using $[D_\mu] = 1$ and $[F_{\mu\nu}] = \frac{d}{2}$ one obtains the dimensions of the $\mathcal{O}_i$'s and thus of the couplings:

$$\begin{aligned}[d_1] = [d_2] =& -2 \\ [d_3] =& -\frac{d}{2} \xrightarrow{d=4} -2\,.\end{aligned} \tag{3.8}$$

The traces over the gauge group indices in case of $\mathcal{O}_1$ and analogously $\mathcal{O}_2$ are performed in the same way as in the Yang-Mills Lagrangian's $-\frac{1}{2}\operatorname{tr}[F^2]$ case:

$$\begin{aligned}\mathcal{O}_1 =& \operatorname{tr}\left[(D_\mu F_{\nu\rho})(D^\mu F^{\nu\rho})\right] \\ =& (D_\mu F_{\nu\rho})^a (D^\mu F^{\nu\rho})^b \underbrace{\operatorname{tr}\left[t^a t^b\right]}_{=\frac{1}{2}\delta^{ab}} \\ =& \tfrac{1}{2}(D_\mu F_{\nu\rho})^a (D^\mu F^{\nu\rho})^a\end{aligned}$$

The trace in $\mathcal{O}_3$ over three field strength tensors yields

$$\begin{aligned}\mathcal{O}_3 =& i\operatorname{tr}\left[F_\alpha{}^\beta F_\beta{}^\gamma F_\gamma{}^\alpha\right] \\ =& iF_\alpha^{a\,\beta} F_\beta^{b\,\gamma} F_\gamma^{c\,\alpha} \operatorname{tr}\left[t^a t^b t^c\right] \\ =& iF_\alpha^{a\,\beta} F_\beta^{b\,\gamma} F_\gamma^{c\,\alpha} \tfrac{1}{2}\operatorname{tr}\left[\{t^a, t^b\}t^c + [t^a, t^b]t^c\right] \\ =& iF_\alpha^{a\,\beta} F_\beta^{b\,\gamma} F_\gamma^{c\,\alpha} \tfrac{1}{4}\left(d^{abc} + if^{abc}\right)\,.\end{aligned}$$

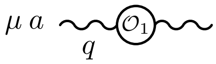
Here is f^{abc} the structure constant of the gauge group and $d^{abc} = \{t^a, t^b\}^c$ originates from the anti-commutator of the generators, so it is symmetric in its

[1]The gravitational part consistits of quadratic terms in the curvature R^2 and $R^{\mu\nu}R_{\mu\nu}$. Also mixed dimension-six terms like $R_{\mu\nu}T^{\mu\nu}$ would appear in an complete consideration as e.ġ. in [6, 4]

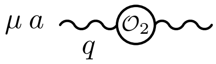
indices. $F_\alpha^{a\,\beta} F_\beta^{b\,\gamma} F_\gamma^{c\,\alpha}$ is antisymmetric in abc, thus only the contribution of f^{abc} remains in $\mathcal{O}_3$:

$$\mathcal{O}_3 = -\tfrac{1}{4} f^{abc} F_\alpha^{a\,\beta} F_\beta^{b\,\gamma} F_\gamma^{c\,\alpha}$$

Now we derive the Feynman rules for $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ in the same manner as we did for the graviton–gluon interactions in chapter 2. The $(D\,F)^2$ terms lead to the two boson vertices:

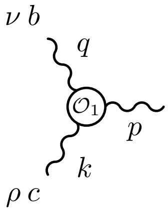


$$= 2d_1 i \delta^{ab} q^2 (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \tag{3.9}$$

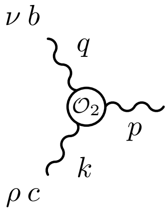


$$= d_2 i \delta^{ab} q^2 (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \tag{3.10}$$

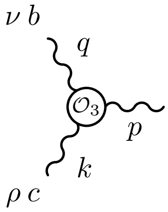
And all dimension-six terms correspond to three boson vertices:



$$\begin{aligned} = d_1 g f^{abc} \big[& \eta^{\mu\nu} (p^\rho (4p \cdot q + 2p \cdot k) - q^\rho (4q \cdot p + 2q \cdot k)) \\ & + \eta^{\nu\rho} (q^\mu (4q \cdot k + 2q \cdot p) - k^\nu (4k \cdot q + 2k \cdot p)) \\ & + \eta^{\rho\mu} (k^\nu (4k \cdot p + 2k \cdot q) - p^\nu (4p \cdot k + 2p \cdot q)) \\ & - 2 \big(k^\mu k^\nu (p^\rho - q^\rho) + p^\nu p^\rho (q^\mu - k^\mu) \\ & \quad + q^\rho q^\mu (k^\nu - p^\nu) \big) \big] \end{aligned} \tag{3.11}$$



$$\begin{aligned} = d_2 g f^{abc} \big[& \eta^{\mu\nu} (p^\rho (2p \cdot q + p \cdot k + 3q \cdot k) \\ & \quad - q^\rho (2q \cdot p + q \cdot k + 3p \cdot k)) \\ & + \eta^{\nu\rho} (q^\mu (2q \cdot k + q \cdot p + 3k \cdot p) \\ & \quad - k^\mu (2k \cdot q + k \cdot p + 3q \cdot p)) \\ & + \eta^{\rho\mu} (k^\nu (2k \cdot p + k \cdot q + 3p \cdot q) \\ & \quad - p^\nu (2p \cdot k + p \cdot q + 3k \cdot q)) \\ & - \big(k^\mu k^\nu (p^\rho - q^\rho) + p^\nu p^\rho (q^\mu - k^\mu) \\ & \quad + q^\rho q^\mu (k^\nu - p^\nu) \big) \\ & - 3 (p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu) \big] \end{aligned} \tag{3.12}$$

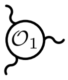


$$\begin{aligned} = \tfrac{3}{2} d_3 f^{abc} \big[& - \eta^{\mu\nu} (p^\rho q \cdot k - q^\rho k \cdot p) \\ & - \eta^{\nu\rho} (q^\mu k \cdot p - k^\nu p \cdot q) \\ & - \eta^{\rho\mu} (k^\nu p \cdot q - p^\nu q \cdot k) \\ & + (p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu) \big] \end{aligned} \tag{3.13}$$

When we match the logarithmic one-loop divergences with the new counterterms, we will see only linear combinations of the contributions of all terms. Thus the results for the sum of all terms are presented in a form which can be compared with the results we will obtain in chapter 4.

$+$

$$= i(2d_1 + d_2) \delta^{ab} q^2 (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \tag{3.14}$$



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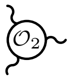

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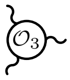


$=$

$$
\begin{aligned}
&f^{abc}\big[\eta^{\mu\nu}(p^\rho((4gd_1+2gd_2)p\cdot q+(2gd_1+gd_2)p\cdot k+(3gd_2-\tfrac{3}{2}d_3)q\cdot k)\\
&\quad - q^\rho((4gd_1+2gd_2)q\cdot p+(2gd_1+gd_2)q\cdot k+(3gd_2-\tfrac{3}{2}d_3)p\cdot k)+\ldots\\
&\qquad - g(2d_1+d_2)\,(k^\mu k^\nu(p^\rho-q^\rho)+p^\nu p^\rho(q^\mu-k^\mu)+q^\rho q^\mu(k^\nu-p^\nu))\\
&\qquad\qquad - (3gd_2-\tfrac{3}{2}d_3)(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big]
\end{aligned}
\tag{3.15}
$$

All combinations of the couplings d_1, d_2 and d_3 are in such a way that a transformation between the terms as in (3.3) would not effect the amplitudes of the combined Feynman graphs. This is the correct behavior required for a physical quantity.

IV

Divergences of the one-loop Diagrams

The aim of this chapter is to diagrammatically calculate the one-loop contributions to the gluon self-energy and the vertex correction. Additional to the contributions of the pure Yang Mills theory, Figure 4.1, in the Einstein-Yang-Mills systems gravitational one-loop diagrams, Figure 4.2, are found.

The lack of multi-photon vertices and electromagnetic ghosts in the Einstein-Maxwell system reduces the number of diagrams dramatically. Only the two gravitational contributions to the photon self-energy – Figure 4.2a) and b) – remain. The values of the pure Yang-Mills diagrams will provide a test for the applied methods by comparing them to the known results from the literature, e. g. [19].

The divegent graphs are calculated in two regularization schemes: First we use a cutoff regulator to see the expected quadratic divergences; but as the calculations will show, all quadratic terms in the cut-off momentum cancel. So we repeated the computation with the more elegant dimensional regularization.

4.1 Feynman Integrals

The one-loop diagrams needed to be evaluated for the renormalization of the theory contain one, two or three propagators.

The momentum integrals with one propagator only need to be Wick rotated and are then computed in the chosen regularization scheme. The other ones first have to be transformed using Feynman parameters. By the right choice of denotation of the propagators only two different combinations of momenta remain:

$$\frac{1}{(k^2+i\varepsilon)((k+q)^2+i\varepsilon)} = \iint_0^1 \mathrm{d}x\mathrm{d}y \frac{\delta(x+y-1)}{(k^2+2k\cdot xq+xq^2+i\varepsilon)^2} \tag{4.1}$$

$$\frac{1}{(k^2+i\varepsilon)((k+q)^2+i\varepsilon)((k-p)^2+i\varepsilon)} = 2\iiint_0^1 \mathrm{d}x\mathrm{d}y\mathrm{d}y \frac{\delta(x+y+z-1)}{(k^2-2k\cdot(yp-xq)+xq^2+yp^2+i\varepsilon)^3}\,. \tag{4.2}$$

The integrals over the parameters x, y and z will be evaluated after the momentum integration is done. The momentum integrals lead to a small number of functions of the parameters. The polynomials were of course easily integrated for the remaining – rational functions and logarithms of polynomials – we used the computer algebra system MATHEMATICA. The two dimensional integrals (4.1)

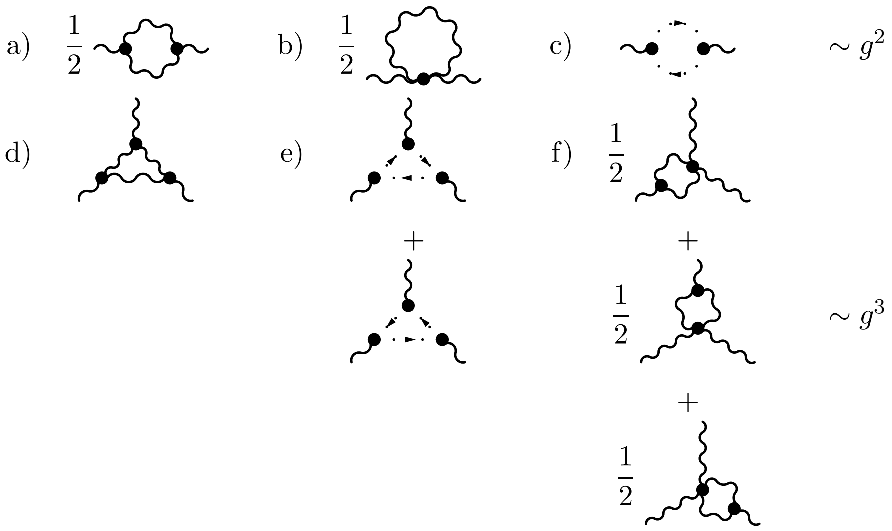


Figure 4.1: One-loop corrections of the pure Yang-Mills theory

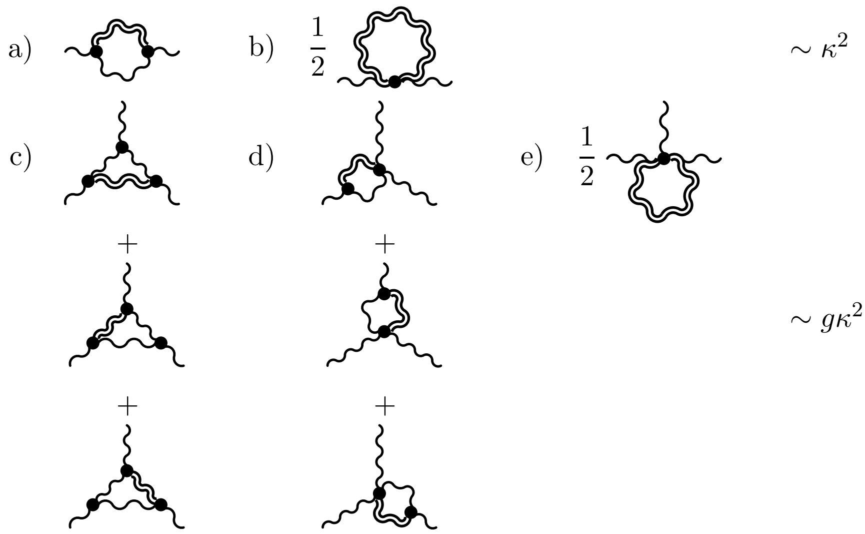


Figure 4.2: Graviton loop corrections

were easily processed. The three dimensional ones (4.2) turned out to be more complicated and were at first not evaluated neither analytically nor numerically. We solved the problem by transforming the parameter integrals as described in Appendix B to a form Mathematica could handle. In the new form we were even able to obtain an analytical result.

Of course most of the integrals are ultra-violet divergent and have to be regulated. This is done by two schemes, each with its advantages and drawbacks. The application of two different regularization methods allows to compare their results and to cross check the calculations.

4.1.1 Cut-off Regularization

The cut-off scheme is the most simple way to regulate divergent integrals in momentum space. The intergals are solved after the time component of the integration variable k_μ is Wick-rotated:

$$\begin{aligned} k_0 &= ik_{\mathrm{E}\,0} \\ \rightarrow k^2 &= -\,k_E^2 \end{aligned}$$

Now we integrate in Euclidean momentum space. The finite valued intergals can be performed directly. The divergent intergals are only taken over finite size sphere with the radius Λ, the so called cut-off momentum.

After introducing the Feynman parameters, the denominators of all momentum integrals take the form $(k^2 - 2k\cdot p + l)^n$. p_μ and l are momentum like and momentum square like expressions respectively depending on the outer momenta and the Feynman parameters. This allows us to apply (4.3)–(4.10).

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2} = \frac{-i}{16\pi^2}\Lambda^2 \tag{4.3}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2-2k\cdot p+l)^2} = \frac{i}{16\pi^2}\left\{\log\frac{\Lambda^2}{p^2-l} - 1\right\} \tag{4.4}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu}{(k^2-2k\cdot p+l)^2} = \frac{i}{16\pi^2}p_\mu\left\{\log\frac{\Lambda^2}{p^2-l} - \frac{3}{2}\right\} \tag{4.5}$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{(k^2-2k\cdot p+l)^2} &= \frac{i}{16\pi^2}\frac{1}{2}\eta_{\mu\nu}\left\{(p^2-l)\log\frac{\Lambda^2}{p^2-l} + \frac{3l-5p^2}{6} - \frac{\Lambda^2}{2}\right\} \\ &\quad + \frac{i}{16\pi^2}p_\mu p_\nu\left\{\log\frac{\Lambda^2}{p^2-l} - \frac{11}{6}\right\}\end{aligned} \tag{4.6}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2-2k\cdot p+l)^3} = \frac{-i}{16\pi^2}\frac{1}{2}\frac{1}{p^2-l} \tag{4.7}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu}{(k^2-2k\cdot p+l)^3} = \frac{-i}{16\pi^2}\frac{1}{2}\frac{k_\mu}{p^2-l} \tag{4.8}$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{(k^2-2k\cdot p+l)^3} &= \frac{i}{16\pi^2}\frac{1}{4}\eta_{\mu\nu}\left\{\log\frac{\Lambda^2}{p^2-l} - \frac{3}{2}\right\} \\ &\quad - \frac{i}{16\pi^2}\frac{1}{2}\frac{p_\mu p_\nu}{p^2-l}\end{aligned} \tag{4.9}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k_\mu k_\nu k_\rho}{(k^2-2k\cdot p+l)^3} = \frac{i}{16\pi^2}\frac{1}{4}\left(\eta_{\mu\nu}p_\rho+\eta_{\nu\rho}p_\mu+\eta_{\rho\mu}p_\nu\right)\left\{\log\frac{\Lambda^2}{p^2-l}-\frac{11}{6}\right\} + \frac{i}{16\pi^2}\frac{1}{2}\frac{p_\mu p_\nu p_\rho}{p^2-l} \tag{4.10}$$

Quadratic appearances of the integration variable are simply written as $k^2 = \eta^{\mu\nu}k_\mu k_\nu$. The formulae are taken from [23] except the last one, which was deduced by derivation from (4.6) and the first one, which is trivial.

In these formulas all momenta, including the integrals at the left hand site, are not Wick-rotated. The Euclidean vectors exist in an intermediate step and do not effect to outer momenta. The Wick rotation becomes only visible in the imaginary factor.

The disadvantage of the cut-off regularization is that the hard cut-off in momentum space violates the symmetries – especially gauge symmetry and Lorentz symmetry. Terms of the regulated expressions which are inconsistent with the symmetries, viz. violating the Slavnov-Taylor-Ward identities, have to be dropped because they are unphysical.

4.1.2 Dimensional Regularization

The considered integrals diverge in 4 dimesions, but yield finite values in $d \neq 4$ dimensions. This is exploited in the dimensional regularization first introduced by 't Hooft and Veltman [24]. In this scheme the momentum k, which is integrated over, is Wick-rotated like in the cut-off regularization. Then the integral is carried out in $d = 4-\epsilon$ dimensions; and finally the obtained expression is expanded in ϵ and all terms of linear and higher order in ϵ are dropped. The divergences now become manifest as poles in ϵ. To the one-loop diagrams we find only poles of first order, so all divergences are terms proportional to $\frac{1}{\epsilon}$.

Because the integrals are taken over the whole now d-dimensional momentum space and not only a finite sphere, the integration variable k can be shifted, so that the denominator depends only on its square k'^2 and is written as $(k'^2-\Delta)^n$. The precise shift is

$$k'_\mu = k_\mu + xq_\mu \qquad \Rightarrow \qquad \Delta = -x(1-x)q^2 \tag{4.11}$$

for the two propagator integral (4.1) and

$$k'_\mu = k_\mu + (xq_\mu - yp_\mu) \qquad \Rightarrow \qquad \Delta = -x(1-x)q^2 - y(1-y)p^2 + 2xyp\cdot q \tag{4.12}$$

for the three propagator integral (4.2).

Due to the symmetry all terms of the numerator linear and cubic in the new k'[1] can be dropped after this substitution. For the same reason the quadratic terms can be simplified by

$$k_\mu k_\nu = \tfrac{1}{d}k^2\eta_{\mu\nu}\,. \tag{4.13}$$

Higher terms in k will not appear in our diagrams.

[1] In the futher text k' will again be denoted as k because no confusion will be possible.

The momentum integrals are now evaluated in d dimensions. The used formulae can also be found in [21]:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \tag{4.14}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2-\Delta)^2} = \frac{-i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-d/2}} \tag{4.15}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2-\Delta)^3} = \frac{-i}{(4\pi)^{d/2}} \frac{1}{2} \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}} \tag{4.16}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^2}{(k^2-\Delta)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{4} \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \tag{4.17}$$

These expressions are all finite for $d = 4 - \epsilon$ as long as $\epsilon \neq 0$. The limit $\epsilon \to 0$ will be taken in the final step and deliver the divergent part $\sim \frac{1}{\epsilon}$ and the finite contributing $\mathcal{O}(1)$ in $d = 4$ dimensions.

The disadvantage of this method is that in the dimensional regularization quadratic divergences disappear:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2} = 0 \tag{4.18}$$

This is especially problematic because the superficial degree of divergence for many of our considered diagrams is two. We solve this problem by the comparison of the results of the two regularization methods: The quadratic divergences[2] in the cut-off regularization cancel each other, so that only logarithmic divergent terms remain. These logarithmical divergences in both schemes will turn out to be identical, if the divergent factors $\log \Lambda^2$ and $\frac{2}{\epsilon}$ are identifed[3].

Finally all expressions containing the dimension d are expanded in small ϵ. The Γ-functions are expanded using:

$$\begin{aligned} \Gamma(1-\tfrac{d}{2}) &= \Gamma(-1+\tfrac{\epsilon}{2}) = -\frac{2}{\epsilon} + \gamma - 1 + \mathcal{O}(\epsilon) \\ \Gamma(2-\tfrac{d}{2}) &= \Gamma(\tfrac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \\ \Gamma(3-\tfrac{d}{2}) &= \Gamma(1+\tfrac{\epsilon}{2}) = 1 + \mathcal{O}(\epsilon) \end{aligned} \tag{4.19}$$

and will provide the divergence $\frac{2}{\epsilon}$. Here γ is the Euler-Mascheroni constant, $\gamma \approx 0.57772$. Additionally only a few functions of the dimension and thus ϵ will appear:

$$\begin{aligned} X^{n-\frac{d}{2}} &= X^{n-2+\frac{\epsilon}{2}} = X^{n-2}(1+\tfrac{1}{2}\epsilon \log X) + \mathcal{O}(\epsilon^2) \\ d^n &= (4-\epsilon)^n = 4^n(1-\tfrac{n}{4}\epsilon) + \mathcal{O}(\epsilon^2) \\ \frac{1}{d-2} &= \frac{1}{2-\epsilon} = \frac{1}{2} + \frac{\epsilon}{4} + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{4.20}$$

In the final result only the terms $\mathcal{O}(\frac{1}{\epsilon})$ and $\mathcal{O}(1)$ are kept.

[2] Here only the terms respecting the Slavnov-Taylor-Ward-identity are meant. The unphysical divergences are dropped anyway.

[3] This identification is reasonable because both have the same mangnitude as the accompaning momentum logarithm $-\log q^2$.

4.2 Technique

The actual computation of the one-loop graphs was mainly done using the computer algebra system FORM. We wrote separate scripts for each calculated graph – only graphs with the same structure like the triangle graphs with gluon and ghost propagators 4.1d) and e) are computed in one file. First the scripts read the vertex formulas for the graviton–gluon vertices from the script described in section 2.1. Each diagram is represented by a FORM expression, which consists of the integrals numerator including the symmetry factors derived by the method described in Appendix A and the imaginary factors i or $-i$ respectively of the propagators. The constants like g, κ and gauge group expressions like f^{abc} were not included in the script because they were not effected by the manipulation done by FORM. The later ones were evaluated by hand using the quite simple rules

$$\delta^{ab}\delta^{bd} = \delta^{ac} \qquad f^{acd}f^{bcd} = C_2\delta^{ab} \qquad f^{lam}f^{mbn}f^{ncl} = -\tfrac{1}{2}C_2 f^{abc}\,. \tag{4.21}$$

In the common case of a $SU(N)$ gauge group The constant $C_2 = N$.

The integral itself including the factor $(2\pi)^{-d}$ and the denominator was also not explicitly written because all expressions in one file have the same denominator and accordingly the integration procedure is the same. E. g. the gravitational one-loop contribution to the gluon self-energy Figure 4.2a) is written as:

```
local PC1=V1gr2gl(q,m,[-K],s,a,b,i1)*P1(a,b,c,d)*V1gr2gl(K,s,[-q],n,c,d,i2);
```

The vertex function `V1gr2gl` depends on the gluon momenta `q`, `K` (vectors), the gluon Lorentz indices `m`, `n` and `s` and the graviton indices `a`–`d`. The additional arguments `i1` and `i2` are needed as indices for internal sums in the vertex functions. Additional to the vertex functions the index structure of the graviton propagator $I^{\alpha\beta,\gamma,\delta} + \frac{1}{2-d}\eta^{\alpha\beta}\eta^{\gamma\delta}$ is represented by a FORM function succeedingly matched by

```
id P1(a?,b?,c?,d?)=1/2*(d_(a,c)*d_(b,d)+d_(a,d)*d_(b,c))+d_(a,b)*d_(c,d)/[2-D];
```

The symbol `[2-D]` is immediately matched with -2 in cut-off regularization. In dimensional regularization the expansion for $D = 4 - \epsilon$ is done at the appropriate time.

For cut-off regularization the dimension is set to four. Now the loop intergal is evaluated. This is done substituting all occurrences of the loop momentum by the solution of the regulated integral (4.3)–(4.10).

For dimensional regularization first the momentum shift (4.11) or (4.12) respectively is implemented. The odd powers of the new integration variable are dropped and the simplification (4.13) for the quadratic terms is utilized. Now the dimension is set to $d = 4 - \epsilon$ and expressions in d including the momentum integrals are expended in small ϵ up to $\mathcal{O}(1)$. For the integrals we used (4.14)–(4.17) and the expansions for the Γ function (4.19).

The remaining process is the same for both regularization schemes. The last step introduced the Feynman parameters in our expressions. The integral in the two propagator case (4.1) is evaluated by substituting each occurrence of a power of x or function of x by the value of the corresponding intergal, which were obtained before using Mathematica.

The three propagator case (4.2) is more complicate. First the integrals over Δ^{-1} and $\log\Delta$ are not solvable for general external momenta. To gain a expression for the complete diagram, we set the momenta to $p^2 = q^2 = k^2 = -E^2$. The divergent part, which is the only important for the counterterms, is additionally computed for not fixed momenta because it is always polynomial in the Feynman parameters. The final evaluation of the integrals is done using the substition presented in Appendix B.

Finally the not totally symmetric three-gluon diagrams, like Figure 4.2c), are summed up by adding all cyclic permutations, viz. $(p,\mu;q,\nu;k,\rho)$ $+(q,\nu;k,\rho;p,\mu) + (k,\rho;p,\mu;q,\nu)$.

4.3 Gluon Self-Energy

The finite part of the diagrams include terms depending logarithmically on the squared momentum. These have a branch cut for negative arguments, which is avoided by the $-i\varepsilon$ term introduced in the propagators. If one wants to evaluate the logarithms – which is no necessary to obtain the results of this work – one has to take the limit $\varepsilon \to +0$ to obtain the correct value:

$$\begin{aligned}\log(q^2 - i\varepsilon) &\xrightarrow[q^2<0]{\varepsilon\to+0} \log(-q^2) - i\pi \\ \log(-q^2 + i\varepsilon) &\xrightarrow[q^2>0]{\varepsilon\to+0} \log q^2 + i\pi\,.\end{aligned}$$

To keep the formulas more readable we drop the $i\varepsilon$ in the results. It can easily restored using that it always appears in the combination $q^2 - i\varepsilon$. Hence one can substitute

$$q^2 \to q^2 - i\varepsilon$$

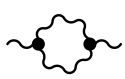
if necessary to evaluate a logarithm.

In the momentum integrals the branch-cut is avoided by the integration in the Euclidean space after Wick-rotation.

4.3.1 Pure Yang-Mills Theory

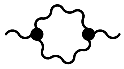
First we deal with graphs which consist only of entities of pure Yang Mills theory, Figure 4.1 a)–c). These calculations can be compared with the long time known results of order g^2 as taken from the literature[25, 19, 26, 21].

Figure 4.1 a)

$\frac{1}{2}$ 

$$= \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} V_{3\mathrm{gl}}^{acd\,\mu\rho\sigma}(q,k,-(k+q))\frac{-i}{k^2+i\varepsilon} \times V_{3\mathrm{gl}\ \ \rho\sigma}^{bcd\nu}(-q,-k,k+q)\frac{-i}{(k+q)^2+i\varepsilon} \tag{4.22}$$

becomes in cut-off regularization:

$\frac{1}{2}$ 
$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Bigg(q^2 \eta^{\mu\nu} \frac{19}{12} \left[\log \Lambda^2 - \log q^2 + \tfrac{71}{114}\right] \\ - q^\mu q^\nu \frac{11}{6} \left[\log \Lambda^2 - \log(-q^2) + \tfrac{61}{66}\right] - \eta^{\mu\nu} \frac{9}{4} \Lambda^2 \Bigg) \tag{4.23}$$

and in dimensional regularization:

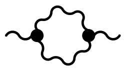
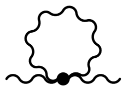
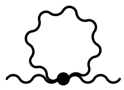
$\frac{1}{2}$ 
$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Bigg(q^2 \eta^{\mu\nu} \frac{19}{12} \left[\frac{2}{\epsilon} - \log(-q^2) - \gamma + \log 4\pi + \tfrac{116}{57}\right] \\ - q^\mu q^\nu \frac{11}{6} \left[\frac{2}{\epsilon} - \log(-q^2) - \gamma + \log 4\pi + \tfrac{67}{33}\right] \Bigg) \tag{4.24}$$

Figure 4.1 b)

$\frac{1}{2}$ 
$$= \frac{1}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} V_{4\mathrm{gl}\,\rho}^{abcc\,\mu\nu\rho} \frac{-i}{k^2 + i\varepsilon} \tag{4.25}$$

yields in cut-off regularization:

$\frac{1}{2}$ 
$$= \frac{i}{16\pi^2} 3 g^2 C_2 \eta^{\mu\nu} \delta^{ab} \Lambda^2 \tag{4.26}$$

in dimensional regularization the all tadpole graphs are zero:

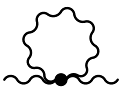
$\frac{1}{2}$ 
$$= -g^2 C_2 \frac{d-1}{2} \eta^{\mu\nu} \delta^{ab} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2} \\ = 0 \tag{4.27}$$

Figure 4.1 c)


$$= - \int \frac{\mathrm{d}^d k}{(2\pi)^d} V_{\mathrm{gl-gh}}^{cad\,\mu}(k+q) \frac{i}{(k+q)^2 + i\varepsilon} V_{\mathrm{gl-gh}}^{dbc\,\nu}(k) \frac{i}{k^2 + i\varepsilon} \tag{4.28}$$

becomes in cut-off regularization:


$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Bigg(q^2 \eta^{\mu\nu} \frac{1}{12} \left[\log \Lambda^2 - \log q^2 + \tfrac{11}{6}\right] \\ + q^\mu q^\nu \frac{1}{6} \left[\log \Lambda^2 - \log(-q^2) + \tfrac{5}{6}\right] + \eta^{\mu\nu} \frac{1}{4} \Lambda^2 \Bigg) \tag{4.29}$$

and in dimensional regularization:

$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \left(q^2 \eta^{\mu\nu} \frac{1}{12} \left[\frac{2}{\epsilon} - \log(-q^2) - \gamma + \log 4\pi + \tfrac{8}{3} \right] + q^\mu q^\nu \frac{1}{6} \left[\frac{2}{\epsilon} - \log(-q^2) - \gamma + \log 4\pi + \tfrac{5}{3} \right] \right) \tag{4.30}$$

Hence the complete gluon self energy in order g^2 in the cut-off regularization:

$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Big[(q^2 \eta^{\mu\nu} - q^\mu q^\nu) \frac{5}{3} \left(\log \Lambda^2 - \log(-q^2) \right) + \tfrac{41}{36} q^2 \eta^{\mu\nu} - \tfrac{14}{9} q^\mu q^\nu + \eta^{\mu\nu} \Lambda^2 \Big] \tag{4.31}$$

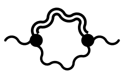
the finite and the quadratically divergent terms are not conform with the Slavnov-Taylor-Ward identity and thus unphysical. So they must be ignored as described in subsection 4.1.1. The result in dimensional regularization

$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \frac{5}{3} \left[\frac{2}{\epsilon} - \log(-q^2) - \gamma + \log 4\pi + \tfrac{31}{15} \right] \tag{4.32}$$

on the other hand is symmetry conform. Both results match with the literature values.

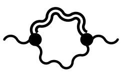
4.3.2 Gravitational Contributions

Figure 4.2 a)



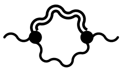
$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} V^{ac\,\mu\sigma,\alpha\beta}_{\text{2gl–1gr}}(q,-k) \frac{-i}{k^2 + i\varepsilon} V^{cb\,\nu,\gamma\delta}_{\text{2gl–1gr}\,\sigma}(k,-q) \frac{i(I_{\alpha\beta,\gamma\delta} - \frac{1}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta})}{(k+q)^2 + i\varepsilon} \tag{4.33}$$

becomes in cut-off regularization:



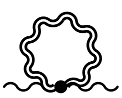
$$= \frac{i}{16\pi^2} \kappa^2 (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \delta^{ab} \left[\frac{3}{2} \Lambda^2 - q^2 \frac{1}{6} \left(\log \Lambda^2 - \log(-q^2) - \tfrac{25}{6} \right) \right] \tag{4.34}$$

and in dimensional regularization:



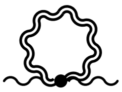
$$= -\frac{i}{16\pi^2} \kappa^2 (q^2 \eta^{\mu\nu} - q^\mu q^\nu) \delta^{ab} \frac{1}{6} q^2 \left[\frac{2}{\epsilon} - \log(-q^2) - \gamma + \log 4\pi + \tfrac{1}{6} \right] \tag{4.35}$$

Figure 4.2 b)

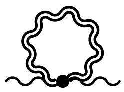
$\frac{1}{2}$ 

$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} V^{ab\,\mu\nu,\alpha\beta\,\gamma\delta}_{\text{2gl–2gr}}(q,-q) \frac{i(I_{\alpha\beta,\gamma\delta} - \frac{1}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta})}{k^2 + i\varepsilon} \tag{4.36}$$

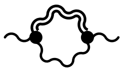
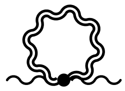
is in cut-off regularization

$\frac{1}{2}$  $$= \frac{-i}{16\pi^2}\frac{3}{2}\kappa^2(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\Lambda^2 \tag{4.37}$$

in dimensional regularization it yields zero as all tadpole graphs:

$\frac{1}{2}$  $$\begin{aligned} &= \frac{3}{2}\kappa^2(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2} \\ &= 0 \end{aligned} \tag{4.38}$$

The sum in cut-off regularization:

 $+\frac{1}{2}$  $$= -\frac{i}{16\pi^2}\frac{1}{6}(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\kappa^2 q^2\left[\log\Lambda^2 - \log(-q^2) - \tfrac{25}{6}\right] \tag{4.39}$$

is in leading order equivalent to 4.2a) in the dimensional calculation, which is the only contribution in this scheme.

4.4 Vertex Corrections

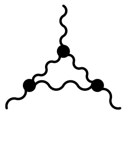
The finite parts of the triangle shaped diagrams Figure 4.1 d), e) and 4.2 c) were evaluated at the regularization point $p^2 = q^2 = k^2 = -E^2$ because otherwise we would not been able to compute all integrals over the Feynman parameters. As mentioned above, the divergent terms depend only polynomially on the parameters. Hence these are additionally calculated for general momenta in order to obtain the right tensor structure for the dimension-six counter-terms.

Except for some of the logarithmic divergent contributions of gravitational loop diagrams, only three, in the momenta and indices antisymmetric structures occur. To enhance readability only the first terms will be written:

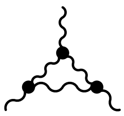
$$\begin{aligned} \eta^{\mu\nu}(p-q)^\rho + \eta^{\nu\rho}(q-k)^\mu + \eta^{\rho\mu}(k-p)^\nu &= \eta^{\mu\nu}(p-q)^\rho + \dots \\ k^\mu k^\nu(p-q)^\rho + p^\nu p^\rho(q-k)^\mu + q^\rho q^\mu(k-p)^\nu &= k^\mu k^\nu(p-q)^\rho + \dots \\ p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu &= p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu \end{aligned}$$

4.4.1 Pure Yang Mills Theory

Figure 4.1 d)

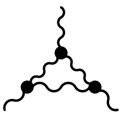
 $$\begin{aligned} &= \int\frac{\mathrm{d}^d k}{(2\pi)^d} V_{3\mathrm{gl}\,\gamma}^{lam\,\mu\alpha}(k-p, p, -k)\frac{-i}{k^2+i\varepsilon} \\ &\quad\times V_{3\mathrm{gl}\,\alpha}^{mbn\,\nu\beta}(k, q, -k-q)\frac{-i}{(k+q)^2+i\varepsilon} \\ &\quad\times V_{3\mathrm{gl}\,\beta}^{ncl\,\rho\gamma}(k+q, -p-q, -k+p)\frac{-i}{(k-p)^2+i\varepsilon} \end{aligned} \tag{4.40}$$

becomes in cut-off regularization:



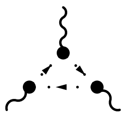
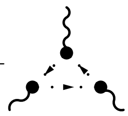
$$= \frac{1}{16\pi^2} g^3 C_2 f^{abc} \Big\{ \frac{13}{8} (\eta^{\mu\nu}(p-q)^\rho + \dots) \Big[\log \Lambda^2 - \log E^2 + \tfrac{85}{78} \Big] - (k^\mu k^\nu (p-q)^\rho + \dots) \left[\tfrac{1}{6} - \tfrac{1}{9}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \right] E^{-2} + (p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu) \left[\tfrac{5}{12} + \tfrac{2}{9}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \right] E^{-2} \Big\} \tag{4.41}$$

and in dimensional regularization:



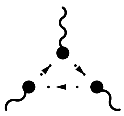
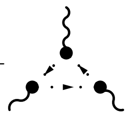
$$= \frac{1}{16\pi^2} g^3 C_2 f^{abc} \Big\{ \frac{13}{8} (\eta^{\mu\nu}(p-q)^\rho + \dots) \Big[\frac{2}{\epsilon} - \log E^2 - \gamma + \log 4\pi + \tfrac{37}{13} - \tfrac{4}{9}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \Big] - (k^\mu k^\nu (p-q)^\rho + \dots) \left[\tfrac{23}{54} + \tfrac{13}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \right] E^{-2} + (p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu) \left[\tfrac{115}{108} - \tfrac{62}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \right] E^{-2} \Big\} \tag{4.42}$$

Figure 4.1 e)

 + 

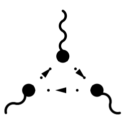
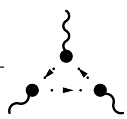
$$= - \int \frac{\mathrm{d}^d k}{(2\pi)^d} V_{\text{gl–gh}}^{lam\,\mu}(k) \frac{i}{k^2 + i\varepsilon} V_{\text{gl–gh}}^{mbn\,\nu}(k+q) \times \frac{i}{(k+q)^2 + i\varepsilon} V_{\text{gl–gh}}^{ncl\,\rho\gamma}(k-p) \frac{i}{(k-p)^2 + i\varepsilon} + V_{\text{gl–gh}}^{mal\,\mu}(-k+p) \frac{i}{k^2 + i\varepsilon} V_{\text{gl–gh}}^{nbm\,\nu}(-k) \times \frac{i}{(k+q)^2 + i\varepsilon} V_{\text{gl–gh}}^{lcn\,\rho\gamma}(-k-q) \frac{i}{(k-p)^2 + i\varepsilon}$$

becomes in cut-off regularization:

 + 

$$= \frac{1}{16\pi^2} g^3 C_2 f^{abc} \Big\{ -\frac{1}{24} (\eta^{\mu\nu}(p-q)^\rho + \dots) \Big[\log \Lambda^2 - \log E^2 + \tfrac{3}{2} - \tfrac{2}{9}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \Big] - (k^\mu k^\nu (p-q)^\rho + \dots) \left[\tfrac{1}{54} - \tfrac{1}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \right] E^{-2} + (p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu) \left[\tfrac{5}{108} + \tfrac{2}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \right] E^{-2} \Big\} \tag{4.43}$$

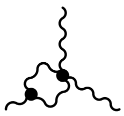
and in dimensional regularization:

 + 

$$= \frac{1}{16\pi^2} g^3 C_2 f^{abc} \Big\{ -\frac{1}{24} (\eta^{\mu\nu}(p-q)^\rho + \dots) \Big[\frac{2}{\epsilon} - \log E^2 - \gamma + \log 4\pi + 3 - \tfrac{2}{9}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \Big]$$

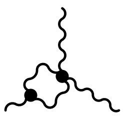
$$-(k^\mu k^\nu (p-q)^\rho + \dots)\left[\tfrac{1}{54} - \tfrac{1}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\right] E^{-2}$$
$$+(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\left[\tfrac{5}{108} + \tfrac{2}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\right] E^{-2}\Big\} \tag{4.44}$$

Figure 4.1 f)

$\frac{1}{2}$ 

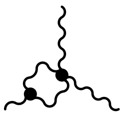
$$= \frac{1}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d} V_{3\mathrm{gl}}^{bmn\,\nu\rho\sigma}(q,k,-k-q)\frac{-i}{k^2+i\varepsilon}$$
$$\times V_{4\mathrm{gl}\,\rho\sigma}^{mnac\,\mu\rho}(-k,k+q,p,-p-q)\frac{-i}{(k+q)^2+i\varepsilon} \tag{4.45}$$

becomes in cut-off regularization:

$\frac{1}{2}$ 

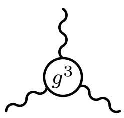
$$+\dots = -\frac{1}{16\pi^2}g^3\frac{9}{4}C_2 f^{abc}\big(\eta^{\mu\nu}\big(p^\rho\left[\log\Lambda^2 - \log(-p^2)+1\right] \tag{4.46}$$
$$-q^\rho\left[\log\Lambda^2 - \log(-q^2)+1\right]\big)+\dots\big)$$

and in dimensional regularization:

$\frac{1}{2}$ 

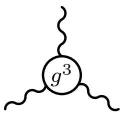
$$+\dots = -\frac{1}{16\pi^2}g^3\frac{9}{4}C_2 f^{abc}\big(\eta^{\mu\nu}\big(p^\rho\left[\frac{2}{\epsilon} - \log(-p^2) - \gamma + \log 4\pi + 2\right]$$
$$-q^\rho\left[\frac{2}{\epsilon} - \log(-q^2) - \gamma + \log 4\pi + 2\right]\big)+\dots\big) \tag{4.47}$$

Hence the sum of the Yang-Mills contributions in cut-off regularization is



$$= \frac{1}{16\pi^2}g^3 C_2 f^{abc}\Big\{-\frac{2}{3}(\eta^{\mu\nu}(p-q)^\rho+\dots)\Big[\log\Lambda^2 - \log E^2$$
$$+\tfrac{13}{54} - \tfrac{1}{243}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\Big] \tag{4.48}$$
$$-(k^\mu k^\nu (p-q)^\rho + \dots)\left[\tfrac{10}{27} + \tfrac{7}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\right] E^{-2}$$
$$+(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\left[\tfrac{25}{27} - \tfrac{41}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\right] E^{-2}\Big\}$$

And the sum in dimensional regularization:

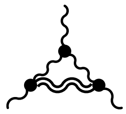


$$= \frac{1}{16\pi^2}g^3 C_2 f^{abc}\Big\{-\frac{2}{3}(\eta^{\mu\nu}(p-q)^\rho+\dots)\Big[\frac{2}{\epsilon} - \log E^2$$
$$-\gamma + \log 4\pi + \tfrac{1}{2} - \tfrac{115}{144}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\Big] \tag{4.49}$$
$$-(k^\mu k^\nu (p-q)^\rho + \dots)\left[\tfrac{10}{27} + \tfrac{7}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\right] E^{-2}$$
$$+(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\left[\tfrac{25}{27} - \tfrac{41}{81}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\right] E^{-2}\Big\}$$

Again the Yang-Mills result reproduces the literature value in amplitude and sign. Thus the applied methods can be regarded as correctly working, also in the gravitational sector.

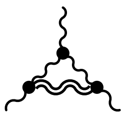
4.4.2 Gravitational Contributions

Figure 4.2 c)



$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} V_{3\mathrm{gl}}^{ade\,\mu\sigma\tau}(p,-k,k-p)\frac{-i}{k^2+i\varepsilon}$$
$$\times V_{2\mathrm{gl-1gr}\,\sigma}^{db\,\nu,\alpha\beta}(k,q)\frac{i(I_{\alpha\beta,\gamma\delta}-\frac{1}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta})}{(k+q)^2+i\varepsilon}$$
$$\times V_{2\mathrm{gl-1gr}\,\tau}^{ec\,\rho,\gamma\delta}(-k+p,-p-q)\frac{-i}{(k-p)^2+i\varepsilon}$$

The divergent part is in both regularization schemes:

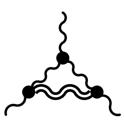


$$\begin{aligned} + \dots = \frac{1}{16\pi^2} g\kappa^2 f^{abc}\Big\{ &\eta^{\mu\nu}\Big[p^\rho(\frac{5}{6}p\cdot q+\frac{1}{4}q\cdot k) \\ &\qquad - q^\rho(\frac{5}{6}q\cdot p+\frac{1}{4}p\cdot k)\Big] \\ &+\eta^{\nu\rho}\Big[q^\mu(\frac{5}{6}q\cdot k+\frac{1}{4}k\cdot p)-k^\mu(\frac{5}{6}k\cdot q+\frac{1}{4}q\cdot p)\Big] \\ &+\eta^{\rho\mu}\Big[k^\nu(\frac{5}{6}k\cdot p+\frac{1}{4}p\cdot q)-p^\nu(\frac{5}{6}p\cdot k+\frac{1}{4}k\cdot q)\Big] \\ &-\frac{5}{6}\left[k^\mu k^\nu (p-q)^\rho+\dots\right] \\ &-\frac{1}{4}\left[p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu\right]\Big\}\mathbf{\Delta} \end{aligned} \tag{4.50}$$

Here the abbreviation

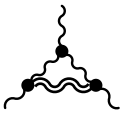
$$\mathbf{\Delta} = \begin{cases} \log\Lambda^2 & \text{in cut-off regularization} \\ \frac{2}{\epsilon} & \text{in dimesional regularization} \end{cases} \tag{4.51}$$

is used. The finite part of the diagram is determinable only at the point $p^2 = q^2 = k^2 = -E^2$. The complete values of the diagram is then in cut-off regularization:



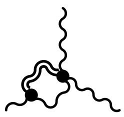
$$\begin{aligned} + \dots = \frac{1}{16\pi^2} g\kappa^2 f^{abc}\Big\{ &\tfrac{13}{24}E^2(\eta^{\mu\nu}(p-q)^\rho+\dots)\big[\log\Lambda^2 \\ &-\log E^2+\tfrac{83}{78}-\tfrac{5}{39}(\psi_1(\tfrac{1}{3})-\psi_1(\tfrac{2}{3}))\big] \\ &-\tfrac{5}{6}(k^\mu k^\nu(p-q)^\rho+\dots)\big[\log\Lambda^2 \\ &-\log E^2+\tfrac{11}{30}+\tfrac{2}{45}(\psi_1(\tfrac{1}{3})-\psi_1(\tfrac{2}{3}))\big] \\ &-\tfrac{1}{4}(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big[\log\Lambda^2 \\ &-\log E^2+\tfrac{61}{72}-\tfrac{19}{27}(\psi_1(\tfrac{1}{3})-\psi_1(\tfrac{2}{3}))\big]\Big\} \end{aligned} \tag{4.52}$$

and in dimensional regularization:



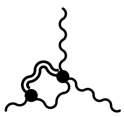
$$
\begin{aligned}
+\cdots = \frac{1}{16\pi^2} g\kappa^2 f^{abc} \Big\{ & \tfrac{13}{24} E^2 (\eta^{\mu\nu}(p-q)^\rho + \dots)\big[\tfrac{2}{\epsilon} - \log E^2 \\
& -\gamma + \log 4\pi - \tfrac{38}{39} - \tfrac{37}{117}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\big] \\
& -\tfrac{5}{6}(k^\mu k^\nu (p-q)^\rho + \dots)\big[\tfrac{2}{\epsilon} - \log E^2 - \gamma + \log 4\pi \\
& +3 - \tfrac{11}{45}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\big] \\
& -\tfrac{1}{4}(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big[\tfrac{2}{\epsilon} - \log E^2 - \gamma + \log 4\pi \\
& -\tfrac{128}{9} + \tfrac{7}{27}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3}))\big]\Big\}
\end{aligned}
\tag{4.53}
$$

Figure 4.2 d)



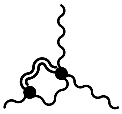
$$
\begin{aligned}
= \int \frac{\mathrm{d}^d k}{(2\pi)^d} V_{\text{2gl–1gr}}^{bd\,\nu\sigma,\alpha\beta}(q,k) \frac{-i}{k^2 + i\varepsilon} \\
\times V_{\text{3gl–1gr}\,\sigma}^{adc\,\mu\sigma\rho,\gamma\delta}(p,-k,-p-q) \frac{i(I_{\alpha\beta,\gamma\delta} - \frac{1}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta})}{(k+q)^2 + i\varepsilon}
\end{aligned}
$$

The divergent part is in cut-off regularization:



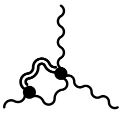
$$
\begin{aligned}
+\cdots = \frac{1}{16\pi^2} g\kappa^2 f^{abc} \Bigg\{ & -\frac{3}{2}\left[\eta^{\mu\nu}(p-q)^\rho + \dots\right]\Lambda^2 \\
& + \Bigg[-\eta^{\mu\nu}\Big[p^\rho(\frac{7}{6}p\cdot q + \frac{1}{6}p\cdot k + \frac{3}{4}q\cdot k) \\
& \qquad - q^\rho(\frac{7}{6}q\cdot p + \frac{1}{6}q\cdot k + \frac{3}{4}p\cdot k)\Big] \\
& \quad -\eta^{\nu\rho}\Big[q^\mu(\frac{7}{6}q\cdot k + \frac{1}{6}q\cdot p + \frac{3}{4}k\cdot p) \\
& \qquad - k^\mu(\frac{7}{6}k\cdot q + \frac{1}{6}k\cdot p + \frac{3}{4}q\cdot p)\Big] \\
& \quad -\eta^{\rho\mu}\Big[k^\nu(\frac{7}{6}k\cdot p + \frac{1}{6}k\cdot q + \frac{3}{4}p\cdot q) \\
& \qquad - p^\nu(\frac{7}{6}p\cdot k + \frac{1}{6}p\cdot q + \frac{3}{4}k\cdot q)\Big] \\
& \quad + \left[k^\mu k^\nu (p-q)^\rho + \dots\right] \\
& \quad + \frac{3}{4}\left[p^\rho q^\mu k^\nu + p^\nu q^\rho k^\mu\right]\Bigg] \log\Lambda^2 \Bigg\}
\end{aligned}
\tag{4.54}
$$

and in dimensional regularization:



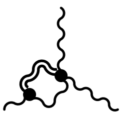
$$
\begin{aligned}
+\cdots = \frac{1}{16\pi^2} g\kappa^2 f^{abc} \Big\{ & -\eta^{\mu\nu}\Big[p^\rho(\frac{7}{6}p\cdot q + \frac{1}{6}p\cdot k + \frac{3}{4}q\cdot k) \\
& \qquad - q^\rho(\frac{7}{6}q\cdot p + \frac{1}{6}q\cdot k + \frac{3}{4}p\cdot k)\Big] \\
& -\eta^{\nu\rho}\Big[q^\mu(\frac{7}{6}q\cdot k + \frac{1}{6}q\cdot p + \frac{3}{4}k\cdot p) \\
& \qquad - k^\mu(\frac{7}{6}k\cdot q + \frac{1}{6}k\cdot p + \frac{3}{4}q\cdot p)\Big] \\
& -\eta^{\rho\mu}\Big[k^\nu(\frac{7}{6}k\cdot p + \frac{1}{6}k\cdot q + \frac{3}{4}p\cdot q) \\
& \qquad - p^\nu(\frac{7}{6}p\cdot k + \frac{1}{6}p\cdot q + \frac{3}{4}k\cdot q)\Big] \\
& + [k^\mu k^\nu (p-q)^\rho + \dots] \\
& + \frac{3}{4}[p^\rho q^\mu k^\nu + p^\nu q^\rho k^\mu] \Big\} \frac{2}{\epsilon}
\end{aligned}
\tag{4.55}
$$

The complete diagram is in cut-off regularization:



$$
\begin{aligned}
+\cdots = \frac{1}{16\pi^2} g\kappa^2 f^{abc} \Big\{ & -\tfrac{3}{2}\left(\eta^{\mu\nu}(p-q)^\rho + \dots\right)\Lambda^2 \\
& + \Big(\eta^{\mu\nu}\big(p^\rho\big[-(\tfrac{7}{6}p\cdot q + \tfrac{1}{6}p\cdot k + \tfrac{3}{4}q\cdot k)\log\Lambda^2 \\
& \qquad - (\tfrac{1}{4}p\cdot k - \tfrac{11}{12}q\cdot k)\log(-p^2) \\
& \qquad - \tfrac{1}{4}k^2\log(-q^2) + \tfrac{1}{2}q\cdot k\log(-k^2) \\
& \qquad + \tfrac{7}{36}p\cdot q + \tfrac{25}{36}p\cdot k - \tfrac{3}{8}q\cdot k\big] \\
& \quad - q^\rho(\dots)\big) + \dots\Big) \\
& + \Big(k^\mu k^\nu\big(p^\rho\big[\log\Lambda^2 - \tfrac{1}{4}\log(-p^2) - \tfrac{3}{4}\log(-k^2) + \tfrac{1}{2}\big] \\
& \quad - q^\rho[\dots]\big) + \dots\Big) \\
& + (p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big[\tfrac{3}{4}\log\Lambda^2 + \tfrac{3}{8} \\
& \quad - \tfrac{1}{4}\left(\log(-p^2) + \log(-q^2) + \log(-k^2)\right)\big]\Big\}
\end{aligned}
\tag{4.56}
$$

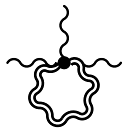
and in dimensional regularization:



$$
\begin{aligned}
+\cdots = \frac{1}{16\pi^2} g\kappa^2 f^{abc} \Big\{ & \Big(\eta^{\mu\nu}\big(p^\rho\big[-(\tfrac{7}{6}p\cdot q + \tfrac{1}{6}p\cdot k \\
& \qquad + \tfrac{3}{4}q\cdot k)[\tfrac{2}{\epsilon} - \gamma + \log 4\pi] \\
& \qquad - (\tfrac{1}{4}p\cdot k - \tfrac{11}{12}q\cdot k)\log(-p^2) \\
& \qquad - \tfrac{1}{4}k^2\log(-q^2) + \tfrac{1}{2}q\cdot k\log(-k^2) \\
& \qquad - \tfrac{23}{18}p\cdot q - \tfrac{1}{36}p\cdot k + \tfrac{3}{4}q\cdot k\big] \\
& \quad - q^\rho(\dots)\big) + \dots\Big)
\end{aligned}
\tag{4.57}
$$

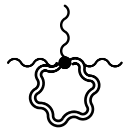
$$
\begin{aligned}
&+ \big(k^\mu k^\nu \big(p^\rho\big[\tfrac{2}{\epsilon} - \tfrac{1}{4}\log(-p^2) - \tfrac{3}{4}\log(-k^2) \\
&\qquad - \gamma + \log 4\pi + \tfrac{5}{4}\big] - q^\rho[\dots]\big) + \dots\big) \\
&+ (p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big[\tfrac{3}{4}\left(\tfrac{2}{\epsilon} - \gamma + \log 4\pi - 1\right) \\
&\qquad - \tfrac{1}{4}\left(\log(-p^2) + \log(-q^2) + \log(-k^2)\right)\big]\Big\}
\end{aligned}
$$

Figure 4.2 e)

$\frac{1}{2}$ 

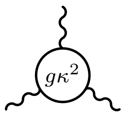
$$
= \int \frac{\mathrm{d}^d k}{(2\pi)^d} V_{\text{3gl–2gr}}^{abc\,\mu\nu\rho,\alpha\beta\,\gamma\delta}(p, q, -p-q) \frac{i(I_{\alpha\beta,\gamma\delta} - \frac{1}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta})}{k^2 + i\varepsilon} \tag{4.58}
$$

is again a tadpole diagram which only contributed in the cut-off scheme

$\frac{1}{2}$ 

$$
= \frac{1}{16\pi^2} g^3 \frac{3}{2} C_2 f^{abc} (\eta^{\mu\nu}(p-q)^\rho + \dots)\Lambda^2 \tag{4.59}
$$

Like in the propagator case the quadratic divergences cancel in the sum of all diagrams. So the remaining logarithmic divergences of all gravitational one-loop corrections to the three-gluon vertex are



$$
\begin{aligned}
= \frac{1}{16\pi^2} g\kappa^2 f^{abc} \Bigg\{ &-\eta^{\mu\nu}\Big[p^\rho(\frac{1}{3}p\cdot q + \frac{1}{6}p\cdot k + \frac{1}{2}q\cdot k) \\
&\qquad - q^\rho(\frac{1}{3}q\cdot p + \frac{1}{6}q\cdot k + \frac{1}{2}p\cdot k)\Big] \\
&-\eta^{\nu\rho}\Big[q^\mu(\frac{1}{3}q\cdot k + \frac{1}{6}q\cdot p + \frac{1}{2}k\cdot p) \\
&\qquad - k^\mu(\frac{1}{3}k\cdot q + \frac{1}{6}k\cdot p + \frac{1}{2}q\cdot p)\Big] \\
&-\eta^{\rho\mu}\Big[k^\nu(\frac{1}{3}k\cdot p + \frac{1}{6}k\cdot q + \frac{1}{2}p\cdot q) \\
&\qquad - p^\nu(\frac{1}{3}p\cdot k + \frac{1}{6}p\cdot q + \frac{1}{2}k\cdot q)\Big] \\
&+\frac{1}{6}\left[k^\mu k^\nu (p-q)^\rho + \dots\right] \\
&+\frac{1}{2}\left[p^\rho q^\mu k^\nu + p^\nu q^\rho k^\mu\right] \Bigg\} \boldsymbol{\Delta}
\end{aligned} \tag{4.60}
$$

in both regularization schemes with $\boldsymbol{\Delta}$ as defined in (4.51).

V

Renormalization

After determining the divergences of the one-loop diagrams, we are able to cancel these by adding infinite counter-terms to the Lagrangian, i. e. renormalizing the theory. We will do this in the minimal subtraction scheme, viz only subtract the leading divergent order.

5.1 The Counter-Term Lagrangian

The effective theory for the considered energy scale is described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{ren.}} + \mathcal{L}_{\text{ct}}$$

plus further corrections of higher order in κ and g, which would result from multi-loop calculations. Here $\mathcal{L}_{\text{ren.}}$ is the renormalized Lagrangian consisting of the Einstein-Hilbert and the dimension-six Lagrangian presented in chapter 3:

$$\mathcal{L}_{\text{ren.}} = \mathcal{L}_{\text{EYM}} + \mathcal{L}_{\text{dim6}}\,.$$

$\mathcal{L}_{\text{ct}}$ collects all counter-terms canceling the determined one-loop divergences from chapter 4. The full Lagrangian can also be expressed by extended Lagrangian $\mathcal{L}_{\text{EYM}} + \mathcal{L}_{\text{dim6}}$ in terms of the bare fields and couplings. For our consideration only the Yang-Mills sector

$$\begin{aligned}\mathcal{L} = -\tfrac{1}{4}F^0_{\mu\nu}F^{0\,\mu\nu} &+ d^0_1\,\mathrm{tr}\left[(D^0_\mu F^0_{\nu\rho})(D^{0\,\mu}F^{0\,\nu\rho})\right]\\ &+ d^0_2\,\mathrm{tr}\left[(D^0_\mu F^{0\,\mu\rho})(D^{0\,\nu}F^0_{\nu\rho})\right] + id^0_3\,\mathrm{tr}\left[F^{0\,\beta}_{\alpha}F^{0\,\gamma}_{\beta}F^{0\,\alpha}_{\gamma}\right]\end{aligned} \tag{5.1}$$

is needed, which is at the examined order:

$$\begin{aligned}\mathcal{L} = &-\partial^\mu A^{0\,a\,\nu}\partial_{[\mu}A^{0\,a}_{\nu]} - g^0 f^{abc}(\partial_\mu A^{0\,a}_\nu)A^{0\,b\,\mu}A^{0\,c\,\nu}\\ &+ 2d^0_1\partial^\mu\partial^\nu A^{0\,a\,\rho}\partial_\mu\partial_{[\nu}A^{0\,a}_{\rho]}\\ &+ 2d^0_1 g^0 f^{abc}\partial^\mu\partial^\nu A^{0\,a\,\rho}(\partial_\mu(A^{0\,b}_\nu A^{0\,c}_\rho) + 2A^{0\,b}_\mu\partial_{[\nu}A^{0\,c}_{\rho]})\\ &+ 2d^0_2\partial^\mu\partial_{[\mu}A^{0\,a}_{\rho]}\partial_\nu\partial^{[\nu}A^{0\,a\,\rho]}\\ &+ 2d^0_2 g^0 f^{abc}\partial_\mu\partial^{[\mu}A^{0\,a\,\rho]}(\partial^\nu A^{0\,b}_\nu A^{0\,c}_\rho) + 2A^{0\,b\,\nu}\partial_{[\nu}A^{0\,c}_{\rho]})\\ &- 2d^0_3 f^{abc}\partial^{[\mu}A^{0\,a\,\nu]}\partial_{[\nu}A^{0\,b}_{\rho]}\partial_{[\sigma}A^{0\,c}_{\mu]}\eta^{\rho\sigma} + \mathcal{O}(A^4)\end{aligned} \tag{5.2}$$

The renormalized Lagrangian $\mathcal{L}_{\text{ren.}}$ is identical to (5.2) with the bare entities substituted by the renormalized quantities. Also the counter-terms must have the structure provided by (5.2). Therefore the counter-term Lagrangian up to order A^3 has to be

$$\begin{aligned}\mathcal{L}_{\mathrm{ct}} = &- \delta_2 \partial^\mu A^{a\,\nu} \partial_{[\mu} A^a_{\nu]} - g\delta_1^g f^{abc} (\partial_\mu A^a_\nu) A^{b\,\mu} A^{c\,\nu} \\ &+ 2\delta_1^{d_1 2} \partial^\mu \partial^\nu A^{a\,\rho} \partial_\mu \partial_{[\nu} A^a_{\rho]} \\ &+ 2g\delta_1^{d_1 3} f^{abc} \partial^\mu \partial^\nu A^{a\,\rho} (\partial_\mu (A^b_\nu A^c_\rho) + 2A^b_\mu \partial_{[\nu} A^c_{\rho]}) \\ &+ 2\delta_1^{d_2 2} \partial^\mu \partial_{[\mu} A^a_{\rho]} \partial_\nu \partial^{[\nu} A^{a\,\rho]} \\ &+ 2g\delta_1^{d_2 3} f^{abc} \partial_\mu \partial^{[\mu} A^{a\,\rho]} (\partial^\nu (A^b_\nu A^c_\rho) + 2A^{b\,\nu} \partial_{[\nu} A^c_{\rho]}) \\ &- 2\delta_1^{d_3} f^{abc} \partial^{[\mu} A^{a\,\nu]} \partial_{[\nu} A^b_{\rho]} \partial_{[\sigma} A^c_{\mu]} \eta^{\rho\sigma} + \mathcal{O}(A^4)\,.\end{aligned} \tag{5.3}$$

The introduced symbols δ_i will carry the divergent expressions canceling the one-loop divergences. Where no confusion with the complete terms is possible these will be denoted as counter-terms as well. δ_1^g and δ_2 are dimensionless by construction and the $\delta_1^{d_i X}$ have obviously the same mass dimension as the corresponding d_i.

The terms of both formulations, bare fields and coupling respectively renormalized quantities plus counter-terms, can be compared utilizing the renormalization of the field strength

$$A^a_\mu = Z_2^{-1/2} A^{a\,0}_\mu \tag{5.4}$$

connecting renormalized and bare fields. Thus the counter-terms and the bare and renormalized couplings are related by

$$\begin{aligned} 1 + \delta_2 &= Z_2 & g(1+\delta_1^g) &= g_0 Z_2^{\frac{3}{2}} \\ d_1 + \delta_1^{d_1 2} &= d_1^0 Z_2 & g(d_1 + \delta^{d_1 3}) &= d_1^0 g_0 Z_2^{\frac{3}{2}} \\ d_2 + \delta_1^{d_2 2} &= d_2^0 Z_2 & g(d_2 + \delta^{d_2 3}) &= d_2^0 g_0 Z_2^{\frac{3}{2}} \\ & & d_3 + \delta^{d_3} &= d_3^0 Z_2^{\frac{3}{2}}\,. \end{aligned} \tag{5.5}$$

These relations will be necessary to derive the β functions and will also reduce the remaining degrees of freedom in the results.

5.2 The Renormalization Scale

Because we want to use only divergences themselves to construct the counter-terms, we have to introduce an energy scale μ in order restore the right mass dimension. The scale μ is the renormalization scale at which the one-loop diagrams are evaluated.

In the cut-off renormalization the leading term of the logarithmic divergences is $\log \Lambda^2$ with the cut-off momentum Λ. Such a term by itself has no defined mass dimension. In order to obtain a dimensionless expression, one has to substitute

$$\log \Lambda^2 \to \log \frac{\Lambda^2}{\mu^2} = \log \Lambda^2 - \log \mu^2$$

in the counter-terms. Because the cut-off renormalization is applied in four dimensions the Yang-Mills coupling g is dimensionless. The only remaining dimensionful quantity in the counter-terms is κ^2,

$$[\kappa^2] = -2\,.$$

Because of the absence of quadratic divergences κ^2 only occurs in the dimension-six counter-terms $\delta_1^{d_i}$. These terms also have in four dimensions the mass dimesion minus two, see (3.8). Therefore the right divergence for the logarithmical counter-terms is again

$$\log\Lambda^2 \to \log\frac{\Lambda^2}{\mu^2} = \log\Lambda^2 - \log\mu^2\,.$$

In combination with the one-loop corrections the term $\log\mu^2$ will cancel the $\log E^2$ term at the renormalization point. For energies near the renormalization point these terms can be combined to $\log\frac{\mu^2}{E^2}$, which is dimensionless as required. Aside of their final canceling, all quadratic divergences we saw were of the form $\kappa^2\Lambda^2$, so they were per se dimensionless and no μ has to be inserted.

In dimensional renormalization the divergences are of the form $\frac{2}{\epsilon}$, hence dimensionless. But now the prefactors will get a non-vanishing mass dimension because of the mass dimension of the couplings in $d = 4-\epsilon$ dimensions. Their mass dimension can be derived by comparing the kinetic and interaction terms of the Lagrangian.[1] The action itself must be dimensionless $[S] = 0$, thus the Lagrangian in d dimensions has the mass dimension $[\mathcal{L}] = d$. The kinetic terms for both appearing bosonic fields are $\sim (\partial A)^2$ respectively $\sim (\partial h)^2$, so the fields must have the mass dimension

$$[A] = [h] = \frac{d}{2} - 1\,. \tag{5.6}$$

From the three-gluon interaction $g(\partial A)A^2$ and the gluon–graviton interaction $\kappa h(\partial A)^2$ one reads:

$$\begin{aligned} [g] + 3[A] + 1 =& d \\ [\kappa] + [h] + 2[A] + 2 =& d \end{aligned}$$

which is solved by

$$[g] = 2 - \frac{d}{2} = \frac{\epsilon}{2} \tag{5.7}$$

$$[\kappa] = 1 - \frac{d}{2} = -1 + \frac{\epsilon}{2}\,. \tag{5.8}$$

The Yang-Mills divergences at one-loop order are proportional to g^2 for the self-energy and g^3 for the vertex correction respectively. So the counter-terms

[1] The dimension of κ can be alternatively determined from the Einstein-Hilbert term $\frac{2}{\kappa^2}\mathbf{R}$ using $\mathbf{R} \sim \partial\partial\mathbf{g} \to [\mathbf{R}] = 2$.

δ_2 and δ_1^g must both be proportional to g^2, hence to keep them dimensionless a factor $\mu^{-\epsilon}$ is required. Therefore the divergent factor $\frac{2}{\epsilon}$ has to be extended to

$$\mu^{-\epsilon}\frac{2}{\epsilon}\,.$$

The gravitational divergences are proportional to κ^2 and $g\kappa^2$ respectively. The counter-terms have the same dimension as the same as the corresonding couplings d_i which were derived in chapter 3 and are

$$\begin{aligned}[\delta_1^{d_1}] = [\delta_1^{d_2}] = &-2\\ [\delta_1^{d_3}] = &-\frac{d}{2} = -2 + \frac{\epsilon}{2}\,.\end{aligned}$$

Hence to achive the correct dimensions of the counter-terms, an additional factor of mass dimension

$$\begin{aligned}[\kappa^2] - [\delta_1^{d_{1,2}}] =& \epsilon\\ [g\kappa^2] - [\delta_1^{d_3}] =& \epsilon\end{aligned}$$

is needed. Thus again, the divergent factor has to be

$$\mu^{-\epsilon}\frac{2}{\epsilon}$$

to achieve a dimensionless counter-term. For small ϵ it can be expanded to

$$\mu^{\epsilon}\frac{2}{\epsilon} = \frac{2}{\epsilon}(1 - \epsilon\log\mu + \mathcal{O}(\epsilon^2)) = \frac{2}{\epsilon} - \log\mu^2 + \mathcal{O}(\epsilon)\,.$$

The divergent $\frac{2}{\epsilon}$ of the one-loop diagrams and the counter-term will cancel. As in the cut-off case the term $\log\mu^2$ will cancel the energy logarithm at the renormalization point.

As expected, the divergences derived by means of cut-off and dimensional regularization are of the same magnitude if we substitute $\log\Lambda^2 \Leftrightarrow \frac{2}{\epsilon}$. Thus the same will hold for the counter-terms. In order to be able to handle the counter-terms in both renormalization schemes at once, we redefine the abbreviation introduced in chapter 4 to the new dimensionless quantity

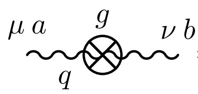
$$\boldsymbol{\Delta} = \begin{cases}\log\dfrac{\Lambda^2}{\mu^2} & \text{in cut-off regularization}\\ \mu^{-\epsilon}\dfrac{2}{\epsilon} & \text{in dimesional regularization}\end{cases} \tag{5.9}$$

including the mass scale μ as described above.

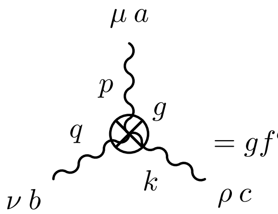
5.3 Determination of the Counter-Terms

The counter-terms must lead to Feynman rules which can reproduce the divergent part of the one-loop diagrams in chapter 4 at tree level. Therefore the next step is the determination of the rules. Here we can use the results of chapter 3 and Appendix C.

The divergences of the pure gauge diagrams have exactly the tensor structure of interactions of the unrenormalized theory. Thus only the counter-terms $\sim \operatorname{tr} F^2$, namely δ_2 and δ_1^g are needed to cancel the divergences. The needlessness of additional counter-terms with a new tensor structure reflects the renormalizability of this sector of the theory. The Feynman rules of the counter-terms are



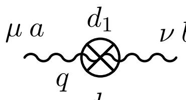
$$= -i\delta^{ab}(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta_2 \tag{5.10}$$



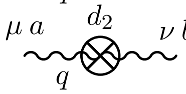
$$= gf^{abc}\left[\eta^{\mu\nu}(p-q)^\rho + \eta^{\nu\rho}(q-k)^\mu + \eta^{\rho\mu}(k-p)^\nu\right]\delta_1^g\,. \tag{5.11}$$

If we saw quadratic divergences in the gravitational sector, i. e. $\sim \kappa^2\Lambda^2$, these would also have the above tensor structure. Thus they would contribute to the counter-terms δ_2 and δ_1^g and hence to the renormalization of g as Robinson and Wilczek found[14]. The absence of these divergences reproduces the results of [16, 17] in diagrammatical approach.

The logarithmic divergences of the gravitational sector all have additional momentum factors, which correspond to higher derivatives in position space, thus in the Lagrangian. The only gauge invariant terms with the right number of derviatives are the dimension-six terms presented in chapter 3. The Feynman rules accord with the rules for the dimension-six terms, but now multiplied by the counter-terms $\delta_1^{d_i}$. They are for two gluons

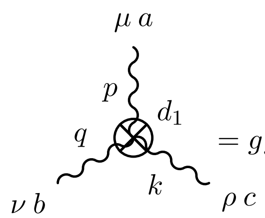


$$= 2i\delta^{ab}q^2(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta_1^{d_1 2} \tag{5.12}$$

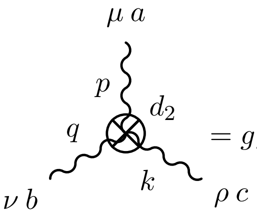


$$= i\delta^{ab}q^2(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta_1^{d_2 2} \tag{5.13}$$

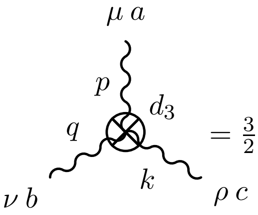
and for three gluons



$$\begin{aligned}= gf^{abc}\big[&\eta^{\mu\nu}(p^\rho(4p\cdot q + 2p\cdot k) - q^\rho(4q\cdot p + 2q\cdot k))\\ &+\eta^{\nu\rho}(q^\mu(4q\cdot k + 2q\cdot p) - k^\nu(4k\cdot q + 2k\cdot p))\\ &+\eta^{\rho\mu}(k^\nu(4k\cdot p + 2k\cdot q) - p^\nu(4p\cdot k + 2p\cdot q))\\ &-2\big(k^\mu k^\nu(p-q)^\rho + p^\nu p^\rho(q-k)^\mu\\ &\quad+q^\rho q^\mu(k-p)^\nu\big)\big]\delta_1^{d_1 3}\end{aligned} \tag{5.14}$$



$$
\begin{aligned}
= g f^{abc}\big[&\eta^{\mu\nu}(p^\rho(2p\cdot q + p\cdot k + 3q\cdot k)\\
&\quad -q^\rho(2q\cdot p + q\cdot k + 3p\cdot k))\\
&+\eta^{\nu\rho}(q^\mu(2q\cdot k + q\cdot p + 3k\cdot p)\\
&\quad -k^\mu(2k\cdot q + k\cdot p + 3q\cdot p))\\
&+\eta^{\rho\mu}(k^\nu(2k\cdot p + k\cdot q + 3p\cdot q)\\
&\quad -p^\nu(2p\cdot k + p\cdot q + 3k\cdot q))\\
&-\big(k^\mu k^\nu(p-q)^\rho + p^\nu p^\rho(q-k)^\mu\\
&\quad +q^\rho q^\mu(k-p)^\nu\big)\\
&-3(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big]\,\delta_1^{d_2}
\end{aligned}
\tag{5.15}
$$



$$
\begin{aligned}
= \tfrac{3}{2} f^{abc}\big[&-\eta^{\mu\nu}(p^\rho q\cdot k - q^\rho k\cdot p)\\
&-\eta^{\nu\rho}(q^\mu k\cdot p - k^\nu p\cdot q)\\
&-\eta^{\rho\mu}(k^\nu p\cdot q - p^\nu q\cdot k)\\
&+(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big]\,\delta_1^{d_3}\,.
\end{aligned}
\tag{5.16}
$$

After determining the tensor structure of the counter-terms, we can now calculate their actual values.

The Yang-Mills counter-terms δ_2 and δ_1^g have to cancel (4.31) and (4.48) in the cut-off regularization scheme and (4.32) and (4.49) in dimensional regularization:

$$
\begin{aligned}
\delta_2 =& \frac{1}{16\pi^2} g^2 C_2 \frac{5}{3}\boldsymbol{\Delta}\\
\delta_1^g =& \frac{1}{16\pi^2} g^2 C_2 \frac{2}{3}\boldsymbol{\Delta}\,.
\end{aligned}
\tag{5.17}
$$

Here $\boldsymbol{\Delta}$ defined in (5.9) is used to abbreviate the parameterization of the divergences in both regularization schemes. This is the well know result, see e. g. [19]. Its reproduction confirms our methods and algorithm.

The two-gluon and three-gluon counter-terms belonging to $\mathcal{O}_1$ and analogously the counter-terms belonging to $\mathcal{O}_2$ are not independent. From (5.5) one sees that

$$
\begin{aligned}
\frac{g^0}{g} Z_2^{\frac{3}{2}} &= 1 + \delta_1^g\\
d_1^0 &= (d_1 + \delta_1^{d_1 2}) Z_2^{-1} = (d_1 + \delta_1^{d_1 2})(1+\delta_2)^{-1}\\
\Rightarrow d_1 + \delta_1^{d_1 3} &= d_1^0 \frac{g^0}{g} Z_2^{\frac{3}{2}}\\
&= (1+\delta_1^g)(d_1 + \delta_1^{d_1 2})(1+\delta_2)^{-1}\,.
\end{aligned}
$$

As we will describe in chapter 6 only linear order in the counter-terms can be used. Thus we obtain

$$
\delta_1^{d_1 3} = \delta_1^{d_1 2} + d_1(\delta_1^g - \delta_2)\,. \tag{5.18}
$$

$\delta_1^{d_1 2}$ and $\delta_1^{d_1 3}$ are purely gravitational[2], thus $\mathcal{O}(\kappa^2)$, and the counter-terms δ_1^g and δ_2 have only Yang-Mills contributions (5.17), $\mathcal{O}(g^2)$. Therefore the two-gluon and three-gluon counter-terms must be identical in leading order and both will the denoted as $\delta_1^{d_1}$ and $\delta_1^{d_2}$ respectively:

$$\begin{aligned} \delta_1^{d_1 2} = \delta_1^{d_1 3} \equiv& \delta_1^{d_1} \\ \text{and analogously:} \quad \delta_1^{d_2 2} = \delta_1^{d_2 3} \equiv& \delta_1^{d_2} \, . \end{aligned} \tag{5.19}$$

Three counter-terms remain when this identity is taken into account. The cancellation of the divergences corresponds to six equations, of which only two are linearly independent. The cancellation of the divergence of the two-gluon amplitude (4.39) or (4.35) requires

$$2\delta_1^{d_1} + \delta_1^{d_2} = \frac{1}{16\pi^2}\frac{1}{6}\kappa^2 \mathbf{\Delta} \tag{5.20}$$

and the cancellation of the three-gluon divergence

$$\begin{aligned} 4g\delta_1^{d_1} + 2\delta_1^{d_2} =& \frac{1}{16\pi^2}\frac{1}{3}g\kappa^2 \mathbf{\Delta} \\ 2g\delta_1^{d_1} + \delta_1^{d_2} =& \frac{1}{16\pi^2}\frac{1}{6}g\kappa^2 \mathbf{\Delta} \\ 3\delta_1^{d_2} - \tfrac{3}{2}\delta_1^{d_3} =& \frac{1}{16\pi^2}\frac{1}{2}g\kappa^2 \mathbf{\Delta} \\ -2g\delta_1^{d_1} - \delta_1^{d_2} =& -\frac{1}{16\pi^2}\frac{1}{6}g\kappa^2 \mathbf{\Delta} \\ -3\delta_1^{d_2} + \tfrac{3}{2}\delta_1^{d_3} =& -\frac{1}{16\pi^2}\frac{1}{2}g\kappa^2 \mathbf{\Delta} \, . \end{aligned} \tag{5.21}$$

Thus the divergences are canceled by these dimension-six counter-terms:

$$\begin{aligned} \delta_1^{d_1} =& \alpha\frac{1}{16\pi^2}\frac{1}{12}\kappa^2 \mathbf{\Delta} \\ \delta_1^{d_2} =& (1-\alpha)\frac{1}{16\pi^2}\frac{1}{6}\kappa^2 \mathbf{\Delta} \\ \delta_1^{d_3} =& -\alpha\frac{1}{16\pi^2}\frac{1}{3}g\kappa^2 \mathbf{\Delta} \end{aligned} \tag{5.22}$$

for any α with $0 \leq \alpha \leq 1$. This indefiniteness of the solutions is in accordance with the dependence of the dimension-six term on each other (3.3) and is not to be ascribed to any insufficiency of the approach. Utilizing this freedom of choice we decide for the most practical solution $\alpha = 0$ leaving only one dimension-six counter-term, namely $\delta_1^{d_2}$:

$$\begin{aligned} \delta_1^{d_1} = \delta_1^{d_3} =& 0 \\ \delta_1^{d_2} =& \frac{1}{16\pi^2}\frac{1}{6}\kappa^2 \mathbf{\Delta} \, . \end{aligned} \tag{5.23}$$

[2]Yang-Mills contributions to these terms would originate from the divergences of one-loop diagrams involving $\mathcal{O}_1$. These graphs were not calculated because they depend on d_1 and hence are not corrections to the classic Einstein-Yang-Mills system.

This solution reproduces the result of Deser et al.[6, 4] if one bears in mind the differing definitions of κ^2 and ϵ. Again, we should note the special nature of $\mathcal{O}_2$. As already mentioned in chapter 3, it vanishes on-shell and can be removed by a field redefinition. Thus no gravitational corrections in the Yang-Mills sector at the computed leading order $\mathcal{O}(\kappa^2)$ remain.

VI
Computing the β-Functions

After obtaining the counter-terms we are now able to compute the Callan-Symanzik β-function for g, d_1, d_2 and d_3. The latter three depend linearly on each other. Therefore we will first calculate the three β-functions for a general combination of possible counter-terms (5.22). Then we will also present the β-function for the solution (5.23), which is distinguished by the special nature of $\mathcal{O}_2$ and its simplicity.

Our results do not allow to compute the β-function for the gravitational coupling κ because it would depend on the counter-terms of the graviton–gluon couplings and the graviton kinetic term. Thus the one-loop divergences of the diagrams with outer graviton would be needed, e. g. graviton selfinteraction, graviton–gluon interaction etc. We did not examine these because the focus of this work is on the influence of gravity to the Yang-Mills renormalization.[1]

The β-function for a coupling λ is defined as

$$\beta_\lambda = \mu \frac{\partial \lambda}{\partial \mu} \,. \tag{6.1}$$

To utilize this definition we have to express the couplings as functions of the energy scale, i. e. in our case of the counter-terms. Since all calculations are done on one-loop level, only expressions linear in the counter-terms must be kept. Higher terms correspond to one-particle-reducible multi-loop diagrams which consist only of the calculated graphs. The complete higher order divergences include the contributions of one-particle-irreducible multi-loop graphs, whose values are not obtained by our calculation.

To compute the Yang-Mills β-function β_g we need the relation

$$(1+\delta_1^g) = \frac{g_0}{g}(1+\delta_2)^{\frac{3}{2}}$$

from (5.5). Neglecting all terms of order $\mathcal{O}(\delta^2)$ and higher, we get the formula

$$g(\mu) = g_0 \frac{(1+\delta_2)^{\frac{3}{2}}}{1+\delta_1^g} = g_0 \left(1 + \frac{3}{2}\delta_2 - \delta_1^g + \mathcal{O}(g^4, \kappa^4, g^2\kappa^2)\right)$$

for the energy dependence of g. Such a formulation is of course not of practical use for reading off $g(\mu)$ directly because it still contains all divergences in g_0 and the δ's, which must cancel for a physical μ.

The bare coupling constant g_0 is of course independent of the energy scale μ,

$$\mu \frac{\partial g_0}{\partial \mu} = 0$$

[1] The results of Deser et al. [6] for these terms are all $\sim R$, thus vanishing in our flat background metric $\eta_{\mu\nu}$.

so the β-function up to order $\mathcal{O}(g^2, \kappa^2)$ is

$$\beta = g_0 \mu \frac{\partial}{\partial \mu} \left(\frac{3}{2} \delta_2 - \delta_1^g \right) .$$

Finally we use the fact that the difference between bare and running coupling

$$g_0 = g - \delta_g$$

will again be of order g^2 or κ^2. So we obtain as the formula for g's β-function

$$\beta_g = g \mu \frac{\partial}{\partial \mu} \left(\frac{3}{2} \delta_2 - \delta_1^g \right) . \tag{6.2}$$

When we now insert the counterterms (5.17), we get

$$\begin{aligned} \beta_g =& g\mu \frac{\partial}{\partial \mu} \left\{ \left[\frac{3}{2} \frac{1}{16\pi^2} \frac{5}{3} g^2 C_2 - \frac{1}{16\pi^2} \frac{2}{3} g^2 C_2 \right] \mathbf{\Delta} \right\} \\ =& \frac{1}{16\pi^2} \frac{11}{6} g^3 C_2 \mu \frac{\partial}{\partial \mu} \mathbf{\Delta} . \end{aligned}$$

And finally, using (5.9)

$$\mu \frac{\partial}{\partial \mu} \mathbf{\Delta} = \mu \frac{\partial}{\partial \mu} \left\{ \begin{matrix} \log \dfrac{\Lambda^2}{\mu^2} \\ \mu^{-\epsilon} \dfrac{2}{\epsilon} \end{matrix} \right\} = -2 + \left\{ \begin{matrix} 0 \\ \mathcal{O}(\epsilon) \end{matrix} \right\} = -2 , \tag{6.3}$$

we obtain the β-function

$$\beta_g = -\frac{1}{16\pi^2} \frac{11}{3} C_2 g^3 . \tag{6.4}$$

The constant C_2 originates from multiple structure constants in the loop and $C_2 = N$ for the gauge group $SU(N)$. This is the famous classical result for the Yang-Mills β-function at one-loop level. Clearly without any gravitational contribution as Robinson and Wilczek found in [14], but in total accordance with Pietrykowski [16] and Toms [17].

At this point we want to add an interesting note: If we had found a gravitational modification of the Yang Mills counter-terms it should be – as mentioned before – quadratically divergent for dimensional reasons, i. e. $\delta_2' = b\kappa^2\Lambda^2$. Obviously such a term is independent of the renormalization scale μ, thus its contribution to the β-function vanishes

$$\Delta\beta_g = \mu \frac{\partial}{\partial \mu} b\kappa^2 \Lambda^2 = 0 .$$

The energy dependence in [15] enters the result as a lower integration limit which is identified with the energy of the background field. In our calculations no such scale is available, due to the abcence of background field in our approach.

Analogously as for β_g (5.5) is used to derive the formulae for $\beta_{d_1,\dots,d_3}$:

$$\beta_{d_1} = d_1 \mu \frac{\partial}{\partial \mu} \delta_2 - \mu \frac{\partial}{\partial \mu} \delta_1^{d_1 2} \tag{6.5}$$

$$\beta_{d_2} = d_2 \mu \frac{\partial}{\partial \mu} \delta_2 - \mu \frac{\partial}{\partial \mu} \delta_1^{d_2 2} \tag{6.6}$$

$$\beta_{d_3} = d_3 \mu \frac{\partial}{\partial \mu} \frac{3}{2} \delta_2 - \mu \frac{\partial}{\partial \mu} \delta_1^{d_3 3} \,. \tag{6.7}$$

As mentioned in the discussion of (5.18) the wavefunction counter-term vanishes at order κ^2, $\delta_2|_{\mathcal{O}(\kappa^2)} = 0$. Therefore δ_2 does not contribute to the β-functions at order κ^2. For a generic combination of counter-terms (5.22) the β-functions take the form:

$$\beta_{d_1} = (1-\alpha)\alpha \frac{1}{16\pi^2} \frac{1}{6} \kappa^2 \tag{6.8}$$

$$\beta_{d_2} = \alpha \frac{1}{16\pi^2} \frac{1}{3} \kappa^2 \tag{6.9}$$

$$\beta_{d_3} = -(1-\alpha) \frac{1}{16\pi^2} \frac{2}{3} \kappa^2 \,. \tag{6.10}$$

As already argued in chapter 5 the counter-term solution (5.23) with $\alpha = 0$ is distinguished. The corresponding β functions are

$$\beta_{d_1} = \beta_{d_3} = 0 \tag{6.11}$$

$$\beta_{d_2} = \frac{1}{16\pi^2} \frac{1}{3} \kappa^2 \tag{6.12}$$

of which only β_{d_2} is non-vanishing.

VII
Summary and Conclusions

We diagrammatically calculated the one-loop corrections of a Yang-Mills system coupled to gravitation. In contrast to the background field approach, the background space time metric was fixed to the flat Minkowski metric in our calculation. Therefore no gravitational corrections $\sim R$, $\sim R^{\mu\nu}$ become visible in our approach. A vacuum background on the other hand avoids necessity of solving the equations of motion – especially the Einstein equation – for background fields.

All Feynman rules were derived and the one-loop diagrams were calculated using the computer algebra system FORM. Only its application allows us to handle the partly very complex expressions.

We applied two different regularization methods: cut-off and dimensional regularization. Therefore we were able to see potential quadratic divergences, which turn out to be absent in the results, but we also took advantage of the fact that dimensional regularization respects the gauge symmetry. Finally the comparison of the results in both regularization schemes allows for an important cross-check of our calculations.

We found as the only gravitational contribution to the Yang-Mills sector

$$\delta\mathcal{L} = \frac{1}{16\pi^2}\frac{\kappa^2}{6}\frac{2\mu^{d-4}}{4-d}\,\mathrm{tr}\left[(D_\mu F^{\mu\rho})(D^\nu F_{\nu\rho})\right] . \tag{7.1}$$

In the case of an Abelian gauge theory, instead of the trace the counter-term is modified by an additional factor $\frac{1}{2}$. Thereby our result is in complete accordance with the result of Deser, Tsao and van Nieuwenhuizen [4, 6]. We especially found no gravitational contribution to the β-function.

The interpretation of the dimension-six counter-term $\mathcal{O}_2$ is still not clarified. The possibility to remove it through field redefinition poses the question of its physical meaning. But we are hopeful that the ongoing discussion will yield an answer.

Contemporary physical models propose extra spatial dimensions[27, 28]. In these models gravity is not confined to the four dimensional space time, which would explain the low coupling strength of gravity. Therefore the gravitational scale would be lower. Gogoladze and Leung pointed out that in such scenarios the effect on the running coupling should be measurable at the Large Hadron Collider (LHC)[29]. Thus the LHC will probably allow us to validate the existence of this quantum gravity effect.

Interesting extensions to this work would be the inclusion of quark fields. The Higgs field, especially for its non-vanishing vacuum expectation value, should also be an important entity of the standard model of particles to include in the calculations. The latter extension is probably connected to the open question of the discrepancy of the cosmological value and the field theoretical prediction of the cosmological constant Λ of 10^{120}.

The combination of supersymmetric gauge theories and effective quantum gravity was no yet examined and should be a fruitful extension, too.

A very interesting topic would be the Einstein-Yang-Mills system with the gravitons propagating in extra dimensions because of the lower gravitational scale in such scenarios.

Appendix A

Symmetry Factors of Feynman Diagrams

To compute the contribution of a Feynman graph, one needs to calculate its weight factor w. This factor originate from its symmetry and is an integral element of the contribution of the considered process. We used a modified version of the formula by Wieczorek et al.[25, 26]:

$$w = \frac{N_O N_I}{\prod\limits_i \vec{\alpha}_i! \prod\limits_j \beta_j!} \tag{A.1}$$

Here the used symbols mean

N_O	Number of possibilities to connect outer lines with lines of the vertices
N_I	Number of possibilities to connect inner lines
$\vec{\alpha}_i$	Number of equivalent lines of vertex i
β_j	Number of vertices of the type j

The $\vec{\alpha}_i$s are index multiplets of the length corresponding to the number of different lines connected to the vertex. The faculty of these multiplets is defined as

$$\vec{\alpha} = \left(\alpha^1, \alpha^2, \dots, \alpha^n\right) \qquad \Longrightarrow \qquad \vec{\alpha}! = \alpha^1! \cdot \alpha^2! \cdots \alpha^n!\,.$$

The easiest way to calculate w is to write down first the vertices and the outer lines without connecting them. It is also practical to enumerate the vertices and their equivalent lines, in order to bear in mind when two of these are permuted. The entities of (A.1) are obtained in the following way:

$\boldsymbol{N_O}$ Take one arbitrary outer line and count the vetrex lines (of all vertices) this line can be connected with. Then count the possible connections of the next line with the remaining vertex lines. Repeat this procedure until all outer lines are connected. Consider always to keep the shape of the diagram, this forbids some combinations the vertex and outer lines (see examples).

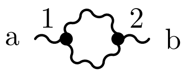
$\boldsymbol{N_I}$ Now count all possibilities for inner lines which are consistent with the diagram shape.

$\boldsymbol{\vec{\alpha}_i}$ Count the lines of each vertex which are of the same kind. If the lines are directed (e. g. fermions or charged particles), in- and outgoing lines are diffent, of course.

$\boldsymbol{\beta_j}$ Count all vertices of the same type.

To illustrate the usage of this formula we will present some examples.

Two Examples from the pure Yang-Mills Theory

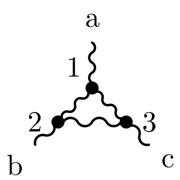


- Two identical vertices $\rightarrow \beta = 2$, both with three equivalent lines $\rightarrow \alpha_1 = \alpha_2 = 3$

- Possibilities for the outer line coming from a: 6 (3 lines on each vertex 1 and 2)
  Possibilities for the outer line coming from b: 3 (after connecting line a to one vertex, i. e. 1, line b must be connected to the other vertex)
  $\rightarrow N_O = 6 \cdot 3 = 18$

- Possibilities for the 1st inner line: 2
  Possibilities for the 2nd inner line: 1
  $\rightarrow N_I = 2$

- $\rightarrow w = \dfrac{18\,2}{2!\,3!\,3!} = \dfrac{1}{2}$



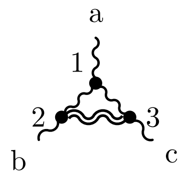
- Three identical vertices $\rightarrow \beta = 3$, both with three equivalent lines $\rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 3$

- Possibilities for the outer line coming from a: 9 (3 lines on each vertex)
  Possibilities for the outer line coming from b: 6 (3 lines on each remaining vertex, i e. 2 and 3)
  Possibilities for the outer line coming from c: 3
  $\rightarrow N_O = 162$

- Possibilities for the 1st inner line: 4 (connect the 1st line of 1 with vertex 2 or 3, e.g. 2+2 possibilities)
  Possibilities for the 2nd inner line: 2 (the 2nd line of 1 must be connected with an different vertex then the 1st one)
  Possibilities for the 3rd inner line: 1
  $\rightarrow N_I = 8$

- $\rightarrow w = \dfrac{162\,8}{(3!)^4} = 1$

Two Examples from the Einstein-Yang-Mills Theory

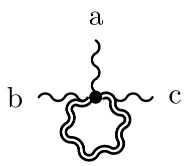


- Two identical vertices $\rightarrow \beta_1 = 1, \beta_2 = 1$, vertex 1 has 3 equivalent lines, 2 and 3 have two
  $\rightarrow \alpha_1 = 3, \vec{\alpha}_2 = \vec{\alpha}_3 = (2, 1)$

- Possibilities for the outer line coming from a: 3 (line a must be connected to 1, otherwise we would get a different graph)
  Possibilities for the outer line coming from b: 4 (3 lines on each vertex 2 and 3)
  Possibilities for the outer line coming from c: 2
  $\rightarrow N_O = 24$

- Possibilities for the inner graviton line: 1
  Possibilities for the 1st inner gluon line: 2
  Possibilities for the 2nd inner gluon line: 1
  $\rightarrow N_I = 2$

- $\rightarrow w = \frac{24\,2}{2!\,3!\,2!\,2!} = 1$



- One vertex $\rightarrow \beta = 1$ with three and two equivalent lines
  $\rightarrow \vec{\alpha} = (3, 2)$

- Possibilities for the outer lines: 3!
  $\rightarrow N_O = 6$

- Possibilities for the inner line: 1
  $\rightarrow N_I = 1$

- $\rightarrow w = \frac{6}{3!\,2!} = \frac{1}{2}$

Appendix B
Feynman Parameters for three Propagator Graphs

After regularizing and calculating the divergent momentum intergrals we have to deal with three types of integrals of the previously introduced Feynman parameters:

$$\iiint_0^1 \delta(x+y+z-1)x^m y^n \mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{B.1}$$

$$\iiint_0^1 \delta(x+y+z-1)\frac{x^m y^n}{x(1-x)+y(1-y)-xy}\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{B.2}$$

$$\text{and} \iiint_0^1 \delta(x+y+z-1)x^m y^n \log\left(x(1-x)+y(1-y)-xy\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{B.3}$$

with $m, n \in \mathbb{N}$.

To calculate these, so we substiute the parameter by three new variables r, a and b

$$x = rab \tag{B.4}$$

$$y = r(1-b) \tag{B.5}$$

$$z = r(1-a)b \tag{B.6}$$

Now the differentials become

$$\mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2 b\mathrm{d}r\mathrm{d}a\mathrm{d}b$$

and the δ-function simplifies to

$$\delta(x+y+z-1) = \delta(r-1)$$

and just fixes $r = 1$. For this value of r the remaining parameters a and b both run independently from 0 to 1 to cover the intergationarea. Thus all triple integrale become such simpler double integrals:

$$\iiint_0^1 \delta(x+y+z-1)f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_0^1 b\tilde{f}(a,b)\mathrm{d}a\mathrm{d}b$$

This means for the appearing intergrals (B.1)–(B.3):

$$\iiint_0^1 \delta(x+y+z-1)x^m y^n \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_0^1 a^{\tilde{m}} b^{\tilde{n}} \mathrm{d}a\mathrm{d}b \tag{B.7}$$

$$\begin{aligned}\iiint_0^1 \delta(x+y+z-1)&\frac{x^m y^n}{x(1-x)+y(1-y)-xy}\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \iint_0^1 \frac{a^{\tilde{m}} b^{\tilde{n}}}{1-(a^2-a+1)b}\mathrm{d}a\mathrm{d}b\end{aligned} \tag{B.8}$$

$$\begin{aligned}\iiint_0^1 \delta(x+y+z-1)x^m y^n \log\left(x(1-x)+y(1-y)-xy\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ = \iint_0^1 a^{\tilde{m}} b^{\tilde{n}} \log b(1-(a^2-a+1)b)\mathrm{d}a\mathrm{d}b\end{aligned} \tag{B.9}$$

For the needed values of $\tilde{m}$ and $\tilde{n}$ the results of the integration are listed in the following tables. The used constant $\mathcal{C}$ is – expressed in terms of the trigamma function ψ_1 or Clausen's integral Cl_2 [30]:

$$\begin{aligned}\mathcal{C} &= \frac{1}{3}(\psi_1(\tfrac{1}{3}) - \psi_1(\tfrac{2}{3})) \\ &= 2\sqrt{3}\,Cl_2(\tfrac{2}{3}\pi) \\ &= 3\sum_{k=1}^{\infty}\left(\frac{1}{(3k-2)^2} - \frac{1}{(3k-1)^2}\right) \\ &\approx 2.34391\end{aligned}$$

m	n	$\int_0^1 \int_0^1 a^m b^n \mathrm{d}a\mathrm{d}b$
0	1	$\frac{1}{2}$
0	2	$\frac{1}{3}$
1	2	$\frac{1}{6}$

Table B.1: The values of intergrals (B.7)

m	n	$\int_0^1 \int_0^1 \frac{a^m b^n}{1-(a^2-a+1)b} \mathrm{d}a\mathrm{d}b$
0	0	$\mathcal{C}$
0	1	$\frac{2}{3}\mathcal{C}$
0	2	$\frac{2}{3}\mathcal{C} - \frac{1}{3}$
0	3	$\frac{20}{27}\mathcal{C} - \frac{19}{27}$
1	1	$\frac{1}{3}\mathcal{C}$
1	2	$\frac{1}{3}\mathcal{C} - \frac{1}{6}$
1	3	$\frac{10}{27}\mathcal{C} - \frac{19}{54}$
2	1	$\frac{2}{3}\mathcal{C} - 1$
2	2	$\frac{1}{3}\mathcal{C} - \frac{1}{3}$
2	3	$\frac{8}{27}\mathcal{C} - \frac{17}{54}$
3	3	$\frac{7}{27}\mathcal{C} - \frac{8}{27}$

Table B.2: The values of intergrals (B.8)

m	n	$\int_0^1 \int_0^1 a^m b^n \log b(1-(a^2-a+1)b)\mathrm{d}a\mathrm{d}b$
0	0	$\mathcal{C} - 4$
0	1	$\frac{1}{3}\mathcal{C} - \frac{3}{2}$
0	2	$\frac{2}{9}\mathcal{C} - 1$
1	2	$\frac{1}{9}\mathcal{C} - \frac{1}{2}$

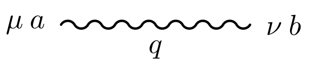
Table B.3: The values of intergrals (B.9)

Appendix C

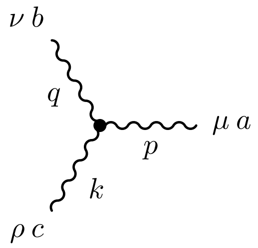
Feynman Rules of the pure Yang Mills Theory

The Feynman rules of the pure Yang Mills theory in Feynman gauge (ξ=1) and without fermions, taken from [21].

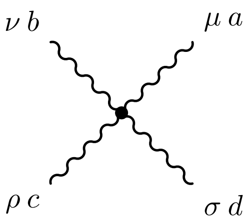
All momenta denoted in the graph are counted inwardly. Including the momentum of the outgoing ghost, which is writen as $-p'$ in the graph, thus p' is the *outgoing* momentum of the ghost.



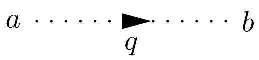
$$= \frac{-i\eta_{\mu\nu}\delta^{ab}}{q^2 + i\varepsilon}$$



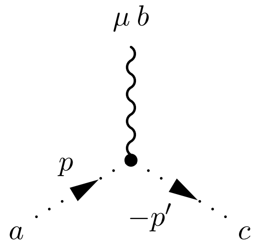
$$= g f^{abc}\left[\eta^{\mu\nu}(p-q)^{\rho} + \eta^{\nu\rho}(q-k)^{\mu} + \eta^{\rho\mu}(k-p)^{\nu}\right]$$



$$\begin{aligned} = -ig^2[\ & f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ & + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ & + f^{ade}f^{cbe}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma})] \end{aligned}$$



$$= \frac{i\delta^{ab}}{q^2 + i\varepsilon}$$



$$= \quad g f^{abc} p'^{\mu}$$

Bibliography

[1] G. 't Hooft and M. J. G. Veltman, "One loop divergencies in the theory of gravitation", *Annales Poincare Phys. Theor.*, vol. A20, pp. 69–94, 1974.

[2] D. M. Capper, G. Leibbrandt and M. Ramon Medrano, "Calculation of the graviton selfenergy using dimensional regularization", *Phys. Rev.*, vol. D8, pp. 4320–4331, 1973.

[3] S. Deser and P. van Nieuwenhuizen, "Nonrenormalizability of the Quantized Einstein-Maxwell System", *Phys. Rev. Lett.*, vol. 32, p. 245, 1973.

[4] S. Deser and P. van Nieuwenhuizen, "One Loop Divergences of Quantized Einstein-Maxwell Fields", *Phys. Rev.*, vol. D10, p. 401, 1974.

[5] S. Deser and P. van Nieuwenhuizen, "Nonrenormalizability of the Quantized Dirac-Einstein System", *Phys. Rev.*, vol. D10, p. 411, 1974.

[6] S. Deser, H.-S. Tsao and P. van Nieuwenhuizen, "One Loop Divergences of the Einstein Yang-Mills System", *Phys. Rev.*, vol. D10, p. 3337, 1974.

[7] O. Lauscher and M. Reuter, "Is quantum Einstein gravity nonperturbatively renormalizable?", *Classical and Quantum Gravity*, vol. 19, no. 3, pp. 483–492, 2002, hep-th/0110021.

[8] M. Reuter and F. Saueressig, "Renormalization group flow of quantum gravity in the Einstein-Hilbert truncation", *Physical Review D*, vol. 65, no. 6, p. 65016, 2002.

[9] O. Lauscher and M. Reuter, "Asymptotic Safety in Quantum Einstein Gravity: nonperturbative renormalizability and fractal spacetime structure", 2005, hep-th/0511260.

[10] M. Reuter and F. Saueressig, "Functional Renormalization Group Equations, Asymptotic Safety, and Quantum Einstein Gravity", 2007, arXiv:0708.1317 [hep-th].

[11] J. Donoghue, "Leading quantum correction to the Newtonian potential", *Physical Review Letters*, vol. 72, no. 19, pp. 2996–2999, 1994.

[12] J. Donoghue, "General relativity as an effective field theory: The leading quantum corrections", *Physical Review D*, vol. 50, no. 6, pp. 3874–3888, 1994.

[13] J. F. Donoghue, "Introduction to the Effective Field Theory Description of Gravity", 1995, gr-qc/9512024.

[14] S. P. Robinson and F. Wilczek, "Gravitational correction to running of gauge couplings", *Phys. Rev. Lett.*, vol. 96, p. 231601, 2006, hep-th/0509050.

[15] S. P. Robinson, *Two Quantum Effects in the Theory of Gravitation.* PhD thesis, Massachusetts Institute of Technology, 2005, `http://web.mit.edu/spatrick/www/PhDThesis/`.

[16] A. R. Pietrykowski, "Gauge dependence of gravitational correction to running of gauge couplings", *Phys. Rev. Lett.*, vol. 98, p. 061801, 2007, hep-th/0606208.

[17] D. J. Toms, "Quantum gravity and charge renormalization", *Physical Review D*, vol. 76, no. 4, p. 45015, 2007, arXiv:0708.2990 [hep-th].

[18] F. Berends and R. Gastmans, "Quantum Gravity and the electron and muon anomalous magnetic moments", *Physics Letters B*, vol. 55, no. 3, pp. 311–312, 1975.

[19] P. Pascual and R. Tarrach, *QCD: Renormalization for the Practitioner*, vol. 194 of *Lect. Notes Phys.* Springer-Verglag, 1984.

[20] L. D. Faddeev and V. N. Popov, "Feynman diagrams for the Yang-Mills field", *Phys. Lett.*, vol. B25, pp. 29–30, 1967.

[21] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory.* Westview Press, 1995.

[22] J. Vermaseren, *Form 3.2.* Available at `http://www.nikhef.nl/~form/`.

[23] A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics*, vol. 11 of *Interscience Moographs and Texts in Physics and Astronomy.* Interscience Publishers, 1965.

[24] G. 't Hooft and M. J. G. Veltman, "Regularization and Renormalization of Gauge Fields", *Nucl. Phys.*, vol. B44, pp. 189–213, 1972.

[25] B. Geyer, D. Robaschik and E. Wieczorek, *Theorie der Tiefinelastischen Lepton-Hadron-Streuung II Quantenchromodynamik.* Zeuthen: Institut für Hochenergiephysik Akademie der Wissenschaften der DDR, 1977.

[26] B. Geyer, J. Kripfganz, J. Ranft, D. Robaschik and E. Wieczorek, *Einführung in die Quantenfeldtheorie der Elementarteilchen.* Scripten zur Physik, Berlin: VEB Deutscher Verlag der Wissenschaften, 1990.

[27] N. Arkani-Hamed and G. Dimopoulos, S.and Dvali, "The hierarchy problem and new dimensions at a millimeter", *Phys. Lett. B*, vol. 429, p. 263, 1998.

[28] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, "Phenomenology, astrophysics, and cosmology of theories with submillimeter dimensions and TeV scale quantum gravity", *Physical Review D*, vol. 59, no. 8, p. 86004, 1999.

[29] I. Gogoladze and C. Leung, "Probing the gravitational scale via running gauge couplings", *Physics Letters B*, vol. 645, no. 5-6, pp. 451–454, 2007.

[30] E. W. Weisstein, "Clausen's integral", *MathWorld – A Wolfram Web Resource*, 2005, `http://mathworld.wolfram.com/ClausensIntegral.html`.

Used Software

- FORM 3.2 by *Jos Vermaseren*
- MATHEMATICA 5.2 by *Wolfram Research*
- LaTeX 2_ε
- `feyn`MP and METAPOST