String Theory in $AdS_5 \times S^5$ Near-Flat Space Limit

Diplomarbeit

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Abstract

In this diploma thesis a special limit of type IIB Superstring Theory in $AdS_5 \times S^5$ is investigated. This so-called *near-flat space limit* was proposed by Maldacena and Swanson [1] and can be motivated by the fact that it interpolates between previously analyzed limits of superstring theory in $AdS_5 \times S^5$: the *giant magnon* regime and the *plane wave* limit. In the former limit the magnon momenta scale like $p \sim \lambda^0$, in the latter they scale like $p \sim \lambda^{-1/2}$, whereas in the near-flat space limit they behave as $p \sim \lambda^{-1/4}$.
We propose a way slightly differing from [1] to derive the gauge fixed Lagrangian. A formalism appropriate for a future analysis of the supersymmetry algebra is used in the calculation. The usual light-cone gauge turns out to be consistent only, if curvature corrections to the world-sheet metric are taken into account. The full gauge fixed model, including these corrections, is given in the present work.

Inhaltsangabe

In dieser Diplomarbeit wird ein spezieller Limes der Typ IIB Superstring Theorie in $AdS_5 \times S^5$ untersucht. Dieser sogenannte *near-flat space* Limes wurde von Maldacena und Swanson in [1] vorgeschlagen und kann dadurch motiviert werden, dass er eine Interpolation zwischen zuvor analysierten Limites der Superstring Theorie in $AdS_5 \times S^5$ darstellt: dem sog. *giant-magnon* Regime und dem *plane wave* Regime. Im Ersteren skalieren die Magnon-Impulse wie $p \sim \lambda^0$, im Letzteren wie $p \sim \lambda^{-1/2}$, im *near-flat space* Limes hingegen verhalten sie sich wie $p \sim \lambda^{-1/4}$.
Der vorgestellte Weg zur Berechnung der eichfixierten Lagrangedichte weicht leicht von dem in [1] benutzten ab. Es wird ein Formalismus verwendet, der eine zukünftige Analyse der Supersymmetrie-Algebra zugänglich macht. Es zeigt sich, dass die verwendete Lichtkegel-Eichung nur konsistent ist, wenn Krümmungskorrekturen zur Weltflächenmetrik berücksichtigt werden. Das volle, diese Korrekturen einschließende, eichfixierte Model wird in der vorliegenden Arbeit vorgestellt.

Keywords:
Near-Flat Space Limit, String Theory in $AdS_5 \times S^5$, AdS/CFT

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Contents

I

Introduction

Physics has always been driven by the desire to find models or laws that are accurate enough to describe phenomena as observed in nature, yet which are as fundamental as possible. All processes which occur and all observations that can be made, are due to interactions between what one might call the fundamental constituents of our world. At present, we know of four *fundamental forces* that describe these interactions

- the gravitational force
- the electromagnetic force
- the weak force
- and the strong force.

Gravity has been known since antiquity and was first accurately described by Isaac Newton. The electromagnetic force - although at first not captured by a single concept - was the next to appear in the arena. The *weak force* being much weaker, and the *strong force* being much stronger than the electromagnetic force, were discovered later at the level of subatomic physics. Nuclear beta decay is governed by the weak force and the strong force is responsible for holding together the constituents of the neutron, the proton and other subnuclear particles.

In the history of physics, it has proved useful to combine or *unify* different laws which describe different phenomena to more general laws that are powerful enough to describe all of these phenomena from a single concept. Such *unifications* have led to a deeper understanding of the underlying principles and, besides explaining the known phenomena, often made the *prediction* of undiscovered phenomena possible.

As a prime example, one may consider the rise of a unified theory of electricity and magnetism. While the phenomena of electrostatics were accurately described by Coulomb's laws (1785), the phenomena of magnetism were described by independent laws, until connections between electricity and magnetism were observed and put into equations which incorporated all of these results. Significant figures along the road to a unified theory of electromagnetism were Oersted, Biot-Savart, Ampère and Faraday. While all of the observed phenomena could be described accurately by a set of equations, these equations contained inconsistencies. When James Clerk Maxwell added a term to these equations, he not only made them consistent, but his addition resulted in the prediction of electromagnetic waves (1865). It was only *after* that, that their existence could be verified and put to use in everyday life.

Einstein's *conceptual unification* of the notions of space and time into a spacetime continuum resulted in a change of paradigm. Besides having many implications for the concepts of space and time, his theory of *special relativity* (1905) led to the famous equivalence between mass and energy. His later theory of *general relativity* resulted in a complete reformulation of the gravitational force in terms of the curvature of spacetime.

By the time of Einsteins discoveries, another extremely important concept had been born. The principles of *quantum mechanics* were developed as an appropriate framework to describe the phenomena occurring in the micro-world. Some important scientists behind these developments were Planck, Bohr, de Broglie, Schrödinger, Heisenberg, Pauli and Dirac. None of today's computer science could be applied or understood without the principles of quantum mechanics.

Again it proved useful, to establish links between the concepts of special relativity and quantum mechanics. A major success of relativistic quantum mechanics was the explanation of an internal property of fundamental particles: the spin. Secondly, it directly led in 1928 to the prediction of anti-particles, which were later discovered in 1932. The theory which incorporates both, special relativity and quantum mechanics, is *quantum field theory* (QFT). The present model of fundamental physics - the standard model of particle physics - makes use of the concepts of QFT successfully and can be regarded as the experimentally best verified theory ever.

Within the concepts of QFT, another unification was made possible in the late 1960s. The quantum theories of electromagnetism and weak interactions were unified within the model of *electroweak* interactions, which was necessary for a consistent and predictive theory of weak interactions. Furthermore, the strong interaction could be described by a QFT called *quantum chromodynamics* (QCD). There have been attempts to find a *grand unified theory* which unifies the electroweak and the strong interaction, but up to date it seems as if these are best described by similar, but separate, QFT's.

The *standard model* accurately describes the microscopic laws of electromagnetic, weak and strong interactions with the methods of QFT. However, the standard model suffers from significant shortcomings: First of all, it does not include gravity. Attempts to describe gravity with the concepts of quantum field theory, i.e. quantum gravity, have failed up to date. At present it is unclear, whether gravity can be described as a quantum theory at all. However, most experts believe that gravity must be turned into a quantum theory. Due to the weakness of the gravitational force, its classical description can be used for most practical calculations. For studying times near the Big Bang or properties of black holes however, quantum effects of gravity become significant and must be incorporated. Secondly, despite its predictive power, the standard model does not seem to be fundamental, as it has a huge number of parameters that have to be put in manually. For instance, there are three families of particles with exactly the same properties but different masses or - by Einstein's famous relation - different energy, and thus look very much like excited states of some fundamental object. There is a description but no explanation of this in the standard model. Thus the standard model does not seem to be the fundamental theory.

In history, it has always proved fruitful to consider unifications of physical laws and it seems promising to continue along the same road. This is where string theory enters the picture. It first appeared as a theory of strong interactions but failed in this endeavour, and made its comeback around 1984 as a candidate for a unified theory of all interactions. Although string theory has not provided us with any predictions and has not been verified or falsified in any manner, as yet, it brought a wealth of new concepts. The quantum version of relativistic strings automatically includes a graviton as a vibrational mode of closed strings. Thus gravity is automatically included and there is evidence that string theory is a good quantum theory: the problems that arise when conventionally quantizing Einstein's theory of gravity do not appear in string theory. Furthermore, although there is no final model, there seems to be enough room to incorporate the standard model in string theory. Additionally, to be a realistic theory that incorporates bosons as well as fermions, string theory requires the concept of supersymmetry, i.e. a symmetry relating fermions to bosons, which also inspired interesting supersymmetric particle models. There is another important difference to the usual theory of the relativistic point particle: superstring theory requires ten spacetime dimensions for a consistent formulation, in contrast to the four spacetime dimensions our world seems to have. Clearly, it is necessary to develop methods that project the results of ten-dimensional superstring theory to a four-dimensional world.

A very interesting area of research is the *Anti-de Sitter / Conformal Field Theory (AdS/CFT) correspondence* that states the equivalence between ten-dimensional superstring theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ space and the four-dimensional $\mathcal{N} = 4$ Super Yang-Mills gauge theory. It thus provides a concept of how to incorporate results from a ten-dimensional theory into a four-dimensional theory and is thus also called a *holographic* correspondence. As will be explained in this thesis, we are dealing with a *strong-weak* coupling duality, i.e. the results of strong coupling on the gauge theory side correspond to a weak coupling limit of string theory and vice versa. This is a remarkable feature: if the correspondence is true, we have a fantastic tool to study the perturbatively inaccessible regime of strong coupling on the gauge theory side by means of a solvable limit of string theory. Inversely, we can get information about the barely accessible regime of string theory by addressing the weak coupling limit of the gauge theory side. But for the same reason, it is extremely difficult to prove the correspondence, since it would be necessary to solve at least one side completely. Fortunately, overlapping regimes of the two theories exist which can be used to test subsectors of the correspondence.

In this thesis, the main focus will be placed on aspects associated with the string theory side. Since superstring theory on known backgrounds is highly non-linear and difficult to deal with, some limiting procedures proved useful. One of the limits widely used in recent years is the so-called *plane wave limit*. The resulting theory is exactly solvable even on the quantized level. The issues concerning the validity of the correspondence have also been studied. Although exactly solvable, some of the important properties and the structure of the original theory are lost due to the limit. One possibility to weaken the plane wave limit and to preserve many of the key features of superstring theory on $AdS_5 \times S^5$ background was proposed by Maldacena and Swanson [1]. We will investigate this *near-flat space limit* of string theory in $AdS_5 \times S^5$ and clearly embed it into the context of current research. This diploma thesis may make a modest contribution to the investigation of the AdS/CFT correspondence. The correspondence in turn might give strong support for the meaning of string theory and develop fascinating tools for the analysis of strongly coupled theories. Thus, in a sense, this thesis potentially contributes to another unified theory of physics.

1.1 Organization of the Content

In chapter 2, some fundamental aspects of the AdS/CFT correspondence are discussed. We begin with a short review of the necessary ingredients of the AdS/CFT correspondence: first, some basic facts on Anti-de-Sitter spaces and the string theory living on this space in 2.1 and 2.2. Then we write down the explicit form of the conformal field theory involved in the correspondence in 2.3, i.e. the $\mathcal{N} = 4$ SYM theory, which enables us to review some of the main statements of the AdS/CFT correspondence in 2.4. To motivate the limit that is investigated in this thesis, we review some of the well studied limits of string theory in $AdS_5 \times S^5$ in chapter 3. Chapter 4 constitutes the main part of the diploma thesis: After a short glance at bosonic string theory in the near-flat space limit we introduce the formalism used to investigate full superstring theory in in 4.3, followed by some useful calculational techniques appropriate for handling the formalism in 4.4. In 4.5, we present the resulting Lagrangian density in conformal gauge. In section 4.7, we comment on the validity of conformal gauge and the necessity of world-sheet curvature corrections in the near-flat space limit. The main text only contains the logic and the main results of the calculations, whereas the technical details were delegated to the appendices as much as possible and may be addressed by the reader who is interested in reproducing the results.

II

The AdS/CFT Correspondence - Elements

The AdS/CFT correspondence is a duality between two seemingly different theories: Type IIB Superstring Theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ Super Yang-Mills (SYM) Theory living on the conformal boundary of Anti-de-Sitter space. Before explicitly stating the correspondence and some of its main features in 2.4, we review some basic facts about Anti-de Sitter space in section 2.1 and the string theory that lives on this space in section 2.2. In 2.3, we comment on the gauge theory side of the correspondence. The presented material is far from being complete and we review only the aspects necessary to understand the special limit investigated in this thesis and its motivation. Thorough introductions to the AdS/CFT correspondence are for instance [2], [3], [4], [5].

2.1 Anti-de Sitter Space

In this chapter, some basic facts about Anti-de Sitter spaces will be reviewed. The $(d+1)$-dimensional Anti-de Sitter space, denoted by AdS_{d+1}, can be represented as the hyperboloid

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2 \,, \tag{2.1}$$

embedded in $(d+2)$-dimensional flat space with metric $\eta = diag(-1, 1, 1, ..., 1, -1)$, i.e.

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 \,. \tag{2.2}$$

This makes the isometry group $SO(2, d)$ explicit. The following choice of coordinates satisfies equation (2.1)

$$\begin{aligned} X_0 &= R\, cosh\, \rho\; cos\, t \\ X_{d+1} &= R\, cosh\, \rho\; sin\, t \\ X_i &= R\, sinh\, \rho\; \Omega_i \end{aligned} \tag{2.3}$$

where the Ω_i parametrize a sphere S^{d-1}, obeying

$$\sum_{i=1}^{d} \Omega_i^2 = 1 \,. \tag{2.4}$$

In the coordinates (2.3), the metric of AdS_{d+1} reads

$$ds^2 = R^2(-cosh^2\rho\, dt^2 + d\rho^2 + sinh^2\rho\, d\Omega_i^2). \quad (2.5)$$

Taking $\rho \geq 0$ and $0 \leq t \leq 2\pi$ the entire hyperboloid is covered once. The coordinates (2.3) are therefore called *global* coordinates of AdS_{d+1}. It is a homogeneous and isotropic space with constant negative curvature $-\frac{d(d+1)}{R^2}$.

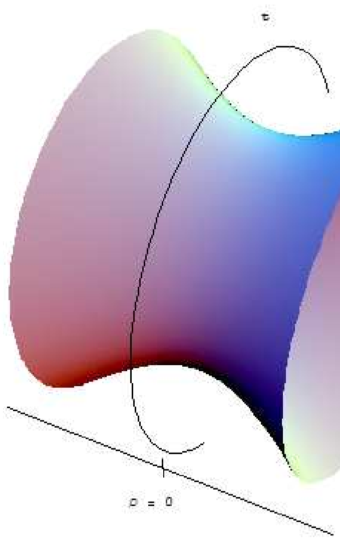

Figure 2.1: AdS_2 embedded in $\mathbb{R}^3$. It may seem misleading to have $\rho = 0$ at the center of the axis as we have $\rho \geq 0$. This is due to the fact that for AdS_2 we have an attached sphere S^0, i.e. two points. Thus, one can represent the hyperboloid as being extended symmetrically in two directions with $\rho \geq 0$ for both of them.

2.1.1 Anti-de Sitter Space in 5 Dimensions

For 5-dimensional Anti-de Sitter space, AdS_5, the metric is

$$ds^2 = R^2(-cosh^2\rho\, dt^2 + d\rho^2 + sinh^2\rho\, d\Omega_3^2) \quad (2.6)$$

where $d\Omega_3^2$ is the part of the metric parameterizing the three-sphere. Thus in 5 dimensions, we have the isometry group $SO(2,4) \simeq SU(2,2)$.

2.1.2 $AdS_5 \times S^5$

Since we want to consider superstring theory in 10 dimensions, we need a 10-dimensional space. We choose the other 5 dimensions to be a 5-sphere S^5, being a homogeneous and isotropic space with constant positive curvature: $\frac{d(d+1)}{R^2}$. Then, for the entire $AdS_5 \times S^5$ space we can write the metric in global coordinates as

$$ds^2 = R^2(-cosh^2\rho\, dt^2 + d\rho^2 + sinh^2\rho\, d\Omega_3^2 + cos^2\theta\, d\phi^2 + d\theta^2 + sin^2\theta\, d\tilde{\Omega}_3^2) \quad (2.7)$$

where $d\Omega_3^2$ resp. $d\tilde{\Omega}_3^2$ denote the two separate 3-spheres and we have chosen Anti-de-Sitter-space and the 5-sphere to have the same *radius* R. The coordinate ϕ and the time coordinate t as well are periodic with period 2π. To get non-periodic time it is necessary to pass to the covering space of AdS, i.e. loosely spoken to unwrap the t-direction and extend it from $-\infty$ to ∞.[1] For later convenience, we reparametrize

$$cosh\,\rho = \frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}, \qquad cos\,\theta = \frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}} \tag{2.8}$$

and thus get the metric

$$\begin{aligned} ds^2 &= R^2\Big[-\Big(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}}\Big)^2 dt^2 + \Big(\frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\Big)^2 d\phi^2 + \frac{dz^2}{(1-\frac{z^2}{4})^2} + \frac{dy^2}{(1+\frac{y^2}{4})^2}\Big] \\ &\equiv R^2\Big[-G_{tt}\,dt^2 + G_{\phi\phi}\,d\phi^2 + G_{zz}\,dz^2 + G_{yy}\,dy^2\Big] \end{aligned} \tag{2.9}$$

where $z^2 = z_a z^a$ with $a = 1,2,3,4$ and $y^2 = y_s y^s$ with $s = 5,6,7,8$. This metric has the full $SO(4,2)\times SO(6)$ symmetry of $AdS_5\times S^5$, but only translational symmetries in t and ϕ (which only appear as dt, $d\phi$) as well as the $SO(4)\times SO(4)$ - symmetry in the transverse coordinates z_a, y_s (which only appear contracted as z^2, y^2) remain explicit.

2.1.3 $AdS_5 \times S^5$ as a Coset Space

In the following, we indicate how AdS_5 and S^5 can be written as *coset spaces* of the form G/H where G is a certain Lie group and H a certain subgroup of G. For a mathematically more precise and more detailed description, [6] is recommended.

Let M be a manifold and G a Lie group acting *transitively* on M, i.e. for any two points $p_1, p_2 \in M$ there exists an element $g \in G$ such that p_1 can be mapped to p_2 by a certain map σ: $G \times M \to M$ such that $\sigma(g, p_1) = p_2$. Let H(p) be the *isotropy group* of $p \in M$, i.e. a subgroup of G that maps every point $p \in M$ to itself[2]. Then - under certain technical requirements - it can be shown that $G/H(p)$ is homeomorphic to M:

$$G/H(p) \cong M\,. \tag{2.10}$$

The d-dimensional sphere S^d can then be written as

$$S^d \cong \frac{SO(d+1)}{SO(d)}\,. \tag{2.11}$$

As an illustrative example, one can consider the unit sphere S^2 embedded in $\mathbb{R}^3$: Without loss of generality we can pick the point $p = (x,y,z) = (0,0,1)$ which

[1] This is explained very detailed in [2].
[2] This is why it is also called the *stabilizer group*.

clearly lies on $M = S^2$. The group of rotations $G = SO(3)$ acting on $\mathbb{R}^3$ maps this point to every other point on the unit sphere, and thus acts transitively on S^2. The subgroup $H = SO(2)$ of $SO(3)$ generates rotations in a plane. Taking the z-axis as rotation-axis the point p is mapped to itself, thus $SO(2)$ is the stabilizer group of $p \in S^2$. In this sense

$$S^2 \cong \frac{SO(3)}{SO(2)} \,. \tag{2.12}$$

AdS_d can be represented as the coset space

$$AdS_d \cong \frac{SO(2,d-1)}{SO(1,d-1)} \,. \tag{2.13}$$

Therefore, we can write $\mathrm{AdS}_5 \times \mathrm{S}^5$ as the coset space

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \cong \frac{SO(2,4) \times SO(6)}{SO(1,4) \times SO(5)} \,. \tag{2.14}$$

For the generalization to the supersymmetric target space, which will be considered soon, the reader may already note that $SO(2,4) \times SO(6) \simeq SU(2,2) \times SU(4)$.

2.2 String Theory

A very basic introduction to string theory can be found in [7]. For a more advanced and deeper introduction [8] is recommended. More up to date but even a bit more advanced and in some points mathematically more rigorous is [9].

2.2.1 Bosonic String Theory

In the relativistic theory of point particles, the action S is proportional to the length of the spacetime-trajectory traced out by the particle, i.e. a 0-dimensional object (see for example [10]). Analogously, the action of a string, i.e. a 1-dimensional object, is taken to be proportional to the *area* that is traced out in spacetime. This leads to the Nambu-Goto form of the action of string theory, which can be shown to be classically equivalent[3] to the *Polyakov action*

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \, \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN} \tag{2.15}$$

where $\gamma^{\alpha\beta}$ is the Weyl invariant combination $\sqrt{-h}h^{\alpha\beta}$ of the world-sheet metric $h^{\alpha\beta}$ and its determinant $h = det\, h_{\alpha\beta}$. Then it is clear that $det\, \gamma^{\alpha\beta} = -1$. G_{MN} is the metric of the background-spacetime that is considered. When working in $\mathrm{AdS}_5 \times \mathrm{S}^5$, we will use it in the form (2.9). The coordinates τ and σ parametrize the string world-sheet and $X^M(\tau, \sigma)$ maps the string into spacetime, $X^M = (t, \phi, z_a, y_s)$.

[3] This means equivalent on the equations of motion of the auxiliary field $h^{\alpha\beta}$.

In flat 10-dimensional space, G_{MN} has the usual 10-dimensional Minkowski signature $\eta = diag(-1, 1, ..., 1)$ and the model can easily be analyzed in the so-called light-cone gauge. First, it should be noted that the action (2.15) possesses the reparametrization invariances $\tau \rightarrow \tilde{\tau}(\tau, \sigma)$, $\sigma \rightarrow \tilde{\sigma}(\tau, \sigma)$ of the world-sheet-parameters as well as a Weyl symmetry $h^{\alpha\beta} \rightarrow e^{\phi(\sigma,\tau)} h^{\alpha\beta}$. The quantization procedure can roughly be summarized as follows and is found in more detail in [8]:

- Reparametrization invariance is used to set the (inverse) world-sheet metric to flat
 2-d Minkowski signature, i.e.

$$\gamma^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.16}$$

- Light-cone coordinates are introduced:

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^9), \qquad X^I \quad with \quad I = 1, ..., 8 . \tag{2.17}$$

- The residual symmetry is used to fix *light-cone gauge*:

$$X^+ = \tau . \tag{2.18}$$

- The equations of motion for $h^{\alpha\beta}$, i.e. $\frac{\delta S}{\delta h^{\alpha\beta}} \stackrel{!}{=} 0$ give two independent equations that can be solved for the derivatives of X^- in terms of transverse coordinates

$$\begin{aligned} \partial_\sigma X^- &= f_1(\partial_\alpha X^I) \\ \partial_\tau X^- &= f_2(\partial_\alpha X^I) . \end{aligned} \tag{2.19}$$

- (2.18) and (2.19) are used to express the action (2.15) in terms of the *transverse, physical fields* X^I

$$S \propto \int d\sigma d\tau \Big((\partial_\tau X^I)^2 - (\partial_\sigma X^I)^2 \Big) . \tag{2.20}$$

The resulting action is the action of a free, massless 2-dimensional field theory with 8 scalar fields X^I. The equations of motion for the fields X^I can then easily be solved by an expansion in Fourier coefficients $a_I, a_I^\dagger$, which by quantization are raised to quantum operators and correspond to annihilation and creation operators. The spectrum of the theory can then be obtained by applying the creation operators to the vacuum state, i.e. excitations on the string are described as states $a_I^\dagger a_J^\dagger |0\rangle$.

2.2.2 Superstring Theory

Above, we have implicitly used that string theory lives in 10 dimensions. To be more precise, this is only true in the case of superstring theory. When considering purely bosonic string theory the critical dimension is $D = 26$. This can be shown by demanding the theory to possess spacetime Lorentz invariance, i.e. invariance under $SO(D-1,1)$ in D-dimensional spacetime. In the light-cone quantization used above, it can be shown that the full Lorentz algebra is not satisfied for arbitrary spacetime-dimension but only for $D = 26$.

Bosonic string theory suffers from different drawbacks. Two obvious ones are the following:

- First of all, bosonic string theory contains a *tachyon* in its spectrum at the level of *zero excitations* on the string. Such particles with negative mass squared cause an unstable *no-excitation vacuum*: A potential term in field theories is of the form $\frac{1}{2}m^2\phi^2$. With negative mass squared the zero excitation state is thus not a stable state, just like the symmetric state in a spontaneously broken theory.

- Secondly, a realistic string theory must contain fermionic states as well, i.e. states that are antisymmetric under the exchange of excitation labels. Such states are not included in bosonic string theory.

These problems can be solved by extending the model to a supersymmetric theory. Therefore, in addition to the scalar fields X^μ, fermionic fields are introduced. For details on this procedure, chapters 4 and 5 of [8] can be recommended. It is then possible to construct an action that possesses supersymmetry, i.e. there exist transformations of the fields that relate bosonic and fermionic fields and leave the action invariant. The action can be quantized in light-cone gauge and as in the bosonic case the theory is not manifestly Lorentz invariant, but it can be shown to be in D=10.

The so-called *Green-Schwarz action*, for details see chapter 5 in [8], is invariant under local supersymmetry transformations[4] and can be written as a sum of two terms

$$S_{superstring} = S_1 + S_2 \tag{2.21}$$

with

$$\begin{aligned} S_1 &= \frac{1}{2\pi}\int d\sigma d\tau \sqrt{-h}h^{\alpha\beta}\Pi_\alpha \Pi_\beta \qquad (2.22)\\ S_2 &= \frac{1}{\pi}\int d\sigma d\tau \epsilon^{\alpha\beta}\left(i\partial_\alpha X^\mu(\bar\theta^1\Gamma_\mu\partial_\beta\theta^1 - \bar\theta^2\Gamma_\mu\partial_\beta\theta^2) - \bar\theta^1\Gamma^\mu\partial_\alpha\theta^1\bar\theta^2\Gamma_\mu\partial_\beta\theta^2\right)\\ \Pi_\alpha^\mu &= \partial_\alpha X^\mu - i\bar\theta^A\Gamma^\mu\partial_\alpha\theta^A\,. \end{aligned}$$

[4] This is often referred to as local κ-symmetry.

The metric of D-dimensional spacetime is kept implicit in the contracted indices μ. Note that S_2 does not depend on the metric $h^{\alpha\beta}$ that describes the internal geometry of the world-sheet. From the world-sheet point of view it is a *topological* term.

The θ^A with A=1,...,N are Majorana-Weyl spinors and Γ^μ are D-dimensional spacetime gamma matrices of appropriate dimension. Local supersymmetry requires $N = 0, 1$ or 2.

Different superstring theories can be formulated by means of (2.22). Setting both, θ^1 and θ^2, to zero, (2.22) reduces to the action (2.15) of bosonic string theory with $\alpha' = \frac{1}{2}$. Setting one of them to zero, describes the $N = 1$ case. In the AdS/CFT correspondence, however, we will be interested in type IIB string theory. II indicates the $N = 2$ case with θ^1 and θ^2 intact and B denotes the case where both spinors have the same handedness χ

$$\Big(\prod_{\mu=0}^{D-1} \Gamma^\mu\Big)\theta^A = \chi\theta^A, \qquad \chi = \pm 1\,. \tag{2.23}$$

As in the bosonic case, the symmetries of the action can be used to reduce the action to the physical degrees of freedom. In flat space, the dynamics of the field content of the resulting gauge fixed superstring action are governed by the free Klein-Gordon equation and the free Dirac equation. Expansion of the fields in oscillator modes and quantization is then as straightforward as in the purely bosonic case. The spectrum of superstring theory can then be formulated without tachyonic states.

2.2.3 String Theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$

Despite the large amount of symmetries of $\mathrm{AdS}_5 \times \mathrm{S}^5$ (as described in section 2.1), string theory has not been quantized exactly in this background up to date. String theory in flat space is easy to solve due to the simple form the action takes after fixing light-cone gauge. As mentioned above, the resulting action (2.20) is that of a free, massless field theory. The transverse fields appear quadratically in the action and thus, a solution for the equations of motion of the transverse fields can easily be found. Due to the non-trivial metric (2.9) this cannot be achieved in $\mathrm{AdS}_5 \times \mathrm{S}^5$. A priori one has an infinite expansion in powers of the transverse fields z_a, y_s and thus a complicated interacting field theory. However, it is possible to analyze different limits of string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime, and we will come back to this in section 3.

2.3 $\mathcal{N} = 4$ Super Yang-Mills

Even though the focus of this thesis lies on the string theory side of the AdS/CFT correspondence, we will mention some important features of the dual gauge theory.

This will enable us to motivate the near-flat space limit in a better way. The dual theory is $\mathcal{N} = 4\,\mathrm{SYM}$ in 4 dimensions and its action is determined uniquely by the coupling constant g_{YM} and the rank N of the gauge group $U(N)$:

$$\begin{aligned} S = \frac{2}{g_{YM}^2}\int d^4x \quad Tr\Big(& \frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi_i)^2 - \frac{1}{4}[\phi_i,\phi_j][\phi_i,\phi_j] \\ + & \frac{1}{2}\bar{\chi}\Gamma^\mu D_\mu\chi - \frac{i}{2}\bar{\chi}\Gamma_i[\phi_i,\chi]\Big) \end{aligned} \qquad (2.24)$$

where $D_\mu = \partial_\mu - i[A_\mu, .]$ is the covariant derivative, A_μ is a gluon field, ϕ_i are six scalars ($i = 1, ..., 6$) and χ represents 4 Majorana gluinos, written as a 16-component Majorana-Weyl spinor with components χ_a ($a = 1, ..., 16$). Γ_μ, Γ_i comprise the 10-d 16×16 Dirac matrices, whose appearance can be understood by constructing the Lagrangian via dimensional reduction of 10-dimensional $\mathcal{N} = 1$ Super Yang-Mills theory [11].

This field theory is invariant under the conformal group in 4 dimensions, i.e. the Poincaré group, dilations and special conformal transformations, together forming the group $SO(2,4) \simeq SU(2,2)$. Another symmetry is the so-called *R-symmetry*, i.e. the $SO(6) \simeq SU(4)$ symmetry group acting as internal rotation of the six scalars and four spinors. Supplemented by the invariance of the Lagrangian under $\mathcal{N} = 4$ Poincaré supersymmetry, the *superconformal symmetry group*[5] $SU(2,2|4)$ is built.

The symmetry group survives quantization and thus the superconformal group $SU(2,2|4)$ is a full quantum mechanical symmetry. The β-function is believed to vanish identically, since no dependence on any renormalization scale is introduced during the renormalization process, i.e. the coupling constant g_{YM} is not renormalized.

It should be noticed that the bosonic part of the symmetry group $SO(2,4)\times SO(6)_R$ precisely matches the isometry group of $\mathrm{AdS}_5 \times \mathrm{S}^5$ (see 2.1.3). Therefore, it is reasonable to label any string state or operator by the eigenvalues of the six Cartan generators of $SO(2,4) \times SO(6)$

$$(E, \underbrace{S_1, S_2}_{S^3}, \underbrace{J_1, J_2, J_3}_{S^5}) \qquad (2.25)$$

where E will be referred to as the energy, S_i as the commuting *spins* on the three sphere within AdS_5 and J_k as the commuting angular momenta of the S^5 sphere. In order to set up a *dictionary* between the string states and the operators, as promised by the AdS/CFT correspondence, it is useful to identify states with identical sets of charges.

[5] The explicit transformation laws of the fields and the algebra of the generators can be found for example in [2], [5].

General conformal field theories (CFT's) possess operators with specific features. In general, we are interested in local, gauge invariant operators of the form $\mathcal{O}_{ij...k}(x) = Tr[\phi_i(x)\phi_j(x)...\phi_k(x)]$, i.e. traces of products of fundamental fields of the theory. A special class of operators are the so-called *conformal primary operators* which have definite *scaling dimensions*. Conformal symmetry constrains the two-point correlation functions of these operators to the form

$$\langle \mathcal{O}_\alpha(x)\mathcal{O}_\beta(y)\rangle = \frac{\delta_{\alpha\beta}}{(x-y)^{2\Delta_\alpha}} \tag{2.26}$$

where Δ_α is the scaling dimension of the operator $\mathcal{O}_\alpha$. Classically, the scaling dimension Δ_0 is the sum of the dimensions of the fields the operator is composed of ($[\chi] = 3/2$, $[A_\mu] = 1$, $[\phi_i] = 1$), whereas in the quantum case the operators aquire *anomalous dimensions*. The anomalous corrections can be organized as a double expansion in the 't Hooft coupling λ and in $1/N^2$

$$\Delta = \Delta_0 + \sum_{k=1}^{\infty} \lambda^k \sum_{l=0}^{\infty} \frac{1}{N^{2l}} \Delta_{k,l} \, . \tag{2.27}$$

The calculation of the scaling dimensions can be cast into the form of the eigenvalue problem of the dilatation operator.

2.3.1 Dilatation Operator, Spin Chain Picture and Bethe Equations

This section is intended to give a brief idea of how the calculation of the scaling dimensions is related to the so-called *Bethe equations*. We will not go into detail though; for a nice review consult [12]. As mentioned, the calculation of the scaling dimensions is simplified by the use of the dilatation operator which was introduced in [13], [14]. It acts on the operator $\mathcal{O}$ and its eigenvalues are the scaling dimensions

$$\mathcal{D} \circ \mathcal{O}_\alpha(x) = \sum_{n=0}^{\infty} \mathcal{D}^{(n)} \circ \mathcal{O}_\alpha(x) = \Delta_{\mathcal{O}_\alpha} \mathcal{O}_\alpha(x) \tag{2.28}$$

where $\mathcal{D}^{(n)}$ is of order $(g_{YM}^2)^n$, i.e. $\mathcal{D}^{(0)}$ yields the tree-level piece, $\mathcal{D}^{(1)}$ corresponds to the one-loop contribution and so on.

Looking at a subsector of the theory with states having non-zero values only for the two angular momenta $(J_1, J_3) = (J, M)$ on S^5, one can directly find, that the planar $\mathcal{D}^{(1)}$-part of the dilatation operator is equivalent to the Hamiltonian of the Heisenberg $XXX_{1/2}$ spin chain [12]. This was first remarked by Minahan and Zarembo [15]. In table 2.1 we picture this relation[6] for the case of a long *operator*

[6] Table 2.1 should not be taken too seriously, since we have only written down part of an operator with $(J_1, J_3) = (J, M)$ as permutations and cyclicity are omitted here. The picture is also true for higher parts of $\mathcal{D}$ when interactions between separated particles in addition to directly neighbored particles are taken into account.

chain which will also be of use for the BMN limit that we consider later. Then we can construct a state with charges $(J_1, J_3) = (J, M)$ by building an operator of J Z-fields and M W-fields with appropriate charges. When we have a chain of diverging length $L = J + M \approx J$ with almost exclusively Z-fields we call the fields W *impurities*. The tree level piece $\mathcal{D}^{(0)}$ simply measures the *length* of the spin chain and yields $\Delta_0 = J + M$.

$\mathcal{D}^{(1)}$ Tr	$[ZZ...Z$	W	$Z..Z$	W	$Z...Z]$	$= \Delta_1$ Tr	$[ZZ...ZWZ..ZWZ...Z]$
		$\updownarrow$		$\updownarrow$			
$H_{\substack{spin\\chain}}$	$\vert\downarrow\downarrow\downarrow ... \downarrow$	$\uparrow$	$\downarrow ... \downarrow$	$\uparrow$	$\downarrow ... \downarrow\rangle$	$= E$	$\vert\downarrow\downarrow\downarrow ... \downarrow\uparrow\downarrow ... \downarrow\uparrow\downarrow ... \downarrow\rangle$

Table 2.1: Analogy Dilatation Operator - Spin Chain Hamiltonian

In the spin chain picture, these impurities correspond to the orientation *spin-up*, whereas the remaining fields correspond to a sea of *spin-down* particles. The scaling dimension of such a state corresponds to the energy of an excited state on the spin-chain side. Therefore, the *impurities* in such operator chains are also referred to as *magnons*. Interestingly, the eigenvalue problem of the planar part of the dilatation operator is translated to the eigenvalue problem of a spin-chain Hamiltonian in this way.

The eigenvalue problem of the Heisenberg spin-chain was solved by Bethe in 1931 [16]. Again, we will not go into detail but rather refer to [12] for a thorough discussion.

For a single excitation, the excited state can be seen as a free particle with momentum p. Thus, a plane wave ansatz for the wave function is appropriate and leads straightforwardly to the energy of the 1-particle state:

$$E = 4sin^2(\frac{p}{2})\,. \tag{2.29}$$

The *Bethe ansatz* for the wave function of the spin-chain system is a superposition ansatz with an incoming and an outgoing (scattered) plane wave. For two excitations (particles with momenta p_1, p_2) the position space ansatz for the wave function is

$$\psi(x_1, x_2) = e^{i(p_1x_1+p_2x_2)} + S(p_2, p_1)e^{i(p_2x_1+p_1x_2)} \tag{2.30}$$

where $S(p_2, p_1)$ is the 2-body scattering matrix (which is a 1×1 *matrix* in the spin-$\frac{1}{2}$-chain analogue of the SU(2) subsector of the gauge theory). In the scattering part of the wave function, the particles have just exchanged their momenta ($p_1 \leftrightarrow p_2$). The *Schrödinger equation* in position space then gives a set of equations. These determine the form of the S-matrix, and the energy to be the sum of the one-particle energies (2.29). Imposing periodicity conditions at the ends of the chain, one gets the *Bethe equations* for the two magnon problem, i.e. two equations that can be solved to determine possible values of the momenta p_1, p_2. The momenta can then be inserted in the expression for the energy. This solves the eigenvalue problem of the spin chain Hamiltonian.

The M-body scattering problem can be solved in an astonishingly simple way, as it factorizes into a sequence of 2-body scatterings, due to the *integrability* of the model. *Integrability* is due to the existence of higher conserved charges which commute with the Hamiltonian and amongst themselves. The resulting conservations laws imply that the particles only exchange their momenta, but the set of momenta $\{p_i\}$ is conserved! Therefore, a generalization of the ansatz (2.30) with 2-body scattering matrices $S(p_i, p_k)$ for all possible interactions ($i \neq k$) is appropriate. This M-magnon Bethe Ansatz[7] leads to a set of M *Bethe equations*

$$e^{ip_k L} = \prod_{i=1, i\neq k}^{M} S(p_i, p_k) \tag{2.31}$$

where $L = M + J$ is the length of the spin chain and $k = 1, 2, ..., M$. $S(p_i, p_k)$ has the same form as in the 2-body case and the energy is the sum of the one-particle energies (2.29)

$$E = \sum_{i=1}^{M} 4 sin^2(\frac{p_i}{2}) \,. \tag{2.32}$$

This solves the M-body scattering problem. Including the factor of proportionality between the spin-chain Hamiltonian and the dilatation operator we get

$$\Delta_1 = \frac{\lambda}{2\pi^2} \sum_{i=1}^{M} sin^2(\frac{p_i}{2}) \,. \tag{2.33}$$

The scaling dimension then is

$$\Delta = \Delta_0 + \Delta_1 + ... = J + M + \frac{\lambda}{2\pi^2} \sum_{i=1}^{M} sin^2(\frac{p_i}{2}) + \mathcal{O}(\lambda^2) \,. \tag{2.34}$$

These are the first terms in an expansion in $\lambda \ll 1$ of an all-loop guess, corresponding to long-range spin chain interactions that was formulated in [17]

$$J + \sum_{i=1}^{M} \sqrt{1 + \frac{\lambda}{\pi^2} sin^2(\frac{p_i}{2})} \,. \tag{2.35}$$

[7] The explicit form can be found in [12].

2.4 The AdS/CFT Correspondence

As mentioned, the AdS/CFT correspondence states the equivalence of two seemingly different theories: Type IIB superstring theory in 10-dimensional $\mathrm{AdS}_5 \times \mathrm{S}^5$ and its gauge theory dual $\mathcal{N} = 4$ SYM in 4 dimensions as described in section 2.2 and 2.3. The 4-dimensional space the gauge theory lives on can be shown to be the conformal boundary of AdS_5. This is to be understood in the sense that the boundary of the conformally compactified AdS_5, is identical to the conformal compactification of the 4-dimensional Minkowski space, as explained in [2].

Saying that the two theories are equivalent implies a precise map between the relevant quantities on both sides, which is summarized in table 2.2

Type IIB S.T.		$\mathcal{N} = 4$ SYM (CFT).
g_s		g_{YM}
	$\leftrightarrow$	
α'		N
	$\boxed{\frac{R^4}{\alpha'^2} = g_{YM}^2 N \equiv \lambda}$	
	$\boxed{4\pi g_s = g_{YM}^2}$	
string states $a_{I_1}^\dagger a_{I_2}^\dagger ... a_{I_n}^\dagger \lvert 0\rangle$	$\leftrightarrow$	local, gauge inv. operators $\mathcal{O}_\alpha(x) = Tr[\phi_{I_1}\phi_{I_2}...\phi_{I_n}]$
string energies	$\boxed{E = \Delta_\alpha}$	scaling dimensions Δ_α

Table 2.2: Parameter matchings in the AdS/CFT correspondence

Type IIB superstring theory is governed by the parameters g_s and α', i.e. by the string coupling constant and the inverse string tension, whereas the relevant parameters on the gauge theory side are the rank of the gauge group N and the Yang-Mills coupling constant g_{YM}. R is the *radius* of both AdS_5 and S^5 and the matching of the parameters is given in table 2.2. λ is the *'t Hooft coupling*.

One of the main statements of the AdS/CFT correspondence is that the energy eigenvalue E of a string state is equal to the scaling dimension Δ_α of a suitable gauge theory operator $\mathcal{O}_\alpha$. These gauge theory operators have to be identified and we will comment on this later.

It can be seen from table 2.3 that we are dealing with a *strong-weak coupling duality*. The AdS/CFT correspondence can be formulated in different versions, which we sort by the *strength* of the statement, corresponding to different values of λ.

$\lambda \to \infty$	$\Leftrightarrow$	$R^2 \gg \alpha'$	$\to$	SUGRA
		$N \to \infty$		
	$\nearrow$	$g_s \to 0$	$\to$	free string theory
λ finite				
	$\searrow$	any values of		
		N, g_{YM}	$\to$	full interacting string theory

Table 2.3: Versions of the AdS/CFT correspondence

When we take $\lambda \to \infty$ the parameter matchings of table 2.2 imply that $R^2 \gg \alpha'$, i.e. in string units the curvature of the space is very small. The string side can then be studied by its low energy effective description in terms of type IIB supergravity. However, this limit corresponds to the strong coupling regime on the gauge theory side and is thus perturbatively inaccessible. The '$\lambda \to \infty$ case' can be considered as the *weakest version* of the AdS/CFT correspondence.
Taking λ to have finite values in order to work in a perturbatively accessible regime on the gauge theory side, we can distinguish two further cases:
Taking $N \to \infty$ and at the same time $g_s \to 0$, the string side can be addressed by *free* IIB superstring theory. Though, even free string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is a complicated 2-dimensional field theory as was explained in 2.2.3.
In the *strongest* version of the correspondence, any values of N and g_{YM} are possible. Then however, on the string side, full interacting superstring theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ has to be considered.

Due to the different regimes of accessibility, a direct proof of the correspondence seems to be out of reach. Either we need a method to treat the gauge theory side non-perturbatively in order to handle the strong-coupling case or we need complete knowledge of superstring theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$, whose exact quantization has not been achieved, as yet, and remains a great challenge. Luckily, both theories turn out to have an overlapping regime that allows for tests of the correspondence. We will discuss this in section 3.2.

On the other hand, if the AdS/CFT correspondence is true, we have a fascinating calculational tool for studying the strongly coupled sector of gauge or string theories!

III

Limits of String Theory in $AdS^5 \times S_5$ Space

As mentioned before, it seems unclear at present how the exact quantum spectrum of super string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ can be obtained. However, there exist *limiting, solvable islands*. In this chapter we will discuss some of them, in order to embed and motivate the special limit that is investigated in this diploma thesis. For simplicity we will only take the bosonic part of the string action here, whereas in the case of the near-flat space limit, we will consider full super string theory.

3.1 Light-Cone Coordinates

We choose the spacetime light-cone coordinates

$$X^{\pm} = x^{\pm} = \frac{1}{2}(\phi \pm t)\,, \qquad X^M = x^M = (z_a, y_s)\,. \tag{3.1}$$

Using these coordinates and the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric (2.9), we can rewrite the Lagrangian[1] of the bosonic action (2.15)

$$\begin{aligned} \mathcal{L} \;=\; \frac{g}{2}\gamma^{\alpha\beta}\Big[& G_{++}\partial_\alpha x^+ \partial_\beta x^+ + G_{--}\partial_\alpha x^- \partial_\beta x^- \\ & + G_{+-}(\partial_\alpha x^+ \partial_\beta x^- + \partial_\alpha x^- \partial_\beta x^+) \\ & + G_{zz}\partial_\alpha z \partial_\beta z + G_{yy}\partial_\alpha y \partial_\beta y \Big] \end{aligned} \tag{3.2}$$

where g is the effective string tension $g = \frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}$ and

$$G_{++} = G_{--} = G_{\phi\phi} - G_{tt}\,, \quad G_{+-} = G_{\phi\phi} + G_{tt}\,. \tag{3.3}$$

3.2 Plane Wave Limit

The first limit we consider, is the plane wave limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$. In the plane wave background, superstring theory can be quantized exactly in light-cone gauge, as noticed by Metsaev and Tseytlin [18], [19]. The plane wave background is one of the three maximally supersymmetric backgrounds of IIB string theory, besides flat

[1] We will usually denote the Lagrangian density just as the Lagrangian since no confusion is to be expected.

Minkowski space and $\mathrm{AdS}_5 \times \mathrm{S}^5$.

Remarkably, the plane wave geometry arises as a Penrose limit of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. By virtue of the AdS/CFT correspondence, there should be a corresponding limit of the gauge theory, which can indeed be found. Despite the fact that we are dealing with a strong-weak coupling duality as explained above, this limit has an overlapping regime which is perturbatively accessible from both sides. This makes it possible to establish a concrete *dictionary* between the string states and the operators of the Super Yang-Mills theory.

The idea is to consider a string that is boosted to light-like momentum along some direction. The local geometry *seen* by the string then is the plane wave geometry as will be explained in the following. A simple choice is to boost along the equator of the S^5, i.e. to consider a light-like trajectory $\phi = t$, $\rho = \theta = 0$ in the coordinates (2.7). As was explained in section 2.1.2, the model has translational invariance in t and ϕ, giving rise to the conserved charges E resp. J. Thus, the considered trajectory corresponds to a state with large angular momentum $J \to \infty$.

A light-like trajectory $t = \phi$ corresponds to $x^- = 0$ in light-cone coordinates. In order to consider fluctuations around this trajectory $x^- \approx 0$, $z^a \approx 0$, $y^s \approx 0$, it is convenient to rescale coordinates[2] in order to *zoom* into the geometry seen by the rotating object

$$x^+ \to x^+ \,, \qquad x^- \to \frac{x^-}{R^2} \,, \qquad z^a \to \frac{z^a}{R} \,, \qquad y^s \to \frac{y^s}{R} \,. \tag{3.4}$$

The first two imply

$$t \to x^+ - \frac{x^-}{R^2} \,, \qquad \phi \to x^+ + \frac{x^-}{R^2} \,. \tag{3.5}$$

The effect of the rescaling can be pictured as in 3.1. The metric (2.9) can then be expanded in powers of R^2. The part proportional to R^2 vanishes and we get

$$ds^2 = 4dx^+dx^- - (z^2 + y^2)dx^+dx^+ + dz^2 + dy^2 + \mathcal{O}(1/R^2) \,. \tag{3.6}$$

This leading R-independent part is the well-known pp-wave metric. Quantization of string theory in this background is straightforward as we will argue soon. But first, we discuss the charges in this limit. As mentioned, E and J are the generators of shifts in time an angular momentum, $E = i\partial_t$, $J = -i\partial_\phi$. In rescaled light-cone coordinates (3.4) we then have the following conserved charges:

$$\begin{aligned} 2p^- &= i\partial_{x^+} = i(\partial_t + \partial_\phi) = (E - J) \\ 2p^+ &= i\partial_{x^-} = \frac{i}{R^2}(\partial_\phi - \partial_t) = -\frac{J + E}{R^2} \,. \end{aligned} \tag{3.7}$$

[2] It would be cleaner to define new coordinates $\tilde{x}$ as $\tilde{x} = \frac{x}{R^2}$ and so on, but we omit the tilde for convenience. The reader should nevertheless keep in mind that after rescaling the coordinates are not the same as before.

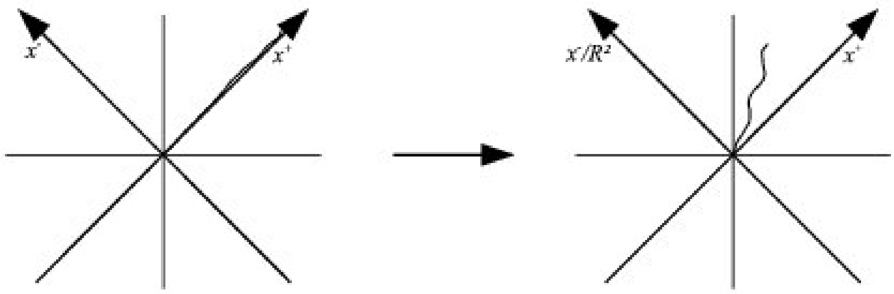

Figure 3.1: The effect of the rescaling is a zoom into the trajectory of the particle. Analogously for a string.

Clearly, we see that in order to get finite values for these momenta in the case $J \to \infty$, we need to consider states with $E \approx J$ and $J \sim R^2 \sim \lambda^{1/2}$. This leads to

$$p^+ = -\frac{J}{R^2}\,. \tag{3.8}$$

The quantization procedure is now essentially the same as in section 2.2.1 but with the metric (3.6). The important difference is that we now have a non-vanishing G_{++}-component in the metric. After gauge fixing $x^+ = p^+\tau$ and solving the Virasoro constraints, we again get an action in terms of the transverse fields $x^i = \{z^a, y^s\}$

$$S \propto \int d\sigma d\tau \Big((\partial_\tau x^i)^2 - (\partial_\sigma x^i)^2 - (x^i)^2 \Big) \tag{3.9}$$

which has the form of a free massive field theory with equations of motion

$$(\partial_\tau^2 - \partial_\sigma^2 + 1)x^i = 0\,. \tag{3.10}$$

Taking into account the closed string boundary conditions $x^i(\tau, \sigma + 2\pi\alpha' p^+) = x^i(\tau, \sigma)$, the most general oscillator mode decomposition for x^i that satisfies the equations of motion (3.10) is

$$x^i(\sigma,\tau) \;=\; \sum_{n=-\infty}^{\infty} \frac{i}{\sqrt{2\omega_n p^+}} \Big(a_n^i e^{-i\omega_n \tau} + a_n^{\dagger i} e^{i\omega_n\tau} \Big) e^{-ik_n\sigma} \tag{3.11}$$

with

$$\begin{aligned} k_n &= \frac{n}{\alpha' p^+} \overset{(3.8)}{=} -\frac{nR^2}{\alpha' J} = -\frac{\sqrt{\lambda}}{J} n \\ \omega_n &= \sqrt{1+k_n^2} = \sqrt{1+\frac{\lambda}{J^2}n^2} = \sqrt{1+\lambda' n^2} \end{aligned} \tag{3.12}$$

where we used the parameter matchings of table 2.2 and introduced the effective parameter $\lambda' = \frac{\lambda}{J^2}$. Thus, in the limit $J \to \infty$, we get finite values for ω_n if λ' takes finite values, i.e. for

$$\boxed{J \sim \lambda^{1/2}} \tag{3.13}$$

as we already noticed above. With the usual procedure one can then calculate the conjugate momentum to (3.11) and evaluate the Poisson brackets. Quantization is achieved by turning the fields into operators and replacing $\{.,.\}_{P.B.} \to -i[.,.]$. The coefficients $a_n^{\dagger i}, a_n^i$ become creation and annihilation operators and their commutators can be calculated straightforwardly. It is then possible to express the light-cone Hamiltonian in terms of these operators:

$$H^{l.c.}_{pp-wave} = \sum_{n=-\infty}^{\infty} \omega_n \sum_{I=1}^{8} a_n^i a_n^{\dagger i} + const. = E - J\,. \tag{3.14}$$

The infinite constant vanishes exactly when taking into account the fermionic part of the action as well. We see, that each oscillator contributes the energy ω_n to the light-cone energy.

We can now compare the string energies with the scaling dimensions that where calculated with the Bethe ansatz on the gauge theory side in section 2.3.1. As mentioned before, we should identify string states with gauge theory operators that have equal sets of charges $(E, S_1, S_2, J_1, J_2, J_3)$. In the plane wave limit we just discussed, we considered states with $J \to \infty$ and $\frac{\lambda}{J^2}$ fixed, i.e. $J \sim \lambda^{1/2}$. On the gauge theory side, this is often referred to as the BMN limit, due to the name of the authors Berenstein, Maldacena and Nastase who first considered this limit in [20]. The BMN limit was the first to incorporate *true stringy* physics in contrast to weaker versions of the correspondence addressed by supergravity. The limit is often denoted as a *double scaling limit* since we consider the case $N \to \infty$, supplemented by $J \sim \sqrt{N}$. In this limit the S-matrix behaves as

$$S(p_k, p_j) \to 1 \tag{3.15}$$

which means that *there is no scattering of the elementary excitations in the BMN limit.* But then the Bethe equations (2.31) simply are

$$e^{ip_k L} = 1 \qquad \Leftrightarrow \qquad p_k = \frac{2\pi n}{L}\,. \tag{3.16}$$

Considering a chain of diverging length $L = J + M \approx J$ with a small number of impurities M, and taking into account (3.13), we therefore see that the magnon momenta scale as

$$\boxed{p_k \sim \lambda^{-1/2}} \tag{3.17}$$

in the plane wave limit. Inserting these solutions for the momenta in the dispersion relation (2.35) we see that

$$\Delta - J = \sum_{k=1}^{M} \sqrt{1 + \frac{\lambda}{\pi^2}\frac{\pi^2 n^2}{J^2}} = \sum_{k=1}^{M} \sqrt{1 + \lambda' n^2} \tag{3.18}$$

which exactly coincides with the energy (3.12) contributed by the string oscillators! It is thus reasonable to identify the string oscillators with the spin chain excitations.

This result confirms the matching of string energies of free non-interacting string theory and the scaling dimensions of planar $\mathcal{N} = 4$ SYM in this limit. This is what was promised by the AdS/CFT correspondence in section 2.4. It should be noted though, that in the BMN limit only a constrained class of operators survive and turn out to correspond to the free string excitations.
If the AdS/CFT correspondence is to hold in its strongest version, one should be able to go beyond free string theory, to the full interacting string theory. This requires to extend the analysis of $\mathcal{N} = 4$ SYM to the non-planar sector. Indeed, in the BMN double scaling limit $J \sim \sqrt{N} \to \infty$, also non-planar graphs that were suppressed with $1/N^2$, survive due to increasing combinatorics of the diagrams with $J \to \infty$. More on this can be found in the reviews [21], [22], [23], [24].

3.3 Near-Plane Wave Limit

The absence of scattering made the strict BMN limit comparatively simple. The situation gets much more involved once curvature corrections to the pp-wave metric are taken into account. Admitting finite but large values for R^2, corrections in $1/R^2$ to the metric (3.6) can be included:

$$\begin{aligned} ds^2 &= 4dx^+dx^- - (z^2+y^2)dx^+dx^+ + dz^2 + dy^2 + \\ &\quad + \frac{1}{2R^2}\Big(4(z^2-y^2)dx^-dx^+ + z^2dz^2 - y^2dy^2 - (z^4-y^4)(dx^+)^2\Big) \\ &\quad + \mathcal{O}(1/R^4)\,. \end{aligned} \tag{3.19}$$

The $1/R^2$-corrections add quartic interactions to the world-sheet theory and lead to first order shifts in the energy spectrum of the string. These should then correspond to $1/J$-corrections of the strict BMN limit results. The limit contains scattering and thus encounters a non-trivial S-matrix. This *near-plane wave* or *near-BMN limit* has been investigated to leading order $1/J$ on the string side, for two excitations ($M = 2$) in [25] and [26]. Three excitations ($M = 3$) were considered in [27]. For both cases intricate but perfect agreement between both sides of the correspondence to two-loop order in λ was found. However, the string and gauge theory predictions for the string energies, respectively the scaling dimensions fail to agree at three loops.

This was a hint for a possible failure of the AdS/CFT correspondence. In a first attempt to explain this phenomenon in [17], it was argued that for getting the results on the string resp. gauge theory side, limits are taken in a different order, which may not commute. Further work along these lines was done, leading to the insight that the discrepancies might be due to a change in the S-matrix (and thus to a phase shift $\theta(p_k, p_j)$) as we go from weak to strong coupling [28]. This would imply that the dispersion relation (2.35) may still hold for arbitrary values of the coupling constant. In [29], Bethe equations that reproduce exactly the results of [25], [26], [27] were proposed.

3.4 Giant Magnons

The strong coupling result for the dispersion relation (2.35) of a single magnon is

$$E(p_{mag}) - J \overset{\lambda \gg 1}{\to} \frac{\sqrt{\lambda}}{\pi} \left| sin(\frac{p_{mag}}{2}) \right| . \tag{3.20}$$

The question is whether we can identify the string dual to a *single* gauge theory magnon. We will see, that we can identify it with a *one-soliton solution of classical string theory on* $\mathbb{R} \times S^2$. Solitons are classical solutions of a field theory that carry finite energy. At first sight, it seems as if the string world-sheet had to be discrete in order to get a periodic dispersion relation as (3.20), just as the periodicity of the gauge theory result came from discreteness of the spin chain. The crucial point, however, that leads to a periodic dispersion relation for a continuous world-sheet, is to identify the magnon momentum with an angle in the considered geometry. This was first done in [30] and their limit can be summarized as follows

$$\begin{aligned} J \to \infty \,, \qquad & \lambda = g_{YM}^2 N = \text{fixed} \,, \\ p = \text{fixed} \,, \qquad & E - J = \text{fixed} \,. \end{aligned} \tag{3.21}$$

This was treated more generally and including finite J corrections in [31]. For the large J limit, their results reduce to those of [30]. To consider the subspace $\mathbb{R} \times S^2$, one picks the following coordinates from the $\text{AdS}_5 \times \text{S}^5$ metric (2.9):

$$t(\tau, \sigma) \in AdS_5, \quad \phi(\tau, \sigma),\, y_1(\tau, \sigma) \in S^5 \,, \tag{3.22}$$

such that the metric of this subspace becomes

$$ds^2 = -dt^2 + \Big(\frac{1 - \frac{y_1^2}{4}}{1 + \frac{y_1^2}{4}} \Big) d\phi^2 + \frac{dy_1^2}{1 - \frac{y_1^2}{4}} \,. \tag{3.23}$$

Reparametrizing[3] $y_1 = \frac{z}{1 + \frac{z^2}{4}}$, this is

$$ds^2 = -dt^2 + \frac{dz^2}{1 - z^2} + (1 - z^2) d\phi^2 \,. \tag{3.24}$$

For $z = cos\,\theta$ it is easy to see that the second part indeed parametrizes S^2:

$$ds^2_{S^2} = d\theta^2 + sin^2\theta d\phi^2 \,. \tag{3.25}$$

Taking t to be the time, this is the metric convenient for the description of a string moving on $\mathbb{R} \times S^2$. The string action (2.15) - here with the metric G_{MN} as in (3.24)

[3] Note that z is not related to the transverse coordinates z_k of AdS_5! We take this parameter to keep the conventions of [31], in order not to confuse the reader interested in studying [31] in detail.

- can be rewritten in first-order formalism leading to a gauge fixed action of the form

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau (p_i \dot{x}^i - \mathcal{H}) \tag{3.26}$$

where

$$\mathcal{H} = -p_-(x^i, x'^i) \tag{3.27}$$

as thoroughly explained in [31]. This action only depends on the transverse field x^i and its derivatives $\dot{x}^i, x'^i$.

The gauge fixed action is invariant under shifts of the world-sheet coordinate σ and the corresponding conserved charge is

$$p_{ws} = -\int d\sigma p_i x'^i \,. \tag{3.28}$$

In the gauge $x^+ = \tau$, $p_+ = 1$, the Virasoro constraints imply $x'^- = -p_i x'^i$. Thus

$$\Delta x^- = \int d\sigma x'^- = -\int d\sigma p_i x'^i = p_{ws} \,. \tag{3.29}$$

Therefore, the *level matching* condition $\Delta x^- = 0$ (closed strings are periodic in x^i, x^-), implies that the total world-sheet momentum vanishes, i.e. we have the same right- and left-moving *levels* of excitation. When we want to consider a single magnon though, it is necessary to drop the level matching condition and consider finite values of the world-sheet momentum $p_{ws} = \Delta x^- \neq 0$.

In the subspace that is investigated here, the only transverse coordinate is z. Eliminating p_z by its equations of motion, the action can be expressed in the form $S(z, \dot{z}, z')$. In order to find a one-soliton solution, a general plane wave ansatz

$$z = z(\sigma - v\tau) \tag{3.30}$$

is used, where v is the velocity of the soliton. Inserting this ansatz into the action leads to a reduced 1-d particle model described by the Lagrangian $L_{red} = L_{red}(z, z')$. Considering σ to be the *time* coordinate of this reduced model, one can express the reduced Hamiltonian as

$$H_{red} = \Pi_z z' - L_{red} = f(\omega) \tag{3.31}$$

where $\Pi_z = \frac{\partial L_{red}}{\partial z'}$ is the *momentum* with repect to time σ. H_{red} is a conserved quantity with respect to shifts $\sigma \to \sigma + const$. The parameter ω was introduced for convenience and is just a reparametrization of $\mathcal{H}_{red}$ as a function $f(\omega)$. This equation can be solved for z'^2 as a function of z, ω, v

$$z'^2 = g(z, \omega, v) \,. \tag{3.32}$$

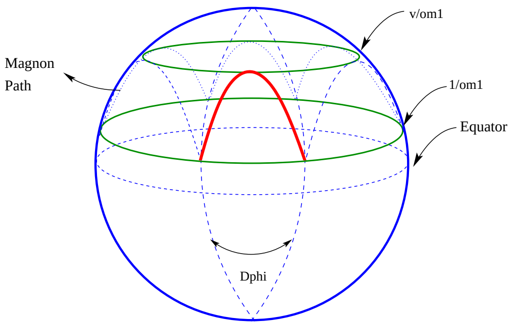

Figure 3.2: Time evolution of the finite J soliton solution on the sphere. The angular separation $\Delta\phi$ of the endpoints remains constant. The string develops a spike during time evolution. This figure was taken from [31].

It is possible to determine the values z_{min}, z_{max} that z can take and to integrate this equation numerically in order to get the profile $z(\sigma)$ for the one-magnon soliton solutions. The evolution of the solution with respect to time t in target space is shown in figure 3.2.

It is interesting to consider the $J \to \infty$ limit. In this case the explicit expression for (3.32) in [31] simplifies and can easily be integrated. The limit corresponds to a decompactification of the world-sheet $-\infty \leq \sigma \leq \infty$ and the relevant quantities explicitly become

$$\begin{aligned} E - J &= \frac{\sqrt{\lambda}}{2\pi}\int_{-\infty}^{\infty} d\sigma\mathcal{H} = \frac{\sqrt{\lambda}}{\pi}\int_{z_{min}}^{z_{max}} dz\frac{\mathcal{H}}{|z'|} \overset{[31]}{=} \frac{\sqrt{\lambda}}{\pi}\int_0^{\sqrt{1-v^2}} \frac{zdz}{\sqrt{1-v^2-z^2}} \qquad (3.33)\\ &= \frac{\sqrt{\lambda}}{\pi}\sqrt{1-v^2}\\ p_{ws} &= -\int_{-\infty}^{\infty} d\sigma p_z z' = 2\int_{z_{min}}^{z_{max}} dz|p_z| \overset{[31]}{=} 2\int_0^{\sqrt{1-v^2}} dz\frac{vz}{(1-z^2)\sqrt{1-v^2-z^2}}\\ &= 2\,arccos\,v\,. \end{aligned}$$

Inverting p_{ws} to $v = cos(\frac{p_{ws}}{2})$ and inserting it into the first equation gives

$$E - J = \frac{\sqrt{\lambda}}{\pi}|sin(\frac{p_{ws}}{2})| \qquad (3.34)$$

which exactly coincides with the strong coupling limit of the gauge theory dispersion relation (3.20), if we identify the world-sheet momentum p_{ws} with the magnon momentum p_{mag}.

The identification of the magnon momentum with an angle, as mentioned in [30], becomes clear once one chooses the coordinates $x^- = \phi - t$, $x^+ = t$ and the gauge $t = \tau$. Then (3.29) becomes $p_{ws} = \Delta x^- = \Delta\phi$ and thus, the identification of the magnon momentum with the world-sheet momentum is equivalent to its identification with the angular separation of the string endpoints. These solutions were named *giant magnons* because for finite values of p_{mag} their size is generically of order of the S^5 radius.

3.5 Near-Flat Space Limit

Recently, Maldacena and Swanson proposed a new limit [1], that interpolates smoothly between the plane wave and giant magnon regimes considered above. It was denoted as the *near-flat space limit*, because it uses the same scaling $J \sim \lambda^{1/4}$, that is used to reproduce the energies of strings in flat space.

The dispersion relation

$$\epsilon(p) = \sqrt{1 + \frac{\lambda}{\pi^2} sin^2(\frac{p_{mag}}{2})} \tag{3.35}$$

at strong coupling $\lambda \gg 1$, is depicted in figure 3.3. For very small values of the magnon momentum, the 1 under the square root plays a significant role. Especially in the plane wave limit the dispersion relation is given by (3.18), whereas for larger values ($p_{mag} \sim \pi$) it can be approximated by (3.34).

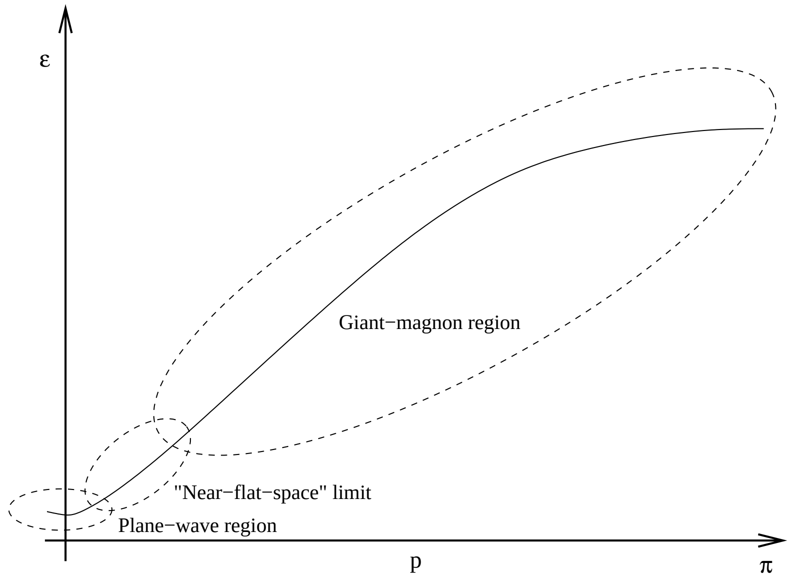

Figure 3.3: Dispersion relation for magnon energies. In the plane wave region, i.e. for small values of p_{mag}, the dispersion relation is given by (3.18). In the giant magnon region, it is given by (3.34). The near-flat space region is the region in between these. The figure was taken from [1].

As we have seen in section 3.2 and 3.4 for $\lambda \to \infty$ the magnon momenta behave as:

$$\begin{aligned} \text{plane wave limit} &: \; p_{mag} \sim \lambda^{-1/2} \\ \text{giant magnons} &: \; p_{mag} \sim \lambda^0 = finite\,. \end{aligned} \tag{3.36}$$

The most obvious way to probe the regime between these two is to scale the magnon momentum as

$$\text{near-flat space limit} \quad : \quad p_{mag} \sim \lambda^{-1/4} \,. \tag{3.37}$$

If one is interested in reproducing the energies of strings in flat space, one considers the strong coupling limit $\lambda \to \infty$ with $J \sim \lambda^{1/4}$. It should be mentioned that one cannot assure that the same asymptotic S-matrix formulas as in the plane wave limit can be applied in the case of the near-flat space limit. Using them anyway, equations (3.15) and (3.16) reappear and lead to the scaling of the magnon momenta

$$p_k \sim J^{-1} \sim \lambda^{-1/4} \,. \tag{3.38}$$

Going back to the Penrose limit of $AdS_5 \times S^5$ discussed at the beginning of section 3.2, we can find an appropriate rescaling, that requires the consideration of states with $J \sim \lambda^{1/4}$, by the same method. Rescaling as

$$x^+ \to Rx^+, \qquad\qquad x^- \to \frac{x^-}{R} \,, \tag{3.39}$$

the corresponding conserved charges become

$$\begin{aligned} 2p^- &= i\partial_{x^+} = R(E - J) \\ 2p^+ &= i\partial_{x^-} = -\frac{1}{R}(E + J) \,. \end{aligned} \tag{3.40}$$

Thus, if we have $E \approx J$ and $J \sim R$, we get non-vanishing results, being finite also for p^- if $E - J \sim R^{-1}$. Keeping in mind the parameter matching of the AdS/CFT correspondence, $\frac{R^4}{\alpha'^2} = \lambda$, this implies

$$\boxed{J \sim \lambda^{1/4}} \,. \tag{3.41}$$

Apparently, the ansatz (3.39) leads to the desired scaling.

However, the scaling used in [1] is motivated somewhat differently. The idea is to expand around a solution of the string sigma model satisfying

$$\dot{t} = 1, \qquad \dot{\phi} = 1 \tag{3.42}$$

where the dot stands for the derivative with respect to σ^0. Such a solution can be realized by

$$t = \phi = \sigma^0 = \frac{\sigma^+ + \sigma^-}{2} \tag{3.43}$$

where $\sigma^\pm = \sigma^0 \pm \sigma^1$ are world-sheet light-cone coordinates. Performing a boost on the world-sheet coordinates $\sigma^\pm \to (4g)^{\pm 1/2}\sigma^\pm$ (with $g \equiv \frac{\sqrt{\lambda}}{4\pi} \sim R^2$) and expanding in small fluctuations around this solution the ansatz can be written as

$$\begin{aligned} t &= \sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}(\frac{\sigma^-}{4} + \text{fluctuations}) = \sqrt{g}\sigma^+ + \frac{\tau}{\sqrt{g}} \\ \phi &= \sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}(\frac{\sigma^-}{4} + \text{fluctuations}) = \sqrt{g}\sigma^+ + \frac{\chi}{\sqrt{g}} \end{aligned} \tag{3.44}$$

where we incorporate the deviations from $t = \phi = \sqrt{g}\sigma^+$ in τ respectively χ. We use the same notations as in [1] and it should be noted that χ and τ are used as fields that depend on the world-sheet coordinates: $\tau = \tau(\sigma^0, \sigma^1)$. This is somewhat confusing since usually τ is reserved for the *time* component σ^0 of the world-sheet coordinates.
Considering the limit $\sqrt{g} \to \infty \Leftrightarrow R \to \infty$, we see that the proposed limit (3.44) indeed is of the form (3.39). Thus, the boost on $\sigma^\pm$ in the chosen coordinates has a similar effect as the Penrose limit of the $AdS_5 \times S^5$-geometry taken in section 3.2: fluctuations around a light-like trajectory $t = \phi$ are studied. Considering trajectories with $t = \phi = \sqrt{g}\sigma^+$ at leading order, leads to a model without diverging terms as we take g to infinity. We will explicitly see this in chapter 4. The relative factor between rescaled x^- and x^+ is R^2, which is the same as in the plane wave limit. We have not mentioned the rescaling of the remaining coordinates yet:

$$\begin{aligned} z &\to \frac{z}{\sqrt{g}}\,, & y &\to \frac{y}{\sqrt{g}}\,, \\ \theta^1 &\to \frac{\theta^1}{g^{1/4}} \equiv \eta_-\,, & \theta^2 &\to \frac{\theta^2}{g^{3/4}} \equiv \eta_+\,, & g &\to \infty\,, \end{aligned} \tag{3.45}$$

where θ^i denote the two 10-d Weyl spinors of type IIB superstring theory after fixing κ-symmetry. Since the fermions are not rescaled in the same manner, the Lagrangian will not have a symmetric structure in the fermions. The η_+ are suppressed with respect to η_-.
The scaling of $x^I = (z_a, y_s)$ is not manifestly different from the plane wave limit, but the *relative factor* between these transverse coordinates and x^+, x^- has changed. This and the scaling of the fermions are the aspects that distinguish the near-flat space limit from the plane wave limit. To find the gauge fixed action, [1] proceeds as follows:

- Derive the equations of motion for χ and τ. They can be written as $\partial_- j_+^\chi = 0$ and $\partial_- j_+^\tau = 0$.
- Fix the gauge by imposing $0 = j_+^\chi + j_+^\tau = \partial_+(\tau + \chi) + \frac{z^2 - y^2}{2}$.
- Change the world-sheet parametrization from $\sigma^\pm$ to $x^+ \equiv \sigma^+$, $x^- \equiv 2(\tau + \chi)$, i.e. $x^\pm$ are the new world-sheet coordinates (and not the spacetime coordinates).

This procedure seems somewhat complicated and we will therefore take a different and more familiar route in the next chapter, which in the end should lead to a model with the same properties.

IV

Near-Flat Space Limit

In this chapter the gauge fixed action for type IIB superstring theory in the near-flat space (NFS) limit, that was motivated in the last chapter, is derived by taking a slightly different way than [1]. It seems however more familiar and straightforward to take the route we present here.

4.1 Light-Cone Coordinates and Gauge-Fixing

We choose the world-sheet light-cone coordinates

$$\sigma^{\pm} = \frac{1}{2}(\sigma^0 \pm \sigma^1), \quad \partial_{\pm} = \partial_0 \pm \partial_1 \tag{4.1}$$

such that conformal gauge

$$\gamma^{\alpha\beta} = \begin{pmatrix} \gamma^{00} & \gamma^{01} \\ \gamma^{10} & \gamma^{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.2}$$

translates to $\gamma^{+-} = \gamma^{-+} = \frac{1}{2}$ and $\gamma^{\pm\pm} = 0$.[1]

We let t and ϕ scale as in [1]:

$$\begin{aligned} t &= \sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}\tau(\sigma^+, \sigma^-) \\ \phi &= \sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}\chi(\sigma^+, \sigma^-)\,, \end{aligned} \tag{4.3}$$

The fields τ and χ describe fluctuations around a light-like trajectory $t = \phi$. Defining $U = (\chi+\tau)$, $V = (\chi-\tau)$ and the spacetime light-cone coordinates $x^{\pm} = \frac{1}{2}(\phi \pm t)$, we get

$$\begin{aligned} x^+ &= \sqrt{g}\sigma^+ + \frac{1}{2\sqrt{g}}U \\ x^- &= \frac{1}{2\sqrt{g}}V\,. \end{aligned} \tag{4.4}$$

Fixing the gauge $U = 2\sigma^-$ we thus get

$$x^+ = \sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}\sigma^- \tag{4.5}$$

[1] With this convention the factor from the Jacobian is absorbed in γ, such that $S = \frac{1}{4\pi\alpha'}\int d\sigma^+ d\sigma^- \gamma^{\alpha\beta}\partial_\alpha x \partial_\beta x$ with $\alpha, \beta = +, -$.

which is nothing but the familiar light-cone gauge in boosted world-sheet coordinates

$$x^+ = \tau = \sigma^+ + \sigma^- \longrightarrow \sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}\sigma^- \,. \tag{4.6}$$

Note that in contrast to [1] here we use $\sigma^\pm \to g^{\pm 1/2}\sigma^\pm$, differing by factors of 2. Except for this and the different gauge choice we scale the other coordinates in exactly the same way as [1], see equation (3.45). Note that the rescaling $\sigma^\pm \to g^{\pm 1/2}\sigma^\pm$ induces a change of the derivatives with respect to the world-sheet coordinates: $\partial_\pm \to g^{\mp 1/2}\partial_\pm$. However, contracted terms $\gamma^{\alpha\beta}\partial_\alpha X \partial_\beta X$ will not be affected by this, since the metric transforms as well under the reparametrization.

4.2 Bosonic Model

The bosonic model can easily be obtained by starting from (3.2), carrying out the scalings (3.45) and using (4.4). In addition, we assume here that it is consistent to take the conformal gauge metric $\gamma^\pm = \frac{1}{2}$, $\gamma^{\pm\pm} = 0$. We will investigate the validity of this light-cone gauge in section 4.7. With the expansions (B.2) of the metric this leads straightforwardly to

$$\begin{aligned} \mathcal{L} &= \frac{g}{2}\partial_- V - \frac{1}{4}(z^2+y^2)\partial_- U + \frac{1}{4}(\partial_+ V \partial_- U + \partial_- V \partial_+ U) \\ &\quad + \frac{1}{4}(z^2-y^2)\partial_- V + \frac{1}{2}(\partial_+ z \partial_- z + \partial_+ y \partial_- y)\,. \end{aligned} \tag{4.7}$$

Using the gauge choice $U = 2\sigma_-$ introduced in (4.5) this reduces to

$$\begin{aligned} \mathcal{L} &= \frac{g}{2}\partial_- V - \frac{1}{2}(z^2+y^2) + \frac{1}{2}\partial_+ V + \frac{1}{4}(z^2-y^2)\partial_- V \\ &\quad + \frac{1}{2}(\partial_+ z \partial_- z + \partial_+ y \partial_- y) \\ &= -\frac{1}{2}(z^2+y^2) + \frac{1}{4}(z^2-y^2)\partial_- V + \frac{1}{2}(\partial_+ z \partial_- z + \partial_+ y \partial_- y)\,, \end{aligned} \tag{4.8}$$

where we dropped total derivatives in the last line. In conformal gauge, the Virasoro constraints read

$$0 \stackrel{!}{=} T_{\alpha\beta} = \partial_\alpha x^M \partial_\beta x^N G_{MN}\,. \tag{4.9}$$

Such that $T_{--} = 0$ can be solved for

$$\partial_- V = -\frac{(\partial_- z)^2 + (\partial_- y)^2}{2}\,. \tag{4.10}$$

Inserting this in (4.8), we get

$$\mathcal{L} = -\frac{1}{2}(z^2+y^2) - \frac{1}{8}(z^2-y^2)((\partial_- z)^2 + (\partial_- y)^2) + \frac{1}{2}(\partial_+ z \partial_- z + \partial_+ y \partial_- y)\,. \tag{4.11}$$

The model has the same type of terms as in [1], but with different numerical factors. This is due to the fact that we perform different rescalings. However, exactly the same model can be achieved by choosing coordinates and a gauge with

$$\begin{aligned} x^+ &= \frac{1}{2}(\sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}\sigma^-) \\ x^- &= \frac{1}{2\sqrt{g}}V \\ x_M &\to \frac{2}{\sqrt{g}}x_M\,. \end{aligned} \tag{4.12}$$

This could be achieved by setting $t = \frac{1}{2}\sqrt{g}\sigma^+ +$ fluctuations, and leads to the Lagrangian

$$\mathcal{L} = 2(\partial_+ z\partial_- z + \partial_+ y\partial_- y - \frac{1}{4}(z^2+y^2) - (z^2-y^2)((\partial_- z)^2 + (\partial_- y)^2)) \tag{4.13}$$

which exactly coincides with the result of [1] up to an overall factor. However, the validity of the chosen light-cone gauge should be investigated before we try to recover the same factors.

4.3 Superstring in $\mathrm{AdS}_5\times\mathrm{S}^5$ as a Supercoset Sigma-Model

In this section, we discuss the *coset construction* of the superstring action. In [32] the covariant κ-symmetric superstring action for a type IIB superstring on $\mathrm{AdS}_5\times\mathrm{S}^5$ background was constructed and was defined as a 2-d non-linear sigma-model on the coset superspace

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}\,. \tag{4.14}$$

The supergroup $PSU(2,2|4)$ with Lie superalgebra $\imath\!\sim\!\approxeq(2,2|4)$ is the isometry group of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superspace, its bosonic part being $SO(2,4)\times SO(6) \simeq SU(2,2)\times SU(4)$ (see section 2.1).

The algebra $\mathfrak{psu}(2,2|4)$ is defined as the quotient algebra of the Lie superalgebra $\mathfrak{su}(2,2|4)$ over the $\mathfrak{u}(1)$ factor of its bosonic decomposition

$$\mathfrak{su}(2,2)\oplus\mathfrak{su}(4)\oplus\mathfrak{u}(1)\,. \tag{4.15}$$

This decomposition gets clear by considering the following definition[2] of the superalgebra $\mathfrak{su}(2,2|4)$:

[2] This definition and the following construction scheme can be found in [33], [34]

A convenient description of the Lie superalgebra $\mathfrak{su}(2,2|4)$ is provided by 8×8 matrices M which can be written in 4×4 blocks as

$$M = \begin{pmatrix} A & X \\ Y & D \end{pmatrix} \tag{4.16}$$

where A, D are Graßmann even and X, Y are Graßmann odd, i.e. depend linearly on anticommuting, fermionic variables. The matrix M is required to have vanishing *supertrace*

$$Str(M) \equiv Tr(A) - Tr(D) = 0 \tag{4.17}$$

and to satisfy the reality condition

$$(HM)^\dagger = -HM \qquad \Leftrightarrow \qquad HM + M^\dagger H = 0 \tag{4.18}$$

where H is a hermitian matrix which we choose as

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & 1 \end{pmatrix} \tag{4.19}$$

with

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{4.20}$$

These choices correspond to the ones used in [34], [35]. Since the eigenvalues of Σ are $(1,1,-1,-1)$ the condition (4.18) implies that A and D span the subalgebras $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$. The algebra $\mathfrak{su}(2,2|4)$ also contains the $\mathfrak{u}(1)$ generator $i1$ as it obeys (4.18) and has vanishing supertrace. Thus the bosonic subalgebra indeed admits the decomposition (4.15). The algebra $\mathfrak{psu}(2,2|4)$ is then the full $\mathfrak{su}(2,2|4)$ algebra over this $\mathfrak{u}(1)$ factor. It makes sense to define it this way, since $\mathfrak{psu}(2,2|4)$ cannot be realized as an 8×8 matrix superalgebra itself[3]. It may be noted that the condition (4.18) implies that the fermionic matrices X, Y are conjugated to each other in the sense $Y = -X^\dagger\Sigma$.

In the following we use the same notations as in [35]. The construction of the superstring uses the $\mathbb{Z}_4$ grading of the superalgebra $\mathfrak{su}(2,2|4)$ defined by the automorphism $M \to \Omega(M)$ with

$$\Omega(M) = \begin{pmatrix} KA^tK & -KY^tK \\ KX^tK & KD^tK \end{pmatrix} \tag{4.21}$$

[3] This is, because even if one omits the 1, it will reappear through the commutators of other elements of the algebra.

where

$$K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.22}$$

obeying $K^2 = -1$. With the help of this automorphism every matrix $M \in \mathfrak{su}(2,2|4)$ can be decomposed into

$$M = \underbrace{M^{(0)} + M^{(2)}}_{even} + \underbrace{M^{(1)} + M^{(3)}}_{odd} \tag{4.23}$$

where the matrices $M^{(l)}$ are the eigenstates of Ω:

$$\Omega(M^{(l)}) = i^l M^{(l)} \,. \tag{4.24}$$

The explicit form of these eigenstates is given in [35] as

$$\begin{aligned} M^{(0)} &= \frac{1}{2}\left(M_{even} + K_8 M_{even}^t K_8\right) \,, \qquad M^{(2)} = \frac{1}{2}\left(M_{even} - K_8 M_{even}^t K_8\right) \,, \\ M^{(1)} &= \frac{1}{2}\left(M_{odd} + i\widetilde{K}_8 M_{odd}^t K_8\right) \,, \qquad M^{(3)} = \frac{1}{2}\left(M_{odd} - i\widetilde{K}_8 M_{odd}^t K_8\right) \end{aligned} \tag{4.25}$$

where we have splitted M into Graßmann even and odd parts

$$M = M_{even} + M_{odd} \,, \quad M_{even} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \,, \quad M_{odd} = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \tag{4.26}$$

and where K_8 and $\widetilde{K}_8$ are defined as

$$K_8 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \,, \qquad \widetilde{K}_8 = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix} \,. \tag{4.27}$$

It can be shown that the matrices $M^{(0)}$ form the $\mathfrak{so}(4,1) \times \mathfrak{so}(5)$ subalgebra. Since we wish to *mod out* this part in the coset, the elements $M^{(0)}$ will not appear in the construction of the Lagrangian in the coset space (4.14). The matrices $M^{(2)}$ form the orthogonal complement to this in $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$. As argued in [35], based on the considerations of [32], one can then represent any matrix $M^{(2)}$ by the use of the Dirac matrices for $so(5)$ and $so(4,1)$. The explicit form of the Dirac matrices γ for $so(5)$ can be found in appendix A

$$\gamma_s \qquad\qquad s = 1,2,3,4 \tag{4.28}$$

$$\begin{aligned} \gamma_5 &\equiv \Sigma \\ \{\gamma_c, \gamma_d\} &= 2\delta_{cd} \qquad c,d = 1,2,3,4,5 \,. \end{aligned} \tag{4.29}$$

The same matrices can be used to build the Dirac matrices γ' for $so(4,1)$

$$\begin{aligned} \gamma'_a &= \gamma_s \qquad a = s = 1,2,3,4 \\ \gamma'_5 &= i\Sigma \\ \{\gamma'_c, \gamma'_d\} &= 2\eta_{cd} \qquad \eta = diag(1,1,1,1,-1) \,. \end{aligned} \tag{4.30}$$

In this basis, every matrix $M^{(2)}$ of $\mathfrak{su}(2,2|4)$ can be written as

$$\begin{aligned} M^{(2)} &= z_a \begin{pmatrix} \gamma_a & 0 \\ 0 & 0 \end{pmatrix} + y_s \begin{pmatrix} 0 & 0 \\ 0 & i\gamma_s \end{pmatrix} + t \begin{pmatrix} i\Sigma & 0 \\ 0 & 0 \end{pmatrix} + \phi \begin{pmatrix} 0 & 0 \\ 0 & i\Sigma \end{pmatrix} + im_0 1_{8\times 8} \\ &= x_M \Sigma_M + ix^+ \Sigma_+ + ix^- \Sigma_- + im_0 1 \end{aligned} \tag{4.31}$$

where z_a, y_s, t, ϕ, m_0 and $x^\pm = \frac{1}{2}(\phi \pm t)$ are real parameters and

$$\begin{aligned} \Sigma_M &= \Big\{ \Sigma_a, \Sigma_s \Big\} = \left\{ \begin{pmatrix} \gamma_a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & i\gamma_s \end{pmatrix} \right\} \\ \Sigma_+ &= \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma \end{pmatrix}, \qquad \Sigma_- = \begin{pmatrix} -\Sigma & 0 \\ 0 & \Sigma \end{pmatrix}. \end{aligned} \tag{4.32}$$

4.3.1 Lagrangian and Charges

We take a group element $g \in PSU(2,2|4)$ and consider the following current

$$A = -g^{-1} dg = \underbrace{A^{(0)} + A^{(2)}}_{even} + \underbrace{A^{(1)} + A^{(3)}}_{odd}. \tag{4.33}$$

With $A_\alpha = -g^{-1}\partial_\alpha g$, the Lagrangian density for the superstring in $AdS_5 \times S^5$ can be written [32] as the sum of a kinetic and a Wess-Zumino term[4]

$$\mathcal{L} = \frac{g}{2} Str\Big(\underbrace{\gamma^{\alpha\beta} A^{(2)}_\alpha A^{(2)}_\beta}_{kinetic\ term} + \underbrace{\kappa \epsilon^{\alpha\beta} A^{(1)}_\alpha A^{(3)}_\beta}_{Wess-Zumino\ term} \Big) \tag{4.34}$$

where κ-symmetry requires $\kappa = \pm 1$ and $\epsilon^{01} = -\epsilon^{10} = 1$. The global invariance of the action under the group $PSU(2,2|4)$ leads to the existence of conserved currents and charges. The conserved currents as derived in [33] are, in the notation of [35],

$$J^\alpha = \sqrt{\lambda} g \Big(\gamma^{\alpha\beta} A^{(2)}_\beta - \frac{\kappa}{2} \epsilon^{\alpha\beta} (A^{(1)}_\beta - A^{(3)}_\beta) \Big) g^{-1} \tag{4.35}$$

where g is the coset element and we have rewritten the coupling constant in terms of λ to avoid confusion. This leads to the conserved charge

$$Q = \int_0^{2\pi} \frac{d\sigma}{2\pi} J^\tau. \tag{4.36}$$

Note that Q is a matrix and the charges corresponding to rotations, dilations, supersymmetry and so on, have to be projected out with appropriate matrices $\mathcal{M}$

$$Q_{\mathcal{M}} = Str(Q\mathcal{M}). \tag{4.37}$$

The Hamiltonian for instance, is projected out with $\mathcal{M} = -\frac{i}{2}\Sigma_+$

$$H = -\frac{i}{2} Str(Q\Sigma_+). \tag{4.38}$$

More on the charges and symmetry algebra in this formalism can be found in [36].

[4] Note that in the expression (4.34) g is the coupling constant not the group element chosen above.

4.3.2 Choice of the Coset Element

There are different ways to parametrize the coset element (4.14). We choose it to have the same form as in [35]

$$g(x,\eta) = \Lambda(x^{\pm}) f(\eta) g(x_M) \tag{4.39}$$

where $g(x_M)$, $\Lambda(x^{\pm})$ describe an embedding of $\mathrm{AdS}_5 \times \mathrm{S}^5$ into $SU(2,2) \times SU(4)$ and $g(\eta)$ incorporates the fermionic degrees of freedom. Explicitly,

$$\begin{aligned} f(\eta) &= \eta + \sqrt{1+\eta^2} \\ \Lambda(x^{\pm}) &= exp\Big[\frac{i}{2}x^+\Sigma_+ + \frac{i}{2}x^-\Sigma_-\Big] \\ g(x_M) &= \begin{pmatrix} g_a(z) & 0 \\ 0 & g_s(y) \end{pmatrix} \end{aligned} \tag{4.40}$$

where

$$g_a(z) = \frac{1}{\sqrt{1-\frac{z^2}{4}}}(1+\frac{1}{2}z_a\gamma_a)\,, \qquad g_s(y) = \frac{1}{\sqrt{1+\frac{y^2}{4}}}(1+\frac{i}{2}y_s\gamma_s) \tag{4.41}$$

with $z^2 = z_a z^a$, $y^2 = y_s y^s$. It is easy to see that $g^{-1}(x_M) = g(-x_M)$, since $g(-x_M)g(x_M) = 1$. Using (4.28), (4.30) we find

$$\{\gamma_s, \Sigma\} = \{\gamma'_a, \Sigma\} = 0 \;\Rightarrow\; \Sigma_{\pm} g^{-1}(x_M) = g(x_M)\Sigma_{\pm}\,. \tag{4.42}$$

As mentioned above, η has the form

$$\eta = \begin{pmatrix} 0 & \theta \\ -\theta^{\dagger}\Sigma & 0 \end{pmatrix}, \qquad \theta = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \\ \theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} \\ \theta_{41} & \theta_{42} & \theta_{43} & \theta_{44} \end{pmatrix} \tag{4.43}$$

and θ_{ij} are complex fermions. The κ-symmetry of the supersymmetric string action can be used to impose the gauge [35]

$$\theta = \begin{pmatrix} 0 & 0 & \theta_{13} & \theta_{14} \\ 0 & 0 & \theta_{23} & \theta_{24} \\ \theta_{31} & \theta_{32} & 0 & 0 \\ \theta_{41} & \theta_{42} & 0 & 0 \end{pmatrix}\,. \tag{4.44}$$

With this, one can directly check that η anticommutes with Σ_+ and commutes with Σ_-

$$\begin{aligned} \Sigma_+\eta &= -\eta\Sigma_+ \\ \Sigma_-\eta &= +\eta\Sigma_-\,. \end{aligned} \tag{4.45}$$

It is easy to see that $f^{-1}(\eta) = f(-\eta)$, since $f(-\eta)f(\eta) = 1$. It follows that

$$\begin{aligned} f^{-1}(\eta)\Sigma_+ &= \Sigma_+ f(\eta) \\ f^{-1}(\eta)\Sigma_- &= \Sigma_- f^{-1}(\eta)\,. \end{aligned} \tag{4.46}$$

The even and odd parts can then be written as in [35]

$$\begin{aligned} A_{even} &= -g^{-1}(x_M)[\frac{i}{2}dx^+\Sigma_+(1+\eta^2) + \frac{i}{2}dx^-\Sigma_-]g(x_M) \\ &\quad -g^{-1}(x_M)[\sqrt{1+\eta^2}d\sqrt{1+\eta^2} - \eta d\eta + dg(x_M)g^{-1}(x_M)]g(x_M) \\ A_{odd} &= -g^{-1}(x_M)[idx^+\Sigma_+\eta\sqrt{1+\eta^2} + \sqrt{1+\eta^2}d\eta - \eta\sqrt{1+\eta^2}]g(x_M)\,. \end{aligned} \tag{4.47}$$

In order to construct the Lagrangian (4.34) we will have to calculate the elements $A_\alpha^{(k)}$ explicitly in the near-flat space limit, by use of (4.25) for $k = 1, 2, 3$ and $\alpha = +, -$. This will be done in the following sections.

4.3.3 Scaling of Fermions

In order to do the rescaling on the fermions, we have to split them up as

$$\eta = \eta_- + \eta_+ \tag{4.48}$$

and can then take the near-flat space limit:

$$\eta_- \to \frac{\eta_-}{g^{1/4}} \quad \eta_+ \to \frac{\eta_+}{g^{3/4}} \quad \text{and} \qquad g \to \infty\,. \tag{4.49}$$

At a first stage, we will not specify what $\eta_\pm$ are and later impose that they belong to the $M^{(1)}$ resp. $M^{(3)}$ subspaces by defining them as in (4.25):

$$\eta_\pm = \frac{1}{2}(\eta \pm i\tilde{K}_8\eta^t K_8)\,. \tag{4.50}$$

It will become clear later that this is a reasonable choice for a well-defined model.

4.4 Calculational Techniques

In the following, some useful calculational techniques for handling supermatrices of the type (4.16) are introduced and will be used throughout the rest of this work.

4.4.1 Cyclicity Properties of the Supertrace

Consider M_1, M_2 to have the structure

$$M_i = \begin{pmatrix} A_i & X_i \\ Y_i & B_i \end{pmatrix} \tag{4.51}$$

where A_i, B_i are Graßmann even and X_i, Y_i Graßmann odd. Then we have

$$\begin{aligned} Str(M_1 M_2) &= Str \begin{pmatrix} A_1 & X_1 \\ Y_1 & B_1 \end{pmatrix} \begin{pmatrix} A_2 & X_2 \\ Y_2 & B_2 \end{pmatrix} \\ &= Tr(A_1 A_2) + Tr(X_1 Y_2) - Tr(B_1 B_2) - Tr(Y_1 X_2) \end{aligned} \tag{4.52}$$

whereas

$$\begin{aligned} Str(M_2 M_1) &= Str \begin{pmatrix} A_2 & X_2 \\ Y_2 & B_2 \end{pmatrix} \begin{pmatrix} A_1 & X_1 \\ Y_1 & B_1 \end{pmatrix} \\ &= Tr(A_2 A_1) + Tr(X_2 Y_1) - Tr(B_2 B_1) - Tr(Y_2 X_1) \\ &= Tr(A_1 A_2) + Tr(X_1 Y_2) - Tr(B_1 B_2) - Tr(Y_1 X_2) \\ &= Str M_1 M_2 \,. \end{aligned} \tag{4.53}$$

Thus we have cyclicity of the supertrace

$$Str(M_1 M_2) = Str(M_2 M_1)\,, \tag{4.54}$$

in contrast to the ordinary trace in the presence of fermions, as used in line 3 of (4.53).

4.4.2 Transposition Properties of the Supertrace

Since it is built out of the usual trace-operator, the supertrace has the property

$$Str(M) = Str(M^t)\,. \tag{4.55}$$

In the following, a matrix of the structure

$$\eta = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \tag{4.56}$$

with X,Y Graßmann odd, will be called a fermion. When transposing a product of n such fermions, one gets

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \tag{4.57}$$

minus signs due to the anticommutation of the Graßmann variables, i.e.

$$Str(\eta_1 \eta_2 ... \eta_n)^t = (-1)^{\frac{n(n-1)}{2}} Str(\eta_n^t ... \eta_2^t \eta_1^t)\,. \tag{4.58}$$

Explicit examples are:

$$\begin{aligned} n = 2: &\qquad Str(\eta_+ \eta_-)^t = -Str(\eta_-^t \eta_+^t) \\ n = 3: &\qquad Str(\eta_+ \eta_- \eta_-)^t = -Str(\eta_-^t \eta_-^t \eta_+^t) \\ n = 4: &\qquad Str(\eta_+ \eta_- \eta_- \eta_-)^t = +Str(\eta_-^t \eta_-^t \eta_-^t \eta_+^t)\,. \end{aligned} \tag{4.59}$$

Application:

How to tranpose objects like

$$Str(X_n A Y_m^t) \tag{4.60}$$

if X_n and Y_m are products of n, resp. m fermions and A is non-fermionic?

We will illustrate this for 3 examples which will be of use in section 4.5.2. We use (4.58) in each line:

$$\begin{aligned}
n=1, m=3: \quad & Str(\eta_1 A(\eta_2\eta_3\eta_4)^t)^t \\
& = -Str(\eta_1 A\eta_4^t\eta_3^t\eta_2^t)^t \\
& = -Str(\eta_2\eta_3\eta_4 A^t\eta_1^t) \\
n=1, m=5: \quad & Str(\eta_1 A(\eta_2\eta_3\eta_4\eta_5\eta_6)^t)^t \\
& = +Str(\eta_1 A\eta_6^t\eta_5^t\eta_4^t\eta_3^t\eta_2^t)^t \\
& = -Str(\eta_2\eta_3\eta_4\eta_5\eta_6 A^t\eta_1^t) \\
n=3, m=3: \quad & Str(\eta_1\eta_2\eta_3 A(\eta_4\eta_5\eta_6)^t)^t \\
& = -Str(\eta_1\eta_2\eta_3 A\eta_6^t\eta_5^t\eta_4^t)^t \\
& = +Str(\eta_4\eta_5\eta_6 A^t\eta_3^t\eta_2^t\eta_1^t) \\
& = -Str(\eta_4\eta_5\eta_6 A^t(\eta_1\eta_2\eta_3)^t)
\end{aligned} \tag{4.61}$$

So for the cases {n,m}={1,3},{1,5},{3,3} we have

$$Str(X_n A Y_m^t)^t = -Str(Y_m A^t X_n^t)^t\,. \tag{4.62}$$

4.4.3 Transposition of Fermions

We will frequently encounter expressions involving combinations of K_8, $\tilde{K}_8$ and transposed fermions η^t. If we choose the fermions $\eta_\pm$ to be of the form (4.50)

$$\eta_\pm = \frac{1}{2}(\eta \pm i\tilde{K}_8\eta^t K_8)\,, \tag{4.63}$$

the following relation can be used:

$$K_8\eta_\pm^t\tilde{K}_8 = \pm i\eta_\pm\,, \qquad \tilde{K}_8\eta_\pm^t K_8 = \mp i\eta_\pm\,. \tag{4.64}$$

Proof:

Using the fact that the fermions are represented by upper-right and lower-left block matrices (4.56) and using the identities

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \tag{4.65}$$

and

$$\tilde{K}_8 K_8 = K_8 \tilde{K}_8 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.66}$$

as well as $K_8^t = -K_8$, $\tilde{K}_8^t = -\tilde{K}_8$, $K_8 K_8 = -1$ and $\tilde{K}_8 \tilde{K}_8 = -1$, this is easily shown:

$$\begin{aligned} K_8 \eta_\pm^t \tilde{K}_8 &= \frac{1}{2}(K_8 \eta^t \tilde{K}_8 \pm i K_8 (K_8^t \eta \tilde{K}_8^t) K_8) \\ &= \frac{1}{2}(K_8 \eta^t \tilde{K}_8 \pm i\eta) \\ &= \pm \frac{i}{2}(\eta \mp i K_8 \eta^t \tilde{K}_8) \\ &\overset{(4.65)}{=} \pm \frac{i}{2}(\eta \pm i K_8 (K_8 \tilde{K}_8) \eta^t (K_8 \tilde{K}_8) \tilde{K}_8) \\ &= \pm \frac{i}{2}(\eta \pm i \tilde{K}_8 \eta^t K_8) \\ &= \pm i \eta_\pm \,. \end{aligned} \tag{4.67}$$

The second relation can then be found as

$$\begin{aligned} K_8 \eta_\pm^t \tilde{K}_8 &\overset{(4.65)}{=} -K_8 (K_8 \tilde{K}_8) \eta_\pm^t (K_8 \tilde{K}_8) \tilde{K}_8 \\ &= -\tilde{K}_8 \eta_\pm^t K_8 \\ \Rightarrow \tilde{K}_8 \eta_\pm^t K_8 &= \mp i \eta_\pm \,. \end{aligned} \tag{4.68}$$

The relations (4.64) complemented by (4.58) can then be used to calculate several higher transpositions of fermions, like

$$\tilde{K}_8 \underbrace{(\eta_\pm ....\eta_\pm)^t}_{odd\ number} K_8 \,, \tag{4.69}$$

by corresponding insertions of $K_8 \tilde{K}_8$ between the fermions and use of (4.65). Terms like

$$K_8 \underbrace{(\eta_\pm ....\eta_\pm)^t}_{even\ number} K_8 \tag{4.70}$$

can be transposed by corresponding insertions of $K_8 K_8 = -1$ and $\tilde{K}_8 \tilde{K}_8 = -1$ between the fermions.

4.5 Lagrangian

As mentioned before, the Lagrangian can be written as the sum of a kinetic term and a Wess-Zumino term

$$\mathcal{L} = \frac{g}{2} Str\Big(\gamma^{\alpha\beta} A_\alpha^{(2)} A_\beta^{(2)} + \kappa \epsilon^{\alpha\beta} A_\alpha^{(1)} A_\beta^{(3)}\Big) \,. \tag{4.71}$$

Using the conventions of section 4.1, assuming that we can work in conformal gauge and using cyclicity of the supertrace the kinetic part is

$$\mathcal{L}_{kinetic} = \frac{g}{2} Str\Big(A_+^{(2)} A_-^{(2)}\Big) \tag{4.72}$$

and the WZ-part is

$$\mathcal{L}_{WZ} = \frac{g}{4}\kappa Str\Big(A_-^{(1)} A_+^{(3)} - A_+^{(1)} A_-^{(3)}\Big) . \tag{4.73}$$

4.5.1 Kinetic Part of the Lagrangian

It should be noted that the consistency of conformal gauge with our light-cone gauge is not trivial and will have to be investigated later. Thus it is not clear whether $\gamma^{\pm\pm}$ can be set to zero. A priori, the kinetic part is

$$\frac{g}{2} Str\Big(2\gamma^{+-} A_+^{(2)} A_-^{(2)} + \gamma^{++} A_+^{(2)} A_+^{(2)} + \gamma^{--} A_-^{(2)} A_-^{(2)}\Big) , \tag{4.74}$$

i.e. we need to calculate $Str(A_+^{(2)} A_-^{(2)})$, $Str(A_+^{(2)} A_+^{(2)})$ and $Str(A_-^{(2)} A_-^{(2)})$. The latter two will also be of use when solving the Virasoro constraints

$$T_{\pm\pm} = Str\Big(A_\pm^{(2)} A_\pm^{(2)}\Big) - \frac{1}{2}\gamma_{\pm\pm} Str\Big(\gamma^{\alpha\beta} A_\alpha^{(2)} A_\beta^{(2)}\Big) . \tag{4.75}$$

With the coset parametrization (4.40) and the equations (4.47) and (4.25) we get

$$\begin{aligned} A^{(2)} &= \frac{1}{2}\{-\frac{i}{2}(dx^+\Sigma_+ + dx^-\Sigma_-)(g_x^2 + g_x^{-2}) \\ &\quad -idx^+(\Sigma_+ g_x \eta^2 g_x - g_x K_8 (\eta^2)^t K_8 g_x \Sigma_+) \\ &\quad +g_x K_8 B^t K_8 g_x^{-1} - g_x^{-1} B g_x - (g_x^{-1} dg_x + dg_x g_x^{-1})\} \end{aligned} \tag{4.76}$$

where $g_x = g(x_M)$ and

$$B = \sqrt{1+\eta^2}\, d\sqrt{1+\eta^2} - \eta d\eta . \tag{4.77}$$

Inserting the spacetime light-cone coordinates defined in section 4.1, gives

$$\begin{aligned} A_-^{(2)} &= \frac{1}{2}\{-\frac{i}{2\sqrt{g}}(\frac{1}{2}\partial_- U\Sigma_+ + \frac{1}{2}\partial_- V\Sigma_-)(g_x^2 + g_x^{-2}) \\ &\quad -\frac{i}{\sqrt{g}}\frac{\partial_- U}{2}(\Sigma_+ g_x \eta^2 g_x - g_x K_8 (\eta^2)^t K_8 g_x \Sigma_+) \\ &\quad +g_x K_8 B_-^t K_8 g_x^{-1} - g_x^{-1} B_- g_x - (g_x^{-1}\partial_- g_x + \partial_- g_x g_x^{-1})\} \end{aligned} \tag{4.78}$$

and

$$\begin{aligned} A_+^{(2)} &= \frac{1}{2}\{-\frac{i}{2}((\sqrt{g} + \frac{1}{2\sqrt{g}}\partial_+ U)\Sigma_+ + \frac{1}{2\sqrt{g}}\partial_+ V\Sigma_-)(g_x^2 + g_x^{-2}) \\ &\quad -i\sqrt{g}(1 + \frac{1}{2g}\partial_+ U)(\Sigma_+ g_x \eta^2 g_x - g_x K_8 (\eta^2)^t K_8 g_x \Sigma_+) \\ &\quad +g_x K_8 B_+^t K_8 g_x^{-1} - g_x^{-1} B_+ g_x - (g_x^{-1}\partial_+ g_x + \partial_+ g_x g_x^{-1})\} . \end{aligned} \tag{4.79}$$

We introduce the short hand notation

$$
\begin{aligned}
-2A_-^{(2)} &= \frac{1}{\sqrt{g}}\alpha_- + \frac{1}{\sqrt{g}}\delta_- + \frac{1}{\sqrt{g}}\beta_- + \gamma_- + \epsilon_- \\
-2A_+^{(2)} &= \sqrt{g}\alpha_+ + \frac{1}{\sqrt{g}}\delta_+ + \sqrt{g}\beta_+ + \gamma_+ + \epsilon_+ \,.
\end{aligned} \tag{4.80}
$$

The introduced symbols $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$, $\delta_\pm$, $\epsilon_\pm$ are defined in equation (C.1). We start by calculating the *conformal gauge part* of the Lagrangian which is determined by

$$
Str\Big(A_+^{(2)}A_-^{(2)}\Big)\,. \tag{4.81}
$$

With the short hand notation (4.80) we get

$$
\begin{aligned}
Str\Big(A_-^{(2)}A_+^{(2)}\Big) \;=\; & \frac{1}{4}Str\Big(\sqrt{g}(\gamma_-\alpha_+ + \epsilon_-\alpha_+ + \gamma_-\beta_+ + \epsilon_-\beta_+) \\
& +(\alpha_-\alpha_+ + \alpha_-\beta_+ + \delta_-\alpha_+ + \delta_-\beta_+ + \beta_-\alpha_+ + \beta_-\beta_+ \\
& +\gamma_-\gamma_+ + \gamma_-\epsilon_+ + \epsilon_-\gamma_+ + \epsilon_-\epsilon_+) \\
& +\frac{1}{\sqrt{g}}(\alpha_-\gamma_+ + \alpha_-\epsilon_+ + \delta_-\gamma_+ + \delta_-\epsilon_+ \\
& +\beta_-\gamma_+ + \beta_-\epsilon_+ + \gamma_-\delta_+ + \epsilon_-\delta_+) \\
& +\frac{1}{g}(\alpha_-\delta_+ + \delta_-\delta_+ + \beta_-\delta_+)\Big)\,.
\end{aligned} \tag{4.82}
$$

This expression has to be expanded to order g^{-1}, since we are interested in the Lagrangian up to order g^0. Terms of the type $\beta^{-1/2}$, $\beta^{-3/2}$ correspond to the terms of β that are proportional to $g^{-1/2}$, $g^{-3/2}$ etc. and can be found explicitly in equation (C.1). Ordering the terms in powers of $g^{-1/2}$, we get

$$
\begin{aligned}
Str\Big(A_-^{(2)}A_+^{(2)}\Big) = \quad & g^0 \quad \frac{1}{4}Str\Big(\gamma_-^{-1/2}\alpha_+^0 + \epsilon_-^{-1/2}\alpha_+^0 + \alpha_-^0\alpha_+^0 + \delta_-^0\alpha_+^0\Big) \\
+ \quad & g^{-1/2} \quad \frac{1}{4}Str\Big(\gamma_-^{-1}\alpha_+^0 + \gamma_-^{-1/2}\beta_+^{-1/2} + \epsilon_-^{-1/2}\beta_+^{-1/2} \\
& \qquad + \alpha_-^0\beta_+^{-1/2} + \delta_-^0\beta_+^{-1/2} + \beta_-^{-1/2}\alpha_+^0\Big) \\
+ \quad & g^{-1} \quad \frac{1}{4}Str\Big(\gamma_-^{-3/2}\alpha_+^0 + \gamma_-^{-1/2}\alpha_+^{-1} + \epsilon_-^{-3/2}\alpha_+^0 + \epsilon_-^{-1/2}\alpha_+^{-1} + \\
& \qquad \gamma_-^{-1}\beta_+^{-1/2} + \gamma_-^{-1/2}\beta_+^{-1} + \epsilon_-^{-1/2}\beta_+^{-1} \\
& \qquad \alpha_-^0\alpha_+^{-1} + \alpha_-^{-1}\alpha_+^0 + \alpha_-^0\beta_+^{-1} + \delta_-^{-1}\alpha_+^0 + \delta_-^0\alpha_+^{-1} \\
& \qquad \delta_-^0\beta_+^{-1} + \beta_-^{-1}\alpha_+^0 + \beta_-^{-1/2}\beta_+^{-1/2} + \gamma_-^{-1/2}\gamma_+^{-1/2} \\
& \qquad +\gamma_-^{-1/2}\epsilon_+^{-1/2} + \epsilon_-^{-1/2}\gamma_+^{-1/2} + \epsilon_-^{-1/2}\epsilon_+^{-1/2} \\
& \qquad +\alpha_-^0\gamma_+^{-1/2} + \alpha_-^0\epsilon_+^{-1/2} + \delta_-^0\gamma_+^{-1/2} + \delta_-^0\epsilon_+^{-1/2} \\
& \qquad +\gamma_-^{-1/2}\delta_+^0 + \epsilon_-^{-1/2}\delta_+^0 + \alpha_-^0\delta_+^0 + \delta_-^0\delta_+^0\Big)\,.
\end{aligned} \tag{4.83}
$$

In the following, expression (4.83) will be evaluated order by order. To leading order we have

$$\frac{g^0}{4} Str\Big(\gamma_-^{-1/2}\alpha_+^0 + \epsilon_-^{-1/2}\alpha_+^0 + \alpha_-^0\alpha_+^0 + \delta_-^0\alpha_+^0\Big)\,. \tag{4.84}$$

The second term vanishes by supertracelessness of $\Sigma_+\Sigma_M$, the third by supertracelessness of $\Sigma_+\Sigma_+$ and the first term simplifies by using cyclicity properties of the supertrace, such that we get

$$\boxed{Str\Big(A_-^{(2)}A_+^{(2)}\Big) = \frac{i}{2} Str(\Sigma_+\partial_-\eta_-\eta_-) + \partial_- V + \mathcal{O}(g^{-1/2})} \tag{4.85}$$

to leading order. The second term can just be dropped as a total derivative. The first term, however, may lead to problems, since it is multiplied with another factor of g in the Lagrangian and thus diverges in the limit $g \to \infty$. As will be seen, exactly the same term reappears in the WZ-part and cancels this one, if we choose the fermions η_+ and η_- to belong to the $M^{(1)}$ and $M^{(3)}$ subspaces respectively, as achieved by the choice (4.50).

The next-to-leading order term in (4.83) is

$$\begin{aligned}
&\frac{1}{4\sqrt{g}} Str\Big(\gamma_-^{-1}\alpha_+^0 + \gamma_-^{-1/2}\beta_+^{-1/2} + \epsilon_-^{-1/2}\beta_+^{-1/2} + \\
&\qquad\qquad \alpha_-^0\beta_+^{-1/2} + \delta_-^0\beta_+^{-1/2} + \beta_-^{-1/2}\alpha_+^0\Big) \\
= \;& \frac{1}{2\sqrt{g}}\Big(iStr(\Sigma_+\partial_-\eta_+\eta_- + \Sigma_+\partial_-\eta_-\eta_+) + 0 + 0 \\
&\qquad - \frac{\partial_- U}{2} Str(\eta_-\eta_-) + 0 - \frac{\partial_- U}{2} Str(\eta_-\eta_-)\Big) \\
= \;& \frac{i}{2\sqrt{g}} Str(\Sigma_+\partial_-\eta_+\eta_-) + \frac{i}{2\sqrt{g}} Str(\Sigma_+\partial_-\eta_-\eta_+) - \frac{1}{\sqrt{g}} Str(\eta_-\eta_-)\frac{\partial_- U}{2}\,.
\end{aligned} \tag{4.86}$$

The second term in the first line vanishes by explicit calculation of the supertrace, whereas the third term vanishes due to supertracelessness of $\Sigma_M\eta_-^2$. The fifth term vanishes since $\Sigma_-\Sigma_+\eta_-^2$ is supertraceless as well. The remaining terms may comprise a problem when multiplied with g in the Lagrangian, since then they are of order $g^{1/2}$ and diverge in the limit $g \to \infty$. However, choosing (4.50), the first two terms combine to zero and the last one vanishes identically.

To order g^{-1} in (4.83) we have a significantly more complicated expression which is calculated in appendix C.1. The result is

$$\begin{aligned} Str\Big(A_+^{(2)}A_-^{(2)}\Big) &= \frac{1}{2g}\Big(iStr\Big(\Sigma_+\partial_-\eta_+\eta_+\Big) + iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big)\frac{\partial_-U}{2} \\ &+ \frac{3}{8}(z^2-y^2)iStr\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) + \partial_-V(z^2-y^2) \\ &- iStr\Big(\Sigma_+\partial_-x_N\Sigma_Nx_M\Sigma_M\eta_-^2\Big) - 2(z^2+y^2) - 4Str\Big(\eta_+\eta_-\Big)\frac{\partial_-U}{2} \\ &+ 2(\partial_-z\partial_+z + \partial_-y\partial_+y) + 2\partial_+V\frac{\partial_-U}{2}\Big)\,. \end{aligned} \tag{4.87}$$

Using the gauge $U = 2\sigma^-$ we get

$$\begin{aligned} Str\Big(A_+^{(2)}A_-^{(2)}\Big) &= g^0(...) + \\ &\quad \frac{1}{2g}\Big(iStr\Big(\Sigma_+\partial_-\eta_+\eta_+\Big) + iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big) + \\ &\quad \frac{3}{8}(z^2-y^2)iStr\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) + \partial_-V(z^2-y^2) \\ &\quad -iStr\Big(\Sigma_+\partial_-x_N\Sigma_Nx_M\Sigma_M\eta_-^2\Big) - 2(z^2+y^2) - 4Str\Big(\eta_+\eta_-\Big) \\ &\quad +2(\partial_-z\partial_+z + \partial_-y\partial_+y) + 2\partial_+V\Big)\,. \end{aligned} \tag{4.88}$$

Next we will be interested in calculating $Str(A_-^{(2)}A_-^{(2)})$. Using (4.80) we get

$$\begin{aligned} Str(A_-^{(2)}A_-^{(2)}) &= \frac{1}{4}Str\Big(g^0(\gamma_-\gamma_- + 2\gamma_-\epsilon_- + \epsilon_-\epsilon_-) \\ &\quad +g^{-1/2}(2\alpha_-\gamma_- + 2\beta_-\gamma_- + 2\delta_-\gamma_- + 2\alpha_-\epsilon_- + 2\delta_-\epsilon_- + 2\beta_-\epsilon_-) \\ &\quad +g^{-1}(\alpha_-\alpha_- + 2\alpha_-\delta_- + 2\alpha_-\beta_- + 2\delta_-\beta_- + \beta_-\beta_- + \delta_-\delta_-)\Big) \\ &= \frac{1}{4g}Str\Big(\gamma_-^{-1/2}\gamma_-^{-1/2} + 2\gamma_-^{-1/2}\epsilon_-^{-1/2} + \epsilon_-^{-1/2}\epsilon_-^{-1/2} \\ &\quad +2\alpha_-^0\gamma_-^{-1/2} + 2\delta_-^0\gamma_-^{-1/2} + 2\alpha_-^0\epsilon_-^{-1/2} \\ &\quad +2\delta_-^0\epsilon_-^{-1/2} + \alpha_-^0\alpha_-^0 + 2\alpha_-^0\delta_-^0 + \delta_-^0\delta_-^0\Big) + \mathcal{O}(g^{-3/2})\,. \end{aligned} \tag{4.89}$$

There are no terms at order g^0 and $g^{-1/2}$ since $\beta_\pm$, $\gamma_\pm$ as well as $\epsilon_\pm$ start at order $g^{-1/2}$. All of these terms have already been calculated in the appendix for $A_+^{(2)}A_-^{(2)}$ (except for different partial derivatives which do not change the matrix structure though). Therefore, one can just write down the result

$$\begin{aligned} Str(A_-^{(2)}A_-^{(2)}) &= \frac{1}{4g}Str\Big(0 + 0 + 4((\partial_-z)^2 + (\partial_-y)^2) \\ &\quad + 4iStr(\Sigma_+\partial_-\eta_-\eta_-)\frac{\partial_-U}{2} + 0 + 0 + 0 + 0 + 8\partial_-V\frac{\partial_-U}{2} + 0\Big) \\ &= \frac{1}{g}Str\Big((\partial_-z)^2 + (\partial_-y)^2 + iStr(\Sigma_+\partial_-\eta_-\eta_-)\frac{\partial_-U}{2} + 2\partial_-V\frac{\partial_-U}{2}\Big)\,. \end{aligned} \tag{4.90}$$

With the gauge $U = 2\sigma_-$ we have

$$\boxed{Str(A_-^{(2)}A_-^{(2)}) = \frac{1}{g}Str\Big((\partial_- z)^2 + (\partial_- y)^2 + iStr(\Sigma_+\partial_-\eta_-\eta_-) + 2\partial_- V\Big)}. \quad (4.91)$$

In conformal gauge, the T_{--} component of the energy momentum tensor (4.75) reduces to

$$T_{--} = Str\Big(A_-^{(2)}A_-^{(2)}\Big). \quad (4.92)$$

Imposing the Virasoro constraint $T_{--} = 0$ is equivalent to

$$\boxed{\partial_- V = -\frac{1}{2}((\partial_- z)^2 + (\partial_- y)^2) - \frac{i}{2}Str(\Sigma_+\partial_-\eta_-\eta_-)}. \quad (4.93)$$

Note however, that this is only true in conformal gauge and will have to be modified when curvature corrections to the world-sheet metric are taken into account.

The next interesting quantity is $Str(A_+^{(2)}A_+^{(2)})$:

$$\begin{aligned}
Str(A_+^{(2)}A_+^{(2)}) &= g\frac{1}{4}Str(\alpha_+\alpha_+ + 2\alpha_+\beta_+ + \beta_+\beta_+) \quad (4.94)\\
&+ \sqrt{g}\frac{1}{4}Str(2\alpha_+\gamma_+ + 2\alpha_+\epsilon_+ + 2\beta_+\gamma + 2\beta_+\epsilon_+)\\
&+ g^0\frac{1}{4}Str(2\alpha_+\delta_+ + 2\beta_+\delta_+ + 2\gamma_+\epsilon_+ + \gamma_+\gamma_+ + \epsilon_+\epsilon_+)\\
&= \frac{1}{4}gStr(\alpha_+^0\alpha_+^0) + \sqrt{g}\frac{1}{4}Str(2\alpha_+^0\beta_+^{-1/2})\\
&+ g^0\frac{1}{4}Str\Big(2\alpha_+^0\alpha_+^{-1} + 2\alpha_+^0\beta_+^{-1} + \beta_+^{-1/2}\beta_+^{-1/2}\\
&\qquad +2\alpha_+^0\gamma_+^{-1/2} + 2\alpha_+^0\epsilon_+^{-1/2} + 2\alpha_+^0\delta_+^0\Big)\\
&= \frac{1}{4}\Big(0 + \sqrt{g}\,4Str\Big(\eta_-\eta_-\Big)\\
&\qquad +g^0(-4(z^2+y^2) - 8Str\Big(\eta_+\eta_-\Big) + 0 +\\
&\qquad 4iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big) + 0 + 8\partial_+ V)\Big),
\end{aligned}$$

where again the calculations for $A_+^{(2)}A_-^{(2)}$ in the appendix, with different partial derivatives, were used. The term to order $\sqrt{g}$ vanishes, if we make use of (4.50) again. Then we have

$$\boxed{Str(A_+^{(2)}A_+^{(2)}) = -(z^2+y^2) - 2Str\Big(\eta_+\eta_-\Big) + iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big) + 2\partial_+ V}. \quad (4.95)$$

This can be used to impose the Virasoro constraint $T_{++} = 0$ and solve for $\partial_+ V$.

Now we are in position to write down the kinetic part of the Lagrangian in *conformal gauge*. We drop total derivatives and replace $\partial_- V$ by (4.93)

$$\begin{aligned}
\mathcal{L}_{kin} &= +\frac{g}{2} Str\Big(\gamma^{\alpha\beta} A^{(2)}_\alpha A^{(2)}_\beta\Big) \\
&\overset{c.f.g.}{=} +\frac{g}{2} Str\Big(A^{(2)}_+ A^{(2)}_-\Big) \\
&= +g\frac{i}{4} Str\Big(\Sigma_+ \partial_- \eta_- \eta_-\Big) \\
&+ \frac{1}{4}\Big(iStr(\Sigma_+\partial_-\eta_+\eta_+) + \frac{i}{8}(z^2 - y^2) Str(\Sigma_+\partial_-\eta_-\eta_-) \\
&\quad + \frac{i}{4}(z^2 - y^2) Str(\Sigma_+\partial_-\eta_-\eta_-) - iStr\Big(\Sigma_+\partial_- x_N \Sigma_N x_M \Sigma_M \eta_-^2\Big) \\
&\quad - 2(z^2 + y^2) - 2Str(\eta_+\eta_-) \\
&\quad + [-\frac{1}{2}((\partial_- z)^2 + (\partial_- y)^2) - \frac{i}{2} Str(\Sigma_+\partial_-\eta_-\eta_-)](z^2 - y^2) \\
&\quad - 2Str(\eta_+\eta_-) + 2(\partial_- z\partial_+ z + \partial_- y \partial_+ y) + iStr(\Sigma_+\partial_+\eta_-\eta_-)\Big),
\end{aligned} \tag{4.96}$$

such that the kinetic part of the gauge fixed Lagrangian (in conformal gauge) is

$$\begin{aligned}
\mathcal{L}_{kin} &= g\frac{i}{4} Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) + \frac{1}{2}(\partial_- z\partial_+ z + \partial_- y\partial_+ y) \\
&\quad - \frac{1}{2}(z^2 + y^2) - \frac{1}{8}(z^2 - y^2)((\partial_- z)^2 + (\partial_- y)^2) \\
&\quad - Str(\eta_+\eta_-) + \frac{i}{4} Str(\Sigma_+\partial_-\eta_+\eta_+) + \frac{i}{4} Str(\Sigma_+\partial_+\eta_-\eta_-) \\
&\quad - \frac{i}{32}(z^2 - y^2) Str(\Sigma_+\partial_-\eta_-\eta_-) - \frac{i}{4} Str\Big(\Sigma_+\partial_- x_N\Sigma_N x_M\Sigma_M\eta_-^2\Big).
\end{aligned} \tag{4.97}$$

The bosonic part is of course exactly the same as in (4.11). The remaining terms have the same structure as in [1], differing by factors due to the different gauge fixing and boost as explained for the purely bosonic Lagrangian in section 4.2. The diverging leading order term will be canceled by a contribution from the Wess-Zumino term as will become clear in the next section.

It is interesting to note that by solving the Virasoro constraints for $\partial_- V$, $\partial_+ V$ we can eliminate terms in the Lagrangian. By use of (4.95) we can replace the terms $-\frac{1}{2}(z^2 + y^2) - Str(\eta_+\eta_-)$ in the above Lagrangian. Up to a total derivative that can be dropped, this changes only the factor in front of $Str(\Sigma_+\partial_+\eta_-\eta_-)$. This leads to a Lagrangian with the same type of interactions, but no mass terms. However, we will not make use of this method, before having investigated the consistency of the gauge choice.

For future considerations, we summarize the expansion scheme of the terms that were calculated in this chapter.

$$\begin{aligned} A_+^{(2)} A_-^{(2)} &= g^0(...) + \frac{1}{g}(...) + ... \\ A_+^{(2)} A_+^{(2)} &= g^0(...) + ... \\ A_-^{(2)} A_-^{(2)} &= \frac{1}{g}(...) + ... \end{aligned} \tag{4.98}$$

4.5.2 Wess-Zumino Part of the Lagrangian

For the WZ-part of the Lagrangian (4.71), we need to evaluate

$$A_-^{(1)} A_+^{(3)} - A_+^{(1)} A_-^{(3)} \tag{4.99}$$

where

$$A^{(1)/(3)} = \frac{1}{2}(A_{odd} \pm i\tilde{K}_8 A_{odd}^t K_8) \tag{4.100}$$

and

$$A_{odd} = -g^{-1}(x_M)[idx^+\Sigma_+\eta\sqrt{1+\eta^2} + \sqrt{1+\eta^2}d\eta - \eta\sqrt{1+\eta^2}]g(x_M)\,. \tag{4.101}$$

Introducing the short hand notations[5]

$$\begin{aligned} \alpha_+ = &\quad i(1 + \frac{\partial_+ U}{2g})g^{-1}(x_M)[\Sigma_+\eta\sqrt{1+\eta^2}]g(x_M) \\ \alpha_- = &\quad i(\frac{\partial_- U}{2})g^{-1}(x_M)[\Sigma_+\eta\sqrt{1+\eta^2}]g(x_M) \\ \beta_\pm = &\quad g^{-1}(x_M)[\sqrt{1+\eta^2}\partial_\pm\eta - \eta\partial_\pm\sqrt{1+\eta^2}]g(x_M) \\ \gamma_\pm = &\quad i\tilde{K}_8\alpha_\pm^t K_8 \\ \epsilon_\pm = &\quad i\tilde{K}_8\beta_\pm^t K_8\,, \end{aligned} \tag{4.102}$$

we can write this as

$$\begin{aligned} A_+^{(1)} &= -\frac{1}{2}(\sqrt{g}\alpha_+ + \beta_+ + \sqrt{g}\gamma_+ + \epsilon_+) \\ A_+^{(3)} &= -\frac{1}{2}(\sqrt{g}\alpha_+ + \beta_+ - \sqrt{g}\gamma_+ - \epsilon_+) \\ A_-^{(1)} &= -\frac{1}{2}(\frac{1}{\sqrt{g}}\alpha_- + \beta_- + \frac{1}{\sqrt{g}}\gamma_- + \epsilon_-) \\ A_-^{(3)} &= -\frac{1}{2}(\frac{1}{\sqrt{g}}\alpha_- + \beta_- - \frac{1}{\sqrt{g}}\gamma_- - \epsilon_-)\,. \end{aligned} \tag{4.103}$$

[5] This notation has nothing to do with the one introduced in section 4.5.1; the α, β etc. are *not* the same.

We thus get

$$A_-^{(1)}A_+^{(3)} - A_+^{(1)}A_-^{(3)} = \tfrac{1}{4}\Big(\sqrt{g}([\beta_-,\alpha_+] - [\epsilon_-,\gamma_+] - \{\beta_-,\gamma_+\} + \{\epsilon_-,\alpha_+\}) \\ + ([\beta_-,\beta_+] + [\epsilon_+,\epsilon_-] - \{\beta_-,\epsilon_+\} + \{\epsilon_-,\beta_+\}) \\ + \tfrac{1}{\sqrt{g}}([\alpha_-,\beta_+] + [\epsilon_+,\gamma_-] - \{\alpha_-,\epsilon_+\} + \{\gamma_-,\beta_+\})\Big)\,. \tag{4.104}$$

By the arguments of section 4.4.1, the commutators vanish under the supertrace and the anti-commutators get just twice its ingredients such that the expression simplifies considerably:

$$\mathcal{L}_{WZ} = \frac{g}{4}\kappa Str(A_-^{(1)}A_+^{(3)} - A_+^{(1)}A_-^{(3)}) \\ = \frac{g}{8}\kappa Str\Big(\sqrt{g}(-\beta_-\gamma_+ + \epsilon_-\alpha_+) + (-\beta_-\epsilon_+ + \epsilon_-\beta_+) + \frac{1}{\sqrt{g}}(-\alpha_-\epsilon_+ + \gamma_-\beta_+)\Big)\,. \tag{4.105}$$

In order to obtain the WZ-part of the Lagrangian to the desired order, we need to expand all terms in powers of g. We find that the introduced symbols have an expansion of the form

$$\alpha_\pm = g^{-1/4}\alpha_\pm^{-1/4} + g^{-3/4}\alpha_\pm^{-3/4} + g^{-5/4}\alpha_\pm^{-5/4} \\ \beta_\pm = g^{-1/4}\beta_\pm^{-1/4} + g^{-3/4}\beta_\pm^{-3/4} + g^{-5/4}\beta_\pm^{-5/4}\,. \tag{4.106}$$

The expansion of $\gamma_\pm$, $\epsilon_\pm$ is then the same. The explicit expressions of $\alpha^{-1/4}$, $\alpha^{-3/4}$ etc. are listed in appendix D.

Collecting the necessary orders, we get

$$\mathcal{L}_{WZ} = g\frac{\kappa}{8}\; Str\Big(-\beta_-^{-1/4}\gamma_+^{-1/4} + \alpha_+^{-1/4}\epsilon_-^{-1/4}\Big) \\ +\sqrt{g}\frac{\kappa}{8}\; Str\Big(-\beta_-^{-3/4}\gamma_+^{-1/4} - \beta_-^{-1/4}\gamma_+^{-3/4} + \alpha_+^{-3/4}\epsilon_-^{-1/4} \\ +\beta_+^{-1/4}\epsilon_-^{-1/4} + \alpha_+^{-1/4}\epsilon_-^{-3/4} - \beta_-^{-1/4}\epsilon_+^{-1/4}\Big) \\ +\frac{\kappa}{8}\; Str\Big(-\beta_-^{-5/4}\gamma_+^{-1/4} + \beta_+^{-1/4}\gamma_-^{-1/4} - \beta_-^{-3/4}\gamma_+^{-3/4} - \beta_-^{-1/4}\gamma_+^{-5/4} \\ +\alpha_+^{-5/4}\epsilon_-^{-1/4} + \beta_+^{-3/4}\epsilon_-^{-1/4} + \alpha_+^{-3/4}\epsilon_-^{-3/4} + \beta_+^{-1/4}\epsilon_-^{-3/4} \\ +\alpha_+^{-1/4}\epsilon_-^{-5/4} - \alpha_-^{-1/4}\epsilon_+^{-1/4} - \beta_-^{-3/4}\epsilon_+^{-1/4} - \beta_-^{-1/4}\epsilon_+^{-3/4}\Big)\,. \tag{4.107}$$

We evaluate this order by order. To leading order we have

$$\mathcal{L}_{WZ}^1 = +g\frac{\kappa}{8}Str\Big(-\beta_-^{-1/4}\gamma_+^{-1/4} + \alpha_+^{-1/4}\epsilon_-^{-1/4}\Big) \\ = +g\frac{\kappa}{4}Str\Big(\Sigma_+\partial_-\eta_-\tilde{K}_8\eta_-^t K_8\Big)\,. \tag{4.108}$$

The details of this calculation can be found in appendix D.1. If we use (4.50), which leads to the identity (4.64), we can write this as

$$\boxed{\mathcal{L}_{WZ}^1 = g\frac{\kappa}{4}iStr\Big(\Sigma_+\partial_-\eta_-\eta_-\Big)}\,. \tag{4.109}$$

But then, for the choice $\kappa = -1$, this compensates the diverging term in the kinetic part (4.97) of the Lagrangian. This confirms that the choice (4.50) leads to a well defined model.

The next-to-leading-order term is

$$\begin{aligned}\mathcal{L}^{1/2}_{WZ} &= g\frac{\kappa}{8}Str\Big(-\beta_-^{-3/4}\gamma_+^{-1/4} - \beta_-^{-1/4}\gamma_+^{-3/4} + \alpha_+^{-3/4}\epsilon_-^{-1/4} \\ &\qquad +\beta_+^{-1/4}\epsilon_-^{-1/4} + \alpha_+^{-1/4}\epsilon_-^{-3/4} - \beta_-^{-1/4}\epsilon_+^{-1/4}\Big) \\ &= \sqrt{g}\frac{\kappa}{8}\Big(4Str\Big(\Sigma_+\partial_-\eta_+\tilde{K}_8\eta_-^t K_8\Big) + Str\Big(\Sigma_+\partial_-\eta_-\eta_-\eta_-\tilde{K}_8\eta_-^t K_8\Big) \\ &\qquad +Str\Big(\Sigma_+\eta_-\eta_-\partial_-\eta_-\tilde{K}_8\eta_-^t K_8\Big)\Big)\,.\end{aligned} \tag{4.110}$$

The details of the calculation can be found in appendix D.2. If we use (4.50), then, as for the kinetic part, all terms to order $g^{-1/2}$ vanish. The g^0 term is

$$\begin{aligned}\mathcal{L}^0_{WZ} = +\frac{\kappa}{8}\quad Str\Big(&-\beta_-^{-5/4}\gamma^{-1/4} + \beta_+^{-1/4}\gamma_+^{-1/4} - \beta_-^{-3/4}\gamma_+^{-3/4} - \beta_-^{-1/4}\gamma_+^{-5/4} \\ &+\alpha_+^{-5/4}\epsilon_-^{-1/4} + \beta_+^{-3/4}\epsilon_-^{-1/4} + \alpha_+^{-3/4}\epsilon_-^{-3/4} + \beta_+^{-1/4}\epsilon_-^{-3/4} \\ &+\alpha_+^{-1/4}\epsilon_-^{-5/4} - \alpha_-^{-1/4}\epsilon_+^{-1/4} - \beta_-^{-3/4}\epsilon_+^{-1/4} - \beta_-^{-1/4}\epsilon_+^{-3/4}\Big)\,.\end{aligned} \tag{4.111}$$

The details of this calculation can be found in appendix D.3. If we use the specific choice (4.50) for the fermions $\eta_\pm$ the result is

$$\begin{aligned}\mathcal{L}^0_{WZ} &= +\frac{\kappa}{4}\Big\{-iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big)\frac{\partial_-U}{2} - iStr\Big(\Sigma_+\partial_-\eta_+\eta_+\Big) \\ &\quad +\frac{i}{2}(z^2-y^2)Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) + \frac{i}{2}Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big)\partial_+U \\ &\quad -iStr\Big(\Sigma_+x_N\Sigma_N\partial_-\eta_-x_M\Sigma_M\eta_-\Big) \\ &\quad +iStr\Big(\Sigma_+(\partial_-\eta_+\eta_-\eta_-\eta_- + \partial_-\eta_-\eta_-\eta_+\eta_-)\Big)\Big\}\,.\end{aligned} \tag{4.112}$$

Using the gauge $U = 2\sigma^-$ we get

$$\begin{aligned}\mathcal{L}^0_{WZ} &= +\frac{\kappa}{4}\Big\{-iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big) - iStr\Big(\Sigma_+\partial_-\eta_+\eta_+\Big) \\ &\quad +\frac{i}{2}(z^2-y^2)Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) - iStr\Big(\Sigma_+x_N\Sigma_N\partial_-\eta_-x_M\Sigma_M\eta_-\Big) \\ &\quad +iStr\Big(\Sigma_+(\partial_-\eta_+\eta_-\eta_-\eta_- + \partial_-\eta_-\eta_-\eta_+\eta_-)\Big)\Big\}\,.\end{aligned} \tag{4.113}$$

To summarize, the Wess-Zumino part provides terms of the same type as in the kinetic part and a new quartic fermionic interaction term.

4.6 Full Conformal Gauge Lagrangian

Adding up the WZ-part and the kinetic part for $\kappa = -1$ and $U = 2\sigma^-$ we get

$$\begin{aligned}
\mathcal{L}_{cf.g.} = & \frac{1}{2}(\partial_- z \partial_+ z + \partial_- y \partial_+ y) - \frac{1}{2}(z^2 + y^2) \\
& -\frac{1}{8}(z^2 - y^2)((\partial_- z)^2 + (\partial_- y)^2) - Str(\eta_+ \eta_-) \\
& +\frac{i}{2} Str(\Sigma_+ \partial_- \eta_+ \eta_+) + \frac{i}{2} Str(\Sigma_+ \partial_+ \eta_- \eta_-) \\
& -i\frac{5}{32}(z^2 - y^2) Str(\Sigma_+ \partial_- \eta_- \eta_-) \\
& +\frac{i}{4} Str\Big(\Sigma_+ x_N \Sigma_N \partial_- \eta_- x_M \Sigma_M \eta_-\Big) \\
& -\frac{i}{4} Str\Big(\Sigma_+ \partial_- x_N \Sigma_N x_M \Sigma_M \eta_-^2\Big) \\
& -\frac{i}{4} Str\Big(\Sigma_+ (\partial_- \eta_+ \eta_- \eta_- \eta_- + \partial_- \eta_- \eta_- \eta_+ \eta_-)\Big) .
\end{aligned} \tag{4.114}$$

As mentioned for the kinetic part, we can eliminate the mass terms through addition of $\partial_+ V$. At first, however, the validity of the chosen light-cone gauge should be investigated.

4.7 Curvature Corrections to the World-Sheet Metric

In genereal curved spaces, the light-cone gauge $x^+ = \tau$, supplemented by a flat world-sheet metric, is not necessarily a valid gauge choice, as it may lead to inconsistent equations of motion for x^-. We follow the discussion of [25] and demand consistency of the x^- equations of motion to be achieved by the introduction of world-sheet curvature corrections. However, there exist different possibilities to achieve a consistent gauge in curved spaces. Before we apply the method of introducing world-sheet curvature corrections in the near-flat space limit, we review the procedure of light-cone gauge fixing in flat space.

4.7.1 Light-Cone Gauge in Flat and Curved Spaces

As an example we take the bosonic Lagrangian

$$\mathcal{L} \propto \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha x^M \partial_\beta x^N G_{MN} \, . \tag{4.115}$$

This Lagrangian is invariant under reparametrizations of $\sigma \to \tilde{\sigma}(\sigma, \tau)$, $\tau \to \tilde{\tau}(\sigma, \tau)$ and Weyl rescalings $h^{\alpha\beta} \to e^{\phi(\sigma,\tau)} h^{\alpha\beta}$. As shown for instance in example 7.9 of [6], it is always possible to transform a general 2-d metric into a conformally flat metric by use of the former symmetries

$$h^{\alpha\beta} \to h'^{\alpha\beta} = e^{\omega(\sigma,\tau)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{4.116}$$

In world-sheet light-cone coordinates this is equivalent to $\gamma^{+-} = \frac{1}{2}$, $\gamma^{\pm\pm} = 0$, for $\gamma^{\alpha\beta} = \sqrt{-h} h^{\alpha\beta}$. Then, the Lagrangian becomes

$$\mathcal{L} \propto \int d\sigma^+ d\sigma^- \partial_+ x^M \partial_- x^N G_{MN} . \tag{4.117}$$

This Lagrangian still has the *residual* symmetry $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$. Then τ can be reparametrized as

$$\begin{aligned} \tau \to \tilde{\tau} \quad = \quad & \frac{1}{2}\Big(\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-)\Big) \\ & \frac{1}{2}\Big(\tilde{\sigma}^+(\tau+\sigma) + \tilde{\sigma}^-(\tau-\sigma)\Big) . \end{aligned} \tag{4.118}$$

This means that any $\tilde{\tau}$ is a solution of the free wave equation

$$(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2})\tilde{\tau} = 0 , \tag{4.119}$$

i.e. any reparametrization of τ achieved by use of the residual symmetry satisfies the free wave equation. The equations of motion for x^- derived from (4.115) are

$$\partial_\alpha\Big(\sqrt{-h} h^{\alpha\beta} \partial_\beta x^M G_{M-}\Big) = 0 . \tag{4.120}$$

In flat space, the only non-vanishing component of G_{M-} is G_{+-}. Using (4.116), we get for (4.120)

$$\partial_+ \partial_- \, x^+ = 0 \qquad or \qquad (\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2})x^+ = 0 , \tag{4.121}$$

i.e. x^+ satisfies the free wave equation and is thus a valid reparametrization for τ such that we can set[6]

$$x^+ = \tau . \tag{4.122}$$

This, however, is not possible in general curved backgrounds. Starting from (4.120) and using the conformal gauge metric (4.116)[7], one will get a more complicated equation which may for example be of the type

$$\partial_+ \partial_- x^+ + f(x^I(\sigma^+, \sigma^-)) = 0 \tag{4.123}$$

[6] We could add a constant, but we will omit it here and in the following.
[7] It is always possible to choose the metric like this, even in curved backgrounds.

where $f(x^I)$ is some function of the transverse fields and its derivatives. In the simplest case it has the form

$$f(x^I) = \partial_+ \partial_- \Big(g(x^I) \Big) \tag{4.124}$$

such that

$$\partial_+ \partial_- \Big(x^+ + g(x^I) \Big) = 0\,. \tag{4.125}$$

This means that $x^+ + g(x^I)$ satisfies the free wave equation and therefore, one can choose the gauge

$$x^+ = \tau - g(x^I)\,. \tag{4.126}$$

In general however, the function f will not be of the form (4.124), such that a light-cone gauge for x^+ will involve a non-local term

$$\partial_+ \partial_- x^+ + f(x^I) = 0 \quad \Leftrightarrow \quad \partial_+ \partial_- \Big(x^+ + \underbrace{\int d\sigma'^+ d\sigma'^- f}_{g(\sigma^+, \sigma^-)} \Big) = 0\,. \tag{4.127}$$

Then, light-cone gauge could be fixed as

$$x^+ = \tau - \int d\sigma'^+ d\sigma'^- f\,. \tag{4.128}$$

However, in the gauge-fixed action, one has to deal with unpleasant terms like the one on the right-hand side then. We will see this explicitly for the case of the near-flat space limit in section 4.9.

There is a second way to impose a light-cone gauge. We could start from (4.120) again and choose the gauge $x^+ = \tau$. As argued above, this requires $\partial_+ \partial_- x^+ = 0$ and this relation does not hold when we take the conformal gauge metric. The way out, is to let the metric differ from its conformal gauge form and determine the metric components in such a way that the equations of motion (4.120) lead to $\partial_+ \partial_- x^+ = 0$ or by using $x^+ = \tau$ from the start, just give consistent equations of motion for x^-. This is the procedure taken in [25] and we will use it in the next section.

4.7.2 Ansatz for Curvature Corrections

Using the standard procedure

$$\frac{\delta \mathcal{L}}{\delta x^-} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu x^-} = 0 \tag{4.129}$$

and noting that the Lagrangian depends only on the derivatives of x^-, the equations of motion read

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu x^-} = 0 \tag{4.130}$$

or explicitly

$$\partial_+ \frac{\delta \mathcal{L}}{\delta \partial_+ x^-} = -\partial_- \frac{\delta \mathcal{L}}{\delta \partial_- x^-} \,. \tag{4.131}$$

These equations may - and will - not be consistent with the light-cone gauge chosen above. In order to satisfy the relations (4.131), we make the following ansatz for the curvature corrected world-sheet metric designated by γ'^{+-}

$$\begin{aligned} \gamma'^{+-} &= \gamma^{+-} + \frac{1}{\sqrt{g}}\tilde{\gamma}^{+-} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-} + O(g^{-3/2}) \\ \gamma'^{\pm\pm} &= \gamma^{\pm\pm} + \frac{1}{\sqrt{g}}\tilde{\gamma}^{\pm\pm} + \frac{1}{g}\tilde{\tilde{\gamma}}^{\pm\pm} + O(g^{-3/2}) \,. \end{aligned} \tag{4.132}$$

We will try to set $\gamma^{+-} = \frac{1}{2}$, and $\gamma^{\pm\pm} = 0$, i.e. we admit $\frac{1}{\sqrt{g}}$ and $\frac{1}{g}$ corrections to the conformal gauge metric and try to determine the components in such a way that the relations (4.131) can be fulfilled. It should be mentioned that we already make the *assumption* then, that the world-sheet metric *can* be chosen to be flat at leading order. It might however turn out, that this is not possible.

We are not interested in higher corrections to the world-sheet metric, since they will not affect the Lagrangian (4.71) to order g^0: The highest contributions of A_+A_-, A_+A_+, A_-A_- come in at order g^0, as was summarized in (4.98). Of course, these corrections will only affect the kinetic part of the Lagrangian, since the topological Wess-Zumino term does not depend on the world-sheet metric.

4.7.3 Virasoro Constraints with Curvature Corrections

In order to get the right Virasoro constraints, we should remember that we work with the Weyl invariant combination $\gamma'^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$ of the world-sheet metric $h^{\alpha\beta}$. Demanding that the variation of the Lagrangian $\mathcal{L} = \frac{g}{2} Str(\sqrt{-h}h^{\alpha\beta}A^{(2)}_\alpha A^{(2)}_\beta)$ with respect to $h^{\alpha\beta}$ vanishes, is equivalent to

$$\begin{aligned} T_{\alpha\beta} &= Str\Big(A^{(2)}_\alpha A^{(2)}_\beta - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}A^{(2)}_\gamma A^{(2)}_\delta\Big) \\ &= Str\Big(A^{(2)}_\alpha A^{(2)}_\beta - \frac{1}{2}\gamma'_{\alpha\beta}\gamma'^{\gamma\delta}A^{(2)}_\gamma A^{(2)}_\delta\Big) = 0 \,. \end{aligned} \tag{4.133}$$

In world-sheet light-cone coordinates we get the two independent Virasoro constraints

$$\begin{aligned} T_{--} &= Str\Big(A^{(2)}_- A^{(2)}_- - \frac{1}{2}\gamma'_{--}(\gamma'^{\alpha\beta}A^{(2)}_\alpha A^{(2)}_\beta)\Big) = 0 \\ T_{++} &= Str\Big(A^{(2)}_+ A^{(2)}_+ - \frac{1}{2}\gamma'_{++}(\gamma'^{\alpha\beta}A^{(2)}_\alpha A^{(2)}_\beta)\Big) = 0 \,. \end{aligned} \tag{4.134}$$

Note, that this involves the metric with lowered components. To determine the components of the curvature corrections explicitly, we need to know how to raise and lower indices, because the metric components with lower indices are functions of the ones with upper indices. The inverse of $\gamma'^{\alpha\beta}$ is $\gamma'_{\alpha\beta}$, i.e.

$$\delta_\alpha^\delta \stackrel{!}{=} \gamma'_{\alpha\beta}\gamma'^{\beta\delta} . \tag{4.135}$$

This leads to the usual equations for the inverse components of a 2×2 matrix

$$\gamma'^{+\beta}\gamma'_{\beta+} = 1 \Rightarrow \gamma'^{+-}\gamma'_{-+} + \gamma'^{++}\gamma'_{++} = 1 \tag{4.136}$$

$$\gamma'^{-\beta}\gamma'_{\beta-} = 1 \Rightarrow \gamma'^{-+}\gamma'_{+-} + \gamma'^{--}\gamma'_{--} = 1 \tag{4.137}$$

$$\begin{aligned} \gamma'^{-\beta}\gamma'_{\beta+} = 0 &\Rightarrow \gamma'^{-+}\gamma'_{++} + \gamma'^{--}\gamma'_{-+} = 0 \\ &\Rightarrow \gamma'_{+-} = -\frac{\gamma'^{+-}\gamma'_{++}}{\gamma'^{--}} \end{aligned} \tag{4.138}$$

$$\begin{aligned} \gamma'^{+\beta}\gamma'_{\beta-} = 0 &\Rightarrow \gamma'^{+-}\gamma'_{--} + \gamma'^{++}\gamma'_{+-} = 0 \\ &\Rightarrow \gamma'_{+-} = -\frac{\gamma'^{+-}\gamma'_{--}}{\gamma'^{++}} . \end{aligned} \tag{4.139}$$

In world-sheet light-cone coordinates, we have

$$det\, \gamma'^{\alpha\beta} = -\frac{1}{4} \quad \Leftrightarrow \quad \gamma'^{++}\gamma'^{--} = (\gamma'^{+-})^2 - \frac{1}{4} . \tag{4.140}$$

Inserting (4.138) in (4.136) as well as (4.139) in (4.137) and using (4.140) leads to

$$\gamma'_{++} = -4\gamma'^{--} , \qquad \gamma'_{--} = -4\gamma'^{++} . \tag{4.141}$$

Inserting this in (4.138) or (4.139) gives

$$\gamma'_{+-} = 4\gamma'^{+-} . \tag{4.142}$$

The transformation of the single curvature corrections - not the full metric - can be obtained as follows

$$\begin{aligned} \delta_\alpha^\delta &\stackrel{!}{=} \gamma'_{\alpha\beta}\gamma'^{\beta\delta} \\ &= (\gamma_{\alpha\beta} + \frac{1}{\sqrt{g}}\tilde{\gamma}_{\alpha\beta} + \frac{1}{g}\tilde{\tilde{\gamma}}_{\alpha\beta} + ...)(\gamma^{\beta\delta} + \frac{1}{\sqrt{g}}\tilde{\gamma}^{\beta\delta} + \frac{1}{g}\tilde{\tilde{\gamma}}^{\beta\delta} + ...) \\ &= \delta_\alpha^\delta + \frac{1}{\sqrt{g}}(\gamma_{\alpha\beta}\tilde{\gamma}^{\beta\delta} + \tilde{\gamma}_{\alpha\beta}\gamma^{\beta\delta}) + \frac{1}{g}(\gamma_{\alpha\beta}\tilde{\tilde{\gamma}}^{\beta\delta} + \tilde{\tilde{\gamma}}_{\alpha\beta}\gamma^{\beta\delta} + \tilde{\gamma}_{\alpha\beta}\tilde{\gamma}^{\beta\delta}) + ... \end{aligned} \tag{4.143}$$

Thus the expressions in brackets have to vanish. This implies for the $\sqrt{g}$-corrections

$$\begin{aligned} \gamma_{\alpha\beta}\tilde{\gamma}^{\beta\delta} &= -\tilde{\gamma}_{\alpha\beta}\gamma^{\beta\delta} \\ &\Leftrightarrow \\ \tilde{\gamma}^{\sigma\delta} &= -\gamma^{\sigma\alpha}\tilde{\gamma}_{\alpha\beta}\gamma^{\beta\delta} . \end{aligned} \tag{4.144}$$

The g-corrections transform as

$$\begin{aligned} \gamma_{\alpha\beta}\tilde{\tilde{\gamma}}^{\beta\delta} &= -\tilde{\tilde{\gamma}}_{\alpha\beta}\gamma^{\beta\delta} - \tilde{\gamma}_{\alpha\beta}\tilde{\gamma}^{\beta\delta} \\ &\Leftrightarrow \\ \tilde{\tilde{\gamma}}^{\sigma\delta} &= -\gamma^{\sigma\alpha}\tilde{\tilde{\gamma}}_{\alpha\beta}\gamma^{\beta\delta} - \gamma^{\sigma\alpha}\tilde{\gamma}_{\alpha\beta}\tilde{\gamma}^{\beta\delta} . \end{aligned} \tag{4.145}$$

4.8 Curvature Corrections for the Bosonic Model

We will now restrict to the bosonic sector of the theory and try to find a solution for the curvature corrections. The analysis for the supersymmetric model does not lead to fundamentally different insights as we will see later. We construct the Lagrangian directly from the $AdS_5 \times S^5$ line element. Our starting point is the action

$$\mathcal{L} = \frac{g}{2}\gamma'^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}\,. \tag{4.146}$$

The transverse components vanish when varying with respect to the derivatives of x^- since G_{MN} are functions of the transverse coordinates and $G_{-i} = 0$ for $i = 1, ..., 8$. Using the same light-cone coordinates as before, we thus get

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= g\gamma'^{+-}\Big[G_{++}(\partial_- x^-) + G_{+-}(\partial_- x^+)\Big] \\
&+ g\gamma'^{++}\Big[G_{++}(\partial_+ x^-) + G_{+-}(\partial_+ x^+)\Big] \\
\frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= g\gamma'^{+-}\Big[G_{++}(\partial_+ x^-) + G_{+-}(\partial_+ x^+)\Big] \\
&+ g\gamma'^{--}\Big[G_{++}(\partial_- x^-) + G_{+-}(\partial_- x^+)\Big]\,.
\end{aligned} \tag{4.147}$$

Using the expansions

$$\begin{aligned}
G_{\pm\pm} &= G_{\phi\phi} - G_{tt} = -\frac{1}{g}(z^2 + y^2) - \frac{1}{2g^2}(z^4 - y^4) + \mathcal{O}(g^{-3}) \\
G_{+-} &= G_{\phi\phi} + G_{tt} = 2 + \frac{1}{g}(z^2 - y^2) + \frac{1}{2g^2}(z^4 + y^4) + \mathcal{O}(g^{-3})
\end{aligned} \tag{4.148}$$

and the gauge we used before

$$\begin{aligned}
x^+ &= \sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}\sigma^- \\
x^- &= \frac{1}{2\sqrt{g}}V
\end{aligned} \tag{4.149}$$

we get

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= \gamma'^{+-}\Big(2g^{1/2} + \frac{1}{\sqrt{g}}\Big((z^2 - y^2) - \frac{1}{2}(z^2 + y^2)\partial_- V\Big)\Big) \\
&+ \gamma'^{++}\Big(2g^{3/2} + g^{1/2}(z^2 - y^2) + \frac{1}{\sqrt{g}}\Big(\frac{1}{2}(z^4 + y^4) - \frac{1}{2}(z^2 + y^2)\partial_+ V\Big)\Big) \\
\frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= \gamma'^{+-}\Big(2g^{3/2} + g^{1/2}(z^2 - y^2) + \frac{1}{\sqrt{g}}\Big(\frac{1}{2}(z^4 + y^4) - \frac{1}{2}(z^2 + y^2)\partial_+ V\Big)\Big) \\
&+ \gamma'^{--}\Big(2g^{1/2} + g^{-1/2}\Big((z^2 - y^2) - \frac{1}{2}(z^2 + y^2)\partial_- V\Big)\Big)\,.
\end{aligned} \tag{4.150}$$

4.8.1 Conformal Gauge at Leading Order

As we would like to calculate *corrections* to the metric, we take the flat conformal gauge metric to leading order, supplemented by terms in powers of $g^{-1/2}$

$$\gamma'^{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + \frac{1}{\sqrt{g}}(...) + ... \tag{4.151}$$

Taking the flat metric $\gamma'^{+-} = \frac{1}{2}$, $\gamma'^{\pm\pm} = 0$, does indeed lead to inconsistencies: to leading order $g^{3/2}$, the equations of motion are correct

$$\partial_+(0) = 0 = \partial_-(g^{3/2})\,, \tag{4.152}$$

as we can directly see from (4.150). But already at order $g^{1/2}$ we get

$$\partial_+(g^{1/2}) = 0 = \partial_-(g^{1/2}\frac{1}{2}(z^2 - y^2))\,, \tag{4.153}$$

which is hardly acceptable.

Since the terms in the equations of motion appear in integer steps of g, i.e. $g^{3/2}$, $g^{1/2}$, $g^{-1/2}$, it seems reasonable to start with an ansatz that does incorporate $1/g$ - corrections, but no $1/\sqrt{g}$ - corrections

$$\gamma'^{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + \frac{1}{g}\begin{pmatrix} \tilde{\tilde{\gamma}}^{++} & \tilde{\tilde{\gamma}}^{+-} \\ \tilde{\tilde{\gamma}}^{+-} & \tilde{\tilde{\gamma}}^{--} \end{pmatrix}\,. \tag{4.154}$$

Then we can rewrite the first orders of (4.150)

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= g^{1/2}(1 + 2\tilde{\tilde{\gamma}}^{++}) \\ \frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= g^{3/2} + g^{1/2}\Big(\frac{1}{2}(z^2 - y^2) + 2\tilde{\tilde{\gamma}}^{+-}\Big)\,. \end{aligned} \tag{4.155}$$

All higher orders incorporate $1/g^2$ - contributions to the world-sheet metric, which we are not interested in for calculating the Lagrangian. It seems that we can easily find a solution by setting

$$\begin{aligned} \tilde{\tilde{\gamma}}^{+-} &= -\frac{1}{4}(z^2 - y^2) \\ \tilde{\tilde{\gamma}}^{\pm\pm} &= 0\,. \end{aligned} \tag{4.156}$$

Further investigations show however, that this solution is only valid in the strict $g \to \infty$ limit: since we are working with the Weyl invariant combination of the metric, we have in world-sheet light-cone coordinates $det\,\gamma^{\alpha\beta} = -\frac{1}{4}$. For our ansatz we have

$$det\begin{pmatrix} \frac{1}{g}\tilde{\tilde{\gamma}}^{++} & \frac{1}{2} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-} \\ \frac{1}{2} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-} & \frac{1}{g}\tilde{\tilde{\gamma}}^{--} \end{pmatrix} = -\frac{1}{4} - \frac{1}{g}\tilde{\tilde{\gamma}}^{+-} + \frac{1}{g^2}(...)\,. \tag{4.157}$$

Since this relation is not multiplied with any higher powers of g one could argue, that in the strict $g \to \infty$ limit we indeed have $det\,\gamma^{\alpha\beta} = -\frac{1}{4}$. However, it seems advantageous - possibly for investigations going beyond the present work - to have this relation exact at the order we determine the world-sheet curvature corrections. Then however, we need a compensation[8] for the $1/g$-term in (4.157). One possibility would be to have the corrections in $\gamma'^{\pm\pm}$ start at $1/\sqrt{g}$

$$det \begin{pmatrix} \frac{1}{\sqrt{g}}\tilde{\gamma}^{++} & \frac{1}{2} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-} \\ \frac{1}{2} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-} & \frac{1}{\sqrt{g}}\tilde{\gamma}^{--} \end{pmatrix} = -\frac{1}{4} + \frac{1}{g}(\tilde{\gamma}^{++}\tilde{\gamma}^{--} - \tilde{\tilde{\gamma}}^{+-}) + \frac{1}{g^2}(...) \tag{4.158}$$

which determines $\tilde{\gamma}^{--}$ as a function of the other components

$$\tilde{\gamma}^{--} = \frac{\tilde{\tilde{\gamma}}^{+-}}{\tilde{\gamma}^{++}}\,. \tag{4.159}$$

Introducing the square root corrections leads to new terms in integer powers of g

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= 2g\tilde{\gamma}^{++} + \text{half-integer powers of g} \qquad (4.160) \\ \frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= 2\tilde{\gamma}^{--} + \text{half-integer powers of g} \end{aligned}$$

which clearly implies that we need $\tilde{\gamma}^{++} = const.$, $\tilde{\gamma}^{--} = const.$ which through (4.159) leads to $\tilde{\tilde{\gamma}}^{+-} = const.$ and therefore clearly doesn't serve our needs.

The upshot is, that - for the chosen light-cone gauge - it is not possible to get consistent x^- equations of motion with a world-sheet metric which is flat at leading order, supplemented by curvature corrections.

4.8.2 A Consistent Gauge

We give up the assumption of taking a conformal gauge metric to leading order and make the following ansatz that does in fact lead to a consistent gauge:

$$\gamma'^{\alpha\beta} = \begin{pmatrix} \frac{1}{g}\tilde{\tilde{\gamma}}^{++} & \frac{1}{2} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-} \\ \frac{1}{2} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-} & \gamma^{--} \end{pmatrix}\,. \tag{4.161}$$

Note that it has contributions differing from conformal gauge already at leading order. The components are not independent since

$$det\,\gamma^{\alpha\beta} = -\frac{1}{4} + \frac{1}{g}(\tilde{\tilde{\gamma}}^{++}\gamma^{--} - \tilde{\tilde{\gamma}}^{+-}) + \frac{1}{g^2}(...) \tag{4.162}$$

[8] The g^2 term and all higher terms have to vanish exactly as well. This gives constraints to the higher corrections of the world-sheet metric comparable to (4.159), but we will not calculate them since they do not enter in the Lagrangian.

requires

$$\gamma^{--} = \frac{\tilde{\tilde{\gamma}}^{+-}}{\tilde{\tilde{\gamma}}^{++}}\,. \tag{4.163}$$

Then we get

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta \partial_+ x^-} &= g^{1/2}(1 + 2\tilde{\tilde{\gamma}}^{++}) + g^{-1/2}(...) \\ \frac{\delta \mathcal{L}}{\delta \partial_- x^-} &= g^{3/2} + g^{1/2}\Big(2\tilde{\tilde{\gamma}}^{+-} + \frac{1}{2}(z^2 - y^2) + 2\gamma^{--}\Big) + g^{-1/2}(...)\,. \end{aligned} \tag{4.164}$$

We get consistent equations of motion for

$$\begin{aligned} \tilde{\tilde{\gamma}}^{++} &= const. \\ 0 &= 2\tilde{\tilde{\gamma}}^{+-} + \frac{1}{2}(z^2 - y^2) + 2\gamma^{--} \\ \Leftrightarrow \tilde{\tilde{\gamma}}^{+-}(1 + \frac{1}{\tilde{\tilde{\gamma}}^{++}}) &= -\frac{1}{4}(z^2 - y^2)\,. \end{aligned} \tag{4.165}$$

where $\tilde{\tilde{\gamma}}^{++}$ is any constant value other than 0 or -1. We can summarize this as

$$\boxed{\tilde{\tilde{\gamma}}^{++} = a, \quad \tilde{\tilde{\gamma}}^{+-} = -\frac{1}{4}\frac{(z^2 - y^2)}{(1 + \frac{1}{a})}, \quad \gamma^{--} = \frac{\tilde{\tilde{\gamma}}^{+-}}{a}} \tag{4.166}$$

where we rewrite $\tilde{\tilde{\gamma}}^{++}$ as the constant a for the sake of better readability. Choosing $a = 1$ for example leads to

$$\gamma'^{\alpha\beta} = \begin{pmatrix} \frac{1}{g} & \frac{1}{2} - \frac{1}{8g}(z^2 - y^2) \\ \frac{1}{2} - \frac{1}{8g}(z^2 - y^2) & -\frac{1}{8}(z^2 - y^2) \end{pmatrix}\,. \tag{4.167}$$

Note that the Virasoro constraint changes as well and gets

$$\begin{aligned} T_{--} &= \partial_- x^N \partial_- x^M G_{MN} \\ &+ \frac{2}{g}\tilde{\tilde{\gamma}}^{++}\Big(\frac{1}{2}(\partial_+ x^N \partial_- x^M + \partial_- x^N \partial_+ x^M) + \gamma^{--}\partial_- x^N \partial_- x^M + \frac{1}{g}(...)\Big)G_{MN}\,. \end{aligned} \tag{4.168}$$

Using

$$\begin{aligned} \partial_- x^N \partial_- x^M G_{MN} &= \frac{1}{g}\Big(2\partial_- V + (\partial_- z)^2 + (\partial_- y)^2\Big) \\ \partial_- x^N \partial_+ x^M G_{MN} &= \partial_- V + \frac{1}{g}(...) \end{aligned} \tag{4.169}$$

we get to leading order

$$T_{--} = \frac{1}{g}\Big(2\partial_- V + (\partial_- z)^2 + (\partial_- y)^2 + 2\tilde{\tilde{\gamma}}^{++}\partial_- V\Big) \stackrel{!}{=} 0 \tag{4.170}$$

$$\boxed{\Rightarrow \partial_- V = -\frac{(\partial_- z)^2 + (\partial_- y)^2}{2(1+a)}} . \tag{4.171}$$

Then the full bosonic Lagrangian is

$$\begin{aligned}
\mathcal{L} &= (\gamma^{+-} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-})(\frac{g}{2}2\partial_+ x^N \partial_- x^M G_{MN}) \\
&\quad + \frac{1}{g}\tilde{\tilde{\gamma}}^{++}\Big(\frac{g}{2}\partial_+ x^N \partial_+ x^M G_{MN}\Big) + \gamma^{--}\Big(\frac{g}{2}\partial_- x^N \partial_- x^M G_{MN}\Big) \\
&= (\gamma^{+-} + \frac{1}{g}\tilde{\tilde{\gamma}}^{+-})\Big(g\partial_- V - (z^2+y^2) + \partial_+ V + \frac{1}{2}(z^2-y^2)\partial_- V \\
&\qquad + (\partial_+ z \partial_- z + \partial_+ y \partial_- y)\Big) \\
&\quad + \frac{1}{g}\tilde{\tilde{\gamma}}^{++}\Big(-\frac{1}{2}g(z^2+y^2) + g\partial_+ V\Big) + \gamma^{--}\frac{1}{2}\Big(2\partial_- V + (\partial_- z)^2 + (\partial_- y)^2\Big) .
\end{aligned} \tag{4.172}$$

If we drop total derivatives we get

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}(z^2+y^2)(1+a) + \frac{1}{2}(\partial_+ z \partial_- z + \partial_+ y \partial_- y) \\
&\quad + \frac{1}{4}(z^2-y^2)\Big(\partial_- V\Big(1 - \frac{1}{1+\frac{1}{a}} - \frac{1}{1+a}\Big) - \frac{1}{2}\frac{(\partial_- z)^2 + (\partial_- y)^2}{a+1}\Big) \\
&= -\frac{1}{2}(z^2+y^2)(1+a) + \frac{1}{2}(\partial_+ z \partial_- z + \partial_+ y \partial_- y) \\
&\quad - \frac{1}{8}\frac{1}{1+a}(z^2-y^2)\Big((\partial_- z)^2 + (\partial_- y)^2\Big) .
\end{aligned} \tag{4.173}$$

Remember we stated above that only for $a \neq 0$, $a \neq -1$ the equations of motion for x^- are consistent. Taking $a \to 0$ leads to the conformal gauge Lagrangian (4.11). Note that the contributions of $\tilde{\tilde{\gamma}}^{+-}$ and γ^{--} exactly compensate each other and the only change comes in through $\tilde{\tilde{\gamma}}^{++}$ by $\gamma'^{++}\partial_+ x^M \partial_+ x^N$ and the changed Virasoro constraint (4.171).

We can recover the relative factors between the terms of [1], by taking an appropriate choice for a. The result of [1], up to an overall factor, is

$$\mathcal{L}'_{kin} = \partial_+ z \partial_- z + \partial_+ y \partial_- y - \frac{1}{4}(z^2+y^2) - (z^2-y^2)((\partial_- z)^2 + (\partial_- y)^2) . \tag{4.174}$$

Taking $a = -\frac{3}{4}$ leads to exactly the same up to an overall factor. However there is a significant difference with respect to [1]. Solving the Virasoro constraint T_{++} for $\partial_+ V$ gives

$$\begin{aligned}
T_{++} &= \Big(\partial_+ x^M \partial_+ x^N + 2\gamma^{--}\Big(\gamma^{\alpha\beta}\partial_\alpha x^M \partial_\beta x^N\Big)\Big)G_{MN} \\
&= -(z^2+y^2) + 2\partial_+ V + 2\gamma^{--}\partial_- V \\
&\Rightarrow \boxed{\partial_+ V = \frac{1}{2}(z^2+y^2) - \frac{1}{8(1+a)^2}(z^2-y^2)((\partial_- z)^2 + (\partial_- y)^2)} .
\end{aligned} \tag{4.175}$$

Since a total derivative does not change the action, $\partial_+ V$ can freely be added to the Lagrangian. This enables us, to eliminate the interaction term and the bosonic theory gets a free massive theory and is thus solvable. On the other hand, by the same method, one could eliminate the mass term and make the theory to a massless interacting theory. However, it is not possible to eliminate the interaction term and the mass term at the same time. Still, this is a remarkable feature of the gauge fixing with metric curvature corrections and gives rise to the question of the validity of this gauge choice, which will be discussed later.

4.9 Alternatives to World-Sheet Metric Curvature Corrections

As mentioned in 4.7.1, the problem of achieving consistent x^- equations of motion can also be addressed by a different approach. We will discuss this method for the case of the near-flat space limit. In fact the necessity of world-sheet curvature corrections was caused by fixing our *boosted* light-cone gauge

$$x^+ = \tau \to \sqrt{g}\sigma^+ + \frac{1}{\sqrt{g}}\sigma^- \,, \tag{4.176}$$

which arose by fixing $U = 2\sigma^-$ in the light-cone coordinates (4.4)

$$\begin{aligned} x^+ &= \sqrt{g} + \frac{1}{2\sqrt{g}}U \\ x^- &= \frac{1}{2\sqrt{g}}V \,. \end{aligned} \tag{4.177}$$

Without fixing the gauge for U, the evaluation of (4.147) in conformal gauge $\gamma^{+-} = \frac{1}{2}$ leads to

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= g^{1/2}\frac{\partial_- U}{2} + g^{-1/2}(...) \\ \frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= g^{3/2} + g^{1/2}\Big(\frac{\partial_+ U}{2} + \frac{1}{4}(z^2 - y^2)\Big) + g^{-1/2}(...) \,. \end{aligned} \tag{4.178}$$

To order $g^{3/2}$ the equations of motion are consistent. To order $g^{1/2}$ we have

$$\partial_+\Big(\partial_- U\Big) = -\partial_-\Big(\partial_+ U + \frac{1}{2}(z^2 - y^2)\Big) \,. \tag{4.179}$$

Choosing the gauge

$$\partial_+ U = -\frac{1}{4}(z^2 - y^2) \tag{4.180}$$

leads to consistent equations of motion. The Lagrangian without specifying U reads

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}(z^2 + y^2)\partial_- U + \frac{1}{4}(\partial_+ V \partial_- U + \partial_- V \partial_+ U) \\ &\quad + \frac{1}{4}(z^2 - y^2)\partial_- V + \frac{1}{2}(\partial_+ z \partial_- z + \partial_+ y \partial_- y) + \frac{g}{2}\partial_- V \,. \end{aligned} \tag{4.181}$$

Then we would also have to address the question of how to treat $\partial_- U$. In [1] this is done by switching to different world-sheet coordinates[9] $\sigma'^{\pm}$, such that terms with the structure $\partial_- U$ do not appear at all.

Genererally, there is always the possibility to find a consistent gauge in the way we did in the last section, i.e. by including corrections to the world-sheet metric *or* by gauge fixing $x^+ = \tau + f(x^I)$ with a suitable function f of the transverse coordinates. As just explained, in [1] they use implicitly the latter procedure without commenting on it. A precise description of this method can be found in [37]. In the present work however, it seems more convenient and straightforward to use the method of world-sheet metric corrections.

4.10 Curvature Corrections for the Full Model

We follow the same logic as in the bosonic model. Calculating the equations of motion requires the evaluation of

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= \frac{g}{2}\frac{\delta Str\left(\gamma'^{\alpha\beta}A^{(2)}_\alpha A^{(2)}_\beta\right)}{\delta\partial_+ x^-} \\
&= \frac{g}{2}\frac{\delta Str\left(\gamma'^{++}A^{(2)}_+ A^{(2)}_+ + 2\gamma'^{+-}A^{(2)}_+ A^{(2)}_-\right)}{\delta\partial_+ x^-} \\
&= gStr\left(\gamma'^{++}A^{(2)}_+\frac{\delta A^{(2)}_+}{\delta\partial_+ x^-} + \gamma'^{+-}\frac{\delta A^{(2)}_+}{\delta\partial_+ x^-}A^{(2)}_-\right) \\
\frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= \frac{g}{2}\frac{\delta Str\left(\gamma'^{--}A^{(2)}_- A^{(2)}_- + 2\gamma'^{+-}A^{(2)}_+ A^{(2)}_-\right)}{\delta\partial_- x^-} \\
&= gStr\left(\gamma'^{--}A^{(2)}_-\frac{\delta A^{(2)}_-}{\delta\partial_- x^-} + \gamma'^{+-}\frac{\delta A^{(2)}_-}{\delta\partial_- x^-}A^{(2)}_+\right).
\end{aligned} \tag{4.182}$$

Using (4.76), it is easy to see that

$$\frac{\delta A^{(2)}_\pm}{\delta(\partial_\pm x^-)} = -\frac{i}{4}\Sigma_-(g_x^2 + g_x^{-2})\,. \tag{4.183}$$

With the help of (B.7) this can be written as

$$\frac{\delta A^{(2)}_\pm}{\delta(\partial_\pm x^-)} = -\frac{i}{2}\Sigma_-(G_+\Sigma_- - G_-\Sigma_+)\,. \tag{4.184}$$

[9] They call the new world-sheet coordinates $x^{\pm}$, which is somewhat confusing in our conventions since we denote the spacetime light-cone coordinates with these letters.

Then, all we need to know is the expansions of $Str(A_\alpha^{(2)}\Sigma_+)$ and $Str(A_\alpha^{(2)}\Sigma_-)$. We calculate these in appendix E and the result is

$$\begin{aligned} Str A_\alpha^{(2)}\Sigma_+ &= \frac{i}{2} Str\Big((\partial_\alpha x^+ G_- - \partial_\alpha x^- G_+)\Sigma_+\Sigma_-\Big) \\ &\quad - Str\Big(G_+\partial_\alpha \eta\eta\Sigma_+\Big) - \frac{i}{2} Str\Big(\partial_+ x^+ G_+ \eta^2\Big) \end{aligned} \tag{4.185}$$

$$\begin{aligned} Str A_\alpha^{(2)}\Sigma_- &= \frac{i}{2} Str\Big((\partial_\alpha x^- G_- - \partial_\alpha x^+ G_+)\Sigma_+\Sigma_-\Big) \\ &\quad + Str\Big(G_-\partial_\alpha \eta\eta\Sigma_+\Big) + \frac{i}{2} Str\Big(\partial_+ x^+ G_- \eta^2\Big). \end{aligned} \tag{4.186}$$

After gauge-fixing we find

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= -\frac{i}{2} g\gamma'^{++}\Big[Str\Big(\frac{i}{2}(\frac{\partial_+ V}{\sqrt{g}} G_+G_- - \sqrt{g}(G_+^2 + G_-^2)\Sigma_+\Sigma_-\Big) \\ &\quad + 2G_+G_- Str\Big(\Sigma_+\partial_+\eta\eta\Big) + iStr\Big(\sqrt{g} G_+G_-\eta^2\Big)\Big] \\ &\quad - \frac{i}{2} g\gamma'^{+-}\Big[Str\Big(\frac{i}{2}(\frac{\partial_- V}{\sqrt{g}} G_+G_- - \frac{1}{\sqrt{g}}(G_+^2 + G_-^2)\Sigma_+\Sigma_-\Big) \\ &\quad + 2G_+G_- Str\Big(\Sigma_+\partial_-\eta\eta\Big) iStr\Big(\frac{1}{\sqrt{g}} G_+G_-\eta^2\Big))\Big]. \end{aligned} \tag{4.187}$$

In the derivative of $\mathcal{L}$ with respect to $\partial_+ x^-$ the expressions in rectangular brackets reappear

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= -\frac{i}{2} g\gamma'^{+-}\Big[Str\Big(\frac{i}{2}(\frac{\partial_+ V}{\sqrt{g}} G_+G_- - \sqrt{g}(G_+^2 + G_-^2)\Sigma_+\Sigma_-\Big) \\ &\quad + 2G_+G_- Str\Big(\Sigma_+\partial_+\eta\eta\Big) + iStr\Big(\sqrt{g} G_+G_-\eta^2\Big)\Big] \\ &\quad - \frac{i}{2} g\gamma'^{--}\Big[Str\Big(\frac{i}{2}(\frac{\partial_- V}{\sqrt{g}} G_+G_- - \frac{1}{\sqrt{g}}(G_+^2 + G_-^2)\Sigma_+\Sigma_-\Big) \\ &\quad + 2G_+G_- Str\Big(\Sigma_+\partial_-\eta\eta\Big) + iStr\Big(\frac{1}{\sqrt{g}} G_+G_-\eta^2\Big)\Big], \end{aligned} \tag{4.188}$$

so all we need to know is the expansions of those. We try to use the same ansatz for the full model, i.e. we take γ'^{--}, γ'^{+-} to be non-vanishing at leading order with corrections coming in for all components at order $1/g$. With the expansions in (B.13), the highest orders are

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= \gamma'^{++}\Big[2g^{3/2} + g^{1/2}(z^2 - y^2) + ...\Big] + \gamma'^{+-}\Big[2g^{1/2} + ...\Big] \\ \frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= \gamma'^{+-}\Big[2g^{3/2} + g^{1/2}(z^2 - y^2) + ...\Big] + \gamma'^{--}\Big[2g^{1/2} + ...\Big]. \end{aligned} \tag{4.189}$$

The fermions do not contribute to this order yet. Inserting the ansatz (4.161) for metric components, we find

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_+ x^-} &= g^{1/2}(1+2\tilde{\tilde{\gamma}}^{++}) + g^{-1/2}(...) \\ \frac{\delta\mathcal{L}}{\delta\partial_- x^-} &= g^{3/2} + g^{1/2}\Big(2\gamma^{--} + 2\tilde{\tilde{\gamma}}^{+-} + \frac{1}{2}(z^2-y^2)\Big) \end{aligned} \tag{4.190}$$

which is exactly the same as in the purely bosonic case and thus has the same solution

$$\tilde{\tilde{\gamma}}^{++} = a, \quad \tilde{\tilde{\gamma}}^{+-} = -\frac{1}{4}\frac{(z^2-y^2)}{(1+\frac{1}{a})}, \quad \gamma^{--} = \frac{\tilde{\tilde{\gamma}}^{+-}}{a} \tag{4.191}$$

where a is any constant other than 0 or -1. Note that equations (4.187) and (4.188) are exact and may be used to determine higher corrections to the world-sheet metric for investigations of this model going beyond this thesis.

4.10.1 Curvature Corrected Virasoro Constraints for the Full Model

Having a look at the expansions of the combinations of $A_+^{(2)}$, $A_-^{(2)}$

$$\begin{aligned} Str\Big(A_-^{(2)}A_-^{(2)}\Big) &= \frac{1}{g}(...) \\ Str\Big(A_+^{(2)}A_-^{(2)}\Big) &= g^0(...) + \frac{1}{g}(...) \\ Str\Big(A_+^{(2)}A_+^{(2)}\Big) &= g^0(...) \end{aligned} \tag{4.192}$$

and using the corrected world-sheet metric, (4.134) becomes

$$T_{--} = Str\Big(A_-^{(2)}A_-^{(2)} + 2\frac{1}{g}\tilde{\tilde{\gamma}}^{++}(A_+^{(2)}A_-^{(2)} + \gamma^{--}A_-^{(2)}A_-^{(2)} + \frac{1}{g}(...)\Big) = 0 \tag{4.193}$$

and to highest order

$$\begin{aligned} T_{--} &= Str\Big(A_-^{(2)}A_-^{(2)}\Big) + 2\frac{1}{g}\tilde{\tilde{\gamma}}^{++}Str\Big(A_+^{(2)}A_-^{(2)} + \mathcal{O}(g^{-1})\Big) \\ &= \frac{1}{g}\Big((\partial_- z)^2 + (\partial_- y)^2 + iStr(\Sigma_+\partial_-\eta_-\eta_-) + 2\partial_- V \\ &\quad +2\tilde{\tilde{\gamma}}^{++}(\frac{i}{2}Str(\Sigma_+\partial_-\eta_-\eta_-) + \partial_- V)\Big) + \mathcal{O}(g^{-2}) \\ &\overset{!}{=} 0\,. \end{aligned} \tag{4.194}$$

This leads to

$$\partial_- V = -\frac{(\partial_- z)^2 + (\partial_- y)^2}{2(1+a)} - \frac{i}{2} Str\Big(\Sigma_+ \partial_- \eta_- \eta_-\Big)\,. \qquad (4.195)$$

In the same way we can impose the second Virasoro constraint

$$\begin{aligned} T_{++} &= Str\Big(A_+^{(2)} A_+^{(2)}\Big) + 2\gamma^{--} Str\Big(A_+^{(2)} A_-^{(2)} + \mathcal{O}(g^{-1})\Big) \qquad (4.196)\\ &= -(z^2+y^2) - 2Str\Big(\eta_+\eta_-\Big) + iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big) + 2\partial_+ V \\ &+ 2\gamma^{--}\Big(\partial_- V + \frac{i}{2} Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big)\Big) + \mathcal{O}(g^{-1}) \\ &\stackrel{!}{=} 0\,. \end{aligned}$$

Inserting (4.195) and (4.191), we can solve for

$$\begin{aligned} \partial_+ V &= \frac{1}{2}(z^2+y^2) + Str\Big(\eta_+\eta_-\Big) - \frac{i}{2} Str\Big(\Sigma_+\partial_+\eta_-\eta_-\Big) \qquad (4.197)\\ &- \frac{(z^2-y^2)}{8(1+a)^2}((\partial_- z)^2 + (\partial_- y)^2)\,. \end{aligned}$$

4.10.2 Full Lagrangian with Curvature Corrections

We are now in position to write down the full *curvature corrected* Lagrangian. Only the kinetic part gets curvature corrections

$$\begin{aligned} \mathcal{L}_{c.c.} &= \frac{g}{2} Str\Big(A_+^{(2)} A_-^{(2)}\Big) + \mathcal{L}_{WZ} \qquad (4.198)\\ &+ \frac{1}{2}\tilde{\tilde{\gamma}}^{+-} Str\Big(2A_+^{(2)} A_-^{(2)}\Big) + \frac{1}{2}\tilde{\tilde{\gamma}}^{++} Str\Big(A_+^{(2)} A_+^{(2)}\Big) + \frac{g}{2}\gamma^{--} A_-^{(2)} A_-^{(2)}\Big) \\ &= \frac{g}{2} Str\Big(A_+^{(2)} A_-^{(2)}\Big) + \mathcal{L}_{WZ} \\ &- \frac{1}{4}\frac{(z^2-y^2)}{(1+\frac{1}{a})}\Big(\frac{i}{2} Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) + \partial_- V\Big) \\ &+ \frac{a}{2}\Big(-(z^2+y^2) - 2Str\Big(\eta_+\eta_-\Big) + iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big) + 2\partial_+ V\Big) \\ &- \frac{1}{8}\frac{(z^2-y^2)}{(1+a)}\Big((\partial_- z)^2 + (\partial_- y)^2 + iStr(\Sigma_+\partial_-\eta_-\eta_-) + 2\partial_- V\Big)\,. \end{aligned}$$

Dropping total derivatives and inserting (4.195) leads to

$$\begin{aligned}
\mathcal{L}_{c.c.} &= \frac{g}{2}Str\Big(A_+^{(2)}A_-^{(2)}\Big) + \mathcal{L}_{WZ} \\
&+ \frac{1}{8}\frac{(z^2-y^2)}{(1+\frac{1}{a})}\frac{(\partial_- z)^2+(\partial_- y)^2}{(1+a)} \\
&+ \frac{a}{2}\Big(-(z^2+y^2) - 2Str\Big(\eta_+\eta_-\Big) + iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big)\Big) \\
&- \frac{1}{8}\frac{(z^2-y^2)}{(1+a)}\Big((\partial_- z)^2+(\partial_- y)^2\Big)\Big(1-\frac{1}{1+a}\Big) \\
&= \frac{g}{2}Str\Big(A_+^{(2)}A_-^{(2)}\Big) + \mathcal{L}_{WZ} \\
&- \frac{a}{2}(z^2+y^2) - aStr\Big(\eta_+\eta_-\Big) - \frac{a}{2}iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big)\,,
\end{aligned} \tag{4.199}$$

i.e. the terms from γ^{--} and $\tilde{\tilde{\gamma}}^{+-}$ compensate each other. Effectively, the Lagrangian is only changed by $\tilde{\tilde{\gamma}}^{++} = a$ through the contributions from $\tilde{\tilde{\gamma}}^{++}A_+^{(2)}A_+^{(2)}$ that also change the Virasoro constraint as in (4.195) and thus effect the $A_+^{(2)}A_-^{(2)}$ part as well. Going back to (4.96) and inserting the modified Virasoro constraint (4.195) leads to

$$\begin{aligned}
\mathcal{L}_{c.c.} &= \frac{1}{2}(\partial_- z\partial_+ z + \partial_- y\partial_+ y) - \frac{1}{2}(1+a)(z^2+y^2) \\
&-\frac{1}{8(1+a)}(z^2-y^2)((\partial_- z)^2+(\partial_- y)^2) - (1+a)Str(\eta_+\eta_-) \\
&+\frac{i}{2}Str(\Sigma_+\partial_-\eta_+\eta_+) + \frac{i}{2}(1+a)Str(\Sigma_+\partial_+\eta_-\eta_-) \\
&-\frac{5}{32}(z^2-y^2)iStr(\Sigma_+\partial_-\eta_-\eta_-) \\
&+\frac{i}{4}Str\Big(\Sigma_+ x_N\Sigma_N\partial_-\eta_- x_M\Sigma_M\eta_-\Big) \\
&-\frac{i}{4}Str\Big(\Sigma_+\partial_- x_N\Sigma_N x_M\Sigma_M\eta_-^2\Big) \\
&-\frac{i}{4}Str\Big(\Sigma_+(\partial_-\eta_+\eta_-\eta_-\eta_- + \partial_-\eta_-\eta_-\eta_+\eta_-)\Big)\,,
\end{aligned} \tag{4.200}$$

with $a \neq 0$, $a \neq -1$. Note, that the terms in line 4, 5 and 6 of this equation are structurally similar interaction terms. We can freely add total derivatives like $\partial_+ V$, $\partial_- V$ to the Lagrangian, since this does not change the action. We already mentioned this procedure above for the conformal gauge case. Using (4.197) and adding $b(\partial_+ V)$, with b being any constant, to the above Lagrangian we get

$$\begin{aligned}
\mathcal{L}'_{c.c.} \;=\; & \frac{1}{2}(\partial_- z\partial_+ z + \partial_- y\partial_+ y) - \frac{1}{2}(1+a-b)(z^2+y^2) \qquad (4.201) \\
& -\Big(\frac{1}{1+a} + \frac{b}{(1+a)^2}\Big)\frac{1}{8}(z^2-y^2)((\partial_- z)^2 + (\partial_- y)^2) - (1+a-b)Str(\eta_+\eta_-) \\
& +\frac{i}{2}Str(\Sigma_+\partial_-\eta_+\eta_+) + \frac{i}{2}(1+a-b)Str(\Sigma_+\partial_+\eta_-\eta_-) \\
& -\frac{5}{32}(z^2-y^2)iStr(\Sigma_+\partial_-\eta_-\eta_-) \\
& +\frac{i}{4}Str\Big(\Sigma_+ x_N\Sigma_N\partial_-\eta_- x_M\Sigma_M\eta_-\Big) \\
& -\frac{i}{4}Str\Big(\Sigma_+\partial_- x_N\Sigma_N x_M\Sigma_M\eta_-^2\Big) \\
& -\frac{i}{4}Str\Big(\Sigma_+(\partial_-\eta_+\eta_-\eta_-\eta_- + \partial_-\eta_-\eta_-\eta_+\eta_-)\Big)\,.
\end{aligned}$$

For the choice $b = (1+a)$ the terms $Str(\eta_+\eta_-)$, (z^2+y^2) and $Str(\Sigma_+\partial_+\eta_-\eta_-)$ vanish. Choosing $b = -(1+a)$ kills the bosonic interaction term. It is a remarkable feature of the theory that it can be written in a form that does not include boson-boson interactions.

This gives rise to the question whether the chosen light-cone gauge really is a consistent gauge. We proceeded in the same way as in [25] and made the equations of motion for x^- consistent through the introduction of world-sheet curvature corrections. Using the corrections (4.191), does indeed lead to consistent equations of motion. Therefore, there at least is no obvious inconsistency. The fundamental question to be investigated in future is, whether it actually is possible to choose the world-sheet metric in the way we did, i.e. by setting the component $\gamma^{+-} = \frac{1}{2} + \frac{a}{g}(z^2-y^2)$ and $\gamma^{++} = 0 + \frac{1}{g}f(a)$ where a is a non-vanishing constant and $f(a)$ a function of a. The remaining component γ^{--} is determined by the others completely. The ansatz of [1], i.e. to perturb around a solution with constant spin density, might turn out to remove more symmetry than is necessary to fix this gauge. This point was neither mentioned in [1] nor in [25]. It remains to be proven that the symmetries of the theory (before light-cone gauge fixing) suffice to tranform the world-sheet metric into the desired form. To do this we propose the following steps, which should be carried out in a mathematically precise manner:

- Show that a general 2-d metric $h^{\alpha\beta}$ can be transformed into $h'^{\alpha\beta}$ by use of the world-sheet symmetries in the near-flat space limit, such that $\gamma'^{\alpha\beta} = \sqrt{-h'}h'^{\alpha\beta}$ has the form used in this thesis. This should be done in the same manner, the proof, that any 2-d metric can be transformed into a conformally flat metric by use of the diffeomorphism invariance, was done. Intuitively, this should be possible, since any 2-d metric is equivalent to any other 2-d metric up to a conformal factor.

- Show that the action has a residual symmetry, sufficient to reparametrize τ

(or its boosted version) in such a way that it can be set to a function of the form of x^+.

4.11 Currents and Charges

In this section, we explain which steps will have to be accomplished in order to derive the conserved currents and charges, closely following the discussion in [36]. We prepare the calculation but we do not carry it out completely. This section may serve as a detailed starting point for future investigations.

The conserved currents with respect to the invariance of the action under the group PSU(2,2|4) are given by[10]

$$J^\alpha = g\Lambda(x^\pm)f(\eta)g(x_M)\Big(\gamma'^{\alpha\beta}A^{(2)}_\beta - \frac{1}{2}\epsilon^{\alpha\beta}(A^{(1)}_\beta - A^{(3)}_\beta)\Big)g^{-1}(x_M)f^{-1}(\eta)\Lambda^{-1}(x^\pm)\,.$$

The corresponding conserved charge is

$$Q = \int_0^{2\pi} \frac{d\sigma}{2\pi} J^\tau\,. \tag{4.202}$$

The charges corresponding to rotations, dilatation, supersymmetry etc. can be projected out by taking the supertrace of the charge Q multiplied with an appropriate 8×8 matrix $\mathcal{M}$ with constant entries

$$Q_\mathcal{M} = Str\Big(Q\mathcal{M}\Big)\,. \tag{4.203}$$

The diagonal and off-diagonal blocks of $\mathcal{M}$ single out the bosonic and fermionic charges, respectively. As discussed in [36] the subalgebra $\mathcal{J} \subset \mathfrak{psu}(2,2|4)$ of generators that Poisson-commute with the light-cone Hamiltonian is singled out by the red (dark) and blue (light) 2×2 blocks in figure 4.1.

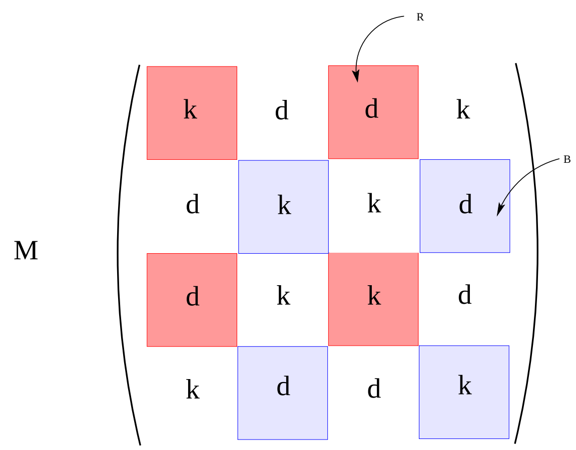

Figure 4.1: The distribution of the kinematical and dynamical charges in the $\mathcal{M}$ supermatrix. The red (dark) and blue (light) blocks correspond to the subalgebra $\mathcal{J}$ of $\mathfrak{psu}(2,2|4)$ which leaves the Hamiltonian invariant. This figure was taken from [36].

[10] The conserved current was originally constructed in [33], but we use the notation of [35].

The blocks denoted by k give rise to kinematical charges (independent of x^-), whereas the ones denoted by d give rise to dynamical charges (depending on x^-). Furthermore, the colored blocks give rise to charges that are independent of x^+ and thus commute with the light-cone Hamiltonian, which is the generator of x^+ translations. These are the charges belonging to $\mathcal{J}$ and if we are only interested in them, we can take the matrix to be of the form

$$\mathcal{M} = \begin{pmatrix} b1 & 0 & f1 & 0 \\ 0 & b2 & 0 & f2 \\ f3 & 0 & b3 & 0 \\ 0 & f4 & 0 & b4 \end{pmatrix} = \mathcal{M}_B + \mathcal{M}_F \tag{4.204}$$

where the 2×2 blocks b_i and f_i single out the bosonic and fermionic subalgebra, repectively. A matrix of the form (4.204) commutes with Σ_+

$$[\mathcal{M}, \Sigma_+] = 0\,. \tag{4.205}$$

Therefore, for the coset parametrization (4.40), the part of $\Lambda(x_\pm)$ which depends on x^+ drops out in conserved currents of the form

$$J^\alpha_{\mathcal{M}} = Str\Big(J^\alpha \mathcal{M}\Big)\,. \tag{4.206}$$

For charges $\in \mathcal{J}$ we can then consider the simplified current

$$J^\alpha_{\mathcal{M}} = gStr\Big(e^{\frac{i}{2}x^-\Sigma_-} f g_x \Big(\gamma'^{\alpha\beta} A^{(2)}_\beta - \frac{1}{2}\epsilon^{\alpha\beta}(A^{(1)}_\beta - A^{(3)}_\beta)\Big) g_x^{-1} f^{-1} e^{-\frac{i}{2}x^-\Sigma_-} \mathcal{M}\Big) \tag{4.207}$$

Furthermore, since $[\mathcal{M}_B, \Sigma_-] = 0$ and $\{\mathcal{M}_F, \Sigma_-\} = 0$, we can distinguish between the bosonic and fermionic currents

$$J^\alpha_{\mathcal{M}_B} = gStr\Big(f g_x \Big(\gamma'^{\alpha\beta} A^{(2)}_\beta - \frac{1}{2}\epsilon^{\alpha\beta}(A^{(1)}_\beta - A^{(3)}_\beta)\Big) g_x^{-1} f^{-1} \mathcal{M}_B\Big) \tag{4.208}$$

$$J^\alpha_{\mathcal{M}_F} = gStr\Big(e^{ix^-\Sigma_-} f g_x \Big(\gamma'^{\alpha\beta} A^{(2)}_\beta - \frac{1}{2}\epsilon^{\alpha\beta}(A^{(1)}_\beta - A^{(3)}_\beta)\Big) g_x^{-1} f^{-1} \mathcal{M}_F\Big) \tag{4.209}$$

For an efficient calculation, the following expansions will be of use

$$g^{\pm 1}(x_M) = 1 \pm g^{-1/2} G^{-1/2} + g^{-1} G^{-1} + \mathcal{O}(g^{-3/2}) \tag{4.210}$$

$$f^{\pm 1}(\eta) = 1 \pm g^{-1/4}\eta_- + g^{-1/2} f^{-1/2} \pm g^{-3/4}\eta_+ + g^{-1} f^{-1} + \mathcal{O}(g^{-6/4}) \tag{4.211}$$

where

$$G^{-1/2} = \frac{1}{2} x_M \Sigma_M, \quad G^{-1} = \frac{1}{8}x^2 = \frac{1}{16}((z^2 - y^2) - (z^2 + y^2)\Sigma_+\Sigma_-) \tag{4.212}$$

$$f^{-1/2} = \frac{1}{2}\eta_-^2, \quad f^{-1} = \frac{1}{2}\{\eta_-, \eta_+\} - \frac{1}{8}\eta_-^4\,.$$

4.11.1 Kinetic Part of the Conserved Current

For the calculation of the kinetic part of the current J^τ, the following term has to be expanded

$$\begin{aligned} & g\, f(\eta) g(x_M)\Big(\gamma'^{0\beta} A^{(2)}_\beta\Big) g^{-1}(x_M) f^{-1}(\eta) \\ = \;& g\, f(\eta) g(x_M)\Big(\sqrt{g}\gamma'^{+\beta} A^{(2)}_\beta + \frac{1}{\sqrt{g}}\gamma'^{-\beta} A^{(2)}_\beta\Big) g^{-1}(x_M) f^{-1}(\eta) \\ = \;& \sqrt{g}\, f(\eta) g(x_M)\Big((g\gamma'^{++} + \gamma'^{-+})A^{(2)}_+ + (g\gamma'^{+-} + \gamma'^{--})A^{(2)}_-\Big) g^{-1}(x_M) f^{-1}(\eta)\,. \end{aligned} \tag{4.213}$$

The short hand notation that was introduced in subsection 4.5.1 for the calculation of the kinetic part of the Lagrangian is

$$\begin{aligned} -2A^{(2)}_- &= \frac{1}{\sqrt{g}}\alpha_- + \frac{1}{\sqrt{g}}\delta_- + \frac{1}{\sqrt{g}}\beta_- + \gamma_- + \epsilon_- \\ -2A^{(2)}_+ &= \sqrt{g}\alpha_+ + \frac{1}{\sqrt{g}}\delta_+ + \sqrt{g}\beta_+ + \gamma_+ + \epsilon_+\,. \end{aligned} \tag{4.214}$$

All of the symbols α, β, γ, δ, ϵ have an expansion in g starting at zeroth order or lower. Since the expansions of both, $f(\eta)$ and $g(x_M)$, start at order g^0, we need the terms inside the capital brackets of (4.213) up to order $g^{-1/2}$, when we are interested in the charges to order g^0.

The remaining freedom in the choice of $\tilde{\tilde{\gamma}}^{++} = a$ may be helpful for the further analysis, since for certain choices parts of the current drop out: e.g. $a = -\frac{1}{2}$, kills a major part of terms coming from $A^{(2)}_+$.

However, since the highest term in $A^{(2)}_+$ is of order $\sqrt{g}$, it is necessary to calculate the g^{-2} corrections to γ'^{++}. An exact form of the equations of motion for x^- was given in (4.187) and (4.188), which can be used to derive the higher curvature corrections. With these corrections at hand, the calculation of the kinetic part of the current can easily be carried out by using the expansions (4.212) as well as the expansions of (4.214) that were introduced earlier.

4.11.2 Wess-Zumino Part of the Conserved Current

For the evaluation of the WZ-part of the current we need to expand

$$\frac{g}{4}\, f(\eta)\, g(x_M)\Big(\sqrt{g}(A^{(1)}_- - A^{(3)}_-) - \frac{1}{\sqrt{g}}(A^{(1)}_+ - A^{(3)}_+)\Big) g^{-1}(x_M) f^{-1}(\eta)\,. \tag{4.215}$$

Using the same short hand notations as for the calculation of the WZ-part and using the gauge $U = 2\sigma^-$ this simplifies drastically

$$
\begin{aligned}
& \frac{1}{4} f(\eta)\, g(x_M) \Big(g^{3/2} (A_-^{(1)} - A_-^{(3)}) - \sqrt{g} (A_+^{(1)} - A_+^{(3)}) \Big) g^{-1}(x_M) f^{-1}(\eta) \quad (4.216) \\
= \quad & \frac{1}{4} f(\eta)\, g(x_M) \Big(- g^{3/2} \epsilon_- - g\gamma_- + g\gamma_+ + \sqrt{g}\epsilon_+ \Big) g^{-1}(x_M) f^{-1}(\eta) \\
\overset{U=2\sigma^-}{=} \quad & \frac{1}{4} f(\eta)\, g(x_M) (-g^{3/2} \epsilon_- + \sqrt{g}\epsilon_+) g^{-1}(x_M) f^{-1}(\eta)\, .
\end{aligned}
$$

The expansion for ϵ has been calculated in subsection 4.5.2

$$
\begin{aligned}
\epsilon_\pm &= g^{-1/4} \epsilon_\pm^{-1/4} + g^{-3/4} \epsilon_\pm^{-3/4} + g^{-5/4} \epsilon_\pm^{-5/4} \qquad (4.217) \\
&= -g^{-1/4} \beta_\pm^{-1/4} + g^{-3/4} \beta_\pm^{-3/4} - g^{-5/4} \beta_\pm^{-5/4}\, .
\end{aligned}
$$

The last line can be obtained by using (4.50) and performing the transpositions in $\epsilon_\pm$ as is shown in appendix F. We calculate the structure of the expansion of (4.216) up to order g^0 in the appendix. However, the result is not complete and the simplifications that take place when the supertrace with $\mathcal{M}_B$ resp. $\mathcal{M}_F$ is taken still need to be evaluated. The result in the appendix may serve as a detailed starting point for future investigations.

V

Summary and Outlook

The results of this diploma thesis can briefly be summarized as follows:

- The recently proposed near-flat space limit was put into the context of present research, after reviewing some of the recent developments in the field of the AdS/CFT correspondence. The limit was motivated by the fact that it interpolates between two well-known limits and, compared to full superstring theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$, has a simple but still non-trivial structure.

- All ingredients for the formulation of the full superstring action were calculated in the supercoset formalism which is convenient for a future analysis of the symmetry algebra of the model. In a first step, we gave the full ingredients of the Lagrangian in an ungauged version, i.e. we left U unfixed and calculated $A_+^{(2)} A_-^{(2)}$ in (4.85), (4.87), $A_-^{(2)} A_-^{(2)}$ in (4.90), $A_+^{(2)} A_+^{(2)}$ in (4.95) as well as the Wess-Zumino part in (4.112). With these results, it is easy to proceed with any desired gauge fixing. For instance, it is possible to continue with the gauge of [1] from this starting point.

- A detailed discussion of the obstacles of light-cone gauge fixing in flat and curved spaces was given in 4.7.1 and two general methods of gauge fixing were explained: the method of [1] and that of [25] which is the one we use as well.

- We discussed a gauge that is different and possibly more intuitive than the one of [1]. We kept the relation $x^+ = \tau$ exact in scaled world-sheet coordinates and did not add any corrections. In contrast to [1], there was no need to do another redefinition of the world-sheet coordinates at the end. We followed the discussion of [25] and allowed the world-sheet metric to differ from the conformal gauge metric in order to get consistent equations of motion for x^-. It proved impossible to keep conformal gauge at leading order, supplemented by curvature corrections in $1/\sqrt{g}$ or $1/g$, and it turned out that the metric must be chosen non-flat already at leading order. The full Lagrangian, including these corrections, was given. The consistency conditions we investigated, do not completely fix the components of the world-sheet metric. It remains an undetermined constant, that can be chosen freely and thus leads to a family of consistent gauges. A clever choice of this constant may be helpful for the further investigation of this model.

- By adding total derivatives to the Lagrangian, we found that one can eliminate the boson-boson-interactions *or* the bosonic and fermionic mass terms, as can be seen from (4.201). This is a remarkable feature but at the same time it gives rise to the question whether we omitted the discussion of an argument that possibly invalidates the gauge choice we have made. Possible reasons were discussed at the end of section 4.10.2. Since this point was not properly discussed in the series of papers [25], [26], [27] either, it seems worth to investigate it in detail. We propose a method for the further investigation of this question.

- As mentioned, the matrix structure was analyzed without imposing a U gauge. Therefore, independently of the gauge that will be chosen in future investigations, this thesis gives a stable fundament for the derivation of the desired Lagrangian and the analysis of the supersymmetry algebra of the model. In section 4.11, we indicated the necessary steps for a continuation of the analysis. We found that it is necessary to consider even higher corrections to the world-sheet metric. We gave a simple form of the exact equations of motion for x^- in (4.187) and (4.188) which can be used to derive the necessary g^{-2} corrections of the world-sheet metric in an effective manner.

- One should pursue, calculating the charges corresponding to rotations, dilations, supersymmetry and so on, as was done in [36] for the plane wave limit. An analysis of the charges would answer the question whether the Lagrangian in the near-flat space limit indeed admits the full extended $SU(2|2)^2 \times R^2$ symmetry algebra as was conjectured in [1].

VI
Appendix

A Dirac Matrices

Throughout this thesis we use the following explicit representation of Dirac matrices

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \Sigma$$

satisfying the $so(5)$ Clifford algebra

$$\{\gamma_c, \gamma_d\} = 2\delta_{cd}\,. \tag{A.1}$$

All of them are hermitian, such that $i\gamma_a$ belongs to $\mathfrak{su}(4)$.

B Expansions

A notational remark: The unit matrix I_8 will just be written as 1 or omitted when no confusion is to be expected.

B.1 Bosonic String Theory

Here we collect the expansions that are used to analyze limits of the bosonic part of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string action.

$$G_{\phi\phi} = \left(\frac{1-\frac{y^2}{4g}}{1+\frac{y^2}{4g}}\right)^2, \qquad G_{tt} = \left(\frac{1+\frac{z^2}{4g}}{1-\frac{z^2}{4g}}\right)^2 \tag{B.1}$$

$$\begin{aligned} G_{\pm\pm} &= G_{\phi\phi} - G_{tt} = -\frac{1}{g}(z^2+y^2) - \frac{1}{2g^2}(z^4-y^4) + \mathcal{O}(g^{-3}) \qquad \text{(B.2)} \\ G_{+-} &= G_{\phi\phi} + G_{tt} = 2 + \frac{1}{g}(z^2-y^2) + \frac{1}{2g^2}(z^4+y^4) + \mathcal{O}(g^{-3}) \end{aligned}$$

B.2 Full Super String Theory

The element

$$g_x \equiv g(x_M) = \begin{pmatrix} g_a(z) & 0 \\ 0 & g_s(y) \end{pmatrix} \tag{B.3}$$

can be decomposed as

$$g(x_M) = g_+ I_8 + g_- \Sigma_8 + g_M \Sigma_M \tag{B.4}$$

where $\Sigma_8 = -\Sigma_+\Sigma_-$, I_8 is the 8-dim. unit matrix and

$$g_\pm = \frac{1}{2}\Big(\frac{1}{\sqrt{1-\frac{z^2}{4}}} \pm \frac{1}{\sqrt{1+\frac{y^2}{4}}}\Big) \tag{B.5}$$

$$g_a = \frac{z_a}{2\sqrt{1-\frac{z^2}{4}}}, \qquad g_s = \frac{y_s}{2\sqrt{1+\frac{y^2}{4}}}\,. \tag{B.6}$$

Furthermore, we can decompose

$$g^2(x_M) = G_+ I_8 + G_- \Sigma_8 + G_M \Sigma_M \tag{B.7}$$

where $M = a, s$ and

$$G_\pm = \frac{1}{2}\Big(\frac{1+\frac{z^2}{4}}{1-\frac{z^2}{4}} \pm \frac{1-\frac{y^2}{4}}{1+\frac{y^2}{4}}\Big) \tag{B.8}$$

$$G_a = \frac{z_a}{1-\frac{z^2}{4}}, \qquad G_s = \frac{y_s}{1-\frac{y^2}{4}} \tag{B.9}$$

after the scalings $z \to \frac{z}{\sqrt{g}}$, $y \to \frac{y}{\sqrt{g}}$ of the transverse coordinates we can expand the expression in powers of g

$$\begin{aligned} g_+ &= 1 + \frac{1}{16g}(z^2 - y^2) + \frac{1}{g^2}(...) \\ g_- &= \frac{1}{16g}(z^2 + y^2) + \frac{1}{g^2}(...) \\ g_a + g_s &= \frac{1}{2\sqrt{g}}(z_a + y_s) + \frac{1}{16g^{3/2}}(z^2 z_a - y^2 y_s) + \frac{1}{g^{5/2}}(...)\,. \end{aligned} \tag{B.10}$$

With this we get

$$\begin{aligned} g(x_M) &= 1 + \frac{1}{2\sqrt{g}}(z_a\Sigma_a + y_s\Sigma_s) + \frac{1}{8g}\frac{1}{2}\Big((z^2+y^2) - (z^2-y^2)\Sigma_+\Sigma_-\Big) \\ &\quad + \frac{1}{16g^{3/2}}(z^2 z_a \Sigma_a - y^2 y_s \Sigma_s) + \frac{1}{g^{5/2}}(...) \\ &= 1 + \frac{1}{2\sqrt{g}} x_M \Sigma_M + \frac{1}{8g} x^2 + \frac{1}{16g^{3/2}} x^2 x_M \Sigma_M + \frac{1}{g^{5/2}}(...) \end{aligned} \tag{B.11}$$

if we define x^2 as a short hand notation for

$$x^2 = \begin{pmatrix} z^2 & 0 \\ 0 & -y^2 \end{pmatrix} = \frac{1}{2}\Big((z^2 - y^2)\Sigma_+ - (z^2 + y^2)\Sigma_-\Big)\Sigma_+ \,. \tag{B.12}$$

Furthermore, we have

$$\begin{aligned} G_+ &= 1 + \frac{1}{4g}(z^2 - y^2) + ... \\ G_- &= \frac{1}{4g}(z^2 + y^2) + ... \\ G_+G_- &= \frac{1}{4g}(z^2 + y^2) + ... \\ G_+^2 - G_-^2 &= 1 + \frac{1}{2g}(z^2 - y^2) + ... \end{aligned} \tag{B.13}$$

With this

$$\begin{aligned} (g_x^2 + g_x^{-2}) &= 2(G_+I_8 + G_-\Sigma_8) \\ &= 2 + \frac{1}{2g}\Big((z^2 - y^2) - (z^2 + y^2)\Sigma_-\Sigma_+\Big) + \frac{1}{g^2}(...) \\ &= 2 + \frac{1}{g}x^2 + \frac{1}{g^2}(...) \,. \end{aligned} \tag{B.14}$$

C Kinetic Part of the Lagrangian

For a structured calculation we define the following short hand notations. We expand these quantities in powers of $g^{-1/2}$. The terms α^0, α^{-1}, ... are the terms in α proportional to g^0, g^{-1}. For the following, the expansions of section B.2 are used.

$$\begin{aligned} \alpha_+ &= \frac{i}{2}(1 + \frac{\partial_+ U}{2g})\Sigma_+(g_x^2 + g_x^{-2}) = \alpha^0 + g^{-1}\alpha^{-1}... \\ &= i\Sigma_+ + g^{-1}\Big(i\Sigma_+(\frac{x^2}{2} + \frac{\partial_+ U}{2})\Big) + g^{-2}... \\ \alpha_- &= \frac{i}{2}\Sigma_+\frac{\partial_- U}{2}(g_x^2 + g_x^{-2}) = \alpha^0 + g^{-1}\alpha^{-1}... \\ &= i\frac{\partial_- U}{2}\Sigma_+ + g^{-1}\Big(i\frac{\partial_- U}{2}\Sigma_+\frac{x^2}{2}\Big) + g^{-2}... \\ \delta_\pm &= \frac{i}{4}\partial_\pm V\Sigma_-(g_x^2 + g_x^{-2}) = \delta_\pm^0 + g^{-1}\delta_\pm^{-1}... \\ &= \frac{i}{2}\partial_\pm V\Sigma_- + g^{-1}\Big(\frac{i}{2}\frac{x^2}{2}\partial_\pm V\Sigma_-\Big) + g^{-2}... \end{aligned} \tag{C.1}$$

$$
\begin{aligned}
\beta_+ &= i(\Sigma_+ g_x \eta^2 g_x - g_x K_8 (\eta^2)^t K_8 g_x \Sigma_+) = g^{-1/2}\beta^{-1/2} + g^{-1}\beta^{-1} + g^{-3/2}...\\
&= g^{-1/2}\Big(i\Sigma_+(\eta_-^2 - K_8 \eta_-^{2\,t} K_8)\Big) + \\
&\quad g^{-1}\Big(i\Sigma_+[\{\eta_+,\eta_-\} - K_8\{\eta_+,\eta_-\}^t K_8] + \frac{i}{2}\Sigma_+\{x_M\Sigma_M, \eta_-^2 + K_8\eta_-^{2\,t}K_8\}\Big) + ...\\
\beta_- &= \frac{\partial_- U}{2}\beta_+\\
\gamma_\pm &= g_x^{-1} B_\pm g_x - g_x K_8 B_\pm^t K_8 g_x^{-1} = g^{-1/2}\gamma_\pm^{-1/2} + g^{-1}\gamma_\pm^{-1} + g^{-3/2}\gamma_\pm^{-3/2} + ...\\
&= g^{-1/2}(B_\pm^{-1/2} - K_8(B_\pm^{-1/2})^t K_8)\\
&+ g^{-1}\Big(B_\pm^{-1} - K_8(B_\pm^{-1})^t K_8 + \frac{1}{2}[x_M\Sigma_M, B_\pm^{-1/2} + K_8(B_\pm^{-1/2})^t K_8]\Big)\\
&+ g^{-3/2}\Big(B_\pm^{-3/2} - K_8(B_\pm^{-3/2})^t K_8 - \frac{1}{2}[x_M\Sigma_M, B_\pm^{-1} + K_8(B_\pm^{-1})^t K_8]\\
&\qquad + \frac{x^2}{4}(B_\pm^{-1/2} - K_8(B_\pm^{-1/2})^t K_8)\\
&\qquad - \frac{1}{4}x_M\Sigma_M(B_\pm^{-1/2} - K_8(B_\pm^{-1/2})^t K_8)x_N\Sigma_N\Big) + ...\\
\epsilon_\pm &= \{\partial_\pm g_x, g_x^{-1}\} = g^{-1/2}\epsilon_\pm^{-1/2} + g^{-1}\epsilon_\pm^{-1} + g^{-3/2}\epsilon_\pm^{-3/2} + ...\\
&= g^{-1/2}\partial_\pm x_M\Sigma_M + 0 + g^{-3/2}\frac{1}{4}x^2\partial_\pm(x_M\Sigma_M)
\end{aligned}
$$

where for $B_\pm$ we use the expansion

$$
\begin{aligned}
B_\pm &= (1 + \frac{1}{2}\eta^2 - \frac{1}{8}\eta^4...)\partial_\pm(1 + \frac{1}{2}\eta^2 - \frac{1}{8}\eta^4 + \frac{1}{16}\eta^6...) - \eta\partial_\pm\eta \qquad (C.2)\\
&= \frac{1}{\sqrt{g}}(\frac{1}{2}[\partial_\pm\eta_-, \eta_-])\\
&+ \frac{1}{g}\Big(\frac{1}{2}[\partial_\pm\eta_+, \eta_-] + \frac{1}{2}[\partial_\pm\eta_-, \eta_+] + \frac{1}{8}[\eta_-\eta_-, \{\eta_-, \partial_\pm\eta_-\}]\Big)\\
&+ \frac{1}{g^{3/2}}\Big(\frac{1}{2}[\partial_\pm\eta_+, \eta_+] + \frac{1}{8}[\eta_-\eta_-, \partial_\pm\{\eta_+, \eta_-\}] + \frac{1}{8}[\{\eta_+, \eta_-\}, \partial_\pm\eta_-^2]\\
&+ \frac{1}{16}[\partial_\pm\eta_-^2, \eta_-^4]\Big) + \mathcal{O}(g^{-2})\\
&= B_\pm^{-1/2} + B_\pm^{-1} + B_\pm^{-3/2}.
\end{aligned}
$$

Note that for the gauge choice $U = 2\sigma^-$ we have $\alpha_- = \alpha_+$, $\beta_- = \beta_+$.

C.1 Conformal Gauge Part

In this subsection, the g^{-1} part of the *conformal gauge piece* $Str\Big(A_+^{(2)}A_-^{(2)}\Big)$ is evaluated. Most of the terms simplify due to algebraic identities as will be indicated. For some terms it is, however, necessary to explicitly evaluate the supertrace of the 8×8 supermatrices. This can be done by hand or by use of Mathematica supplemented with the Grassmann package mentioned in the section *Hilfsmittel* at the end of this document.

Using the expansions of α, β, γ, δ, ϵ and collecting the necessary orders we get

$$\begin{aligned} Str\Big(A_+^{(2)}A_-^{(2)}\Big) &= g^0(...) + g^{-1/2}(...) + \\ &\frac{1}{4g}Str\Big(\gamma_-^{-3/2}\alpha_+^0 + \gamma_-^{-1/2}\alpha_+^{-1} + \epsilon_-^{-3/2}\alpha_+^0 + \epsilon_-^{-1/2}\alpha_+^{-1} + \\ &\quad \gamma_-^{-1}\beta_+^{-1/2} + \gamma_-^{-1/2}\beta_+^{-1} + \epsilon_-^{-1/2}\beta_+^{-1} \\ &\quad \alpha_-^0\alpha_+^{-1} + \alpha_-^{-1}\alpha_+^0 + \alpha_-^0\beta_+^{-1} + \delta_-^{-1}\alpha_+^0 + \delta_-^0\alpha_+^{-1} \\ &\quad \delta_-^0\beta_+^{-1} + \beta_-^{-1}\alpha_+^0 + \beta_-^{-1/2}\beta_+^{-1/2} + \gamma_-^{-1/2}\gamma_+^{-1/2} \\ &\quad +\gamma_-^{-1/2}\epsilon_+^{-1/2} + \epsilon_-^{-1/2}\gamma_+^{-1/2} + \epsilon_-^{-1/2}\epsilon_+^{-1/2} \\ &\quad +\alpha_-^0\gamma_+^{-1/2} + \alpha_-^0\epsilon_+^{-1/2} + \delta_-^0\gamma_+^{-1/2} + \delta_-^0\epsilon_+^{-1/2} \\ &\quad +\gamma_-^{-1/2}\delta_+^0 + \epsilon_-^{-1/2}\delta_+^0 + \alpha_-^0\delta_+^0 + \delta_-^0\delta_+^0\Big)\,. \end{aligned} \tag{C.3}$$

We evaluate this term by term. First we have

$$\begin{aligned} Str\Big(\gamma_-^{-3/2}\alpha_+^0\Big) &= Str\Big(\Big(B_-^{-3/2} - K_8(B_-^{-3/2})^tK_8 - \\ &\frac{1}{2}[x_M\Sigma_M, B_-^{-1} + K_8(B_-^{-1})^tK_8] + \frac{x^2}{4}(B_-^{-1/2} - K_8(B_-^{-1/2})^tK_8) \\ &-\frac{1}{4}x_M\Sigma_M(B_-^{-1/2} - K_8(B_-^{-1/2})^tK_8)x_N\Sigma_N\Big)i\Sigma_+\Big) \\ &= 2Str\Big((B_-^{-3/2}) + \frac{x^2}{4}(B_-^{-1/2}))i\Sigma_+\Big) \\ &= 2iStr\Big(\Sigma_+\partial_-\eta_+\eta_+\Big) + \frac{i}{4}(z^2-y^2)Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big)\,. \end{aligned} \tag{C.4}$$

The next term is

$$\begin{aligned} Str\Big(\gamma_-^{-1/2}\alpha_+^{-1}\Big) &= Str\Big((B_-^{-1/2} - K_8(B_-^{-1/2})^tK_8)i\Sigma_+(\frac{x^2}{2} + \frac{\partial_+U}{2})\Big) \\ &= 2iStr\Big(B_-^{-1/2}\Sigma_+\frac{x^2}{2}\Big) + i\partial_+UStr\Big(\Sigma_+B_-^{-1/2}\Big) \\ &= iStr\Big(B_-^{-1/2}\frac{1}{2}((z^2-y^2)\Sigma_+ - (z^2+y^2)\Sigma_-)\Big) \\ &\quad +i\partial_+UStr\Big(\Sigma_+B_-^{-1/2}\Big) \\ &= \frac{i}{2}(z^2-y^2)Str\Big(\Sigma_+B_-^{-1/2}\Big) + i\partial_+UStr\Big(\Sigma_+B_-^{-1/2}\Big) \\ &= \frac{i}{2}(z^2-y^2)Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) + i\partial_+UStr\Big(\Sigma_+\partial_-\eta_-\eta_-\Big)\,. \end{aligned} \tag{C.5}$$

The following term

$$Str\Big(\epsilon_-^{-3/2}\alpha_+^0\Big) = Str\Big(\frac{x^2}{4}\partial_-x_M\Sigma_Mi\Sigma_+\Big) = 0 \tag{C.6}$$

vanishes, because $\Sigma_\pm\Sigma_M$ are supertraceless. The term thereafter

$$Str\Big(\epsilon_-^{-1/2}\alpha_+^{-1}\Big) = Str\Big(i\Sigma_+(\frac{x^2}{2}+\frac{\partial_+ U}{2})\partial_- x_M\Sigma_M\Big) = 0 \tag{C.7}$$

has exactly the same structure an thus vanishes as well. Then we check (Mathematica) that

$$\begin{aligned} Str\Big(\gamma_-^{-1}\beta_+^{-1/2}\Big) &= Str\Big(\Big(B_-^{-1} - K_8(B_-^{-1})^t K_8 \\ &+ \frac{1}{2}[x_M\Sigma_M, B_-^{-1/2} + K_8(B_-^{-1/2})^t K_8]\Big)\Big(i\Sigma_+(\eta_-^2 - K_8 {\eta_-^2}^t K_8)\Big)\Big) \\ &= 0 \end{aligned} \tag{C.8}$$

and

$$\begin{aligned} Str\Big(\gamma_-^{-1/2}\beta_+^{-1}\Big) &= Str\Big(\Big(B_-^{-1/2} - K_8(B_-^{-1/2})^t K_8\Big)\Big(i\Sigma_+(\{\eta_+,\eta_-\} \\ &\quad -K_8\{\eta_+,\eta_-\}^t K_8) + \frac{i}{2}\Sigma_+\{x_M\Sigma_M, \eta_-^2 + K_8{\eta_-^2}^t K_8\}\Big)\Big) \\ &= 0\,. \end{aligned} \tag{C.9}$$

The next one is

$$\begin{aligned} Str\Big(\epsilon_-^{-1/2}\beta_+^{-1}\Big) &= Str\Big(\partial_- x_N\Sigma_N\Big(i\Sigma_+(\{\eta_+,\eta_-\} \\ &\quad -K_8\{\eta_+,\eta_-\}^t K_8) + \frac{i}{2}\Sigma_+\{x_M\Sigma_M, \eta_-^2 + K_8{\eta_-^2}^t K_8\}\Big)\Big) \\ &= Str\Big(\partial_- x_N\Sigma_N\Big(\frac{i}{2}\Sigma_+\{x_M\Sigma_M, \eta_-^2 + K_8{\eta_-^2}^t K_8\}\Big)\Big) \\ &= -iStr\Big(\Sigma_+\partial_- x_N\Sigma_N\{x_M\Sigma_M,\eta_-^2\}\Big) \\ &= -i\partial_- x_N x_M Str\Big(\Sigma_+[\Sigma_N,\Sigma_M]\eta_-^2\Big) \\ &= -2i\partial_- x_N x_M Str\Big(\Sigma_+\Sigma_N\Sigma_M\eta_-^2\Big) + iStr\Big(\Sigma_+\Sigma_8\eta_-^2\Big) \\ &= -2iStr\Big(\Sigma_+\partial_- x_N\Sigma_N x_M\Sigma_M\eta_-^2\Big) \end{aligned} \tag{C.10}$$

where the last term in the penultimate line vanished due to use of the fermion choice (4.50). The next two terms give the same result

$$\begin{aligned} Str\Big(\alpha_-^0\alpha_+^{-1}\Big) &= Str\Big(i\Sigma_+\frac{\partial_- U}{2}(i\Sigma_+(\frac{x^2}{2}+\frac{\partial_+ U}{2})\Big) \\ &= -\frac{\partial_- U}{2}Str\Big(\frac{x^2}{2}+\frac{\partial_+ U}{2}\Big) \\ &= -\frac{\partial_- U}{8}Str\Big((z^2-y^2)-(z^2+y^2)\Sigma_+\Sigma_-\Big) \\ &= -\partial_- U(z^2+y^2) \end{aligned} \tag{C.11}$$

$$\begin{aligned} Str\Big(\alpha_-^{-1}\alpha_+^0\Big) &= Str\Big(i\Sigma_+\frac{\partial_- U}{2}\frac{x^2}{2}i\Sigma_+)\Big) \\ &= -\frac{\partial_- U}{8}Str\Big((z^2-y^2)-(z^2+y^2)\Sigma_+\Sigma_-\Big) \\ &= -\partial_- U(z^2+y^2)\,. \end{aligned} \tag{C.12}$$

Next we have to evaluate

$$\begin{aligned} Str\Big(\alpha_-^0\beta_+^{-1}\Big) &= i\frac{\partial_- U}{2}Str\Big(\Sigma_+\Big(i\Sigma_+(\{\eta_+,\eta_-\} \\ &\quad -K_8\{\eta_+,\eta_-\}^t K_8)+\frac{i}{2}\Sigma_+\{x_M\Sigma_M,\eta_-^2+K_8\eta_-^{2\,t}K_8\}\Big)\Big) \\ &= iStr\Big(\Sigma_+(i\Sigma_+[\{\eta_+,\eta_-\}-K_8\{\eta_+,\eta_-\}^t K_8])\Big)\frac{\partial_- U}{2} \\ &= -4Str\Big(\eta_+\eta_-\Big)\frac{\partial_- U}{2}\,. \end{aligned} \tag{C.13}$$

The next two have the same structure

$$\begin{aligned} Str\Big(\delta_-^{-1}\alpha_+^0+\delta_-^0\alpha_+^{-1}\Big) &= Str\Big(\frac{i}{2}\frac{x^2}{2}\partial_- V\Sigma_-\frac{i}{2}\Sigma_+ + i\partial_- V\Sigma_- i\Sigma_+(\frac{x^2}{2}+\frac{\partial_+ U}{2})\Big) \\ &= -\frac{1}{4}\partial_- V(z^2-y^2)Str\Big(\Sigma_-\Sigma_+\Big)-\frac{1}{2}\partial_- V\partial_+ U Str\Big(\Sigma_-\Sigma_+\Big) \\ &= 2\partial_- V(z^2-y^2)+4\partial_- V\partial_+ U\,. \end{aligned} \tag{C.14}$$

Using $\{\Sigma_\pm,\Sigma_M\}=0$ and $\Sigma_\pm\eta_\pm=\mp\eta_\pm\Sigma_\pm$ one easily sees that the following term vanishes

$$\begin{aligned} Str\Big(\delta_-^0\beta_+^{-1}\Big) &= Str\Big(i\partial_- V\Sigma_-\Big(i\Sigma_+(\{\eta_+,\eta_-\}-K_8\{\eta_+,\eta_-\}^t K_8) \\ &\quad +\frac{i}{2}\Sigma_+\{x_M\Sigma_M,\eta_-^2+K_8\eta_-^{2\,t}K_8\}\Big)\Big) \\ &= 0\,. \end{aligned} \tag{C.15}$$

Due to cyclicity of the supertrace, the next term was already calculated above

$$Str\Big(\beta_-^{-1}\alpha_+^0\Big) = -4Str\Big(\eta_+\eta_-\Big)\frac{\partial_- U}{2}\,. \tag{C.16}$$

The next two vanish by anticommutation of $\eta_\pm$ with Σ_+ and/or use of (4.64)

$$\begin{aligned} Str\Big(\beta_-^{-1/2}\beta_+^{-1/2}\Big) &= \frac{\partial_- U}{2}Str\Big(i\Sigma_+(\eta_-^2-K_8\eta_-^{2\,t}K_8) \\ &\quad i\Sigma_+(\eta_-^2-K_8\eta_-^{2\,t}K_8)\Big) \\ &= 0 \end{aligned} \tag{C.17}$$

$$\begin{aligned} Str\Big(\gamma_-^{-1/2}\gamma_+^{-1/2}\Big) &= Str\Big((B_-^{-1/2}-K_8(B_-^{-1/2})^t K_8) \\ &\quad (B_+^{-1/2}-K_8(B_+^{-1/2})^t K_8)\Big) \\ &= 0\,. \end{aligned} \tag{C.18}$$

The following two terms vanish due to the structure of $\Sigma_M \eta^2$

$$Str\Big(\gamma_-^{-1/2}\epsilon_+^{-1/2}\Big) = Str\Big((B_-^{-1/2} - K_8(B_-^{-1/2})^t K_8)\partial_+ x_M \Sigma_M\Big) = 0 \tag{C.19}$$

$$Str\Big(\epsilon_-^{-1/2}\gamma_+^{-1/2}\Big) = Str\Big(\partial_- x_M \Sigma_M (B_+^{-1/2} - K_8(B_+^{-1/2})^t K_8)\Big) = 0\,. \tag{C.20}$$

The next term is

$$\begin{aligned} Str\Big(\epsilon_-^{-1/2}\epsilon_+^{-1/2}\Big) &= Str\Big(\partial_- x_M \Sigma_M \partial_+ x_M \Sigma_M\Big) \\ &= 4(\partial_- z \partial_+ z + \partial_- y \partial_+ y)\,, \end{aligned} \tag{C.21}$$

followed by

$$\begin{aligned} Str\Big(\alpha_-^0 \gamma_+^{-1/2}\Big) &= \frac{\partial_- U}{2} Str\Big(i\Sigma_+ (B_+^{-1/2} - K_8(B_+^{-1/2})^t K_8)\Big) \\ &= 2i Str\Big(\Sigma_+ \partial_+ \eta_- \eta_-\Big)\frac{\partial_- U}{2} \end{aligned} \tag{C.22}$$

and

$$Str\Big(\alpha_-^0 \epsilon_+^{-1/2}\Big) = \frac{\partial_- U}{2} Str\Big(i\Sigma_+ \partial_+ x_M \Sigma_M\Big) = 0 \tag{C.23}$$

which vanishes by supertracelessness of $\Sigma_+ \Sigma_M$. Then, we have to evaluate the same expressions with δ_-^0 instead of α:

$$Str\Big(\delta_-^0 \gamma_+^{-1/2}\Big) = Str\Big(\frac{i}{2}\partial_- V \Sigma_- (B_{-1/2}^+ - K_8 B_{-1/2}^{+t} K_8)\Big) = 0 \tag{C.24}$$

and

$$Str\Big(\delta_-^0 \epsilon_+^{-1/2}\Big) = Str\Big(\frac{i}{2}\partial_- V \Sigma_- \partial_+ x_M \Sigma_M\Big) = 0\,. \tag{C.25}$$

The following two expressions have essentially the same matrix structure as the ones calculated in the last step and therefore vanish as well

$$Str\Big(\gamma_-^{-1/2}\delta_+^0\Big) = 0 \tag{C.26}$$

$$Str\Big(\epsilon_-^{-1/2}\delta_+^0\Big) = 0\,. \tag{C.27}$$

We have two more terms to evaluate, the first being

$$\begin{aligned} Str\Big(\alpha_-^0 \delta_+^0\Big) &= Str\Big(i\Sigma_+ \frac{i}{2}\partial_+ V \Sigma_-\Big)\frac{\partial_- U}{2} \\ &= 4\partial_+ V \frac{\partial_- U}{2}\,. \end{aligned} \tag{C.28}$$

The last term vanishes

$$Str\Big(\delta^0_-\delta^0_+\Big) = Str\Big(\frac{i}{2}\partial_- V\Sigma_- \frac{i}{2}\partial_+ V\Sigma_-\Big) = 0 \tag{C.29}$$

due to supertracelessness of $\Sigma_-\Sigma_-$. Writing down all these terms we get

$$\begin{aligned} Str\Big(A^{(2)}_+ A^{(2)}_-\Big) &= g^0(...) + g^{-1/2}(...) \\ &+ \frac{1}{2g}\Big(iStr\Big(\Sigma_+\partial_-\eta_+\eta_+\Big) + \frac{i}{8}(z^2-y^2)Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) \\ &+ \frac{i}{4}((z^2-y^2)+2\partial_+U)Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) \\ &- iStr\Big(\Sigma_+\partial_- x_N\Sigma_N x_M\Sigma_M\eta_-^2\Big) - 2(z^2+y^2) - 2Str\Big(\eta_+\eta_-\Big)\frac{\partial_- U}{2} \\ &+ \partial_- V(z^2-y^2) + 2\partial_- V\partial_+ U - 2Str\Big(\eta_+\eta_-\Big)\frac{\partial_- U}{2} \\ &+ 2(\partial_- z\partial_+ z + \partial_- y\partial_+ y) + iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big)\frac{\partial_- U}{2} + 2\partial_+ V\frac{\partial_- U}{2}\Big)\,. \end{aligned} \tag{C.30}$$

D Wess-Zumino Part of the Lagrangian

In this section the details of the expansion of the WZ-part can be found. We calculate (4.107) order by order. In the following calculations, the relation (4.62) will come to use several times. For a better overview we introduce symbols for the expansion of (B.11)

$$g^{\pm 1}(x_M) = 1 \pm g^{-1/2}G^{-1/2} + g^{-1}G^{-1} + \mathcal{O}(g^{-3/2}) \tag{D.1}$$

with

$$G^{-1/2} = \frac{1}{2}x_M\Sigma_M, \quad G^{-1} = \frac{1}{8}x^2\,. \tag{D.2}$$

First, note that

$$
\begin{aligned}
& i g_x^{-1}\Sigma_+\,\eta\sqrt{1+\eta^2}\,g_x && \text{(D.3)}\\
= \;& i\Sigma_+ g_x(\eta((1+\frac{1}{2}\eta^2-\frac{1}{8}\eta^4+...)))g_x\\
= \;& i\Sigma_+ g_x\Big(g^{-1/4}\eta_- + g^{-3/4}(\eta_+ + \frac{1}{2}\eta_-\eta_-\eta_-) +\\
& g^{-5/4}(\frac{1}{2}(\eta_+\eta_-\eta_- + \eta_-\eta_+\eta_- + \eta_-\eta_-\eta_+)) + ...\Big)g_x\\
= \;& i\Sigma_+(1+g^{-1/2}G^{-1/2}+g^{-1}G^{-1}+...)\\
& (g^{-1/4}\eta_- + g^{-3/4}c^{-3/4} + g^{-5/4}c^{-5/4}+...)\\
& (1+g^{-1/2}G^{-1/2}+g^{-1}G^{-1}+...)\\
= \;& i\Sigma_+(g^{-1/4}\eta_- + g^{-3/4}(\{G^{-1/2},\eta_-\}+c^{-3/4}) +\\
& \quad g^{-5/4}(\{G^{-1},\eta_-\}+\{G^{-1/2},c^{-3/4}\}+G^{-1/2}\eta_- G^{-1/2}+c^{-5/4}))\\
= \;& g^{-1/4}i\Sigma_+\eta_- + g^{-3/4}i\Sigma_+(\{\frac{1}{2}x_M\Sigma_M,\eta_-\}+\eta_+ + \frac{1}{2}\eta_-^3)\\
& +g^{-5/4}i\Sigma_+\Big(\{\frac{x^2}{8},\eta_-\}+\{\frac{1}{2}x_M\Sigma_M,\eta_+ + \frac{1}{2}\eta_-^3\}+\frac{1}{4}x_M\Sigma_M\eta_- x_N\Sigma_N\\
& +\frac{1}{2}(\eta_+\eta_-\eta_- + \eta_-\eta_+\eta_- + \eta_-\eta_-\eta_+) - \frac{1}{8}\eta_-^5\}\Big)\,.
\end{aligned}
$$

Therefore, we get for $\alpha_\pm$

$$
\begin{aligned}
\alpha_- \;=\;& i\frac{\partial_- U}{2}g_x^{-1}\,\Sigma_+\,\eta\sqrt{1+\eta^2}\,g_x = g^{-1/4}\alpha_-^{-1/4} + g^{-3/4}\alpha_-^{-3/4} + g^{-5/4}\alpha_-^{-5/4} && \text{(D.4)}\\
=\;& g^{-1/4}(i\frac{\partial_- U}{2}\Sigma_+\eta_-) + g^{-3/4}i\frac{\partial_- U}{2}\Sigma_+(\{\frac{1}{2}x_M\Sigma_M,\eta_-\}+\eta_+ + \frac{1}{2}\eta_-^3)\\
+\;& g^{-5/4}i\frac{\partial_- U}{2}\Sigma_+\Big(\{\frac{x^2}{8},\eta_-\}+\{\frac{1}{2}x_M\Sigma_M,\eta_+ + \frac{1}{2}\eta_-^3\}+\frac{1}{4}x_M\Sigma_M\eta_- x_N\Sigma_N\\
+\;& \frac{1}{2}(\eta_+\eta_-\eta_- + \eta_-\eta_+\eta_- + \eta_-\eta_-\eta_+) - \frac{1}{8}\eta_-^5\Big)\\
\alpha_+ \;=\;& i(1+\frac{\partial_+ U}{2g})g_x^{-1}\Sigma_+\,\eta\sqrt{1+\eta^2}\,g_x = g^{-1/4}\alpha_+^{-1/4} + g^{-3/4}\alpha_+^{-3/4} + g^{-5/4}\alpha_+^{-5/4}\\
=\;& g^{-1/4}(i\Sigma_+\eta_-) + g^{-3/4}i\Sigma_+(\{\frac{1}{2}x_M\Sigma_M,\eta_-\}+\eta_+ + \frac{1}{2}\eta_-^3)\\
+\;& g^{-5/4}i\Sigma_+\Big(\{\frac{x^2}{8},\eta_-\}+\{\frac{1}{2}x_M\Sigma_M,\eta_+ + \frac{1}{2}\eta_-^3\}+\frac{1}{4}x_M\Sigma_M\eta_- x_N\Sigma_N\\
+\;& \frac{1}{2}(\eta_+\eta_-\eta_- + \eta_-\eta_+\eta_- + \eta_-\eta_-\eta_+) - \frac{1}{8}\eta_-^5 + \frac{1}{2}\eta_-\partial_+ U\Big)\,.
\end{aligned}
$$

The expansion for $\beta_\pm$ up to $\mathcal{O}(g^{-5/4})$ is

$$
\begin{aligned}
\beta_\pm &= g^{-1}(x_M)[\sqrt{1+\eta^2}\partial_\pm\eta - \eta\partial_\pm\sqrt{1+\eta^2}]g(x_M) \\
&= g^{-1}(x_M)[(1+\frac{1}{2}\eta^2-\frac{1}{8}\eta^4+...)\partial_\pm\eta - \eta\partial_\pm(1+\frac{1}{2}\eta^2-\frac{1}{8}\eta^4+...)]g(x_M) \\
&= g^{-1}(x_M)\Big(g^{-1/4}\partial_\pm\eta_- + g^{-3/4}(\partial_\pm\eta_+ - \frac{1}{2}\eta_-\partial_\pm\eta_-\eta_-) \\
&\quad +g^{-5/4}(-\frac{1}{2}(\eta_+\partial_\pm\eta_-\eta_- + \eta_-\partial_\pm\eta_+\eta_- + \eta_-\partial_\pm\eta_-\eta_+) \\
&\quad +\frac{1}{8}(\eta_-\partial_\pm\eta_-\eta_-^3 + \eta_-^2\partial_\pm\eta_-\eta_-^2 + \eta_-^3\partial_\pm\eta_-\eta_-))\Big)g(x_M) \\
&= (1-g^{-1/2}G^{-1/2}+g^{-1}G^{-1}+...) \\
&\quad (g^{-1/4}\partial_\pm\eta_- + g^{-1/4}b^{-3/4} + g^{-1/4}b^{-5/4}+..) \\
&\quad (1+g^{-1/2}G^{-1/2}+g^{-1}G^{-1}+...) \\
&= g^{-1/4}\partial_\pm\eta_- + g^{-3/4}([\partial_\pm\eta_-, G^{-1/2}] + b^{-3/4}) \\
&\quad +g^{-5/4}(\{G^{-1},\partial_\pm\eta_-\} - G^{-1/2}\partial_\pm\eta_- G^{-1/2} + [b^{-3/4}, G^{-1/2}] + b^{-5/4}) \\
&= g^{-1/4}\partial_\pm\eta_- + g^{-3/4}\Big([\partial_\pm\eta_-, \frac{1}{2}x_M\Sigma_M] + \partial_\pm\eta_+ - \frac{1}{2}\eta_-\partial_\pm\eta_-\eta_-\Big) \\
&+ g^{-5/4}\Big(\{\frac{x^2}{8},\partial_\pm\eta_-\} - \frac{1}{2}x_M\Sigma_M\partial_\pm\eta_-\frac{1}{2}x_N\Sigma_N \\
&\qquad +[\partial_\pm\eta_+ - \frac{1}{2}\eta_-\partial_-\eta_-\eta_-, \frac{1}{2}x_M\Sigma_M]) \\
&\qquad -\frac{1}{2}(\eta_+\partial_\pm\eta_-\eta_- + \eta_-\partial_\pm\eta_+\eta_- + \eta_-\partial_\pm\eta_-\eta_+) \\
&\qquad +\frac{1}{8}(\eta_-\partial_\pm\eta_-\eta_-^3 + \eta_-^2\partial_\pm\eta_-\eta_-^2 + \eta_-^3\partial_\pm\eta_-\eta_-\Big) \\
&= g^{-1/4}\beta^{-1/4} + g^{-3/4}\beta^{-3/4} + g^{-5/4}\beta^{-5/4}\,.
\end{aligned}
\tag{D.5}
$$

With this the expansions of $\gamma_\pm$, $\epsilon_\pm$ are clear.

D.1 Leading Order

The leading order term is

$$
\begin{aligned}
\mathcal{L}^1_{WZ} &= -g\frac{\kappa}{8}Str\Big(-\beta_-^{-1/4}\gamma_+^{-1/4} + \alpha_+^{-1/4}\epsilon_-^{-1/4}\Big) \\
&= -g\frac{\kappa}{8}Str\Big(\partial_-\eta_-\tilde{K}_8(\Sigma_+\eta_-)^t K_8 - \Sigma_+\eta_-\tilde{K}_8\partial_-\eta_-^t K_8\Big) \\
&= -g\frac{\kappa}{8}Str\Big(\partial_-\eta_-\tilde{K}_8\eta_-^t\Sigma_+^t K_8 + K_8^t\partial_-\eta_-\tilde{K}_8{}^t\eta_-^t\Sigma_+^t\Big) \\
&= -g\frac{\kappa}{4}Str\Big(\Sigma_+\partial_-\eta_-\tilde{K}_8\eta_-^t K_8\Big)
\end{aligned}
\tag{D.6}
$$

where we have used that $Str() = Str()^t$ in the third line.

D.2 Next-to-leading Order

Next-to-leading order we have

$$\begin{aligned}\mathcal{L}_{WZ}^{1/2} \;=\; & +\sqrt{g}\frac{\kappa}{8}Str\Big(-\beta_-^{-3/4}\gamma_+^{-1/4}-\beta_-^{-1/4}\gamma_+^{-3/4}+\alpha_+^{-3/4}\epsilon_-^{-1/4} \\ & +\beta_+^{-1/4}\epsilon_-^{-1/4}+\alpha_+^{-1/4}\epsilon_-^{-3/4}-\beta_-^{-1/4}\epsilon_+^{-1/4}\Big)\,.\end{aligned} \tag{D.7}$$

We calculate similar terms in pairs of two

$$\begin{aligned} & Str\Big(-\beta_-^{-3/4}\gamma_+^{-1/4}+\alpha_+^{-1/4}\epsilon_-^{-3/4}\Big) \\ =\; & Str\Big(-\beta_-^{-3/4}i\tilde{K}_8(\alpha_+^{-1/4})^tK_8+\alpha_+^{-1/4}i\tilde{K}_8(\beta_-^{-1/4})^tK_8\Big) \\ =\; & Str\Big(-\beta_-^{-3/4}i\tilde{K}_8(\alpha_+^{-1/4})^tK_8-iK_8^t\beta_-^{-1/4}\tilde{K}_8^{\,t}(\alpha_+^{-1/4})^t\Big) \\ =\; & -2iStr\Big(\beta_-^{-3/4}\tilde{K}_8(\alpha_+^{-1/4})^tK_8\Big) \\ =\; & -2iStr\Big(([\partial_-\eta_-,\frac{1}{2}x_M\Sigma_M]+\partial_-\eta_+-\frac{1}{2}\eta_-\partial_-\eta_-\eta_-)\tilde{K}_8(i\Sigma_+\eta_-)^tK_8\Big) \\ =\; & -2Str\Big(\Sigma_+\partial_-\eta_+\tilde{K}_8\eta_-^tK_8\Big)+Str\Big(\Sigma_+\eta_-\partial_-\eta_-\eta_-\tilde{K}_8\eta_-^tK_8\Big)\end{aligned} \tag{D.8, D.9}$$

where the transposition of the second part of the second line involved one minus sign, because α includes one fermion and β_- consists of one resp. three fermions (see the calculational techniques in (4.62)). The expression involving Σ_M is super-traceless and thus vanishes. Using the same arguments we can group the following terms and get

$$\begin{aligned} & Str\Big(-\beta_-^{-1/4}\gamma_+^{-3/4}+\alpha_+^{-3/4}\epsilon_-^{-1/4}\Big) \\ =\; & 2iStr\Big(\alpha_+^{-3/4}\tilde{K}_8(\beta_-^{-1/4})^tK_8) \\ =\; & -2iStr\Big(i\Sigma_+(\{\frac{1}{2}x_M\Sigma_M,\eta_-\}+\eta_++\frac{1}{2}\eta_-^3)\tilde{K}_8\partial_-\eta_-^tK_8\Big) \\ =\; & 2Str\Big(\Sigma_+\eta_+\tilde{K}_8\partial_-\eta_-^tK_8\Big)+Str\Big(\Sigma_+\eta_-^3\tilde{K}_8\partial_-\eta_-^tK_8\Big)\,.\end{aligned} \tag{D.10}$$

The remaining terms are

$$\begin{aligned} & Str\Big(\beta_+^{-1/4}\epsilon_-^{-1/4}-\beta_-^{-1/4}\epsilon_+^{-1/4}\Big) \\ =\; & Str\Big(\beta_+^{-1/4}i\tilde{K}_8(\beta_-^{-1/4})^tK_8-\beta_-^{-1/4}i\tilde{K}_8(\beta_+^{-1/4})^tK_8\Big) \\ =\; & Str\Big(\partial_+\eta_-i\tilde{K}_8(\partial_-\eta_-)^tK_8-\partial_-\eta_-i\tilde{K}_8(\partial_+\eta_-)^tK_8\Big) \\ =\; & Str\Big(\partial_-(\partial_+\eta_-i\tilde{K}_8(\eta_-)^tK_8)-\partial_+(\partial_-\eta_-i\tilde{K}_8(\eta_-)^tK_8) \\ & +\partial_+\partial_-\eta_-i\tilde{K}_8(\eta_-)^tK_8-\partial_+\partial_-\eta_-i\tilde{K}_8(\eta_-)^tK_8\Big) \\ =\; & 0\,,\end{aligned} \tag{D.11}$$

if we drop total derivatives. Adding all these terms up we get for the next-to-leading order term

$$\begin{aligned}\mathcal{L}_{WZ}^{1/2} &= +\sqrt{g}\frac{\kappa}{8}\Big(-2Str\Big(\Sigma_+\partial_-\eta_+\tilde{K}_8\eta_-^tK_8\Big) + Str\Big(\Sigma_+\eta_-\partial_-\eta_-\eta_-\tilde{K}_8\eta_-^tK_8\Big) \\ &\qquad +2Str\Big(\Sigma_+\eta_+\tilde{K}_8\partial_-\eta_-^tK_8\Big) + Str\Big(\Sigma_+\eta_-^3\tilde{K}_8\partial_-\eta_-^tK_8\Big)\Big) \\ &= -\sqrt{g}\frac{\kappa}{8}\Big(4Str\Big(\Sigma_+\partial_-\eta_+\tilde{K}_8\eta_-^tK_8\Big) + Str\Big(\Sigma_+\partial_-\eta_-\eta_-\eta_-\tilde{K}_8\eta_-^tK_8\Big) \\ &\qquad +Str\Big(\Sigma_+\eta_-\eta_-\partial_-\eta_-\tilde{K}_8\eta_-^tK_8\Big)\Big)\,,\end{aligned} \tag{D.12}$$

if we drop total derivatives again.

D.3 Next-to-next-to-leading Order

As for the kinetic part, the contributions to order g^0 are more complicated

$$\begin{aligned}\mathcal{L}_{WZ}^{0} = +\frac{\kappa}{8}\ Str\Big(&-\beta_-^{-5/4}\gamma_+^{-1/4} + \beta_+^{-1/4}\gamma_-^{-1/4} - \beta_-^{-3/4}\gamma_+^{-3/4} - \beta_-^{-1/4}\gamma_+^{-5/4} \\ &+\alpha_+^{-5/4}\epsilon_-^{-1/4} + \beta_+^{-3/4}\epsilon_-^{-1/4} + \alpha_+^{-3/4}\epsilon_-^{-3/4} + \beta_+^{-1/4}\epsilon_-^{-3/4} \\ &+\alpha_+^{-1/4}\epsilon_-^{-5/4} - \alpha_-^{-1/4}\epsilon_+^{-1/4} - \beta_-^{-3/4}\epsilon_+^{-1/4} - \beta_-^{-1/4}\epsilon_+^{-3/4}\Big)\,.\end{aligned} \tag{D.13}$$

Using (4.62), $Str() = Str()^t$, $\tilde{K}_8^{\,t} = -\tilde{K}_8$ and $K_8^t = -K_8$, each term appears twice, so the 12 terms reduce to 6:

$$\begin{aligned}\mathcal{L}_{WZ}^{0} = +\frac{\kappa}{4}\ Str\Big(&\beta_+^{-1/4}\gamma_-^{-1/4} + +\beta_+^{-3/4}\epsilon_-^{-1/4} + \beta_+^{-1/4}\epsilon_-^{-3/4} \\ &-\beta_-^{-3/4}\gamma_+^{-3/4} - \beta_-^{-1/4}\gamma_+^{-5/4} - \beta_-^{-5/4}\gamma_+^{-1/4}\Big)\,.\end{aligned} \tag{D.14}$$

We evaluate this term by term. The first term is

$$\begin{aligned}Str\Big(\beta_+^{-1/4}\gamma_-^{-1/4}\Big) &= Str\Big(\partial_+\eta_-i\tilde{K}_8i(\Sigma_+\eta_-)^tK_8\Big)\frac{\partial_-U}{2} \\ &= -Str\Big(\Sigma_+\partial_+\eta_-\tilde{K}_8\eta_-^tK_8\Big)\frac{\partial_-U}{2}\,.\end{aligned} \tag{D.15}$$

The following are

$$\begin{aligned}&Str\Big(\beta_+^{-3/4}\epsilon_-^{-1/4} + \beta_+^{-1/4}\epsilon_-^{-3/4}\Big) \\ =\ &Str\Big(([\partial_+\eta_-,\frac{1}{2}x_M\Sigma_M] + \partial_+\eta_+ - \frac{1}{2}\eta_-\partial_+\eta_-\eta_-)i\tilde{K}_8\partial_-\eta_-^tK_8 \\ &\quad -([\partial_-\eta_-,\frac{1}{2}x_M\Sigma_M] + \partial_-\eta_+ - \frac{1}{2}\eta_-\partial_-\eta_-\eta_-)i\tilde{K}_8\partial_+\eta_-^t\Big) \\ =\ &Str\Big((\partial_+\eta_+ - \frac{1}{2}\eta_-\partial_+\eta_-\eta_-)i\tilde{K}_8\partial_-\eta_-^tK_8 \\ &\quad -(\partial_-\eta_+ - \frac{1}{2}\eta_-\partial_-\eta_-\eta_-)i\tilde{K}_8\partial_+\eta_-^tK_8\Big) \\ =\ &-\frac{i}{2}Str\Big(\eta_-\partial_+\eta_-\eta_-\tilde{K}_8\partial_-\eta_-^tK_8\Big) + \frac{i}{2}Str\Big(\eta_-\partial_-\eta_-\eta_-\tilde{K}_8\partial_+\eta_-^tK_8\Big)\,,\end{aligned} \tag{D.16}$$

if we use $\partial_+\eta_+\tilde{K_8}\partial_-\eta_-^t K_8 - \partial_-\eta_+\tilde{K_8}\partial_+\eta_-^t K_8 = \partial_-(...) - \partial_+(...) + 0$ and drop the total derivatives.

Next we have

$$\begin{aligned}
& Str\Big(-\beta_-^{-3/4}\gamma_+^{-3/4}\Big) \qquad (D.17)\\
= \;& -Str\Big(([\partial_-\eta_-, \frac{1}{2}x_M\Sigma_M] + \partial_-\eta_+ - \frac{1}{2}\eta_-\partial_-\eta_-\eta_-)\\
& \quad i\tilde{K_8}(i\Sigma_+(\{\frac{1}{2}x_M\Sigma_M, \eta_-\} + \eta_+ + \frac{1}{2}\eta_-^3))^t K_8\Big)\\
= \;& +Str\Big(([\partial_-\eta_-, \frac{1}{2}x_M\Sigma_M]\tilde{K_8}(\Sigma_+(\{\frac{1}{2}x_M\Sigma_M, \eta_-\})^t K_8\Big)\\
& +Str\Big((\partial_-\eta_+ - \frac{1}{2}\eta_-\partial_-\eta_-\eta_-)\tilde{K_8}(\Sigma_+(\eta_+ + \frac{1}{2}\eta_-^3))^t K_8\Big)\\
= \;& +\frac{1}{4}Str\Big((z^2 - y^2)\partial_-\eta_-\tilde{K_8}\eta_-^t K_8\Sigma_+ - 2x_N\Sigma_N\partial_-\eta_-x_M\Sigma_M\tilde{K_8}\eta_-^t K_8\Sigma_+\Big)\\
& +Str\Big(\Sigma_+\partial_-\eta_+\tilde{K_8}\eta_+^t K_8\Big) - \frac{1}{2}Str\Big(\Sigma_+\eta_-^3\tilde{K_8}\partial_-\eta_+^t K_8\Big)\\
& -\frac{1}{2}Str\Big(\Sigma_+\eta_-\partial_-\eta_-\eta_-\tilde{K_8}\eta_+^t K_8\Big) - \frac{1}{4}Str\Big(\Sigma_+\eta_-\partial_-\eta_-\eta_-\tilde{K_8}(\eta_-^3)^t K_8\Big)\,.
\end{aligned}$$

The last two terms are

$$\begin{aligned}
& Str\Big(-\beta_-^{-1/4}\gamma_+^{-5/4} - \beta_-^{-5/4}\gamma_+^{-1/4}\Big) \qquad (D.18)\\
= \;& Str\Big(\alpha_+^{-5/4}\epsilon_-^{-1/4} - \beta_-^{-5/4}\gamma^{-1/4}\Big)\\
= \;& -Str\Big(\Sigma_+\Big(\{\frac{x^2}{8}, \eta_-\} + \{\frac{1}{2}x_M\Sigma_M, \eta_+ + \frac{1}{2}\eta_-^3\} + \frac{1}{4}x_M\Sigma_M\eta_-x_N\Sigma_N\\
& \quad +\frac{1}{2}(\eta_+\eta_-\eta_- + \eta_-\eta_+\eta_- + \eta_-\eta_-\eta_+) - \frac{1}{8}\eta_-^5 + \frac{1}{2}\eta_-\partial_+U\Big)\tilde{K_8}\partial_-\eta_-^t K_8\Big)\\
& +Str\Big(\Sigma_+\Big(\{\frac{x^2}{8}, \partial_-\eta_-\} - \frac{1}{2}x_M\Sigma_M\partial_-\eta_-\frac{1}{2}x_N\Sigma_N\\
& +[\partial_-\eta_+ - \frac{1}{2}\eta_-\partial_-\eta_-\eta_-, \frac{1}{2}x_M\Sigma_M]\\
& \quad -\frac{1}{2}(\eta_+\partial_-\eta_-\eta_- + \eta_-\partial_-\eta_+\eta_- + \eta_-\partial_-\eta_-\eta_+)\\
& \quad +\frac{1}{8}(\eta_-\partial_-\eta_-\eta_-^3 + \eta_-^2\partial_-\eta_-\eta_-^2 + \eta_-^3\partial_-\eta_-\eta_-)\Big)\tilde{K_8}\eta_-^t K_8\Big)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}(z^2-y^2)Str\Big(\Sigma_+\eta_-\tilde{K}_8\partial_-\eta_-^tK_8\Big) - \frac{1}{2}Str\Big(\Sigma_+x_M\Sigma_M\partial_-\eta_-x_N\Sigma_N\tilde{K}_8\eta_-^tK_8\Big) \qquad \text{(D.19)}\\
&\quad -\frac{1}{2}Str\Big(\Sigma_+(\eta_+\eta_-\eta_- + \eta_-\eta_+\eta_- + \eta_-\eta_-\eta_+)\tilde{K}_8\partial_-\eta_-^tK_8\Big)\\
&\quad -\frac{1}{2}Str\Big(\Sigma_+(\eta_+\partial_-\eta_-\eta_- + \eta_-\partial_-\eta_+\eta_- + \eta_-\partial_-\eta_-\eta_+)\tilde{K}_8\partial_-\eta_-^tK_8\Big)\\
&\quad +\frac{1}{8}Str\Big(\Sigma_+\eta_-^5\tilde{K}_8\partial_-\eta_-^tK_8\Big) - \frac{\partial_+U}{2}Str\Big(\Sigma_+\eta_-\tilde{K}_8\partial_-\eta_-^tK_8\Big)\\
&\quad +\frac{1}{8}Str\Big(\Sigma_+(\eta_-\partial_-\eta_-\eta_-^3 + \eta_-^2\partial_-\eta_-\eta_-^2 + \eta_-^3\partial_-\eta_-\eta_-)\tilde{K}_8\eta_-^tK_8\Big)\\
&= -\frac{1}{4}(z^2-y^2)Str\Big(\Sigma_+\eta_-\tilde{K}_8\partial_-\eta_-^tK_8\Big) - \frac{1}{2}Str\Big(\Sigma_+x_M\Sigma_M\partial_-\eta_-x_N\Sigma_N\tilde{K}_8\eta_-^tK_8\Big)\\
&\quad +\frac{1}{2}Str\Big(\Sigma_+(\partial_-\eta_+\eta_-\eta_- + \partial_-\eta_-\eta_+\eta_- + \partial_-\eta_-\eta_-\eta_+\\
&\quad +\eta_+\eta_-\partial_-\eta_- + \eta_-\eta_+\partial_-\eta_- + \eta_-\eta_-\partial_-\eta_+)\tilde{K}_8\eta_-^tK_8\Big)\\
&\quad -\frac{1}{8}Str\Big(\Sigma_+\{\partial_-\eta_-,\eta_-^4\}\tilde{K}_8\eta_-^tK_8\Big) - \frac{\partial_+U}{2}Str\Big(\Sigma_+\eta_-\tilde{K}_8\partial_-\eta_-^tK_8\Big)\,,
\end{aligned}
$$

if we drop total derivatives after partial integration.

So the full order g^0 term is

$$
\begin{aligned}
\mathcal{L}^0_{WZ} &= +\frac{\kappa}{4}\Big\{ - Str\Big(\Sigma_+\partial_+\eta_-\tilde{K}_8\eta_-^tK_8\Big)\frac{\partial_-U}{2} \qquad \text{(D.20)}\\
&\quad -\frac{i}{2}Str\Big(\eta_-\partial_+\eta_-\eta_-\tilde{K}_8\partial_-\eta_-^tK_8\Big) + \frac{i}{2}Str\Big(\eta_-\partial_-\eta_-\eta_-\tilde{K}_8\partial_+\eta_-^tK_8\Big)\\
&\quad +\frac{1}{2}(z^2-y^2)Str\Big(\partial_-\eta_-\tilde{K}_8\eta_-^tK_8\Sigma_+\Big) - \frac{\partial_+U}{2}Str\Big(\Sigma_+\eta_-\tilde{K}_8\partial_-\eta_-^tK_8\Big)\\
&\quad -Str\Big(\Sigma_+x_N\Sigma_N\partial_-\eta_-x_M\Sigma_M\tilde{K}_8\eta_-^tK_8\Big)\\
&\quad +Str\Big(\Sigma_+\partial_-\eta_+\tilde{K}_8\eta_+^tK_8\Big) - \frac{1}{2}Str\Big(\Sigma_+\eta_-^3\tilde{K}_8\partial_-\eta_+^tK_8\Big)\\
&\quad -\frac{1}{2}Str\Big(\Sigma_+\eta_-\partial_-\eta_-\eta_-\tilde{K}_8\eta_+^tK_8\Big)\\
&\quad -\frac{1}{4}Str\Big(\Sigma_+\eta_-\partial_-\eta_-\eta_-\tilde{K}_8(\eta_-^3)^tK_8\Big)\\
&\quad +\frac{1}{2}Str\Big(\Sigma_+(\partial_-\eta_+\eta_-\eta_- + \partial_-\eta_-\eta_+\eta_- + \partial_-\eta_-\eta_-\eta_+\\
&\quad +\eta_+\eta_-\partial_-\eta_- + \eta_-\eta_+\partial_-\eta_- + \eta_-\eta_-\partial_-\eta_+)\tilde{K}_8\eta_-^tK_8\Big)\\
&\quad -\frac{1}{8}Str\Big(\Sigma_+\{\partial_-\eta_-,\eta_-^4\}\tilde{K}_8\eta_-^tK_8\Big)\Big\}\,.
\end{aligned}
$$

Using the choice (4.50) for the fermions, the 6th order terms vanish and a lot of terms can be combined or cancel such that we get

$$\begin{aligned}\mathcal{L}^0_{WZ} &= +\frac{\kappa}{4}\Big\{ - iStr\Big(\Sigma_+\partial_+\eta_-\eta_-\Big)\frac{\partial_- U}{2} - iStr\Big(\Sigma_+\partial_-\eta_+\eta_+\Big) \\ &+\frac{i}{2}(z^2-y^2)Str\Big(\Sigma_+\partial_-\eta_-\eta_-\Big) - iStr\Big(\Sigma_+ x_N\Sigma_N\partial_-\eta_- x_M\Sigma_M\eta_-\Big) \\ &+iStr\Big(\Sigma_+(\partial_-\eta_+\eta_-\eta_-\eta_- + \partial_-\eta_-\eta_-\eta_+\eta_-)\Big) + \frac{\partial_+ U}{2} iStr\Big(\Sigma_+\partial_-\eta_-\eta_-\Big)\Big\}\,.\end{aligned} \tag{D.21}$$

E World-Sheet Curvature Corrections

In this section we calculate the necessary ingredients to evaluate (4.182). Therefore we need

$$Str\Big(A^{(2)}_\alpha\Sigma_\pm\Big)\,. \tag{E.1}$$

We start by noticing that by use of $[K_8,\Sigma_\pm]=0$ we have

$$\begin{aligned}Str\Big(A^{(2)}\Sigma_\pm\Big) &= Str\Big(\frac{1}{2}(A_{even} - K_8(A_{even})^t K_8)\Sigma_\pm\Big) \\ &= Str\Big(A_{even}\Sigma_\pm\Big)\,.\end{aligned} \tag{E.2}$$

With the definition (4.47) of A_{even} and $g(x_M)^{-1}\Sigma_\pm = \Sigma_\pm g(x_M)$, we get

$$\begin{aligned}Str(A_{even}\Sigma_\pm) &= -\frac{i}{2}Str\Big((dx^+\Sigma_+ + dx^-\Sigma_-)g^2\Sigma_\pm)\Big) \\ &\quad -\frac{i}{2}Str\Big(dx^+\Sigma_+\eta^2 g^2\Sigma_\pm\Big) - Str\Big(Bg^2\Sigma_\pm\Big) - Str\Big(g^{-1}dg\Sigma_\pm\Big)\end{aligned} \tag{E.3}$$

where B is the same as defined in the calculation of the kinetic part. We evaluate this part by part. The first term in (E.3) is easy to evaluate by use of (B.7)

$$-\frac{i}{2}Str\Big((dx^+\Sigma_+ + dx^-\Sigma_-)(G_+ I_8 + G_-\Sigma_8 + G_M\Sigma_M)\Sigma_\pm\Big)\,. \tag{E.4}$$

Since Σ_M, $\Sigma_M\Sigma_\pm$, $\Sigma_\pm$ are supertraceless, this term becomes

$$\begin{aligned}Str\Big(...\Sigma_+\Big) &= \frac{i}{2}Str\Big((dx^+G_- - dx^-G_+)\Sigma_+\Sigma_-\Big) \\ Str\Big(...\Sigma_-\Big) &= \frac{i}{2}Str\Big((dx^-G_- - dx^+G_+)\Sigma_+\Sigma_-\Big)\,.\end{aligned} \tag{E.5}$$

The second term in (E.3) is

$$-\frac{i}{2}Str\Big(dx^+\Sigma_+\eta^2(G_+ I_8 + G_-\Sigma_8 + G_M\Sigma_M)\Sigma_\pm\Big)\,. \tag{E.6}$$

All terms that include Σ_M can be checked to have vanishing supertrace for our choice (4.50). The terms including $\Sigma_+\Sigma_-\eta^2$ vanish due to the anticommutation of η and Σ_+. The remaining terms are

$$\begin{aligned} Str\Big(...\Sigma_+\Big) &= -\frac{i}{2}Str\Big(dx^+G_+\eta^2\Big) \\ Str\Big(...\Sigma_-\Big) &= +\frac{i}{2}Str\Big(dx^+G_-\eta^2\Big)\,. \end{aligned} \tag{E.7}$$

Finally, only terms with $\eta_-\eta_+$ survive the supertrace for the fermion choice (4.50). The third term in (E.3) is

$$-Str\Big(Bg^2\Sigma_\pm\Big) = -Str\Big(B(G_+I_8 + G_-\Sigma_8 + G_M\Sigma_M)\Sigma_\pm\Big)\,. \tag{E.8}$$

Again all terms with Σ_M can be checked to vanish in the choice (4.50) and what remains is

$$-Str\Big(B(G_+\Sigma_\pm - G_-\Sigma_\mp)\Big)\,. \tag{E.9}$$

Having a glance at (C.2) we see that at least for the listed order (and we won't need higher terms) we have

$$Str(\Sigma_+B) = Str(\Sigma_+d\eta\eta), \qquad Str(\Sigma_-B) = 0\,. \tag{E.10}$$

This is due to anticommutation of η with Σ_+ and commutation with Σ_-. As shown in the appendix of [35] the relation is true in general. So we get for the Σ_+ resp. Σ_- part

$$\begin{aligned} Str\Big(...\Sigma_+\Big) &= -G_+Str\Big(d\eta\eta\Sigma_+\Big) \\ Str\Big(...\Sigma_-\Big) &= +G_-Str\Big(d\eta\eta\Sigma_+\Big)\,. \end{aligned} \tag{E.11}$$

The last term in (E.3) vanishes identically

$$\begin{aligned} -Str\Big(g^{-1}dg\Sigma_\pm\Big) &= -Str\Big((g_+I_8 + g_-\Sigma_8 - g_M\Sigma_M) \\ &\qquad (dg_+I_8 + dg_-\Sigma_8 + dg_M\Sigma_M)\Sigma_\pm\Big) = 0\,. \end{aligned} \tag{E.12}$$

since only supertraceless matrix structures $\Sigma_M\Sigma_\pm$, $\Sigma_\pm$ appear. Alltogether, we have

$$\begin{aligned} Str\Big(A^{(2)}_\alpha\Sigma_+\Big) &= \frac{i}{2}Str\Big((dx^+G_- - dx^-G_+)\Sigma_+\Sigma_-\Big) - \frac{i}{2}Str\Big(dx^+G_+\eta^2\Big) \\ &\quad - G_+Str\Big(d\eta\eta\Sigma_+\Big) \\ Str\Big(A^{(2)}_\alpha\Sigma_-\Big) &= \frac{i}{2}Str\Big((dx^-G_- - dx^+G_+)\Sigma_+\Sigma_-\Big) + \frac{i}{2}Str\Big(dx^+G_-\eta^2\Big) \\ &\quad + G_-Str\Big(d\eta\eta\Sigma_+\Big)\,. \end{aligned} \tag{E.13}$$

F Currents and Charges

In this section, we list expansions that prepare a future discussion of the conserved charges of the near-flat space model.

F.1 Wess-Zumino Part of the Current

For the Wess-Zumino part of the charges the expansion of ϵ is needed. It can be rewritten in terms of β

$$\epsilon_\pm^{-1/4} = i\tilde{K}_8 \partial_\pm \eta_-^t K_8 = -\partial_\pm \eta_- = -\beta_\pm^{1/4} \tag{F.1}$$

$$\begin{aligned}
\epsilon_\pm^{-3/4} &= i\tilde{K}_8(\partial_\pm \eta_+^t + \frac{1}{2}\eta_-^t \partial_\pm \eta_-^t \eta_-^t + \frac{1}{2} x_M [\Sigma_M^t, \partial_\pm \eta_-^t]) K_8 \\
&= i\tilde{K}_8 \partial_\pm \eta_+^t K_8 - \frac{i}{2} \tilde{K}_8 \eta_-^t K_8 \tilde{K}_8 \partial_\pm \eta_-^t K_8 \tilde{K}_8 \eta_-^t K_8 \\
&\quad + \frac{i}{2} x_M [\Sigma_M, \tilde{K}_8 \partial_\pm \eta_-^t K_8]) \\
&= \partial_\pm \eta_+ - \frac{1}{2}\eta_- \partial_\pm \eta_- \eta_- - \frac{1}{2} x_M [\Sigma_M, \partial_\pm \eta_-] \\
&= +\beta_\pm^{3/4}
\end{aligned} \tag{F.2}$$

$$\begin{aligned}
\epsilon_\pm^{-5/4} &= i\tilde{K}_8 (\frac{1}{8}(\eta_-^3 \partial_\pm \eta_- \eta_- + \eta_-^2 \partial_\pm \eta_- \eta_-^2 + \eta_- \partial_\pm \eta_- \eta_-^3) \\
&\quad - \frac{1}{2}(\eta_+ \partial_\pm \eta_- \eta_- + \eta_- \partial_\pm \eta_+ \eta_- + \eta_- \partial_\pm \eta_- \eta_+) \\
&\quad + \frac{1}{8}(z^2 - y^2)\partial_\pm \eta_- - \frac{1}{2} x_M [\Sigma_M, (\partial_\pm \eta_+ - \frac{1}{2}\eta_- \partial_\pm \eta_- \eta_-)] \\
&\quad - \frac{1}{4} x_M x_N \Sigma_M \partial_\pm \eta_- \Sigma_N)^t K_8 \\
&= -\frac{1}{8}(\eta_- \partial_\pm \eta_- \eta_-^3 + \eta_-^2 \partial_\pm \eta_- \eta_-^2 + \eta_-^3 \partial_\pm \eta_- \eta_-) \\
&\quad + \frac{1}{2}(\eta_- \partial_\pm \eta_- \eta_+ + \eta_- \partial_\pm \eta_+ \eta_- + \eta_+ \partial_\pm \eta_- \eta_-) \\
&\quad - \frac{1}{8}(z^2 - y^2)\partial_\pm \eta_- + \frac{1}{2} x_M [\Sigma_M, (\partial_\pm \eta_+ - \frac{1}{2}\eta_- \partial_\pm \eta_- \eta_-)] \\
&\quad + \frac{1}{4} x_M x_N \Sigma_M \partial_\pm \eta_- \Sigma_N \\
&= -\beta_\pm^{5/4}
\end{aligned} \tag{F.3}$$

where we used the fermion choice (4.50) as well as the transposition rules introduced in section (4.4).

So we have

$$
\begin{aligned}
& \frac{1}{4} f(\eta)\, g(x)(-g^{3/2}\epsilon_- + \sqrt{g}\epsilon_+) g^{-1}(x) f^{-1}(\eta) \qquad \text{(F.4)} \\
= \;& \frac{1}{4}(1 + g^{-1/4}\eta_- + g^{-1/2} f^{-1/2} + g^{-3/4}\eta_+ + g^{-1} f^{-1} + ...) \\
& (1 + g^{-1/2} G^{-1/2} + g^{-1} G^{-1} + ...) \\
& (-g^{5/4}\epsilon_-^{-1/4} - g^{3/4}\epsilon_-^{-3/4} - g^{1/4}\epsilon_-^{-5/4} + g^{1/4}\epsilon_+^{-1/4} + g^{-1/4}\epsilon_+^{-3/4} + g^{-3/4}\epsilon_+^{-5/4}) \\
& (1 - g^{-1/2} G^{-1/2} + g^{-1} G^{-1} + ...) \\
& (1 - g^{-1/4}\eta_- + g^{-1/2} f^{-1/2} - g^{-3/4}\eta_+ + g^{-1} f^{-1}) \\
= \;& \frac{1}{4}\Big(- g^{5/4}\epsilon_-^{-1/4} + g[\epsilon_-^{-1/4}, \eta_-] \\
& + g^{3/4}\Big(- \epsilon_-^{-3/4} + \eta_-\epsilon_-^{-1/4}\eta_- + [\epsilon_-^{-1/4}, G^{-1/2}] - \{f^{-1/2}, \epsilon_-^{-1/4}\}\Big) \\
& + g^{1/2}\Big([\epsilon_-^{-1/4}, \eta_+] + [\eta_-, [\epsilon_-^{-1/4}, G^{-1/2}]] \\
& \quad + f^{-1/2}\epsilon_-^{-1/4}\eta_- - \eta_-\epsilon_-^{-1/4} f^{-1/2} + [\epsilon_-^{-3/4}, \eta_-]\Big) \\
& + g^{1/4}\Big(\epsilon_+^{-1/4} - \epsilon_-^{-5/4} - \{f^{-1/2}, \epsilon_-^{-3/4}\} + \eta_-\epsilon_-^{-3/4}\eta_- - [G^{-1/2}, \epsilon_-^{-3/4}] \\
& \quad - \{f^{-1}, \epsilon_-^{-1/4}\} - \{G^{-1}, \epsilon_-^{-1/4}\} - \{[G^{-1/2}, \epsilon_-^{-1/4}], f^{-1/2}\} \\
& \quad + \eta_-[G^{-1/2}, \epsilon_-^{-1/4}]\eta_- - \eta_-\{f^{-1/2}, \epsilon_-^{-1/4}\}\eta_- + \eta_-\epsilon_-^{-1/4}\eta_+ + \eta_+\epsilon_-^{-1/4}\eta_- \\
& \quad - f^{-1/2}\epsilon_-^{-1/4} f^{-1/4} + G^{-1/2}\epsilon_-^{-1/4} G^{-1/4}\Big) \\
& + g^0\Big([\eta_-, \epsilon_+^{-1/4} - \epsilon_-^{-5/4}] - [\eta_-, [G^{-1/2}, \epsilon_-^{-3/4}]] - [\eta_+, \epsilon_-^{-3/4}] \\
& \quad - \eta_-\epsilon_-^{-3/4} f^{-1/2} + f^{-1/2}\epsilon_-^{-3/4}\eta_- \\
& \quad - [\eta_-, \{G^{-1}, \epsilon_-^{-1/4}\}] - [\eta_+, [G^{-1/2}, \epsilon_-^{-1/4}]] \\
& \quad - \eta_+\epsilon_-^{-1/4} f^{-1/2} + f^{-1/2}\epsilon_-^{-1/4}\eta_+ \\
& \quad - \eta_-\epsilon_-^{-1/4} f^{-1} + f^{-1}\epsilon_-^{-1/4}\eta_- - \eta_-[G^{-1/2}, \epsilon_-^{-1/4}] f^{-1/2} \\
& \quad + f^{-1/2}[G^{-1/2}, \epsilon_-^{-1/4}]\eta_- + [\eta_-, G^{-1/2}\epsilon_-^{-1/4} G^{-1/2}]\Big)\Big) \, .
\end{aligned}
$$

To pursue, one should evaluate the supertrace of this expression, multiplied with $\mathcal{M}_B$ respectively $\mathcal{M}_F$, term by term.

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Hilfsmittel

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Berlin, 4. Juni 2008 Konstantin Wiegandt