HUMBOLDT-UNIVERSITÄT ZU BERLIN

Mathematisch–Naturwissenschaftliche Fakultät I

Institut für Physik
AG Quantenfeldtheorie und Stringtheorie

Diplomarbeit
zur Erlangung des akademischen Grades des
Diplom-Physikers (Dipl.-Phys.)

Lee-Wick Gauge Theory and Effective Quantum Gravity

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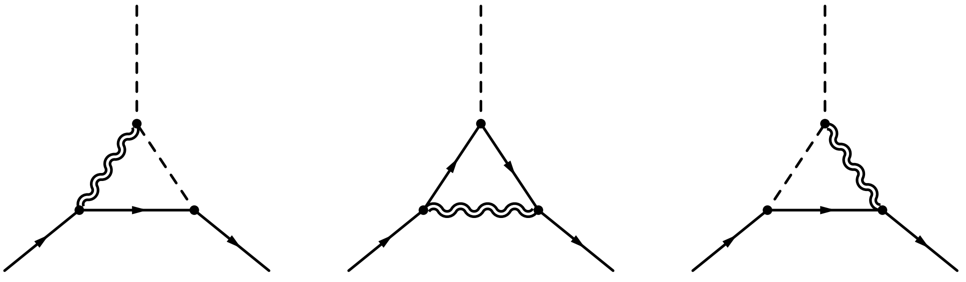
geboren am 26.10.1983 in Berlin



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Berlin, den 30. Dezember 2008
(revised version of August 20, 2009)

Zusammenfassung

Im ersten Abschnitt dieser Arbeit wird die Ein-Schleifen-Renormierung der Lee-Wick-Eichtheorie, einer Verallgemeinerung der Yang-Mills-Theorie mit höheren kovarianten Ableitungen, mit und ohne Verwendung der Hintergrundfeldmethode untersucht. Es wird gezeigt, dass die Theorie asymptotisch frei ist. Anschließend wird der Limes zur Yang-Mills-Theorie durchgeführt.

Im zweiten Abschnitt werden die Ein-Schleifen-Konterterme in der Einstein-Yang-Mills-Theorie untersucht. Dabei wird gezeigt, dass die fermionischen und skalaren Terme höherer Ableitungen im Gegensatz zum Eichsektor nicht mit den Termen höherer Ableitungen des Lee-Wick-Standardmodells übereinstimmen.

Des Weiteren werden die Gravitationsbeiträge niedrigster Ordnung zu den β-Funktionen der Yukawa- und φ^4-Theorie bestimmt. Es wird gezeigt, wie die Gravitation das Laufen der Kopplungskonstanten bei niedrigen Energien modifiziert. Eine Extrapolation der Resultate zu hohen Energien zeigt, dass bei *massiven* Teilchen die gravitativen Wechselwirkungen zu asymptotischer Freiheit der Yukawa- und φ^4-Theorie führen können.

Abstract

In the first part of this thesis we investigate the one-loop renormalization of Lee-Wick gauge theory, a higher covariant derivative generalization of Yang-Mills theory, with and without applying the background field method. We show that this theory is asymptotically free and we perform the limit to Yang-Mills theory.

In the second part the one-loop counterterms in Einstein Yang-Mills theory are examined. We show that, in contrast to the gauge sector, the fermionic and scalar higher-derivative counterterms do not coincide with the higher-derivative terms in the Lee-Wick Standard Model.

Furthermore we determine the lowest order gravitational contributions to the β functions of Yukawa and φ^4 theory. We show how gravity modifies the running of the couplings at low energies. Extrapolating our results to high energies, we find that for *massive* particles the gravitational interactions can lead to asymptotic freedom of Yukawa and φ^4 theory.

Acknowledgments

First of all I would like to use this opportunity to thank my parents Frank and Christine Schuster for their constant love and support throughout my life.

I wish to express my gratitude to my supervisor Prof. Jan Plefka for his time, helpful advises and encouragement, as well as for giving me the opportunity to work in the Quantum Field Theory and String Theory group at the Humboldt University Berlin.

Furthermore, I would like to thank Prof. Ebert for drawing my attention to the topics of this thesis and for dedicating plenty of time to support this work.

I am especially grateful to Prof. Slavnov for the numerous discussions about quantum field theory during his stay at the Humboldt University.

I also wish to thank Andreas Rodigast for the fruitful collaboration on the topic of higher-derivative counterterms.

Last but not least I would like to thank the former and present members of the Quantum Field Theory and String Theory group for the friendly and open atmosphere, especially thanking my office colleagues and fellow graduands Nikolai Beck, Volker Branding, Charlotte Grosse Wiesmann, Ralf Sattler, Per Sudin and Konstantin Wiegandt for stimulating discussions on physical and non-physical topics. Cordial thanks to Sylvia Richter for her administrative assistance.

I am very obliged to Charlotte Grosse Wiesmann, Andreas Rodigast, Franz Schuster and Konstantin Wiegandt for comments on the manuscript.

This diploma thesis is available at
`http://qft.physik.hu-berlin.de`.

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Notations and Conventions

We use the metric signature $(+---)$, hence for Minkowski space we have

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Derivatives with respect to covariant (x_μ) and contravariant (x^μ) coordinates are abbreviated as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \text{and} \qquad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} .$$

Summation over repeated indices is understood in all equations and for contractions we use the notations

$$V_\mu W^\mu = VW = V \cdot W \qquad \text{and} \qquad V_\mu V^\mu = V^2 .$$

For total derivatives we use the abbreviation t.d. when their explicit form is irrelevant.
To indicate symmetrization or anti-symmetrization of indices we use round or square brackets respectively:

$$A^{(\mu_1 \cdots \mu_n)} = \frac{1}{n!} \sum_{\sigma \in S_n} A^{\mu_{\sigma(1)} \cdots \mu_{\sigma(n)}}$$
$$A^{[\mu_1 \cdots \mu_n]} = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) A^{\mu_{\sigma(1)} \cdots \mu_{\sigma(n)}} .$$

We denote the gauge field by $A_\mu = A^a_\mu T^a$, where the Hermitian generators of the fundamental representation are normalized to

$$\text{tr}\{T^a T^b\} = \tfrac{1}{2}\delta^{ab} .$$

The structure constants f^{abc} of the Lie algebra are defined by

$$[T^a, T^b] = i f^{abc} T^c$$

and the matrices of the adjoint representation are $(T^b_{\text{ad}})^{ac} = i f^{abc}$.
We use natural units, where $c = \hbar = 1$. In this system

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1} .$$

We will omit the hat on the Fourier transform $\hat{f}(k)$ of a function $f(x)$ when there is no potential for confusion. $f(x)$ and $\hat{f}(k)$ are related by

$$f(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ikx} \hat{f}(k) \qquad \int \mathrm{d}^4 x \, e^{ikx} = (2\pi)^4 \delta(k) .$$

In all Feynman rules the momenta of gauge bosons and real scalars are counted as ingoing.

Chapter 1
Introduction

The most important tool in understanding and describing the microscopic world is quantum field theory[1]. It is capable of combining three of the major themes of modern physics: quantum mechanics, the field concept and the principle of relativity. Originally introduced to describe quantum electrodynamics, quantum field theory has become the basis of modern elementary particle physics.

It provides the framework for the formulation of the standard model, which describes the electroweak and strong interactions of elementary particles. Despite its great success, the standard model can only be regarded as an effective low-energy theory because it does not include gravity. We expect the effects of quantum gravity to become large at the Planck scale, corresponding to energies of $\sim 10^{19}$GeV or distances of $\sim 10^{-33}$cm.

Another issue of the standard model is the hierarchy puzzle. The mass of the Higgs boson acquires quadratically divergent radiative corrections. In order to keep the Higgs mass small compared to the Planck scale a delicate cancellation has to happen, which requires an extreme fine-tuning. This motivated among others a supersymmetric extension of the standard model.

In 2007, Grinstein, O'Connel, and Wise [4] suggested an alternative solution to the hierarchy puzzle. Their proposition is based on the ideas of Lee and Wick [5, 6] who studied the consequences of the assumption that the modification of the photon propagator

$$\left(\frac{-i\eta_{\mu\nu}}{p^2}\right)_{\text{reg}} = \overbrace{\frac{-i\eta_{\mu\nu}}{p^2}}^{\text{photon}} + \overbrace{\frac{i\eta_{\mu\nu}}{p^2 - M^2}}^{\text{Lee-Wick photon}} = \frac{i\eta_{\mu\nu}M^2}{p^2(p^2-M^2)} , \tag{1.1}$$

in the Pauli-Villars regularization [7] of quantum electrodynamics, corresponds to a physical degree of freedom with the mass M. The modified propagator (1.1) has the improved UV behaviour p^{-4}, which can be achieved by adding the higher-derivative term $-\frac{1}{4M^2}F_{\mu\nu}\partial^2 F^{\mu\nu}$ to the Lagrangian.

Grinstein, O'Connel, and Wise extended the standard model to include a massive Lee-Wick partner for each particle. The corresponding higher-derivative terms are

$$\begin{aligned} &\frac{1}{M_A^2}\,\mathrm{tr}\{(D^\mu F_{\mu\nu})^2\} && \text{for gauge fields,} \\ &\frac{1}{M_\phi^2}(D^2\phi)^\dagger(D^2\phi) && \text{for scalars (the Higgs), and} \\ &\frac{1}{M_\psi^2}\overline{\psi} i \not{D}^3 \psi && \text{for fermions.} \end{aligned} \tag{1.2}$$

This extension, known as the Lee-Wick standard model, is free of the problematic quadratic divergences and is therefore one possible solution to the hierarchy

[1] The textbooks on quantum field theory used here are [1], [2], [3].

puzzle. Several recent papers investigated the properties of the Lee-Wick standard model [8, 9, 10, 11]. What we will examine here, is how the gauge field higher-derivative term (1.2) effects the renormalization and thus the running of the gauge coupling.

As we already indicated, of the four fundamental forces gravity is excluded from the standard model because up to now the quantum theory of gravity is still unknown.

It has been shown by 't Hooft and Veltman [12] that quantized general relativity coupled to scalars is a non-renormalizable theory. Also its coupling to fermions as well as Abelian or non-Abelian gauge fields results in non-renormalizable field theories, as has been established by Deser, van Nieuwenhuizen et al. [13, 14, 15].

In the class of renormalizable quantum field theories, low energy physics is perfectly shielded from the arbitrary high energies of the quantum fluctuations because these high-energy effects only occur in the renormalization of a small number of parameters. However, also non-renormalizable theories can be renormalized at each loop order and reliable predictions can be made, if they are treated in the general enough framework of effective field theories [16, 17].

In this context it has been shown by Ebert, Plefka, and Rodigast that Einstein Yang-Mills theory can be renormalized at one-loop order by adding the dimension six counterterm $d_2 \mathrm{Tr}\{(D_\mu F^{\mu\nu})^2\}$ to the original Lagrangian [18, 19]. This is exactly the Lee-Wick term for gauge fields (1.2) and it arises the interesting question wether the Lee-Wick terms of scalars and fermions are also related to gravitational counterterms.

In 2006 Robinson and Wilczek [20] initiated an intriguing discussion on the influence of gravity on the running of gauge couplings, calculated in the framework of effective field theories. However, Pietrykowsky proved [21] that the background field method they used yields gauge dependent results. Using the gauge invariant and gauge condition independent Vilkovisky-DeWitt effective action, Toms [22] showed that there is no gravitational contribution to the running of gauge couplings. This result has been confirmed by diagrammatic calculations of Ebert, Plefka, and Rodigast in [18, 19, 23].

Nevertheless, there might be gravitational corrections to the running of the coupling in other theories. Since they are part of the standard model, natural candidates are the Yukawa and φ^4 interactions. As we will see, their investigation yields astonishing results.

Outline

This thesis is organized as follows: In Chapter 2 we present a diagrammatic calculation of the β function of Lee-Wick gauge theory. First we introduce Lee -Wick gauge theory by investigating some of its classical properties. We review the path integral quantization of gauge theories to verify the use of a special gauge fixing term and prove the gauge invariance of the wave function renormalization by power counting. For later use we investigate the shift invariance of cut-off integrals and state some general formulas used in dimensional regularization. We compute the divergent parts of the diagrams for the proper two- and three-point function and use them to determine the one-loop β function. We conclude this chapter by performing the limit to Yang-Mills theory, thus obtaining the well known Yang-Mills β function.

In Chapter 3 we apply the background field method to determine the β function of Lee-Wick gauge theory. We give a short review of the method and first apply it to Yang-Mills Theory. Then we use two different gauge fixing terms to reproduce the result of Chapter 2.

Chapter 4 is devoted to the one-loop counterterms in Einstein Yang-Mills theory and their relation to the Lee-Wick standard model. We begin with the quantization of general relativity and its treatment as an effective field theory. Before showing explicitly that the dimension six counterterm in the gauge sector is given by the higher-derivative term of Lee-Wick gauge theory, we establish relations between the gravitational contributions to the various Z factors. We review the coupling of spinors to gravity using the vielbein formalism and determine the fermionic higher-derivative counterterm by computing the appropriate Feynman diagrams. We complete the chapter by showing that there is no higher-derivative counterterm involving scalars.

In Chapter 5 we investigate the gravitational contributions to the Yukawa and the φ^4 coupling. Beginning with Yukawa theory we determine the wave-function and vertex renormalization by computing all the contributing proper diagrams. We derive an expression for the one-loop β function and integrate it. We compare the thus obtained running coupling to the result in the absence of gravity and carry through the same analysis for φ^4 theory.

We summarize our results in Chapter 6 and give an outlook.

The Feynman rules of Yang-Mills theory can be found in Appendix A and in Appendix B we give a list of cut-off integrals including their derivation.

Part I | Lee-Wick Gauge Theory

We are going to investigate Yang-Mills theory with the higher covariant derivative term $d_2(D_\mu F^{\mu\nu})^2$. This generalization of ordinary Yang-Mills theory is called Lee-Wick gauge theory. Of special interest is the one-loop renormalization of this super-renormalizable theory and in particular the modification of the β function of the coupling g.

Chapter 2

Diagrammatic Approach

Lee-Wick gauge theory is a special higher-derivative gauge theory. The most general gauge invariant dimension six term is given by $d_1 \,\mathrm{tr}\{F^{\mu}{}_{\nu}F^{\nu}{}_{\rho}F^{\rho}{}_{\mu}\} + d_2 \,\mathrm{tr}\{(D_\mu F^{\mu\nu})^2\}$ but only Lee-Wick gauge theory has an equivalent formulation in which all operators are of dimension four or less and fulfills the constraints of perturbative unitarity [24].

In this chapter we will, beside a modification of the gauge fixing term, use the conventional Feynman diagram technique to renormalize the theory at one-loop order.

2.1 Classical Equations of Motion

Before tackling the quantum theory, it is worth to have a short look at the classical theory. The action of Lee-Wick gauge theory is given by

$$S = \int \mathrm{d}^4x \,\mathcal{L} = \int \mathrm{d}^4x \;\mathrm{tr}\{-\tfrac{1}{2}(F^{\mu\nu})^2 + d_2(D_\mu F^{\mu\nu})^2\} \,. \tag{2.1}$$

The equations of motion are

$$\frac{\delta S}{\delta A_\mu} = 0 \tag{2.2}$$

and can be easily derived by starting from

$$\begin{aligned} \delta F^{\mu\nu} &= D^\mu \delta A^\nu - D^\nu \delta A^\mu \\ \delta D_\mu F^{\mu\nu} &= D^2 \delta A^\nu - D_\mu D^\nu \delta A^\mu - ig[\delta A_\mu, F^{\mu\nu}] \\ &= D^2 \delta A^\nu - D^\nu D_\mu \delta A^\mu + 2ig[F^{\mu\nu}, \delta A_\mu] \end{aligned} \tag{2.3}$$

which immediately yields

$$\mathrm{tr}\{\delta F^2_{\mu\nu}\} = 2\,\mathrm{tr}\{F_{\mu\nu}\delta F^{\mu\nu}\} = 4\,\mathrm{tr}\{F_{\mu\nu}D^\mu \delta A^\nu\} = -4\,\mathrm{tr}\{(D_\mu F^{\mu\nu})\delta A_\nu\} + \text{t.d.} \tag{2.4}$$

as well as

$$\begin{aligned} &\mathrm{tr}\{\delta(D_\mu F^{\mu\nu})^2\} \\ &\quad = 2\,\mathrm{tr}\{D_\mu F^{\mu\nu}(D^2\delta A_\nu - D_\nu D_\rho \delta A^\rho + 2ig[F_{\rho\nu}, \delta A^\rho])\} \\ &\quad = 2\,\mathrm{tr}\{(D^2 D_\mu F^{\mu\nu} - D^\nu D_\rho D_\mu F^{\mu\rho} + 2ig[D_\mu F^{\mu\rho}, F^{\nu}{}_{\rho}])\delta A_\nu\} + \text{t.d.} \\ &\quad = 2\,\mathrm{tr}\{(D^2 D_\mu F^{\mu\nu} - \tfrac{ig}{2} D^\nu [F_{\mu\rho}, F^{\mu\rho}] + 2ig[D_\mu F^{\mu\rho}, F^{\nu}{}_{\rho}])\delta A_\nu\} + \text{t.d.} \\ &\quad = 2\,\mathrm{tr}\{(D^2 D_\mu F^{\mu\nu} + 2ig[D_\mu F^{\mu\rho}, F^{\nu}{}_{\rho}])\delta A_\nu\} + \text{t.d.} \end{aligned} \tag{2.5}$$

Hence the classical equations of motion are therefore given by

$$\frac{\delta S}{\delta A_\nu} = (1 + d_2 D^2) D_\mu F^{\mu\nu} - 2igd_2[F^{\nu}{}_{\rho}, D_\mu F^{\mu\rho}] = 0 \tag{2.6}$$

This is a fourth order partial differential equation which reveals the fact that the gauge field in Lee-Wick gauge theory contains two degrees of freedom. Interestingly every solution to the ordinary Yang-Mills (YM) equations of motion is also a solution to this equation which also follows directly from (2.1) because the Lee-Wick term is the square of the YM equations of motion.

2.2 Degrees of Freedom

To make the two degrees of freedom of Lee-Wick gauge theory manifest we show how to obtain an equivalent formulation of the theory containing only dimension four operators. As has also been done in [4], we remove the higher-derivative term by introducing the auxiliary field $\widetilde{A}$:

$$\mathcal{L} = \mathrm{tr}\{-\tfrac{1}{2}F_{\mu\nu}^2 - \tfrac{1}{d_2}\widetilde{A}^2 + 2F^{\mu\nu}D_\mu\widetilde{A}_\nu\} \; . \tag{2.7}$$

At this intermediate step $\widetilde{A}$ is a non-dynamical field in the adjoint representation. Integrating it out or inserting its algebraic equations of motion which are nothing but a constraint

$$\frac{\delta S}{\delta \widetilde{A}_\nu} = -\tfrac{1}{d_2}\widetilde{A}^\nu - D_\mu F^{\mu\nu} = 0 \tag{2.8}$$

gives the original Lagrangian. Now one translates the gauge field A by $\widetilde{A}$ i.e. $A \to A + \widetilde{A}$ to make $\widetilde{A}$ a dynamical field and arrives at the new Lagrangian

$$\begin{aligned}\mathcal{L}(A,\widetilde{A}) = \mathrm{tr}\{ &- \tfrac{1}{2}(F_{\mu\nu} + D_\mu\widetilde{A}_\nu - D_\nu\widetilde{A}_\mu - ig[\widetilde{A}_\mu,\widetilde{A}_\nu])^2 - \tfrac{1}{d_2}\widetilde{A}^2 \\ &+ 2(F_{\mu\nu} + D_\mu\widetilde{A}_\nu - D_\nu\widetilde{A}_\mu - ig[\widetilde{A}_\mu,\widetilde{A}_\nu])(D^\mu\widetilde{A}^\nu - ig[\widetilde{A}^\mu,\widetilde{A}^\nu])\} \\ = \mathrm{tr}\{ &- \tfrac{1}{2}F_{\mu\nu}^2 + \tfrac{1}{2}(D_\mu\widetilde{A}_\nu - D_\nu\widetilde{A}_\mu)^2 \\ &- 4ig[\widetilde{A}_\mu,\widetilde{A}_\nu]D^\mu\widetilde{A}^\nu - \tfrac{3}{2}g^2[\widetilde{A}_\mu,\widetilde{A}_\nu]^2 - ig[\widetilde{A}_\mu,\widetilde{A}_\nu]F^{\mu\nu} - \tfrac{1}{d_2}\widetilde{A}^2\} \; .\end{aligned} \tag{2.9}$$

The mixing terms $\mathrm{tr}\{F_{\mu\nu}D^\mu\widetilde{A}^\nu\}$ exactly cancel. Obviously now A is a massless gauge field and $\widetilde{A}$ is a field in the adjoint representation of mass $1/\sqrt{d_2}$. The classical equations of motion for the fields are

$$0 = \frac{\delta S}{\delta A_\nu} = D_\mu F^{\mu\nu} + ig[\widetilde{A}_\mu, D^\mu\widetilde{A}^\nu - D^\nu\widetilde{A}^\mu] + 2g^2[\widetilde{A}_\mu,[\widetilde{A}^\mu,\widetilde{A}^\nu]] + igD_\mu[\widetilde{A}^\mu,\widetilde{A}^\nu] \tag{2.10}$$

and

$$\begin{aligned}0 = \frac{\delta S}{\delta \widetilde{A}_\nu} = &- D_\mu(D^\mu\widetilde{A}^\nu - D^\nu\widetilde{A}^\mu) + 2ig[\widetilde{A}_\mu, D^\mu\widetilde{A}^\nu - D^\nu\widetilde{A}^\mu] + \\ &+ 3g^2[\widetilde{A}_\mu,[\widetilde{A}^\mu,\widetilde{A}^\nu]] + ig[\widetilde{A}_\mu, F^{\mu\nu}] - \tfrac{1}{d_2}\widetilde{A}\end{aligned} \tag{2.11}$$

The equations of motion are quite complicated, but at least they are of second order as we are used to. The more important observation is that the kinetic term of the massive field $\widetilde{A}$ has the wrong sign. On the classical level this indicates a instability of the theory. In the quantum theory wrong sign kinetic terms lead to negative norm states in the Hilbert space which results in problems with the unitarity of the theory. This and also the potential violations of causality have been extensively investigated in [5], [6], [25] and recently in [26] and will be no topic of this thesis.

Note that there is one subtlety in this derivation. In general one could have added the complete square $-\frac{1}{d_2}(\widetilde{A}_\mu \pm d_2 D^\nu F_{\nu\mu})^2$ to the Lee-Wick Lagrangian in order to remove the higher-derivative term. Shifting the gauge field $A \to A \pm \widetilde{A}$ afterwards we end up with the Lagrangian $\mathcal{L}(A, \pm\widetilde{A})$ of equation (2.9). In this sense the two particle formulation is not unique because $\widetilde{A} \to -\widetilde{A}$ is no symmetry of (2.9).

2.3 Quantization

To verify a special choice of gauge fixing term we will now go through the steps of the path integral quantization of gauge theories. Starting point is a action $\mathcal{L}(A)$ which is invariant under gauge transformations

$$A \to A^u = UAU^{-1} - \tfrac{i}{g}\mathrm{d}UU^{-1} \qquad \text{with} \qquad U = e^{iu} \ . \tag{2.12}$$

Consider the functional integral

$$\int \mathcal{D}A \exp\left[i\int \mathrm{d}^4x \mathcal{L}(A)\right] \ . \tag{2.13}$$

To avoid the overcount of gauge equivalent configurations one has to apply a gauge fixing condition $G(A) = w$. Following Faddeev and Popov [27] this constraint can be introduced by inserting the identity

$$1 = \Delta(A)\int \mathcal{D}U\delta(G(A^u) - w) \ . \tag{2.14}$$

Here $\mathcal{D}U$ is the invariant measure of the gauge group. For arbitrary U_0 it fulfills:

$$\mathcal{D}(UU_0) = \mathcal{D}(U_0U) = \mathcal{D}U \ . \tag{2.15}$$

Obviously the functional $\Delta(A)$ is gauge invariant.

$$\int \mathcal{D}A\int \mathcal{D}U\Delta(A)\delta(G(A^u) - w)\exp\left[i\int \mathrm{d}^4x\mathcal{L}(A)\right] \tag{2.16}$$

Making the change of variables $A \to A^{-u}$ the Jacobian of which is one and exploiting gauge invariance we arrive at

$$\left(\int \mathcal{D}U\right)\int \mathcal{D}A\Delta(A)\delta(G(A) - w)\exp\left[i\int \mathrm{d}^4x\mathcal{L}(A)\right] \ . \tag{2.17}$$

Now the group volume has factored out and is just a normalization factor which cancels in the computations of correlation functions. On the surface $G(A) = w$ the solution to $G(A^u) = w$ is $U = 1$ and the functional $\Delta(A)$ is given by

$$\Delta(A)|_{G(A)=w} = \det(\mathcal{M}) \qquad \text{with} \qquad \mathcal{M} = \left.\frac{\delta G(A^u)}{\delta u}\right|_{u=0} \tag{2.18}$$

To get rid of the arbitrary field w we can integrate over it with an in principle arbitrary weight. One choice could be

$$\exp\left[-i\int \mathrm{d}^4x\,\mathrm{tr}\{\tfrac{1}{\alpha}wf(\partial^2)w\}\right]\;. \tag{2.19}$$

Performing the integration over w we arrive at

$$\int\mathcal{D}A\det(\mathcal{M})\exp\left[i\int\mathrm{d}^4x(\mathcal{L}(A)-\mathrm{tr}\{\tfrac{1}{\alpha}G(A)f(\partial^2)G(A)\})\right]\;. \tag{2.20}$$

Introducing ghost fields $c, \bar{c}$ this can be written as

$$\int\mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}\exp\left[i\int\mathrm{d}^4x\left(\mathcal{L}(A)-\mathrm{tr}\{\tfrac{1}{\alpha}G(A)f(\partial^2)G(A)+2\bar{c}\,\mathcal{M}\,c\}\right)\right]\;. \tag{2.21}$$

A convenient choice of gauge fixing condition is

$$G(A)=\partial\cdot A \qquad\Rightarrow\qquad \mathcal{M}=\tfrac{1}{g}\partial\cdot D\;. \tag{2.22}$$

Absorbing the factor g^{-1} into the normalization of the ghost fields we end up with the gauge fixed Lagrangian

$$\mathcal{L}_{gf}=\mathrm{tr}\{-\tfrac{1}{2}(F^{\mu\nu})^2+d_2(D_\mu F^{\mu\nu})^2-\tfrac{1}{\alpha}\partial\cdot A\,f(\partial^2)\,\partial\cdot A+2\bar{c}(-\partial\cdot D)c\}\;. \tag{2.23}$$

The quadratic part of the action is given by

$$\int\mathrm{d}^4x\mathcal{L}|_{A^2}=\frac{1}{2}\int\mathrm{d}^4xA^a_\mu\underbrace{\left[(1+d_2\partial^2)(\partial^2\eta^{\mu\nu}-\partial^\mu\partial^\nu)+\tfrac{1}{\alpha}f(\partial^2)\partial^\mu\partial^\nu\right]\delta^{ab}}_{=i\Delta^{\mu\nu}_{ab}}A^b_\nu\;, \tag{2.24}$$

or equivalently in momentum space

$$\begin{aligned}=\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}A^a_\mu(k)\Big[&(1-d_2k^2)(k^\mu k^\nu-k^2\eta^{\mu\nu})\\ &-\tfrac{1}{\alpha}f(-k^2)k^\mu k^\nu\Big]\delta^{ab}A^b_\nu(-k)\;.\end{aligned} \tag{2.25}$$

The propagator is the inverse of the fluctuation operator $\Delta^{\mu\nu}_{ab}$ and thus defined by the equation

$$\begin{aligned}&\left[(1+d_2\partial^2)(\partial^2\eta^{\mu\nu}-\partial^\mu\partial^\nu)+\tfrac{1}{\alpha}f(\partial^2)\partial^\mu\partial^\nu\right]D^{ab}_{\nu\rho}(x-y)=i\delta^\rho_\mu\delta^{ab}\delta(x-y)\\ \text{or}\quad&\left[(1-d_2k^2)(k^\mu k^\nu-k^2\eta^{\mu\nu})-\tfrac{1}{\alpha}f(-k^2)k^\mu k^\nu\right]D^{ab}_{\nu\rho}(k)=i\delta^\rho_\mu\delta^{ab}\;.\end{aligned} \tag{2.26}$$

The propagator can be easily obtained by plugging the ansatz

$$D^{ab}_{\mu\nu}(k)=A(k^2)\eta_{\mu\nu}+B(k^2)k_\mu k_\nu \tag{2.27}$$

into the above equation:

$$D^{ab}_{\mu\nu}(k)=\delta^{ab}\left[\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{-i}{k^2(1-d_2k^2)}-\frac{i\alpha k_\mu k_\nu}{k^4f(-k^2)}\right]\;. \tag{2.28}$$

A convenient choice of the arbitrary function f is $f(-k^2) = 1 - d_2 k^2$

$$\Rightarrow \qquad D^{ab}_{\mu\nu}(k) = \delta^{ab}\left(\eta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)\frac{-i}{k^2(1-d_2k^2)} \tag{2.29}$$

because with this choice the UV behavior of the propagator is k^{-4}.

If one quantizes the theory in the two field formulation and fixes the gauge in the usual way by adding the term $\frac{1}{\alpha}(\partial \cdot A)^2$, the propagators are

$$D^{\mu\nu} = \frac{-i}{p^2}\left(\eta^{\mu\nu} - (1-\alpha)\frac{p^\mu p^\nu}{p^2}\right) \tag{2.30}$$

$$\widetilde{D}^{\mu\nu} = \frac{i}{p^2 - d_2^{-1}}\left(\eta^{\mu\nu} - d_2 p^\mu p^\nu\right) \tag{2.31}$$

It is clear that the two field formulation should yield the same results [28] but the higher-derivative formulation with the special choice of gauge fixing term provides the easiest calculation which will become clear in what follows.

2.4 Power Counting

As stated above the Lee-Wick gauge theory is a super-renormalizable theory. This can be seen by power counting. Let N_A, N_c denote the number of internal gauge and ghost lines, V_n the number of n gauge field vertices and V_c the number of ghost vertices in a given diagram, then the superficial degree of divergence of a diagram is given by:

$$\omega = 4L - 4N_A - 2N_c + \sum_{n=3}^{6}(6-n)V_n + V_c\ . \tag{2.32}$$

The number of loops is given by

$$L = N_A + N_c - \sum_{n=3}^{6} V_n - V_c + 1\ . \tag{2.33}$$

The number of external particles is given by

$$E_A = \sum_{n=3}^{6} nV_n + V_c - 2N_A\ , \qquad\qquad E_c = 2V_c - 2N_c\ . \tag{2.34}$$

This gives the superficial degree of divergence

$$\omega = 6 - 2L - E_A - 2E_c\ . \tag{2.35}$$

For ordinary Yang-Mills theory one obtains by a similar calculation the degree of divergence

$$\omega_{YM} = 4 - E_c - E_A\ . \tag{2.36}$$

From this expression it is obvious that the Lee-Wick gauge theory is super-renormalizable. The highest divergence is a potential quadratic divergence in the gauge field 2-point function which is ruled out by the Slavnov-Taylor identity. At two-loop only the gauge field 2-point function diverges. All diagrams with external ghost fields or more than two loops converge. The only candidate for a divergence in a diagram with external ghost fields is the ghost 2-point function with $\omega = 0$ at one-loop. But since one derivative of the vertices acts on an external leg the diagram is finite. As a consequence the counterterms are gauge independent in the minimal subtraction scheme and it is sufficient to calculate the wave function renormalization to determine the β function. This can be seen as follows:

No matter which vertex we take to define the bare coupling, the Slavnov-Taylor identities [29], [30] or for example [2], tell us that the result has to be the same:

$$Z_g = \frac{g_0}{g} = Z_{A^3} Z_A^{-\frac{3}{2}} = Z_{A^4}^{\frac{1}{2}} Z_A^{-1} = Z_{\bar{c}Ac} Z_c^{-1} Z_A^{-\frac{1}{2}} \; . \tag{2.37}$$

This can be brought into the more familiar form

$$\frac{Z_{A^4}}{Z_{A^3}} = \frac{Z_{A^3}}{Z_A} = \frac{Z_{\bar{c}Ac}}{Z_c} \; . \tag{2.38}$$

From powercounting we know that $Z_{\bar{c}Ac} = Z_c = 1$, what can also be found in [31], and we get

$$Z_A = Z_{A^3} = Z_{A^4} \tag{2.39}$$

Hence the wavefunction renormalization is gauge independent and related to the gauge coupling renormalization as well as to the renormalization of d_2:

$$Z_g = Z_A^{-\frac{1}{2}} \qquad\qquad Z_{d_2} = Z_g^2 \; . \tag{2.40}$$

All these properties are consequences of the higher derivatives in the Lagrangian and the choice of gauge fixing term which ensures the k^{-4} UV behavior of the propagator. As a direct consequence all one has to calculate is Z_A and it is in contrast to the case of ordinary Yang-Mills theory no advantage to use the background field method. If one takes the limit $d_2 \to 0$ only (2.38) holds and the background field method has the advantage that Z_A is gauge independent.

2.5 Derivation of the Feynman Rules

In what follows we split all the vertices in their ordinary Yang-Mills part which is represented in Feynman graphs by dots and the higher-derivative part represented by circles. Sticking to this convention it is clear that it will increase the number of diagrams. However, not all of these diagrams are divergent and simple power counting tells which can be thrown away. The diagrams contributing to the proper two-point and three-point function are listed in figure 2.1 and 2.2.

To calculate the one-loop diagrams we need to determine the new Feynman rules corresponding to the higher-derivative term:

$$\mathrm{tr}\{(D_\mu F^{\mu\rho})^2\} = \frac{1}{2}(D_\mu F^{\mu\rho})^a (D_\mu F^{\mu}{}_{\rho})^a = \frac{1}{2}(\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b\mu} F^c_{\mu\nu})^2 \tag{2.41}$$

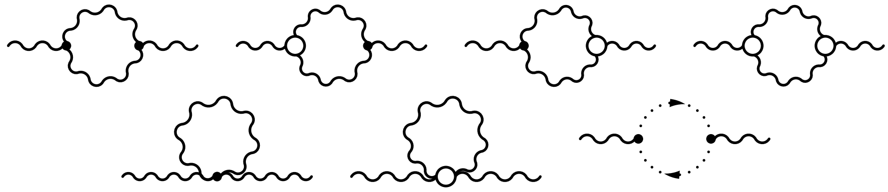


Figure 2.1: One-loop diagrams for the proper two-point function.

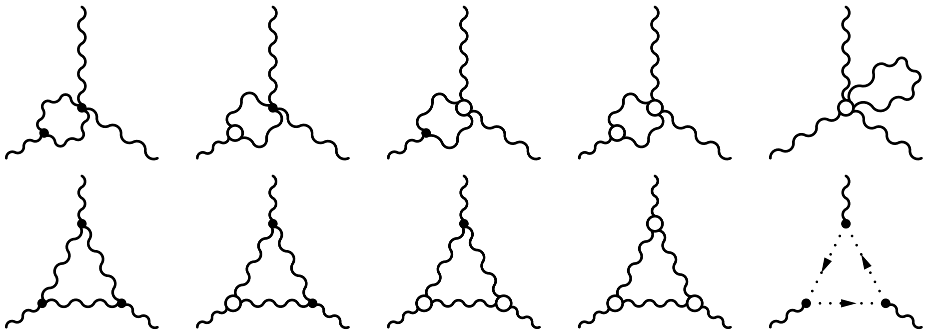


Figure 2.2: One-loop diagrams for the proper three gauge filed vertex not only differing by a permutation of outer legs.

We derive the Feynman rules in momentum space and begin with the quadratic part of the Lee-Wick term which in this calculation is involved in the propagator, but the corresponding Feynman rules will be necessary to determine the one-loop counterterms of Einstein Yang-Mills theory in Section 4.4.

$$
\begin{aligned}
i\int \mathrm{d}^4x\ d_2\,\mathrm{tr}\{(D_\mu F^{\mu\nu})^2\}|_{A^2} &= \frac{i}{2}\int \mathrm{d}^4x\ d_2(\partial^2 A^a_\nu - \partial^\mu\partial_\nu A^a_\mu)^2 \\
&= \frac{i}{2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\ d_2\delta^{ab}q^2(\eta^{\mu\nu}q^2 - q^\mu q^\nu)A^a_\mu(q)A^b_\nu(-q)
\end{aligned}
$$

Both possible contractions with two external gauge particles give the same contribution. And we obtain

$$ = id_2\delta^{ab}q^2(\eta^{\mu\nu}q^2 - q^\mu q^\nu)\ . \tag{2.42}$$

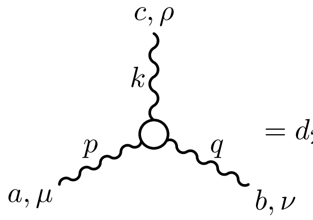
Let us go on with the three gauge field vertex. The three gauge field part of the Lee-Wick term is given by

$$
\begin{aligned}
& i\int \mathrm{d}^4x\ d_2\,\mathrm{tr}\{(D_\mu F^{\mu\nu})^2\}|_{A^3} \\
&= igd_2f^{abc}\int \mathrm{d}^4x(\partial^2A^a_\nu - \partial^\mu\partial_\nu A^a_\mu)(\partial^\mu A^b_\mu A^{c\nu} + A^{b\mu}(\partial_\mu A^{c\nu} - \partial^\nu A^c_\mu)) \\
&= -\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int \mathrm{d}^4k\ \delta(p{+}q{+}k)\times \\
&\qquad\qquad \times\, gd_2f^{abc}[p^2\delta^\mu_\alpha - p^\mu p_\alpha][(2k+q)^\nu\eta^{\rho\alpha} - \eta^{\nu\rho}q^\alpha]A^a_\mu(p)A^b_\nu(q)A^c_\rho(k) \\
&= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int \mathrm{d}^4k\ \delta(p{+}q{+}k)V^{\mu\nu\rho\ abc}_{3gl}(p,q,k)A^a_\mu(p)A^b_\nu(q)A^c_\rho(k)\ .
\end{aligned}
$$

From this expression one can easily read off the three gauge field vertex. One of the $3! = 6$ contraction gives

$$V_{3gl}^{\mu\nu\rho\;abc}(p,q,k) = d_2 g f^{abc}[\eta^{\nu\rho}(q^\mu p^2 - p^\mu pq) + (2k+q)^\nu(p^\mu p^\rho - \eta^{\mu\rho}p^2)]$$

and we have to sum up all them. Doing so we obtain the result



$$\begin{aligned} = d_2 g f^{abc}\big[&+ \eta^{\mu\nu}\{p^\rho(2pq + pk + 3qk) - q^\rho(2pq + qk + 3pk)\} \\ &+ \eta^{\mu\rho}\{k^\nu(2pk + kq + 3pq) - p^\nu(2pk + pq + 3qk)\} \\ &+ \eta^{\nu\rho}\{q^\mu(2kq + pq + 3pk) - k^\mu(2kq + pk + 3pq)\} \\ &- 2(k^\mu k^\nu(p-q)^\rho + p^\nu p^\rho(q-k)^\mu + q^\rho q^\mu(k-p)^\nu) \\ &- 3(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big] \; . \end{aligned} \tag{2.43}$$

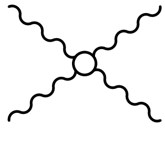
Next is the four gauge field vertex and therefore the four gauge field part of the Lee-Wick term:

$$\begin{aligned} & i\int \mathrm{d}^4x \; d_2 \,\mathrm{tr}\{(D_\mu F^{\mu\nu})^2\}\big|_{A^4} \\ &= i g^2 d_2 f^{abe} f^{ecd} \int \mathrm{d}^4x [(\partial^2 A^a_\nu - \partial^\mu\partial_\nu A^a_\mu)A^b_\mu A^{c\mu} A^{d\nu} \\ &\qquad + \frac{1}{2}(\partial^\mu A^a_\mu A^b_\nu + A^{a\mu}(\partial_\mu A^b_\nu - \partial_\nu A^b_\mu))(\partial^\mu A^c_\mu A^{d\nu} + A^{c\mu}(\partial_\mu A^{d\nu} - \partial^\nu A^d_\mu))] \\ &= -i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int \mathrm{d}^4l \;\delta(\sum_{i=1}^4 p_i) g^2 d_2 f^{abe} f^{ecd}\{(p^2\delta^\mu_\alpha - p^\mu p_\alpha)\eta^{\nu\rho}\eta^{\alpha\sigma} \\ &\qquad + \tfrac{1}{2}[(2q+p)^\mu\delta^\nu_\alpha - \eta^{\mu\nu}q_\alpha][(2l+k)^\rho\eta^{\sigma\alpha} - \eta^{\rho\sigma}l^\alpha]\}A^a_\mu(p)A^b_\nu(q)A^c_\rho(k)A^d_\sigma(l) \; . \end{aligned}$$

One contraction gives

$$\begin{aligned} V_{4gl}^{\mu\nu\rho\sigma\;abcd}(p,q,k,l) = -\frac{i}{2} g^2 d_2 f^{abe} f^{ecd}\{&2\eta^{\nu\rho}(\eta^{\mu\sigma}p^2 - p^\mu p^\sigma) \\ &+ (2q+p)^\mu[(2l+k)^\rho\eta^{\sigma\nu} - \eta^{\rho\sigma}l^\nu] - \eta^{\mu\nu}[(2l+k)^\rho q^\sigma - \eta^{\rho\sigma}ql]\} \; , \end{aligned}$$

and the vertex is given by the sum of all 4!=24 possible contractions:



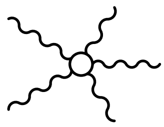
$$= \sum_{i,j,k,l} |\varepsilon^{ijkl}| V_{4gl}^{\mu_i\mu_j\mu_k\mu_l\;a_i a_j a_k a_l}(p_i,p_j,p_k,p_l) \tag{2.44}$$

This is a rather formal expression but by far more illuminating than the at least one page long explicit expression.

The last vertex we need is the five gauge field vertex. The five gauge field part of the Lee-Wick term is

$$
\begin{aligned}
& i\int \mathrm{d}^4x \; d_2 \, \mathrm{tr}\{(D_\mu F^{\mu\nu})^2\}|_{A^5} \\
&= i\int \mathrm{d}^4x \; f^{abm} f^{mcn} f^{nde} (\partial^\mu A^a_\mu A^{b\nu} + A^{a\mu}(\partial_\mu A^{b\nu} - \partial^\nu A^b_\mu)) A^{c\mu} A^d_\mu A^e_\nu \\
&= \int \prod_{i=1}^{5} \frac{\mathrm{d}^4 p_i}{(4\pi)^4} (4\pi)^4 \delta(\sum_{i=1}^{5} p_i) \times \\
&\qquad \times \underbrace{f^{abm} f^{mcn} f^{nde} \eta^{\mu_3\mu_4} [(p_1+2p_2)^{\mu_1} \eta^{\mu_5\mu_2} - p_2^{\mu_5} \eta^{\mu_1\mu_2}]}_{=V_{5gl}^{\mu\nu\rho\sigma\gamma\; abcde}} \prod_{i=1}^{5} A^{a_i}_{\mu_i}(p_i) \; .
\end{aligned}
$$

The formal expression for the five gluon vertex is given by



$$
= \sum_{i,j,k,l,m} |\varepsilon^{ijklm}| V_{5gl}^{\mu_i\mu_j\mu_k\mu_l\mu_m \; a_i a_j a_k a_l a_m}(p_i, p_j, p_k, p_l, p_m) \; . \tag{2.45}
$$

Again we do not give the explicit expression because it is to long. Obviously there is also a six gauge field vertex, but since it is not necessary to determine the one-loop renormalization of Lee-Wick gauge theory we do not derive it.

2.6 Regularization

Appart from some exceptions one is usually faced with divergent momentum integrals appearing in the perturbation expansion. Fortunately this is not an obstacle in extracting meanigfull quantities out of a quantum field theory. In order to get rid of these infinities we have, as an intermediate step, to introduce a regulator which renders the integrals finite and parametrizes its divergent parts. Now we absorb the divergences into a renormalization of the parameters of our theory and if done properly we can calculate physical quantities which are independent of the regularization prescription used.

2.6.1 Cut-Off Regularization

The simplest regularization is just to cut off the integral at some large but finite momentum Λ. The domain of integration is obviously a Lorentz invariant subspace, hence cut-off regularization preserves Lorentz invariance. However it is well known that it breaks gauge invariance and will therefore be of minor importance here and in Chapter 3. Nevertheless, since we will be interested in polynomial divergences in Part II we point out some properties of cut-off regularization which will become important there.

Shift Invariance of Logarithmic Divergences

Beside its non-gauge-invariance there is an additional drawback of cut-off regularization. If the leading divergence of an integral is more than quadratic even its divergent part is not invariant under a shift of the loop momentum. Only lagarithmic divergent integrals yield shift invariant results.

In the following proof of the statement above all momenta are Euclidean. Let $f(k)$ be an arbitrary function which may depend on further momenta or be tensor valued and $B_\Lambda(q) := \{k \mid (k-q)^2 \leq \Lambda^2\}$, $B_\Lambda(0) = B_\Lambda$. We can give the estimate

$$\begin{aligned}
\int_{B_\Lambda} \mathrm{d}^4k\, f(k) &= \int_{B_\Lambda(q)} \mathrm{d}^4k\, f(k-q) \\
&= \int_{B_\Lambda} \mathrm{d}^4k\, f(k-q) + \overbrace{\int_{B_\Lambda(q)\setminus B_\Lambda} \mathrm{d}^4k\, f(k-q) - \int_{B_\Lambda\setminus B_\Lambda(q)} \mathrm{d}^4k\, f(k-q)}^{=\Delta} \\
\Rightarrow \qquad |\Delta| &\leq \sup_{[B_\Lambda(q)\setminus B_\Lambda]\cup[B_\Lambda\setminus B_\Lambda(q)]} |f(k-q)| \int_{[B_\Lambda(q)\setminus B_\Lambda]\cup[B_\Lambda\setminus B_\Lambda(q)]} \mathrm{d}^4k\ .
\end{aligned} \tag{2.46}$$

The easiest way to calculate the volume, is to calculate it indirectly:

$$\begin{aligned}
\mathrm{Vol}(B_\Lambda(q) \setminus B_\Lambda) + \mathrm{Vol}(B_\Lambda \setminus B_\Lambda(q)) &= 2\mathrm{Vol}(B_\Lambda(q) \setminus B_\Lambda) \\
&= 2(\mathrm{Vol}(B_\Lambda) - \mathrm{Vol}(B_\Lambda(q) \cap B_\Lambda))\ .
\end{aligned} \tag{2.47}$$

and to use cylinder coordinates around the direction of q to calculate the volume $\mathrm{Vol}(B_\Lambda(q) \cap B_\Lambda)$:

$$\begin{aligned}
\int_{B_\Lambda(q)\cap B_\Lambda} \mathrm{d}^4k &= \int_{B_\Lambda(q)\cap B_\Lambda} \mathrm{d}z\, r^2 \sin(\theta)\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \\
&= 4\pi \int_{-\Lambda+|q|}^{|q|/2} \mathrm{d}z \int_0^{\sqrt{\Lambda^2-(z-|q|)^2}} r^2\mathrm{d}r + 4\pi \int_{|q|/2}^{\Lambda} \mathrm{d}z \int_0^{\sqrt{\Lambda^2-z^2}} r^2\mathrm{d}r \\
&= \frac{8\pi}{3} \int_{|q|/2}^{\Lambda} \mathrm{d}z(\Lambda^2 - z^2)^{\frac{3}{2}} \\
&= \frac{\pi^2}{2}\Lambda^4 - \pi\Lambda^4 \left[\arcsin\left(\frac{|q|}{2\Lambda}\right) + \left(\frac{5}{6}\frac{|q|}{\Lambda} - \frac{1}{12}\left(\frac{|q|}{\Lambda}\right)^3\right)\sqrt{1-\left(\frac{|q|}{2\Lambda}\right)^2}\,\right]\ .
\end{aligned}$$

And consequently

$$\begin{aligned}
2\mathrm{Vol}(B_\Lambda(q) \setminus B_\Lambda) &= 2\pi\Lambda^4 \left[\arcsin\left(\frac{|q|}{2\Lambda}\right) + \left(\frac{5}{6}\frac{|q|}{\Lambda} - \frac{1}{12}\left(\frac{|q|}{\Lambda}\right)^3\right)\sqrt{1-\left(\frac{|q|}{2\Lambda}\right)^2}\,\right] \\
&= \frac{8\pi}{3}|q|\Lambda^3 - \frac{\pi}{3}|q|^3\Lambda + \mathcal{O}(|q|^5\Lambda^{-1})\ .
\end{aligned}$$

This gives the indeed rough estimate

$$|\Delta| \leq \sup_{[B_\Lambda(q)\setminus B_\Lambda]\cup[B_\Lambda\setminus B_\Lambda(q)]} |f(k-q)| \frac{8\pi}{3}|q|\Lambda^3\ . \tag{2.48}$$

Nevertheless this suffices for our purpose. For a logarithmic divergent Integral $\sup|f(k-q)|$ is of order Λ^{-4}. Therefore the fault one makes by neglecting the shift of the domain of integration is at most of order $|q|\Lambda^{-1}$.

In d dimensions it looks quite similar. We only do calculations in four dimensions here, but with regard to extradimensional scenarios it is interesting to study the d dimensional case as well. In d dimensions we have

$$\int\limits_{B_\Lambda(q)\cap B_\Lambda} \mathrm{d}^d k = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int\limits_{B_\Lambda(q)\cap B_\Lambda} \mathrm{d}z\, r^{n-2}\mathrm{d}r = \frac{4\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int\limits_{|q|/2}^{\Lambda} dz \int\limits_{0}^{\sqrt{\Lambda^2-z^2}} r^{n-2} dr$$

$$= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \int\limits_{|q|/2}^{\Lambda} \mathrm{d}z (\Lambda^2 - z^2)^{\frac{d-1}{2}} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \Lambda^d \int\limits_{\arcsin\frac{|q|}{2\Lambda}}^{\frac{\pi}{2}} \mathrm{d}\varphi \cos^d\varphi \,.$$

Taylor expanding the remaining integral gives

$$\int\limits_{\arcsin\frac{|q|}{2\Lambda}}^{\frac{\pi}{2}} \mathrm{d}\varphi \cos^d\varphi = \int\limits_{0}^{\frac{\pi}{2}} \mathrm{d}\varphi \cos^d\varphi - \frac{|q|}{2\Lambda} + \frac{d-1}{6}\left(\frac{|q|}{2\Lambda}\right)^3 + \mathcal{O}\left(\left(\frac{|q|}{\Lambda}\right)^5\right) .$$

Integration by parts yields the recursion formula

$$\int\limits_{0}^{\frac{\pi}{2}} \mathrm{d}\varphi \cos^d\varphi = \frac{d-1}{d} \int\limits_{0}^{\frac{\pi}{2}} \mathrm{d}\varphi \cos^{d-2}\varphi \,,$$

which is solved by

$$\int\limits_{0}^{\frac{\pi}{2}} \mathrm{d}\varphi \cos^d\varphi = \prod_{k=0}^{\left[\frac{d}{2}\right]-1} \frac{d-2k-1}{d-2k} \int\limits_{0}^{\frac{\pi}{2}} \mathrm{d}\varphi \cos^{d-2\left[\frac{d}{2}\right]}\varphi$$

$$= \left(\frac{\pi}{2}\right)^{-d+2\left[\frac{d}{2}\right]+1} \prod_{k=0}^{\left[\frac{d}{2}\right]-1} \frac{\frac{d}{2}-k-\frac{1}{2}}{\frac{d}{2}-k}$$

$$= \left(\frac{\pi}{2}\right)^{-d+2\left[\frac{d}{2}\right]+1} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2}+1)} \frac{\Gamma(\frac{d}{2}-\left[\frac{d}{2}\right]+1)}{\Gamma(\frac{d}{2}-\left[\frac{d}{2}\right]+\frac{1}{2})}$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2}+1)} \,.$$

This must have been the result because in the limit $|q| \to 0$ we have to get the volume of the d dimensional ball. Therefore we arrive at the estimate

$$|\Delta| \leq \sup_{[B_\Lambda(q)\setminus B_\Lambda]\cup[B_\Lambda)\setminus B_\Lambda(q)]} |f(k-q)| \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} |q|\Lambda^{d-1}$$

The result for $d = 4$ is reproduced and obviously also in d dimensions the logarithmic divergences are shift invariant. If we have an integral of divergence index ω, then the fault is at most of order $|q|\Lambda^{\omega-1}$, independent of the dimension. A list of cut-off integrals and their derivation can be found in Appendix B.

2.6.2 Dimensional Regularilzation

In contrast to other regularization schemes as for example cut-off regularization, dimensional regularization [32] has no meaning outside perturbation theory. However it is the regularization which leads to the simplest perturbative calculations.

In dimensional regularization one analytically continues the Feynman diagrams to an arbitrary value of the spacetime dimension $d = 4 - \epsilon$. After subtracting the poles at $d = 4$ which correspond to logarithmic divergences one can send ϵ to zero and obtains finite Green functions.

Beside the calculational advantages dimensional regularization has the usefull property that it respects all properties of a theory which are not sensitive to the number of spacetime dimensions, e.g. gauge invariance. The only drawback with respect to the calculations in this theses is that dimensional regularization provides no straightforward way to determine polynomial divergences.

We will now gather some formulas, frequently used in our calculation. Starting from the simple expression

$$\frac{1}{A_1 A_2 \dots A_n} = \int\limits_{(\mathbb{R}_+)^n} \mathrm{d}^n s \; \exp\left(-\sum_{i=1}^{n} s_i A_i\right) \tag{2.49}$$

we can make the change of variables

$$s_i = r x_i \quad \text{with} \quad x_n = 1 - \sum_{i=1}^{n-1} x_i \quad \text{and} \quad x_i \in [0\,,1]\,, \qquad r \in \mathbb{R}_+\,, \tag{2.50}$$

whose Jacobian is equal to r^{n-1}, to end up with

$$\frac{1}{A_1 A_2 \dots A_n} = \int\limits_{[0,1]^n} \mathrm{d}^n x\, \delta(1 - \textstyle\sum x_i) \frac{(n-1)!}{(\sum x_i A_i)^n}\,. \tag{2.51}$$

By repeated differentiation of (2.51) one obtains the more general formula

$$\frac{1}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}} = \int\limits_{[0,1]^n} \mathrm{d}^n x\, \delta(1 - \textstyle\sum x_i) \frac{\prod x_i^{m_i - 1}}{[\sum x_i A_i]^{\sum m_i}} \frac{\Gamma(\sum m_i)}{\prod \Gamma(m_i)}\,, \tag{2.52}$$

which can be used to combine denominators in Feynman integrals. After shifting the loop momentum to complete the square in the denominator, dropping all odd terms and making the symmetry replacements

$$\int \mathrm{d}^d p \; f(p^2) p^\mu p^\nu = \frac{\eta^{\mu\nu}}{d} \int \mathrm{d}^d p \; f(p^2) p^2$$

$$\int \mathrm{d}^d p \; f(p^2) p^\mu p^\nu p^\rho p^\sigma = \frac{1}{d(d+2)} (\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \int \mathrm{d}^d p \; f(p^2) p^4$$

all integrals have the form:

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^{2n}}{(l^2 - \Delta)^m} \tag{2.53}$$

and can be solved by Wick rotating to Euclidean momenta

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^{2n}}{(l^2-\Delta)^m} = \frac{i}{(2\pi)^d}(-1)^{n+m}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2})}\int\limits_0^\infty \frac{y^{d/2+n-1}}{(y+\Delta)^n}\mathrm{d}y \,.$$

Substituting $z = \frac{\Delta}{y+\Delta}$ gives

$$= \frac{i}{(4\pi)^{d/2}}(-1)^{n+m}\frac{1}{\Gamma(\frac{d}{2})}\Delta^{n+d/2-m}\int\limits_0^1 \mathrm{d}z\; z^{m-d/2-n} z^{d/2+n-1} \,.$$

The remaining integral is nothing but the Euler beta function, which can be written more conveniently as a combination of Gamma functions. To see this, consider

$$\Gamma(x)\Gamma(y) = \int\limits_0^\infty \mathrm{d}s \int\limits_0^\infty \mathrm{d}t\; s^{x-1}t^{y-1}e^{-(s+t)}$$

and make the change of variables $s = rz$, $t = r(1-z)$. One immediately obtains the well known result

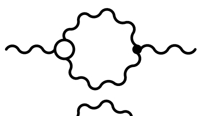
$$B(x,y) = \int\limits_0^1 \mathrm{d}z\; z^{x-1}(1-z)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \,.$$

Putting everything together we end up with the master formula in dimensional regularization:

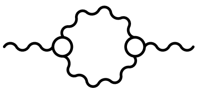
$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^{2n}}{(l^2-\Delta)^m} = \frac{i}{(4\pi)^{d/2}}(-1)^{n+m}\frac{\Gamma(m-\frac{d}{2}-n)\Gamma(\frac{d}{2}+n)}{\Gamma(\frac{d}{2})\Gamma(m)}\Delta^{n+d/2-m} \,. \quad (2.54)$$

2.7 One-Loop Divergences

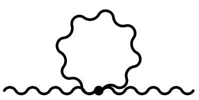
Now let us determine the one-loop renormalization of Lee-Wick gauge theory. The divergent parts of the diagrams of figure 2.1 are



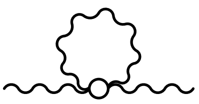
$$= \frac{i}{16\pi^2}C_2 g^2\delta^{ab}\frac{2}{\epsilon}\left[-\frac{\eta^{\mu\nu}}{d_2}\left(6+\frac{3}{2}\alpha\right)\right] \quad (2.55)$$

$\frac{1}{2}$ 

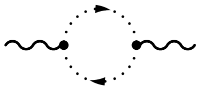
$$= \frac{i}{16\pi^2}C_2 g^2\delta^{ab}\frac{2}{\epsilon}\left[\frac{\eta^{\mu\nu}}{d_2}\left(12+\frac{3}{2}\alpha\right) + \eta^{\mu\nu}q^2\left(\frac{46}{3}+\frac{11}{4}\alpha\right) - q^\mu q^\nu\left(\frac{40}{3}+2\alpha\right)\right] \quad (2.56)$$

$\frac{1}{2}$ 

$$= \frac{i}{16\pi^2}C_2 g^2\delta^{ab}\frac{2}{\epsilon}\left[\frac{\eta^{\mu\nu}}{d_2}\left(\frac{9}{4}+\frac{3}{4}\alpha\right)\right] \quad (2.57)$$

$\frac{1}{2}$ 

$$= \frac{i}{16\pi^2}C_2 g^2\delta^{ab}\frac{2}{\epsilon}\left[-\frac{\eta^{\mu\nu}}{d_2}\left(\frac{33}{4}+\frac{3}{4}\alpha\right) - \eta^{\mu\nu}q^2\left(\frac{33}{4}+\frac{11}{4}\alpha\right) + q^\mu q^\nu\left(6+2\alpha\right)\right] \quad (2.58)$$



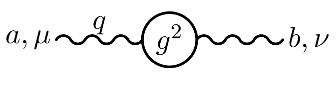
$$= \frac{i}{16\pi^2} C_2 g^2 \delta^{ab} \frac{2}{\epsilon} \left[\eta^{\mu\nu} q^2 \frac{1}{12} + q^\mu q^\nu \frac{1}{6} \right] . \tag{2.59}$$

Here the constant C_2 denotes the Casimir operator of the adjoint representation of the gauge group.

$$T^a_{\mathrm{ad}} T^a_{\mathrm{ad}} = C_2 \qquad \Leftrightarrow \qquad if^{acd} if^{dcb} = C_2 \delta^{ab} = \mathrm{tr}\{T^a_{\mathrm{ad}} T^b_{\mathrm{ad}}\} \tag{2.60}$$

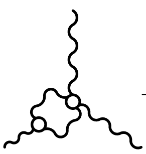
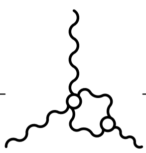
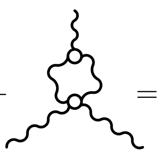
In the case of $SU(N)$ it is for example given by $C_2 = N$.

Summation of all the diagrams yields

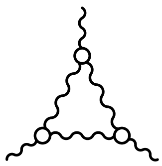


$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \frac{2}{\epsilon} \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu \right) \frac{43}{6} . \tag{2.61}$$

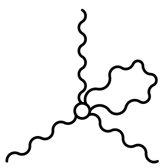
All terms proportional to d_2^{-1} cancel due to the Slavnov-Taylor identity for the two-point function. There is no renormalization of the longitudinal part. From Section 2.4 we know that we could stop here because the divergent part of the proper two-point function is gauge independent and completely determines the one-loop renormalization of the theory. The two requirements of transversality and independence of α are obviously fulfilled by (2.61) in contrast to the individual diagrams. Nevertheless we will also determine the proper three gauge field vertex to cross check the calculation. For the divergent diagrams we obtain

 +  + 

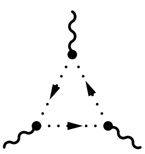
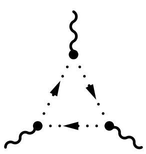
$$= \frac{C_2 g^3}{16\pi^2} f^{abc} \frac{2}{\epsilon} \left(\tfrac{49}{8}\alpha - \tfrac{85}{4} \right) \left[\eta^{\mu\nu}(p^\rho - q^\rho) + ... \right] \tag{2.62}$$



$$= \frac{C_2 g^3}{16\pi^2} f^{abc} \frac{2}{\epsilon} \left(4 + \tfrac{11}{4}\alpha \right) \left[\eta^{\mu\nu}(p^\rho - q^\rho) + ... \right] \tag{2.63}$$



$$= \frac{C_2 g^3}{16\pi^2} f^{abc} \frac{2}{\epsilon} \left(\tfrac{81}{8} + \tfrac{27}{8}\alpha \right) \left[\eta^{\mu\nu}(p^\rho - q^\rho) + ... \right] \tag{2.64}$$

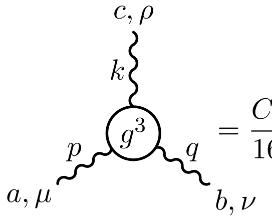
 + 

$$= \frac{C_2 g^3}{16\pi^2} f^{abc} \frac{2}{\epsilon} \left(-\frac{1}{24} \right) \left[\eta^{\mu\nu}(p^\rho - q^\rho) + ... \right] . \tag{2.65}$$

All other diagrams are finite by powercounting. To obtain these results we made use of

$$\begin{aligned} if^{lam} if^{mbn} if^{ncl} &= \mathrm{tr}\{T^a_{\mathrm{ad}} T^b_{\mathrm{ad}} T^c_{\mathrm{ad}}\} = if^{abe}\, \mathrm{tr}\{T^e_{\mathrm{ad}} T^c_{\mathrm{ad}}\} + \mathrm{tr}\{T^b_{\mathrm{ad}} T^a_{\mathrm{ad}} T^c_{\mathrm{ad}}\} \\ &= \tfrac{i}{2} C_2 f^{abc} \end{aligned} \tag{2.66}$$

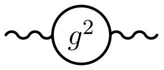
Summation of all diagrams yields the divergent part of the proper three gauge field vertex.



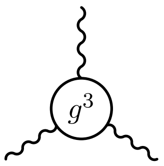
$$= \frac{C_2 g^3}{16\pi^2} f^{abc} \frac{2}{\epsilon} \left(-\frac{43}{6}\right) \left[\eta^{\mu\nu}(p^\rho - q^\rho) + \eta^{\mu\rho}(k^\nu - p^\nu) + \eta^{\nu\rho}(q^\mu - k^\mu)\right] \tag{2.67}$$

As expected this agrees perfectly with the result for the proper two-point function.

Since it provides an additional check, we also performed the calculation using cut-off regularization and obtained



$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \left[\frac{5}{2}\eta_{\mu\nu}\left(\Lambda^2 - \mu^2\right) + \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu\right) \frac{43}{6} \ln \frac{\Lambda^2}{\mu^2}\right] \tag{2.68}$$



$$= \frac{1}{16\pi^2} f^{abc} C_2 g^3 \ln \frac{\Lambda^2}{\mu^2} \left(-\frac{43}{6}\right) \left[\eta_{\mu\nu}(p_\rho - q_\rho) + ...\right] . \tag{2.69}$$

Because of the non gauge invariance of the cut-off regularization the quadratic divergences of the proper gauge field two-point function do not cancel, similar to ordinary Yang-Mills theory. This is a crucial point. We have to neglect the quadratic divergence because it leads to a mass counterterm which contradicts gauge invariance. In Part II we are faced with quadratic divergences which cannot be ruled out by any symmetry of the theory and hence cannot be thrown away.

2.8 The β Function

2.8.1 Lee-Wick Gauge Theory

In the minimal subtraction scheme the counterterm corresponding to the divergences we have found is

$$\delta\mathcal{L}_{\text{c.t.}} = (Z_A - 1)\text{tr}\left\{-\tfrac{1}{2}F_{\mu\nu}^2\right\} \tag{2.70}$$

and the wave function renormalization factor, compare (2.61) and (2.67), has been determined to be

$$Z_A - 1 = Z_{A^3} - 1 = \frac{g^2 C_2}{16\pi^2} \frac{43}{6} \frac{2}{\epsilon} . \tag{2.71}$$

The bare coupling g_0 is defined by

$$g_0 = g\mu^{2-d/2} Z_g = g\mu^{2-d/2} Z_A^{-1/2} . \tag{2.72}$$

Differentiation with respect to the renormalization scale yields

$$0 = (2 - \tfrac{d}{2}) g Z_g + \beta \frac{\partial}{\partial g} (g Z_g) \ . \tag{2.73}$$

Therefore the β function is given by

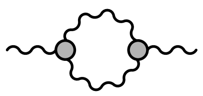
$$\beta = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} g = -(2 - \tfrac{d}{2}) \left(\frac{\partial \ln(g Z_g)}{\partial g} \right)^{-1} = -(2 - \tfrac{d}{2}) \left(\frac{\partial \ln(g Z_A^{-1/2})}{\partial g} \right)^{-1} \ . \tag{2.74}$$

Expanding this result in powers of g and removing the regulator gives the one-loop β function

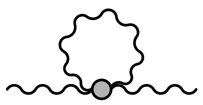
$$\beta = -\frac{g^3 C_2}{16\pi^2} \frac{43}{6} \tag{2.75}$$

This result is independent of d_2 and different from the well known $-\frac{11}{3}$ for Yang-Mills theory. As the minus sign indicates, Lee-Wick gauge theory is also asymptotically free. However, we see that the coupling constant runs approximately twice as fast as in Yang-Mills theory, which one also could have guessed because of the two particles corresponding to the Lee-Wick gauge field. The fact that we got only approximately a factor of two has to be addressed to the interactions between the two particles in Lee-Wick gauge theory.

As we figured out after completing the calculation of the β function (2.75), the one-loop β function of higher-derivative gauge theories has already been investigated in [28] and [33], using the background field method. In order to compare with these existing results, we also determined the β function in the case of a general dimension six term $d_2 \operatorname{tr}\{(D^\mu F_{\mu\nu})^2 + \xi i g F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\mu\}$. We just state our results:

$\frac{1}{2}$ 

$$= \frac{i C_2 g^2}{16\pi^2} \delta^{ab} \frac{2}{\epsilon} \Bigg[-\frac{\eta^{\mu\nu}}{d_2} 6 + \eta^{\mu\nu} q^2 \left(\frac{185}{12} + \frac{11}{4}\alpha - 9\xi + \frac{9}{8}\xi^2 \right) - q^\mu q^\nu \left(\frac{79}{6} + 2\alpha - 9\xi + \frac{9}{8}\xi^2 \right) \Bigg] \tag{2.76}$$

$\frac{1}{2}$ 

$$= \frac{i C_2 g^2}{16\pi^2} \delta^{ab} \frac{2}{\epsilon} \Bigg[-\frac{\eta^{\mu\nu}}{d_2} 6 - \eta^{\mu\nu} q^2 \left(\frac{33}{4} + \frac{11}{4}\alpha \right) + q^\mu q^\nu (6 + 2\alpha) \Bigg] \ . \tag{2.77}$$

Here we did not split the vertices into the dimension four and dimension six part and the shaded circles denote the full vertices. From the sum of both diagrams we obtain

$$Z_A - 1 = \frac{g^2 C_2}{16\pi^2} \frac{2}{\epsilon} \left(\frac{43}{6} - 9\xi + \frac{9}{8}\xi^2 \right) \quad \Rightarrow \quad \beta = -\frac{g^3 C_2}{16\pi^2} \left(\frac{43}{6} - 9\xi + \frac{9}{8}\xi^2 \right) \ . \tag{2.78}$$

We note that this expression agrees with the recently obtained result of Grinstein and O'Connell [28], but differs from the β function found by Fradkin and Tseytlin in appendix C of [33].

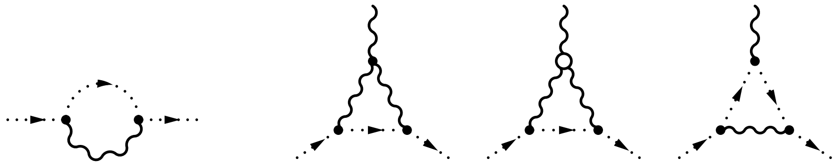


Figure 2.3: One-loop diagrams for the proper ghost two-point function and the vertex.

2.8.2 The Limit to Yang-Mills Theory

As the result (2.75) is apparently independent of d_2 it seem interesting to perform the limit $d_2 \to 0$. In order to do this one has to include the divergences which arise in this limit into the counterterms. The β function obtained in this way is of course the well known Yang-Mills β function because $\frac{1}{d_2} \to \infty$ is nothing but a gauge invariant regulator. However, it is not sufficient to make all diagrams finite and has to to be complemented by a second invariant regulator to render the remaining divergent one- and two-loop diagrams finite.

This time it is more convenient to compute the running of g at the ghost vertex because there are less contributing diagrams and all of them have no poles at $d = 4$. As we showed before, the contribution from dimensional regularization to the Z factors is gauge independent. However, this is not the case for the $\ln d_2^{-1}$ contributions. Only Z_g is gauge independent in the limit $d_2 \to 0$. We performed the calculation for arbitrary α to make the gauge dependence explicit and to see that only Z_g is independent of it.

It is convenient to use the following parameterization of the gauge field propagator:

$$\begin{aligned} D^{ab}_{\mu\nu} &= i\delta^{ab}\left(\eta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)\frac{\frac{1}{d_2}}{k^2(k^2 - \frac{1}{d_2})} \\ &= i\delta^{ab}\left(\eta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)\int\limits_0^{\frac{1}{d_2}} \frac{\mathrm{d}m}{(k^2 - m)^2}\ . \end{aligned} \tag{2.79}$$

In this way one can extract without difficulties the divergent parts of the Feynman parameter integrals which remain after momentum integration. The integrals over the mass square parameters in the gauge field propagators are all of the form

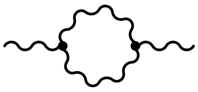
$$\int \mathrm{d}\, m \frac{1}{(A + xm)^n} \qquad \text{or} \qquad \int \mathrm{d}\, m (A + xm)^{n-1} \ln(A + xm) \qquad \text{with} \qquad n > 0 \tag{2.80}$$

and easy to perform.

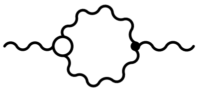
The diagrams for the proper ghost vertex and two-point function are listed in figure 2.3 and the diagrams for the proper gauge field two-point function have already been listed in figure 2.1.

Let us begin with the proper gauge field two-point function.

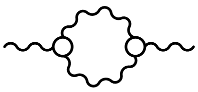
For the individual diagrams we get

$\frac{1}{2}$ 

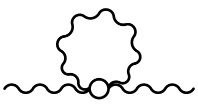
$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Bigg[-\left(\frac{3}{2} + \frac{3}{4}\alpha\right) \frac{\eta_{\mu\nu}}{d_2} + \left\{ \left(\frac{25}{12} - \frac{\alpha}{2}\right) \eta^{\mu\nu} q^2 + \left(\frac{\alpha}{2} - \frac{7}{3}\right) q^\mu q^\nu \right\} \ln \frac{d_2^{-1}}{\mu^2} \Bigg] \tag{2.81}$$



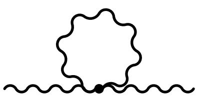
$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Bigg[-\left(6 + \frac{3}{2}\alpha\right) \frac{\eta_{\mu\nu}}{d_2} \frac{2}{\epsilon} + \left\{ 1 + \frac{\alpha}{4} + \left(6 + \frac{3}{2}\alpha\right) \left(\gamma + \ln \frac{d_2^{-1}}{\mu^2}\right) \right\} \frac{\eta_{\mu\nu}}{d_2} \Bigg] \tag{2.82}$$

$\frac{1}{2}$ 

$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Bigg[\left(12 + \frac{3}{2}\alpha\right) \frac{\eta_{\mu\nu}}{d_2} \frac{2}{\epsilon} + \left\{ 4 + \frac{\alpha}{2} - \left(12 + \frac{3}{2}\alpha\right) \left(\gamma + \ln \frac{d_2^{-1}}{\mu^2}\right) \right\} \frac{\eta_{\mu\nu}}{d_2} + \left\{ \left(\frac{46}{3} + \frac{11}{4}\alpha\right) \eta^{\mu\nu} q^2 - \left(\frac{40}{3} + 2\alpha\right) q^\mu q^\nu \right\} \left(\frac{2}{\epsilon} - \ln \frac{d_2^{-1}}{\mu^2}\right) \Bigg] \tag{2.83}$$

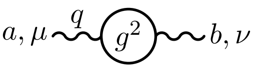
$\frac{1}{2}$ 

$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Bigg[-\left(\frac{33}{4} + \frac{3}{4}\alpha\right) \frac{\eta_{\mu\nu}}{d_2} \frac{2}{\epsilon} + \left\{ -\frac{31}{8} - \frac{5}{8}\alpha + \left(\frac{33}{4} + \frac{3}{4}\alpha\right) \left(\gamma + \ln \frac{d_2^{-1}}{\mu^2}\right) \right\} \frac{\eta_{\mu\nu}}{d_2} + \left\{ \left(\frac{33}{4} + \frac{11}{4}\alpha\right) \eta^{\mu\nu} q^2 - (6 + 2\alpha) q^\mu q^\nu \right\} \left(\ln \frac{d_2^{-1}}{\mu^2} - \frac{2}{\epsilon}\right) \Bigg] \tag{2.84}$$

$\frac{1}{2}$ 

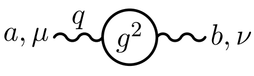
$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \Bigg[\left(\frac{9}{4} + \frac{3}{4}\alpha\right) \frac{\eta_{\mu\nu}}{d_2} \frac{2}{\epsilon} + \left\{ \frac{3}{8} + \frac{5}{8}\alpha - \left(\frac{9}{4} + \frac{3}{4}\alpha\right) \left(\gamma + \ln \frac{d_2^{-1}}{\mu^2}\right) \right\} \frac{\eta_{\mu\nu}}{d_2} \Bigg] . \tag{2.85}$$

Again all terms proportional to $\frac{1}{d_2}$ cancel in the result due to the Slavnov-Taylor identities and we obtain



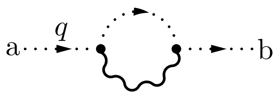
$$= \frac{ig^2 C_2}{16\pi^2} \delta^{ab} \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu\right) \left[\frac{43}{6}\frac{2}{\epsilon} - \left(5 + \frac{\alpha}{2}\right) \ln \frac{d_2^{-1}}{\mu^2}\right] . \quad (2.86)$$

If we use cut-off regularization instead of the dimensional regularization and throw away the unphysical quadratic divergences we get



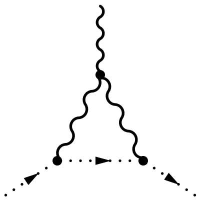
$$= \frac{ig^2 C_2}{16\pi^2} \delta^{ab} \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu\right) \left[\frac{43}{6} \ln \frac{\Lambda^2}{d_2^{-1}} + \left(\frac{13}{6} - \frac{\alpha}{2}\right) \ln \frac{d_2^{-1}}{\mu^2}\right]. \quad (2.87)$$

The divergent part of the one-loop contribution to the proper ghost two-point function has been calculated to be

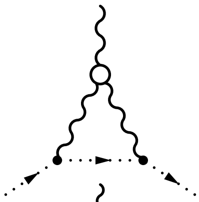


$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} q^2 \left(\frac{\alpha}{4} - \frac{3}{4}\right) \ln \frac{d_2^{-1}}{\mu^2} . \quad (2.88)$$

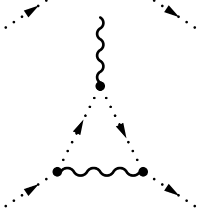
For the proper ghost vertex we get



$$= \frac{g^3 C_2}{16\pi^2} f^{abc} q^\rho \left(-\frac{3}{8} \alpha \ln \frac{d_2^{-1}}{\mu^2}\right) \quad (2.89)$$

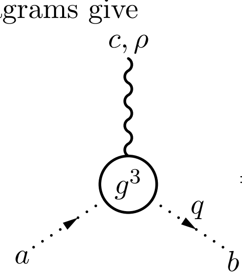


$$= 0 \quad (2.90)$$



$$= \frac{g^3 C_2}{16\pi^2} f^{abc} q^\rho \left(-\frac{1}{8} \alpha \ln \frac{d_2^{-1}}{\mu^2}\right) . \quad (2.91)$$

The above diagrams give



$$= \frac{g^3 C_2}{16\pi^2} f^{abc} q^\rho \left(-\frac{\alpha}{2} \ln \frac{d_2^{-1}}{\mu^2}\right) . \quad (2.92)$$

The divergences of these diagrams determine the one-loop contributions to the folowing counterterms:

$$\delta\mathcal{L}_{A^2} = (Z_A - 1) \,\mathrm{tr}\{-\tfrac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\} \quad (2.93)$$

$$\delta\mathcal{L}_{\bar{c}c} = (Z_c - 1)\{-\bar{c}^a\partial^2 c^a\} \tag{2.94}$$

$$\delta\mathcal{L}_{\bar{c}Ac} = (Z_{\bar{c}Ac} - 1)\{g f^{abc} A^a_\rho (\partial^\rho \bar{c}^c) c^b\} \ . \tag{2.95}$$

The Z factors can be read off from the above results. In minimal subtraction scheme one gets:

$$Z_A - 1 = \frac{g^2 C_2}{16\pi^2}\left(\frac{43}{6}\frac{2}{\epsilon} - \left(5 + \frac{\alpha}{2}\right)\ln\frac{d_2^{-1}}{\mu^2}\right) \tag{2.96}$$

$$Z_c - 1 = \frac{g^2 C_2}{16\pi^2}\left(\frac{3}{4} - \frac{\alpha}{4}\right)\ln\frac{d_2^{-1}}{\mu^2} \tag{2.97}$$

$$Z_{\bar{c}Ac} - 1 = -\frac{g^2 C_2}{16\pi^2}\frac{\alpha}{2}\ln\frac{d_2^{-1}}{\mu^2} \ . \tag{2.98}$$

These gauge dependent Z factors combine to the gauge independent coupling constant renormalization

$$Z_g = Z_{\bar{c}Ac} Z_c^{-1} Z_A^{-\frac{1}{2}} = 1 - \frac{g^2 C_2}{16\pi^2}\left(\frac{43}{12}\frac{2}{\epsilon} - \frac{7}{4}\ln\frac{d_2^{-1}}{\mu^2}\right) \tag{2.99}$$

or with cut-off regularization

$$Z_g = 1 - \frac{g^2 C_2}{16\pi^2}\left(\frac{43}{12}\ln\frac{\Lambda^2}{d_2^{-1}} + \frac{11}{6}\ln\frac{d_2^{-1}}{\mu^2}\right) \ . \tag{2.100}$$

It is interesting to notice that the dependence $Z_{\bar{c}Ac} \sim \alpha$ is fixed by the observation that in Lorenz gauge

$$Z_{\bar{c}Ac}|_{\alpha=0} = 1 \tag{2.101}$$

to all orders. From the powercounting analysis 2.36 we know that the superficial degree of divergence of the ghost-vector vertex is one. Because one derivative acts on an external ghost leg the degree of divergence is zero. The transversality of the gauge field propagator reduces the degree of divergence further and the ghost-vector vertex is in fact finite in Lorenz gauge.
The bare coupling is defined as

$$g_0 = g\mu^{2-d/2} Z_g = g\mu^{2-d/2} Z_{\bar{c}Ac} Z_c^{-1} Z_A^{-\frac{1}{2}} \ . \tag{2.102}$$

Differentiating this equation with respect to the renormalization scale μ gives

$$0 = (2 - \tfrac{d}{2}) g Z_g + \left(\beta\frac{\partial}{\partial g} - 2\frac{\partial}{\partial \ln\frac{\Lambda^2}{\mu^2}}\right) g Z_g \ . \tag{2.103}$$

Therefore the β function is given by

$$\beta = \mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = -(2 - \tfrac{d}{2})\left(\frac{\partial \ln(g Z_g}{\partial g}\right)^{-1} + 2\frac{\frac{\partial}{\partial \ln \Lambda^2/\mu^2} g Z_g}{\frac{\partial}{\partial g} g Z_g} \ . \tag{2.104}$$

The first term in this expression is just the contribution to the β function from dimensional regularization and the second term, which arises because of the explicit dependence of the Z factors on μ through $\ln(d_2^{-1}/\mu^2)$, is the contribution of the higher-derivative regulator. Taking the ordered limit $\epsilon \to 0$, $d_2 \to 0$ of the expansion in powers of g one obtains the well known result

$$\beta = -\frac{g^3 C_2}{16\pi^2}\frac{11}{3} \tag{2.105}$$

Note that we could have written this down without calculating anything. What we have done by performing the limit $d_2 \to 0$ was nothing but a higher covariant derivative regularization [31], supplemented by a dimensional regularization. The result is also clear from the two particle formulation of Lee-Wick gauge theory. If we send d_2 to zero, or equivalently the mass $d_2^{-1/2}$ to infinity, the massive Lee-Wick field decouples according to the Appelquist-Carazzone decoupling theorem [34].

Chapter 3
Background Field Method

We will now apply the background field method to determine the β function of the Lee-Wick gauge theory and of Yang-Mills theory. As stated at the beginning of the previous chapter, in the case of Lee-Wick gauge theory this is no advantage. In addition the background field calculation for Lee-Wick gauge theory has already been done by Grinstein and O'Connell [28] with the same result as in Chapter 2. Hence our motivation to redo the calculation is solely educational.

3.1 The Formalism

At first let us briefly review the background field method following the lines of [35]. The generating functional of correlation functions is defined by

$$Z(J) = e^{iW(J)} = \int \mathcal{D}Q \, \det(\partial \cdot D) \exp\left[i \int \mathrm{d}^4x \left(\mathcal{L}(Q) - \tfrac{1}{2\alpha}(\partial \cdot Q^a)^2 + J^a \cdot Q^a \right) \right] \tag{3.1}$$

and the conventional effective action Γ is the Legendre transform of the generating functional of connected Green's functions $W(J)$.

$$\Gamma(\widehat{Q}) = W(J) - \int \mathrm{d}^4x \; J^a \cdot \widehat{Q}^a \tag{3.2}$$

where

$$\widehat{Q}^a_\mu = \frac{\delta W}{\delta J^{a\mu}} \; . \tag{3.3}$$

Note that we have chosen the gauge fixing condition $G^a = \partial \cdot Q^a$ and of course one could also use another one without any impact on physical quantities.

$\Gamma(\widehat{Q})$ is the generating functional of proper Green's functions, hence its derivatives with respect to $\widehat{Q}$ give the one-particle-irreducible vertices. It therefore plays an essential role in the theory of renormalization.

Denoting by A the background field and by Q the quantum fluctuation we define the background field generating functional

$$\widetilde{Z}(J, A) = \int \mathcal{D}Q \, \det(D \cdot \widetilde{D}) \exp\left[i \int \mathrm{d}^4x \left(\mathcal{L}(A{+}Q) - \tfrac{1}{2\alpha}(D \cdot Q^a)^2 + J^a \cdot Q^a \right) \right] . \tag{3.4}$$

Here and throughout this chapter $D_\mu = \partial_\mu - igA_\mu$ denotes the background field covariant derivative and $\widetilde{D}_\mu = D_\mu - igQ_\mu$ is the covariant derivative corresponding to $A + Q$.

Note that the chosen gauge fixing condition is gauge covariant with respect to the background field. We define the background effective action

$$\widetilde{\Gamma}(\widetilde{Q}, A) = \widetilde{W}(J, A) - \int \mathrm{d}^4x \, J^a \cdot \widetilde{Q}^a \tag{3.5}$$

with

$$\widetilde{W}(J,A) = -i \ln \widetilde{Z}(J,A) \qquad \text{and} \qquad \widetilde{Q}^a_\mu = \frac{\delta \widetilde{W}}{\delta J^{a\mu}} \,. \tag{3.6}$$

It is obvious that $\widetilde{Z}$ and $\widetilde{W}$ are invariant under a gauge transformation of the background field

$$A \to A^u = UAU^{-1} - \tfrac{i}{g}\mathrm{d}UU^{-1} \qquad J \to UJU^{-1} \tag{3.7}$$

with the source transforming homogeneously. Consequently, the background effective action has the property

$$\widetilde{\Gamma}(U\widetilde{Q}U^{-1}, A^u) = \widetilde{\Gamma}(\widetilde{Q}, A) \,. \tag{3.8}$$

What we found thereby is that $\widetilde{\Gamma}(0,A)$ is a gauge invariant functional of A. $\widetilde{\Gamma}(0,A)$ is the gauge-invariant effective action which one computes in the background field method. Making the change of variables $Q \to Q - A$ in (3.4) it follows that W and $\widetilde{W}$ are related by

$$\widetilde{W}(J,A) = W(J) - \int \mathrm{d}^4x\, J^a \cdot A^a \tag{3.9}$$

where $W(J)$ is the conventional generating functional with the background field dependent gauge fixing condition $G = D^\mu(Q_\mu - A_\mu)$. We also find

$$\widetilde{Q} = \widehat{Q} - A \qquad \text{and} \qquad \widetilde{\Gamma}(\widetilde{Q}, A) = \Gamma(\widetilde{Q} + A), \tag{3.10}$$

and thus obtain the desired relation

$$\widetilde{\Gamma}(0,A) = \Gamma(A) \,. \tag{3.11}$$

The gauge-invariant effective action is equal to the conventional effective action evaluated in an unconventional A-dependent gauge.

To compute $\widetilde{\Gamma}(0,A)$ we have to sum up all one-particle-irreducible diagrams with A fields on external legs and Q and ghost fields inside loops.

In fact, if we are interested in the β function, we only need to calculate the proper diagrams with two external background fields, because as a direct consequence of the gauge invariance of $\widetilde{\Gamma}(0,A)$ the divergences must have the manifestly gauge invariant form

$$-\tfrac{1}{2}(Z_A - 1)\,\mathrm{tr}\left(F^2_{\mu\nu}\right) \tag{3.12}$$

which implies the relation

$$Z_g = Z_A^{-\frac{1}{2}} \tag{3.13}$$

between the coupling constant and background field renormalization factors. This is exactly the relation we obtained in Section 2.4 without the background field method only by choosing an unconventional gauge fixing term.

3.2 Yang-Mills Theory

Before we do the background field calculation for Lee-Wick gauge theory, let us do the much simpler calculation for Yang-Mills theory to get familiar with the method. This calculation can be found for example in [35] and [3].

The starting point is the Lagrangian

$$\mathcal{L}(A,Q,\bar{c},c) = \mathrm{tr}\{-\tfrac{1}{2}(F_{\mu\nu} + D_\mu Q_\nu - D_\nu Q_\mu - ig[Q_\mu,Q_\nu])^2 - \tfrac{1}{\alpha}(D_\mu Q^\mu)^2\} \\ + \bar{c}^a[-D\cdot\widetilde{D}]^{ab}c^b \; . \tag{3.14}$$

Its quadratic part in the quantum field is

$$\begin{aligned}\mathcal{L}|_{Q^2} &= \mathrm{tr}\{-\tfrac{1}{2}(D_\mu Q_\nu - D_\nu Q_\mu)^2 + igF_{\mu\nu}[Q^\mu,Q^\nu] - \tfrac{1}{\alpha}(D_\mu Q^\mu)^2\} \\ &= \mathrm{tr}\{Q_\mu(D^2\eta^{\mu\nu} - D^\nu D^\mu + \tfrac{1}{\alpha}D^\mu D^\nu)Q_\nu + igF_{\mu\nu}[Q^\mu,Q^\nu]\} \; .\end{aligned} \tag{3.15}$$

Because of $[D_\mu, D_\nu] = -igF_{\mu\nu}$ it is convenient to set $\alpha = 1$ and further simplify the above expression by using

$$D^\mu D^\nu Q = [D^\mu,[D^\nu,Q]] = [D^\nu,[D^\mu,Q]] - ig[F^{\mu\nu},Q] \; . \tag{3.16}$$

To keep the notation simple it is also convenient to make use of the fact that if A, B and C are fields in the adjoint representation then the following equations hold.

$$\begin{aligned}[][A,B]^a &= if^{abc}A^bB^c = A^b(T^b_{\mathrm{ad}})^{ac}B^c = \mathrm{ad}(A)^{ac}B^c \\ \mathrm{tr}\{A[B,C]\} &= \tfrac{1}{2}A^a\,\mathrm{ad}(B)^{ab}C^b\end{aligned} \tag{3.17}$$

In what follows we will stick to the convention $B^{ab} := \mathrm{ad}(B)^{ab}$.

Applying all this to equation (3.15) we find

$$\begin{aligned}\mathcal{L}|_{Q^2} &= \mathrm{tr}\{Q_\mu(D^2\eta^{\mu\nu})Q_\nu - 2igQ_\mu[F^{\mu\nu},Q_\nu]\} \\ &= -\tfrac{1}{2}Q^a_\mu[-D^2\eta^{\mu\nu} + 2igF^{\mu\nu}]^{ab}Q^b_\nu \\ &= -\tfrac{1}{2}Q^a_\mu\Delta^{\mu\nu}_{ab}Q^b_\nu \; .\end{aligned} \tag{3.18}$$

The quadratic term in ghost fields is simply

$$\mathcal{L}|_{c^2} = \bar{c}^a[-(D^2)^{ab}]c^b \; . \tag{3.19}$$

To one-loop order the effective action is given by

$$e^{i\Gamma[A]} = \exp\left[i\int \mathrm{d}^4x \;\, \mathrm{tr}\{-\tfrac{1}{2}F^2_{\mu\nu}\}\right](\det\Delta)^{-1/2}\det(-D^2) \; . \tag{3.20}$$

In order to compute the determinant of a gauge covariant operator Δ we expand it in powers of A and exploit its gauge invariance. Δ_n is of order n in the background field.

$$\begin{aligned}\ln\det\Delta &= \mathrm{Tr}\{\ln(\Delta_0 + \Delta_1 + \Delta_2 + \dots)\} \\ &= \ln\det\Delta_0 + \mathrm{Tr}\{\ln[1 + \Delta_0^{-1}(\Delta_1 + \Delta_2 + \dots)]\} \\ &= \ln\det\Delta_0 + \mathrm{Tr}\{\Delta_0^{-1}\Delta_2\} - \tfrac{1}{2}\,\mathrm{Tr}\{\Delta_0^{-1}\Delta_1\Delta_0^{-1}\Delta_1\} + \dots\end{aligned} \tag{3.21}$$

We can drop $\det \Delta_0$ because it is just an infinite constant.

In our case

$$\Delta_0^{\mu\nu}(x-y) = -\partial^2 \eta^{\mu\nu}\delta(x-y) \tag{3.22}$$

and because of

$$\begin{aligned} -D^2 &= -\partial^2 + ig(\partial^\mu A_\mu + A_\mu \partial^\mu) + g^2 A^2 \\ F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]\,, \end{aligned} \tag{3.23}$$

Δ_1 and Δ_2 are given by

$$\begin{aligned} \Delta_1^{\mu\nu}(x-y) &= ig\big[(\partial\cdot A + A\cdot\partial)\eta^{\mu\nu} + 2((\partial^\mu A^\nu) - (\partial^\nu A^\mu))\big]\delta(x-y) \\ &= g\int\frac{\mathrm{d}^4k}{(2\pi)^4}A_\rho(k)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\Big[(k+2p)^\rho\eta^{\mu\nu} \\ &\qquad + 2(k^\mu\eta^{\rho\nu} - k^\nu\eta^{\rho\mu})\Big]e^{-ikx-ip(x-y)} \end{aligned} \tag{3.24}$$

$$\begin{aligned} \Delta_2^{\mu\nu}(x-y) &= \big[g^2A^2\eta^{\mu\nu} + \text{traceless}\big]\delta(x-y) \\ &= g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int\frac{\mathrm{d}^4p}{(2\pi)^4}A_\rho(k)A_\sigma(q)\eta^{\rho\sigma}\eta^{\mu\nu}e^{-i(k+q)x-ip(x-y)}\,. \end{aligned} \tag{3.25}$$

Here we dropped traceless terms because they do not contribute at one-loop level.

Let us first calculate the ghost determinant. According to (3.21) the two contributions are

$$\mathrm{Tr}[(-\partial^2)^{-1}gA^2] = g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{tr}\{A_\mu(k)A_\nu(-k)\}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{\eta^{\mu\nu}}{p^2} = 0 \tag{3.26}$$

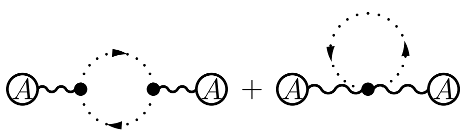
and

$$\begin{aligned} &-\frac{1}{2}\,\mathrm{Tr}[\{(-\partial^2)^{-1}ig(\partial\cdot A + A\cdot\partial)\}^2 \\ &\quad= -\frac{g^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{tr}\{A_\mu(k)A_\nu(-k)\}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{(2p+k)^\mu(2p+k)^\nu}{p^2(p+k)^2} \\ &\quad= \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}A_\mu^a(k)A_\nu^b(-k)\delta^{ab}(k^2\eta^{\mu\nu} - k^\mu k^\nu)\left[\frac{i}{16\pi^2}\frac{g^2C_2}{3}\frac{2}{\epsilon} + \dots\right]\,. \end{aligned} \tag{3.27}$$

Therefore we get for the divergent part of the ghost determinant

$$\ln\det(-D^2)|_{A^2} = \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}A_\mu^a(k)A_\nu^b(-k)\delta^{ab}(k^2\eta^{\mu\nu} - k^\mu k^\nu)\frac{i}{16\pi^2}\frac{g^2C_2}{3}\frac{2}{\epsilon}\,. \tag{3.28}$$

In terms of diagrams the A^2 part of the ghost determinant is given by

 (3.29)

and of course the sum of both diagrams yields the same result.

Let us proceed with the calculation of $\det \Delta$. In the following expression the trace over spacetime indices is already performed.

$$
\begin{aligned}
&-\frac{1}{2}\,\mathrm{Tr}[(\partial^2)^{-1}(\Delta_1)_{\mu\nu}(\partial^2)^{-1}\Delta_1^{\nu\mu}] \qquad (3.30)\\
&= \frac{-g^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{tr}\{A_\mu(k)A_\nu(-k)\}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\frac{d(2p+k)^\mu(2p+k)^\nu+8(k^2\eta^{\mu\nu}-k^\mu k^\nu)}{p^2(p+k)^2}\\
&= -\int\frac{\mathrm{d}^4k}{(2\pi)^4}A_\mu^a(k)A_\nu^b(-k)\delta^{ab}(k^2\eta^{\mu\nu}-k^\mu k^\nu)\left[\frac{ig^2C_2}{16\pi^2}\frac{10}{3}\frac{2}{\epsilon}+\dots\right]
\end{aligned}
$$

Consequently the divergent contribution of the determinant of Δ to the effective action is given by

$$
-\frac{1}{2}\ln\det\Delta|_{A^2} = \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}A_\mu^a(k)A_\nu^b(-k)\delta^{ab}(k^2\eta^{\mu\nu}-k^\mu k^\nu)\frac{i}{16\pi^2}g^2C_2\frac{10}{3}\frac{2}{\epsilon} \qquad (3.31)
$$

and the background field renormalization is

$$
Z_g^{-2} = Z_A = 1+\frac{g^2C_2}{16\pi^2}\frac{11}{3}\frac{2}{\epsilon}\ , \qquad (3.32)
$$

leading to the Yang-Mills β function

$$
\beta = -\frac{g^3C_2}{16\pi^2}\frac{11}{3}\ . \qquad (3.33)
$$

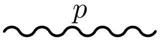
While the gauge invariance of the background field effective action follows directly from its definition, the gauge independence is not obvious. For example Z_A has to be independent of α if α is kept arbitrary throughout the calculation, which is indeed the case. We have

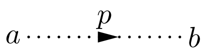
$$
\begin{aligned}
\Delta_0^{\mu\nu}(x-y) &= (P^{-1})^{\mu\nu}(x-y) = \left[-\partial^2\eta^{\mu\nu}+\tfrac{1}{\alpha}\partial^\mu\partial^\nu\right]\delta(x-y)\\
\Rightarrow\quad P_{\mu\nu} &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2}\left(\eta_{\mu\nu}-(1-\alpha)\tfrac{p_\mu p_\nu}{p^2}\right)e^{-ip(x-y)}\ ,
\end{aligned} \qquad (3.34)
$$

$$
\begin{aligned}
\Delta_1^{\mu\nu}(x-y) &= ig\big[(\partial\cdot A+A\cdot\partial)\eta^{\mu\nu}-(\partial^\nu A^\mu+A^\nu\partial^\mu)\\
&\quad+\tfrac{1}{\alpha}(\partial^\mu A^\nu+A^\mu\partial^\nu)+(\partial^\mu A^\nu)-(\partial^\nu A^\mu)\big]\delta(x-y)\\
&= g\int\frac{\mathrm{d}^4k}{(2\pi)^4}A_\rho(k)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\Big[(k+2p)^\rho\eta^{\mu\nu}-\eta^{\mu\rho}(p+2k-\tfrac{1}{\alpha}p)^\nu\\
&\qquad+\eta^{\nu\rho}(k-p+\tfrac{1}{\alpha}(p+k))^\mu\Big]e^{-ikx-ip(x-y)}\ .
\end{aligned} \qquad (3.35)
$$

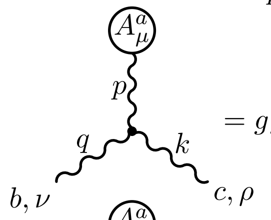
Proceeding as in the case $\alpha = 1$, we end up with the same result, which we don't give here again.

Beside the calculation presented above it is also possible to apply the usual Feynman diagram technique.
The Feynman rules we need are

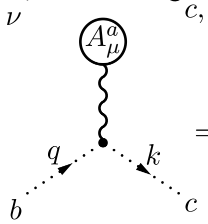
a,μ  $b,\nu = \dfrac{-i\delta^{ab}}{p^2}\left(\eta_{\mu\nu}-(1-\alpha)\tfrac{p_\mu p_\nu}{p^2}\right)$ (3.36)



$$= \frac{i\delta^{ab}}{p^2} \tag{3.37}$$

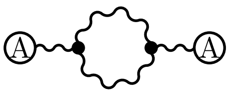


$$= gf^{abc}[\eta^{\nu\rho}(q-k)^\mu + \eta^{\mu\rho}(k-p+\tfrac{1}{\alpha}q)^\nu + \eta^{\mu\nu}(p-q-\tfrac{1}{\alpha}k)^\rho] \tag{3.38}$$

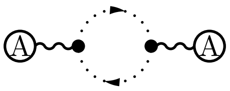


$$= -gf^{abc}(q+k)^\mu \tag{3.39}$$

All these Feynman rules are straightforward to obtain from equation (3.14). For example one contraction of the one-background field two-quantum fields vertex can be directly read off from expression (3.35). Of course the results are the same and independent of α:

$\frac{1}{2}$ 

$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \frac{2}{\epsilon} \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu\right) \frac{10}{3} \tag{3.40}$$

$\frac{1}{2}$ 

$$= \frac{i}{16\pi^2} g^2 C_2 \delta^{ab} \frac{2}{\epsilon} \left(\eta^{\mu\nu} q^2 - q^\mu q^\nu\right) \frac{1}{3} \; . \tag{3.41}$$

3.3 Lee-Wick Gauge Theory

After this short Yang-Mills theory warm-up to get familiar with the background field method let us apply it to Lee-Wick gauge theory. We start with the Lagrangian

$$\begin{aligned}
\mathcal{L}(A,Q,\bar{c},c) = \operatorname{tr}\{ &-\tfrac{1}{2}(F_{\mu\nu} + D_\mu Q_\nu - D_\nu Q_\mu - ig[Q_\mu, Q_\nu])^2 - \tfrac{1}{\alpha}(D_\mu Q^\mu)^2\} \\
&+ d_2\big(D^\mu F_{\mu\nu} + D^\mu(D_\mu Q_\nu - D_\nu Q_\mu - ig[Q_\mu, Q_\nu]) \\
&\qquad - ig[Q^\mu, F_{\mu\nu} + D_\mu Q_\nu - D_\nu Q_\mu - ig[Q_\mu, Q_\nu]]\big)^2 \\
&+ \bar{c}^a[-D\cdot\widetilde{D}]^{ab} c^b \; .
\end{aligned} \tag{3.42}$$

The quadratic part in the quantum field is

$$\begin{aligned}
\mathcal{L}|_{Q^2} &= \operatorname{tr}\{ -\tfrac{1}{2}(D_\mu Q_\nu - D_\nu Q_\mu)^2 + igF_{\mu\nu}[Q^\mu, Q^\nu] - \tfrac{1}{\alpha}(D_\mu Q^\mu)^2\} \\
&\quad + d_2 \operatorname{tr}\{(D^\mu(D_\mu Q_\nu - D_\nu Q_\mu) - ig[Q^\mu, F_{\mu\nu}])^2 \\
&\quad - 2ig(D_\rho F^{\rho\nu})(D^\sigma[Q_\sigma, Q_\nu] + [Q^\mu, D_\mu Q_\nu - D_\nu Q_\mu])\} \\
&= \operatorname{tr}\{Q_\mu(D^2\eta^{\mu\nu} - D^\nu D^\mu + \tfrac{1}{\alpha}D^\mu D^\nu)Q_\nu + igF_{\mu\nu}[Q^\mu, Q^\nu]\} \\
&\quad + d_2 \operatorname{tr}\{(D^\mu(D_\mu Q_\nu - D_\nu Q_\mu))^2 - 2igQ_\mu[F^{\mu\nu}, D^2 Q_\nu] \\
&\qquad + 2igQ_\mu[F^{\mu\rho}, D^\nu D_\rho Q_\nu] + g^2 Q_\mu[F^{\mu\rho}, [F^\nu{}_\rho, Q_\nu]] \\
&\qquad + 2ig(D^\sigma D_\rho F^{\rho\nu})[Q_\sigma, Q_\nu] \\
&\qquad + 2igQ_\mu[(D_\rho F^{\rho\nu}), D^\mu Q_\nu] - 2igQ_\mu[(D_\rho F^{\rho\nu}), D_\nu Q^\mu]\} \; .
\end{aligned} \tag{3.43}$$

Again we set $\alpha = 1$ for convenience and further simplify the above expression by making use of $[D_\mu, D_\nu] = -igF_{\mu\nu}$:

$$\begin{aligned}(D^2 Q_\mu - D^\rho D_\mu Q_\rho)^2 &= Q_\mu(D^4\eta^{\mu\nu} - D^2 D^\nu D^\mu)Q_\nu \\ &\quad - Q_\mu(D^\nu D^\mu D^2 - D^\rho D^\mu D^\nu D_\rho)Q_\nu \\ &= Q_\mu(D^4\eta^{\mu\nu} - D^2 D^\nu D^\mu)Q_\nu - igQ_\mu[F^{\rho\mu}, D^\nu D_\rho Q_\nu] \\ &\quad - igQ_\mu(+D^\mu[F^{\rho\nu}, D_\rho Q_\nu] + [F^{\mu\nu}, D^2 Q_\nu]) \ .\end{aligned} \tag{3.44}$$

Plugging this into the previous equation we get

$$\begin{aligned}\mathcal{L}|_{Q^2} &= \mathrm{tr}\{Q_\mu(D^2\eta^{\mu\nu})Q_\nu - 2igQ_\mu[F^{\mu\nu}, Q_\nu]\} \\ &\quad + d_2\,\mathrm{tr}\{Q_\mu(D^4\eta^{\mu\nu} - D^2 D^\nu D^\mu)Q_\nu \\ &\qquad + g^2 Q_\mu[F^{\mu\rho}, [F^\nu{}_\rho, Q_\nu]] - 2igQ_\mu[(D^\mu D_\rho F^{\rho\nu}), Q_\nu] \\ &\qquad - igQ_\mu(3[F^{\rho\mu}, D^\nu D_\rho Q_\nu] + D^\mu[F^{\rho\nu}, D_\rho Q_\nu] + 3[F^{\mu\nu}, D^2 Q_\nu]) \\ &\qquad + 2igQ_\mu[(D_\rho F^{\rho\nu}), D^\mu Q_\nu] - 2igQ_\mu[(D_\rho F^{\rho\nu}), D_\nu Q^\mu]\} \\ &= -\tfrac{1}{2}Q^a_\mu[-D^2\eta^{\mu\nu} + 2igF^{\mu\nu}]^{ab}Q^b_\nu - \tfrac{d_2}{2}Q^a_\mu[-D^4\eta^{\mu\nu} + D^2 D^\nu D^\mu]^{ab}Q^b_\nu \\ &\quad - \tfrac{d_2}{2}Q^a_\mu[-g^2 F^{\mu\rho}F^\nu{}_\rho + 3igF^{\rho\mu}D^\nu D_\rho + igD^\mu F^{\rho\nu}D_\rho + 3igF^{\mu\nu}D^2 \\ &\quad + 2ig(D^\mu D_\rho F^{\rho\nu}) + 2ig(D^\rho F_{\rho\sigma})(\eta^{\mu\nu}D^\sigma - \eta^{\nu\sigma}D^\mu)]^{ab}Q^b_\nu \\ &= -\tfrac{1}{2}Q^a_\mu \Delta^{\mu\nu}_{ab} Q^b_\nu\end{aligned} \tag{3.45}$$

The one-loop effective action is now given by

$$e^{i\Gamma[A]} = \exp\left[i\int \mathrm{d}^4x\,\mathrm{tr}\{-\tfrac{1}{2}F^2_{\mu\nu} + d_2(D^\mu F_{\mu\nu})^2\}\right](\det\Delta)^{-1/2}\det(-D^2) \tag{3.46}$$

To calculate the determinant of Δ we will apply the usual Feynman diagram technique. Both determinants are determinants of gauge covariant operators and thus individually gauge invariant.

To determine the required Feynman rules we need the expansion in powers of the background field of all the operators which Δ consists of. We have

$$\Delta^{\mu\nu} = -(\partial^2 + d_2\partial^4)\eta^{\mu\nu} + d_2\partial^2\partial^\mu\partial^\nu + \Delta_1^{\mu\nu} + \Delta_2^{\mu\nu} + \Delta_3^{\mu\nu} + \Delta_4^{\mu\nu} \ . \tag{3.47}$$

Here Δ_n is of order n in the background field. The appearing operators are

$$\begin{aligned}D^4 &= \partial^4 - ig[\partial^2(\partial\cdot A + A\cdot\partial) + (\partial\cdot A + A\cdot\partial)\partial^2] \\ &\quad - g^2[(\partial\cdot A + A\cdot\partial)^2 + (\partial^2 A^2 + A^2\partial^2)] + \mathcal{O}(A^3)\end{aligned} \tag{3.48}$$

$$\begin{aligned}D^2 D^\nu D^\mu &= \partial^2\partial^\nu\partial^\mu - ig[(\partial\cdot A + A\cdot\partial)\partial^\nu\partial^\mu \\ &\qquad + \partial^2(\partial^\nu A^\mu + A^\nu\partial^\mu)] \\ &\quad - g^2[(\partial\cdot A + A\cdot\partial)(\partial^\nu A^\mu + A^\nu\partial^\mu) \\ &\qquad + \partial^2 A^\nu A^\mu + A^2\partial^\nu\partial^\mu] + \mathcal{O}(A^3)\end{aligned} \tag{3.49}$$

$$\begin{aligned}(D^\mu D_\rho F^{\rho\nu}) &= \partial^\mu\partial_\rho(\partial^\rho A^\nu - \partial^\nu A^\rho) \\ &\quad - ig\big(\partial^\mu[A_\rho, F^{\rho\nu}] + [A^\mu, \partial_\rho F^{\rho\nu}] \\ &\qquad + \partial^\mu\partial_\rho[A^\rho, A^\nu]\big) + \mathcal{O}(A^3)\end{aligned} \tag{3.50}$$

$$
\begin{aligned}
F^{\mu\rho}F^{\nu}{}_{\rho} &= (\partial^\mu A^\rho - \partial^\rho A^\mu)(\partial^\nu A_\rho - \partial_\rho A^\nu) + \mathcal{O}(A^3) && (3.51)\\
D^\mu F^{\rho\nu} D_\rho &= \partial^\mu[(\partial^\rho A^\nu) - (\partial^\nu A^\rho)]\partial_\rho && (3.52)\\
&\quad - igA^\mu[(\partial^\rho A^\nu) - (\partial^\nu A^\rho)]\partial_\rho \\
&\quad - ig\partial^\mu[(\partial^\rho A^\nu) - (\partial^\nu A^\rho)]A_\rho - ig\partial^\mu[A^\rho, A^\nu]\partial_\rho + \mathcal{O}(A^3)\\
F^{\rho\mu}D^\nu D_\rho &= [(\partial^\rho A^\mu) - (\partial^\mu A^\rho)]\partial^\nu\partial_\rho && (3.53)\\
&\quad -ig[(\partial^\rho A^\mu) - (\partial^\mu A^\rho)](\partial^\nu A_\rho + A^\nu\partial_\rho) - ig[A^\rho, A^\mu] + \mathcal{O}(A^3)\\
F^{\mu\nu}D^2 &= [(\partial^\mu A^\nu) - (\partial^\nu A^\mu)]\partial^2 - ig[A^\mu, A^\nu]\partial^2 && (3.54)\\
&\quad -ig[(\partial^\mu A^\nu) - (\partial^\nu A^\mu)](\partial\cdot A + A\cdot\partial) + \mathcal{O}(A^3)\\
(D^\rho F_{\rho\sigma})2\eta^{\nu[\mu}D^{\sigma]} &= [(\partial^2 A_\sigma) - (\partial^\rho\partial_\sigma A_\rho)](\eta^{\mu\nu}\partial^\sigma - \eta^{\nu\sigma}\partial^\mu) && (3.55)\\
&\quad -ig(\partial^\rho[A_\rho, A_\sigma] + [A^\rho, \partial_\rho A_\sigma - \partial_\sigma A_\rho])(\eta^{\mu\nu}\partial^\sigma - \eta^{\nu\sigma}\partial^\mu)\\
&\quad -ig[(\partial^2 A_\sigma) - (\partial^\rho\partial_\sigma A_\rho)](\eta^{\mu\nu}A^\sigma - \eta^{\nu\sigma}A^\mu) + \mathcal{O}(A^3)\ .
\end{aligned}
$$

The propagator of the field Q is given by

$$
\begin{aligned}
(D^{-1})^{\mu\nu}_{ab} &= -i\delta_{ab}(\partial^2 + d_2\partial^4)\eta^{\mu\nu} + id_2\delta_{ab}\partial^2\partial^\mu\partial^\nu \\
D^{ab}_{\mu\nu} &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \delta^{ab}\frac{-i(\eta_{\mu\nu} - d_2 p_\mu p_\nu)}{p^2(1-d_2p^2)}\, e^{-ip(x-y)}\ .
\end{aligned}
\qquad (3.56)
$$

Note that because the UV behavior of the propagator is only k^{-2}, the superficial degree of divergence of the diagrams (3.62) and (3.63) is four, but as we will see all the higher divergences cancel and they are only logarithmically divergent.

To obtain the one and two background, two quantum fields vertices we need Δ_1 and Δ_2.

$$
\begin{aligned}
\Delta_1^{\mu\nu}(x-y) &= g\Big[- i(\partial\cdot A + A\cdot\partial)\eta^{\mu\nu} + 2i((\partial^\mu A^\nu) - (\partial^\nu A^\mu)) && (3.57)\\
&\quad + id_2\eta^{\mu\nu}[\partial^2(\partial\cdot A + A\cdot\partial) + (\partial\cdot A + A\cdot\partial)\partial^2]\\
&\quad - id_2[(\partial\cdot A + A\cdot\partial)\partial^\nu\partial^\mu + \partial^2(\partial^\nu A^\mu + A^\nu\partial^\mu)]\\
&\quad + 2id_2[\partial^\mu\partial_\rho(\partial^\rho A^\nu - \partial^\nu A^\rho)] + 3ig[(\partial^\rho A^\mu) - (\partial^\mu A^\rho)]\partial^\nu\partial_\rho\\
&\quad + id_2\partial^\mu[(\partial^\rho A^\nu) - (\partial^\nu A^\rho)]\partial_\rho + 3ig[(\partial^\mu A^\nu) - (\partial^\nu A^\mu)]\partial^2\\
&\quad + 2i[(\partial^2 A_\sigma) - (\partial^\rho\partial_\sigma A_\rho)](\eta^{\mu\nu}\partial^\sigma - \eta^{\nu\sigma}\partial^\mu)\Big]\delta(x-y)\\
&= g\int\frac{\mathrm{d}^4 k}{(2\pi)^4}A_\rho(k)\int\frac{\mathrm{d}^4 p}{(2\pi)^4}\Big[(k+2p)^\rho\eta^{\mu\nu} + 2(k^\mu\eta^{\rho\nu} - k^\nu\eta^{\rho\mu}) && (3.58)\\
&\quad + d_2\{ - ((p+k)^2 + p^2)(k+2p)^\rho\eta^{\mu\nu} + (k+2p)^\rho p^\mu p^\nu\\
&\qquad + (p+k)^2((k+p)^\nu\eta^{\rho\mu} + p^\mu\eta^{\rho\nu}) - 2k^\mu(k^2\eta^{\nu\rho} - k^\nu k^\rho)\\
&\qquad - 3p^\nu(kp\eta^{\mu\rho} - k^\mu p^\rho) - (k+p)^\mu(kp\eta^{\nu\rho} - k^\nu p^\rho)\\
&\qquad - 3(k^\mu\eta^{\nu\rho} - k^\nu\eta^{\mu\rho})p^2 - 2\eta^{\mu\nu}(p^\rho k^2 - k^\rho kp)\\
&\qquad + 2(\eta^{\nu\rho}k^2 - k^\nu k^\rho)p^\mu\}\Big]e^{-ikx-ip(x-y)}
\end{aligned}
$$

As in Yang-Mills theory we can easily read off one contraction of the one-background field two-quantum fields vertex. We don't give the explicit expression because it does not contain more information than equation (3.59).

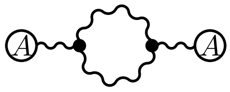
$$\begin{aligned}
\Delta_2^{\mu\nu}(x-y) = g^2\Big[&A^2\eta^{\mu\nu} + d_2\eta^{\mu\nu}[(\partial\cdot A + A\cdot\partial)^2 + (\partial^2 A^2 + A^2\partial^2)] \qquad (3.59)\\
&- d_2[(\partial\cdot A + A\cdot\partial)(\partial^\nu A^\mu + A^\nu\partial^\mu) + \partial^2 A^\nu A^\mu + A^2\partial^\nu\partial^\mu]\\
&- d_2[(\partial^\mu A^\rho) - (\partial^\rho A^\mu)][(\partial^\nu A_\rho) - (\partial_\rho A^\nu)]\\
&+ 3d_2[(\partial^\rho A^\mu) - (\partial^\mu A^\rho)](\partial^\nu A_\rho + A^\nu\partial_\rho)\\
&+ d_2[A^\mu[(\partial^\rho A^\nu) - (\partial^\nu A^\rho)]\partial_\rho + \partial^\mu[(\partial^\rho A^\nu) - (\partial^\nu A^\rho)]A_\rho]\\
&+ 3d_2[(\partial^\mu A^\nu) - (\partial^\nu A^\mu)](\partial\cdot A + A\cdot\partial)\\
&+ 2d_2[(\partial^2 A_\sigma) - (\partial^\rho\partial_\sigma A_\rho)](\eta^{\mu\nu}A^\sigma - \eta^{\nu\sigma}A^\mu)\\
&+ \text{traceless}\Big]\delta(x-y)\\
= g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}&\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int\frac{\mathrm{d}^4p}{(2\pi)^4}A_\rho(k)A_\sigma(q)\Big[\eta^{\rho\sigma}\eta^{\mu\nu} \qquad (3.60)\\
&- d_2[\eta^{\mu\nu}(k+2q+2p)^\rho(q+2p)^\sigma + \eta^{\mu\nu}\eta^{\rho\sigma}((k+q+p)^2 + p^2)]\\
&+ d_2[(k+2q+2p)^\rho((p+q)^\nu\eta^{\mu\sigma} + p^\mu\eta^{\nu\sigma}) + (p+q+k)^2\eta^{\nu\rho}\eta^{\mu\sigma}\\
&+ d_2[\eta^{\rho\sigma}p^\mu p^\nu]k^\mu(q^\nu\eta^{\rho\sigma} - q^\rho\eta^{\nu\sigma}) + \eta^{\mu\rho}(qk\eta^{\nu\sigma} - q^\nu k^\sigma)]\\
&- 3d_2[\eta^{\mu\rho}(kp\eta^{\nu\sigma} + (q+p)^\nu k^\sigma) - k^\mu((q+p)^\nu\eta^{\rho\sigma} + p^\rho\eta^{\nu\sigma})]\\
&- d_2[\eta^{\mu\rho}(qp\eta^{\nu\sigma} - q^\nu p^\sigma) + (k+p+q)^\mu(k^\sigma\eta^{\nu\rho} - k^\nu\eta^{\rho\sigma})]\\
&- 3d_2[(k^\mu\eta^{\nu\rho} - k^\nu\eta^{\mu\rho})(q+2p)^\sigma]\\
&- 2d_2[(k^2\eta^{\rho\sigma} - k^\rho k^\sigma)\eta^{\mu\nu} - (k^2\eta^{\nu\rho} - k^\nu k^\rho)\eta^{\mu\sigma}\Big]e^{-i(k+q)x-ip(x-y)}
\end{aligned}$$

In the above expression we omitted the traceless part because it clearly does not contribute at one-loop level and is therefore irrelevant in our calculation. Again we can read off one contraction of the two-background two-quantum fields vertex from the expression for Δ_2 given in (3.60).

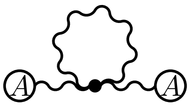
Since we didn't change the gauge fixing condition the contribution of the ghost fields is the same as in Yang-Mills theory and we just state the result obtained in Section 3.2

$$\ln\det(-D^2)|_{A^2} = \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}A^a_\mu(k)A^b_\nu(-k)\delta^{ab}(k^2\eta^{\mu\nu} - k^\mu k^\nu)\frac{i}{16\pi^2}\frac{g^2C_2}{3}\frac{2}{\epsilon}\,. \qquad (3.61)$$

It still remains to compute the determinant of the fluctuation operator Δ. The contribution of $\det\Delta$ to the generating functional of proper vertices is determined by the following two Feynman diagrams:

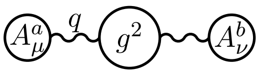
$\frac{1}{2}$ 

$$\begin{aligned}
= \frac{ig^2C_2}{16\pi^2}\delta^{ab}\Big[&\eta^{\mu\nu}\Big\{\frac{181}{12}q^2\frac{2}{\epsilon} - \frac{47}{4}\ln\frac{d_2^{-1}}{\mu^2}\\
&+ \frac{1}{d_2}\left(\frac{23}{8} + \frac{21}{4}\frac{2}{\epsilon} - \frac{21}{4}\left(\gamma + \ln\frac{d_2^{-1}}{\mu^2}\right)\right)\Big\}\\
&+ q^\mu q^\nu\left\{-\frac{77}{6}\frac{2}{\epsilon} + \frac{19}{2}\ln\frac{d_2^{-1}}{\mu^2}\right\}\Big] \qquad (3.62)
\end{aligned}$$

$\frac{1}{2}$ 

$$= \frac{ig^2 C_2}{16\pi^2}\delta^{ab}\Bigg[\eta^{\mu\nu}q^2\Bigg\{-\frac{33}{4}\frac{2}{\epsilon}+\frac{33}{4}\ln\frac{d_2^{-1}}{\mu^2} - \frac{1}{d_2}\left(\frac{23}{8}+\frac{21}{4}\frac{2}{\epsilon}-\frac{21}{4}\left(\gamma+\ln\frac{d_2^{-1}}{\mu^2}\right)\right)\Bigg\} + q^\mu q^\nu\left\{6\frac{2}{\epsilon}-6\ln\frac{d_2^{-1}}{\mu^2}\right\}\Bigg] . \tag{3.63}$$

Their sum is



$$= \frac{i}{16\pi^2}g^2 C_2\delta^{ab}\frac{2}{\epsilon}\left(\eta^{\mu\nu}q^2-q^\mu q^\nu\right)\left[\frac{41}{6}-\frac{7}{2}\ln\frac{d_2^{-1}}{\mu^2}\right] . \tag{3.64}$$

Together with the ghost determinant we get

$$Z_g^{-2} = Z_3 = 1+\frac{g^2}{16\pi^2}C_2\frac{43}{6}\frac{2}{\epsilon} , \tag{3.65}$$

similar to Section 2.8. And in the limit $d_2 \to 0$

$$Z_g = 1-\frac{g^2}{16\pi^2}C_2\left(\frac{43}{12}\frac{2}{\epsilon}-\frac{7}{4}\ln\frac{d_2^{-1}}{\mu^2}\right) , \tag{3.66}$$

which exactly reproduces our previous result (2.99).

3.3.1 Improving the Convergence

As in [28] we will now use the gauge fixing term

$$-\operatorname{tr}\{(D_\mu Q^\mu)(1+d_2D^2)(D_\nu Q^\nu)\} \tag{3.67}$$

to improve the convergence of the integrals. This is nothing but the background field analogue of the gauge fixing term we used in Chapter 2 however in contrast to Section 2.3 the weight corresponding to the counterterm depends on the background field and thus we have to include the A dependent normalization factor $(\det(1+d_2D^2))^{1/2}$.

The Lagrangian we start with this time is

$$\begin{aligned}\mathcal{L}(A,Q,\bar{c},c) = \operatorname{tr}\{ &-\tfrac{1}{2}(F_{\mu\nu}+D_\mu Q_\nu - D_\nu Q_\mu - ig[Q_\mu,Q_\nu])^2 \\ &+ d_2(D^\mu F_{\mu\nu} + D^\mu(D_\mu Q_\nu - D_\nu Q_\mu - ig[Q_\mu,Q_\nu]) \\ &- ig[Q^\mu, F_{\mu\nu}+D_\mu Q_\nu - D_\nu Q_\mu - ig[Q_\mu,Q_\nu]])^2 \\ &- (D_\mu Q^\mu)(1+d_2D^2)(D_\nu Q^\nu)\} + \bar{c}^a[-D\cdot\widetilde{D}]^{ab}c^b\end{aligned} . \tag{3.68}$$

The quadratic part in the quantum field is

$$\begin{aligned}\mathcal{L}|_{Q^2} = \operatorname{tr}\{&Q_\mu(D^2\eta^{\mu\nu})Q_\nu - 2igQ_\mu[F^{\mu\nu},Q_\nu]\} \\ &+ d_2\operatorname{tr}\{Q_\mu(D^4\eta^{\mu\nu} - D^2D^\nu D^\mu + D^\mu D^2 D^\nu)Q_\nu \\ &+ g^2Q_\mu[F^{\mu\rho},[F^\nu{}_\rho,Q_\nu]] - 2igQ_\mu[(D^\mu D_\rho F^{\rho\nu}),Q_\nu] \\ &- igQ_\mu(3[F^{\rho\mu},D^\nu D_\rho Q_\nu] + D^\mu[F^{\rho\nu},D_\rho Q_\nu] + 3[F^{\mu\nu},D^2Q_\nu]) \\ &+ 2igQ_\mu[(D_\rho F^{\rho\nu}),D^\mu Q_\nu] - 2igQ_\mu[(D_\rho F^{\rho\nu}),D_\nu Q^\mu]\} .\end{aligned} \tag{3.69}$$

Again one can use $[D_\mu, D_\nu] = -igF_{\mu\nu}$ to further simplify the above expression.

$$\begin{aligned} D^\mu D^2 D^\nu Q =& -ig[F^{\mu\rho}, D_\rho D^\nu Q] - igD_\rho[F^{\mu\rho}, D^\nu Q] \\ & -igD^2[F^{\mu\nu}, Q] + D^2 D^\nu D^\mu Q \\ [F^{\mu\rho}, D_\rho D^\nu Q] =& \, ig[F^{\mu\rho}, [F^\nu{}_\rho, Q] - [F^{\rho\mu}, D^\nu D_\rho Q] \\ \mathrm{tr}\{Q_\mu D^2[F^{\mu\nu}, Q_\nu]\} =& \, \mathrm{tr}\{Q_\mu[F^{\mu\nu}, D^2 Q_\nu]\} + \text{t.d.} \end{aligned} \tag{3.70}$$

Applying this to the last equation we get

$$\begin{aligned} \mathcal{L}|_{Q^2} =& \, \mathrm{tr}\{Q_\mu(D^2\eta^{\mu\nu})Q_\nu - 2igQ_\mu[F^{\mu\nu}, Q_\nu]\} \qquad (3.71) \\ & + d_2\,\mathrm{tr}\{Q_\mu(D^4\eta^{\mu\nu}) + 2g^2 Q_\mu[F^{\mu\rho}, [F^\nu{}_\rho, Q_\nu]] - 2igQ_\mu[(D^\mu D_\rho F^{\rho\nu}), Q_\nu] \\ & \quad - igQ_\mu(2[F^{\rho\mu}, D^\nu D_\rho Q_\nu] + D_\rho[F^{\mu\rho}, D^\nu Q_\nu]) \\ & \quad - igQ_\mu(D^\mu[F^{\rho\nu}, D_\rho Q_\nu] + 4[F^{\mu\nu}, D^2 Q_\nu]) \\ & \quad + 2igQ_\mu[(D_\rho F^{\rho\nu}), D^\mu Q_\nu] - 2igQ_\mu[(D_\rho F^{\rho\nu}), D_\nu Q^\mu]\} \\ =& -\tfrac{1}{2}Q^a_\mu[-D^2\eta^{\mu\nu} + 2igF^{\mu\nu}]^{ab}Q^b_\nu - \tfrac{d_2}{2}Q^a_\mu[-D^4\eta^{\mu\nu}]^{ab}Q^b_\nu \qquad (3.72) \\ & -\tfrac{d_2}{2}Q^a_\mu[-2g^2F^{\mu\rho}F^\nu{}_\rho + 2F^{\rho\mu}D^\nu D_\rho + igD^\mu F^{\rho\nu}D_\rho + igD_\rho F^{\mu\rho}D^\nu \\ & +4igF^{\mu\nu}D^2 + 2ig(D^\mu D_\rho F^{\rho\nu}) + 2ig(D^\rho F_{\rho\sigma})(\eta^{\mu\nu}D^\sigma - \eta^{\nu\sigma}D^\mu)]^{ab}Q^b_\nu \\ =& -\tfrac{1}{2}Q^a_\mu \Delta^{\mu\nu}_{ab} Q^b_\nu \qquad (3.73) \end{aligned}$$

The one-loop effective action is given by

$$e^{i\Gamma[A]} = \exp\left[i\int \mathrm{d}^4x\,(\mathcal{L}(A) + \mathcal{L}_{c.t.})\right] \frac{\det(-D^2)(\det(1{+}d_2D^2))^{\frac{1}{2}}}{(\det\Delta)^{\frac{1}{2}}} \tag{3.74}$$

where $\mathcal{L}(A)$ is the Lee-Wick Lagrangian (2.1). Again all the determinants are individually gauge invariant. As stated at the beginning of this section the factor $(\det(1{+}d_2D^2))^{1/2}$ has to be included because of the additional background field dependence of the gauge fixing term. To calculate this determinant diagrammaticly one has to introduce further ghost fields, but we will not do this here.

Let us begin with the calculation of the determinant coming from the gauge fixing term. We have

$$1 + d_2D^2 = 1 + d_2\partial^2 - igd_2(\partial\cdot A + A\cdot\partial) - d_2g^2A^2\,. \tag{3.75}$$

According to the expansion (3.21) there are the two contributions

$$\begin{aligned} & -\mathrm{Tr}\{(1{+}d_2\partial^2)^{-1}d_2g^2A^2\} \qquad (3.76) \\ & = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{tr}\{A_\mu(k)A_\nu(-k)\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{g^2\eta^{\mu\nu}}{p^2 - d_2^{-1}} \\ & = \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{tr}\{A_\mu(k)A_\nu(-k)\}\left[\frac{i}{16\pi^2}\eta^{\mu\nu}(-2)g^2d_2^{-1}\Gamma(1-\tfrac{d}{2})\left(\frac{d_2^{-1}}{\mu^2}\right)^{\frac{d}{2}-2}\right] \end{aligned}$$

and

$$\frac{g^2 d_2^2}{2} \operatorname{Tr}\{[(1+d_2\partial^2)^{-1}(\partial\cdot A + A\cdot\partial)]^2\} \tag{3.77}$$
$$= -\frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\operatorname{tr}\{A_\mu(k)A_\nu(-k)\}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{g^2(2p+k)^\mu(2p+k)^\nu}{(p^2-d_2^{-1})((p+k)^2-d_2^{-1})}$$
$$= \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\operatorname{tr}\{A_\mu(k)A_\nu(-k)\}\frac{ig^2}{16\pi^2}\Bigg[(\eta^{\mu\nu}k^2-k^\mu k^\nu)\frac{1}{3}\Gamma(2-\tfrac{d}{2})\left(\frac{d_2^{-1}}{\mu^2}\right)^{\frac{d}{2}-2}$$
$$+ 2\eta^{\mu\nu}d_2^{-1}\left(\frac{d_2^{-1}}{\mu^2}\right)^{\frac{d}{2}-2}\Gamma(1-\tfrac{d}{2})+\ldots\Bigg].$$

They sum up to give the divergent part of the determinant

$$\frac{1}{2}\ln\det(1+d_2D^2)|_{A^2} \tag{3.78}$$
$$= \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}A_\mu^a(k)A_\nu^b(-k)\delta^{ab}(\eta^{\mu\nu}k^2-k^\mu k^\nu)\left[\frac{i}{16\pi^2}\frac{g^2C_2}{6}\Gamma(2-\tfrac{d}{2})\left(\frac{d_2^{-1}}{\mu^2}\right)^{\frac{d}{2}-2}\right].$$

It is quite obvious that this expression does not contribute to the β function in the limit d_2 to zero, no matter wether we perform the limit before or after sending $d \to 4$.

As in the previous section we will calculate the determinant of Δ using the Feynman diagram technique. Hence again the expansion of Δ in powers of the background field is needed:

$$\Delta^{\mu\nu} = -(\partial^2 + d_2\partial^4)\eta^{\mu\nu} + \Delta_1^{\mu\nu} + \Delta_2^{\mu\nu} + \Delta_3^{\mu\nu} + \Delta_4^{\mu\nu}\,. \tag{3.79}$$

Beside the operators (3.49)-(3.55) we also need

$$D_\rho F^{\mu\rho}D^\nu = \partial_\rho[(\partial^\mu A^\rho)-(\partial^\rho A^\mu)]\partial^\nu - igA_\rho[(\partial^\mu A^\rho)-(\partial^\rho A^\mu)]\partial^\nu \tag{3.80}$$
$$- ig\partial_\rho[(\partial^\mu A^\rho)-(\partial^\rho A^\mu)]A^\nu - ig\partial_\rho[A^\mu,A^\rho]\partial^\nu + \mathcal{O}(A^3)$$

The quantum field propagator is given by

$$\begin{aligned}(D^{-1})^{\mu\nu}_{ab} &= -i\delta_{ab}(\partial^2+d_2\partial^4)\eta^{\mu\nu}\\ D^{ab}_{\mu\nu} &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\delta^{ab}\,\frac{-i\eta_{\mu\nu}}{p^2(1-d_2p^2)}e^{-ip(x-y)}\end{aligned} \tag{3.81}$$

and now has the UV behaviour k^{-4}.

Similar to the previous section the necessary Feynman rules can be obtained from Δ_1 and Δ_2.

$$\begin{aligned}\Delta_1^{\mu\nu}(x-y) = ig\Big[&(\partial\cdot A + A\cdot\partial)\eta^{\mu\nu} + 2((\partial^\mu A^\nu)-(\partial^\nu A^\mu))\\ &+ d_2\eta^{\mu\nu}[\partial^2(\partial\cdot A + A\cdot\partial)+(\partial\cdot A + A\cdot\partial)\partial^2]\\ &+ 2d_2[\partial^\mu\partial_\rho(\partial^\rho A^\nu - \partial^\nu A^\rho)] + d_2\partial^\mu[(\partial^\rho A^\nu)-(\partial^\nu A^\rho)]\partial_\rho\\ &+ d_2\partial_\rho[(\partial^\mu A^\rho)-(\partial^\rho A^\mu)]\partial^\nu + 4d_2[(\partial^\mu A^\nu)-(\partial^\nu A^\mu)]\partial^2\\ &+ 2d_2[(\partial^\rho A^\mu)-(\partial^\mu A^\rho)]\partial^\nu\partial_\rho\\ &+ 2d_2[(\partial^2A_\sigma - \partial^\rho\partial_\sigma A_\rho](\eta^{\mu\nu}\partial^\sigma - \eta^{\nu\sigma}\partial^\mu)\Big]\delta(x-y)\end{aligned} \tag{3.82}$$

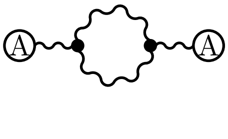
$$
\begin{aligned}
&= g\int\frac{\mathrm{d}^4k}{(2\pi)^4}A_\rho(k)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\Big[(k+2p)^\rho\eta^{\mu\nu}+2(k^\mu\eta^{\rho\nu}-k^\nu\eta^{\rho\mu}) \qquad (3.83)\\
&\quad +d_2\{-((p+k)^2+p^2)(k+2p)^\rho\eta^{\mu\nu}-2k^\mu(k^2\eta^{\nu\rho}-k^\nu k^\rho)\\
&\qquad -(k+p)^\mu(kp\eta^{\nu\rho}-k^\nu p^\rho)-((k+p)^\rho k^\mu-\eta^{\mu\rho}(k^2+pk))p^\nu\\
&\qquad -4(k^\mu\eta^{\nu\rho}-k^\nu\eta^{\mu\rho})p^2-2p^\nu(kp\eta^{\mu\rho}-k^\mu p^\rho)\\
&\qquad -2\eta^{\mu\nu}(p^\rho k^2-k^\rho kp)+2(\eta^{\nu\rho}k^2-k^\nu k^\rho)p^\mu\}\Big]e^{-ikx-ip(x-y)}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_2^{\mu\nu}(x-y) &= g^2\Big[A^2\eta^{\mu\nu}+d_2\eta^{\mu\nu}[(\partial\cdot A+A\cdot\partial)^2+(\partial^2A^2+A^2\partial^2)] \qquad (3.84)\\
&\quad -2d_2[(\partial^\mu A^\rho)-(\partial^\rho A^\mu)][(\partial^\nu A_\rho)-(\partial_\rho A^\nu)]\\
&\quad +d_2[A^\mu[(\partial^\rho A^\nu)-(\partial^\nu A^\rho)]\partial_\rho+\partial^\mu[(\partial^\rho A^\nu)-(\partial^\nu A^\rho)]A_\rho]\\
&\quad +d_2[A_\rho[(\partial^\mu A^\rho)-(\partial^\rho A^\mu)]\partial^\nu+\partial_\rho[(\partial^\mu A^\rho)-(\partial^\rho A^\mu)]A^\nu]\\
&\quad +4d_2(\partial\cdot A+A\cdot\partial)[(\partial^\mu A^\nu)-(\partial^\nu A^\mu)]\\
&\quad +2d_2[(\partial^\rho A^\mu)-(\partial^\mu A^\rho)](\partial^\nu A_\rho+A^\nu\partial_\rho)\\
&\quad +2d_2g[(\partial^2A_\sigma)-(\partial^\rho\partial_\sigma A_\rho)](\eta^{\mu\nu}A^\sigma-\eta^{\nu\sigma}A^\mu)\\
&\quad +\text{traceless terms}\Big]\delta(x-y)
\end{aligned}
$$

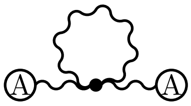
$$
\begin{aligned}
&= g^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int\frac{\mathrm{d}^4p}{(2\pi)^4}A_\rho(k)A_\sigma(q)\Big[\eta^{\rho\sigma}\eta^{\mu\nu} \qquad (3.85)\\
&\quad -d_2[\eta^{\mu\nu}(k+2q+2p)^\rho(q+2p)^\sigma+\eta^{\mu\nu}\eta^{\rho\sigma}((k+q+p)^2+p^2)]\\
&\quad +2d_2[k^\mu(q^\nu\eta^{\rho\sigma}-q^\rho\eta^{\nu\sigma})+\eta^{\mu\rho}(qk\eta^{\nu\sigma}-q^\nu k^\sigma)]\\
&\quad -d_2[\eta^{\mu\rho}(qp\eta^{\nu\sigma}-q^\nu p^\sigma)+(k+p+q)^\mu(k^\sigma\eta^{\nu\rho}-k^\nu\eta^{\rho\sigma})]\\
&\quad -d_2[(q^\mu\eta^{\rho\sigma}-q^\rho\eta^{\mu\sigma})p^\nu+((p+k+q)^\rho k^\mu-k(k{+}p{+}q)\eta^{\mu\rho})\eta^{\nu\sigma}]\\
&\quad -d_2[(k^\mu\eta^{\nu\rho}-k^\nu\eta^{\mu\rho})(q+2p)^\sigma+(k+2p+2q)^\rho(q^\mu\eta^{\nu\sigma}-q^\nu\eta^{\mu\sigma})]\\
&\quad -2d_2[\eta^{\mu\rho}(kp\eta^{\nu\sigma}+(q+p)^\nu k^\sigma)-k^\mu((q+p)^\nu\eta^{\rho\sigma}+p^\rho\eta^{\nu\sigma})]\\
&\quad -2d_2[(k^2\eta^{\rho\sigma}-k^\rho k^\sigma)\eta^{\mu\nu}-(k^2\eta^{\nu\rho}-k^\nu k^\rho)\eta^{\mu\sigma}]\Big]e^{-i(k+q)x-ip(x-y)}
\end{aligned}
$$

One contraction of the one and two background, two quantum fields vertices can be directly read off from (3.83) and (3.85) respectively. We don't give the explicit expressions as they appeared only as intermediate steps in our calculation and do not contain more information than (3.83) and (3.85) whereas being much longer.

Now we have everything together to determine the contribution of the determinant of Δ to the A^2 part of the effective action. It is determined by the following two diagrams:

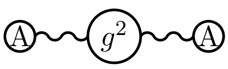
$\frac{1}{2}$ 

$$
\begin{aligned}
&= \frac{ig^2C_2}{16\pi^2}\delta^{ab}\Big[\eta^{\mu\nu}q^2\Big\{\frac{56}{3}\frac{2}{\epsilon}-\frac{46}{3}\ln\frac{d_2^{-1}}{\mu^2}\\
&\qquad +\frac{1}{d_2}\left(6+8\frac{2}{\epsilon}-8\left(\gamma+\ln\frac{d_2^{-1}}{\mu^2}\right)\right)\Big\}\\
&\quad +q^\mu q^\nu\left\{-\frac{44}{3}\frac{2}{\epsilon}+\frac{34}{3}\ln\frac{d_2^{-1}}{\mu^2}\right\}\Big] \qquad (3.86)
\end{aligned}
$$

$$\frac{1}{2}$$ 

$$= \frac{ig^2 C_2}{16\pi^2}\delta^{ab}\Bigg[\eta^{\mu\nu}q^2\Bigg\{-12\frac{2}{\epsilon}+12\ln\frac{d_2^{-1}}{\mu^2} - \frac{1}{d_2}\left(6+8\frac{2}{\epsilon}-8\left(\gamma+\ln\frac{d_2^{-1}}{\mu^2}\right)\right)\Bigg\} + q^\mu q^\nu\left\{8\frac{2}{\epsilon}-8\ln\frac{d_2^{-1}}{\mu^2}\right\}\Bigg] \tag{3.87}$$

Their sum is



$$= \frac{i}{16\pi^2}g^2 C_2\delta^{ab}\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\left(\frac{20}{3}\frac{2}{\epsilon}-\frac{10}{3}\ln\frac{d_2^{-1}}{\mu^2}\right). \tag{3.88}$$

All terms contradicting gauge invariance exactly cancel.

If we add up the contributions (3.61), (3.78) and (3.88) we get the same result as several times before (3.65), (3.66).

Part II | Gravitational Corrections

We investigate the one-loop counterterms of Yang-Mills theory minimally coupled to gravity, including fermionic and scalar matter, to find out wether or not the fermionic and the scalar higher-derivative counterterms coincide with the higher-derivative terms of the Lee-Wick Standard Model. Furthermore, we determine the gravitational corrections to the running of the φ^4 and Yukawa coupling.

Chapter 4
Lee-Wick Fields out of Gravity

Motivated by the works of Grinstein, O'Connel, and Wise [4] and by Ebert, Plefka and Rodigast [18], we investigate the one-loop counterterms of Einstein Yang-Mills theory. We include fermionic and scalar multiplets to find out wether or not the fermionic and the scalar higher-derivative counterterms coincide with the higher-derivative terms of the Lee-Wick Standard Model, as is the case in the gauge sector [18].

A similar analysis has been done by Wu and Zhong with positive outcome, but the results they present in [36] are questionable. They only calculate two-point functions which, as we will show explicitly, alone do not determine the higher-derivative counterterms.

4.1 General Relativity as an Effective Field Theory

Up to now, the true theory of quantum gravity is still unknown and the general wisdom is that general relativity and quantum mechanics are presently incompatible.

It is a well known fact that the quantization of general relativity leads to a non-renormalizable quantum field theory. Loop diagrams generate divergences which cannot be absorbed into a renormalization of the original Lagrangian. Instead, we have to add an increasing number of new terms to the Lagrangian in order to renormalize the theory at a given loop order.

Nevertheless, treated as an effective field theory, quantized general relativity should lead to the correct low energy quantum corrections because the correct degrees of freedom and the correct vertices at low energies are used. It has been established by Donoghue [37, 38, 39] that general relativity naturally fits into the framework of effective field theories and therefore calculations made with quantized general relativity can be used to determine genuine low-energy predictions of quantum gravity.

Since the gravitational interactions are proportional to the energy, they can easily be organized into an energy expansion. Furthermore, as has been shown in [40], higher order loop diagrams always generate higher orders in the energy expansion. Thus general relativity is well suited to organize all calculations in a systematic expansion in the energy, which forms the basis of an effective field theory treatment.

However, it is crucial that the quantum corrections obtained within the framework of effective field theory only have to coincide with the predictions of the correct theory of quantum gravity at energies well below the Planck mass $M_p \sim 10^{19}$GeV, where we expect the theory to break down. Hence, any extrapolation of the thus obtained results to energies comparable to the Planck mass or greater is speculative.

4.2 The Graviton

The general theory of relativity is described by the Einstein-Hilbert Lagrangian given by

$$\mathcal{L}_{\text{EH}} = \frac{2}{\kappa^2}\sqrt{-\boldsymbol{g}}\boldsymbol{R} \,, \tag{4.1}$$

where κ is the gravitational coupling related to Newtons constant by $\kappa^2 = 32\pi G$, $\boldsymbol{g}$ is the determinant of the metric $\boldsymbol{g}_{\mu\nu}$ and $\boldsymbol{R} = \boldsymbol{g}^{\mu\nu}R_{\mu\nu}$ is the scalar curvature defined by

$$\begin{aligned}\boldsymbol{R}_{\mu\nu} &= \partial_\mu\Gamma^\rho_{\rho\nu} - \partial_\rho\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu} - \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu}\\ \Gamma^\rho_{\mu\nu} &= \tfrac{1}{2}\boldsymbol{g}^{\rho\sigma}(\partial_\mu\boldsymbol{g}_{\nu\sigma} + \partial_\nu\boldsymbol{g}_{\mu\sigma} - \partial_\sigma\boldsymbol{g}_{\mu\nu}) \,.\end{aligned} \tag{4.2}$$

In order to quantize gravity, we expand the metric $\boldsymbol{g}_{\mu\nu}$ around the flat background $\eta_{\mu\nu}$

$$\boldsymbol{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}. \tag{4.3}$$

Starting from this expression we expand the Einstein-Hilbert Lagrangian in powers of the symmetric tensor field $h_{\mu\nu}$, the graviton. The inverse metric becomes

$$\boldsymbol{g}^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha}h^\nu_\alpha + \mathcal{O}(\kappa^3) \tag{4.4}$$

and the expansion of the measure is given by

$$\sqrt{-\boldsymbol{g}} = 1 + \tfrac{\kappa}{2}h + \tfrac{\kappa^2}{8}(h^2 - 2h^{\alpha\beta}h_{\alpha\beta}) + \mathcal{O}(\kappa^3) \,. \tag{4.5}$$

Here and in the following indices are raised and lowered with $\eta_{\mu\nu}$ and $h = h^\mu_\mu$. As an intermediate step we give the expansion of the affine connection

$$\Gamma^\rho_{\mu\nu} = \tfrac{\kappa}{2}(\partial_\mu h^\rho_\nu + \partial_\nu h^\rho_\mu - \partial^\rho h_{\mu\nu}) - \tfrac{\kappa^2}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \mathcal{O}(\kappa^3) \,. \tag{4.6}$$

which together with (4.4) can be plugged into the definition of the curvature scalar

$$\begin{aligned}\boldsymbol{R} = \kappa(\partial^2 h - \partial^\alpha\partial^\beta h_{\alpha\beta}) - \tfrac{\kappa^2}{2}\big[&h^{\alpha\beta}(\partial^2 h_{\alpha\beta} + \partial_\alpha\partial_\beta h - 2\partial_\rho\partial_\alpha h^\rho_\beta)\\ &+ \partial_\alpha h\partial_\beta h^{\alpha\beta} - (\partial_\alpha h)^2 + \tfrac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta}\\ &- \partial_\alpha h_{\mu\beta}\partial^\beta h^{\mu\alpha} + \text{t.d.}\big] + \mathcal{O}(\kappa^3) \,.\end{aligned} \tag{4.7}$$

To lowest order in κ, the Einstein Hilbert action is given by:

$$\mathcal{L}_{\text{EH}} = \partial_\alpha h\partial_\beta h^{\alpha\beta} - \tfrac{1}{2}(\partial_\mu h)^2 + \tfrac{1}{2}(\partial_\mu h_{\alpha\beta})^2 - \partial_\alpha h_{\mu\beta}\partial^\beta h^{\mu\alpha} + \text{t.d.} + \mathcal{O}(\kappa) \,. \tag{4.8}$$

It is not necessary to expand the action any further because we are only interested in one-loop diagrams with no external gravitons. General coordinate invariance implies that the action is invariant under

$$\delta h_{\mu\nu} = 2h_{\sigma(\mu}\partial_{\nu)}\xi^\sigma + \xi^\sigma\partial_\sigma h_{\mu\nu} + \tfrac{2}{\kappa}\partial_{(\mu}\xi_{\nu)} \,. \tag{4.9}$$

To fix this freedom we proceed exactly as in Section 2.3 and use the Faddeev-Popov trick [27], in order to get a gauge fixed Lagrangian. We take the harmonic (de Donder) gauge fixing condition

$$G_\mu = \partial^\nu h_{\mu\nu} - \tfrac{1}{2}\partial_\mu h \tag{4.10}$$

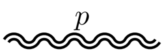
with the gauge fixing term

$$\mathcal{L}_{\text{gf}} = (\partial^\nu h_{\mu\nu} - \tfrac{1}{2}\partial_\mu h)^2 = \tfrac{1}{4}(\partial_\mu h)^2 + (\partial^\nu h_{\mu\nu})^2 - \partial_\alpha h \partial_\beta h^{\alpha\beta} \tag{4.11}$$

This is a convenient choice because the quadratic part of the Einstein Hilbert action simplifies to

$$\mathcal{L}_{\text{EH}}|_{h^2} + \mathcal{L}_{\text{gf}} = \tfrac{1}{2} h_{\alpha\beta}(\tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta} - \eta^{\alpha(\gamma}\eta^{\delta)\beta})\partial^2 h_{\gamma\delta} \; . \tag{4.12}$$

This leads to the graviton propagator in d dimensions

$\alpha\ \beta$ 

$$\gamma\ \delta = i\big(\eta_{\alpha(\gamma}\eta_{\delta)\beta} - \tfrac{1}{d-2}\eta_{\alpha\beta}\eta_{\gamma\delta}\big)\frac{1}{p^2} \; . \tag{4.13}$$

Note that with the gauge fixing condition we have chosen the operator $\frac{\delta G_\mu}{\delta \xi^\nu}$ depends on the graviton.

$$\frac{\delta G_\mu}{\delta \xi^\nu} = \partial^\rho h_{\mu\nu}\partial_\rho + \partial^\rho h_{\nu\rho}\partial_\mu + \partial^\rho(\partial_\nu h_{\mu\rho}) - \partial_\mu h_{\nu\rho}\partial^\rho - \tfrac{1}{2}\partial_\mu(\partial_\nu h) + \tfrac{1}{\kappa}\eta_{\mu\nu}\partial^2 \tag{4.14}$$

Hence, we have to include its determinant, which can be written in terms of a local Lagrangian by introducing gravitational ghosts

$$\mathcal{L}_{gh} = -\bar{b}^\mu \left(\kappa \frac{\delta G_\mu}{\delta \xi^\nu}\right) b^\nu \; . \tag{4.15}$$

However, since we are only interested in one-loop diagrams with no external gravitons, we do not need the graviton ghosts in our calculation.

4.3 Renormalization of the YM Coupling Constant

Before starting to calculate loop integrals it is helpful to have a closer look at the renormalization factors and to exploit the relations between them. From Slavnov-Taylor identities, we know that the universality of the gauge coupling is preserved by renormalization, which as in Section 2.4 implies

$$Z_g = \frac{g_0}{g} = Z_{A^3} Z_A^{-\frac{3}{2}} = Z_{A^4}^{\frac{1}{2}} Z_A^{-1} = Z_{\bar{c}Ac} Z_c^{-1} Z_A^{-\frac{1}{2}} \tag{4.16}$$

or equivalently

$$\frac{Z_{A^4}}{Z_{A^3}} = \frac{Z_{A^3}}{Z_A} = \frac{Z_{\bar{c}Ac}}{Z_c} \; . \tag{4.17}$$

Since the graviton does not couple to the ghosts of the gauge fields, it follows immediately that

$$Z_{\bar{c}Ac}\big|_{\mathcal{O}(\kappa^2)} = Z_c\big|_{\mathcal{O}(\kappa^2)} = 0 \tag{4.18}$$

and hence

$$\frac{Z_{A^4}}{Z_{A^3}}\bigg|_{\mathcal{O}(\kappa^2)} = \frac{Z_{A^3}}{Z_A}\bigg|_{\mathcal{O}(\kappa^2)} = 0 \; . \tag{4.19}$$

At one-loop level this means

$$Z_A\big|_{\mathcal{O}(\kappa^2)} = Z_{A^3}\big|_{\mathcal{O}(\kappa^2)} = Z_{A^4}\big|_{\mathcal{O}(\kappa^2)}, \tag{4.20}$$

and therefore

$$Z_g\big|_{\mathcal{O}(\kappa^2)} = -\tfrac{1}{2} Z_A\big|_{\mathcal{O}(\kappa^2)} \; . \tag{4.21}$$

This simple observation leads to an interesting fact: if there is a gravitational contribution to the running of the gauge coupling, then it is solely determined by the wavefunction renormalization Z_A of the gauge field.

In the case of fermions ψ and scalars ϕ, we know that

$$\frac{g_0}{g} = Z_g = Z_A^{-\frac{1}{2}} Z_{\overline{\psi}A\psi} Z_\psi^{-1} = Z_A^{-\frac{1}{2}} Z_{\phi^\dagger A\phi} Z_\phi^{-1} = Z_A^{-\frac{1}{2}} Z_{\phi^\dagger A^2\phi}^{\frac{1}{2}} Z_\phi^{-\frac{1}{2}} \tag{4.22}$$

holds, which together with (4.20) results in

$$\begin{aligned} \left.\frac{Z_{\overline{\psi}A\psi}}{Z_\psi}\right|_{\mathcal{O}(\kappa^2)} &= 0 \\ \left.\frac{Z_{\phi^\dagger A\phi}}{Z_\phi}\right|_{\mathcal{O}(\kappa^2)} &= \left.\frac{Z_{\phi^\dagger A^2\phi}}{Z_{\phi^\dagger A\phi}}\right|_{\mathcal{O}(\kappa^2)} = 0 \; . \end{aligned} \tag{4.23}$$

At one-loop level this further simplifies to

$$\begin{aligned} Z_\psi\big|_{\mathcal{O}(\kappa^2)} &= Z_{\overline{\psi}A\psi}\big|_{\mathcal{O}(\kappa^2)} \\ Z_\phi\big|_{\mathcal{O}(\kappa^2)} &= Z_{\phi^\dagger A\phi}\big|_{\mathcal{O}(\kappa^2)} = Z_{\phi^\dagger A^2\phi}\big|_{\mathcal{O}(\kappa^2)} \; . \end{aligned} \tag{4.24}$$

All these relations between the various Z factors provide nontrivial tests for the following calculations and hence will be very useful.

4.4 Gauge Fields

Before treating fermions and scalars we are going to investigate the one-loop divergences in the gauge sector. This has already been done in [18] in order to determine the gravitational contribution to the running of the gauge coupling. Indeed, the result obtained in [18] was the starting point of this thesis and hence, in order to substantiate the motivation for the investigation of Lee-Wick gauge theory in Chapter 2, we will show explicitly how the Lee-Wick term arises as a one-loop counterterm of Einstein Yang-Mills theory.

It is straightforward to couple gauge fields to gravity. The evident generalization of the flat space Yang-Mills Lagrangian is given by

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{2}\sqrt{-\boldsymbol{g}}\,\boldsymbol{g}^{\mu\rho}\boldsymbol{g}^{\nu\sigma}\,\mathrm{tr}\{F_{\mu\nu}F_{\rho\sigma}\} \; . \tag{4.25}$$

The theory to be investigated in this section, as well as in Sections 4.5 and 4.6, is the non-renormalizable Einstein Yang-Mills theory including fermionic and scalar multiplets, described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_f + \mathcal{L}_s \; . \tag{4.26}$$

Similar to the case of the Einstein-Hilbert action we expand the Yang-Mills Lagrangian in orders of κ and obtain

$$\begin{aligned}\mathcal{L}_{\rm YM} = &-\tfrac{1}{2}\,\mathrm{tr}\{F_{\mu\nu}^2\} + \tfrac{\kappa}{2}\big[\eta^{\mu\rho}h^{\nu\sigma} + \eta^{\nu\sigma}h^{\mu\rho} - \tfrac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma}h\big]\,\mathrm{tr}\{F_{\mu\nu}F_{\rho\sigma}\}\\ &+ \tfrac{\kappa^2}{2}\big[-\tfrac{1}{8}(h^2 - 2h^{\alpha\beta}h_{\alpha\beta})\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\rho}h^{\nu\alpha}h^{\sigma}_{\alpha} - \eta^{\nu\sigma}h^{\mu\alpha}h^{\rho}_{\alpha}\\ &\quad + \tfrac{1}{2}h(\eta^{\mu\rho}h^{\nu\sigma} + \eta^{\nu\sigma}h^{\mu\rho}) - h^{\mu\rho}h^{\nu\sigma}\big]\,\mathrm{tr}\{F_{\mu\nu}F_{\rho\sigma}\} + \mathcal{O}(\kappa^3)\ .\end{aligned} \tag{4.27}$$

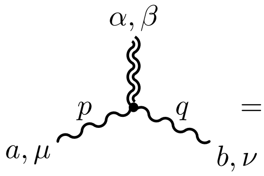
Now we fix the gauge freedom as usual by adding to the Lagrangian a gauge fixing term and the corresponding ghost Lagrangian.

$$\mathcal{L}_{\rm YM} \to \mathcal{L}_{\rm YM} - \tfrac{1}{\alpha}\,\mathrm{tr}\{(\partial^\mu A_\mu)^2\} - 2\,\mathrm{tr}\{\bar{c}\,\partial\!\cdot\! Dc\} \tag{4.28}$$

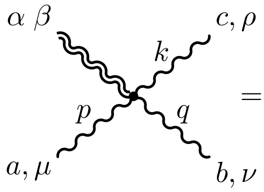
This leads to the well known gauge field propagator and Feynman rules including only gauge fields and their ghosts, which are listed in Appendix A. An arbitrary number of gravitons can couple to each of these vertices. However, for a one-loop calculation we only need vertices with less than three gravitons. To get the Feynman rules we proceed exactly as in Section 2.5. For some intermediate steps have a look at [19].

Note that in order to fix the gauge freedom in the Lagrangian (4.26) we have to impose five independent conditions. However, physical quantities do not dependent on them and we can fix this freedom in a convenient way for our calculations. With the gauge fixing conditions we have chosen (4.10) and (4.28), Lorentz invariance is manifest. Additionally, there is no coupling between gravitons and the ghosts of the gauge fields (4.28) and the vector ghost fields (4.15) coming from the harmonic gauge fixing condition are irrelevant in our calculation, which is a great simplification.

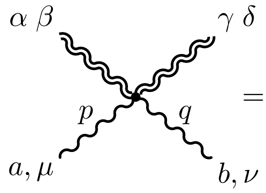
Some of the Feynman rules we need are quite long and therefore we will write them down in such a way that at least the symmetries of the vertices are evident. To do so, we define $P^{\mu\nu\,\alpha\beta} := \eta^{\mu(\alpha}\eta^{\beta)\nu} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}$. Note that $P^{\mu\nu\,\alpha\beta}$ is symmetric in each pair of indices and under exchange of the pairs.



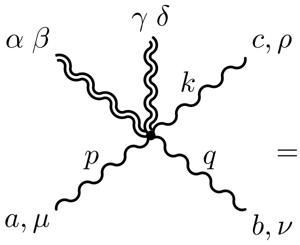
$$= -i\kappa\delta^{ab}\big[P^{\mu\nu\,\alpha\beta}qp + \eta^{\mu\nu}q^{(\alpha}p^{\beta)} - \eta^{\mu(\alpha}q^{\beta)}p^{\nu} - \eta^{\nu(\alpha}p^{\beta)}q^{\mu} + \tfrac{1}{2}\eta^{\alpha\beta}q^{\mu}p^{\nu}\big] \tag{4.29}$$



$$\begin{aligned}= \kappa g f^{abc}\big[\ & P^{\alpha\beta\,\mu\nu}(q-p)^{\rho}\\ &+ P^{\alpha\beta\,\mu\rho}(p-k)^{\nu}\\ &+ P^{\alpha\beta\,\nu\rho}(k-q)^{\mu}\\ &+ \eta^{\mu\nu}\eta^{\rho(\alpha}(q-p)^{\beta)}\\ &+ \eta^{\mu\rho}\eta^{\nu(\alpha}(p-k)^{\beta)}\\ &+ \eta^{\nu\rho}\eta^{\mu(\alpha}(k-q)^{\beta)}\big]\end{aligned} \tag{4.30}$$



$$= i\kappa^2\delta^{ab}\big[\tfrac{1}{2}(p^\nu q^\mu - \eta^{\mu\nu}pq)P^{\alpha\beta\,\gamma\delta} \tag{4.31}$$
$$+ qk\big(\eta^{\mu(\alpha}\eta^{\beta)(\gamma}\eta^{\delta)\nu} + \eta^{\mu(\gamma}\eta^{\delta)(\alpha}\eta^{\beta)\nu}$$
$$- \tfrac{1}{2}\eta^{\mu(\alpha}\eta^{\beta)\nu}\eta^{\gamma\delta} - \tfrac{1}{2}\eta^{\mu(\gamma}\eta^{\delta)\nu}\eta^{\alpha\beta}\big)$$
$$+ \eta^{\mu\nu}p^{(\alpha}\eta^{\beta)(\gamma}q^{\delta)} + \eta^{\mu\nu}p^{(\gamma}\eta^{\delta)(\alpha}q^{\beta)}$$
$$+ \eta^{\mu(\alpha}\eta^{\beta)\nu}p^{(\gamma}q^{\delta)} + \eta^{\mu(\gamma}\eta^{\delta)\nu}p^{(\alpha}q^{\beta)}$$
$$+ \eta^{\mu(\alpha}q^{\beta)}\eta^{\nu(\gamma}p^{\delta)} + \eta^{\mu(\gamma}q^{\delta)}\eta^{\nu(\alpha}p^{\beta)}$$
$$+ \eta^{\mu(\alpha}\eta^{\beta)(\gamma}q^{\delta)}p^\nu + \eta^{\mu(\gamma}\eta^{\delta)(\alpha}q^{\beta)}p^\nu$$
$$+ \eta^{\nu(\alpha}\eta^{\beta)(\gamma}p^{\delta)}q^\mu + \eta^{\nu(\gamma}\eta^{\delta)(\alpha}p^{\beta)}q^\mu$$
$$- \tfrac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}p^{(\gamma}q^{\delta)} - \tfrac{1}{2}\eta^{\mu\nu}\eta^{\gamma\delta}p^{(\alpha}q^{\beta)}$$
$$- \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\mu(\gamma}q^{\delta)}p^\nu - \tfrac{1}{2}\eta^{\gamma\delta}\eta^{\mu(\alpha}q^{\beta)}p^\nu$$
$$- \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\nu(\gamma}p^{\delta)}q^\mu - \tfrac{1}{2}\eta^{\gamma\delta}\eta^{\nu(\alpha}p^{\beta)}q^\mu\big]$$



$$= \kappa^2 g f^{abc}\Big[\quad (k-q)^\mu\big(\tfrac{1}{2}\eta^{\nu\rho}P^{\alpha\beta\,\gamma\delta} + \tfrac{1}{2}\eta^{\nu(\alpha}\eta^{\beta)\rho}\eta^{\gamma\delta} \tag{4.32}$$
$$+ \tfrac{1}{2}\eta^{\nu(\gamma}\eta^{\delta)\rho}\eta^{\alpha\beta} - \eta^{\nu(\alpha}\eta^{\beta)(\gamma}\eta^{\delta)\rho}$$
$$- \eta^{\nu(\gamma}\eta^{\delta)(\alpha}\eta^{\beta)\rho}\big)$$
$$+ (p-k)^\nu\big(\tfrac{1}{2}\eta^{\mu\rho}P^{\alpha\beta\,\gamma\delta} + \tfrac{1}{2}\eta^{\mu(\alpha}\eta^{\beta)\rho}\eta^{\gamma\delta}$$
$$+ \tfrac{1}{2}\eta^{\mu(\gamma}\eta^{\delta)\rho}\eta^{\alpha\beta} - \eta^{\mu(\alpha}\eta^{\beta)(\gamma}\eta^{\delta)\rho}$$
$$- \eta^{\mu(\gamma}\eta^{\delta)(\alpha}\eta^{\beta)\rho}\big)$$
$$+ (q-p)^\rho\big(\tfrac{1}{2}\eta^{\mu\nu}P^{\alpha\beta\,\gamma\delta} + \tfrac{1}{2}\eta^{\mu(\alpha}\eta^{\beta)\nu}\eta^{\gamma\delta}$$
$$+ \tfrac{1}{2}\eta^{\mu(\gamma}\eta^{\delta)\nu}\eta^{\alpha\beta} - \eta^{\mu(\alpha}\eta^{\beta)(\gamma}\eta^{\delta)\nu}$$
$$- \eta^{\mu(\gamma}\eta^{\delta)(\alpha}\eta^{\beta)\nu}\big)$$
$$+ (q-p)^{(\alpha}\big(\tfrac{1}{2}\eta^{\beta)\rho}\eta^{\mu\nu}\eta^{\gamma\delta} - \eta^{\beta)(\gamma}\eta^{\delta)\rho}\eta^{\mu\nu}$$
$$- \eta^{\beta)\rho}\eta^{\mu(\gamma}\eta^{\delta)\nu}\big)$$
$$+ (q-p)^{(\gamma}\big(\tfrac{1}{2}\eta^{\delta)\rho}\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\delta)(\alpha}\eta^{\beta)\rho}\eta^{\mu\nu}$$
$$- \eta^{\delta)\rho}\eta^{\mu(\alpha}\eta^{\beta)\nu}\big)$$
$$+ (p-k)^{(\alpha}\big(\tfrac{1}{2}\eta^{\beta)\nu}\eta\gamma\delta\eta^{\mu\rho} - \eta^{\beta)(\gamma}\eta^{\delta)\nu}\eta^{\mu\rho}$$
$$- \eta^{\beta)\nu}\eta^{\mu(\gamma}\eta^{\delta)\rho}\big)$$
$$+ (p-k)^{(\gamma}\big(\tfrac{1}{2}\eta^{\delta)\nu}\eta^{\alpha\beta}\eta^{\mu\rho} - \eta^{\delta)(\alpha}\eta^{\beta)\nu}\eta^{\mu\rho}$$
$$- \eta^{\delta)\nu}\eta^{\mu(\alpha}\eta^{\beta)\rho}\big)\Big]$$

Note that the two gauge field one graviton vertex (4.29) is transverse with respect to p_μ and q_ν. Hence, without calculating we already know that the divergent parts of the one-loop diagrams listed in figures 4.1 and 4.2 are independent of the arbitrary parameter α in the gauge fixing term (4.28) because only the

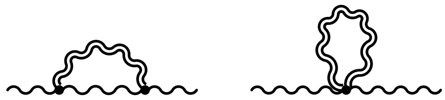


Figure 4.1: Order κ^2 one-loop diagrams for the gauge boson 2-point function.

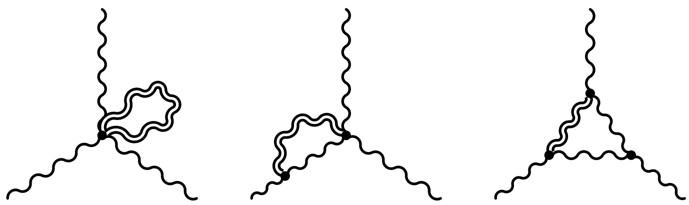
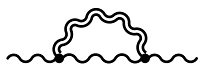


Figure 4.2: Order $g\kappa^2$ one-loop diagrams for the proper gauge field 3-point function not only differing by a permutation of outer legs.

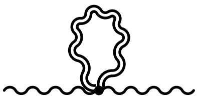
longitudinal part ($\alpha = 1$) of the propagator survives.

All diagrams were computed using both cut-off and dimensional regularization and agreement was found.

Let us begin with the order κ^2 gravitational contribution to the proper gauge field two-point function:

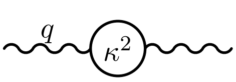


$$= \frac{\kappa^2}{16\pi^2}\delta^{ab}\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\left[\tfrac{3}{2}\eta^{\mu\nu}\left(\Lambda^2-\mu^2\right) - \tfrac{q^2}{6}\ln\tfrac{\Lambda^2}{\mu^2}\right] \tag{4.33}$$

$\frac{1}{2}$ 

$$= \frac{\kappa^2}{16\pi^2}\delta^{ab}\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\left[-\tfrac{3}{2}\eta^{\mu\nu}\left(\Lambda^2-\mu^2\right)\right]. \tag{4.34}$$

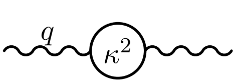
In contrast to 2.68, we are now faced with quadratic divergences, which do not contradict gauge invariance, but astonishingly, they exactly cancel each other. Hence, as a consequence of (4.20), we find that there is no gravitational contribution to the running of the gauge coupling. This contradicts the result of Robinson and Wilczec [20]. However, it has been shown by Pietrykowski in [21], that the result of Robinson and Wilczec is gauge dependent and therefore cannot be correct. Furthermore, the calculation of Toms [22], using the Vilkovisky-DeWitt effective action, agrees with our result.

a, μ 

b, ν

$$= \frac{\kappa^2}{16\pi^2}\delta^{ab}q^2\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\left(-\frac{1}{6}\ln\frac{\Lambda^2}{\mu^2}\right) \tag{4.35}$$

or in dimensional regularization

a, μ 

b, ν

$$= \frac{\kappa^2}{16\pi^2}\delta^{ab}q^2\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\left(-\frac{1}{6}\frac{2}{\epsilon}\right). \tag{4.36}$$

The result is proportional to the Lee-Wick term $\mathrm{tr}\{(D_\mu F^{\mu\nu})^2\}$ equation (2.42) in Lee-Wick gauge theory. However, at this point it is not clear that this will be the counterterm. The possible gauge and Lorentz invariant candidates for the dimension six counterterm are

$$\begin{aligned}
&\mathrm{tr}\{D_\mu F_{\nu\rho} D^\mu F^{\nu\rho}\} = 2\,\mathrm{tr}\{D_\mu F_{\nu\rho} D^\nu F^{\mu\rho}\} \\
&\mathrm{tr}\{D_\mu F^{\mu\nu} D^\rho F_{\rho\nu}\} \qquad\qquad\qquad\qquad . \\
&ig\,\mathrm{tr}\{F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\mu\}
\end{aligned} \tag{4.37}$$

These three terms are not independent of one another:

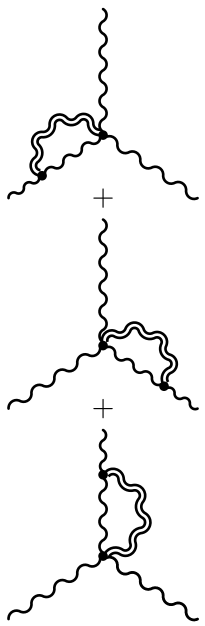
$$\begin{aligned}
\mathrm{tr}\{D_\mu F^{\mu\nu} D^\rho F_{\rho\nu}\} &= -\,\mathrm{tr}\{F^{\mu\nu}[D_\mu,[D^\rho,F_{\rho\nu}]\} + \text{t.d.} \\
&= -\,\mathrm{tr}\{F^{\mu\nu}([D_\rho,[D^\mu,F_{\rho\nu}] + [F_{\rho\nu},[D^\rho,D_\mu]])\} + \text{t.d.} \\
&= \mathrm{tr}\{D_\rho F_{\mu\nu} D^\mu F^{\rho\nu}\} + ig\,\mathrm{tr}\{F^{\mu\nu}[F_{\rho\nu},F^\rho{}_\mu]\} + \text{t.d.} \\
&= \tfrac{1}{2}\,\mathrm{tr}\{D_\mu F_{\nu\rho} D^\mu F^{\nu\rho}\} - 2ig\,\mathrm{tr}\{F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\mu\} + \text{t.d.} \quad ,
\end{aligned} \tag{4.38}$$

which implies that the counterterm has to be a linear combination of the form

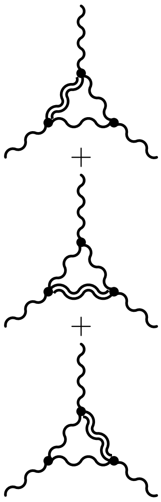
$$d_1\, ig\,\mathrm{tr}\{F^\mu{}_\nu F^\nu{}_\rho F^\rho{}_\mu\} + d_2\ \mathrm{tr}\{D_\mu F^{\mu\nu} D^\rho F_{\rho\nu}\}\ , \tag{4.39}$$

where d_1 and d_2 have mass dimension minus two. From the result for the two-point function (4.35) we already know $d_2 = \frac{1}{16\pi^2}\frac{\kappa^2}{6}\ln\frac{\Lambda^2}{\mu^2}$.

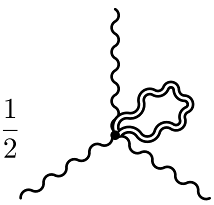
To determine the value of d_1, we have to calculate the gravitational one-loop contribution to the proper three gauge field vertex. As a consequence of (4.20), the quadratic divergences of the vertex correction have to cancel as well in order to be consistent with the two-point function result (4.35).



$$\begin{aligned}
= \frac{g\kappa^2}{16\pi^2} f^{abc}\Bigg[&-\tfrac{3}{2}\left(\Lambda^2-\mu^2\right)\big[\ \ \eta^{\mu\nu}(p-q)^\rho + \eta^{\mu\rho}(k-p)^\nu \\
&\qquad\qquad\qquad\quad + \eta^{\nu\rho}(q-k)^\mu\big] \\
&+ \ln\tfrac{\Lambda^2}{\mu^2}\Big[\ \ p^\nu p^\rho (q-k)^\mu + q^\mu q^\rho (k-p)^\nu \\
&\quad + k^\mu k^\nu (p-q)^\rho \\
&\quad - \tfrac{3}{4} p^\nu q^\rho k^\mu + \tfrac{3}{4} p^\rho q^\mu k^\nu \\
&\quad + \eta^{\mu\nu}\{\ \ q^\rho(\tfrac{7}{6}pq + \tfrac{1}{6}qk + \tfrac{3}{4}pk) \\
&\qquad\qquad - p^\rho(\tfrac{7}{6}pq + \tfrac{1}{6}pk + \tfrac{3}{4}qk)\} \\
&\quad + \eta^{\mu\rho}\{\ \ p^\nu(\tfrac{7}{6}pk + \tfrac{1}{6}pq + \tfrac{3}{4}qk) \\
&\qquad\qquad - k^\nu(\tfrac{7}{6}pk + \tfrac{1}{6}qk + \tfrac{3}{4}pq)\} \\
&\quad + \eta^{\nu\rho}\{\ \ k^\mu(\tfrac{7}{6}qk + \tfrac{1}{6}pk + \tfrac{3}{4}pq) \\
&\qquad\qquad - q^\mu(\tfrac{7}{6}qk + \tfrac{1}{6}pq + \tfrac{3}{4}pk)\}\Big]\Bigg]
\end{aligned} \tag{4.40}$$

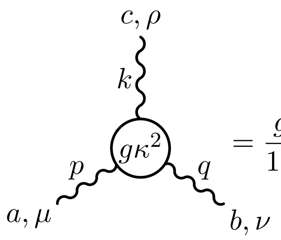


$$= \frac{g\kappa^2}{16\pi^2} f^{abc} \ln\tfrac{\Lambda^2}{\mu^2}\Big[-\tfrac{5}{6}p^\nu p^\rho (q-k)^\mu - \tfrac{5}{6}q^\mu q^\rho (k-p)^\nu - \tfrac{5}{6}k^\mu k^\nu (p-q)^\rho + \tfrac{1}{4}p^\nu q^\rho k^\mu + \tfrac{1}{4}p^\rho q^\mu k^\nu + \eta^{\mu\nu}\big(p^\rho(\tfrac{5}{6}pq+\tfrac{1}{4}qk) - q^\rho(\tfrac{5}{6}pq+\tfrac{1}{4}pk)\big) + \eta^{\mu\rho}\big(k^\nu(\tfrac{5}{6}pk+\tfrac{1}{4}pq) - p^\nu(\tfrac{5}{6}pk+\tfrac{1}{4}qk)\big) + \eta^{\nu\rho}\big(q^\mu(\tfrac{5}{6}qk+\tfrac{1}{4}pk) - k^\mu(\tfrac{5}{6}qk+\tfrac{1}{4}pq)\big)\Big] \tag{4.41}$$

$\frac{1}{2}$ 

$$= \frac{g\kappa^2}{16\pi^2} f^{abc}\Big[\tfrac{3}{2}\left(\Lambda^2-\mu^2\right)\big[\ \eta^{\mu\nu}(p-q)^\rho + \eta^{\mu\rho}(k-p)^\nu + \eta^{\nu\rho}(q-k)^\mu\big]\Big] \tag{4.42}$$

Summation of all the diagrams yields



$$= \frac{g\kappa^2}{16\pi^2} f^{abc}\big[\ \eta^{\mu\nu}\{p^\rho(2pq+pk+3qk) - q^\rho(2pq+qk+3pk)\} + \eta^{\mu\rho}\{k^\nu(2pk+kq+3pq) - p^\nu(2pk+pq+3qk)\} + \eta^{\nu\rho}\{q^\mu(2kq+pq+3pk) - k^\mu(2kq+pk+3pq)\} - 2(k^\mu k^\nu (p-q)^\rho + p^\nu p^\rho (q-k)^\mu + q^\rho q^\mu (k-p)^\nu) - 3(p^\rho q^\mu k^\nu - p^\nu q^\rho k^\mu)\big]\left(-\tfrac{1}{6}\ln\tfrac{\Lambda^2}{\mu^2}\right)\ . \tag{4.43}$$

This is proportional to the higher-derivative part of the three gauge field vertex (2.43) of Lee-Wick gauge theory and the factor of $-\frac{1}{6}\ln\frac{\Lambda^2}{\mu^2}$ is consistent with the result for the two-point function (4.35). Consequently, the one-loop higher-derivative counterterm is exactly given by

$$d_2\,\mathrm{tr}\{(D_\mu F^{\mu\nu})^2\} \qquad \text{with} \qquad d_2 = \frac{1}{16\pi^2}\frac{\kappa^2}{6}\ln\tfrac{\Lambda^2}{\mu^2}\ , \tag{4.44}$$

which agrees with the results of [19] and [41, 42].

It is quite interesting that we only get the Lee-Wick term which as we have seen in Section 2.2 corresponds to a massive vector field, and not the linear combination (4.39), which does not allow for a two particle formulation in which all operators are of dimension four or less. This observation was the main motivation for our investigation of Lee-Wick gauge theory in Part I.

4.5 Fermions

While the coupling of gravity to scalars and gauge bosons is straightforward, the coupling to fermions leads to an immediate difficulty: *there are no finite dimensional spinor representations of GL(d)*. To circumvent this problem, one has to make use of the fact that there are spinor representations of the Lorentz group. First it is necessary to introduce a local frame in the tangent space of the manifold. The vector fields

$$e_a = e_a{}^{\mu} \frac{\partial}{\partial x^{\mu}} \tag{4.45}$$

are called vielbeins. If the manifold has Lorentz signature, the frame can be chosen such that

$$\boldsymbol{g}(e_a, e_b) = e_a{}^{\mu} e_b{}^{\nu} \boldsymbol{g}_{\mu\nu} = \eta_{ab} \ . \tag{4.46}$$

The inverse vielbeins $\theta^a = e^a{}_{\mu} \mathrm{d}x^{\mu}$ are the dual one-forms of the vielbeins:

$$\theta^a(e_b) = \delta^a_b \ . \tag{4.47}$$

Their components are given by

$$e^a{}_{\mu} = \boldsymbol{g}_{\mu\nu} \eta^{ab} e_b{}^{\nu}, \tag{4.48}$$

and are the matrix inverses of the vielbeins. This can be used to express the metric in terms of the inverse vielbeins

$$\begin{aligned} \boldsymbol{g} = \boldsymbol{g}_{\mu\nu} \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} &= \eta_{ab}\, \theta^a \otimes \theta^b \\ \text{or} \qquad \boldsymbol{g}_{\mu\nu} &= \eta_{ab} e^a{}_{\mu} e^b{}_{\nu} \ . \end{aligned} \tag{4.49}$$

The metric is oviously invariant under $O(1, d-1)$ rotations of the vielbeins. Let $\Lambda^a{}_b(x)$ be such a local Lorentz transformation. Then the metric is invariant under the transformations

$$\theta^a \to \Lambda^a{}_b \theta^b \qquad \Longleftrightarrow \qquad e_a \to (\Lambda^{-1})^b{}_a e_b \ . \tag{4.50}$$

A local Lorentz transformation can be written as

$$\Lambda = \exp(-\tfrac{i}{2} \lambda_{ab} \mathcal{J}^{ab}) = e^{\lambda} \ , \tag{4.51}$$

where the matrices

$$(\mathcal{J}^{ab})_{cd} = i(\delta^a_c \delta^b_d - \delta^a_d \delta^b_c) \tag{4.52}$$

fulfill the Lorentz algebra

$$[L^{ab}, L^{cd}] = i\big(\eta^{bc} L^{ad} - \eta^{ac} L^{bd} - \eta^{bd} L^{ac} + \eta^{ad} L^{bc}\big) \tag{4.53}$$

and where λ satisfies

$$\lambda^T = -\eta \lambda \eta^{-1} \qquad \Leftrightarrow \qquad \lambda_{ab} = -\lambda_{ba} \ . \tag{4.54}$$

The fermions have to transform under the spinor representation ρ of this local Lorentz symmetry, which is given by

$$\rho(\Lambda) = \exp(-\tfrac{i}{2}\lambda_{ab}S^{ab}) \ . \tag{4.55}$$

Here S_{ab} is the spinor representation of the Lorentz algebra (4.53)

$$S_{ab} = \tfrac{i}{4}[\gamma_a, \gamma_b] = \tfrac{i}{2}\gamma_{ab} \tag{4.56}$$

and the spinors transform as

$$\psi \to \rho(\Lambda)\psi \qquad\qquad \overline{\psi} \to \overline{\psi}\rho(\Lambda)^{-1} \ . \tag{4.57}$$

The space time gamma matrices can be defined by

$$\boldsymbol{\gamma}^\mu = \gamma^a {e_a}^\mu \tag{4.58}$$

and fulfill the Clifford algebra

$$\{\boldsymbol{\gamma}^\mu, \boldsymbol{\gamma}^\nu\} = {e_a}^\mu {e_b}^\nu \{\gamma^a, \gamma^b\} = 2\boldsymbol{g}^{\mu\nu} \tag{4.59}$$

To write down a Lagrangian, we have to define the covariant derivative of a spinor such that the vector $\boldsymbol{\mathcal{D}}_\mu\psi$ transforms as a spinor:

$$\boldsymbol{\mathcal{D}}_\mu\psi \to (\boldsymbol{\mathcal{D}}_\mu\psi)' = \rho(\Lambda)\boldsymbol{\mathcal{D}}_\mu\psi \ . \tag{4.60}$$

In order to construct this covariant derivative, we need the components of the Levi-Civita connection with respect to the vielbeins. They are determined by the equation

$$\nabla_\mu e_a = \omega^b_{\mu a} e_b \ . \tag{4.61}$$

From the orthogonality of the vielbeins it follows directly that

$$\omega_{ab} = \boldsymbol{g}(e_a, \nabla e_b) \ . \tag{4.62}$$

Because of the metric compatibility of the covariant derivative, ω_{ab} is antisymmetric. Plugging $e_a = {e_a}^\mu \partial_\mu$ into equation (4.61) one can obtain a relation to the coordinate components of the connection:

$$\nabla_\mu {e_a}^\nu = \partial_\mu {e_a}^\nu + \Gamma^\nu_{\mu\rho}{e_a}^\rho = \omega^b_{\mu a}{e_b}^\nu \ . \tag{4.63}$$

We therefore arrive at

$$\omega^{ab}_\mu = {e^a}_\nu \nabla_\mu e^{b\nu} = {e^a}_\nu \partial_\mu e^{b\nu} + {e^a}_\nu e^{b\sigma}\Gamma^\nu_{\mu\sigma} \ , \tag{4.64}$$

or equivalently by inserting the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ from equation (4.2) and making use of ${e^a}_\nu \partial_\mu e^{b\nu} = -{e^b}_\nu \partial_\mu e^{a\nu} - e^{a\rho}e^{b\nu}\partial_\mu \boldsymbol{g}_{\nu\rho}$

$$\omega^{ab}_\mu = \tfrac{1}{2}{e^a}_\nu \partial_\mu e^{b\nu} + \tfrac{1}{2}e^{a\nu}e^{b\sigma}\partial_\sigma \boldsymbol{g}_{\mu\nu} - (a \leftrightarrow b) \ . \tag{4.65}$$

With all these ingredients at hand one can write down the Lagrangian for fermions coupled to gravity.

$$\mathcal{L}_f = \sqrt{-\boldsymbol{g}}\ \overline{\psi}(i\boldsymbol{\not{\mathcal{D}}} - m_\psi)\psi \tag{4.66}$$

Here, $\boldsymbol{\not{\mathcal{D}}} = \boldsymbol{\gamma}^\mu \boldsymbol{\mathcal{D}}_\mu = \gamma^a \boldsymbol{\mathcal{D}}_a$ means contraction with a gamma matrix. The covariant derivative acting on a spinor multiplet is given by

$$\boldsymbol{\mathcal{D}}_\mu = D_\mu - i\Omega_\mu = \partial_\mu - i\Omega_\mu - igA_\mu \ , \tag{4.67}$$

were Ω_μ is the spin connection

$$\Omega_\mu = \tfrac{1}{2} S_{ab}\, \omega_\mu^{ab} \ . \tag{4.68}$$

The explicit form of the fermion Lagrangian therefore is

$$\mathcal{L}_f = \sqrt{-\boldsymbol{g}}\ \overline{\psi}\big(\gamma^a i e_a{}^\mu (\partial_\mu + \tfrac{1}{4}\gamma_{bc}\omega_\mu^{bc} - igA_\mu) - m_\psi\big)\psi. \tag{4.69}$$

It remains to be shown that this is indeed a scalar under coordinate and local Lorentz transformations. Because of

$$\rho(\Lambda)^{-1}\gamma^a\rho(\Lambda) = \Lambda^a{}_b\gamma^b \qquad \text{or} \qquad \rho(\Lambda)\gamma_a\rho(\Lambda)^{-1} = \Lambda^b{}_a\gamma_b \ , \tag{4.70}$$

the vector $\overline{\psi}\gamma^a e_a \psi$ is a scalar under local Lorentz transformations. The spin connection has to transform inhomogeneously

$$\Omega \to \rho(\Lambda)\Omega\rho(\Lambda)^{-1} - i\mathrm{d}\rho(\Lambda)\rho(\Lambda)^{-1} \ . \tag{4.71}$$

From equation (4.62) it is easy to see that under a local Lorentz transformation the connection transforms as

$$\omega^{ab} \to \Lambda^a{}_c\Lambda^b{}_d\omega^{cd} + \eta^{cd}\Lambda^a{}_c\mathrm{d}\Lambda^b{}_d \ . \tag{4.72}$$

Inserting this into the definition of the spin connection gives

$$\begin{aligned}\Omega \to \Omega' &= \tfrac{i}{8}[\gamma_a,\gamma_b]\left(\Lambda^a{}_c\Lambda^b{}_d\omega^{cd} + \eta^{cd}\Lambda^a{}_c\mathrm{d}\Lambda^b{}_d\right)\\ &= \rho(\Lambda)\omega\rho(\Lambda)^{-1} + \tfrac{i}{8}[\rho(\Lambda)\gamma_a\rho(\Lambda)^{-1}, \mathrm{d}\rho(\Lambda)\gamma^a\rho(\Lambda)^{-1} + \rho(\Lambda)\gamma^a\mathrm{d}\rho(\Lambda)^{-1}]\\ &= \rho(\Lambda)\omega\rho(\Lambda)^{-1} + \tfrac{i}{4}(-d\,\mathrm{d}\rho(\Lambda)\rho(\Lambda)^{-1} + \rho(\Lambda)\gamma_a\rho(\Lambda)^{-1}\mathrm{d}\rho(\Lambda)\gamma^a\rho(\Lambda)^{-1})\\ &= \rho(\Lambda)\omega\rho(\Lambda)^{-1} - i\mathrm{d}\rho(\Lambda)\rho(\Lambda)^{-1}\end{aligned} \tag{4.73}$$

To verify the last step in the above calculation we consider $\gamma_a\rho(\Lambda)^{-1}\mathrm{d}\rho(\Lambda)\gamma^a$ and make use of $\gamma_a\gamma_b\gamma_c\gamma^a = 2\eta_{bc} + (d-4)\gamma_b\gamma_c$:

$$\begin{aligned}\gamma_a\rho(\Lambda)^{-1}\mathrm{d}\rho(\Lambda)\gamma^a &= -\tfrac{i}{2}\gamma_a S_{bc}\gamma^a\mathrm{d}\lambda^{bc}\\ &= -\tfrac{i}{2}(d-4)S_{bc}\mathrm{d}\lambda^{bc}\\ &= (d-4)\rho(\Lambda)^{-1}\mathrm{d}\rho(\Lambda).\end{aligned} \tag{4.74}$$

Now that we have established how to couple fermions to gravity we can proceed as before and expand the Lagrangian (4.69) around flat space. We begin with the components of the vielbeins

$$e_a{}^\mu = \delta_a^\mu - \tfrac{\kappa}{2}h_a^\mu + \tfrac{3\kappa^2}{8}h_\rho^\mu h_a^\rho + \mathcal{O}(\kappa^3) \tag{4.75}$$

and their inverses

$$e^a{}_\mu = \delta^a_\mu + \tfrac{\kappa}{2} h^a_\mu - \tfrac{\kappa^2}{8} h^a_\rho h^\rho_\mu + \mathcal{O}(\kappa^3)\ . \tag{4.76}$$

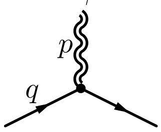
The expansion of the spin connection is given by

$$\omega^{ab}_\mu = \tfrac{\kappa}{2}\partial^b h^a_\mu + \tfrac{\kappa^2}{4} h^{\nu b}(\partial^a h_{\mu\nu} - \partial_\nu h^a_\mu + \tfrac{1}{2}\partial_\mu h^a_\nu) - (a \leftrightarrow b) + \mathcal{O}(\kappa^3)\ . \tag{4.77}$$

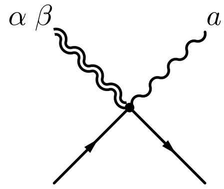
Plugging everything together, we get the expansion of the Lagrangian up to order κ^2:

$$\begin{aligned}\mathcal{L}_f = {} & \overline{\psi}(i\not{D} - m_\psi)\psi + i\tfrac{\kappa}{2}\overline{\psi}[h(\not{D} + i m_\psi) - \gamma^a h^\mu_a D_\mu + \tfrac{1}{2}\partial_b \not{h}_a \gamma^{ab}]\psi \\ & + i\tfrac{\kappa^2}{8}\overline{\psi}\big[(h^2 - 2h^{\alpha\beta}h_{\alpha\beta})(\not{D} + i m_\psi) \\ & \qquad + (3\gamma^a h^\mu_\rho h^\rho_a - 2h\gamma^a h^\mu_a) D_\mu \\ & \qquad + h^\nu_b(\partial_a \not{h}_\nu - \partial_\nu \not{h}_a + \tfrac{1}{2}\not{\partial} h_{\nu a})\gamma^{ab} \\ & \qquad + h\partial_b \not{h}_a \gamma^{ab} - \gamma^a h^\mu_a \partial_c h_{\mu b}\gamma^{bc}\big]\psi \qquad + \mathcal{O}(\kappa^3)\ .\end{aligned} \tag{4.78}$$

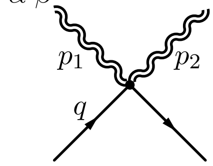
To compute the one-loop diagrams contributing to the proper fermion two point function and the proper fermion, gauge boson vertex, listed in figures 4.3 and 4.4, we need the following Feynman rules:



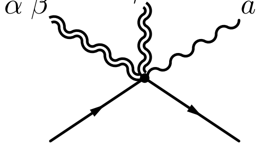
$$= \tfrac{i\kappa}{2}[\eta^{\alpha\beta}(\not{q} - m_\psi + \tfrac{1}{2}\not{p}) - \gamma^{(\alpha}(q + \tfrac{1}{2}p)^{\beta)}] \tag{4.79}$$



$$= \tfrac{ig\kappa}{2}[\eta^{\alpha\beta}\gamma^\mu - \eta^{\mu(\alpha}\gamma^{\beta)}]t^a \tag{4.80}$$



$$\begin{aligned}= i\kappa^2\Big[& (\not{p}_1 + \not{p}_2)(\tfrac{5}{16}\gamma^{(\alpha}\eta^{\beta)(\gamma}\gamma^{\delta)} + \tfrac{5}{16}\gamma^{(\gamma}\eta^{\delta)(\alpha}\gamma^{\beta)} + \tfrac{1}{8}\eta^{\alpha\beta}\eta^{\gamma\delta}) \\ & + \gamma^{(\alpha}\eta^{\beta)(\gamma}(\tfrac{3}{8}q + \tfrac{1}{2}p_1 - \tfrac{3}{8}p_2)^{\delta)} - \gamma^{(\alpha}(\tfrac{1}{4}q + \tfrac{1}{8}p_2)^{\beta)}\eta^{\gamma\delta} \\ & + \gamma^{(\gamma}\eta^{\delta)(\alpha}(\tfrac{3}{8}q + \tfrac{1}{2}p_2 - \tfrac{3}{8}p_1)^{\beta)} - \gamma^{(\gamma}(\tfrac{1}{4}q + \tfrac{1}{8}p_1)^{\delta)}\eta^{\alpha\beta} \\ & - \tfrac{1}{2}(\not{q} - m_\psi)P^{\alpha\beta\,\gamma\delta}\Big]\end{aligned} \tag{4.81}$$



$$\begin{aligned}= ig\tfrac{\kappa^2}{2}\Big[& \tfrac{3}{4}\big(\gamma^{(\alpha}\eta^{\beta)(\gamma}\eta^{\delta)\mu} + \gamma^{(\gamma}\eta^{\delta)(\alpha}\eta^{\beta)\mu}\big) \\ & - \tfrac{1}{2}\big(\gamma^{(\alpha}\eta^{\beta)\mu}\eta^{\gamma\delta} + \gamma^{(\gamma}\eta^{\delta)\mu}\eta^{\alpha\beta}\big) - \gamma^\mu P^{\alpha\beta\,\gamma\delta}\Big]t^a\end{aligned} \tag{4.82}$$

Here t^a are the generators in the irreducible representation of the gauge group which the fermion multiplet belongs to.

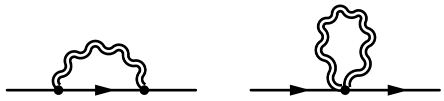


Figure 4.3: Order κ^2 diagrams for the proper fermion two-point function.

It is straight forward to obtain the Feynman rules from the expansion of the Lagrangian (4.78). One can for instance read off the $h\overline{\psi}\psi$ and the $h\overline{\psi}A\psi$ vertex from the following two expressions

$$\begin{aligned} i\int \mathrm{d}^4x \mathcal{L}_f|_{h\overline{\psi}\psi} &= -\int \mathrm{d}^4x \tfrac{\kappa}{2}\,\overline{\psi}\,[h(\not{\partial} + im_\psi) - \gamma^a h_a^\mu \partial_\mu + \tfrac{1}{2}\partial_b \not{h}_a \gamma^{ab}]\psi \\ &= \int \frac{\mathrm{d}^4q}{(2\pi)^4}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\int \mathrm{d}^4p\ \delta(q{+}p{-}k)\times \\ &\quad \times \tfrac{i\kappa}{2} h_{\mu\nu}(p)\overline{\psi}(k)[\eta^{\mu\nu}(\not{q} - m_\psi) - \gamma^{(\mu}q^{\nu)} + \tfrac{1}{2}p_\rho \gamma^{(\mu}\gamma^{\nu)\rho}]\psi(q) \end{aligned} \tag{4.83}$$

and

$$\begin{aligned} i\int \mathrm{d}^4x \mathcal{L}_f|_{h\overline{\psi}A\psi} &= i\int \mathrm{d}^4x \tfrac{g\kappa}{2}\,\overline{\psi}\,[h\not{A} - \gamma^a h_a^\mu A_\mu]\psi \\ &= \int \frac{\mathrm{d}^4q}{(2\pi)^4}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\int \mathrm{d}^4l\ \delta(q{+}p{+}l{-}k)\times \\ &\quad \times \tfrac{ig\kappa}{2} h_{\alpha\beta}(l)\overline{\psi}(k)[\eta^{\alpha\beta}\gamma^\mu - \eta^{\mu(\alpha}\gamma^{\beta)}]t^a\psi(q)A_\mu^a(p)\ . \end{aligned} \tag{4.84}$$

As the lowest order gravitational corrections are proportional to κ^2 and the combination $\kappa^2 m_\psi$ has mass dimension minus one, we expect counterterms containing dimension three and dimension two operators to appear. Possible operators of dimension three are

$$\begin{array}{lcllll} & & gF^{\mu\nu}\gamma_\nu D_\mu & & & \\ i\not{D}\not{D}\not{D} & & gD_\mu F^{\mu\nu}\gamma_\nu & & & \\ i\not{D}D^2 & & g[D_\mu, F^{\mu\nu}]\gamma_\nu & & & \\ iD^2\not{D} & \text{and} & g\gamma_{\mu\nu}F^{\mu\nu}\not{D} & \text{or} & gF^{\mu\nu}\not{D}\gamma_{\mu\nu} & . \\ iD_\mu\not{D}D^\mu & & g\not{D}F^{\mu\nu}\gamma_{\mu\nu} & \text{or} & g\gamma_{\mu\nu}\not{D}F^{\mu\nu} & \\ & & g[\not{D}, F^{\mu\nu}]\gamma_{\mu\nu} & \text{or} & g\gamma_{\mu\nu}[\not{D}, F^{\mu\nu}] & \end{array} \tag{4.85}$$

Only four of these terms are independent of one another. Because of the simple identity

$$[\gamma^{\mu\nu}, \gamma^\rho] = \eta^{\sigma\rho}(\gamma^{[\mu}\gamma^{\nu]}\gamma_\sigma + \gamma^{[\mu}\gamma_\sigma\gamma^{\nu]}) - 2\gamma^{[\nu}\eta^{\mu]\rho} = 4\gamma^{[\mu}\eta^{\nu]\rho} \tag{4.86}$$

we have

$$\begin{aligned} g\gamma_{\mu\nu}F^{\mu\nu}\not{D} &= gF^{\mu\nu}\not{D}\gamma_{\mu\nu} + 4gF^{\mu\nu}\gamma_\mu D_\nu \\ g\not{D}F^{\mu\nu}\gamma_{\mu\nu} &= g\gamma_{\mu\nu}\not{D}F^{\mu\nu} + 4gD_\mu F^{\mu\nu}\gamma_\nu \\ g[\not{D}, F^{\mu\nu}]\gamma_{\mu\nu} &= g\gamma_{\mu\nu}[\not{D}, F^{\mu\nu}] + 4g[D_\mu, F^{\mu\nu}]\gamma_\nu\ . \end{aligned} \tag{4.87}$$

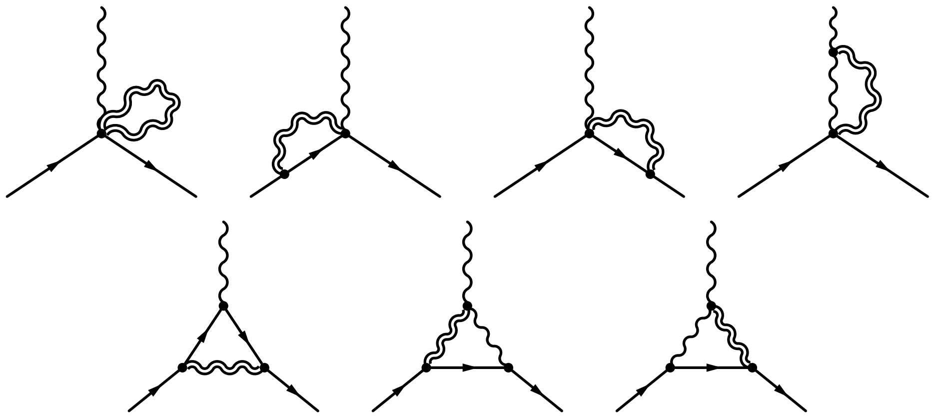


Figure 4.4: One-loop diagrams for the proper two fermion one gauge boson vertex.

Furthermore, using the identity $[D^\mu, D^\nu] = -igF^{\mu\nu}$, we get the relations

$$\begin{aligned}
igF^{\mu\nu}\gamma_\nu D_\mu &= \not{D} D^2 - D_\mu \not{D} D^\mu \\
igD_\mu F^{\mu\nu}\gamma_\nu &= -D^2\not{D} + D_\mu \not{D} D^\mu \\
[\not{D}, F^{\mu\nu}]\gamma_{\mu\nu} &= 2[D_\mu, F^{\mu\nu}]\gamma_\nu - \gamma_{\mu\nu}[\not{D}, F^{\mu\nu}] - [\not{D}, F^{\mu\nu}]\gamma_{\mu\nu} \\
&= 2[D_\mu, F^{\mu\nu}]\gamma_\nu \\
ig[D_\mu, F^{\mu\nu}]\gamma_\nu &= -D^2\not{D} - \not{D}D^2 + 2D_\mu\not{D}D^\mu \\
ig\gamma_{\mu\nu}F^{\mu\nu}\not{D} &= 2D^2\not{D} - 2\not{D}\not{D}\not{D} \\
ig\not{D}F^{\mu\nu}\gamma_{\mu\nu} &= 2\not{D}D^2 - 2\not{D}\not{D}\not{D}\ .
\end{aligned} \tag{4.88}$$

Possible operators of dimension two are

$$D^2, \qquad \not{D}^2, \qquad \gamma^\mu\not{D}D_\mu \qquad \text{and} \qquad igF_{\mu\nu}\gamma^{\mu\nu}. \tag{4.89}$$

They are related by

$$\not{D}^2 = 2D^2 - \gamma^\mu\not{D}D_\mu = D^2 - i\tfrac{g}{2}F_{\mu\nu}\gamma^{\mu\nu}\ . \tag{4.90}$$

The basis we choose is $\not{D}^2$, $igF_{\mu\nu}\gamma^{\mu\nu}$, $i\not{D}^3$, $i\not{D}D^2$, $iD^2\not{D}$ and $iD_\mu\not{D}D^\mu$. The Feynman rules belonging to these higher-derivative operators are determined by the following equations:

$$\begin{aligned}
i\int d^4x\,\overline{\psi}(\not{D}^2)\big|_A\psi = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\int d^4k\;\delta(p+q-k)\times \\
\times\,\overline{\psi}(k)\big[-ig(\not{p}\gamma^\mu + 2q^\mu)t^a\big]\psi(q)A^a_\mu(p)
\end{aligned} \tag{4.91}$$

$$\begin{aligned}
i\int d^4x\,\overline{\psi}(igF_{\mu\nu}\gamma^{\mu\nu})\big|_A\psi = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\int d^4k\;\delta(p+q-k)\times \\
\times\,\overline{\psi}(k)\big[ig(\not{p}\gamma^\mu - p^\mu)t^a\big]\psi(q)A^a_\mu(p)
\end{aligned} \tag{4.92}$$

$$\begin{aligned}
i\int d^4x\,\overline{\psi}(i\not{D}^3)\big|_A\psi = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\int d^4k\;\delta(p+q-k)\times \\
\times\,\overline{\psi}(k)\big[-ig\{\gamma^\mu(q^2+p^2) \\
+ 2q^\mu(\not{p}+\not{q}) + \not{q}\not{p}\gamma^\mu\}t^a\big]\psi(q)A^a_\mu(p)
\end{aligned} \tag{4.93}$$

$$i\int \mathrm{d}^4x\, \overline{\psi}(iD^2 \not{D})\big|_A \psi = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4q}{(2\pi)^4} \int \mathrm{d}^4k\ \delta(p+q-k)\times \qquad (4.94)$$

$$\times\, \overline{\psi}(k)\big[-ig\{(2q+p)^\mu \not{q} + (q+p)^2\gamma^\mu\}t^a\big]\psi(q)A^a_\mu(p)$$

$$i\int \mathrm{d}^4x\, \overline{\psi}(i\not{D} D^2)\big|_A \psi = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4q}{(2\pi)^4} \int \mathrm{d}^4k\ \delta(p+q-k)\times \qquad (4.95)$$

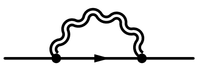
$$\times\, \overline{\psi}(k)\big[-ig\{(2q+p)^\mu(\not{p}+\not{q}) + q^2\gamma^\mu\}t^a\big]\psi(q)A^a_\mu(p)$$

$$i\int \mathrm{d}^4x\, \overline{\psi}(iD_\mu \not{D} D^\mu)\big|_A \psi = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4q}{(2\pi)^4} \int \mathrm{d}^4k\ \delta(p+q-k)\times \qquad (4.96)$$

$$\times\, \overline{\psi}(k)\big[-ig\{(2q+p)^\mu \not{q}$$

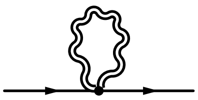
$$+ (q+p)^\mu \not{p} + q(q+p)\gamma^\mu\}t^a\big]\psi(q)A^a_\mu(p)$$

Now that we have investigated the structure of the higher-derivative counterterms, let us turn to the evaluation of the one-loop diagrams, starting with the fermion two-point function.



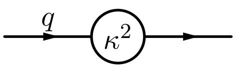
$$= \frac{i}{16\pi^2}\kappa^2\Big[\big\{\tfrac{15}{32}\not{q} - \tfrac{3}{4}m_\psi\big\}(\Lambda^2-\mu^2) \qquad (4.97)$$

$$+ \big\{\tfrac{3}{8}m_\psi q^2 + \tfrac{1}{4}m_\psi^2\not{q} - \tfrac{1}{4}m_\psi^3 - \tfrac{1}{8}\not{q}q^2\big\}\ln\tfrac{\Lambda^2}{\mu^2}\Big]$$

$\frac{1}{2}$ 

$$= \frac{i}{16\pi^2}\kappa^2\big\{\tfrac{5}{2}m_\psi - \tfrac{3}{2}\not{q}\big\}(\Lambda^2-\mu^2) \qquad (4.98)$$

The sum is



$$= \frac{i}{16\pi^2}\kappa^2\Big[\big\{\tfrac{7}{4}m_\psi - \tfrac{33}{32}\not{q}\big\}(\Lambda^2-\mu^2) \qquad (4.99)$$

$$+ \big\{\tfrac{3}{8}m_\psi q^2 + \tfrac{1}{4}m_\psi^2\not{q} - \tfrac{1}{4}m_\psi^3 - \tfrac{1}{8}\not{q}q^2\big\}\ln\tfrac{\Lambda^2}{\mu^2}\Big]$$

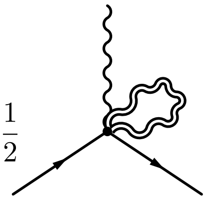
It is important to notice that the degree of divergence of the diagram (4.97) is three. Therefore, the quadratic divergence of the proper fermion two-point function cannot be determined using the non shift invariant cut-off regularization. Because of (2.48), this leads to a dependence of the quadratic divergence on the parametrization of the loop integral. However, there is an easy way to circumvent this problem.

As we have figured out in equation (4.24), the gravitational contributions to the Z factors are related. The vertex function is only quadratically divergent and therefore can be used to determine the quadratic divergence of the two-point function.

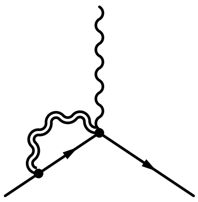
Interestingly, the "natural" parametrization, in which the ingoing fermion momentum flows along the fermion line, leads to the right quadratic divergence.

Another thing to notice is that the m_ψ^2-term of (4.99) yields a logarithmic contribution to the fermion wavefunction renormalization and thus a potential contribution to the running of the gauge coupling. However we know from (4.24) as well as from the fact that at order κ^2, no m_ψ^2-term can emerge in the gauge sector, that this contribution has to be canceled by an analogous term in the vertex renormalization.

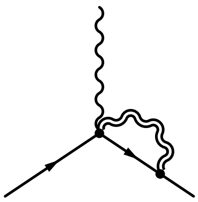
For the diagrams of figure 4.4 we get

$\frac{1}{2}$ 

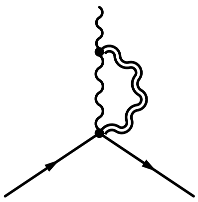
$$= \frac{i}{16\pi^2} g\kappa^2 t^a \gamma^\mu \Big[-\tfrac{3}{2}(\Lambda^2 - \mu^2)\Big] \ln \tfrac{\Lambda^2}{\mu^2} \tag{4.100}$$



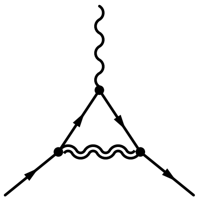
$$\begin{aligned} = \frac{i}{16\pi^2} g\kappa^2 t^a \Big[&\tfrac{9}{32}(\Lambda^2 - \mu^2)\gamma^\mu \\ &+ \big\{ \tfrac{9}{16} m_\psi q^\mu - \tfrac{3}{16} m_\psi \not{q}\gamma^\mu + \tfrac{15}{32} m_\psi^2 \gamma^\mu \\ &\quad - \tfrac{5}{48}\not{q} q^\mu - \tfrac{11}{96}\gamma^\mu q^2 \big\} \ln \tfrac{\Lambda^2}{\mu^2} \Big] \end{aligned} \tag{4.101}$$



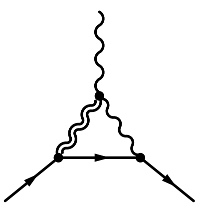
$$\begin{aligned} = \frac{i}{16\pi^2} g\kappa^2 t^a \Big[&\tfrac{9}{32}(\Lambda^2 - \mu^2)\gamma^\mu \\ &+ \big\{ \tfrac{9}{16} m_\psi k^\mu - \tfrac{3}{16} m_\psi \gamma^\mu \not{k} + \tfrac{15}{32} m_\psi^2 \gamma^\mu \\ &\quad - \tfrac{5}{48}\not{k} k^\mu - \tfrac{11}{96}\gamma^\mu k^2 \big\} \ln \tfrac{\Lambda^2}{\mu^2} \Big] \end{aligned} \tag{4.102}$$



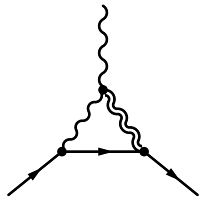
$$= \frac{i}{16\pi^2} g\kappa^2 t^a \Big[\tfrac{3}{8}\gamma^\mu p^2 - \tfrac{3}{8}\not{p} p^\mu \Big] \ln \tfrac{\Lambda^2}{\mu^2} \tag{4.103}$$



$$\begin{aligned} = \frac{i}{16\pi^2} g\kappa^2 t^a \Big[&-\tfrac{3}{32}(\Lambda^2 - \mu^2)\gamma^\mu \\ &+ \big\{ \tfrac{3}{16} m_\psi p^\mu - \tfrac{3}{16} m_\psi \not{p}\gamma^\mu - \tfrac{11}{16} m_\psi^2 \gamma^\mu \\ &\quad - \tfrac{1}{24}\not{q} q^\mu - \tfrac{13}{48}\not{q} p^\mu + \tfrac{1}{4}\not{q}\not{p}\gamma^\mu + \tfrac{11}{48}\not{p} q^\mu - \tfrac{1}{16}\not{p} p^\mu \\ &\quad + \tfrac{5}{48}\gamma^\mu q^2 - \tfrac{7}{48}\gamma^\mu qp + \tfrac{5}{32}\gamma^\mu p^2 \big\} \ln \tfrac{\Lambda^2}{\mu^2} \Big] \end{aligned} \tag{4.104}$$

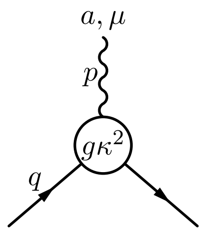


$$\begin{aligned} = \frac{i}{16\pi^2} g\kappa^2 t^a \Big[&-\tfrac{3}{16} m_\psi p^\mu + \tfrac{3}{16} m_\psi \not{p}\gamma^\mu \\ &+ \tfrac{7}{16}\not{q} p^\mu - \tfrac{7}{16}\not{q}\not{p}\gamma^\mu - \tfrac{3}{4}\not{p} q^\mu \\ &- \tfrac{1}{16}\not{p} p^\mu + \tfrac{3}{4}\gamma^\mu qp + \tfrac{1}{16}\gamma^\mu p^2 \Big] \ln \tfrac{\Lambda^2}{\mu^2} \end{aligned} \tag{4.105}$$



$$= \frac{i}{16\pi^2} g\kappa^2 t^a \Big[-\tfrac{3}{16} m_\psi p^\mu + \tfrac{3}{16} m_\psi \not{p}\gamma^\mu + \tfrac{7}{16}\not{q} p^\mu - \tfrac{7}{16}\not{q}\not{p}\gamma^\mu - \tfrac{1}{8}\not{p} q^\mu + \tfrac{1}{4}\not{p} p^\mu + \tfrac{1}{8}\gamma^\mu qp - \tfrac{1}{4}\gamma^\mu p^2 \Big] \ln\tfrac{\Lambda^2}{\mu^2} \, . \tag{4.106}$$

These diagrams sum up to



$$= \frac{i}{16\pi^2} g\kappa^2 t^a \Big[-\tfrac{33}{32}(\Lambda^2-\mu^2)\gamma^\mu + \big\{ \tfrac{3}{4} m_\psi q^\mu + \tfrac{3}{8} m_\psi \not{p}\gamma^\mu + \tfrac{1}{4} m_\psi^2 \gamma^\mu - \tfrac{1}{4}\not{q} q^\mu + \tfrac{1}{2}\not{q} p^\mu - \tfrac{5}{8}\not{q}\not{p}\gamma^\mu - \tfrac{3}{4}\not{p} q^\mu - \tfrac{17}{48}\not{p} p^\mu - \tfrac{1}{8}\gamma^\mu q^2 + \tfrac{1}{2}\gamma^\mu qp + \tfrac{11}{48}\gamma^\mu p^2 \big\} \ln\tfrac{\Lambda^2}{\mu^2} \Big] \, . \tag{4.107}$$

Comparing this with the higher-derivative vertices (4.91)-(4.96), we determine the corresponding higher-derivative counterterm to be

$$\mathcal{L}_{c.t.} = \frac{\kappa^2}{16\pi^2} \ln\tfrac{\Lambda^2}{\mu^2} \overline{\psi} \Big[\tfrac{3}{8} m_\psi \not{D}^2 - \tfrac{5}{8} i\not{D}\not{D}\not{D} + \tfrac{41}{48} iD^2\not{D} - \tfrac{29}{24} i\not{D}D^2 + \tfrac{41}{48} iD_\mu \not{D} D^\mu \Big] \psi \, . \tag{4.108}$$

At the level of the two-point function, all the dimension three operators are indistinguishable. The only thing we can conclude from the two-point function result (4.99) is that their coefficients necessarily add up to $-\frac{1}{8}$, which is indeed the case. As predicted the m_ψ^2-terms of (4.107) and (4.99) are exactly such that $Z_\psi|_{\mathcal{O}(\kappa^2)} = Z_{\overline{\psi}A\psi}|_{\mathcal{O}(\kappa^2)}$. Everything else would imply that we had made a serious mistake.

Coming back to our initial goal, we see that even in the massless case we do not get the single dimension six higher-derivative term $\overline{\psi} i\not{D}^3 \psi$, as has been claimed in [36]. Instead we found a linear combination of all possible terms.

4.6 Scalars

The coupling of a massive, charged scalar multiplet to gravity is straight forward. The evident generalization of the flat space Lagrangian is

$$\mathcal{L}_s = \sqrt{-\boldsymbol{g}} \Big[\boldsymbol{g}^{\mu\nu} (D_\mu\phi)^\dagger D_\nu\phi - m_\phi^2 \, \phi^\dagger\phi \Big] \, . \tag{4.109}$$

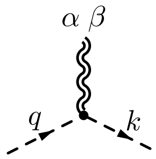
The multiplet ϕ belongs to an irreducible representation of the gauge group whose generators we, as in the case of fermions, denote by t^a, allthough they not necessarily belong to the same representation.

Figure 4.5: Order κ^2 diagrams for the proper scalar two-point function.

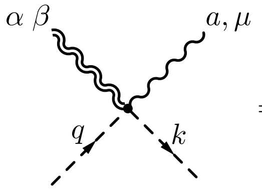
In order to derive the Feynman rules involving gravitons we expand the Lagrangian in orders of κ:

$$\mathcal{L}_s = -\phi^\dagger(D^2+m_\phi^2)\phi + \kappa\left[\tfrac{1}{2}\eta^{\mu\nu}h - h^{\mu\nu}\right](D_\mu\phi)^\dagger D_\nu\phi - \tfrac{\kappa}{2}hm_\phi^2\phi^\dagger\phi$$
$$+\kappa^2\left[\tfrac{1}{8}(h^2-2h^{\alpha\beta}h_{\alpha\beta})\eta^{\mu\nu} + h^{\mu\alpha}h_\alpha^\nu - \tfrac{1}{2}hh^{\mu\nu}\right](D_\mu\phi)^\dagger D_\nu\phi + \mathcal{O}(\kappa^3)\ . \tag{4.110}$$

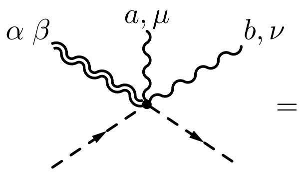
To determine all the higher-derivative counterterms, we have to calculate the gravitational contribution to the proper scalar two-point function and the proper two-scalar, one- and two-gauge field vertices. The corresponding one-loop diagrams are listed in figures 4.5, 4.6 and 4.7. For their calculation we need the following Feynman rules:



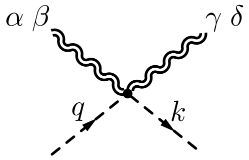
$$= i\kappa\left[\tfrac{1}{2}\eta^{\alpha\beta}(qk-m_\varphi^2) - q^{(\alpha}k^{\beta)}\right] \tag{4.111}$$



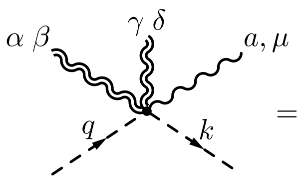
$$= i\kappa g\left[\tfrac{1}{2}\eta^{\alpha\beta}(q+k)^\mu - \eta^{\mu(\alpha}(q+k)^{\beta)}\right]t^a \tag{4.112}$$



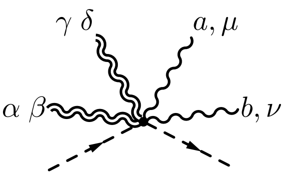
$$= i\kappa g^2\left[\tfrac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu(\alpha}\eta^{\beta)\nu}\right]\{t^a,t^b\} = -i\kappa g^2 P^{\alpha\beta\,\mu\nu}\{t^a,t^b\} \tag{4.113}$$



$$= -i\kappa^2\Big[\tfrac{1}{2}P^{\alpha\beta\,\gamma\delta}(qk-m_\varphi^2) + \tfrac{1}{2}\eta^{\alpha\beta}q^{(\gamma}k^{\delta)} + \tfrac{1}{2}\eta^{\gamma\delta}q^{(\alpha}k^{\beta)}$$
$$- q^{(\alpha}\eta^{\beta)(\gamma}k^{\delta)} - q^{(\gamma}\eta^{\delta)(\alpha}k^{\beta)}\Big] \tag{4.114}$$



$$= -ig\kappa^2\Big[\tfrac{1}{2}P^{\alpha\beta\,\gamma\delta}(q+k)^\mu + P^{\gamma\delta\,\mu(\alpha}(q+k)^{\beta)}$$
$$+ P^{\alpha\beta\,\mu(\gamma}(q+k)^{\delta)}\Big]t^a \tag{4.115}$$



$$= -ig^2\kappa^2\Big[\tfrac{1}{2}\eta^{\mu\nu}P^{\alpha\beta\,\gamma\delta} + P^{\gamma\delta\,\nu(\alpha}\eta^{\beta)\mu} + P^{\alpha\beta\,\nu(\gamma}\eta^{\delta)\mu}\Big]\{t^a,t^b\} \tag{4.116}$$

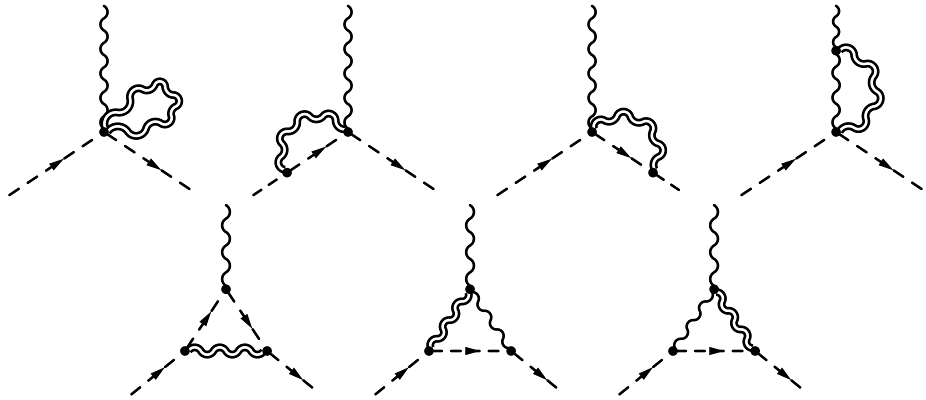


Figure 4.6: Order $g\kappa^2$ diagrams for the two-scalar one-gauge boson vertex.

As in the case of fermions and gauge fields, we expect higher-derivative counterterms to appear. The corresponding higher-derivative operators must have mass dimension four because the gravitational corrections come along with a factor of κ^2. In contrast to the fermions, the square of the scalar mass and not the mass itself is a parameter of the theory and therefore, no operators of dimension three can appear. The dimension four operators are

$$\begin{array}{lcl} (D^2)^2 & & igD_\mu F^{\mu\nu} D_\nu \\ D_\mu D^2 D^\mu & , & ig[D_\mu, F^{\mu\nu}]D_\nu \quad \text{and} \quad g^2 F^{\mu\nu}F_{\mu\nu}\ . \\ D_\mu D_\nu D^\mu D^\nu & & igD_\nu [D_\mu, F^{\mu\nu}] \end{array} \tag{4.117}$$

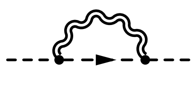
Only three of these terms are independent of one another. The relations between them are

$$\begin{aligned} D_\mu D^2 D^\mu &= \tfrac{g^2}{2} F_{\mu\nu}F^{\mu\nu} + D_\mu D_\nu D^\mu D^\nu \\ igD_\mu F^{\mu\nu} D_\nu &= D_\mu D_\nu D^\mu D^\nu - (D^2)^2 \\ igD_\nu [D_\mu, F^{\mu\nu}] &= ig[D_\mu, F^{\mu\nu}]D_\nu \\ ig[D_\mu, F^{\mu\nu}]D_\nu &= 2D_\mu D_\nu D^\mu D^\nu - (D^2)^2 - D_\mu D^2 D^\mu\ . \end{aligned} \tag{4.118}$$

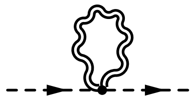
A possible basis is $(D^2)^2$, $igD_\mu F^{\mu\nu} D_\nu$ and $F^{\mu\nu}F_{\mu\nu}$. The one-loop counterterm will therefore have the general form

$$\mathcal{L}_{c.t.} = \phi^\dagger \left(\alpha_1 (D^2)^2 + \alpha_2 igD_\mu F^{\mu\nu} D_\nu + \alpha_3 g^2 F^{\mu\nu}F_{\mu\nu}\right) \phi\ . \tag{4.119}$$

We begin with the proper scalar two-point function. For the two diagrams of figure 4.5 we obtain

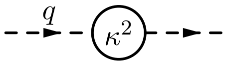


$$= \frac{i}{16\pi^2}\kappa^2 \left[-\tfrac{1}{2}q^2\left(\Lambda^2-\mu^2\right) + \left\{m_\phi^2 q^2 - m_\phi^4\right\}\ln\tfrac{\Lambda^2}{\mu^2}\right] \tag{4.120}$$



$$= \frac{i}{16\pi^2}\kappa^2 \tfrac{5}{2} m_\phi^2 \left(\Lambda^2-\mu^2\right)\ . \tag{4.121}$$

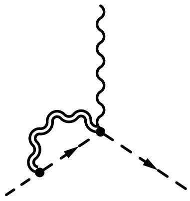
The one-loop result is



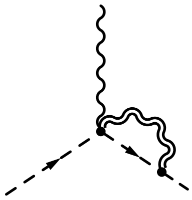
$$= \frac{i}{16\pi^2}\kappa^2\Big[\big\{\tfrac{5}{2}m_\phi^2 - \tfrac{1}{2}q^2\big\}\big(\Lambda^2-\mu^2\big) + \big\{m_\phi^2 q^2 - m_\phi^4\big\}\ln\tfrac{\Lambda^2}{\mu^2}\Big]\,. \tag{4.122}$$

The terms containing no momenta lead to a quadratic and a logarithmic mass renormalization. The terms proportional to q^2 correspond to a quadratic and a logarithmic wave function renormalization. Since there is no logarithmically divergent term proportional to q^4, we do not need a dimension six counterterm involving only covariant derivatives and no field strength tensor. In the chosen basis (4.119), this means $\alpha_1 = 0$ and hence there is no Lee-Wick term $\phi^\dagger D^4\phi$.

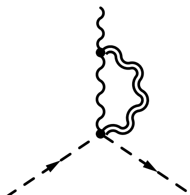
We proceed with the diagrams of figure 4.6.



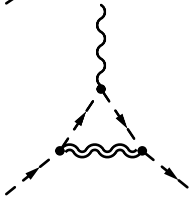
$$\begin{aligned}= \frac{i}{16\pi^2}\kappa^2 g t^a\Big[&-\tfrac{1}{2}q^\mu\left(\Lambda^2-\mu^2\right)\\ &+\big\{(\tfrac{3}{4}q^\mu+\tfrac{1}{2}k^\mu)m_\phi^2\\ &\quad+\tfrac{1}{2}q^\mu qk - \tfrac{1}{4}k^\mu q^2\big\}\ln\tfrac{\Lambda^2}{\mu^2}\Big]\end{aligned} \tag{4.123}$$



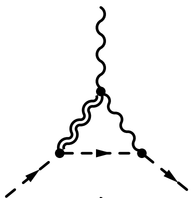
$$\begin{aligned}= \frac{i}{16\pi^2}\kappa^2 g t^a\Big[&-\tfrac{1}{2}q^\mu\left(\Lambda^2-\mu^2\right)\\ &+\big\{(\tfrac{3}{4}k^\mu+\tfrac{1}{2}q^\mu)m_\phi^2\\ &\quad+\tfrac{1}{2}k^\mu qk - \tfrac{1}{4}q^\mu k^2\big\}\ln\tfrac{\Lambda^2}{\mu^2}\Big]\end{aligned} \tag{4.124}$$



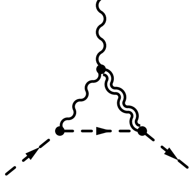
$$= \frac{i}{16\pi^2}\kappa^2 g t^a\Big[\tfrac{3}{2}q^\mu(k-q)k + \tfrac{3}{2}k^\mu(q-k)q\Big]\ln\tfrac{\Lambda^2}{\mu^2} \tag{4.125}$$



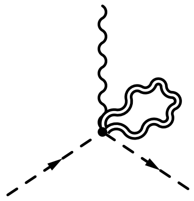
$$\begin{aligned}= \frac{i}{16\pi^2}\kappa^2 g t^a\Big[&-\tfrac{1}{4}(q+k)^\mu m_\phi^2\\ &+\tfrac{1}{4}q^\mu(k-2q)k + \tfrac{1}{4}k^\mu(q-2k)q\Big]\ln\tfrac{\Lambda^2}{\mu^2}\end{aligned} \tag{4.126}$$



$$= \frac{i}{16\pi^2}\kappa^2 g t^a\Big[\tfrac{3}{4}q^\mu(q-k)k + \tfrac{3}{4}k^\mu(k-q)q\Big]\ln\tfrac{\Lambda^2}{\mu^2} \tag{4.127}$$

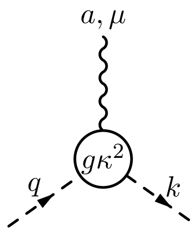


$$= \frac{i}{16\pi^2}\kappa^2 g t^a\Big[\tfrac{3}{4}q^\mu(q-k)k + \tfrac{3}{4}k^\mu(k-q)q\Big]\ln\tfrac{\Lambda^2}{\mu^2} \tag{4.128}$$



$$= 0 \tag{4.129}$$

All these diagrams sum up to



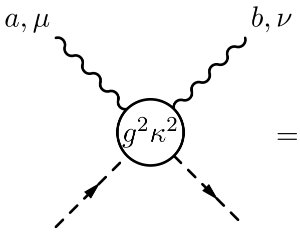
$$= \frac{i}{16\pi^2}\kappa^2 g t^a (q+k)^\mu \left[-\tfrac{1}{2}\left(\Lambda^2-\mu^2\right) + m_\phi^2 \ln \tfrac{\Lambda^2}{\mu^2}\right] + \text{finite}\ . \tag{4.130}$$

This simply is a quadratic and logarithmic vertex renormalization. Consequently, there is no counterterm involving one field strength tensor and α_2 is zero as well.

In order to find out wether or not there is a counterterm involving two field strength tensors, it is necessary to investigate the one-loop divergences of the $\phi^\dagger A^2 \phi$-vertex. From the calculation above we know that the result has to be $\alpha_3 g^2 \phi^\dagger F^{\mu\nu} F_{\mu\nu} \phi$.

Note that our result for the scalar two-point function already showed, that the one-loop counterterms do not coincide with the Lee-Wick term $\phi^\dagger D^4 \phi$. Nevertheless, to complete the investigation of the counterterms in the scalar sector, we will also give the result for the proper $\phi^\dagger A^2 \phi$-vertex. To simplify this calculation we considered only the Abelian case. This reduces the number of diagrams compared to the non-Abelian ones listed in figure 4.7. However, gravity is insensitive to the gauge group and we can generalize our result to the non-Abelian case.

The result for the proper $\phi^\dagger A^2 \phi$-vertex is



$$= \frac{ig^2\kappa^2}{16\pi^2}\,\eta^{\mu\nu}\{t^a, t^b\}\left[-\tfrac{1}{2}(\Lambda^2-\mu^2) + m_\phi^2 \ln \tfrac{\Lambda^2}{\mu^2}\right] + \text{finite}\ . \tag{4.131}$$

This agrees with our previous results (4.122), (4.130) and yields $\alpha_3 = 0$. Hence there is no higher derivative counterterm at all and similar to the fermions we find no connection to the Lee-Wick standard Model.

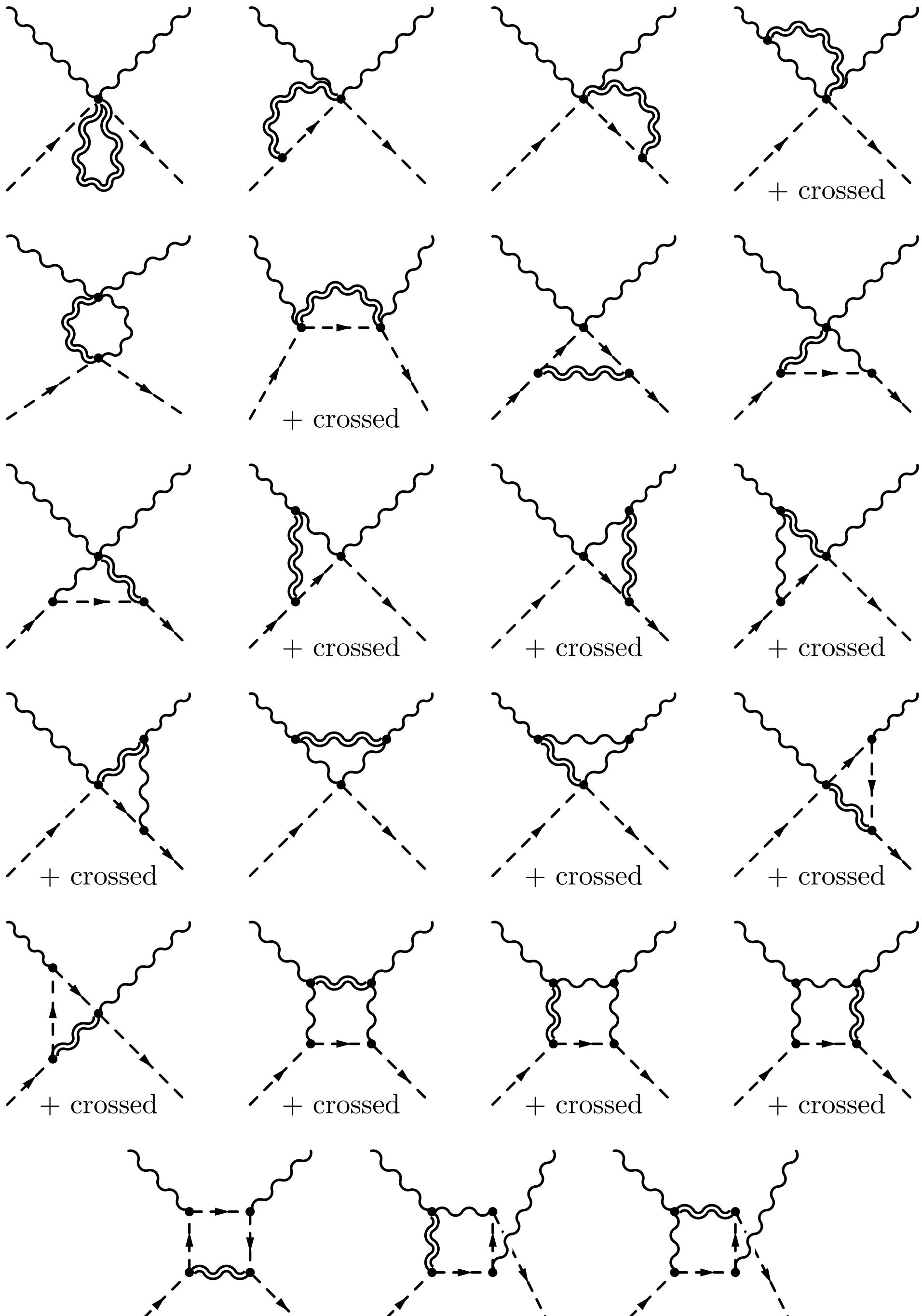


Figure 4.7: One-loop diagrams for the proper two-scalar two-gauge boson vertex. Here crossed refers to the corresponding diagram with exchanged outer gauge boson legs.

Chapter 5

Running of the Yukawa and φ^4 Couplings

Motivated by Robinson [43], we are going to determine the gravitational contributions to the running of Yukawa and φ^4 coupling.

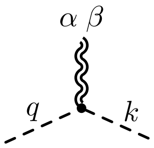
5.1 Yukawa Theory

Yukawa interactions play an essential role in the Standard Model of elementary particles. Therefore it is interesting to investigate how gravity modifies the running of the Yukawa coupling.

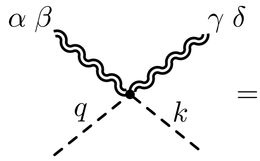
The starting point is the Lagrangian of a fermion ψ and a real scalar φ, minimally coupled to gravity and interacting via a Yukawa term.

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{-\boldsymbol{g}}\boldsymbol{R} + \sqrt{-\boldsymbol{g}}\left(\bar{\psi}(i\boldsymbol{\not{D}} - m_\psi)\psi + \frac{1}{2}\boldsymbol{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m_\varphi\varphi^2 - \lambda\varphi\bar{\psi}\psi\right) \quad (5.1)$$

The Feynman rules for the fermion graviton vertices are the same as in Section 4.5. The rules including the real scalar field φ are:

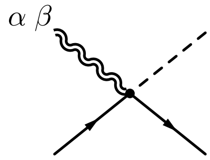


$$= -i\kappa\left[\frac{1}{2}\eta^{\alpha\beta}(qk + m_\varphi^2) - q^{(\alpha}k^{\beta)}\right] \quad (5.2)$$

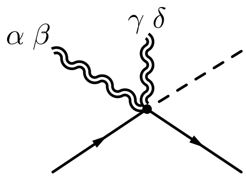


$$= i\kappa^2\Big[\frac{1}{2}P^{\alpha\beta\,\gamma\delta}(qk + m_\varphi^2) + \frac{1}{2}\eta^{\alpha\beta}q^{(\gamma}k^{\delta)} + \frac{1}{2}\eta^{\gamma\delta}q^{(\alpha}k^{\beta)} - q^{(\alpha}\eta^{\beta)(\gamma}k^{\delta)} - q^{(\gamma}\eta^{\delta)(\alpha}k^{\beta)}\Big]\,. \quad (5.3)$$

Now also k points into the vertex.



$$= -i\frac{\kappa}{2}\lambda\eta^{\alpha\beta} \quad (5.4)$$



$$= -i\frac{\kappa^2}{4}\lambda\left[\eta^{\alpha\beta}\eta^{\gamma\delta} - 2\eta^{\alpha(\gamma}\eta^{\delta)\beta}\right] \quad (5.5)$$

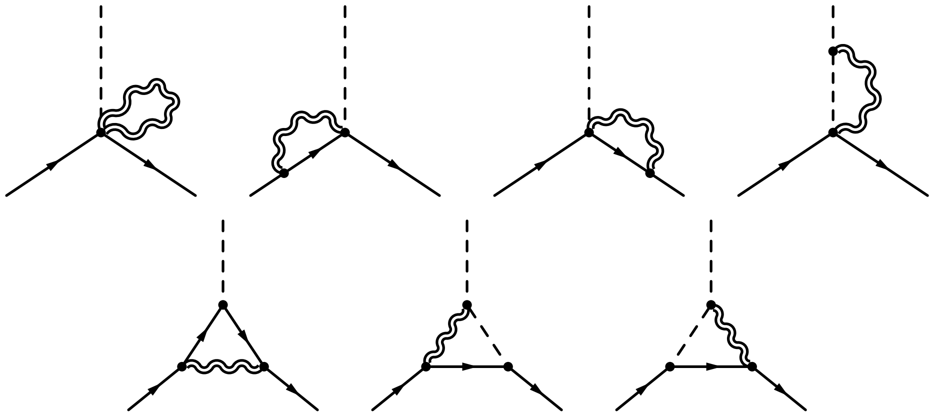


Figure 5.1: Order $\lambda\kappa^2$ diagrams of the proper Yukawa vertex.

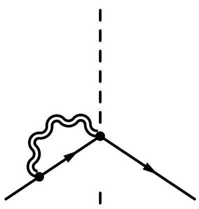
The one-loop diagrams determining the gravitational contribution to the proper Yukawa vertex are listed in figure 5.1.

Again we expect to find higher-derivative counterterms. The possible conterterms containing two derivatives or one derivative are:

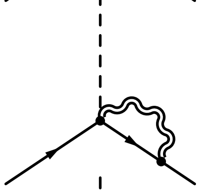
$$\begin{matrix} \overline{\psi}\psi\partial^2\varphi \\ \varphi\overline{\psi}\partial^2\psi \\ \overline{\psi}\partial^\mu\psi\partial_\mu\varphi \\ \overline{\psi}\partial\!\!\!/\gamma^\mu\psi\partial_\mu\varphi \end{matrix} \quad \text{and} \quad \begin{matrix} i\overline{\psi}\gamma^\mu\psi\partial_\mu\varphi \\ i\varphi\overline{\psi}\partial\!\!\!/\psi \end{matrix} . \tag{5.6}$$

This time the counterterms only have to be Lorentz invariant and there is no gauge invariance, which limits the number of possible higher-derivative interactions.

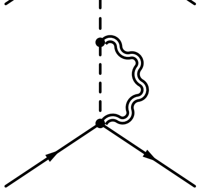
We calculated all diagrams in dimensional and in cut-off regularization, but for the individual diagrams we only give the cut-off result because the result from dimensional regularization is just the cut-off result with the quardratic divergences dropped and $\ln\frac{\Lambda^2}{\mu^2}$ replaced by $\frac{2}{\epsilon}$.



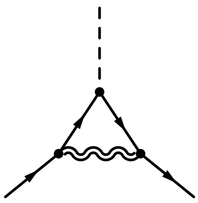
$$= \frac{i}{16\pi^2}\kappa^2\lambda\Big[-\tfrac{3}{8}(\Lambda^2-\mu^2) + \big\{-\tfrac{5}{8}m_\psi^2 + \tfrac{1}{16}m_\psi q\!\!\!/ + \tfrac{3}{16}q^2\big\}\ln\tfrac{\Lambda^2}{\mu^2}\Big] \tag{5.7}$$



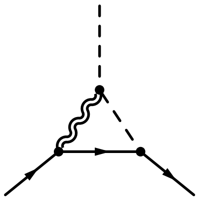
$$= \frac{i}{16\pi^2}\kappa^2\lambda\Big[-\tfrac{3}{8}(\Lambda^2-\mu^2) + \big\{-\tfrac{5}{8}m_\psi^2 + \tfrac{1}{16}m_\psi k\!\!\!/ + \tfrac{3}{16}k^2\big\}\ln\tfrac{\Lambda^2}{\mu^2}\Big] \tag{5.8}$$



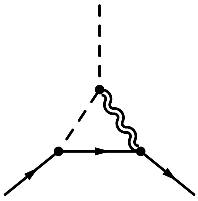
$$= \frac{i}{16\pi^2}\kappa^2\lambda\Big[-m_\phi^2 + \tfrac{1}{4}p^2\Big]\ln\tfrac{\Lambda^2}{\mu^2} \tag{5.9}$$



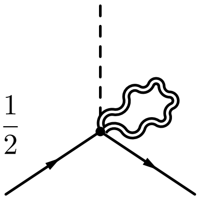
$$= \frac{i}{16\pi^2}\kappa^2\lambda\Big[\tfrac{1}{2}m_\psi^2 + \tfrac{3}{16}m_\psi(\not{q}+\not{k}) - \tfrac{9}{32}q^2 + \tfrac{7}{8}qk - \tfrac{9}{32}k^2 - \tfrac{5}{16}\not{q}\not{k}\Big]\ln\tfrac{\Lambda^2}{\mu^2} \tag{5.10}$$



$$= \frac{i}{16\pi^2}\kappa^2\lambda\Big[\tfrac{3}{8}m_\phi^2 + \tfrac{7}{32}m_\psi(\not{q}-\not{k}) + \tfrac{3}{32}q^2 - \tfrac{1}{8}qk - \tfrac{3}{16}k^2 + \tfrac{7}{32}\not{q}\not{k}\Big]\ln\tfrac{\Lambda^2}{\mu^2} \tag{5.11}$$

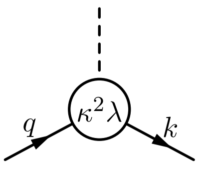


$$= \frac{i}{16\pi^2}\kappa^2\lambda\Big[\tfrac{3}{8}m_\phi^2 + \tfrac{7}{32}m_\psi(\not{k}-\not{q}) + \tfrac{3}{32}k^2 - \tfrac{1}{8}qk - \tfrac{3}{16}q^2 + \tfrac{7}{32}\not{q}\not{k}\Big]\ln\tfrac{\Lambda^2}{\mu^2} \tag{5.12}$$

$\frac{1}{2}$ 

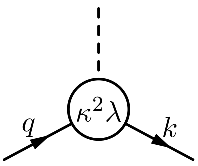
$$= \frac{i}{16\pi^2}\kappa^2\lambda\Big[\tfrac{5}{2}(\Lambda^2-\mu^2)\Big] \tag{5.13}$$

The one-loop result is



$$= \frac{i}{16\pi^2}\kappa^2\lambda\Big[\tfrac{7}{4}(\Lambda^2-\mu^2) + \big\{-\tfrac{3}{4}m_\psi^2 - \tfrac{1}{4}m_\phi^2 + \tfrac{1}{4}m_\psi(\not{q}+\not{k}) + \tfrac{1}{16}(q+k)^2 + \tfrac{1}{8}\not{q}\not{k}\big\}\ln\tfrac{\Lambda^2}{\mu^2}\Big]\,, \tag{5.14}$$

or with dimensional regularization



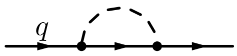
$$= \frac{i\lambda\kappa^2}{16\pi^2}\Big[-\tfrac{3}{4}m_\psi^2 - \tfrac{1}{4}m_\phi^2 + \tfrac{1}{4}m_\psi(\not{q}+\not{k}) + \tfrac{1}{16}(q+k)^2 + \tfrac{1}{8}\not{q}\not{k}\Big]\frac{2}{\epsilon}\,. \tag{5.15}$$

The higher-derivative counterterm is

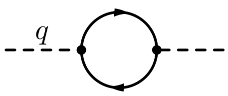
$$\mathcal{L}_{c.t.} = \frac{\kappa^2\lambda}{16\pi^2}\ln\tfrac{\Lambda^2}{\mu^2}\Big[-\tfrac{1}{4}m_\psi i\overline{\psi}\gamma^\mu\psi\partial_\mu\varphi + \tfrac{1}{2}m_\psi i\varphi\overline{\psi}\not{\partial}\psi - \tfrac{5}{16}\overline{\psi}\psi\partial^2\varphi + \tfrac{3}{8}\varphi\overline{\psi}\partial^2\psi + \tfrac{1}{4}\overline{\psi}\partial^\mu\psi\partial_\mu\varphi + \tfrac{1}{8}\overline{\psi}\not{\partial}\gamma^\mu\psi\partial_\mu\varphi\Big]\,. \tag{5.16}$$

This is exactly what one expects because of the non-renormalizability of the theory, but we are not interested in higher-derivative counterterms now.

The order λ^2 contribution to the wave function renormalization of the scalar and the fermion are determined by the logarithmic divergences of the following two diagrams:

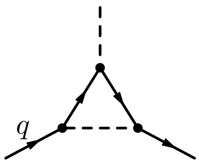


$$= \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{\not{k} + \not{q} + m_\psi}{((k+q)^2 - m_\psi^2)(k^2 - m_\phi^2)}$$
$$= \frac{i\lambda^2}{16\pi^2} \{ \tfrac{1}{2}\not{q} + m_\psi \} \ln \tfrac{\Lambda^2}{\mu^2} \tag{5.17}$$



$$= -\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{tr}\{(\not{k} + \not{q} + m_\psi)(\not{k} + m_\psi)\}}{((k+q)^2 - m_\psi)(k^2 - m_\psi)}$$
$$= \frac{i}{16\pi^2}\lambda^2 \left[4(\Lambda^2 - \mu^2) + \{2q^2 - 12m_\psi^2\} \ln \tfrac{\Lambda^2}{\mu^2} \right] . \tag{5.18}$$

The order λ^2 contribution to the vertex renormalization factor is determined by:



$$= \lambda^3 \int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} + \not{q} + m_\psi)(\not{k} + \not{p} + m_\psi)}{((k+q)^2 - m_\psi^2)((k+p)^2 - m_\psi)(k^2 - m_\phi)}$$
$$= \frac{i}{16\pi^2}\lambda^3 \ln \tfrac{\Lambda^2}{\mu^2} . \tag{5.19}$$

So far we have computed all diagrams that are necessary to determine the one-loop renormalization of the coupling constant. However, it is not clear to us how to proceed now. The first possibility is to use the results from dimensional regularization to determine the running of the coupling. In this way only the logarithmic divergences contribute to the β function. If we adapt the Wilsonian point of view instead [44], then the evolution of the coupling is determined by integrating out a momentum shell from μ to Λ, where Λ is the cutoff of our theory. In this scheme it is obvious that quadratic divergences contribute to the dependence of the coupling on μ and thus contribute to the β function. In what follows we will investigate both possibilities and their consequences.

The Z factors in the minimal subtraction scheme are

$$\begin{aligned}
Z_{\varphi\bar{\psi}\psi} - 1 &= \frac{\lambda^2}{16\pi^2} \ln \tfrac{\Lambda^2}{\mu^2} + \frac{\kappa^2}{16\pi^2} \tfrac{7}{4}(\Lambda^2 - \mu^2) - \frac{\kappa^2}{16\pi^2}(\tfrac{3}{4}m_\psi^2 + \tfrac{1}{4}m_\phi^2) \ln \tfrac{\Lambda^2}{\mu^2} \\
Z_\psi - 1 &= -\frac{\lambda^2}{16\pi^2}\frac{1}{2} \ln \tfrac{\Lambda^2}{\mu^2} + \frac{\kappa^2}{16\pi^2}\frac{33}{32}(\Lambda^2 - \mu^2) - \frac{\kappa^2}{16\pi^2} \tfrac{1}{4} m_\psi^2 \ln \tfrac{\Lambda^2}{\mu^2} \\
Z_\varphi - 1 &= -\frac{\lambda^2}{16\pi^2} 2 \ln \tfrac{\Lambda^2}{\mu^2} + \frac{\kappa^2}{16\pi^2}\frac{1}{2}(\Lambda^2 - \mu^2) - \frac{\kappa^2}{16\pi^2} m_\phi^2 \ln \tfrac{\Lambda^2}{\mu^2} .
\end{aligned} \tag{5.20}$$

The bare couplings are defined by

$$\begin{aligned}
\lambda_0 &= \lambda Z_{\varphi\bar{\psi}\psi} Z_\psi^{-1} Z_\varphi^{-\frac{1}{2}} = \lambda Z_\lambda \\
\kappa_0 &= \kappa Z_\kappa .
\end{aligned} \tag{5.21}$$

We are now going to derive an expression for the β function of λ. Even though it is intuitively clear that at one-loop there is no influence between the running of κ and λ we will show explicitly that the flows decouple at one-loop.

The starting point is the μ independence of the bare couplings λ_0 and κ_0. Differentiating both equations (5.21) with respect to μ and using the chain rule

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} = \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_\kappa\frac{\partial}{\partial\kappa} + \mu\frac{\partial}{\partial\mu}\ , \tag{5.22}$$

one obtains the equations

$$\begin{aligned} 0 &= \beta_\lambda\frac{\partial}{\partial\lambda}\left(\lambda Z_\lambda\right) + \lambda\beta_\kappa\frac{\partial Z_\lambda}{\partial\kappa} + \lambda\mu\frac{\partial Z_\lambda}{\partial\mu} \\ 0 &= \kappa\beta_\lambda\frac{\partial Z_\kappa}{\partial\lambda} + \beta_\kappa\frac{\partial}{\partial\kappa}(\kappa Z_\kappa) + \kappa\mu\frac{\partial Z_\kappa}{\partial\mu}\ , \end{aligned} \tag{5.23}$$

which can be easily solved for β_λ.

$$\beta_\lambda = -\lambda\frac{\mu\frac{\partial Z_\lambda}{\partial\mu}\frac{\partial}{\partial\kappa}(\kappa Z_\kappa) - \kappa\frac{\partial Z_\lambda}{\partial\kappa}\mu\frac{\partial Z_\kappa}{\partial\mu}}{\frac{\partial}{\partial\lambda}(\lambda Z_\lambda)\frac{\partial}{\partial\kappa}(\kappa Z_\kappa) - \lambda\frac{\partial Z_\kappa}{\partial\lambda}\frac{\partial Z_\lambda}{\partial\kappa}} \tag{5.24}$$

As allready stated we expect that the running of κ does not contribute to the one-loop result for β_λ. Observing that at one-loop level one has

$$\begin{aligned} &\kappa\frac{\partial Z_\lambda}{\partial\kappa} \sim \kappa^2 && \frac{\partial}{\partial\kappa}(\kappa Z_\kappa) = \mathcal{O}(1) \\ &\mu\frac{\partial Z_\kappa}{\partial\mu} \sim \kappa^2 && \frac{\partial Z_\kappa}{\partial\lambda} = 0\ , \end{aligned} \tag{5.25}$$

the one-loop β function simplifies to

$$\beta_\lambda = -\lambda\frac{\mu\frac{\partial Z_\lambda}{\partial\mu}}{\frac{\partial}{\partial\lambda}\left(\lambda Z_\lambda\right)} \tag{5.26}$$

This is the result one would obtain by setting $\beta_\kappa = 0$ at the beginning. At one-loop order this can be simplified further to

$$\begin{aligned} \beta_\lambda &= -\lambda\mu\frac{\partial Z_\lambda}{\partial\mu} = -\lambda\mu\frac{\partial}{\partial\mu}\left(Z_{\varphi\bar\psi\psi}Z_\psi^{-1}Z_\varphi^{-\frac{1}{2}}\right) \\ &= \underbrace{\frac{5}{16\pi^2}}_{=a}\lambda^3 - \underbrace{\frac{\kappa^2}{16\pi^2}(m_\psi^2 - \tfrac{1}{2}m_\phi^2)}_{=b}\lambda + \underbrace{\frac{\kappa^2}{16\pi^2}\frac{15}{16}}_{=c}\lambda\mu^2\ . \end{aligned} \tag{5.27}$$

Now that we have found the gravitational corrections to the β function we will try to figure out their consequences. In order to do this, we determine the solution of the differential equation (5.27) assuming κ, m_ψ and m_ϕ and thus a,b and c to be constant. This simplification should nevertheless yield the right qualitative behavior in the region where our perturbative calculations are valid. Especially at low energies the predictions should be trustworthy.

To solve the equation, it is convenient to rewrite it as a one-form and try to find an integrating factor.

$$0 = -\frac{1}{\lambda^3}\mathrm{d}\lambda + \frac{a}{\mu}\mathrm{d}\mu - \frac{b}{\mu\lambda^2}\mathrm{d}\mu + \frac{c}{\lambda^2}\mu\mathrm{d}\mu \tag{5.28}$$

An integrating factor f, which is only μ dependent, has to satisfy

$$\frac{\mathrm{d}f}{\mathrm{d}\mu} = 2\big(c\mu - \tfrac{b}{\mu}\big) f \ . \tag{5.29}$$

This is solved by $f(\mu) = \big(\frac{\mu}{\mu_0}\big)^{-2b} \exp\big(c\mu^2\big)$. Now we can easily write down the running coupling:

$$\lambda^2(\mu) = \frac{\lambda^2(\mu_0)}{\big(\frac{\mu}{\mu_0}\big)^{2b}\exp[c(\mu_0^2-\mu^2)] - 2a\lambda^2(\mu_0)\mu^{2b}\int\limits_{\mu_0}^{\mu}\mathrm{d}k\; k^{-2b-1}\exp[c(k^2-\mu^2)]} \ . \tag{5.30}$$

This has to be compared with the result in the absence of gravity ($c = 0$, $b = 0$):

$$\lambda^2(\mu) = \frac{\lambda^2(\mu_0)}{1 - a\lambda^2(\mu_0)\ln\frac{\mu^2}{\mu_0^2}} \qquad \text{with}$$
$$\frac{\partial\lambda}{\partial\mu} > 0 \ , \qquad \mu_{\text{pole}} = \mu_0 \exp\left(\frac{1}{2a\lambda^2(\mu_0)}\right) \qquad \text{and} \qquad \lim_{\mu\to 0}\lambda = 0 \ . \tag{5.31}$$

Here μ_{pole} denotes the Landau pole. Before considering the Wilsonian running with non-zero masses ($b, c \neq 0$), let us investigate the result in dimensional regularization ($c = 0$):

$$\lambda^2(\mu) = \frac{\lambda^2(\mu_0)}{\big(\frac{\mu}{\mu_0}\big)^{2b}\big(1 - \lambda^2(\mu_0)\frac{a}{b}\big) + \lambda^2(\mu_0)\frac{a}{b}} \tag{5.32}$$

There are two regions of distinct behaviour. They are seperated by the fixpoint $\lambda_\star^2 = \frac{b}{a}$.

- $\mathbf{0 < \lambda(\mu_0) < \lambda_\star}$

  In this region we get asymptotic freedom and λ approches the fix point when μ tends to zero:

$$\frac{\partial\lambda}{\partial\mu} < 0 \qquad \lim_{\mu\to\infty}\lambda(\mu) = 0 \qquad \lim_{\mu\to 0}\lambda(\mu) = \lambda_\star \ . \tag{5.33}$$

- $\mathbf{\lambda(\mu_0) > \lambda_\star}$

  In this region λ grows with energy and is bounded from below by the fixed point. There is a pole at μ_{pole}:

$$\frac{\partial\lambda}{\partial\mu} > 0 \qquad \mu_{\text{pole}} = \mu_0\left(1 - \frac{b}{a\lambda^2(\mu_0)}\right)^{-\frac{1}{2b}} \qquad \lim_{\mu\to 0}\lambda(\mu) = \lambda_\star \ . \tag{5.34}$$

The fact that we at least get a region of asymptotic freedom is particularly interesting. Of course we expect this region to be small and far away from any resonable values of λ, as $\sqrt{b}$ is essentially the ratio of the masses and the Planck scale. Nevetheless, asymptotic freedom of the interaction between two *massive* fields as a consequence of gravity is astonishing.

The region $\lambda(\mu_0) > \lambda_\star$ apparently resembles the non-gravitational behaviour. The main difference is that gravity prevents the coupling to exactly vanish in the infrared. The pole is slightly shifted

$$\mu_{\text{pole}} = \mu_0 \exp\left(\frac{1}{2a\lambda^2(\mu_0)}\right)\left(1 + \frac{b}{4}\frac{1}{a^2\lambda^4(\mu_0)} + \mathcal{O}(b^2)\right) . \tag{5.35}$$

The differences to the absence of gravity are controlled by the small value of b and therefore are probably out of experimental reach.

Let us come back to the Wilsonian running of the coupling (5.30). If we write it down in the form

$$\lambda^2(\mu) = \frac{\lambda^2(\mu_0)}{\left(\frac{\mu}{\mu_0}\right)^{2b} \exp(-c\mu^2)\left[\exp(c\mu_0^2) - 2a\lambda^2(\mu_0)\mu_0^{2b} \int\limits_{\mu_0}^{\mu} \mathrm{d}k\, k^{-2b-1} \exp(ck^2)\right]} , \tag{5.36}$$

then it is obvious that we have a pole determined by

$$\int\limits_{\mu_0}^{\mu_{\text{pole}}} \mathrm{d}k\, k^{-2b-1} \exp(ck^2) = \frac{\mu_0^{-2b} \exp(c\mu_0^2)}{2a\lambda^2(\mu_0)} . \tag{5.37}$$

Applying the rule of l'Hôspital we find that

$$\lim_{\mu \to 0} \mu^{2b} \int\limits_{\mu_0}^{\mu} \mathrm{d}k\, k^{-2b-1} \exp(ck^2) = -\frac{1}{2b} \tag{5.38}$$

and hence

$$\lim_{\mu \to 0} \lambda(\mu) = \lambda_\star = \sqrt{\frac{b}{a}} . \tag{5.39}$$

Again the coupling does not vanish in the infrared if at least one of the masses is nonzero. For the massless case ($b = 0$) and $\mu \to 0$ the Yukawa coupling (5.30) tends to zero as

$$\lambda^2 \quad \stackrel{\mu \to 0}{\longrightarrow} \quad \frac{\lambda^2(\mu_0)}{\exp\left(c\mu_0^2\right) + 2a\lambda^2(\mu_0) \int\limits_0^{\mu_0} \frac{\mathrm{d}k}{k}(\exp(ck^2) - 1) - a\lambda^2(\mu_0) \ln \frac{\mu^2}{\mu_0^2}} \tag{5.40}$$

Interesting is the small region $\lambda^2(\mu) < \lambda_\star^2 - \frac{c}{a}\mu^2$ because λ has negative slope there. If we start at $\lambda^2(\mu_0) < \lambda_\star^2 - \frac{c}{a}\mu_0^2$, then the coupling decreases with increasing energy at most until $\mu = \sqrt{b/c}$, which is of order of the masses. If one of the masses would be bigger than any energy accessable in experiments, this would look like asymptotic freedom.

5.2 φ^4 Theory

One of the simplest interacting field theories is φ^4 theory. The most important particle physics example of a φ^4 interaction in a real world model is the self interaction of the Higgs field in standard electroweak theory.

We are now going to investigate the gravitational contributions to the running of the φ^4 coupling. The starting point is the Lagrangian of a massive, real scalar field with a φ^4 selfinteraction term, minimally coupled to gravity:

$$\mathcal{L} = \tfrac{2}{\kappa^2}\sqrt{-\boldsymbol{g}}\boldsymbol{R} + \sqrt{-\boldsymbol{g}}\left(\tfrac{1}{2}\boldsymbol{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}m_\varphi^2\varphi^2 - \tfrac{\lambda}{4!}\varphi^4\right) \ . \tag{5.41}$$

As several times before we expand the Lagrangian around flat space. Beside the Feynman rules including two scalars (5.2) and (5.3), we need the Feynman rules for the four-scalar one- and two-graviton vertices to calculate the diagrams of figure 5.2

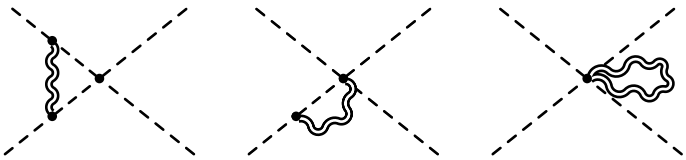
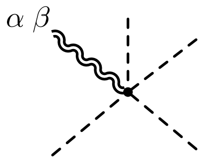
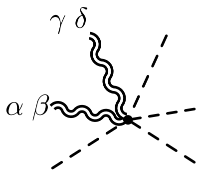


Figure 5.2: Order $\lambda\kappa^2$ diagrams for the proper four-scalar vertex, which not only differ in a permutation of outer legs.



$$= -i\lambda\tfrac{\kappa}{2}\eta^{\alpha\beta} \tag{5.42}$$

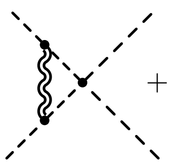


$$= i\lambda\tfrac{\kappa^2}{2}\left[\eta^{\alpha(\gamma}\eta^{\delta)\beta} - \tfrac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}\right] \ . \tag{5.43}$$

This time there is only one possible higher-derivative counterterm:

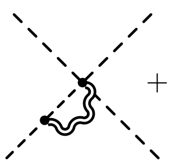
$$\varphi^3\partial^2\varphi = -3\varphi^2(\partial_\mu\varphi)^2 + \text{t.d.} \tag{5.44}$$

The divergent order $\lambda\kappa^2$ contribution to the proper φ^4-vertex is given by



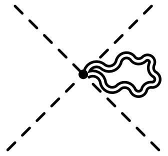
+ 5 diagrams corresponding to permutations of (p_1, p_2, p_3, p_4)

$$= \frac{i}{16\pi^2}\lambda\kappa^2\tfrac{1}{4}\left(\sum p_i^2\right)\ln\tfrac{\Lambda^2}{\mu^2} \tag{5.45}$$

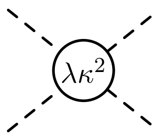


+ 3 diagrams corresponding to permutations of (p_1, p_2, p_3, p_4)

$$= \frac{i}{16\pi^2}\lambda\kappa^2\Big[-4m_\varphi + \tfrac{1}{4}\left(\sum p_i^2\right)\Big]\ln\tfrac{\Lambda^2}{\mu^2} \tag{5.46}$$

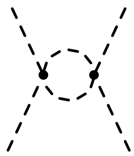
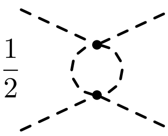
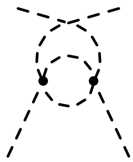
$\frac{1}{2}$  $= \frac{i}{16\pi^2}\lambda\kappa^2 \frac{5}{2}\left(\Lambda^2 - \mu^2\right)\ .$ (5.47)

These one-loop diagrams add up to



$$\begin{aligned} &= \frac{i}{16\pi^2}\lambda\kappa^2\left[\tfrac{5}{2}\left(\Lambda^2-\mu^2\right) - 4m_\varphi^2 \ln\tfrac{\Lambda^2}{\mu^2} + \tfrac{1}{2}\left(\sum p_i^2\right)\ln\tfrac{\Lambda^2}{\mu^2}\right] \\ &= \frac{i}{16\pi^2}\lambda\kappa^2\left[\tfrac{5}{2}\left(\Lambda^2-\mu^2\right) - 4m_\varphi^2 \ln\tfrac{\Lambda^2}{\mu^2} - \left(\sum_{i<j} p_i p_j\right)\ln\tfrac{\Lambda^2}{\mu^2}\right]\ . \end{aligned} \tag{5.48}$$

The order λ contribution to the vertex renormalization factor is determined by the diagrams:

$\frac{1}{2}$  $+\frac{1}{2}$  $+\frac{1}{2}$  $= \frac{i}{16\pi^2}\frac{3}{2}\lambda^2 \ln\frac{\Lambda^2}{\mu^2}$ (5.49)

There is no order λ contribution to Z_φ, because there is only the tadpole diagram. We obtain the Z factors

$$Z_{\varphi^4} - 1 = \frac{\lambda}{16\pi^2}\frac{3}{2}\ln\frac{\Lambda^2}{\mu^2} + \frac{\kappa^2}{16\pi^2}\frac{5}{2}\left(\Lambda^2-\mu^2\right) - \frac{\kappa^2}{16\pi^2}4m_\phi^2\ln\frac{\Lambda^2}{\mu^2} \tag{5.50}$$

$$Z_\varphi - 1 = \frac{\kappa^2}{16\pi^2}\frac{1}{2}(\Lambda^2-\mu^2) - \frac{\kappa^2}{16\pi^2}m_\phi^2\ln\frac{\Lambda^2}{\mu^2}\ . \tag{5.51}$$

The bare coupling is defined by

$$\lambda_0 = Z_{\varphi^4}Z_\varphi^{-2}\lambda\ . \tag{5.52}$$

Differentiating with respect to μ and neglecting the μ dependence of κ we arrive, analogous to Section 5.1, at the following expression for the β function

$$\beta_\lambda = -\lambda\frac{\mu\frac{\partial}{\partial\mu}Z_{\varphi^4}Z_\varphi^{-2}}{\frac{\partial}{\partial\lambda}(\lambda Z_{\varphi^4}Z_\varphi^{-2})} \tag{5.53}$$

which at one-loop simplifies to

$$\begin{aligned} \beta_\lambda &= -\lambda\mu\frac{\partial}{\partial\mu}\left(Z_{\varphi^4}Z_\varphi^{-2}\right) \\ &= \underbrace{\frac{3}{16\pi^2}}_{=:a}\lambda^2 - \underbrace{\frac{1}{4\pi^2}\kappa^2 m_\phi^2}_{=:b}\lambda + \underbrace{\frac{3\kappa^2}{16\pi^2}}_{=:c}\lambda\mu^2\ . \end{aligned} \tag{5.54}$$

By the same reasoning as in Section 5.1 we treat a, b and c as constants in the following investigation. Rewriting this differential equation as a one-form

$$0 = -\frac{1}{\lambda^2}\mathrm{d}\lambda + \frac{a}{\mu}\mathrm{d}\mu - \frac{b}{\mu\lambda}\mathrm{d}\mu + \frac{c}{\lambda}\mu\mathrm{d}\mu\ , \tag{5.55}$$

we find the integrating factor

$$\frac{\mathrm{d}f}{\mathrm{d}\mu} = \left(c\mu - \tfrac{b}{\mu}\right)f \qquad \Rightarrow \qquad f = \left(\tfrac{\mu}{\mu_0}\right)^{-b} \exp\left(\tfrac{c}{2}\mu^2\right) \tag{5.56}$$

and thus obtain the solution

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{\left(\frac{\mu}{\mu_0}\right)^b \exp[\frac{c}{2}(\mu_0^2 - \mu^2)] - a\lambda(\mu_0)\mu^b \int\limits_{\mu_0}^{\mu} \mathrm{d}k\; k^{-b-1} \exp[\frac{c}{2}(k^2 - \mu^2)]} \;. \tag{5.57}$$

This has to be compared to

$$\begin{aligned} &\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - a\lambda(\mu_0)\ln\frac{\mu}{\mu_0}} \qquad \text{with} \\ \frac{\partial\lambda}{\partial\mu} > 0\;, \qquad &\mu_{\mathrm{pole}} = \mu_0 \exp\left(\frac{1}{a\lambda(\mu_0)}\right) \qquad \text{and} \qquad \lim_{\mu\to 0}\lambda = 0\;. \end{aligned} \tag{5.58}$$

in the absence of gravity. Let us begin with the examination of the result in dimensional regularization ($c = 0$):

$$\lambda = \frac{\lambda(\mu_0)}{\left(\frac{\mu}{\mu_0}\right)^b\left(1 - \lambda(\mu_0)\frac{a}{b}\right) + \lambda(\mu_0)\frac{a}{b}} \tag{5.59}$$

Similar to the Yukawa coupling we have two regions of distinct behavior, seperated by the fixpoint $\lambda_\star = \frac{b}{a}$.

- $\boldsymbol{0 < \lambda(\mu_0) < \lambda_\star}$

  In this region we get asymptotic freedom and λ approches the fixpoint when μ tends to zero:

$$\frac{\partial\lambda}{\partial\mu} < 0 \qquad \lim_{\mu\to\infty}\lambda(\mu) = 0 \qquad \lim_{\mu\to 0}\lambda(\mu) = \lambda_\star\;. \tag{5.60}$$

- $\boldsymbol{\lambda(\mu_0) > \lambda_\star}$

  In this region λ grows with energy and is bounded from below by the fixed point. There is a pole at μ_{pole}.

$$\frac{\partial\lambda}{\partial\mu} > 0 \qquad \mu_{\mathrm{pole}} = \mu_0\left(1 - \frac{b}{a\lambda(\mu_0)}\right)^{-\frac{1}{b}} \qquad \lim_{\mu\to 0}\lambda(\mu) = \lambda_\star \tag{5.61}$$

Astonishingly, we see again that gravity can lead to asymptotic freedom in the *massive* case. The region of asymptotic freedom is very small since $\lambda_\star$ is essentially the ratio of the mass and the Planck scale.

Above the fixed point we get allmost the non-gravitational behaviour. Similar to the Yukawa interaction gravity prevents the coupling to exactly vanish in the infrared and the pole is slightly shifted

$$\mu_{\mathrm{pole}} = \mu_0 \exp\left(\frac{1}{a\lambda(\mu_0)}\right)\left(1 + \frac{b}{2}\frac{1}{a^2\lambda^2(\mu_0)} + \mathcal{O}(b^2)\right)\;. \tag{5.62}$$

The differences to the absence of gravity are controlled by the small value of b and therefore are probably out of experimental reach.

Let us come back to the Wilsonian running of the coupling (5.57). Writing the coupling down in following form

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{\left(\frac{\mu}{\mu_0}\right)^b \exp(-\frac{c}{2}\mu^2)\left[\exp(\frac{c}{2}\mu_0^2) - a\lambda(\mu_0)\mu_0^b \int\limits_{\mu_0}^{\mu} \mathrm{d}k\; k^{-b-1}\exp(\frac{c}{2}k^2)\right]}\,, \tag{5.63}$$

it is obvious that we have a pole determined by

$$\int\limits_{\mu_0}^{\mu_{\text{pole}}} \mathrm{d}k\; k^{-b-1}\exp(\tfrac{c}{2}k^2) = \frac{\mu_0^{-b}\exp(\frac{c}{2}\mu_0^2)}{a\lambda(\mu_0)}\,. \tag{5.64}$$

Applying the rule of l'Hôspital we find that

$$\lim_{\mu\to 0}\mu^b \int\limits_{\mu_0}^{\mu} \mathrm{d}k\; k^{-b-1}\exp(\tfrac{c}{2}k^2) = -\frac{1}{b} \tag{5.65}$$

and hence

$$\lim_{\mu\to 0}\lambda(\mu) = \lambda_\star = \frac{b}{a}\,. \tag{5.66}$$

Again the coupling does not vanish in the infrared if the mass is nonzero. For the massless case ($b = 0$) and $\mu \to 0$ the Yukawa coupling (5.30) tends to zero as

$$\lambda \quad \overset{\mu\to 0}{\longrightarrow} \quad \frac{\lambda(\mu_0)}{\exp\left(\frac{c}{2}\mu_0^2\right) + a\lambda(\mu_0)\int\limits_0^{\mu_0}\frac{\mathrm{d}k}{k}(\exp(\frac{c}{2}k^2) - 1) - a\lambda(\mu_0)\ln\frac{\mu}{\mu_0}} \tag{5.67}$$

Interesting is the small region $\lambda(\mu) < \lambda_\star - \frac{c}{a}\mu^2$, because λ has negative slope there. If we start at $\lambda(\mu_0) < \lambda_\star - \frac{c}{a}\mu_0^2$, then the coupling decreases with increasing energy at most until $\mu = \sqrt{b/c}$, which is of order of the mass of the scalar. If this mass would be bigger than any energy accessable in experiments, this would look like asymptotic freedom. On the other hand we have $\lambda < \lambda_\star$ in this region and this is probably to small to be measured.

Chapter 6
Summary and Conclusions

Motivated by the recently proposed Lee-Wick standard model [4] and the observation that Einstein Yang-Mills theory can be renormalized at one-loop order by adding the Lee-Wick term $d_2 \mathrm{Tr}\{(D_\mu F^{\mu\nu})^2\}$ to the Lagrangian, we investigated Lee-Wick gauge theory in Part I.

We established in Chapter 2 that the renormalization of Lee-Wick gauge theory is completely determined by the wavefunction renormalization of the gauge field, if a modified gauge fixing term is used.

Performing a diagrammatic calculation, we determined the one-loop β function and found agreement with the results of [28], obtained by applying the background field method. In accordance with the general statement of Coleman and Gross [45], that among renormalizable quantum field theories in four dimensions, only non-Abelian gauge theories are asymptotically free, we found asymptotic freedom for Lee-Wick gauge theory. The coupling runs approximately twice as fast as in Yang-Mills theory. Hence, bearing in mind that the higher-derivative gauge field corresponds to two degrees of freedom, this almost confirms the intuitive guess.

One might hope that the modified gauge field contribution to the running of the coupling results in improved unification properties of the Lee-Wick standard model, but according to [28] this is not the case.

Furthermore, we performed the limit to Yang-Mills theory, which equals sending the mass of the Lee-Wick field to infinity. Thus we obtained the well known Yang-Mills β function in agreement with the decoupling theorem [34].

In Chapter 3 we applied the background field method and reproduced all the previous results. In comparison we consider the diagrammatic calculation of Chapter 2 to be more appropriate for loop calculations in higher-derivative gauge theories.

Gravitational corrections, obtained in the framework of effective field theories, were the topic of Part II.

In Chapter 4 we investigated the one-loop divergences of Einstein Yang-Mills theory, to find out wether the fermionic and scalar dimension six counterterms coincide with the higher-derivative terms in the Lee-Wick standard model.

First we established that the gravitational contributions to the dimension four counterterms are related, which provided a consistency check for the following computations.

Because it was the motivation for our investigations we reperformed the calculation of [18], to show that the gauge field dimension six counterterm is given by the Lee-Wick term. But our results show that this is special to the gauge fields.

We determined the fermionic dimension six counterterm to be a linear combination of the four possible terms and found that there is no dimension six counterterm for scalars, thereby proving that there is *no* general connection between the higher-derivative counterterms and the Lee-Wick terms. This result is in agreement with the calculation of Rodigast [46], who investigated the same question in an extra-dimensional scenario.

Although the gauge field counterterm is given by the Lee-Wick term, it is important to notice that this is just one term in an infinite series of counterterms of a non-renormalizable field theory. One might argue that the higher order terms are exactly such that they correspond to further new particles, as in the higher-derivative Lee-Wick standard model recently proposed by Carone [47].

However, we do not think that it is in general possible to conclude the existence of further particles from the appearance of special counterterms in an effective field theory, because these terms are only the residual low-energy effects of the unknown physics at high energies.

In Chapter 5 we determined the gravitational contributions to the β functions of the Yukawa and the φ^4 coupling. We calculated the Wilsonian running couplings as well as the running of the couplings in dimensional regularization and showed how gravity modifies their behavior at energies well below the Planck scale.

We found that all gravitational corrections are probably undetectable because they are exceedingly small. However, this does not mean that they are unimportant because it underlines, that the exclusion of gravity from the standard model is not a poor substitute for a unknown quantum theory but rather an extreme good approximation.

Furthermore, the quantum corrections we found reveal interesting properties of the quantum theory. For non-zero *masses* gravity causes the couplings not to vanish in the infrared and leads to regions of negative slope. Extrapolating our results to high energies, we found that gravity can lead to asymptotic freedom of the Yukawa and φ^4 coupling if the fields are *massive*. Of course we expect the region of asymptotic freedom to be small and far away from any reasonable values of the couplings, as its size is controlled by the ratio of the masses and the Planck scale. Nevetheless, asymptotic freedom of the interaction between *massive* fields as a consequence of gravity is astonishing and worthy of further investigations.

Recently Toms showed in [48] that a non-vanishing positive cosmological constant leads to a similar contribution to the β function of quantum electrodynamics as we found for Yukawa and φ^4 theory.

The possibility, that quantum gravity renders a theory asymptotic free which can not be asymptotically free in four dimensions without gravity, is intriguing and should be investigated further.

One possible program to confirm our results is to use the gauge invariant and gauge condition independent Vilkovisky-DeWitt effective action [49, 50] to determine the quatum gravity contribution to the β functions of Yukawa and φ^4 theory.

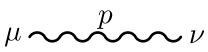
Appendix A
Feynman Rules of YM Theory

In Chapters 2 and 4 we need some of the Feynman rules of Yang-Mills theory with fermionic and scalar multiplets defined by the gauge fixed Lagrangian

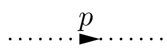
$$\mathcal{L}_{\mathrm{YM}} = \mathrm{tr}\{-\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} - \phi^\dagger(D^2 + m_\phi^2)\phi + \overline{\psi}(i\not{D} - m_\psi)\psi - \tfrac{1}{\alpha}(\partial\cdot A)^2 - 2\bar{c}\,\partial\cdot D\,c\}\ .$$

Indices of the group representations, as well as spinor indices are omitted whenever possible.

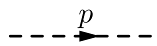
Propagators

gauge field 

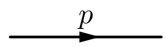
$$= \frac{-i}{p^2}\left(\eta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right)$$

ghost 

$$= \frac{i}{p^2}$$

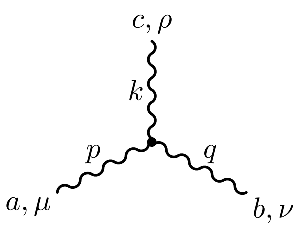
scalars 

$$= \frac{i}{p^2 - m_\phi^2}$$

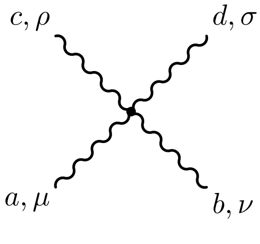
fermions 

$$= \frac{i}{\not{p} - m_\psi}$$

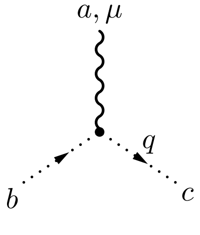
Vertices



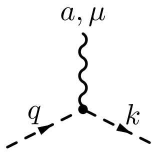
$$= gf^{abc}\big[\eta^{\mu\nu}(p^\rho - q^\rho) + \eta^{\mu\rho}(k^\nu - p^\nu) + \eta^{\nu\rho}(q^\mu - k^\mu)\big]$$



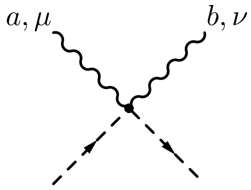
$$\begin{aligned} = \quad -ig^2\big[\ & f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ &+ f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\ &+ f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\big] \end{aligned}$$



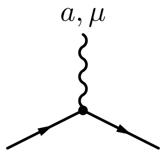
$$= -gf^{abc}q^\mu$$



$$= ig(q+k)^\mu t^a$$



$$= ig^2\eta^{\mu\nu}\{t^a, t^b\}$$



$$= ig\gamma^\mu$$

Appendix B
Cut-Off Integrals

In Section 2.6.1 we showed that in cut-off regularization logarithmically divergent integrals are shift invariant. This can be used to determine the correct asymptotics of arbitrary cut-off integrals. We give a complete list of one-loop integrals with two propogators up to divergence index of four and of three propagator integrals up to divergence index of three.

Logarithmically Divergent Integrals

We start with the easiest of the integrals:

$$D_0 = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2 + 2pk + D)^2} \; .$$

This logarithmic divergent integral can be calculated without difficulties. Shift the integration variable $k \to k + p$, Wick rotate and integrate over all Euclidean momenta fulfilling $k_\mathrm{E}^2 \leq \Lambda^2$. In all calculations y is defined by $y = k_\mathrm{E}^2$. We get

$$\begin{aligned} D_0 &= \frac{i}{16\pi^2} \int_0^{\Lambda^2} \mathrm{d}y \, \frac{y}{(y + p^2 - D)^2} \\ &= \frac{i}{16\pi^2} \left[\ln\left(\frac{\Lambda^2}{p^2 - D} \right) - 1 \right] \; . \end{aligned}$$

Linearly Divergent Integrals

Throughout the following calculations we will frequently use the simple formula:

$$\frac{1}{\alpha^n} - \frac{1}{\beta^n} = -\int_0^1 \mathrm{d}x \, \frac{n(\alpha - \beta)}{((\alpha - \beta)x + \beta)^{n+1}} \tag{B.1}$$

to combine fractions. Integrals of divergence index one have the general form

$$\begin{aligned} D_1^\mu &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^\mu}{(k^2 + 2pk + D)^2} \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \left[\frac{k^\mu}{(k^2 + 2pk + D)^2} - \overbrace{\frac{k^\mu}{(k^2 + D)^2}}^{=0} \right] \\ &\overset{(B.1)}{=} -4p_\nu \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + 2zpk + D)^3} \end{aligned}$$

The remaining momentum integral is only logarithmically divergent and we can shift $k \to k + zp$, perform a Wick rotation and integrate over the ball of radius Λ arround the origin.

$$\begin{aligned} D_1^\mu &= \frac{i}{16\pi^2} p_\nu \int_0^1 \mathrm{d}z \int_0^{\Lambda^2} \mathrm{d}y \, \frac{p^\mu p^\nu 4z^2 y - \eta^{\mu\nu} y^2}{(y + z^2p^2 - D)^3} \\ &= \frac{i}{16\pi^2} p^\mu \int_0^1 \mathrm{d}z \left\{ \frac{2z^2p^2}{(z^2p^2 - D)} - \ln\left(\frac{\Lambda^2}{p^2z^2 - D}\right) + \frac{3}{2} \right\} \\ &= \frac{i}{16\pi^2} p^\mu \left[\frac{3}{2} - \ln\left(\frac{\Lambda^2}{p^2 - D}\right) \right] \end{aligned}$$

Quadratically Divergent Integrals

All integrals of divergence index two have the form:

$$\begin{aligned} D_2^{\mu\nu} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + 2pk + D)^2} \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{k^\mu k^\nu}{(k^2 + 2pk + D)^2} - \frac{k^\mu k^\nu}{(k^2 + D)^2} \right] + \overbrace{\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + D)^2}}^{=A_2^{\mu\nu}(q)} \; . \end{aligned}$$

$A_2^{\mu\nu}$ can be calculated easily since its denominator is a function of k^2. We can make the replacement $k^\mu k^\nu \to \frac{1}{4}\eta^{\mu\nu}$ and get

$$A_2^{\mu\nu} = -\frac{i}{16\pi^2} \frac{\eta^{\mu\nu}}{4} \int_0^{\Lambda^2} \frac{y^2}{(y - D)^2} = \frac{i}{16\pi^2} \eta^{\mu\nu} \left[-\frac{\Lambda^2}{4} - \frac{D}{2} \ln\left(-\frac{\Lambda^2}{D}\right) + \frac{D}{4} \right] \; .$$

For the sake of simplicity we will use the condensed notation $\prod_{i=1}^{n} k^{\mu_i} := k^{\mu_1 \cdots \mu_n}$ if n is greater than three. Let us continue with:

$$\begin{aligned} D_2^{\mu\nu} - A_2^{\mu\nu} &= -4p_\sigma \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{k^\mu k^\nu k^\sigma}{(k^2 + 2zpk + D)^3} - \frac{k^\mu k^\nu k^\sigma}{(k^2 + D)^3} \right] \\ &= 24 p_\sigma p_\rho \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{z k^{\mu\nu\sigma\rho}}{(k^2 + 2zwpk + D)^4} \\ &= 24 p_\sigma p_\rho \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int \frac{\mathrm{d}^4l}{(2\pi)^4} z \frac{l^{\mu\nu\sigma\rho} + 6z^2w^2 p^{(\mu} p^\nu l^\sigma l^{\rho)} + z^4 w^4 p^{\mu\nu\sigma\rho}}{(l^2 - z^2w^2p^2 + D)^4} \; . \end{aligned}$$

Again due to symmetry we can make the replacement $l^{\mu\nu\rho\sigma} \to C\eta^{(\mu\nu}\eta^{\rho\sigma)} l^4$. The constant C can be determined by contracting all indices. To be more explicit we make the replacement $l^{\mu\nu\rho\sigma} \to \frac{1}{24}(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})l^4$. Hence we get

$$
\begin{aligned}
&D_2^{\mu\nu} - A_2^{\mu\nu} \\
&= \frac{i24p_\sigma p_\rho}{16\pi^2}\int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^{\Lambda^2} \mathrm{d}y\, z \frac{\frac{1}{8}\eta^{(\mu\nu}\eta^{\rho\sigma)}y^3 - z^2w^2\frac{3}{2}p^{(\mu}p^\nu\eta^{\rho\sigma)}y^2 + z^4w^4p^{\mu\nu\rho\sigma}y}{(y+z^2w^2p^2-D)^4} \\
&= \frac{i}{16\pi^2}\int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w\, z\Bigg[\eta^{\mu\nu}p^2\left\{\ln\left(\frac{\Lambda^2}{z^2w^2p^2-D}\right) - \frac{2z^2w^2p^2}{z^2w^2p^2-D} - \frac{11}{6}\right\} \\
&\quad +p^\mu p^\nu\left\{2\ln\left(\frac{\Lambda^2}{z^2w^2p^2-D}\right) + \frac{4z^4w^4p^4}{(z^2w^2p^2-D)^2} - \frac{10z^2w^2p^2}{z^2w^2p^2-D} - \frac{11}{3}\right\}\Bigg] \\
&= \frac{i}{16\pi^2}\int_0^1 \mathrm{d}z\, z\Bigg[\eta^{\mu\nu}p^2\left\{\ln\left(\frac{\Lambda^2}{z^2p^2-D}\right) - \frac{11}{6}\right\} \\
&\qquad + p^\mu p^\nu\left\{2\ln\left(\frac{\Lambda^2}{z^2p^2-D}\right) - \frac{2D}{z^2p^2-D} - \frac{17}{3}\right\}\Bigg] \\
&= \frac{i}{16\pi^2}\Bigg[\eta^{\mu\nu}p^2\left\{\frac{1}{2}\ln\left(\frac{\Lambda^2}{p^2-D}\right) + \frac{D}{2p^2}\ln\left(\frac{p^2-D}{-D}\right) - \frac{5}{12}\right\} \\
&\qquad + p^\mu p^\nu\left\{\ln\left(\frac{\Lambda^2}{p^2-D}\right) - \frac{11}{6}\right\}\Bigg] .
\end{aligned}
$$

The result is

$$
\begin{aligned}
\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^\mu k^\nu}{(k^2+2pk+D)^2} = \frac{i}{16\pi^2}\Bigg[&\eta^{\mu\nu}\bigg\{-\frac{\Lambda^2}{4} + \frac{1}{2}(p^2-D)\ln\left(\frac{\Lambda^2}{p^2-D}\right) \\
&\quad - \frac{5}{12}p^2 + \frac{D}{4}\bigg\} \\
&+ p^\mu p^\nu\left\{\ln\left(\frac{\Lambda^2}{p^2-D}\right) - \frac{11}{6}\right\}\Bigg] .
\end{aligned}
$$

Cubically Divergent Integrals

Integrals of divergence index three have the general form

$$
\begin{aligned}
D_3^{\mu\nu\rho} &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^\mu k^\nu k^\rho}{(k^2+2pk+D)^2} \\
&= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{k^\mu k^\nu k^\rho}{(k^2+2pk+D)^2} - \frac{k^\mu k^\nu k^\rho}{(k^2+D)^2}\right] \\
&= -4p_\sigma\int_0^1 \mathrm{d}z\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{k^{\mu\nu\rho\sigma}}{(k^2+2zpk+D)^3} - \frac{k^{\mu\nu\rho\sigma}}{(k^2+D)^3}\right] + A_3^{\mu\nu\rho} .
\end{aligned}
$$

Here $A_3^{\mu\nu\rho}$ is given by

$$
A_3^{\mu\nu\rho} = 4p_\sigma\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-k^{\mu\nu\rho\sigma}}{(k^2+D)^3} = \frac{i}{16\pi^2}p_\sigma\frac{1}{2}\eta^{(\mu\nu}\eta^{\rho\sigma)}\int_0^{\Lambda^2}\mathrm{d}y\frac{y^3}{(y-D)^3}
$$

$$
\begin{aligned}
A_3^{\mu\nu\rho} &= \frac{i}{16\pi^2} q_\sigma \frac{1}{2} \eta^{(\mu\nu}\eta^{\rho\sigma)} \left[\Lambda^2 + 3D\ln\left(\frac{\Lambda^2}{-D}\right) - \frac{5}{2}D\right] \\
&= \frac{i}{16\pi^2}\left(\eta^{\mu\nu}p^\rho + \eta^{\mu\rho}p^\nu + \eta^{\nu\rho}p^\mu\right)\left[\frac{\Lambda^2}{6} + \frac{D}{2}\ln\left(\frac{\Lambda^2}{-D}\right) - \frac{5}{12}D\right] .
\end{aligned}
$$

We go on with the computation of

$$
\begin{aligned}
D_3^{\mu\nu\rho} - A_3^{\mu\nu\rho} &= 24 p_\sigma p_\alpha \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{z k^{\mu\nu\rho\sigma\alpha}}{(k^2+2zwpk+D)^4} - \frac{z k^{\mu\nu\rho\sigma\alpha}}{(k^2+D)^4}\right] \\
&= -192 p_\sigma p_\alpha p_\beta \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{z^2 w k^{\mu\nu\rho\sigma\alpha\beta}}{(k^2+2zwvpk+D)^5} \\
&= -192 p_\sigma p_\alpha p_\beta \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^4l}{(2\pi)^4} z^2 w \times \\
&\quad \times \left[\frac{l^{\mu\nu\rho\sigma\alpha\beta} + 15(zwv)^2 p^{(\mu}p^\nu l^{\rho\sigma\alpha\beta)}}{(l^2-(zwv)^2p^2+D)^5}\right. \\
&\quad\quad \left. + \frac{15(zwv)^4 l^{(\mu}l^\nu p^{\rho\sigma\alpha\beta)} + (zwv)^6 p^{\mu\nu\rho\sigma\alpha\beta}}{(l^2-(zwv)^2p^2+D)^5}\right] .
\end{aligned}
$$

Now we can make the replacement $l^{\mu\nu\rho\sigma\alpha\beta} \to \frac{5}{64}\eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\beta)}$ due to symmetry. To ceep it clear we define the tensors:

$$
\begin{aligned}
I_1^{\mu\nu\rho} &:= \frac{5}{64} p_\sigma p_\alpha p_\beta \eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\beta)} = \frac{1}{192}\left[3p^2\left(\eta^{\mu\nu}p^\rho + \eta^{\mu\rho}p^\nu + \eta^{\nu\rho}p^\mu\right) + 6p^\mu p^\nu p^\rho\right] \\
I_2^{\mu\nu\rho} &:= \frac{15}{8} p_\sigma p_\alpha p_\beta p^{(\mu}p^\nu\eta^{(\rho\sigma}\eta^{\alpha\beta))} = \frac{p^2}{24}\left[6p^2\left(\eta^{\mu\nu}p^\rho + \eta^{\mu\rho}p^\nu + \eta^{\nu\rho}p^\mu\right) + 27p^\mu p^\nu p^\rho\right] \\
I_3^{\mu\nu\rho} &:= \frac{15}{4} p_\sigma p_\alpha p_\beta p^{(\mu}p^\nu p^\rho p^\sigma \eta^{\alpha\beta)} = \frac{p^4}{4}\left[p^2\left(\eta^{\mu\nu}p^\rho + \eta^{\mu\rho}p^\nu + \eta^{\nu\rho}p^\mu\right) + 12p^\mu p^\nu p^\rho\right] .
\end{aligned}
$$

With this convention we get

$$
\begin{aligned}
&D_3^{\mu\nu\rho} - A_3^{\mu\nu\rho} \\
&= \frac{i}{16\pi^2} 192 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v \int_0^{\Lambda^2} \mathrm{d}y\, z^2 w \times \\
&\quad \times \frac{-I_1^{\mu\nu\rho} y^4 + z^2w^2v^2 I_2^{\mu\nu\rho} y^3 - z^4w^4v^4 I_3^{\mu\nu\rho} y^2 + z^6w^6v^6 p^{\mu\nu\rho} p^6 y}{(y + z^2w^2v^2p^2 - D)^5} \\
&= \frac{i}{16\pi^2} 192 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v\, z^2 w \left[I_1^{\mu\nu\rho}\left\{-\ln\left(\frac{\Lambda^2}{z^2w^2v^2p^2 - D}\right) + \frac{25}{12}\right\}\right. \\
&\quad \left. + \frac{I_2^{\mu\nu\rho}}{4}\frac{z^2w^2v^2}{z^2w^2v^2p^2 - D} - \frac{I_3^{\mu\nu\rho}}{12}\frac{z^4w^4v^4}{(z^2w^2v^2p^2 - D)^2} + \frac{p^{\mu\nu\rho}}{12}\frac{z^6w^6v^6p^6}{(z^2w^2v^2p^2 - D)^3}\right]
\end{aligned}
$$

$$
\begin{aligned}
&D_3^{\mu\nu\rho} - A_3^{\mu\nu\rho} \\
&= \frac{i}{16\pi^2} \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v \; z^2 w \\
&\quad \times \Bigg[3\eta^{(\mu\nu} p^{\rho)} p^2 \Bigg\{ -3 \ln\left(\frac{\Lambda^2}{z^2w^2v^2p^2 - D} \right) + 12 \frac{z^2w^2v^2p^2}{z^2w^2v^2p^2 - D} \\
&\qquad - 4 \frac{z^4w^4v^4}{(z^2w^2v^2p^2 - D)^2} + \frac{25}{4} \Bigg\} \\
&\qquad + p^\mu p^\nu p^\rho \Bigg\{ -6 \ln\left(\frac{\Lambda^2}{z^2w^2v^2p^2 - D} \right) + \frac{54 z^2w^2v^2p^2}{z^2w^2v^2p^2 - D} \\
&\qquad - \frac{48 z^4w^4v^4}{(z^2w^2v^2p^2 - D)^2} + \frac{16 z^6w^6v^6p^6}{(z^2w^2v^2p^2 - D)^3} + \frac{25}{2} \Bigg\} \Bigg] \\
&= \frac{i}{16\pi^2} \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \; z^2 w \Bigg[3\eta^{(\mu\nu} p^{\rho)} p^2 \Bigg\{ -3 \ln\left(\frac{\Lambda^2}{z^2w^2p^2 - D} \right) \\
&\qquad\qquad + \frac{2D}{z^2w^2p^2 - D} + \frac{33}{4} \Bigg\} \\
&\quad - p^\mu p^\nu p^\rho \left\{ 6 \ln\left(\frac{\Lambda^2}{z^2w^2p^2 - D} \right) - \frac{6D}{z^2w^2p^2 - D} + \frac{4D^2}{(z^2w^2p^2 - D)^2} - \frac{45}{2} \right\} \Bigg] \\
&= \frac{i}{16\pi^2} \int_0^1 \mathrm{d}z \; z^2 \Bigg[3\eta^{(\mu\nu} p^{\rho)} p^2 \left\{ -\frac{3}{2} \ln\left(\frac{\Lambda^2}{z^2p^2 - D} \right) - \frac{D}{2p^2} \ln\left(\frac{z^2p^2 - D}{-D} \right) + \frac{21}{8} \right\} \\
&\quad + p^\mu p^\nu p^\rho \left\{ -3 \ln\left(\frac{\Lambda^2}{z^2p^2 - D} \right) + \frac{2D}{z^2w^2p^2 - D} + \frac{33}{4} \right\} \Bigg] \\
&= \frac{i}{16\pi^2} \Bigg[3\eta^{(\mu\nu} p^{\rho)} p^2 \left\{ -\frac{1}{2} \ln\left(\frac{\Lambda^2}{p^2 - D} \right) - \frac{D}{2p^2} \ln\left(\frac{p^2 - D}{-D} \right) + \frac{13}{24} \right\} \\
&\qquad + p^\mu p^\nu p^\rho \left\{ -\ln\left(\frac{\Lambda^2}{p^2 - D} \right) + \frac{25}{12} \right\} \Bigg]
\end{aligned}
$$

The result we obtain is

$$
\begin{aligned}
\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho}{(k^2 + 2pk + D)^2} = \frac{i}{16\pi^2} \Bigg[& 3\eta^{(\mu\nu} p^{\rho)} \Bigg\{ \frac{\Lambda^2}{6} - \frac{1}{2}(p^2 - D) \ln\left(\frac{\Lambda^2}{p^2 - D} \right) \\
& + \frac{13}{24} p^2 - \frac{5}{12} D \Bigg\} \\
& - p^\mu p^\nu p^\rho \left\{ \ln\left(\frac{\Lambda^2}{p^2 - D} \right) - \frac{25}{12} \right\} \Bigg] \, .
\end{aligned}
$$

Quartically Divergent Integrals

The general form of an integral of divergence index four is

$$
D_4^{\mu\nu\rho\sigma} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2 + 2pk + D)^2} \, .
$$

$$D_4^{\mu\nu\rho\sigma} = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{k^{\mu\nu\rho\sigma}}{(k^2+2pk+D)^2} - \frac{k^{\mu\nu\rho\sigma}}{(k^2+D)^2} \right] + A_4^{\mu\nu\rho\sigma}$$

Here $A_4^{\mu\nu\rho\sigma}$ is given by

$$\begin{aligned} A_4^{\mu\nu\rho\sigma} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^{\mu\nu\rho\sigma}}{(k^2+D)^2} = \frac{i}{16\pi^2} \frac{1}{8} \eta^{(\mu\nu}\eta^{\rho\sigma)} \int_0^{\Lambda^2} \mathrm{d}y \frac{y^3}{(y-D)^2} \\ &= \frac{i}{16\pi^2} 3\eta^{(\mu\nu}\eta^{\rho\sigma)} \left\{ \frac{\Lambda^4}{48} + \frac{D}{12}\Lambda^2 + \frac{D^2}{8} \ln\left(\frac{\Lambda^2}{-D}\right) - \frac{D^2}{24} \right\} \end{aligned}$$

and we go on with

$$\begin{aligned} &D_4^{\mu\nu\rho\sigma} - A_4^{\mu\nu\rho\sigma} \\ &= -4p_\kappa \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{k^{\mu\nu\rho\sigma\kappa}}{(k^2+2xpk+D)^3} - \frac{k^{\mu\nu\rho\sigma\kappa}}{(k^2+D)^3} \right] \\ &= 24p_\kappa p_\alpha \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{xk^{\mu\nu\rho\sigma\kappa\alpha}}{(k^2+2xzpk+D)^4} - \frac{xk^{\mu\nu\rho\sigma\kappa\alpha}}{(k^2+D)^4} \right] + B_4^{\mu\nu\rho\sigma} , \end{aligned}$$

where $B_4^{\mu\nu\rho\sigma}$ is given

$$\begin{aligned} B_4^{\mu\nu\rho\sigma} &= 12p_\kappa p_\alpha \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^{\mu\nu\rho\sigma\kappa\alpha}}{(k^2+D)^4} \\ &= \frac{i}{16\pi^2} \frac{1}{16} \left(3\eta^{(\mu\nu}\eta^{\rho\sigma)}p^2 + 12p^{(\mu}p^{\nu}\eta^{\rho\sigma)} \right) \int_0^{\Lambda^2} \mathrm{d}y \frac{-y^4}{(y-D)^4} \\ &= \frac{i}{16\pi^2} \left(3\eta^{(\mu\nu}\eta^{\rho\sigma)}p^2 + 12p^{(\mu}p^{\nu}\eta^{\rho\sigma)} \right) \left[-\frac{\Lambda^2}{16} - \frac{D}{4} \ln\left(\frac{\Lambda^2}{-D}\right) + \frac{13}{48}D \right] . \end{aligned}$$

It remains to compute

$$\begin{aligned} &D_4^{\mu\nu\rho\sigma} - A_4^{\mu\nu\rho\sigma} - B_4^{\mu\nu\rho\sigma} \\ &= 192p_\kappa p_\alpha p_\beta \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left[\frac{x^2zk^{\mu\nu\rho\sigma\kappa\alpha\beta}}{(k^2+2xzwpk+D)^5} - \frac{x^2zk^{\mu\nu\rho\sigma\kappa\alpha\beta}}{(k^2+D)^5} \right] \\ &= 1920p_{\kappa\alpha\beta\gamma} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{x^3z^2wk^{\mu\nu\rho\sigma\kappa\alpha\beta\gamma}}{(k^2+2xzwvpk+D)^6} \\ &= 1920p_{\kappa\alpha\beta\gamma} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v \int \frac{\mathrm{d}^4k}{(2\pi)^4} x^3z^2w \\ &\quad \times \left[\frac{l^{\mu\nu\rho\sigma\kappa\alpha\beta\gamma} + 28x^2z^2w^2v^2p^{(\mu}p^{\nu}l^{\rho\sigma\kappa\alpha\beta\gamma)} + 70x^4z^4w^4v^4p^{(\mu\nu\rho\sigma}l^{\kappa\alpha\beta\gamma)}}{(l^2-x^2z^2w^2v^2p^2+D)^6} \right. \\ &\qquad \left. + \frac{28x^6z^6w^6v^6l^{(\mu}l^{\nu}p^{\rho\sigma\kappa\alpha\beta\gamma)} + x^8z^8w^8v^8p^{\mu\nu\rho\sigma\kappa\alpha\beta\gamma}}{(l^2-x^2z^2w^2v^2p^2+D)^6} \right] \end{aligned}$$

With the definitions

$$I_1^{\mu\nu\rho\sigma} := \frac{7}{128}\, p_{\kappa\alpha\beta\gamma}\eta^{(\mu\nu}\eta^{\rho\sigma}\eta^{\kappa\alpha}\eta^{\beta\gamma)} \qquad I_3^{\mu\nu\rho\sigma} := \frac{35}{4}\, p_{\kappa\alpha\beta\gamma}p^{(\mu\nu\rho\sigma}\eta^{(\kappa\alpha}\eta^{\beta\gamma))}$$

and

$$I_2^{\mu\nu\rho\sigma} := \frac{35}{16}\, p_{\kappa\alpha\beta\gamma}p^{(\mu}p^{\nu}\eta^{(\rho\sigma}\eta^{\kappa\alpha}\eta^{\beta\gamma))} \qquad I_4^{\mu\nu\rho\sigma} := 7\, p_{\kappa\alpha\beta\gamma}p^{(\mu\nu\rho\sigma\kappa\alpha}\eta^{\beta\gamma)}$$

we get

$$\begin{aligned}
&D_4^{\mu\nu\rho\sigma} - A_4^{\mu\nu\rho\sigma} - B_4^{\mu\nu\rho\sigma} \\
&= \frac{i}{16\pi^2} 1920 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v \int_0^{\Lambda^2} \mathrm{d}y\; x^3 z^2 w \times \\
&\quad \times \Bigg[\frac{I_1^{\mu\nu\rho\sigma} y^5 - x^2z^2w^2v^2 I_2^{\mu\nu\rho\sigma} y^4 + x^4z^4w^4v^4 I_3^{\mu\nu\rho\sigma} y^3}{(y + x^2z^2w^2v^2p^2 - D)^6} \\
&\qquad + \frac{-x^6z^6w^6v^6 I_4^{\mu\nu\rho\sigma} y^2 + x^8z^8w^8v^8 p^{\mu\nu\rho\sigma} p^8 y}{(y + x^2z^2w^2v^2p^2 - D)^6} \\
&= \frac{i}{16\pi^2} 1920 \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v\; x^3 z^2 w \times \\
&\quad \times \Bigg[I_1^{\mu\nu\rho\sigma} \left(\ln\left(\frac{\Lambda^2}{x^2z^2w^2v^2p^2 - D} \right) - \frac{137}{60} \right) \\
&\qquad - \frac{1}{5} I_2^{\mu\nu\rho\sigma} \frac{x^2z^2w^2v^2}{x^2z^2w^2v^2p^2 - L} + \frac{1}{20} I_3^{\mu\nu\rho\sigma} \frac{x^4z^4w^4v^4}{(x^2z^2w^2v^2p^2 - L)^2} \\
&\qquad - \frac{1}{30} I_4^{\mu\nu\rho\sigma} \frac{x^6z^6w^6v^6}{(x^2z^2w^2v^2p^2 - L)^3} + \frac{1}{20} p^{\mu\nu\rho\sigma} p^8 \frac{x^8z^8w^8v^8}{(x^2z^2w^2v^2p^2 - L)^4} \Bigg] \\
&= \frac{i}{16\pi^2} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}w \int_0^1 \mathrm{d}v\; x^3 z^2 w \times \\
&\quad \times \Bigg[3p^4 \eta^{(\mu\nu}\eta^{\rho\sigma)} \Bigg\{ 3 \ln\left(\frac{\Lambda^2}{x^2z^2w^2v^2p^2 - D} \right) - 12 \frac{x^2z^2w^2v^2p^2}{x^2z^2w^2v^2p^2 - L} \\
&\qquad\qquad + 4 \frac{x^4z^4w^4v^4p^4}{(x^2z^2w^2v^2p^2 - L)^2} - \frac{137}{20} \Bigg\} \\
&\qquad + 6\eta^{(\mu\nu}p^\rho p^{\sigma)} p^2 \Bigg\{ 12 \ln\left(\frac{\Lambda^2}{x^2z^2w^2v^2p^2 - D} \right) - 78 \frac{x^2z^2w^2v^2p^2}{x^2z^2w^2v^2p^2 - L} - \frac{137}{5} \\
&\qquad\qquad + 56 \frac{x^4z^4w^4v^4p^4}{(x^2z^2w^2v^2p^2 - L)^2} - 16 \frac{x^6z^6w^6v^6p^6}{(x^2z^2w^2v^2p^2 - L)^3} \Bigg\} \\
&\qquad + p^{\mu\nu\rho\sigma} \Bigg\{ 24 \ln\left(\frac{\Lambda^2}{x^2z^2w^2v^2p^2 - D} \right) \\
&\qquad\qquad - 336 \frac{x^2z^2w^2v^2p^2}{x^2z^2w^2v^2p^2 - L} + 492 \frac{x^4z^4w^4v^4p^4}{(x^2z^2w^2v^2p^2 - L)^2} \\
&\qquad\qquad - 352 \frac{x^6z^6w^6v^6p^6}{(x^2z^2w^2v^2p^2 - L)^3} + 96 \frac{x^8z^8w^8v^8p^8}{(x^2z^2w^2v^2p^2 - L)^4} - \frac{274}{5} \Bigg\} \Bigg] .
\end{aligned}$$

Performing the remaining integrations yields

$$
\begin{aligned}
&D_4^{\mu\nu\rho\sigma} - A_4^{\mu\nu\rho\sigma} - B_4^{\mu\nu\rho\sigma} \\
&= \frac{i}{16\pi^2}\Bigg[3p^4\eta^{(\mu\nu}\eta^{\rho\sigma)}\Bigg\{\frac{1}{8}\ln\left(\frac{\Lambda^2}{p^2-D}\right) - \frac{D^2}{8p^4}\ln\left(\frac{p^2-D}{-D}\right) \\
&\qquad\qquad + \frac{D}{4p^2}\ln\left(\frac{p^2-D}{-D}\right) - \frac{47}{480} - \frac{1}{8}\frac{D}{p^2}\Bigg\} \\
&\qquad + 6\eta^{(\mu\nu}p^\rho p^{\sigma)}p^2\Bigg\{\frac{1}{2}\ln\left(\frac{\Lambda^2}{p^2-D}\right) + \frac{D}{2p^2}\ln\left(\frac{p^2-D}{-D}\right) - \frac{77}{120}\Bigg\} \\
&\qquad + p^{\mu\nu\rho\sigma}\Bigg\{\ln\left(\frac{\Lambda^2}{p^2-D}\right) - \frac{137}{60}\Bigg\}\Bigg]\,.
\end{aligned}
$$

Our final result is

$$
\begin{aligned}
\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2+2pk+D)^2} = \frac{i}{16\pi^2}\Bigg[&3\eta^{(\mu\nu}\eta^{\rho\sigma)}\Bigg\{\frac{\Lambda^4}{48} + \left(\frac{D}{12} - \frac{p^2}{16}\right)\Lambda^2 \\
&\qquad + \frac{1}{8}(p^2-D)^2\ln\left(\frac{\Lambda^2}{p^2-D}\right) \\
&\qquad - \frac{47}{480}p^4 + \frac{7}{48}p^2D - \frac{D^2}{24}\Bigg\} \\
&+ 6\eta^{(\mu\nu}p^\rho p^{\sigma)}\Bigg\{-\frac{\Lambda^2}{8} + \frac{1}{2}(p^2-D)\ln\left(\frac{\Lambda^2}{p^2-D}\right) \\
&\qquad - \frac{77}{120}p^2 + \frac{13}{24}D\Bigg\} \\
&+ p^{\mu\nu\rho\sigma}\Bigg\{\ln\left(\frac{\Lambda^2}{p^2-D}\right) - \frac{137}{60}\Bigg\}\Bigg]\,.
\end{aligned}
$$

Differentiating a general two propagator integral with divergence index n with respect to p, we obtain a three propagator integral with index $n-1$. In this way we can also obtain integrals with further propagators, but since we did not need them in our calculation, we leave it at three propagators.

B.1 List Of Cut-Off Integrals

General Two Propagator Integrals

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2+2pk+D)^2} = \frac{i}{16\pi^2}\left[\ln\frac{\Lambda^2}{p^2-D} - 1\right]$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu}{(k^2+2pk+D)^2} = \frac{i}{16\pi^2}p^\mu\left[\frac{3}{2} - \ln\frac{\Lambda^2}{p^2-D}\right]$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2+2pk+D)^2} = \frac{i}{16\pi^2}\Bigg[\eta^{\mu\nu}\left\{-\frac{\Lambda^2}{4} + \frac{1}{2}(p^2-D)\ln\frac{\Lambda^2}{p^2-D} - \frac{5p^2}{12} + \frac{D}{4}\right\} + p^\mu p^\nu\left\{\ln\frac{\Lambda^2}{p^2-D} - \frac{11}{6}\right\}\Bigg]$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho}{(k^2+2pk+D)^2} = \frac{i}{16\pi^2}\Bigg[3\eta^{(\mu\nu}p^{\rho)}\left\{\frac{\Lambda^2}{6} - \frac{1}{2}(p^2-D)\ln\frac{\Lambda^2}{p^2-D} + \frac{13}{24}p^2 - \frac{5}{12}D\right\} - p^\mu p^\nu p^\rho\left\{\ln\frac{\Lambda^2}{p^2-D} - \frac{25}{12}\right\}\Bigg]$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2+2pk+D)^2} = \frac{i}{16\pi^2}\Bigg[&3\eta^{(\mu\nu}\eta^{\rho\sigma)}\bigg\{\frac{\Lambda^4}{48} + \left(\frac{D}{12} - \frac{p^2}{16}\right)\Lambda^2 \\ &\quad + \frac{1}{8}(p^2-D)^2\ln\frac{\Lambda^2}{p^2-D} \\ &\quad - \frac{47}{480}p^4 + \frac{7}{48}p^2 D - \frac{D^2}{24}\bigg\} \\ &- 6\eta^{(\mu\nu}p^\rho p^{\sigma)}\bigg\{\frac{\Lambda^2}{8} - \frac{1}{2}(p^2-D)\ln\frac{\Lambda^2}{p^2-D} \\ &\quad + \frac{77}{120}p^2 - \frac{13}{24}D\bigg\} \\ &+ p^{\mu\nu\rho\sigma}\left\{\ln\frac{\Lambda^2}{p^2-D} - \frac{137}{60}\right\}\Bigg]\end{aligned}$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^2}{(k^2+2pk+D)^2} = \frac{i}{16\pi^2}\Bigg[&\eta^{\mu\nu}\bigg\{\frac{\Lambda^4}{8} - \frac{1}{2}\left(p^2-D\right)\Lambda^2 \\ &\quad + \left(\frac{3}{4}(p^2-D)^2 + \frac{1}{2}(p^2-D)p^2\right)\ln\frac{\Lambda^2}{p^2-D} \\ &\quad - \frac{59}{48}p^4 + \frac{17}{12}p^2 D - \frac{D^2}{4}\bigg\} \\ &- p^\mu p^\nu\bigg\{\Lambda^2 - (5p^2-4D)\ln\frac{\Lambda^2}{p^2-D} \\ &\quad + \frac{89}{12}p^2 - \frac{13}{3}D\bigg\}\Bigg]\end{aligned}$$

Massless Two Propagator Integrals

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2(q+k)^2} = \frac{i}{16\pi^2}\left[\ln\frac{\Lambda^2}{-q^2}+1\right]$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu}{k^2(q+k)^2} = \frac{i}{16\pi^2}q^\mu\left[-\frac{1}{2}\ln\frac{\Lambda^2}{-q^2}-\frac{1}{4}\right]$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{k^2(q+k)^2} = \frac{i}{16\pi^2}\left[\eta^{\mu\nu}\left\{-\frac{\Lambda^2}{4}-\frac{q^2}{12}\ln\frac{\Lambda^2}{-q^2}-q^2\frac{11}{72}\right\}\right.$$
$$\left. + q^\mu q^\nu\left\{\frac{1}{3}\ln\frac{\Lambda^2}{-q^2}+\frac{1}{9}\right\}\right]$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho}{k^2(q+k)^2} = \frac{i}{16\pi^2}\left[3\eta^{(\mu\nu}q^{\rho)}\left\{\frac{\Lambda^2}{12}+\frac{q^2}{24}\ln\frac{\Lambda^2}{-q^2}+\frac{19}{288}q^2\right\}\right.$$
$$\left. - q^\mu q^\nu q^\rho\left\{\frac{1}{4}\ln\frac{\Lambda^2}{-q^2}+\frac{1}{16}\right\}\right]$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^2 k^\mu k^\nu}{k^2(q+k)^2} = \frac{i}{16\pi^2}\left[\eta^{\mu\nu}\left\{\frac{\Lambda^4}{8}+\frac{q^2}{12}\Lambda^2+\frac{1}{48}q^4\right\}-q^\mu q^\nu\left\{\frac{\Lambda^2}{3}+\frac{q^2}{4}\right\}\right]$$

General Three Propagator Integrals

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2+2pk+D)^3} = \frac{i}{16\pi^2}\frac{1}{2}\frac{1}{p^2-D}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu}{(k^2+2pk+D)^3} = \frac{i}{16\pi^2}\frac{1}{2}\frac{p^\mu}{p^2-D}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2+2pk+D)^3} = \frac{i}{16\pi^2}\left[\eta^{\mu\nu}\left\{\frac{1}{4}\ln\left(\frac{\Lambda^2}{p^2-D}\right)-\frac{3}{8}\right\}-\frac{1}{2}\frac{p^\mu p^\nu}{p^2-D}\right]$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho}{(k^2+2pk+D)^3} = \frac{i}{16\pi^2}\left[3\eta^{(\mu\nu}p^{\rho)}\left\{\frac{11}{24}-\frac{1}{4}\ln\left(\frac{\Lambda^2}{p^2-D}\right)\right\}+\frac{1}{2}\frac{p^\nu p^\nu p^\rho}{p^2-D}\right]$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2+2pk+D)^3} = \frac{i}{16\pi^2}\Bigg[&3\eta^{(\mu\nu}\eta^{\rho\sigma)}\Bigg\{-\frac{\Lambda^2}{24}+\frac{1}{8}(p^2-D)\ln\left(\frac{\Lambda^2}{p^2-D}\right)\\ &\qquad -\frac{13}{96}p^2+\frac{5}{48}D\Bigg\}\\ &+6\eta^{(\mu\nu}p^\rho p^{\sigma)}\left\{\frac{1}{4}\ln\left(\frac{\Lambda^2}{p^2-D}\right)-\frac{25}{48}\right\}\\ &-\frac{1}{2}\frac{p^{\mu\nu\rho\sigma}}{p^2-D}\Bigg]\end{aligned}$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho k^\sigma k^\alpha}{(k^2+2pk+D)^3} = \frac{i}{16\pi^2}\Bigg[&15\eta^{(\mu\nu}\eta^{\rho\sigma}p^{\alpha)}\Bigg\{\frac{\Lambda^2}{24}-\frac{1}{8}(p^2-D)\ln\left(\frac{\Lambda^2}{p^2-D}\right)\\ &\qquad +\frac{77}{480}p^2+\frac{13}{96}D\Bigg\}\\ &+10\eta^{(\mu\nu}p^\rho p^\sigma p^{\alpha)}\left\{-\frac{1}{4}\ln\left(\frac{\Lambda^2}{p^2-D}\right)+\frac{137}{240}\right\}\\ &+\frac{1}{2}\frac{p^{\mu\nu\rho\sigma\alpha}}{p^2-D}\Bigg]\end{aligned}$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{k^2 k^\mu k^\nu k^\rho}{(k^2+2pk+D)^3} = \frac{i}{16\pi^2}\Bigg[&3\eta^{(\mu\nu}p^{\rho)}\Bigg\{\frac{\Lambda^2}{4}-\frac{5p^2-4D}{4}\ln\left(\frac{\Lambda^2}{p^2-D}\right)\\ &\qquad +\frac{89}{48}p^2-\frac{13}{12}D\Bigg\}\\ &-p^\mu p^\nu p^\rho\left\{\frac{5}{2}\ln\left(\frac{\Lambda^2}{p^2-D}\right)+\frac{D-\frac{5}{4}p^2}{p^2-D}-\frac{89}{24}\right\}\Bigg]\end{aligned}$$

Massless Three Propagator Integrals

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^\mu k^\nu}{k^2(k+q)^2(k-p)^2} = \frac{i}{16\pi^2}\eta^{\mu\nu}\frac{1}{4}\ln\left(\frac{\Lambda^2}{\mu^2}\right) \qquad + \text{finite}$$

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^\mu k^\nu k^\rho}{k^2(k+q)^2(k-p)^2} = -\frac{i}{16\pi^2}\eta^{(\mu\nu}(q-p)^{\rho)}\frac{1}{4}\ln\left(\frac{\Lambda^2}{\mu^2}\right) \qquad + \text{finite}$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^\mu k^\nu k^\rho k^\sigma}{k^2(k+q)^2(k-p)^2} = \frac{i}{16\pi^2}\Bigg[&\eta^{(\mu\nu}\eta^{\rho\sigma)}\left\{-\frac{\Lambda^2}{8}-\frac{1}{16}(q^2+qp+p^2)\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right\} \\ &+\frac{1}{4}\big(\eta^{(\mu\nu}q^\rho q^{\sigma)}+\eta^{(\mu\nu}p^\rho p^{\sigma)} \\ &-\eta^{(\mu\nu}q^\rho p^{\sigma)}\big)\ln\left(\frac{\Lambda^2}{\mu^2}\right)\Bigg] + \text{finite}\end{aligned}$$

$$\begin{aligned}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2 k^\mu k^\nu k^\rho}{k^2(k+q)^2(k-p)^2} = \frac{i}{16\pi^2}\Bigg[&\eta^{(\mu\nu}(q-p)^{\rho)}\left\{\frac{\Lambda^2}{4}+\frac{1}{8}(p+q)^2\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right\} \\ &-\frac{1}{4}(q^\mu q^\rho q^\sigma - p^\mu p^\rho p^\sigma \\ &+\eta^{(\mu\nu}(q-p)^{\rho)})\ln\left(\frac{\Lambda^2}{\mu^2}\right)\Bigg] + \text{finite}\end{aligned}$$

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Hilfsmittel

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Hiermit erkläre ich, die vorliegende Diplomarbeit selbständig sowie ohne unerlaubte fremde Hilfe verfasst und nur die angegebenen Quellen und Hilfsmittel verwendet zu haben.

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Berlin, den 30. Dezember 2008

Theodor Schuster