HUMBOLDT-UNIVERSITÄT ZU BERLIN

Three-Point Functions in Superconformal Field Theories

Diplomarbeit (korrigierte Version)

Humboldt-Universität zu Berlin
Mathematisch-Naturwissenschaftliche Fakultät I
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eingereicht von

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geb. am 9. Oktober 1984 in Offenburg

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Berlin, Juli 2010

„In einem Zeitalter, wo man Früchte oft vor der Blüte erwartet und vieles darum zu verachten scheint, weil es nicht unmittelbar Wunden heilt, den Acker düngt, oder Mühlräder treibt, ... vergisst man, dass Wissenschaften einen inneren Zweck haben und verliert das eigentlich literarische Interesse, das Streben nach Erkenntnis, als Erkenntnis, aus dem Auge."

Alexander von Humboldt
– „Über die Freiheit des Menschen.
Auf der Suche nach Wahrheit"

Abstract

Three-point functions as well as two-point functions in conformal field theories hold a simple structure that depends only on the scaling dimension and the structure constants. While the two-point functions in maximally supersymmetric conformal Yang-Mills theory are well understood, much less is known about three-point functions.

We gather the information about the general structure of these three-point functions and state a way of diagrammatically calculating the structure constants at one-loop level. The structure constants for short scalar single-trace operators are computed numerically. We also calculate some structure constants for BMN operators. Three-point functions containing at least one Konishi operator take a very simple form at one-loop level that is proven explicitly.

Zusammenfassung in deutscher Sprache

Zwei- und Dreipunktfunktionen in konformen Feldtheorien haben eine einfache Struktur, die alleine durch die Skalendimension und die Strukturkonstanten bestimmt ist. Während Zweipunktfunktionen in der maximal supersymmetrischen konformen Yang-Mills-Theorie bereits gut verstanden sind, ist über die Dreipunktfunktionen sehr viel weniger bekannt.

Es wird ein Überblick darüber gegeben, was über die allgemeine Form dieser Dreipunktfunktionen bekannt ist, und eine Methode der diagrammatischen Berechnung der Einschleifen-Strukturkonstanten beschrieben. Die Strukturkonstanten kurzer skalarer Spuroperatoren werden numerisch bestimmt. Es werden außerdem einige Strukturkonstanten für BMN Operatoren berechnet. Dreipunktfunktionen mit mindestens einem Konishi-Operator nehmen auf Einschleifen-Ebene eine sehr einfache Form an, für welche ein detaillierter Beweis gegeben wird.

Acknowledgements

First of all, I wish to express my sincere thanks to Professor Dr. Jan Plefka for being an out- and understanding supervisor. His precisely formulated explanations were a great help as well as his excellent lecture on quantum field theory.

I would like to thank Dr. Harald Dorn for an instructive lecture on differential geometry and mathematical methods in physics, as well as Professor Dr. Robert Denk and Professor Dr. Claus Scheiderer from the University of Konstanz for a magnificent introduction to analysis and linear algebra during my basic studies. Furthermore, my thanks go to Professor Dr. Dietmar Ebert for illuminating philosophical discussions and to the whole quantum field theory group—including the secretary Sylvia Richter—just for being very nice people altogether.

I am much obliged to my parents for backing me both materially and immaterially throughout my whole studies and my sister as well as all my friends for reminding me from time to time that there is more in life than physics.

Last but not least, I would like to thank Rouven Frassek for helpful discussions and sharing my sorrows, Ralf Sattler and Konstantin Wiegandt for accommodating me in their room, Andreas Rodigast for always reminding me of lunch on time and Kika Ernst for protracted coffee breaks. Special thanks go to Andreas Rodigast and Konstantin Wiegandt for comments on form and content of the manuscript and to Kika Ernst for linguistic corrections.

To play it safe—at the risk of being redundant—I want to thank everyone for everything.

Selbstständigkeitserklärung

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Berlin, den 15. Dezember 2009
(korrigierte Version vom 16. Juli 2010)

André Großardt

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Chapter 1

Preliminaries

1.1 The Unfinished Revolution

At the beginning of the twentieth century our view of the world was radically changed by the advent of two fundamentally new theories. On the one hand, there was the discovery of quantum mechanics, whose foundations were established in 1900 with Max Planck's work about the spectrum of heat radiation. Its further developments finally resulted in quantum field theory and the standard model of elementary-particle physics, which describe the to date smallest and most fundamental parts of nature we know: the constituents of matter, namely the fermions as well as the gauge bosons, which carry the fundamental forces.

On the other hand, in 1916 there was Albert Einstein's discovery of general relativity. While quantum mechanics forced us to completely modify our conception of causality, matter and measurement, general relativity deeply changed our picture of space, time and the gravitational field. Both quantum mechanics and general relativity led to predictions that are in greater accordance with experiment than any physical theory had been before. In fact, there has not been a single experimental result that yields certain evidence that either quantum mechanics or general relativity might be wrong.[1]

With the completion of the standard model in the 1970s there were two efficient theories, but while the standard model was formulated in "old" (that means special relativistic) terms of time and space, general relativity was still a classical theory that did not account for quantum mechanics. Therefore attempts were made to unify them to a *quantum theory of gravity*. Such unifications have led to some of the most striking advances in physics. For example, the combination of Newtonian mechanics with Maxwell's theory led to special relativity, combining special relativity with Newtonian gravity led to general relativity, combining special relativity with nonrela-

[1] Although there is no *certain* evidence, there are some observations that seem to be inexplicable by contemporary physics, such as the Pioneer anomaly or the anisotropy of the cosmic microwave background, but these are not sufficiently backed by experimental data yet.

tivistic quantum mechanics led to the discovery of antiparticles, and so on. Hence we should have a great interest in also unifying quantum field theory and general relativity. But our interest is more than just a philosophical one. When our thoughts reach regimes where we have both small distances and huge masses, as in the case of a black hole or the early universe, we can neglect neither quantum mechanics nor relativity. So we cannot confidently tell anything about these regimes unless we find a quantum theory of gravity.

As general relativity is a classical field theory, we can try to quantise it in the canonical way. But when we do so, we are faced with a problem: The resulting quantum theory turns out to be non-renormalisable. A lot of attempts were made during the last decades to find a quantum theory of gravity but so far none of them was successful. In addition to the problem of quantum gravity the standard model suffers from the large number of eighteen[2] free parameters which can be arbitrarily chosen and have to be determined by experimental data.

String theory, which arose in the early 1970s as a theory for the strong interaction and was developed further to a candidate for a unified theory around 1984, solves these problems in an elegant way. In string theory elementary point particles are replaced by extended one-dimensional objects. This leads to a unified theory containing a large amount of phenomenology, including fermions and gauge fields. In particular it automatically includes gravity in the form of the graviton, a vibrational mode of closed strings. Unfortunately this comes at a high price. String theory is laden with a gigantic baggage of additional physics, particularly supersymmetry and extra dimensions, of which nothing has shown up in experiments so far. As well as all other candidates for a quantum theory of gravity, string theory does not make any falsifiable predictions so far which are accessible for current experiments.

1.2 The AdS/CFT Correspondence

One of the most interesting active areas of research in string theory is the AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence which was first proposed in Maldacena's 1997 paper [34]. The correspondence states that a type IIB superstring theory in ten dimensions, namely in the product space of a five dimensional anti-de Sitter space with a five sphere ($AdS_5 \times S^5$), is equivalent to the four-dimensional maximally supersymmet-

[2] or even more, if we take neutrino masses into account

ric Yang-Mills gauge theory ($\mathcal{N} = 4$ *super Yang-Mills* or *SYM* in short) which is a conformal field theory. Therefore it endows us with a very helpful instrument to perform calculations either on the string theory side or on the field theory side and simultaneously learn something about the other side.

While the free parameters of the string theory are the string coupling g_S and the effective string tension R^2/α', the parameters on the gauge theory side are the rank N of the gauge group $SU(N)$, whose physical meaning is the number of colours, and the coupling constant g_{YM} which are combined to the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$. The correspondence relates these parameters by the identifications

$$\frac{4\pi\,\lambda}{N} = g_S \qquad \text{and} \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}. \tag{1.1}$$

The energy eigenstates of the strings are identified with operators in the gauge theory and the energy eigenvalues E correspond to the so-called scaling dimensions Δ of the gauge theory operators that describe the behaviour of these operators under dilatations.

It is both fascinating and disadvantageous, that the domain best understood on the string theory side is the weakly curved limit, i. e. $\sqrt{\lambda} \gg 1$, because it can be considered an effective supergravity theory, while on the gauge theory side only the weakly coupled regime where $\lambda \ll 1$ is perturbatively accessible. This, on one hand, makes sophisticated string calculations a lot easier when performed on the gauge theory side and vice versa, and thus it endows us with a tool to access so far inaccessible regimes of both gauge and string theory. On the other hand it complicates checking the proposition of the correspondence for which there is no rigorous proof yet.

Obviously AdS/CFT can teach us a lot about string theory and might be of great use if string theory turned out to describe physics rightly. But even if string theory would not be the correct quantum theory of gravity the correspondence could still be a useful tool for calculations in quantum field theory.

1.3 Correlation Functions in Superconformal Yang-Mills Theory

If we want to get quantum field theoretical results that are actually measurable we need to consider cross sections and thus have to calculate S-matrix elements. But, although not experimentally accessible, all the information about interactions of the theory is already encoded in the n-point *correlation functions*

$$\langle \Omega \mid T\phi(x_1)\cdots\phi(x_n) \mid \Omega \rangle \,, \tag{1.2}$$

where T denotes time ordering, $\phi(x)$ is a (for the sake of simplicity scalar) field and Ω is the ground state of the interacting theory.[3] So, if we want to explore the physical content of a quantum field theory and compare it with others' the somewhat abstract correlation functions endow us with an easy way to approach this goal.

The invariance under conformal transformations is a beautiful property that appears in many classical field theories such as the massless Yang-Mills theory. One of the most striking features of conformal invariance is that the form of two- and three-point functions of operators in the theory is highly restricted. The two-point functions are completely fixed by the *scaling dimension* Δ. The three-point functions take the form

$$\langle \mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)\mathcal{O}_\gamma(x_3)\rangle = \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}\,|x_{23}|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}\,|x_{13}|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta}} \tag{1.3}$$

and therefore depend only on the scaling dimensions and the scalar factor $C_{\alpha\beta\gamma}$ called *structure constant.*

In $\mathcal{N} = 4$ super Yang-Mills theory, due to supersymmetry, conformal invariance survives the quantisation process. The simple structure (1.3) then applies to the three-point functions of the quantum operators and we can expand both the scaling dimensions and the structure constants in the 't Hooft coupling λ

$$\Delta = \Delta^{(0)} + \lambda\gamma + O(\lambda^2), \tag{1.4}$$

$$C_{\alpha\beta\gamma} = C^{(0)}_{\alpha\beta\gamma} + \lambda C^{(1)}_{\alpha\beta\gamma} + O(\lambda^2). \tag{1.5}$$

The *anomalous dimensions* γ are known or can be calculated with the help of a powerful mechanism developed by Beisert, Kristjansen, Plefka and Staudacher in [5] and [6] that makes use of the dilatation operator instead of explicitly calculating the two-point functions. As the tree-level structure constants $C^{(0)}_{\alpha\beta\gamma}$ are not hard to determine, the quantum corrections of the structure constants remain the unknown entities in the three-point functions. Gathering information about the structure of the one-loop corrections $C^{(1)}_{\alpha\beta\gamma}$ will be the main subject of this thesis.

[3]In the following we will omit the time ordering and the states and just write the fields in angle brackets to denote correlation functions.

1.4 An Outline of this Thesis

Starting with the derivation of the structure of two- and three-point functions in conformal field theories in chapter 2, we continue with a short review of the $\mathcal{N} = 4$ super Yang-Mills theory including a description of the spin chain picture and the dilatation operator in chapter 3. In chapter 4 we apply the results from chapter 2 to the quantum field theory and thus obtain the one-loop corrections to the scaling dimensions and structure constants. We state the diagrammatic dressing formulae that we use to calculate structure constants at one-loop level.

First results from the application of the dressing formulae are given in chapter 5. Beside some short operators we calculate three-point functions for a class of twisted operators. In chapter 6 we regard the transformation between non-diagonal and diagonal bases and apply this to calculate structure constants computationally. The very simple general structure of three-point functions containing a Konishi operator is proven and both results obtained for $SO(6)$ operators by numerical calculations and results for structure constants of BMN operators are listed.

We conclude with summarising the results and stating prospects for further considerations on the topic.

Chapter 2

Conformal Invariance and Correlation Functions

2.1 The Conformal Group

A conformal transformation is an invertible mapping $x \to x'$ of the coordinates which leaves the metric tensor invariant up to a scale factor:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda(x)\, g_{\mu\nu}(x). \tag{2.1}$$

For the special case $\Lambda(x) \equiv 1$ we get the Poincaré group as a subgroup of the conformal group.

These transformations form the conformal group consisting of the following types of transformations:

Translation:	$x^\mu \to x'^\mu = x^\mu + a^\mu$
Dilatation:	$x^\mu \to x'^\mu = \lambda\, x^\mu$
Rotation:	$x^\mu \to x'^\mu = M^\mu{}_\nu\, x^\nu$
Special Conformal Transformation:	$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2}$

where $M^\mu{}_\nu = -M^\nu{}_\mu$, λ is a constant and a^μ, b^μ are arbitrary constant vectors.

By definition spinless *quasi-primary fields* transform under general conformal transformations as

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta}{d}} \phi(x) \tag{2.2}$$

where d is the space-time dimension, Δ the scaling dimension of the field and $|\partial x'/\partial x|$ the Jacobian. For dilatations the Jacobian is λ^d and thus ϕ transforms as $\phi'(\lambda x) = \lambda^{-\Delta}\phi(x)$.

Note that when speaking of fields we do not necessarily mean objects that are integrated over in the functional integral. We also mean composite objects such as for example derivatives of physical fields.

2.2 The Structure of Correlation Functions in Conformal Field Theories

In quantum field theory the n-point correlation function for a number of fields is the vacuum expectation value of the time ordered product of these fields. In the path integral formalism it is given by

$$\begin{aligned}\langle\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\rangle &= \langle\Omega \mid T(\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)) \mid \Omega\rangle \\ &= \lim_{\varepsilon\to 0}\frac{\int\mathcal{D}\Phi\,\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)e^{\mathrm{i}S_\varepsilon[\Phi(x)]}}{\int\mathcal{D}\Phi\,e^{\mathrm{i}S_\varepsilon[\Phi(x)]}}\end{aligned} \tag{2.3}$$

where the ϕ_i are arbitrary fields (not necessarily distinct), Φ denotes the set of all fields in the theory including the ϕ_i, $|\Omega\rangle$ is the ground state of the interacting theory and S_ε is the action with complex time obtained by replacing t by $t(1-\mathrm{i}\varepsilon)$.

2.2.1 Transformation of Correlation Functions under Arbitrary Transformations

Let us consider a coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \omega_a\frac{\delta x^\mu}{\delta\omega_a}, \tag{2.4}$$

where ω_a is infinitesimal. Let $\mathcal{F}$ be a function defined by $\mathcal{F}(\phi(x)) = \phi'(x')$. Then the fields transform like

$$\phi(x) \to \phi'(x') = \phi(x) + \omega_a\frac{\delta\mathcal{F}}{\delta\omega_a}(x). \tag{2.5}$$

If we have an action $S[\Phi]$ that is invariant under the given transformation then we can easily derive the transformation of the correlation functions:

$$\begin{aligned}\left\langle\phi_1(x_1')\cdots\phi_n(x_n')\right\rangle &\overset{\text{by def.}}{=} \frac{1}{Z}\int\mathcal{D}\Phi\;\phi_1(x_1')\cdots\phi_n(x_n')\,e^{-S[\Phi]} \\ &\overset{\phi\to\phi'}{=} \frac{1}{Z}\int\mathcal{D}\Phi'\;\phi_1'(x_1')\cdots\phi_n'(x_n')\,e^{-S[\Phi']} \\ &\overset{S\text{ invar.}}{=} \frac{1}{Z}\int\mathcal{D}\Phi\;\mathcal{F}(\phi_1(x_1))\cdots\mathcal{F}(\phi_n(x_n))\,e^{-S[\Phi]} \\ &= \left\langle\mathcal{F}(\phi_1(x_1))\cdots\mathcal{F}(\phi_n(x_n))\right\rangle.\end{aligned} \tag{2.6}$$

2.2.2 Two-Point Functions

Let $\phi_i(x)$ be spinless quasi-primary fields. With (2.6) and (2.2) we get

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\frac{\Delta_2}{d}} \left\langle\phi_1(x_1')\phi_2(x_2')\right\rangle. \tag{2.7}$$

Due to rotational and translational invariance, the two-point function has to be a function of the absolute value of the distance $x_1 - x_2$

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|). \tag{2.8}$$

The invariance under scaling transformations $x \to \lambda x$ requires

$$f(|x_1 - x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda\,|x_1 - x_2|). \tag{2.9}$$

This means

$$\begin{aligned} & f(\lambda x) = \frac{1}{\lambda^{\Delta_1+\Delta_2}} f(x) \\ \Rightarrow \quad & f(\lambda) = \frac{1}{\lambda^{\Delta_1+\Delta_2}} f(1) =: \frac{C_{12}}{\lambda^{\Delta_1+\Delta_2}} \\ \Rightarrow \quad & \langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}. \end{aligned} \tag{2.10}$$

In addition to scaling, translational and rotational invariance, we have to demand invariance under special conformal transformations. In a lengthy but straightforward calculation we get the Jacobian for those transformations

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 - 2b \cdot x + b^2 x^2)^d}. \tag{2.11}$$

We define $\gamma_i := 1 - 2b \cdot x_i + b^2 x_i^2$, then the distance transforms as

$$|x_i - x_j| \overset{\text{SCT}}{\longrightarrow} \left|x_i' - x_j'\right| = \frac{|x_i - x_j|}{\sqrt{\gamma_i \gamma_j}}. \tag{2.12}$$

Applying this to the correlation function we get

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\frac{\Delta_2}{d}} \left\langle\phi_1(x_1')\phi_2(x_2')\right\rangle$$

$$
\begin{aligned}
&= \gamma_1^{-d\frac{\Delta_1}{d}} \gamma_2^{-d\frac{\Delta_2}{d}} \frac{C_{12}}{|x_1' - x_2'|^{\Delta_1+\Delta_2}} \\
&= \frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{C_{12}\,(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \\
&= \frac{(\gamma_1\gamma_2)^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \left\langle \phi_1(x_1)\phi_2(x_2) \right\rangle . \qquad (2.13)
\end{aligned}
$$

This has to be true for arbitrary b^μ and thus for arbitrary $\gamma_{1,2}$ which can be achieved only if $\Delta_1 = \Delta_2 =: \Delta$. Our final result is

$$
\boxed{\left\langle \phi_1(x_1)\phi_2(x_2) \right\rangle = \delta_{\Delta_1,\Delta_2} \frac{C_{12}}{|x_1 - x_2|^{2\Delta}}.} \qquad (2.14)
$$

2.2.3 Three-Point Functions

The structure of the three-point functions can be obtained in the same way. From now on we write $x_{ij} = x_i - x_j$ for the distance.

First, by demanding rotational and translational invariance, we obtain

$$
\left\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \right\rangle = f(|x_{12}|, |x_{13}|, |x_{23}|) \qquad (2.15)
$$

and because of the scaling invariance we get

$$
f(x, y, z) = \lambda^{\Delta_1+\Delta_2+\Delta_3} f(\lambda x, \lambda y, \lambda z). \qquad (2.16)
$$

This is true if and only if

$$
x^a y^b z^c f(x, y, z) = (\lambda x)^a (\lambda y)^b (\lambda z)^c f(\lambda x, \lambda y, \lambda z) \qquad (2.17)
$$

for some a, b, c with $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$. Thus the left hand side has to be a constant with respect to x, y and z and we end up with

$$
\left\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \right\rangle = \frac{C_{123}^{abc}}{|x_{12}|^a |x_{13}|^b |x_{23}|^c}. \qquad (2.18)
$$

If we now consider special conformal transformations we get

$$
\begin{aligned}
\left\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \right\rangle &= \gamma_1^{-\Delta_1}\gamma_2^{-\Delta_2}\gamma_3^{-\Delta_3} \\
&\quad \times \frac{C_{123}^{abc}}{(\gamma_1\gamma_2)^{-\frac{a}{2}} |x_{12}|^a (\gamma_1\gamma_3)^{-\frac{b}{2}} |x_{13}|^b (\gamma_2\gamma_3)^{-\frac{c}{2}} |x_{23}|^c}
\end{aligned}
$$

$$= \gamma_1^{-\Delta_1+\frac{a+b}{2}} \gamma_2^{-\Delta_2+\frac{a+c}{2}} \gamma_3^{-\Delta_3+\frac{b+c}{2}} \langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle \,. \tag{2.19}$$

As for the two-point functions this has to be true for arbitrary $\gamma_{1,2,3}$. This means

$$2\,\Delta_1 = a+b, \qquad 2\,\Delta_2 = a+c, \qquad 2\,\Delta_3 = b+c \tag{2.20}$$

which can be uniquely solved and leaves us with

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 \\ b &= \Delta_1 + \Delta_3 - \Delta_2 \\ c &= \Delta_2 + \Delta_3 - \Delta_1 . \end{aligned} \tag{2.21}$$

The final result is

$$\boxed{\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}\,|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}.} \tag{2.22}$$

2.3 The Operator Product Expansion

A common method in quantum field theory is to replace complex interactions by single effective vertices. The operator product expansion (OPE) endows us with a formalism to describe this procedure. Consider two operators $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ at separate but close points x_1 and x_2 and suppose that any other field is located much farther away. Then the product $\mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)$ can be described by a local operator at x_2 that can be expanded in a basis of operators. In conformal field theories the operator product expansion takes the form

$$\mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2) \sim \sum_\gamma \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}} \mathcal{O}_\gamma(x_2) \tag{2.23}$$

where "$\sim$" denotes that both sides of the equation show the same divergent behaviour in the limit $x_1 \to x_2$ but may differ by finite terms. The coefficients $C_{\alpha\beta\gamma}$ are the structure constants that also appear in the three-point functions.

Chapter 3

$\mathcal{N} = 4$ Super Yang-Mills as a Superconformal Field Theory

3.1 The $\mathcal{N} = 4$ Super Yang-Mills Action

The maximally supersymmetric Yang-Mills theory contains a gluon field $A_\mu(x)$, six scalar fields $\phi_i(x)$ $(i = 1, \ldots, 6)$ and four gluinos that can be written as a ten-dimensional Majorana-Weyl spinor with sixteen components $\chi_\alpha(x)$ $(\alpha = 1, \ldots, 16)$. The fields are in the adjoint representation of the gauge group.

If we define the covariant derivative to be $D_\mu = \partial_\mu - \mathrm{i}\,[A_\mu, \cdot]$, the ten-dimensional Dirac matrices to be (Γ_μ, Γ_i) $(\mu = 0, \ldots, 3,\ i = 1, \ldots, 6)$ and the conjugate spinor to be $\overline{\chi} = \chi^\dagger \Gamma_0$, according to [42] the action of $\mathcal{N} = 4$ super Yang-Mills takes the form

$$S = \frac{2}{g_{\mathrm{YM}}^2} \int \mathrm{d}^4x\, \mathrm{Tr} \Bigg(\frac{1}{4}\,(F_{\mu\nu})^2 + \frac{1}{2}\,(D_\mu \phi_i)^2 - \frac{1}{4}\,[\phi_i, \phi_j]\,[\phi_i, \phi_j] + \frac{1}{2}\,\overline{\chi}\,\Gamma_\mu D^\mu\,\chi - \frac{\mathrm{i}}{2}\,\overline{\chi}\,\Gamma_i\,[\phi_i, \chi] \Bigg) \tag{3.1}$$

and is uniquely determined by the coupling constant g_{YM} and the rank N of the gauge group $SU(N)$. The β-function of the theory is believed to vanish to all orders.[1] This is equivalent to the statement that conformal invariance is maintained even after renormalisation.

3.2 Operators in $\mathcal{N} = 4$ Super Yang-Mills

Let us introduce some common classifications for operators in the superconformal theory. We conclude with the local single-trace operators made up of scalar fields whose three-point functions will be the subject of our research.

[1] This was shown up to three-loop order in [3, 10, 23] and there are several arguments that it should hold to all loop orders [9, 26–28, 35].

3.2.1 The Superconformal Algebra

The superconformal algebra is a *graded* algebra. It is generated by the fifteen generators of the four-dimensional conformal algebra, which are the four generators of space-time translations P_μ and the six generators of Lorentz transformations $M_{\mu\nu}$ obeying the Poincaré algebra, as well as the generator of scaling transformations D and the four generators of special conformal transformations K_μ, obeying the algebra[2]

$$\begin{aligned} [D,P_\mu] &= -\mathrm{i}\,P_\mu & [K_\mu,K_\nu] &= 0 \\ [D,K_\mu] &= \mathrm{i}\,K_\mu & [P_\mu,K_\nu] &= 2\mathrm{i}\,(M_{\mu\nu}-\eta_{\mu\nu}D) \\ [D,M_{\mu\nu}] &= 0 & [M_{\mu\nu},K_\rho] &= \mathrm{i}\,(\eta_{\rho\nu}K_\mu-\eta_{\mu\rho}K_\nu), \end{aligned} \tag{3.2}$$

together with the eight supercharges Q^a_α and their conjugates $\widetilde{Q}^{\overline{a}}_{\dot\alpha}$ that satisfy the anti-commutation relations

$$\left\{Q^a_\alpha,\widetilde{Q}^{\overline{b}}_{\dot\alpha}\right\} = \gamma^\mu_{\alpha\dot\alpha}\,\delta^{a\overline{b}}\,P_\mu \quad\text{and}\quad \left\{Q^a_\alpha,Q^b_\beta\right\} = \left\{\widetilde{Q}^{\overline{a}}_{\dot\alpha},\widetilde{Q}^{\overline{b}}_{\dot\beta}\right\} = 0. \tag{3.3}$$

$\alpha=1,2$ and $\dot\alpha=1,2$ index the two $SU(2)$ algebras making up the Lorentz algebra and $a=1,\ldots,4$ and $\overline{a}=1,\ldots,4$ are indices for the internal R-symmetry.

Both algebras are combined by the commutators

$$\begin{aligned} \left[P_\mu,Q^a_\alpha\right] &= 0 & \left[P_\mu,\widetilde{Q}^{\overline{a}}_{\dot\alpha}\right] &= 0 \\ \left[D,Q^a_\alpha\right] &= -\frac{\mathrm{i}}{2}\,Q^a_\alpha & \left[D,\widetilde{Q}^{\overline{a}}_{\dot\alpha}\right] &= -\frac{\mathrm{i}}{2}\,\widetilde{Q}^{\overline{a}}_{\dot\alpha} \\ \left[M^{\mu\nu},Q^a_\alpha\right] &= \mathrm{i}\sigma^{\mu\nu}_{\alpha\beta}\,\epsilon^{\beta\gamma}\,Q^a_\gamma & \left[M^{\mu\nu},\widetilde{Q}^{\overline{a}}_{\dot\alpha}\right] &= \mathrm{i}\sigma^{\mu\nu}_{\dot\alpha\dot\beta}\,\epsilon^{\dot\beta\dot\gamma}\,\widetilde{Q}^{\overline{a}}_{\dot\gamma} \\ \left[K^\mu,Q^a_\alpha\right] &= \gamma^\mu_{\alpha\dot\alpha}\,\epsilon^{\dot\alpha\dot\beta}\,\widetilde{S}^a_{\dot\beta} & \left[K^\mu,\widetilde{Q}^{\overline{a}}_{\dot\alpha}\right] &= \gamma^\mu_{\alpha\dot\alpha}\,\epsilon^{\alpha\beta}\,S^{\overline{a}}_\beta. \end{aligned} \tag{3.4}$$

$S^{\overline{a}}_\alpha$ and $\widetilde{S}^a_{\dot\alpha}$, called *special conformal supercharges*, obey anti-commutation relations similar to those of the supercharges:

$$\left\{S^{\overline{a}}_\alpha,\widetilde{S}^b_{\dot\alpha}\right\} = \gamma^\mu_{\alpha\dot\alpha}\,\delta^{\overline{a}b}\,K_\mu \quad\text{and}\quad \left\{S^{\overline{a}}_\alpha,S^{\overline{b}}_\beta\right\} = \left\{\widetilde{S}^a_{\dot\alpha},\widetilde{S}^b_{\dot\beta}\right\} = 0. \tag{3.5}$$

The anti-commutation relations still missing are those of the supercharges

[2] See [38] for any of the relations stated in this section.

with the special conformal supercharges. These are given by

$$\begin{aligned}\{Q^a_\alpha, S^{\bar b}_\beta\} &= \sigma^{\mu\nu}_{\alpha\beta}\,\delta^{a\bar b}\,M_{\mu\nu} - \epsilon_{\alpha\beta}\,\delta^{a\bar b}\,D - \mathrm{i}\epsilon_{\alpha\beta}\,\sigma^{ij}_{a\bar b}\,R_{ij}\\ \{\widetilde{Q}^{\bar a}_{\dot\alpha}, \widetilde{S}^{b}_{\dot\beta}\} &= \sigma^{\mu\nu}_{\dot\alpha\dot\beta}\,\delta^{\bar a b}\,M_{\mu\nu} - \epsilon_{\dot\alpha\dot\beta}\,\delta^{\bar a b}\,D + \mathrm{i}\epsilon_{\dot\alpha\dot\beta}\,\sigma^{ij}_{\bar a b}\,R_{ij}\end{aligned} \tag{3.6}$$

where R_{ij} $(i,j=1,\ldots,6)$ denote the generators of the $SO(6)$ R-symmetry. The $\sigma^{ij}_{a\bar b}$ are the $SO(6)$ generators in the fundamental representation. Obviously the supercharges are spinors under the R-symmetry. All other generators commute with R_{ij}.

$SU(4) \simeq SO(6)$ has three commuting generators with corresponding charges $J_{1,2,3}$. Fields can then be classified by sextuplets of charges

$$(\Delta^{(0)}, S_1, S_2; J_1, J_2, J_3)$$

with the bare scaling dimension $\Delta^{(0)}$ and the two charges $S_{1,2}$ of the Lorentz group.

3.2.2 Primaries and Chiral Primaries

Let $\mathcal{O}(x)$ denote a local operator of the theory. By definition of the *scaling dimension* Δ of $\mathcal{O}(x)$, under a dilatation[3] $x \to \lambda x$ the operator scales as $\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x)$. These dilatations are generated by D, also called the *dilatation operator*, acting on $\mathcal{O}(x)$ by

$$[D, \mathcal{O}(x)] = \mathrm{i}\left(-\Delta + x\frac{\partial}{\partial x}\right)\mathcal{O}(x). \tag{3.7}$$

The action of D on the commutator of K_μ and $\mathcal{O}(0)$ can be found using the Jacobi identity and is

$$[D, [K_\mu, \mathcal{O}(0)]] = \mathrm{i}(-\Delta + 1)\,[K_\mu, \mathcal{O}(0)]\,. \tag{3.8}$$

The scaling dimension is therefore lowered by one.

Since unitarity requires the scaling dimensions of local operators to be positive, there must be operators that cannot be lowered any further by action of K_μ, i. e.

$$[K_\mu, \mathcal{O}(0)] = 0. \tag{3.9}$$

These are the *primary operators*. The operators that follow by acting on the primaries with the generators of the superconformal algebra are called

[3] Note that *dilatation* and *scaling transformation* are just two words for the same thing.

descendants.

Only primary operators can commute with all the special conformal supercharges. If, additionally, they commute with at least one of the supercharges they are called *chiral primaries*. A special class are the operators with one R-charge J and the other R-charges equal to zero. These are chiral primaries if $\Delta = J$ and are then known as *BPS operators*[4]. Chiral primaries are *protected*, meaning that their scaling dimension gets no quantum corrections, i. e. $\Delta = \Delta^{(0)}$.

3.2.3 Gauge Invariant Operators

Gauge invariant operators can be constructed by taking traces over products of the fields. If ψ_k denotes one of the fields A_μ, ϕ_i, χ_α or any derivative of these we can have single-trace operators $\mathrm{Tr}\,(\psi_1\psi_2\cdots\psi_L)\,(x)$, double-trace operators $\mathrm{Tr}\,(\psi_1\psi_2\cdots\psi_{L_1})\,\mathrm{Tr}\,(\psi_{L_1+1}\psi_{L_1+2}\cdots\psi_{L_1+L_2})\,(x)$ and so on.

From now on we consider only operators of the scalar fields. These are usually referred to as the $SO(6)$ sector. The six scalar fields can be combined to three complex fields

$$Z = \phi_1 + \mathrm{i}\,\phi_2, \qquad W = \phi_3 + \mathrm{i}\,\phi_4 \quad \text{and} \quad X = \phi_5 + \mathrm{i}\,\phi_6 \tag{3.10}$$

together with their complex conjugates. If we restrict the operators to such made up only of traces of Z and W we get what is known as the $SU(2)$ sector. This sector is closed under operator mixing to all orders. Although this does not hold for the $SO(6)$ sector, this sector is at least closed at one-loop order.

In the large N limit that will be explained in the next section, multi-trace operators are suppressed and it is therefore possible to restrict our considerations at one-loop order to scalar single-trace operators.

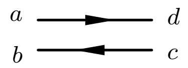
3.3 Large N Expansion and Planar Limit

It was first proposed by 't Hooft [25], originally as a method for the strong interaction, to treat the rank N of the gauge group as a parameter of the theory and expand the theory with respect to it. In quantum chromodynamics (QCD) confinement defines a fundamental scale in the theory, namely the confinement scale Λ_{QCD}, associated with physical effects. Therefore it is natural to keep this scale fixed in an expansion. This can be achieved by

[4]The notion of chiral primary operators (CPO) is often used as a synonym for BPS operators, too.

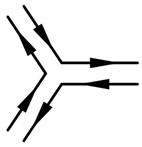
keeping the product $\lambda := g_{\mathrm{YM}}^2 N$, called the 't Hooft coupling, fixed while taking the limit $N \to \infty$.

If we work in the adjoint representation we can use 't Hooft's double line notation. Each field holds two indices both of which can be connected with a line. The propagators may then be depicted by double lines

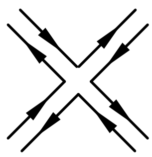


$$\sim g_{\mathrm{YM}}^2 \tag{3.11}$$

and are proportional to $g_{\mathrm{YM}}^2 = \lambda/N$ while the vertices



$$\sim \frac{1}{g_{\mathrm{YM}}^2} \quad \text{and}$$



$$\sim \frac{1}{g_{\mathrm{YM}}^2} \tag{3.12}$$

are proportional to $1/g_{\mathrm{YM}}^2 = N/\lambda$. We wrote down only the propagator and the vertices for scalar fields here but those for gluons and fermions could in principle be depicted the same way.

Each index loop appearing in a Feynman diagram gives rise to an extra factor of N that comes from summing over the group indices. The number of index loops equals the number of faces F of the diagram which is one more than the number of loops of the Feynman diagram, because we take the trace over all fields or, diagrammatically speaking, we also have to count the outer face which is no genuine loop of the Feynman diagram. A typical diagram with P propagators, V vertices and F faces is then associated with a factor

$$\lambda^{P-V} N^{V-P+F}. \tag{3.13}$$

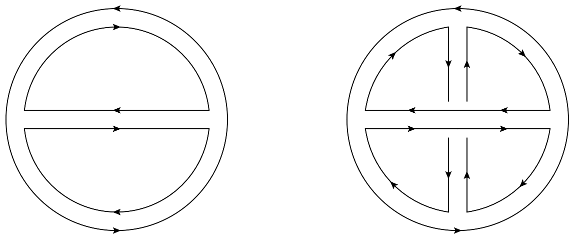
The exponent can be substituted by the topological invariant

$$\chi = V - P + F = 2 - 2h \tag{3.14}$$

of a simplicial complex with V vertices, P edges and F faces called *Euler characteristic*. h denotes the genus of the complex that corresponds to the number of handles. We can then decompose physical quantities of the theory in a double expansion in the 't Hooft coupling λ and $1/N^2$

$$\sum_{h=0}^{\infty} N^{2-2h} \sum_{n=0}^{\infty} c_{h,n}\, \lambda^n. \tag{3.15}$$

It is easy to see that when taking the limit $N \to \infty$ the dominant contributions come from the diagrams of lowest genus. These are the diagrams that



(a) Planar diagram **(b)** Non-planar diagram

Figure 3.1: Examples of a planar and a non-planar diagram.

can be drawn in a plane without crossing lines, referred to as *planar* for obvious reasons. An example for both a planar and a non-planar diagram is shown in figure 3.1.

As it is a conformal theory there is no natural scale for $\mathcal{N} = 4$ super Yang-Mills. In particular, there is no confinement because the β-function vanishes for all values of g_{YM}. Thus there is no value that has to remain fixed and other limits than the 't Hooft limit can also be taken. Among these the *BMN limit*[5] is worth mentioning where we take $N \to \infty$ and the R-charge $J \to \infty$ while keeping J^2/N fixed. This limit corresponds to the plane wave limit of string theory. The corresponding *BMN operators* constitute long strings of Z-fields with a small number of other fields called impurities. For an introduction to this topic see for example [43] or [45].

3.4 The Integrable Spin Chain Picture

Let us at first restrict our considerations to the $SU(2)$ sector, more precisely to single-trace operators that are built only from the complex scalars Z and W. The trace is invariant under cyclic permutations. If we regard Z as a spin "down" $|\downarrow\rangle$ and W as spin "up" $|\uparrow\rangle$ we can therefore depict such an operator as a ring of up- and down-spins.

This picture can be generalised to the $SO(6)$ sector as well. Instead of two-dimensional $SU(2)$ spins we must then endow the spin chain with six-dimensional vectors transforming under $SO(6)$.

If we now want to calculate for example the tree-level contribution to the two-point correlation function of two such operators we have to contract each field of the first with a field of the second operator. This is pictured in figure 3.2. In principle we could have any possible contraction,

[5]named after Berenstein, Maldacena and Nastase, introduced in [8]

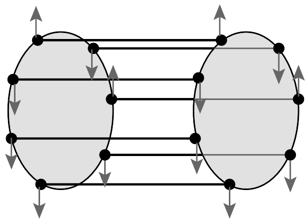


Figure 3.2: Tree-level two-point function in the spin chain picture.

but as we consider operators in the large N limit we need to regard only planar contractions. Thus, by choosing the first contraction, all others are uniquely determined. We just have to sum over all cyclic permutations for one operator. As the tree-level scaling dimension $\Delta^{(0)}$ equals the length of the operator, i. e. the number of fields in the operator, operators of different dimensions cannot be fully contracted and therefore do not mix on tree-level.

The operators of equal length hold a high degeneracy that is broken for the full quantum operators. The scaling dimension then gets quantum corrections

$$\Delta = \Delta^{(0)} + \lambda\gamma + O(\lambda^2) \tag{3.16}$$

where γ ist called the *anomalous dimension* of the operator. For an arbitrary basis of bare operators with equal tree-level dimension $\Delta^{(0)}$ there is an operator mixing at one-loop level. The anomalous dimension for these operators is then ill-defined.

What we want to have is a basis of operators for which the two-point functions are also diagonal at higher loop orders. The anomalous dimensions could then be read off directly. This diagonalisation may be done loop order by loop order, and we will carry this out in detail for one-loop order in the next chapter, but there is a far more elegant way introduced in [5, 6] that makes use of the dilatation operator.

3.4.1 The Dilatation Operator

The dilatation operator was already introduced when we first met the superconformal algebra. It is the generator D of scaling transformations. Its feature is that the gauge invariant local trace operators $\mathcal{O}_\alpha(x)$ are eigenstates of the dilatation operator and its eigenvalues are the scaling dimensions

$$D\,\mathcal{O}_\alpha(x) = \Delta_\alpha\,\mathcal{O}_\alpha(x). \tag{3.17}$$

The operator can be expanded in powers of the coupling constant

$$D = \sum_{k=0}^{\infty} \left(\frac{g_{\mathrm{YM}}^2}{16\pi^2} \right)^k D_{2k} \tag{3.18}$$

and according to [6] the first two orders of D for the scalar $SO(6)$ sector are

$$D_0 = \mathrm{Tr}\left(\phi_m \check{\phi}_m\right) \tag{3.19}$$

$$D_2 = - : \mathrm{Tr}\left([\phi_m, \phi_n]\left[\check{\phi}_m, \check{\phi}_n\right]\right) : -\frac{1}{2} : \mathrm{Tr}\left(\left[\phi_m, \check{\phi}_n\right]\left[\phi_m, \check{\phi}_n\right]\right) : . \tag{3.20}$$

The colon denotes normal ordering, meaning that derivatives do not act on the enclosed fields, and we use the notation

$$\check{\phi}_m = \frac{\delta}{\delta\phi_m} = T^a \frac{\delta}{\delta\phi_m^{(a)}}, \tag{3.21}$$

where T^a denote the generators of the $SU(N)$ gauge group.

Instead of calculating all two-point functions and diagonalising them loop order by loop order, we can now directly diagonalise the dilatation operator to get the basis that is diagonal to a given order.

This gets even more interesting if we consider the $SU(2)$ sector in the planar limit. The one-loop dilatation operator acting on a trace operator of length L then takes the form

$$D_2^{\mathrm{planar}} = \sum_{i=1}^{L} \left(\mathbb{1}_{i,i+1} - P_{i,i+1}\right) \tag{3.22}$$

where $P_{i,j}$ is the permutation of the fields at position i and j and P is cyclically periodic ($L+1 \equiv 1$). This is exactly the Hamiltonian of a ferromagnetic $\mathrm{XXX}_{1/2}$ Heisenberg spin chain. The powerful tool of the Bethe ansatz which in condensed matter physics is well-known to solve the Heisenberg spin chain, can therefore be used to diagonalise the dilatation operator.

3.4.2 Computation of Anomalous Dimensions with the Coordinate Bethe Ansatz

Let us roughly outline the general idea of the Bethe ansatz. One of the crucial ingredients is the notion of *integrability*. We can define an R-Matrix, an operator that acts in the product of two auxiliar spaces $V_a \otimes V_b$, which for the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain takes the form

$$R_{ab}(u) = u\,\mathbb{1}_{ab} + \mathrm{i}\,P_{ab}. \tag{3.23}$$

The criterion for integrability is then stated by the *Yang-Baxter equation*

$$R_{12}(u)\,R_{13}(u+v)\,R_{23}(v) = R_{23}(v)\,R_{13}(u+v)\,R_{12}(u). \tag{3.24}$$

The ground state for the spin-chain is $|{\downarrow\downarrow}\cdots{\downarrow\downarrow}\rangle$. We call a state with M up-spins a M-magnon state and write $|x_1, x_2, \ldots, x_J\rangle_L$ for a length L state with up-spins at positions $x_1, x_2, \ldots, x_J$, e. g.

$$|1,3,4,7\rangle_{L=8} = |{\uparrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow}\rangle. \tag{3.25}$$

The dilatation operator (or Hamiltonian) D_2 does not change the magnon number. We can thus diagonalise the M-magnon states for each M separately. This is trivial for the one-magnon states $|x\rangle$. The diagonal states are just the Fourier transformations

$$|\psi(p_1)\rangle = \sum_{x=1}^{L} e^{\mathrm{i}p_1 x}\ |x\rangle \qquad \text{with} \quad D_2\,|\psi(p_1)\rangle = 4\sin^2\frac{p_1}{2}\ |\psi(p_1)\rangle$$
$$\text{and} \quad p_1 = \frac{2\pi\,k}{L} \quad (k \in \mathbb{Z}). \tag{3.26}$$

For the two-magnon states we use Bethe's ansatz

$$\psi(x_1, x_2) = e^{\mathrm{i}(p_1 x_1 + p_2 x_2)} + S(p_1, p_2)\, e^{\mathrm{i}(p_2 x_1 + p_1 x_2)}, \tag{3.27}$$

i. e. a superposition of an incoming and an outgoing plane wave. $S(p_1, p_2)$ denotes the S-matrix. The Schrödinger equation for the two-magnon state

$$E_2\,|\psi(p_1, p_2)\rangle = D_2\,|\psi(p_1, p_2)\rangle \tag{3.28}$$

leads to an expression for the total energy

$$E_2 = 4\sin^2\frac{p_1}{2} + 4\sin^2\frac{p_2}{2} \tag{3.29}$$

and determines the form of the S-matrix

$$S(p_1, p_2) = \frac{\cot\frac{p_1}{2} - \cot\frac{p_2}{2} + 2\mathrm{i}}{\cot\frac{p_1}{2} - \cot\frac{p_2}{2} - 2\mathrm{i}}. \tag{3.30}$$

Using the periodicity of the spin chain we get the two-magnon *Bethe equations*

$$e^{\mathrm{i}p_1 L} = S(p_1, p_2) \quad \text{and} \quad e^{\mathrm{i}p_2 L} = S(p_2, p_1). \tag{3.31}$$

Integrability is special in that the general M-magnon states factorise into sequences of two-magnon states. This yields a set of M Bethe equations

$$e^{\mathrm{i}p_k L} = \prod_{\substack{i=1 \\ i \neq k}}^{M} S(p_k, p_i) \tag{3.32}$$

together with the S-matrix (3.30) and the total energy

$$E = \sum_{i=1}^{M} 4 \sin^2 \frac{p_i}{2}. \tag{3.33}$$

We can further take account of cyclic invariance of the trace by constraining the total momentum to zero

$$\sum_{i=1}^{M} p_i = 0. \tag{3.34}$$

For a general introduction to the Bethe ansatz and a precise definition of the somewhat difficult to manage notion of the R-matrix see [22], for its application to $\mathcal{N}=4$ super Yang-Mills see [37, 42].

Chapter 4

Three-Point Functions at One-Loop

4.1 General Form of Two- and Three-Point Functions at One-Loop

4.1.1 Two-Point Functions

We will derive the general form of the two- and three-point functions in $\mathcal{N}=4$ super Yang-Mills following [40]. We start with a set of bare primary operators $\mathcal{O}^B$ whose two-point functions to first order in λ take the form

$$\left\langle \mathcal{O}^B_\alpha(x_1)\,\mathcal{O}^B_\beta(x_2)\right\rangle = \frac{\delta_{\Delta^{(0)}_\alpha,\Delta^{(0)}_\beta}}{x_{12}^{2\Delta^{(0)}_\alpha}}\left(t_{\alpha\beta} - \lambda\gamma_{\alpha\beta}\ln|x_{12}\Lambda|^2\right), \tag{4.1}$$

with the ultraviolet momentum cutoff Λ and the free scaling dimensions $\Delta^{(0)}_\alpha$. $t_{\alpha\beta}$ may contain terms of order λ.

We can simultaneously diagonalise both $t_{\alpha\beta}$ and $\gamma_{\alpha\beta}$. Let M be the change-of-basis matrix from the bare operators $\mathcal{O}^B_\alpha$ to the diagonal operators

$$\mathcal{O}_\alpha = M_{\alpha\beta}\mathcal{O}^B_\beta. \tag{4.2}$$

The two-point functions of these operators take the form

$$\begin{aligned}\langle \mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\rangle &= M_{\alpha\gamma}M_{\beta\delta}\left\langle \mathcal{O}^B_\gamma(x_1)\mathcal{O}^B_\delta(x_2)\right\rangle \\ &= \frac{\delta_{\Delta^{(0)}_\alpha,\Delta^{(0)}_\beta}}{x_{12}^{2\Delta^{(0)}_\alpha}}\left(M_{\alpha\gamma}t_{\gamma\delta}M^T_{\delta\beta} - \lambda M_{\alpha\gamma}\gamma_{\gamma\delta}M^T_{\delta\beta}\ln|x_{12}\Lambda|^2\right).\end{aligned} \tag{4.3}$$

We want these two-point functions to become diagonal, which means

$$\langle \mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)\rangle = \frac{\delta_{\alpha\beta}\,N_\alpha}{x_{12}^{2\Delta^{(0)}_\alpha}}\left(1 - \lambda\gamma_\alpha\ln|x_{12}\Lambda|^2\right) \tag{4.4}$$

with a normalisation constant N_α that may contain terms of order λ. Comparing (4.3) and (4.4) we obtain the following matrix equations:

$$(MtM^T)_{\alpha\beta} = \delta_{\alpha\beta} N_\alpha \tag{4.5}$$

$$(M\gamma M^T)_{\alpha\beta} = \delta_{\alpha\beta} N_\alpha \gamma_\alpha . \tag{4.6}$$

We normalise the two-point functions in such a way that

$$N_\alpha = 1 + \lambda g_\alpha + O(\lambda^2). \tag{4.7}$$

The two-point functions for the diagonal operators are then

$$\langle \mathcal{O}_\alpha(x_1)\, \mathcal{O}_\beta(x_2)\rangle = \frac{\delta_{\alpha\beta}}{x_{12}^{2\Delta_\alpha^{(0)}}} \left(1 + \lambda g_\alpha - \lambda\gamma_\alpha \ln |x_{12}\Lambda|^2\right). \tag{4.8}$$

If we now define the renormalised operators as

$$\widetilde{\mathcal{O}}_\alpha = \mathcal{O}_\alpha \left(1 - \frac{\lambda}{2} g_\alpha + \lambda\gamma_\alpha \ln \left|\frac{\Lambda}{\mu}\right| + O(\lambda^2)\right) \tag{4.9}$$

with renormalisation scale μ, we obtain two-point functions that show the correct structure as demanded by conformal invariance, with respect to the one-loop scaling dimensions $\Delta_\alpha = \Delta_\alpha^{(0)} + \lambda\gamma_\alpha$

$$\begin{aligned}\left\langle \widetilde{\mathcal{O}}_\alpha(x_1)\, \widetilde{\mathcal{O}}_\beta(x_2)\right\rangle &= \frac{\delta_{\alpha\beta}}{|x_{12}|^{2\Delta_\alpha^{(0)}}} \left(1 - \lambda\gamma_\alpha \ln |x_{12}\mu|^2 + O(\lambda^2)\right) \\ &= \frac{\delta_{\alpha\beta}}{|x_{12}|^{2\Delta_\alpha^{(0)}}}\, \mathrm{e}^{-\lambda\gamma_\alpha \ln |x_{12}\mu|^2} + O(\lambda^2) \\ &= \frac{\delta_{\alpha\beta}}{|x_{12}|^{2\Delta_\alpha^{(0)}}\, |x_{12}\mu|^{2\lambda\gamma_\alpha}}. \end{aligned} \tag{4.10}$$

4.1.2 Three-Point Functions

As derived in chapter 2, by conformal invariance the three-point functions for the renormalised operators take the general form (2.22)

$$\begin{aligned}&\left\langle \widetilde{\mathcal{O}}_\alpha(x_1)\, \widetilde{\mathcal{O}}_\beta(x_2)\, \widetilde{\mathcal{O}}_\gamma(x_3)\right\rangle \\ &= \frac{C_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}\, |x_{23}|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}\, |x_{13}|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta}\, |\mu|^{\lambda(\gamma_\alpha+\gamma_\beta+\gamma_\gamma)}}. \end{aligned} \tag{4.11}$$

We expand the structure constant as

$$C_{\alpha\beta\gamma} = C^{(0)}_{\alpha\beta\gamma} + \lambda C^{(1)}_{\alpha\beta\gamma} + O(\lambda^2) \tag{4.12}$$

and substitute the renormalised operators (4.9) into (4.11). Thus, we obtain for the three-point functions of the unrenormalised operators

$$\begin{aligned}
&\langle \mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\,\mathcal{O}_\gamma(x_3)\rangle \\
&= \frac{C^{(0)}_{\alpha\beta\gamma}\left(1+\frac{\lambda}{2}\left(g_\alpha+g_\beta+g_\gamma\right)\right)+\lambda C^{(1)}_{\alpha\beta\gamma}}{|x_{12}|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma}\,|x_{23}|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}\,|x_{13}|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta}\,|\Lambda|^{\lambda(\gamma_\alpha+\gamma_\beta+\gamma_\gamma)}} \\
&= \frac{1}{|x_{12}|^{\Delta^{(0)}_\alpha+\Delta^{(0)}_\beta-\Delta^{(0)}_\gamma}\,|x_{23}|^{\Delta^{(0)}_\beta+\Delta^{(0)}_\gamma-\Delta^{(0)}_\alpha}\,|x_{13}|^{\Delta^{(0)}_\alpha+\Delta^{(0)}_\gamma-\Delta^{(0)}_\beta}} \\
&\times\Bigg[C^{(0)}_{\alpha\beta\gamma}\left(1-\lambda\gamma_\alpha\ln\left|\frac{x_{12}x_{13}\Lambda}{x_{23}}\right|-\lambda\gamma_\beta\ln\left|\frac{x_{12}x_{23}\Lambda}{x_{13}}\right|-\lambda\gamma_\gamma\ln\left|\frac{x_{13}x_{23}\Lambda}{x_{12}}\right|\right) \\
&+\lambda\underbrace{\left(C^{(1)}_{\alpha\beta\gamma}+\frac{1}{2}C^{(0)}_{\alpha\beta\gamma}\left(g_\alpha+g_\beta+g_\gamma\right)\right)}_{=:\widetilde{C}^{(1)}_{\alpha\beta\gamma}}\Bigg].
\end{aligned} \tag{4.13}$$

The finite part of the one-loop correction to the three-point functions is given by

$$\widetilde{C}^{(1)}_{\alpha\beta\gamma} = C^{(1)}_{\alpha\beta\gamma} + \frac{1}{2}\,C^{(0)}_{\alpha\beta\gamma}\,\left(g_\alpha+g_\beta+g_\gamma\right). \tag{4.14}$$

While $C^{(1)}_{\alpha\beta\gamma}$ is renormalisation scheme independent, $\widetilde{C}^{(1)}_{\alpha\beta\gamma}$ is not.

4.2 Derivation of the One-Loop Dressing Formulae

4.2.1 Handling of the $SO(6)$-Indices

We consider operators that are linear combinations of the single-trace operators

$$\mathrm{Tr}\left(\phi^{I_1}\cdots\phi^{I_N}\right)(x) \tag{4.15}$$

built of scalar fields ϕ^I. We will handle the $SO(6)$ indices by attaching six-dimensional vectors u_I to each ϕ^I, leaving us with operators

$$\mathcal{O}^N(u_1,\ldots,u_N)(x) = \mathrm{Tr}\left(u_1\cdot\phi\cdots u_N\cdot\phi\right)(x). \tag{4.16}$$

As is described in [20] we can obtain operators that are protected, i. e. have anomalous dimension $\gamma_{\mathcal{O}^N} = 0$, by attaching complex null vectors with

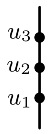
$u_I u_I = 0$ and $u_I \overline{u}_I = 1$.

Most of the time we will be faced with linear combinations of various operators taking the form (4.16). For example the Konishi operator is

$$\mathcal{K}(x) = \mathrm{Tr}\left(\phi^I \phi^I\right)(x) = \sum_{i=1}^{6} \mathcal{O}^2(e_i, e_i)(x) \tag{4.17}$$

where e_i is the $\mathbb{R}^6$ unity vector in i-th direction.

The operators (4.16) can be depicted in the spin chain picture with a ring of $SO(6)$ vectors. We choose a more compact and clear form and draw the ring as a line



,

keeping in mind that the ends of the line have to be identified.

4.2.2 Point-Splitting Regularisation

We make use of the point-splitting regularisation scheme. The general idea underlying this method is simple. We consider composite operators which are built from elementary fields, originally located at the same space-time point, and let those space-time points differ by a little distance ε. We can then expand our regularised expressions in ε and isolate the ultraviolet divergences. This corresponds to an ultraviolet momentum cutoff $\Lambda = \varepsilon^{-1}$.

In this thesis we omit the details of this procedure because they are not important for the ensuing discussion. Anyway, the interested reader is refered to [39].

We can switch to another renormalisation scheme by a rescaling $\varepsilon \to \lambda\varepsilon$ with a scaling factor of λ. This is used in the further discussion because the choice of a suitable renormalisation scheme simplifies the calculation of the one-loop structure constants.

4.2.3 Propagator and Fundamental Tree Functions

According to appendix A.2 of [4] we use the following short hand notations:

$$I_{12} = \frac{1}{(2\pi)^2 x_{12}^2}, \tag{4.18}$$

$$Y_{123} = \int \mathrm{d}^4 w \, I_{1w} I_{2w} I_{3w}, \tag{4.19}$$

$$X_{1234} = \int \mathrm{d}^4 w \, I_{1w} I_{2w} I_{3w} I_{4w}, \tag{4.20}$$

$$H_{12,34} = \int \mathrm{d}^4 v \, \mathrm{d}^4 w \, I_{1v} I_{2v} I_{vw} I_{3w} I_{4w}. \tag{4.21}$$

While the function H occurs only in the combination

$$\begin{aligned} F_{12,34} &= \frac{(\partial_1 - \partial_2) \cdot (\partial_3 - \partial_4) H_{12,34}}{I_{12} I_{34}} \\ &= \frac{X_{1234}}{I_{13} I_{24}} - \frac{X_{1234}}{I_{14} I_{23}} + \frac{Y_{134}}{I_{14}} - \frac{Y_{134}}{I_{13}} - \frac{Y_{234}}{I_{24}} + \frac{Y_{234}}{I_{23}} \\ &\quad + \frac{Y_{123}}{I_{13}} - \frac{Y_{123}}{I_{23}} - \frac{Y_{124}}{I_{14}} + \frac{Y_{124}}{I_{24}}, \end{aligned} \tag{4.22}$$

X and Y can be evaluated explicitly. Therefore we define

$$r = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad s = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

and use the function

$$\Phi(r,s) = \frac{1}{A} \operatorname{Im} \left[\operatorname{Li}_2 \left(e^{\mathrm{i}\varphi} \frac{\sqrt{r}}{\sqrt{s}} \right) + \ln \frac{\sqrt{r}}{\sqrt{s}} \ln \frac{\sqrt{s} - e^{\mathrm{i}\varphi} \sqrt{r}}{\sqrt{s}} \right] \tag{4.23}$$

where $e^{\mathrm{i}\varphi}$ and A are defined by

$$\begin{aligned} e^{\mathrm{i}\varphi} &= \mathrm{i} \sqrt{ - \frac{1 - r - s - 4\mathrm{i}A}{1 - r - s + 4\mathrm{i}A} }, \\ A &= \frac{1}{4} \sqrt{4rs - (1 - r - s)^2}. \end{aligned}$$

The properties of the function $\Phi(r,s)$ are described in appendix B of [1].

With these notations X and Y take the form

$$X_{1234} = \frac{\pi^2 \Phi(r,s)}{(2\pi)^8 x_{13}^2 x_{24}^2} \tag{4.24}$$

$$Y_{123} = \lim_{x_4 \to \infty} x_4^2 \, X_{1234}. \tag{4.25}$$

In point-splitting regularisation, where we take $x_1 \to x_2$ and define $\varepsilon := x_{12}$,

r and s can be expanded as

$$r = \frac{x_{23}^2 \varepsilon^2}{x_{12}^2 x_{13}^2} + O(\varepsilon^3), \tag{4.26}$$

$$s = 1 + 2\frac{x_{12} - x_{13}}{x_{12}x_{13}}\varepsilon + \frac{6x_{12}^2 + 2x_{13}^2 - 8x_{12}x_{13}}{x_{12}^2 x_{13}^2}\varepsilon^2 + O(\varepsilon^3) \tag{4.27}$$

and we can use the expansion for $\Phi(r, s)$ given in [1]:

$$\begin{aligned} \Phi(r,s) &= -\sum_{l=0}^{\infty} \frac{1}{l+1}(1-s)^l \ln r + 2\frac{(1-s)^l}{(l+1)^2} \\ &\approx -\ln r + 2 + O(\varepsilon) \\ &= -\left(\ln \frac{x_{23}^2 \varepsilon^2}{x_{12}^2 x_{13}^2} - 2\right). \end{aligned} \tag{4.28}$$

This yields the following limits in point-splitting regularisation:

$$X_{1123} = -\frac{1}{16\pi^2} I_{12} I_{13} \left(\ln \frac{x_{23}^2 \varepsilon^2}{x_{12}^2 x_{13}^2} - 2\right), \tag{4.29}$$

$$Y_{112} = -\frac{1}{16\pi^2} I_{12} \left(\ln \frac{\varepsilon^2}{x_{12}^2} - 2\right) = Y_{122}, \tag{4.30}$$

$$F_{12,13} = -\frac{1}{16\pi^2}\left(\ln \frac{\varepsilon^2}{x_{23}^2} - 2\right) + Y_{123}\left(\frac{1}{I_{12}} + \frac{1}{I_{13}} - \frac{2}{I_{23}}\right), \tag{4.31}$$

$$X_{1122} = -\frac{1}{8\pi^2} I_{12}^2 \left(\ln \frac{\varepsilon^2}{x_{12}^2} - 1\right), \tag{4.32}$$

$$F_{12,12} = -\frac{1}{8\pi^2}\left(\ln \frac{\varepsilon^2}{x_{12}^2} - 3\right). \tag{4.33}$$

4.2.4 Basic Interactions at One-Loop Level

We introduce a graphical symbol for the propagator and normalise the scalar propagators such that

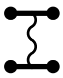
$$\left\langle \phi^I(x_1)\phi^J(x_2)\right\rangle_{\text{tree}} u_1^I u_2^J =$$

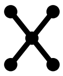
$$= (u_1 \cdot u_2) I_{12}, \tag{4.34}$$

where we handle the $SO(6)$-indices with vectors u_1 and u_2 as described above.

The one-loop corrections are then built of the following three components:

u_1  u_2 $$= -\lambda(u_1 \cdot u_2)\, I_{12}\, \frac{Y_{112} + Y_{122}}{I_{12}} \qquad \text{(self-energy)}, \qquad (4.35)$$

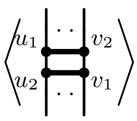
u_1 u_2  u_3 u_4 $$= \frac{\lambda}{2}(u_1 \cdot u_2)(u_3 \cdot u_4)\, I_{12}\, I_{34}\, F_{12,34} \qquad \text{(gluon)}, \qquad (4.36)$$

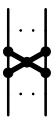
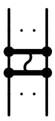
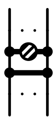
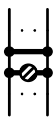
u_1 u_2  u_3 u_4 $$= \frac{\lambda}{2}\big[2(u_2 \cdot u_3)(u_1 \cdot u_4) - (u_2 \cdot u_4)(u_1 \cdot u_3) - (u_1 \cdot u_2)(u_3 \cdot u_4)\big]\, X_{1234} \qquad \text{(vertex)}. \qquad (4.37)$$

With these basic interactions we can diagrammatically state the essential parts in which the two- and three-point correlation functions factorise: the 2-gon and the 3-gon.

4.2.5 The Dressing for the 2-Gon

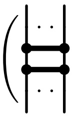
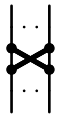
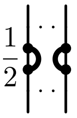
By combining the basic interactions (4.35)–(4.37) using the limits (4.29)–(4.33) in point-splitting regularisation we can easily derive the one-loop dressing formula for the 2-gon graph. For this purpose we calculate the sum of the three possible corrections, regarding that the self-energy term has to be shared between neighbouring lines. We obtain



$_{\text{1-loop}}$ $=$  $+$  $+\frac{1}{2}$  $+\frac{1}{2}$ 

$$= \frac{\lambda}{2} X_{1122}\,(2u_1 \cdot v_1\, u_2 \cdot v_2 - u_1 \cdot v_2\, u_2 \cdot v_1 - u_1 \cdot u_2\, v_1 \cdot v_2)$$
$$+ \frac{\lambda}{2}\, u_1 \cdot v_2\, u_2 \cdot v_1\, I_{12}^2\, F_{12,12}$$
$$- \lambda\, u_1 \cdot v_2\, u_2 \cdot v_1\, I_{12}\,(Y_{112} + Y_{122})$$
$$= I_{12}^2\, \frac{\lambda}{8\pi^2}\Bigg[\left(\ln\frac{\varepsilon^2}{x_{12}^2} - 1\right)\Big(- u_1 \cdot v_1\, u_2 \cdot v_2$$

$$+\frac{1}{2}u_1\cdot v_2\,u_2\cdot v_1+\frac{1}{2}\,u_1\cdot u_2\,v_1\cdot v_2\Big)$$
$$-\frac{1}{2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-3\right)u_1\cdot v_2\,u_2\cdot v_1$$
$$+\left(\ln\frac{\varepsilon^2}{x_{12}^2}-2\right)u_1\cdot v_2\,u_2\cdot v_1\Bigg]$$

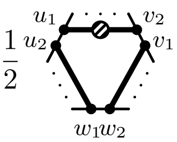
$$=I_{12}^2\,\frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\Bigg($$

$-$

$+\frac{1}{2}$

$\Bigg).$ (4.38)

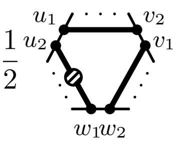
The diagrams in the last line just stand for the index contractions, not for propagators.

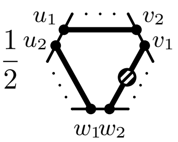
4.2.6 The Dressing for the 3-Gon

We consider the three contributions to the one-loop correction of the 3-gon separately.

Self-Energy Correction Like for the 2-gon, the self-energy term has to be shared between neighbouring lines. Thus we get

(SE) $=\frac{1}{2}$


$+\frac{1}{2}$


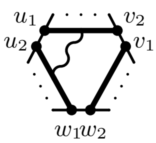
$+\frac{1}{2}$


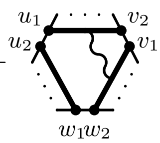
$$=\nabla\times\frac{-\lambda}{2}\left(\frac{Y_{112}+Y_{122}}{I_{12}}+\frac{Y_{113}+Y_{133}}{I_{13}}+\frac{Y_{223}+Y_{233}}{I_{23}}\right)$$

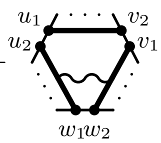
$$=\nabla\times\frac{\lambda}{16\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}+\ln\frac{\varepsilon^2}{x_{13}^2}+\ln\frac{\varepsilon^2}{x_{23}^2}-2-2-2\right)\qquad(4.39)$$

where we write ∇ as an abbreviation for the tree-level 3-gon.

Gluon Exchange For the gluon exchange terms we get

(GL) $=$


$+$


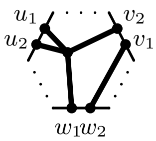
$+$


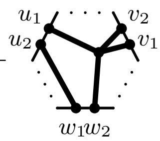
$$= \nabla \times \frac{\lambda}{2} \left(F_{12,13} + F_{12,23} + F_{13,23} \right)$$

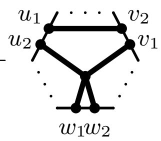
$$= \nabla \times \frac{-\lambda}{32\pi^2} \left(\ln \frac{\varepsilon^2}{x_{12}^2} + \ln \frac{\varepsilon^2}{x_{13}^2} + \ln \frac{\varepsilon^2}{x_{23}^2} - 2 - 2 - 2 \right), \qquad (4.40)$$

utilising that the Y_{123} terms in $F_{ij,ik}$ add up to zero contribution.

Vertex Correction The contribution of scalar vertices to the one-loop correction of the 3-gon is

(VX) = 

\+ 

\+ 

$$= I_{12} I_{13} I_{23} \times \frac{\lambda}{2}$$

$$\times \Bigg(\frac{X_{1123}}{I_{12} I_{13}} \left(2 u_1 \cdot w_1 \, u_2 \cdot v_2 - u_1 \cdot v_2 \, u_2 \cdot w_1 - u_1 \cdot u_2 \, v_2 \cdot w_1 \right) v_1 \cdot w_2$$

$$+ \frac{X_{1223}}{I_{12} I_{23}} \left(2 v_1 \cdot u_1 \, v_2 \cdot w_2 - v_1 \cdot w_2 \, v_2 \cdot u_1 - v_1 \cdot v_2 \, w_2 \cdot u_1 \right) u_2 \cdot w_1$$

$$+ \frac{X_{1233}}{I_{13} I_{23}} \left(2 w_1 \cdot v_1 \, w_2 \cdot u_2 - w_1 \cdot u_2 \, w_2 \cdot v_1 - w_1 \cdot w_2 \, u_2 \cdot v_1 \right) u_1 \cdot v_2 \Bigg)$$

$$= I_{12} I_{13} I_{23} \times \frac{\lambda}{32\pi^2} \Bigg[\left(\ln \frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2 \right) \big(u_1 \cdot v_2 \, u_2 \cdot w_1$$

$$- 2 \, u_1 \cdot w_1 \, u_2 \cdot v_2 + u_1 \cdot u_2 \, v_2 \cdot w_1 \big) v_1 \cdot w_2 + \left(\ln \frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 2 \right)$$

$$\times \big(v_1 \cdot w_2 \, v_2 \cdot u_1 - 2 v_1 \cdot u_1 \, v_2 \cdot w_2 + v_1 \cdot v_2 \, w_2 \cdot u_1 \big) u_2 \cdot w_1$$

$$+ \left(\ln \frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2 \right) \big(w_1 \cdot u_2 \, w_2 \cdot v_1 - 2 w_1 \cdot v_1 \, w_2 \cdot u_2$$

$$+ w_1 \cdot w_2 \, u_2 \cdot v_1 \big) u_1 \cdot v_2 \Bigg]. \qquad (4.41)$$

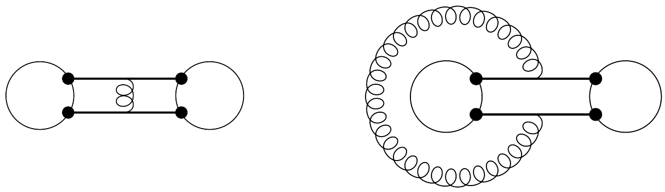


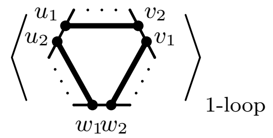
Figure 4.1: Correction *inside* and *outside* the 2-gon for length two operators.

The Sum of All Corrections Now, taking into account that

$$\ln\frac{\varepsilon^2}{x_{12}^2}+\ln\frac{\varepsilon^2}{x_{13}^2}+\ln\frac{\varepsilon^2}{x_{23}^2}=\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2x_{23}^2}+\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2x_{23}^2}+\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2x_{13}^2}, \tag{4.42}$$

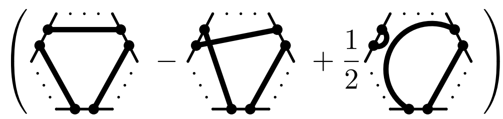
we can add all three contributions.

(SE) and (GL) are both proportional to the tree-level contraction terms of (VX) only. Therefore by adding (SE) and (GL) to (VX) these terms are just doubled. The final result is

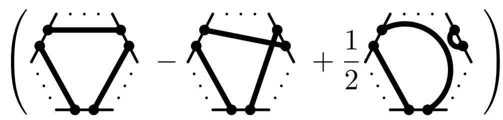


$$= I_{12}I_{13}I_{23}\times\frac{\lambda}{16\pi^2}$$

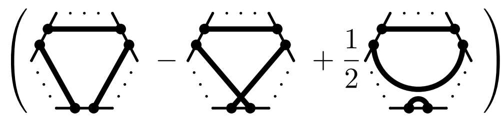
$$\times\Bigg[\left(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2x_{13}^2}-2\right)$$



$$+\left(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2x_{23}^2}-2\right)$$



$$+\left(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2x_{23}^2}-2\right)$$



$$\Bigg]. \tag{4.43}$$

As above, the graphs just depict the index contractions and not the propagators.

4.2.7 Operators of Length Two

For length two operators we have to take into account corrections inside and outside of both the 2-gon and the 3-gon. In figure 4.1 this is pictured for the 2-gon. This yields a factor of two for such operators.

Additionally, the calculation of one-loop corrections simplifies a lot for

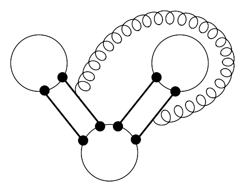


Figure 4.2: Additional Feynman-Graphs for extremal three-point functions

length two operators. By taking the sum over all permutations the permutated straightly contracted graph cancels out the non-permutated crossed graph of the one-loop correction. This holds for both the 2-gon and 3-gon calculation, and leaves us with just the third term of the one-loop dressing, namely the self-contraction term whose prefactor of one half cancels with the factor of two coming from the inner and outer corrections.

4.2.8 Extremal Three-Point Functions

Three-point functions of operators with lengths $\Delta_1^{(0)}$, $\Delta_2^{(0)}$ and $\Delta_3^{(0)}$ where $\Delta_1^{(0)} + \Delta_2^{(0)} = \Delta_3^{(0)}$ are called extremal. For these extremal functions the formulae above do not hold any longer for two reasons: First of all, there appear additional diagrams with a gluon exchange or a vertex between non-nearest neighbours as the one in figure 4.2. These non-nearest neighbour interactions lead to additional terms in the dressing formulae. Second of all, unlike non-extremal ones, extremal three-point functions with double-trace operators contain the same factor of N, the number of colours, than those with single-trace operators. This results in an operator mixing with such double-trace operators at tree level. This is described in detail in [15, 40].

We therefore regard only non-extremal correlators during the further discussion. Fortunately the structure constants of extremal three-point functions take a very simple form that will be established in section 4.3.1.

4.3 The Renormalisation Scheme Independent Structure Constants

As discussed in [40], supersymmetry endows us with non-renormalisation theorems for the two- and three-point functions. The point-splitting regulated fundamental functions $F_{ij,kl}$, X_{ijkl} and Y_{ijk} given in (4.29)–(4.33) are in an arbitrary renormalisation scheme related by

$$0 = I_{12}^2 F_{12,12} + X_{1122} - 2I_{12}(Y_{112} + Y_{122}) \tag{4.44}$$

$$\begin{aligned} 0 = {} & I_{12}I_{13}I_{23}(F_{12,13} + F_{12,23} + F_{13,23}) + I_{23}X_{1123} + I_{13}X_{1223} + I_{12}X_{1233} \\ & - I_{13}I_{23}(Y_{112} + Y_{122}) - I_{12}I_{13}(Y_{223} + Y_{233}) - I_{12}I_{23}(Y_{113} + Y_{133}). \end{aligned} \tag{4.45}$$

It can be shown that the two- and three-point functions are independent of $F_{ij,kl}$ and Y_{ijk} and depend only on X_{ijkl} for which, still following [40], we make the ansatz

$$\frac{X_{1122}}{I_{12}^2} = a_0 - \frac{1}{8\pi^2} \ln \frac{\varepsilon^2}{x_{12}^2} \tag{4.46}$$

$$\frac{X_{1123}}{I_{12}I_{13}} = b_0 - \frac{1}{16\pi^2} \ln \frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2}. \tag{4.47}$$

The constants a_0 and b_0 are determined by the renormalisation scheme. We can thus choose either a_0 or b_0 to be zero.

Because the two-point functions can only depend on X_{1122} they take the form

$$\begin{aligned} \langle \mathcal{O}_\alpha(x_1)\, \mathcal{O}_\alpha(x_2) \rangle &= \frac{1}{x_{12}^{2\Delta_\alpha^{(0)}}} \left(1 + \lambda a_\alpha \frac{X_{1122}}{I_{12}^2} \right) \\ &= \frac{1}{x_{12}^{2\Delta_\alpha^{(0)}}} \left(1 + \lambda a_\alpha a_0 - \frac{\lambda\, a_\alpha}{8\pi^2} \ln \frac{\varepsilon^2}{x_{12}^2} \right). \end{aligned} \tag{4.48}$$

Comparing this to equation (4.8), regarding that $\Lambda = \varepsilon^{-1}$ we obtain

$$g_\alpha = a_\alpha a_0 \tag{4.49}$$

$$\gamma_\alpha = -\frac{a_\alpha}{8\pi^2} \tag{4.50}$$

$$\Rightarrow \gamma_\alpha = -\frac{g_\alpha}{8\pi^2\, a_0}. \tag{4.51}$$

For the three-point functions we start with

$$\begin{aligned} & \langle \mathcal{O}_\alpha(x_1)\, \mathcal{O}_\beta(x_2)\, \mathcal{O}_\gamma(x_3) \rangle \\ & = \frac{1}{|x_{12}|^{\Delta_\alpha^{(0)} + \Delta_\beta^{(0)} - \Delta_\gamma^{(0)}}\, |x_{23}|^{\Delta_\beta^{(0)} + \Delta_\gamma^{(0)} - \Delta_\alpha^{(0)}}\, |x_{13}|^{\Delta_\alpha^{(0)} + \Delta_\gamma^{(0)} - \Delta_\beta^{(0)}}} \end{aligned}$$

$$\times\left(C^{(0)}_{\alpha\beta\gamma}+\lambda\left[a_{12}\frac{X_{1122}}{I_{12}^2}+a_{13}\frac{X_{1133}}{I_{13}^2}+a_{23}\frac{X_{2233}}{I_{23}^2}\right.\right.$$
$$\left.\left.+b^1_{23}\frac{X_{1123}}{I_{12}I_{13}}+b^2_{13}\frac{X_{1223}}{I_{12}I_{23}}+b^3_{12}\frac{X_{1233}}{I_{13}I_{23}}\right]\right)$$
$$=\frac{1}{|x_{12}|^{\Delta^{(0)}_\alpha+\Delta^{(0)}_\beta-\Delta^{(0)}_\gamma}\,|x_{23}|^{\Delta^{(0)}_\beta+\Delta^{(0)}_\gamma-\Delta^{(0)}_\alpha}\,|x_{13}|^{\Delta^{(0)}_\alpha+\Delta^{(0)}_\gamma-\Delta^{(0)}_\beta}}$$
$$\times\left(C^{(0)}_{\alpha\beta\gamma}+\lambda\left[(a_{12}+a_{13}+a_{23})a_0+(b^1_{23}+b^2_{13}+b^3_{12})b_0\right]\right.$$
$$-\frac{\lambda}{16\pi^2}\left[2a_{12}\ln\frac{\varepsilon^2}{x_{12}^2}+2a_{13}\ln\frac{\varepsilon^2}{x_{13}^2}+2a_{23}\ln\frac{\varepsilon^2}{x_{23}^2}\right.$$
$$\left.\left.+b^1_{23}\ln\frac{\varepsilon^2x_{23}^2}{x_{12}^2x_{13}^2}+b^2_{13}\ln\frac{\varepsilon^2x_{13}^2}{x_{12}^2x_{23}^2}+b^3_{12}\ln\frac{\varepsilon^2x_{12}^2}{x_{13}^2x_{23}^2}\right]\right). \tag{4.52}$$

Note that

$$\ln\frac{\varepsilon^2}{x_{12}^2}=\frac{1}{2}\left(\ln\frac{\varepsilon^2x_{23}^2}{x_{12}^2x_{13}^2}+\ln\frac{\varepsilon^2x_{13}^2}{x_{12}^2x_{23}^2}\right) \tag{4.53}$$

and the same applies to the other logarithms accordingly. Then in comparison to equation (4.13) we see that

$$C^{(0)}_{\alpha\beta\gamma}\,\gamma_\alpha=-\frac{1}{8\pi^2}(a_{12}+a_{13}+b^1_{23})=-\frac{C^{(0)}_{\alpha\beta\gamma}\,g_\alpha}{8\pi^2\,a_0}=-\frac{C^{(0)}_{\alpha\beta\gamma}\,a_\alpha}{8\pi^2} \tag{4.54}$$

$$C^{(0)}_{\alpha\beta\gamma}\,\gamma_\beta=-\frac{1}{8\pi^2}(a_{12}+a_{23}+b^2_{13})=-\frac{C^{(0)}_{\alpha\beta\gamma}\,g_\beta}{8\pi^2\,a_0}=-\frac{C^{(0)}_{\alpha\beta\gamma}\,a_\beta}{8\pi^2} \tag{4.55}$$

$$C^{(0)}_{\alpha\beta\gamma}\,\gamma_\gamma=-\frac{1}{8\pi^2}(a_{13}+a_{23}+b^3_{12})=-\frac{C^{(0)}_{\alpha\beta\gamma}\,g_\gamma}{8\pi^2\,a_0}=-\frac{C^{(0)}_{\alpha\beta\gamma}\,a_\gamma}{8\pi^2} \tag{4.56}$$

and from this follows

$$b^1_{23}=C^{(0)}_{\alpha\beta\gamma}\,a_\alpha-a_{12}-a_{13} \tag{4.57}$$

$$b^2_{13}=C^{(0)}_{\alpha\beta\gamma}\,a_\beta-a_{12}-a_{23} \tag{4.58}$$

$$b^3_{12}=C^{(0)}_{\alpha\beta\gamma}\,a_\gamma-a_{13}-a_{23}. \tag{4.59}$$

The finite contribution to the three-point function takes the form

$$\widetilde{C}^{(1)}_{\alpha\beta\gamma} = (a_{12} + a_{13} + a_{23})a_0 + (b^1_{23} + b^2_{13} + b^3_{12})b_0. \tag{4.60}$$

Using relations (4.49) and (4.57)–(4.59) we get the renormalisation scheme independent structure constants

$$\begin{aligned} C^{(1)}_{\alpha\beta\gamma} &= \widetilde{C}^{(1)}_{\alpha\beta\gamma} - \frac{1}{2}\, C^{(0)}_{\alpha\beta\gamma}(g_\alpha + g_\beta + g_\gamma) \\ &= \frac{2b_0 - a_0}{2}\left(b^1_{23} + b^2_{13} + b^3_{12}\right). \end{aligned} \tag{4.61}$$

With the help of formulae (4.46) and (4.47) we can check that this expression is indeed independent of the renormalisation scheme by calculating

$$2b_0 - a_0 = \frac{X_{1123}}{I_{12}I_{13}} + \frac{X_{1223}}{I_{12}I_{23}} - \frac{X_{1122}}{I^2_{12}} = \frac{1}{8\pi^2}. \tag{4.62}$$

Our final result for the structure constant is then

$$\boxed{C^{(1)}_{\alpha\beta\gamma} = \frac{1}{16\pi^2}\left(b^1_{23} + b^2_{13} + b^3_{12}\right)} \tag{4.63}$$

which only depends on the constants b^i_{jk}. Alternatively we could express the structure constant by the constants a_{ij} and the anomalous dimensions

$$C^{(1)}_{\alpha\beta\gamma} = -\frac{1}{8\pi^2}(a_{12} + a_{13} + a_{23}) - \frac{1}{2}\, C^{(0)}_{\alpha\beta\gamma}\,(\gamma_\alpha + \gamma_\beta + \gamma_\gamma). \tag{4.64}$$

4.3.1 Extremal Correlators

For extremal correlation functions with $\Delta^{(0)}_\alpha + \Delta^{(0)}_\beta = \Delta^{(0)}_\gamma$ there are no contractions between the operators $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$. Therefore we have

$$a_{12} = b^1_{23} = b^2_{13} = 0 \quad \Rightarrow \quad b^3_{12} = C^{(0)}_{\alpha\beta\gamma}\, a_\gamma - a_{13} - a_{23} \tag{4.65}$$

$$a_{13} = C^{(0)}_{\alpha\beta\gamma}\, a_\alpha \tag{4.66}$$

$$a_{23} = C^{(0)}_{\alpha\beta\gamma}\, a_\beta \tag{4.67}$$

$$\Rightarrow \quad b^3_{12} = 8\pi^2\, C^{(0)}_{\alpha\beta\gamma}\,(\gamma_\alpha + \gamma_\beta - \gamma_\gamma) \tag{4.68}$$

and the structure constants take the very simple form

$$\boxed{C^{(1)}_{\alpha\beta\gamma} = \frac{1}{2}\, C^{(0)}_{\alpha\beta\gamma}\,(\gamma_\alpha + \gamma_\beta - \gamma_\gamma)\,.} \tag{4.69}$$

4.3.2 Structure Constants as 3-Gon Dressing

As we saw, we can simplify the form of the correlators by choosing a convenient renormalisation scheme. Let us now apply this to our diagrammatic dressing formulae.

First we change the renormalisation by the transformation

$$\begin{aligned} && \varepsilon &\rightarrow \sqrt{e}\varepsilon && (4.70)\\ \Rightarrow && \ln\frac{\varepsilon^2}{x_{ij}^2} - 1 &\rightarrow \ln\frac{\varepsilon^2}{x_{ij}^2} \\ \Rightarrow && \ln\frac{\varepsilon^2 x_{ij}^2}{x_{ik}^2 x_{jk}^2} - 2 &\rightarrow \ln\frac{\varepsilon^2 x_{ij}^2}{x_{ik}^2 x_{jk}^2} - 1. \end{aligned}$$

In this scheme the 2-gon dressing (4.38) holds only logarithmic terms and therefore the finite part of the one-loop correction for the two-point functions vanishes

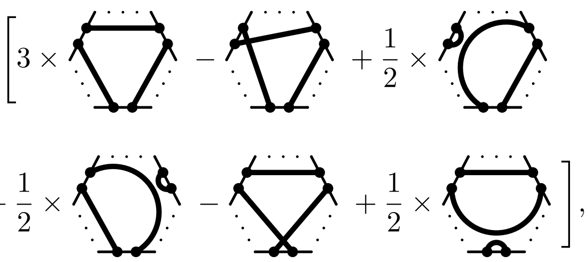
$$g_\alpha = 0, \tag{4.71}$$

and the finite contributions to the three-point functions equal the renormalisation scheme independent structure constants

$$\widetilde{C}^{(1)}_{\alpha\beta\gamma} = C^{(1)}_{\alpha\beta\gamma}. \tag{4.72}$$

Only the 3-gon dressings contribute to the renormalisation scheme independent structure constants that can then be schematically depicted as

$$C^{(1)}_{\alpha\beta\gamma} = -\frac{1}{16\pi^2}\sum_{\substack{\text{cyclic}\\ \text{perm.}}}$$



(4.73)

where we have to sum over all $\Delta_\alpha^{(0)} \times \Delta_\beta^{(0)} \times \Delta_\gamma^{(0)}$ cyclic permutations and take into account a factor of two for length two operators.

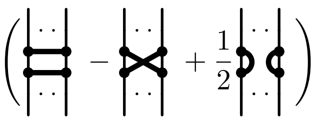
4.3.3 Structure Constants as 2-Gon Dressing

By changing the renormalisation scheme as

$$\begin{aligned} \varepsilon &\to e\varepsilon \\ \Rightarrow \quad \ln\frac{\varepsilon^2}{x_{ij}^2} - 1 &\to \ln\frac{\varepsilon^2}{x_{ij}^2} + 1 \\ \Rightarrow \quad \ln\frac{\varepsilon^2 x_{ij}^2}{x_{ik}^2 x_{jk}^2} - 2 &\to \ln\frac{\varepsilon^2 x_{ij}^2}{x_{ik}^2 x_{jk}^2} \end{aligned} \tag{4.74}$$

we can achieve that the 3-gons do not hold finite contributions and therefore the finite contributions to the three-point functions take the form

$$\widetilde{C}_{\alpha\beta\gamma}^{(1)} = \frac{1}{8\pi^2} \sum_{\substack{\text{cyclic}\\ \text{perm.}}} \sum_{\substack{\text{all}\\ \text{2-gons}}} \tag{4.75}$$



The scheme independent constants can then be calculated using equation (4.14) with $g_\alpha = \gamma_\alpha$. They are

$$C_{\alpha\beta\gamma}^{(1)} = \widetilde{C}_{\alpha\beta\gamma}^{(1)} - \frac{1}{2} C_{\alpha\beta\gamma}^{(0)} \left(\gamma_\alpha + \gamma_\beta + \gamma_\gamma\right). \tag{4.76}$$

Chapter 5

First Steps with the Dressing Formulae

5.1 A Class of Simple Short Operators

As a first example of how to apply the dressing formulae derived in the previous chapter we now want to calculate the two- and three-point functions for the following set of operators[1]:

$$\mathcal{O}_a(x) = \mathrm{Tr}\left(\phi^n \phi^n\right)(x) = \mathcal{K}(x), \tag{5.1}$$

$$\mathcal{O}_b(x) = \mathrm{Tr}\left(\phi^n \phi^n \phi^p\right)(x), \tag{5.2}$$

$$\mathcal{O}_c(x) = \mathrm{Tr}\left(\phi^n \phi^n \left[\phi^p, \phi^q\right]\right)(x). \tag{5.3}$$

These operators are part of a basis in which the two-point functions are diagonal.

In order to perform the calculation of two- and three-point functions we first derive an expression for general operators

$$\mathcal{O}^N(u_1, \ldots, u_N)(x) = \mathrm{Tr}\left(u_1 \cdot \phi \cdots u_N \cdot \phi\right)(x). \tag{5.4}$$

By inserting particular vectors we can specify the given operators. We obtain

$$\mathcal{O}_a(x) = \sum_{n=1}^{6} \mathcal{O}^2(e_n, e_n)(x), \tag{5.5}$$

$$\mathcal{O}_b(x) = \sum_{n=1}^{6} \mathcal{O}^3(e_n, e_n, e_p)(x), \tag{5.6}$$

$$\mathcal{O}_c(x) = \sum_{n=1}^{6} \mathcal{O}^4(e_n, e_n, e_{[p}, e_{q]})(x), \tag{5.7}$$

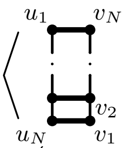
where e_i denote the $\mathbb{R}^6$ unity vectors.

[1] See also [6] for definition of the operators and two-loop results for the scaling dimensions.

5.1.1 The Two-Point Functions

Generally, by conformal invariance two-point functions of operators of different length vanish. Therefore we only have to calculate the one-loop corrections

$$\left\langle \mathcal{O}^N(u_1,\ldots,u_N)\mathcal{O}^N(v_1,\ldots,v_N)\right\rangle_{\text{1-loop}} = \left\langle \quad \right\rangle_1 + \text{cyclic perm.}$$



$$= I_{12}^N \frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right) \sum_{\substack{j=1\\ \text{mod } N}}^{N} \sum_{\substack{k=1\\ \text{mod } N}}^{N} \left(\prod_{\substack{l=1\\ l\neq k, l\neq k+1}}^{N} u_l\cdot v_{N+j-l}\right)$$
$$\times\Big(u_k\cdot v_{N+j-k}\,u_{k+1}\cdot v_{N+j-k-1} - u_k\cdot v_{N+j-k-1}\,u_{k+1}\cdot v_{N+j-k}$$
$$+\frac{1}{2}\,u_k\cdot u_{k+1}\,v_{N+j-k}\cdot v_{N+j-k-1}\Big). \tag{5.8}$$

Now we can straightforwardly obtain all three two-point functions by inserting the corresponding vectors. They are

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_a(x_2)\rangle = 12 I_{12}^2\left(1+\frac{3\lambda}{4\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right) \tag{5.9}$$

$$\left\langle \mathcal{O}_b^p(x_1)\mathcal{O}_b^r(x_2)\right\rangle = 8\delta^{pr} I_{12}^3\left(1+\frac{\lambda}{2\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right) \tag{5.10}$$

$$\langle \mathcal{O}_c^{pq}(x_1)\mathcal{O}_c^{rs}(x_2)\rangle = 8(\delta^{pr}\delta^{qs}-\delta^{qr}\delta^{ps}) I_{12}^4\left(1+\frac{3\lambda}{4\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right). \tag{5.11}$$

Thus we get the following values for the associated anomalous dimensions:

$$\gamma_a = \frac{3}{4\pi^2}, \qquad \gamma_b = \frac{1}{2\pi^2}, \qquad \gamma_c = \frac{3}{4\pi^2}. \tag{5.12}$$

5.1.2 The Three-Point Functions

For the three-point functions

$$\left\langle \mathcal{O}^L(u_1,\ldots,u_L)\mathcal{O}^M(v_1,\ldots,v_M)\mathcal{O}^N(w_1,\ldots,w_N)\right\rangle \tag{5.13}$$

we cannot derive a general relation like (5.8) for operators of arbitrary length because we have to sum over all permutations on every point and take into account all 2-gon and 3-gon corrections. Thus we calculate (5.13) for every combination (L, M, N) separately. This leaves us with lengthy expressions in which we can insert the corresponding vectors. We get:

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_a(x_2)\mathcal{O}_a(x_3)\rangle = 48 I_{12} I_{13} I_{23} \left(1 + \frac{3\lambda}{8\pi^2}\left(\ln \frac{\varepsilon^6}{x_{12}^2 x_{13}^2 x_{23}^2} - 6\right)\right) \tag{5.14}$$

$$\begin{aligned}\left\langle \mathcal{O}_a(x_1)\mathcal{O}_b^p(x_2)\mathcal{O}_b^r(x_3)\right\rangle = 48\delta^{pr} I_{12} I_{13} I_{23}^2 \Bigg(1 &+ \frac{3\lambda}{8\pi^2}\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} \\ &+ \frac{\lambda}{2\pi^2}\ln\frac{\varepsilon^2}{x_{23}^2} - \frac{19\lambda}{12\pi^2}\Bigg)\end{aligned} \tag{5.15}$$

$$\begin{aligned}\langle \mathcal{O}_a(x_1)\mathcal{O}_c^{pq}(x_2)\mathcal{O}_c^{rs}(x_3)\rangle &= -64(\delta^{pr}\delta^{qs} - \delta^{qr}\delta^{ps}) I_{12} I_{13} I_{23}^3 \\ &\times \left(1 + \frac{3\lambda}{8\pi^2}\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} + \frac{3\lambda}{4\pi^2}\ln\frac{\varepsilon^2}{x_{23}^2} - \frac{15\lambda}{8\pi^2}\right)\end{aligned} \tag{5.16}$$

All other three-point functions are zero. Comparing with the general form (4.13) of the three-point function we get the one-loop structure constants

$$\widetilde{C}^{(1)}_{aaa} = -\frac{27}{16\pi^8}, \qquad \widetilde{C}^{(1)}_{abb} = -\frac{19}{64\pi^{10}}, \qquad \widetilde{C}^{(1)}_{acc} = -\frac{15}{128\pi^{12}}. \tag{5.17}$$

The tree-level structure constants are

$$C^{(0)}_{aaa} = \frac{3}{4\pi^6}, \qquad C^{(0)}_{abb} = \frac{3}{16\pi^8}, \qquad C^{(0)}_{acc} = \frac{1}{16\pi^{10}}. \tag{5.18}$$

Using equation (4.14) the renormalisation scheme independent one-loop structure constants can be obtained as

$$C^{(1)}_{\alpha\beta\gamma} = \widetilde{C}^{(1)}_{\alpha\beta\gamma} + \frac{1}{2}(\gamma_\alpha + \gamma_\beta + \gamma_\gamma) C^{(0)}_{\alpha\beta\gamma}. \tag{5.19}$$

They are

$$C^{(1)}_{aaa} = -\frac{27}{32\pi^8} = -\frac{9}{8\pi^2} C^{(0)}_{aaa}, \tag{5.20}$$

$$C^{(1)}_{abb} = -\frac{17}{128\pi^{10}} = -\frac{17}{24\pi^2} C^{(0)}_{abb}, \tag{5.21}$$

$$C^{(1)}_{acc} = -\frac{3}{64\pi^{12}} = -\frac{3}{4\pi^2} C^{(0)}_{acc}. \tag{5.22}$$

5.2 Correlation Functions for Twisted Operators

Let us consider another example. We define the operators

$$B(x) := \Phi_5(x) + \mathrm{i}\Phi_6(x) \tag{5.23}$$

$$C(x) := \Phi_5(x) - \mathrm{i}\Phi_6(x) + 2\,x \cdot \Phi(x) - x^2\,B(x) \tag{5.24}$$

$$V_a(x) := a^\mu V_\mu(x) = a^\mu(\Phi_\mu(x) - x_\mu B(x)). \tag{5.25}$$

These operators are part of a non-diagonal basis. They arise from an operator twisting—that is an embedding of the conformal group in the bosonic symmetry group—and hold an explicit x dependence. In particular, the two-point function of the operator $C(x)$ is a constant in space-time. For a more thorough treatment of these operators see [36].

Again, we can write these operators as contractions with six-vectors:

$$B(x) = u_B^I(x)\Phi_I(x) \qquad u_B^I(x) = (0,0,0,0,1,\mathrm{i}) \tag{5.26}$$

$$C(x) = u_C^I(x)\Phi_I(x) \qquad u_C^I(x) = \left(2x_\mu, 1-x^2, -\mathrm{i}(1+x^2)\right) \tag{5.27}$$

$$V_a(x) = u_{V_a}^I(x)\Phi_I(x) \qquad u_{V_a}^I(x) = (a_\mu, -a\cdot x, -\mathrm{i}a\cdot x) \tag{5.28}$$

with the contractions

$$\begin{aligned} u_B^I(x_1)\ u_B^I(x_2)\ &= 0, & u_B^I(x_1)\ u_C^I(x_2)\ &= 2,\\ u_B^I(x_1)\ u_{V_a}^I(x_2)\ &= 0, & u_C^I(x_1)\ u_C^I(x_2)\ &= -2x_{12}^2,\\ u_C^I(x_1)\ u_{V_a}^I(x_2)\ &= 2x_{12}\cdot a, & u_{V_a}^I(x_1)\ u_{V_b}^I(x_2)\ &= a\cdot b. \end{aligned} \tag{5.29}$$

From these length one operators we can build single-trace operators of arbitrary length. Correlation functions of such trace operators are always zero if there are more B's than C's in them, because every B has to be contracted with a C in order not to give a zero contribution.

5.2.1 Length Two Operators

We can construct the following length two trace operators:

$$\begin{aligned} &\mathrm{Tr}\left(B^2\right)(x), && \mathrm{Tr}\left(BC\right)(x), && \mathrm{Tr}\left(BV_a\right)(x),\\ &\mathrm{Tr}\left(C^2\right)(x), && \mathrm{Tr}\left(CV_a\right)(x), && \mathrm{Tr}\left(V_aV_b\right)(x). \end{aligned}$$

The results for all the two- and three-point functions can be found in appendix A.1. We just draw a heuristic picture here.

Two-Point Functions The two-point functions are zero if there are more B's than C's in it. They can also be zero if they contain some V_a and V_b with a and b orthogonal. There are three two-point functions that are not protected, i. e. get one-loop corrections. These are

$$\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\,(BC)\,(x_2)\ \rangle = \frac{1}{4\pi^4 x_{12}^4}\left(1+\frac{\lambda}{4\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right) \quad (5.30)$$

$$\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\,(V_aV_b)\,(x_2)\ \rangle = \frac{\lambda\, a\cdot b}{32\pi^6 x_{12}^4}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right) \quad (5.31)$$

$$\langle \mathrm{Tr}\,(V_aV_b)\,(x_1)\ \mathrm{Tr}\,(V_cV_d)\,(x_2)\ \rangle$$
$$= \frac{1}{16\pi^4 x_{12}^4}\left(a\cdot c\, b\cdot d + a\cdot d\, b\cdot c + \frac{\lambda\, a\cdot b\, c\cdot d}{4\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right). \quad (5.32)$$

Three-Point Functions We can divide the three-point functions into four classes.

- Three-point functions with a total number of B's larger than the number of C's are always zero.
- Three-point functions consisting of two of the operators C^2 or CV and one of C^2, CV, B^2 or BV, as well as the three-point function $\left\langle \mathrm{Tr}\,(BV)\,(x_1)\,\mathrm{Tr}\,(BV)\,(x_2)\,\mathrm{Tr}\left(C^2\right)(x_3)\right\rangle$, get no one-loop corrections.
- The three-point functions

$$\left\langle \mathrm{Tr}\,(V_aV_b)\,(x_1)\,\mathrm{Tr}\left(B^2\right)(x_2)\,\mathrm{Tr}\left(C^2\right)(x_3)\right\rangle,$$
$$\langle \mathrm{Tr}\,(V_aV_b)\,(x_1)\,\mathrm{Tr}\,(BC)\,(x_2)\,\mathrm{Tr}\,(BC)\,(x_3)\rangle \quad \text{and}$$
$$\langle \mathrm{Tr}\,(V_aV_b)\,(x_1)\,\mathrm{Tr}\,(V_cV_d)\,(x_2)\,\mathrm{Tr}\,(BC)\,(x_3)\rangle$$

  yield no tree-level contribution but do get one-loop corrections.
- All others yield both tree-level and one-loop contributions.

Additionally, for all of the functions containing more than one of the operators V_a, some or all contributions are zero if the attached vectors of some V's are orthogonal.

5.2.2 Length Three Operators

These are all operators of length three:

$$\begin{array}{lll}
\mathrm{Tr}\left(B^3\right)(x), & \mathrm{Tr}\left(B^2C\right)(x), & \mathrm{Tr}\left(B^2V_a\right)(x), \\
\mathrm{Tr}\left(BC^2\right)(x), & \mathrm{Tr}\left(BCV_a\right)(x), & \mathrm{Tr}\left(BV_aV_b\right)(x), \\
\mathrm{Tr}\left(CBV_a\right)(x), & \mathrm{Tr}\left(C^3\right)(x), & \mathrm{Tr}\left(C^2V_a\right)(x), \\
\mathrm{Tr}\left(CV_aV_b\right)(x), & \mathrm{Tr}\left(V_aV_bV_c\right)(x). &
\end{array}$$

The results for some of the the two- and three-point functions can be found in appendix A.1. There are fifteen two-point functions that are not protected.

In summary we conclude that these correlation functions seem too complicated to learn something about their general structure. Due to the space-time dependency of the operators, the structure constants are space-time dependent and because of the non-diagonality of this basis it is unclear how the structure constants of these correlators should be interpreted. In the following, we thus consider operators in diagonal bases.

Chapter 6

Structure Constants in Diagonal Bases

6.1 Transformation between Diagonal and Non-Diagonal Bases

6.1.1 Two-Point Functions

We consider operators $\mathcal{O}_\alpha(x)$ ($\alpha = 1, \ldots, d$) of arbitrary length that form a d-dimensional, non-diagonal basis. Let $\{\mathcal{D}_\alpha(x)\}$ denote a basis in which the two-point functions are diagonal

$$\langle \mathcal{D}_\alpha(x_1)\mathcal{D}_\beta(x_2)\rangle = \frac{\delta_{\alpha\beta}}{x_{12}^{2\Delta_\alpha^{(0)}}}\left(1 + \lambda\gamma_\alpha\left(\ln\frac{\varepsilon^2}{x_{12}^2} - 1\right)\right). \tag{6.1}$$

We work in the fixed renormalisation scheme from section 4.2 where

$$g_\alpha = -\gamma_\alpha. \tag{6.2}$$

Let M be the change-of-basis matrix from the operators $\mathcal{O}_\alpha$ to $\mathcal{D}_\alpha$

$$\mathcal{D}_\alpha = M_{\alpha\beta}\mathcal{O}_\beta, \tag{6.3}$$

where we suppose M to be block diagonal with respect to the tree-level scaling dimensions. The two-point functions of the $\mathcal{O}_\alpha$ take the general form

$$\langle \mathcal{O}_\alpha(x_1)\mathcal{O}_\beta(x_2)\rangle = \frac{\delta_{\Delta_\alpha^{(0)}\Delta_\beta^{(0)}}}{x_{12}^{2\Delta_\alpha^{(0)}}}\left(t_{\alpha\beta}^{(0)} + \lambda\gamma_{\alpha\beta}\left(\ln\frac{\varepsilon^2}{x_{12}^2} - 1\right)\right). \tag{6.4}$$

This means

$$\begin{aligned}\langle \mathcal{D}_\alpha(x_1)\mathcal{D}_\beta(x_2)\rangle &= M_{\alpha\gamma}M_{\beta\delta}\,\langle \mathcal{O}_\gamma(x_1)\mathcal{O}_\delta(x_2)\rangle \\ &= \frac{\delta_{\Delta_\alpha^{(0)}\Delta_\beta^{(0)}}}{x_{12}^{2\Delta_\alpha^{(0)}}}\left(M_{\alpha\gamma}t_{\gamma\delta}^{(0)}M_{\delta\beta}^T + \lambda M_{\alpha\gamma}\gamma_{\gamma\delta}M_{\delta\beta}^T\left(\ln\frac{\varepsilon^2}{x_{12}^2} - 1\right)\right).\end{aligned} \tag{6.5}$$

Comparing (6.1) and (6.5) leads to the following matrix equations:

$$M t^{(0)} M^T \stackrel{!}{=} \mathbb{1} \qquad \Rightarrow M^T = (t^{(0)})^{-1} M^{-1} \tag{6.6}$$

$$M \gamma M^T \stackrel{!}{=} \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_d) =: D \tag{6.7}$$

$$\Rightarrow \quad D = M \gamma (t^{(0)})^{-1} M^{-1}. \tag{6.8}$$

As we have seen, the change-of-basis matrix M is the matrix diagonalising $\gamma (t^{(0)})^{-1}$. The eigenvalues are the anomalous dimensions γ_α. γ and $t^{(0)}$ can be directly read off from the two-point functions of the non-diagonal operators.

6.1.2 Three-Point Functions

Equation (4.13) together with (6.2) gives us the form of the three-point functions in the diagonal basis

$$\begin{aligned}
&\langle \mathcal{D}_\alpha(x_1) \mathcal{D}_\beta(x_2) \mathcal{D}_\gamma(x_3) \rangle \\
&\quad = \frac{1}{|x_{12}|^{\Delta_\alpha^{(0)} + \Delta_\beta^{(0)} - \Delta_\gamma^{(0)}} |x_{13}|^{\Delta_\alpha^{(0)} + \Delta_\gamma^{(0)} - \Delta_\beta^{(0)}} |x_{23}|^{\Delta_\beta^{(0)} + \Delta_\gamma^{(0)} - \Delta_\alpha^{(0)}}} \\
&\quad \times \left[C^{(0)}_{\alpha\beta\gamma} \left(1 + \frac{\lambda}{2} \left\{ \gamma_\alpha \left(\ln \frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 1 \right) + \gamma_\beta \left(\ln \frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 1 \right) \right.\right.\right. \\
&\quad \left.\left.\left. + \gamma_\gamma \left(\ln \frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 1 \right) \right\} \right) + \lambda C^{(1)}_{\alpha\beta\gamma} \right].
\end{aligned} \tag{6.9}$$

We perform the change of basis and obtain

$$\begin{aligned}
&\langle \mathcal{O}_\alpha(x_1) \mathcal{O}_\beta(x_2) \mathcal{O}_\gamma(x_3) \rangle = M^{-1}_{\alpha\rho} M^{-1}_{\beta\sigma} M^{-1}_{\gamma\tau} \langle \mathcal{D}_\rho(x_1) \mathcal{D}_\sigma(x_2) \mathcal{D}_\tau(x_3) \rangle \\
&= \frac{1}{|x_{12}|^{\Delta_\alpha^{(0)} + \Delta_\beta^{(0)} - \Delta_\gamma^{(0)}} |x_{13}|^{\Delta_\alpha^{(0)} + \Delta_\gamma^{(0)} - \Delta_\beta^{(0)}} |x_{23}|^{\Delta_\beta^{(0)} + \Delta_\gamma^{(0)} - \Delta_\alpha^{(0)}}} \Big[\underbrace{M^{-1}_{\alpha\rho} M^{-1}_{\beta\sigma} M^{-1}_{\gamma\tau} C^{(0)}_{\rho\sigma\tau}}_{=: \overline{C}^{(0)}_{\alpha\beta\gamma}} \\
&+ \frac{\lambda}{2} \Big\{ M^{-1}_{\alpha\rho} M^{-1}_{\beta\sigma} M^{-1}_{\gamma\tau} \gamma_\rho \left(\ln \frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 1 \right) + M^{-1}_{\alpha\rho} M^{-1}_{\beta\sigma} M^{-1}_{\gamma\tau} \gamma_\sigma \left(\ln \frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 1 \right) \\
&+ M^{-1}_{\alpha\rho} M^{-1}_{\beta\sigma} M^{-1}_{\gamma\tau} \gamma_\tau \left(\ln \frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 1 \right) \Big\} + \lambda \underbrace{M^{-1}_{\alpha\rho} M^{-1}_{\beta\sigma} M^{-1}_{\gamma\tau} C^{(1)}_{\rho\sigma\tau}}_{=: \overline{C}^{(1)}_{\alpha\beta\gamma}} \Big].
\end{aligned} \tag{6.10}$$

Using this formula we can extract the non-diagonal constants $\overline{C}^{(0)}_{\alpha\beta\gamma}$ and

$\overline{C}^{(1)}_{\alpha\beta\gamma}$ from the three-point functions of the non-diagonal operators. Together with the change-of-basis matrix M obtained from the two-point functions, the renormalisation scheme independent structure constants for the diagonal basis can be calculated as

$$C^{(0)}_{\alpha\beta\gamma} = M_{\alpha\rho} M_{\beta\sigma} M_{\gamma\tau} \overline{C}^{(0)}_{\rho\sigma\tau} \tag{6.11}$$

$$C^{(1)}_{\alpha\beta\gamma} = M_{\alpha\rho} M_{\beta\sigma} M_{\gamma\tau} \overline{C}^{(1)}_{\rho\sigma\tau}. \tag{6.12}$$

6.1.3 Degenerate Subspaces

If the considered basis holds linearly independent operators of both the same tree-level and anomalous scaling dimension, there is an ambiguity in the determination of the diagonal basis. It can then only be determined up to orthogonal transformations within the subspaces of identical scaling dimension. The structure constants are therefore only determined up to these orthogonal transformations, too.

Suppose that the one-loop structure constant in some subspace $V \subset \{\mathcal{O}_\alpha\}$ can be written as

$$C^{(1)}_{\alpha\beta\gamma} = c_V \, C^{(0)}_{\alpha\beta\gamma} \tag{6.13}$$

where c_V is a constant for all operators in V.[1] This constant is then invariant under arbitrary basis-transformations *within* V:

$$\frac{C^{(1)}_{\alpha\beta\gamma}}{C^{(0)}_{\alpha\beta\gamma}} = c_V \quad \to \quad \frac{M_{\alpha\rho} M_{\beta\sigma} M_{\gamma\tau} C^{(1)}_{\rho\sigma\tau}}{M_{\alpha\rho} M_{\beta\sigma} M_{\gamma\tau} C^{(0)}_{\rho\sigma\tau}} = \frac{M_{\alpha\rho} M_{\beta\sigma} M_{\gamma\tau} c_V \, C^{(0)}_{\rho\sigma\tau}}{M_{\alpha\rho} M_{\beta\sigma} M_{\gamma\tau} C^{(0)}_{\rho\sigma\tau}} = c_V. \tag{6.14}$$

If the quotients $C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$ differ in any subspace they must differ in every basis for this subspace. For the anomalous dimension eigenspaces this indicates an additional degeneracy that should be broken on higher loop levels.

6.1.4 Summary

As we know, operators of different lengths are automatically orthogonal and the scalar-single trace operators do not mix with any other operators at one-loop level. For a given non-diagonal basis of length L operators $\mathcal{O}_\alpha$ we can thus determine the basis-independent quotients of the renormali-

[1] The following considerations are in fact true for arbitrary subspaces, but we expect this situation to appear only in eigenspaces for the same anomalous dimension.

sation scheme independent one-loop structure constants and the tree-level constants, as well as the anomalous dimensions, as follows:

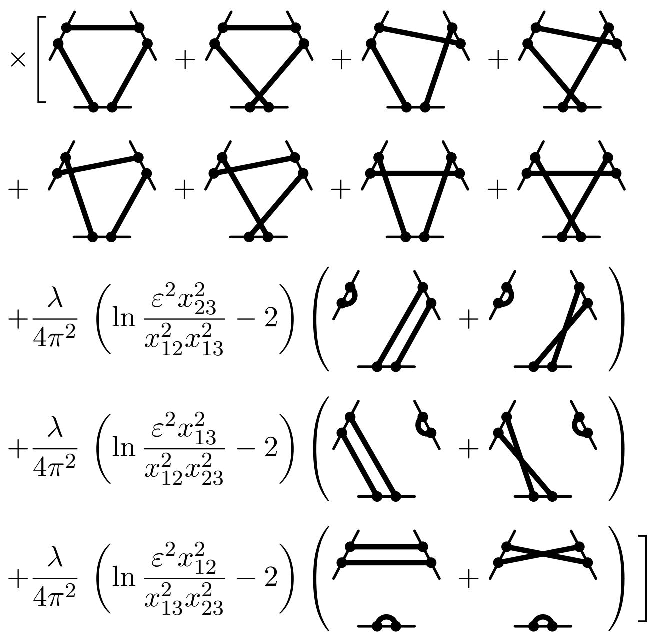
1. Determine all possible two- and three-point functions of the non-diagonal operators.
2. Extract $t^{(0)}_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ from the two-point functions.
3. Extract $\overline{C}^{(0)}_{\alpha\beta\gamma}$ and $\overline{C}^{(1)}_{\alpha\beta\gamma}$ from the three-point functions.
4. Diagonalise $\gamma(t^{(0)})^{-1}$. If $M\gamma(t^{(0)})^{-1}M^{-1}$ is diagonal, then M is the demanded change-of-basis matrix. The eigenvalues are the anomalous dimensions γ_α.
5. Calculate $C^{(0)}_{\alpha\beta\gamma}$ and $C^{(1)}_{\alpha\beta\gamma}$ using (6.11) and (6.12).
6. Calculate the quotients $C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$. If there is no additional degeneracy, these results will be unique within the γ_α eigenspaces.

This procedure serves as a starting point for the computational calculations of structure constants in section 6.5.

6.2 Three-Point Functions with only Length Two Operators

Three-point functions with all three operators of length two are in some respects less complicated than those of longer operators for two reasons. First of all, there are no 2-gons appearing in the three-point function. So not only the structure constants but the full correlation functions can be calculated by considering only the 3-gon corrections. Second of all, as described in section 4.2.7, only the self contraction terms in the dressing formula (4.43) contribute. The full three-point function up to one-loop order can then be depicted as

$$\langle \mathcal{O}_\alpha(x_1)\, \mathcal{O}_\beta(x_2)\, \mathcal{O}_\gamma(x_3)\rangle = I_{12}I_{13}I_{23}$$



$$+\frac{\lambda}{4\pi^2}\left(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2x_{13}^2}-2\right)(\ldots) +\frac{\lambda}{4\pi^2}\left(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2x_{23}^2}-2\right)(\ldots) +\frac{\lambda}{4\pi^2}\left(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2x_{23}^2}-2\right)(\ldots) \Bigg]. \tag{6.15}$$

We could also handle the $SO(6)$ indices with symmetric matrices and denote operators by

$$\mathcal{O}_\alpha(x) = U^\alpha_{IJ}\mathrm{Tr}\left(\phi^I\phi^J\right)(x). \tag{6.16}$$

Unlike the notation with vectors attached to the fields, this notation allows us to write any scalar length two trace operator as a single matrix. The Konishi operator then corresponds to the unity matrix. This leaves us with formulae for the two- and three-point functions that contain nothing but traces of products of these matrices:

$$\langle \mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\rangle = \frac{1}{8\pi^4\,x_{12}^4}\left[\mathrm{Tr}\big(U^\alpha U^\beta\big) + \frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\mathrm{Tr}\big(U^\alpha\big)\mathrm{Tr}\big(U^\beta\big)\right] \tag{6.17}$$

$$\begin{aligned}\langle \mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\,\mathcal{O}_\gamma(x_3)\rangle = \frac{1}{8\pi^6\,x_{12}^2x_{13}^2x_{23}^2}\Bigg[&\mathrm{Tr}\big(U^\alpha U^\beta U^\gamma\big) + \frac{\lambda}{16\pi^2}\\ &\times\Bigg\{\left(\ln\frac{\varepsilon^2x_{23}^2}{x_{12}^2x_{13}^2}-2\right)\mathrm{Tr}\big(U^\alpha\big)\mathrm{Tr}\big(U^\beta U^\gamma\big)\\ &+\left(\ln\frac{\varepsilon^2x_{13}^2}{x_{12}^2x_{23}^2}-2\right)\mathrm{Tr}\big(U^\beta\big)\mathrm{Tr}\big(U^\alpha U^\gamma\big)\\ &+\left(\ln\frac{\varepsilon^2x_{12}^2}{x_{13}^2x_{23}^2}-2\right)\mathrm{Tr}\big(U^\gamma\big)\mathrm{Tr}\big(U^\alpha U^\beta\big)\Bigg\}\Bigg]\end{aligned} \tag{6.18}$$

We are interested in a basis in which the two-point functions become diagonal. This means that $\mathrm{Tr}\big(U^\alpha\big)\mathrm{Tr}\big(U^\beta\big) \sim \delta^{\alpha\beta}$ and $\mathrm{Tr}\left(U^\alpha U^\beta\right) \sim \delta^{\alpha\beta}$. Therefore all except one of the matrices have to be traceless. We have

$$\begin{aligned}\mathrm{Tr}\big(U^0\big) &= \gamma_0 \\ \mathrm{Tr}\big(U^\alpha\big) &= 0 &&(\alpha = 1,\dots,20)\\ \mathrm{Tr}\big((U^\alpha)^2\big) &= c_\alpha &&(\alpha = 0,\dots,20)\\ \mathrm{Tr}\big(U^\alpha U^\beta\big) &= 0 &&(\alpha \neq \beta),\end{aligned} \tag{6.19}$$

where without loss of generality we choose $\mathcal{O}_0$ to be the operator with a non-vanishing trace, which is known to be the Konishi operator $\mathcal{K}$ corresponding to the unity matrix. The only non-vanishing three-point functions are then those of one Konishi operator with two identical operators. We obtain the structure constants

$$C^{(1)}_{\mathcal{K}\mathcal{K}\mathcal{K}} = -\frac{9}{8\pi^2}C^{(0)}_{\mathcal{K}\mathcal{K}\mathcal{K}} \tag{6.20}$$

$$C^{(1)}_{\mathcal{K}\mathcal{O}_\alpha\mathcal{O}_\alpha} = -\frac{3}{8\pi^2}C^{(0)}_{\mathcal{K}\mathcal{O}_\alpha\mathcal{O}_\alpha}, \tag{6.21}$$

in accordance with equation (4.76), that for $\widetilde{C}^{(1)}_{\alpha\beta\gamma} = 0$, i. e. no 2-gons, yields the general form

$$C^{(1)}_{\mathcal{O}_\alpha\mathcal{O}_\beta\mathcal{O}_\gamma} = -\frac{1}{2}\left(\gamma_\alpha + \gamma_\beta + \gamma_\gamma\right) C^{(0)}_{\mathcal{O}_\alpha\mathcal{O}_\beta\mathcal{O}_\gamma}. \tag{6.22}$$

6.3 Three-Point Functions with a Konishi Operator

The only possibilities for non-vanishing three-point functions

$$\langle \mathcal{O}_\alpha(x_1)\, \mathcal{O}_\beta(x_2)\, \mathcal{K}(x_3)\rangle \tag{6.23}$$

containing a Konishi operator are extremal ones with $\Delta_\alpha^{(0)} = \Delta_\beta^{(0)} \pm 2$ whose structure constants are given by equation (4.69) and such with two operators of equal length.

We will show that for the latter, i. e. three-point functions of a Konishi operator with any two operators of a diagonal length $\Delta^{(0)}$ basis, the structure constants take the form

$$\begin{aligned} C^{(1)}_{\alpha\beta\mathcal{K}} &= -\left(\frac{\gamma_\alpha}{\Delta_\alpha^{(0)}} + \frac{\gamma_\beta}{\Delta_\beta^{(0)}} + \frac{\gamma_\mathcal{K}}{\Delta_\mathcal{K}^{(0)}}\right) C^{(0)}_{\alpha\beta\mathcal{K}} \\ &= -\frac{\delta_{\alpha\beta}}{4\pi^2\sqrt{3}}\left(2\gamma_\alpha + \frac{3}{8\pi^2}\Delta_\alpha^{(0)}\right). \end{aligned} \tag{6.24}$$

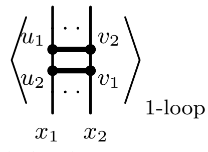
Let $\mathcal{K}$ be the length two Konishi operator and the set $\{\mathcal{O}_\alpha\}$ an arbitrary non-diagonal basis for the operators of length $\Delta^{(0)}$ that can be written in terms of attached vectors, namely

$$\mathcal{K} = \frac{1}{\sqrt{12}} \sum_i \operatorname{Tr}\left(\phi^i \phi^i\right) \tag{6.25}$$

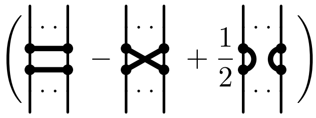
$$\mathcal{O}_\alpha = \operatorname{Tr}\left(u_1^\alpha \cdot \phi \cdots u_{\Delta^{(0)}}^\alpha \cdot \phi\right) \qquad (\Delta^{(0)} > 2). \tag{6.26}$$

Let $Z_k \subset S_k$ denote the set of cyclic permutations of $(1, 2, \ldots, k)$.

We choose the renormalisation scheme $\varepsilon \to e\varepsilon$ in which only the 2-gons hold finite contributions



$$= I_{12}^2 \frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2} + 1\right)$$



(6.27)

while the 3-gons only contribute to the logarithmic terms. For the two-point functions we get

$$
\begin{aligned}
\langle \mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\rangle = I_{12}^{\Delta^{(0)}} \sum_{\sigma\in Z_{\Delta^{(0)}}} \Bigg[& \prod_{i=1}^{\Delta^{(0)}} u_i^\alpha \cdot u_{\sigma(i)}^\beta + \frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}+1\right) \\
& \times \sum_{\tau\in Z_{\Delta^{(0)}}} \Big(u^\alpha_{\tau(1)}\cdot u^\beta_{\tau\circ\sigma(1)}\, u^\alpha_{\tau(2)}\cdot u^\beta_{\tau\circ\sigma(2)} - u^\alpha_{\tau(1)}\cdot u^\beta_{\tau\circ\sigma(2)} \\
& \times u^\alpha_{\tau(2)}\cdot u^\beta_{\tau\circ\sigma(1)} + \frac{1}{2}\, u^\alpha_{\tau(1)}\cdot u^\alpha_{\tau(2)}\, u^\beta_{\tau\circ\sigma(1)}\cdot u^\beta_{\tau\circ\sigma(2)}\Big) \\
& \times \prod_{i=3}^{\Delta^{(0)}} u^\alpha_{\tau(i)}\cdot u^\beta_{\tau\circ\sigma(i)} \Bigg]. \qquad (6.28)
\end{aligned}
$$

Now let $\mathcal{D}_\alpha = M_{\alpha\beta}\,\mathcal{O}_\beta$ denote a diagonal basis of the length $\Delta^{(0)}$ subspace. Then

$$
\begin{aligned}
\langle \mathcal{D}_\alpha(x_1)\,\mathcal{D}_\beta(x_2)\rangle &= \frac{1}{x_{12}^{2\Delta^{(0)}}}\left(\delta_{\alpha\beta} + \lambda g_{\alpha\beta} + \lambda\gamma_\alpha\delta_{\alpha\beta}\ln\frac{\varepsilon^2}{x_{12}^2}\right) \\
&= M_{\alpha\gamma}\,M_{\beta\delta}\,\langle \mathcal{O}_\gamma(x_1)\,\mathcal{O}_\delta(x_2)\rangle \qquad (6.29)
\end{aligned}
$$

from which we immediately get the condition for tree-level diagonality

$$
\sum_{\sigma\in Z_{\Delta^{(0)}}} M_{\alpha\gamma}\,M_{\beta\delta} \prod_{i=1}^{\Delta^{(0)}} u_i^\gamma\cdot u^\delta_{\sigma(i)} = (2\pi)^{2\Delta^{(0)}}\,\delta_{\alpha\beta}. \qquad (6.30)
$$

Using this result we obtain

$$
\begin{aligned}
&\langle \mathcal{D}_\alpha(x_1)\,\mathcal{D}_\beta(x_2)\rangle \\
&\quad= \frac{1}{x_{12}^{2\Delta^{(0)}}}\Bigg(\delta_{\alpha\beta} + \frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}+1\right)\Bigg[\Delta^{(0)}\,\delta_{\alpha\beta} - \frac{1}{(2\pi)^{2\Delta^{(0)}}} \\
&\qquad\times \sum_{\sigma\in Z_{\Delta^{(0)}}}\sum_{\tau\in Z_{\Delta^{(0)}}} M_{\alpha\gamma}\,M_{\beta\delta}\Big(u^\gamma_{\tau(1)}\cdot u^\delta_{\tau\circ\sigma(2)}\,u^\gamma_{\tau(2)}\cdot u^\delta_{\tau\circ\sigma(1)} \\
&\qquad -\frac{1}{2}u^\gamma_{\tau(1)}\cdot u^\gamma_{\tau(2)}\,u^\delta_{\tau\circ\sigma(1)}\cdot u^\delta_{\tau\circ\sigma(2)}\Big) \times \prod_{i=3}^{\Delta^{(0)}} u^\gamma_{\tau(i)}\cdot u^\delta_{\tau\circ\sigma(i)}\Bigg]\Bigg) \qquad (6.31)
\end{aligned}
$$

and thus the condition for one-loop diagonality

$$(2\pi)^{2\Delta^{(0)}}\,\delta_{\alpha\beta}\,\left(\Delta^{(0)}-8\pi^2\,\gamma_\alpha\right)$$
$$=\sum_{\sigma\in Z_{\Delta^{(0)}}}\sum_{\tau\in Z_{\Delta^{(0)}}} M_{\alpha\gamma}\,M_{\beta\delta}\Big(u^\gamma_{\tau(1)}\cdot u^\delta_{\tau\circ\sigma(2)}\,u^\gamma_{\tau(2)}\cdot u^\delta_{\tau\circ\sigma(1)}$$
$$-\frac{1}{2}u^\gamma_{\tau(1)}\cdot u^\gamma_{\tau(2)}\,u^\delta_{\tau\circ\sigma(1)}\cdot u^\delta_{\tau\circ\sigma(2)}\Big)\prod_{i=3}^{\Delta^{(0)}}u^\gamma_{\tau(i)}\cdot u^\delta_{\tau\circ\sigma(i)} \tag{6.32}$$

and

$$\boxed{g_\alpha=\gamma_\alpha.} \tag{6.33}$$

The three-point functions are

$$\langle\mathcal{D}_\alpha(x_1)\,\mathcal{D}_\beta(x_2)\mathcal{K}(x_3)\rangle = M_{\alpha\gamma}\,M_{\beta\delta}\;\langle\mathcal{O}_\alpha(x_1)\,\mathcal{O}_\beta(x_2)\mathcal{K}(x_3)\rangle$$
$$=\frac{1}{(2\pi)^{2\Delta^{(0)}+2}\sqrt{3}\,x_{12}^{2\Delta^{(0)}-2}\,x_{13}^2\,x_{23}^2}\sum_{\sigma\in Z_{\Delta^{(0)}}}\sum_{\tau\in Z_{\Delta^{(0)}}}M_{\alpha\gamma}\,M_{\beta\delta}$$
$$\times\Bigg[\prod_{i=1}^{\Delta^{(0)}}u^\gamma_{\sigma(i)}\cdot u^\delta_{\tau(i)}+\frac{\lambda}{8\pi^2}\sum_{\rho\in Z_{\Delta^{(0)}-2}}\Big(u^\gamma_{\sigma\circ\rho(1)}\cdot u^\delta_{\tau\circ\rho(1)}\,u^\gamma_{\sigma\circ\rho(2)}\cdot u^\delta_{\tau\circ\rho(2)}$$
$$-u^\gamma_{\sigma\circ\rho(1)}\cdot u^\delta_{\tau\circ\rho(2)}\,u^\gamma_{\sigma\circ\rho(2)}\cdot u^\delta_{\tau\circ\rho(1)}+\frac{1}{2}u^\gamma_{\sigma\circ\rho(1)}\cdot u^\gamma_{\sigma\circ\rho(2)}\,u^\delta_{\tau\circ\rho(1)}\cdot u^\delta_{\tau\circ\rho(2)}\Big)$$
$$\times\prod_{i=3}^{\Delta^{(0)}-2}\Big(u^\gamma_{\sigma\circ\rho(i)}\cdot u^\delta_{\tau\circ\rho(i)}\Big)\times u^\gamma_{\sigma(\Delta^{(0)}-1)}\cdot u^\delta_{\tau(\Delta^{(0)}-1)}\,u^\gamma_{\sigma(\Delta^{(0)})}\cdot u^\delta_{\tau(\Delta^{(0)})}$$
$$+\lambda\times\text{logs}\Bigg]$$
$$\stackrel{!}{=}\frac{1}{x_{12}^{2\Delta^{(0)}-2}\,x_{13}^2\,x_{23}^2}\Big(C^{(0)}_{\alpha\beta\mathcal{K}}+\lambda\,\widetilde{C}^{(1)}_{\alpha\beta\mathcal{K}}+\lambda\times\text{logs}\Big) \tag{6.34}$$

and we obtain the tree-level structure constant

$$C^{(0)}_{\alpha\beta\mathcal{K}}=\frac{1}{(2\pi)^{2\Delta^{(0)}+2}\sqrt{3}}\sum_{\sigma\in Z_{\Delta^{(0)}}}\sum_{\tau\in Z_{\Delta^{(0)}}}M_{\alpha\gamma}\,M_{\beta\delta}\prod_{i=1}^{\Delta^{(0)}}u^\gamma_{\sigma(i)}\cdot u^\delta_{\tau(i)}$$

$$= \frac{\Delta^{(0)}}{(2\pi)^{2\Delta^{(0)}+2}\,\sqrt{3}} \sum_{\tau \in Z_{\Delta^{(0)}}} M_{\alpha\gamma}\, M_{\beta\delta} \prod_{i=1}^{\Delta^{(0)}} u_i^{\gamma} \cdot u_{\tau(i)}^{\delta}, \tag{6.35}$$

where we omitted one sum over all permutations in the second line because the first sum already delivers all possible contractions.

Using equation (6.30) we get

$$\boxed{C^{(0)}_{\alpha\beta\mathcal{K}} = \frac{\Delta^{(0)}}{4\pi^2\,\sqrt{3}}\,\delta_{\alpha\beta}.} \tag{6.36}$$

The one-loop structure constant is

$$\begin{aligned}
\widetilde{C}^{(1)}_{\alpha\beta\mathcal{K}} &= \frac{1}{(2\pi)^{2\Delta^{(0)}+4}\,\sqrt{12}} \sum_{\sigma \in Z_{\Delta^{(0)}}} \sum_{\tau \in Z_{\Delta^{(0)}}} \sum_{\rho \in Z_{\Delta^{(0)}-2}} M_{\alpha\gamma}\, M_{\beta\delta} \\
&\times \Bigg[\prod_{i=1}^{\Delta^{(0)}-2} \Big(u^{\gamma}_{\sigma\circ\rho(i)} \cdot u^{\delta}_{\tau\circ\rho(i)} \Big) \times u^{\gamma}_{\sigma(\Delta^{(0)}-1)} \cdot u^{\delta}_{\tau(\Delta^{(0)}-1)}\, u^{\gamma}_{\sigma(\Delta^{(0)})} \cdot u^{\delta}_{\tau(\Delta^{(0)})} \\
&- \Big(u^{\gamma}_{\sigma\circ\rho(1)} \cdot u^{\delta}_{\tau\circ\rho(2)}\, u^{\gamma}_{\sigma\circ\rho(2)} \cdot u^{\delta}_{\tau\circ\rho(1)} - \frac{1}{2}\, u^{\gamma}_{\sigma\circ\rho(1)} \cdot u^{\gamma}_{\sigma\circ\rho(2)} u^{\delta}_{\tau\circ\rho(1)} \cdot u^{\delta}_{\tau\circ\rho(2)} \Big) \\
&\times \prod_{i=3}^{\Delta^{(0)}-2} \Big(u^{\gamma}_{\sigma\circ\rho(i)} \cdot u^{\delta}_{\tau\circ\rho(i)} \Big) \times u^{\gamma}_{\sigma(\Delta^{(0)}-1)} \cdot u^{\delta}_{\tau(\Delta^{(0)}-1)}\, u^{\gamma}_{\sigma(\Delta^{(0)})} \cdot u^{\delta}_{\tau(\Delta^{(0)})} \Bigg] \\
&= \frac{\delta_{\alpha\beta}}{(2\pi)^4\,\sqrt{12}} \Big[(\Delta^{(0)}-2)\,\Delta^{(0)} - (\Delta^{(0)}-2)\,(\Delta^{(0)} - 8\pi^2\,\gamma_\alpha) \Big] \\
&= \frac{(\Delta^{(0)}-2)\,\gamma_\alpha}{4\pi^2\,\sqrt{3}}\,\delta_{\alpha\beta},
\end{aligned} \tag{6.37}$$

where the sum over the ρ-permutations gives only a factor of $(\Delta^{(0)}-2)$ and we made use of equations (6.30) and (6.32) in the second step.

The renormalisation scheme independent structure constants are given by equation (4.14) as

$$C^{(1)}_{\alpha\beta\gamma} = \widetilde{C}^{(1)}_{\alpha\beta\gamma} - \frac{1}{2}\, C^{(0)}_{\alpha\beta\gamma}\,\left(g_\alpha + g_\beta + g_\gamma\right). \tag{6.38}$$

Regarding (6.33) we obtain the desired result

$$C^{(1)}_{\alpha\beta\mathcal{K}} = \widetilde{C}^{(1)}_{\alpha\beta\mathcal{K}} - \frac{1}{2}\, C^{(0)}_{\alpha\beta\mathcal{K}} \left(\gamma_\alpha + \gamma_\beta + \frac{3}{4\pi^2}\right)$$

$$
\begin{aligned}
&= \delta_{\alpha\beta}\,\frac{1}{4\pi^2\sqrt{3}}\left[(\Delta^{(0)}-2)\,\gamma_\alpha - \Delta^{(0)}\,\gamma_\alpha - \Delta^{(0)}\,\frac{3}{8\pi^2}\right] \\
&= -\left(\frac{2\,\gamma_\alpha}{\Delta^{(0)}} + \frac{3}{8\pi^2}\right) C^{(0)}_{\alpha\beta\mathcal{K}} \\
&= -\left(\frac{\gamma_\alpha}{{\Delta^{(0)}}_\alpha} + \frac{\gamma_\beta}{{\Delta^{(0)}}_\beta} + \frac{\gamma_\mathcal{K}}{{\Delta^{(0)}}_\mathcal{K}}\right) C^{(0)}_{\alpha\beta\mathcal{K}}.
\end{aligned} \tag{6.39}
$$

6.4 Three-Point Functions of $SU(2)$ Operators

We recall the complex fields

$$Z = \phi_1 + \mathrm{i}\,\phi_2, \qquad W = \phi_3 + \mathrm{i}\,\phi_4 \quad \text{and} \quad X = \phi_5 + \mathrm{i}\,\phi_6 \tag{6.40}$$

and consider the $SU(2)$ operators $\mathrm{Tr}(Z^r \overline{W}^s)$, $\mathrm{Tr}(\overline{Z}^r X^t)$ and $\mathrm{Tr}(\overline{X}^t W^s)$. Let us denote the permutations of the fields by $[Z^r \overline{W}^s]_a$ where the index a runs over all permutations of the fields that are genuinely different modulo cyclic permutations. Now consider operators of a diagonal basis that we denote by

$$\mathrm{Tr}\big(Z^r \overline{W}^s\big)_{\mathrm{diag}} = \sum_a c_a^1 \,\mathrm{Tr}\big([Z^r \overline{W}^s]_a\big), \tag{6.41}$$

$$\mathrm{Tr}\big(\overline{Z}^r X^t\big)_{\mathrm{diag}} = \sum_b c_b^2 \,\mathrm{Tr}\big([\overline{Z}^r X^t]_b\big), \tag{6.42}$$

$$\mathrm{Tr}\big(\overline{X}^t W^s\big)_{\mathrm{diag}} = \sum_c c_c^3 \,\mathrm{Tr}\big([\overline{X}^t W^s]_c\big), \tag{6.43}$$

where $c_{a,b,c}^{1,2,3}$ are arbitrary coefficients. The three-point function of these three operators is

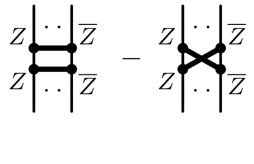
$$\Big\langle \mathrm{Tr}\big(Z^r \overline{W}^s\big)_{\mathrm{diag}}(x_1)\, \mathrm{Tr}\big(\overline{Z}^r X^t\big)_{\mathrm{diag}}(x_2)\, \mathrm{Tr}\big(\overline{X}^t W^s\big)_{\mathrm{diag}}(x_3) \Big\rangle = \sum_{a,b,c} c_a^1 c_b^2 c_c^3$$
$$\times \Big\langle \mathrm{Tr}\big([Z^r \overline{W}^s]_a\big)(x_1)\, \mathrm{Tr}\big([\overline{Z}^r X^t]_b\big)(x_2)\, \mathrm{Tr}\big([\overline{X}^t W^s]_c\big)(x_3) \Big\rangle. \tag{6.44}$$

There is only one non-vanishing tree-level diagram, pictured in figure 6.1.

We consider again the renormalisation scheme in which only the 2-gons contribute and in which the scheme independent structure constant is

$$C^{(1)}_{\alpha\beta\gamma} = \widetilde{C}^{(1)}_{\alpha\beta\gamma} - \frac{1}{2}\,(\gamma_\alpha + \gamma_\beta + \gamma_\gamma)\; C^{(0)}_{\alpha\beta\gamma}. \tag{6.45}$$

All self-contractions on these operators yield zero and 2-gons that contribute at tree-level do not contribute at one-loop level, because



$$= 0. \tag{6.46}$$

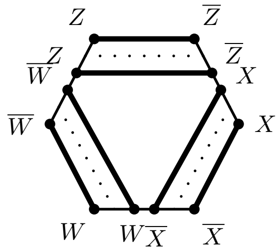
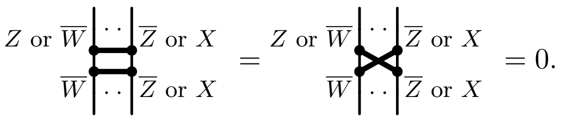


Figure 6.1: The only non-vanishing tree-level diagram.

But neither do the other permutations:



(6.47)

This argument equally holds for all possible types of 2-gons and therefore there are no 2-gon contributions at all, i. e.

$$\widetilde{C}^{(1)}_{\alpha\beta\gamma} = 0. \tag{6.48}$$

The renormalisation scheme independent structure constants are

$$\boxed{C^{(1)}_{\alpha\beta\gamma} = -\frac{1}{2}\,(\gamma_\alpha + \gamma_\beta + \gamma_\gamma)\,C^{(0)}_{\alpha\beta\gamma}.} \tag{6.49}$$

6.5 Computational Calculation of Structure Constants

Using the dressing formulae given in section 4.2 we can in principle straightforwardly calculate arbitrary three-point functions. Nevertheless, because of the summation over all permutations, these calculations become very lengthy even for short operators. For operators of lengths L, M and N there are LMN tree-level and $LMN(L+M+N+1)$ one-loop diagrams that have to be calculated.

To handle this problem we calculate the structure constants computationally on two different levels. The first program simply performs the summation over permutations and can be used as a tool to calculate the structure constants of diagonal operators by hand. The second program that we introduce is used to calculate both the diagonal bases themselves and the structure constants for the whole set of operators of a given length by "brute force".

6.5.1 Summation of Permutations

The Matlab® programs `zpf.m` and `dpf.m` listed in appendix B.1 can be used to calculate anomalous dimensions and structure constants. They perform the summation of 2-gons and the summation of 3-gons given in equation (4.73) respectively. Length L operators

$$\mathcal{O}_\alpha = u^\alpha_{i_1} \cdots u^\alpha_{i_L} \operatorname{Tr}\left(\phi^{i_1} \cdots \phi^{i_L}\right) \tag{6.50}$$

are represented by $6 \times L$ matrices where each row corresponds to one of the attached vectors u^α_i.

Anomalous Dimensions The program `zpf.m` takes two such matrices as parameters and returns two values `tree` and `loop`. If the parameters are integer matrices, the return values are integers, too. Then, in the standard renormalisation scheme of section 4.2 the two-point function of the operators corresponding to the parameter matrices is given by

$$\langle \mathcal{O}_\alpha(x_1)\, \mathcal{O}_\beta(x_2) \rangle = I^L_{12} \left(\texttt{tree} + \frac{\lambda}{16\pi^2} \left(\ln \frac{\varepsilon^2}{x^2_{12}} - 1 \right) \times \texttt{loop} \right). \tag{6.51}$$

In general, diagonal operators are represented by linear combinations

$$\widetilde{\mathcal{O}}_\alpha = \sum_k a_k \mathcal{O}^k_\alpha \tag{6.52}$$

of operators of the form (6.50). The anomalous dimension of $\widetilde{\mathcal{O}}_\alpha$ is then given by

$$\gamma_\alpha = \frac{1}{16\pi^2} \frac{\sum_{k,l} a_k\, a_l \times \texttt{loop}_{kl}}{\sum_{k,l} a_k\, a_l \times \texttt{tree}_{kl}} \tag{6.53}$$

where $\texttt{loop}_{kl}$ and $\texttt{tree}_{kl}$ are calculated for $\mathcal{O}^k_\alpha$ and $\mathcal{O}^l_\alpha$.

Structure Constants The program `dpf.m` takes three matrices, representing $\mathcal{O}^k_\alpha$, $\mathcal{O}^l_\beta$ and $\mathcal{O}^m_\gamma$, as parameters and returns two values $\texttt{c0}_{klm}$ and $\texttt{c1}_{klm}$. The renormalisation scheme independent structure constant is given by

$$C^{(1)}_{\alpha\beta\gamma} = \frac{1}{32\pi^2} \frac{\sum_{k,l,m} a_k\, a_l\, a_m \times \texttt{c1}_{klm}}{\sum_{k,l,m} a_k\, a_l\, a_m \times \texttt{c0}_{klm}}\, C^{(0)}_{\alpha\beta\gamma}. \tag{6.54}$$

6.5.2 Diagonal Structure Constants from Non-Diagonal Bases

To calculate the diagonal structure constants from non-diagonal bases by brute force, one needs to follow the steps in section 6.1.4. We start with a standard basis for the operators of a given length and calculate the change-of-basis matrix diagonalising these operators. In a second step all non-diagonal structure constants for the standard basis are calculated. Finally the diagonal structure constants are obtained by summation using the change-of-basis matrix.

For operators of length two and three we can perform these calculations with Mathematica® and obtain exact algebraic results. Unfortunately Mathematica® fails at diagonalising the 336×336 matrix for length four algebraically. Numerical results for operators of length four and five are therefore calculated with Matlab®.

As the calculations of all diagonal structure constants would have taken several months, we calculate them for each eigenspace separately and restrict our calculation to small samples of about 250 randomly chosen data points.

The program codes for both the Mathematica® and Matlab® routines can be found in appendix B.2. All numerical results are listed in appendix A.2. In the next section we analyse the results for which an algebraic form can be found.

6.6 Short $SO(6)$ Operators in Diagonal Bases

We depict the operators

$$\mathrm{Tr}\left(u_{i_1}\cdot\phi^{i_1}\cdots u_{i_n}\cdot\phi^{i_n}\right)(x) \tag{6.55}$$

by n-tuples $(u_{i_1},\ldots,u_{i_n})$ and regard non-diagonal bases

$$\begin{aligned}\mathcal{B}_2 =&\frac{1}{\sqrt{2}}(e_1,e_1),(e_1,e_2),\ldots,\frac{1}{\sqrt{2}}(e_6,e_6)\\ \mathcal{B}_3 =&\frac{1}{\sqrt{3}}(e_1,e_1,e_1),(e_1,e_1,e_2),\ldots,\frac{1}{\sqrt{3}}(e_6,e_6,e_6)\\ &\vdots\end{aligned} \tag{6.56}$$

where e_i is the i-th vector of the $\mathbb{R}^6$ standard basis. These serve as a starting point for the programs described in the previous section. Exact results can be obtained with Mathematica® up to length three and by hand for the length four singlets.

We already know the form of structure constants for extremal correlators (4.69). We have also derived the general form of structure constants for three length two operators in section 6.2 and of those that include a Konishi operator in section 6.3. As far as they were calculated, these results could be confirmed and will not be listed here.

For correlators that contain operators of lengths larger than three we calculate the structure constants numerically. All numerical results are listed in appendix A.2. In this section we state only results that are non-zero and for which we can reasonably guess the exact values based upon the numerical results. All possible classes of non-extremal correlation functions up to length five are calculated, except those of one length four and two length five operators.

In addition to the results that are given below, we found several classes of operators for which the results vary strongly. As shown in section 6.1.3, if they vary in one basis they have to vary in every basis. Therefore there has to be an additional degeneracy for these operators that is broken by the three-point functions. This will be shown explicitly for the protected length two operators in the next section where we calculate structure constants of BMN operators.

In appendix A.2.3 we list the qualitative structure of the three-point functions, i. e. whether they take a definite value, indicate an additional degeneracy, or vanish.

6.6.1 Anomalous Dimension Eigenspaces

The diagonalisation provides us with the anomalous dimensions. The diagonal bases can be decomposed into eigenspaces of operators with the same anomalous dimension. We can therefore classify the operators by these eigenspaces. This classification is given in the following table including the dimensions of the eigenspaces.

Length	Class	Dimension (degeneracy)	Anomalous dimension	BMN operators in this class
2	2A	1	$\gamma = \frac{3}{4\pi^2}$	
	2B	20	$\gamma = 0$	
3	3A	20	$\gamma = \frac{3}{4\pi^2}$	
	3B	6	$\gamma = \frac{1}{2\pi^2}$	$\mathcal{B}^{1,1}_{\text{tr}}$
	3C	50	$\gamma = 0$	$\mathcal{B}^{1,0}_{(ij)}$
4	4A	1	$\gamma = \frac{13+\sqrt{41}}{16\pi^2}$	
	4B	20	$\gamma = \frac{5+\sqrt{5}}{8\pi^2}$	$\mathcal{B}^{2,2}_{\text{tr}}$
	4C	99	$\gamma = \frac{3}{4\pi^2}$	$\mathcal{B}^{2,1}_{(ij)}$
	4D	90	$\gamma = \frac{1}{2\pi^2}$	$\mathcal{B}^{2,1}_{[ij]}$
	4E	1	$\gamma = \frac{13-\sqrt{41}}{16\pi^2}$	
	4F	20	$\gamma = \frac{5-\sqrt{5}}{8\pi^2}$	$\mathcal{B}^{2,1}_{\text{tr}}$
	4G	105	$\gamma = 0$	$\mathcal{B}^{2,0}_{(ij)}$
5	5A	20	$\gamma = \frac{7+\sqrt{13}}{8\pi^2}$	
	5B	12	$\gamma = \frac{5}{4\pi^2}$	
	5C	128	$\gamma = \frac{15}{16\pi^2}$	
	5D	258	$\gamma = \frac{5+\sqrt{5}}{8\pi^2}$	$\mathcal{B}^{3,2}_{[ij]}$
	5E	190	$\gamma = \frac{3}{4\pi^2}$	$\mathcal{B}^{3,2}_{\text{tr}}$
	5F	128	$\gamma = \frac{5}{8\pi^2}$	
	5G	300	$\gamma = \frac{1}{2\pi^2}$	$\mathcal{B}^{3,1}_{(ij)}$
	5H	20	$\gamma = \frac{7-\sqrt{13}}{8\pi^2}$	
	5I	258	$\gamma = \frac{5-\sqrt{5}}{8\pi^2}$	$\mathcal{B}^{3,1}_{[ij]}$
	5J	50	$\gamma = \frac{1}{4\pi^2}$	$\mathcal{B}^{3,1}_{\text{tr}}$
	5K	196	$\gamma = 0$	$\mathcal{B}^{3,0}_{(ij)}$

6.6.2 Operators up to Length Three

For operators of length two and three the following results were obtained with Mathematica®:

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
2B	3A	3A	$-\frac{1}{2\,\pi^2}$	$-2\,\widetilde{\gamma}_{\beta=\gamma}$, $(\widetilde{\gamma}_\alpha = 0)$
2B	3B	3B	$-\frac{1}{6\,\pi^2}$	$-\widetilde{\gamma}_{\beta=\gamma}$, $(\widetilde{\gamma}_\alpha = 0)$
2B	3B	3C	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\beta$, $(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
2B	3C	3C	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = \widetilde{\gamma}_\gamma = 0)$

Here as well as in the following, we propose a function of the scaling dimensions—where possible—that reproduces the structure constant similar to the form (6.39) for the Konishi operator, i. e. a function

$$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma} = p\,\widetilde{\gamma}_\alpha + q\,\widetilde{\gamma}_\beta + r\,\widetilde{\gamma}_\gamma, \tag{6.57}$$

with p, q, $r \in \mathbb{Q}$. We denote the quotient of the anomalous dimension and the tree-level scaling dimension by

$$\widetilde{\gamma}_\alpha = \frac{\gamma_\alpha}{\Delta^{(0)}_\alpha}. \tag{6.58}$$

Note that this structure is just a suggestion and for some of the following results it is rather questionable whether this suggestion is anywhere close to the truth. Where we cannot guess the complete form of the structure constant, we use ξ to symbolise a free parameter that can take values in the interval $[0,1]$.

6.6.3 Exact Results for Length Four Singlets

The diagonal bases for length two and three found with Mathematica® are:

Length Two

$$\mathcal{O}_{2A} = \sum_{i=1}^{6} \mathrm{Tr}\left(\phi^i\phi^i\right) = \mathcal{K} \tag{6.59}$$

$$\mathcal{O}_{2B,(ij)} = \mathrm{Tr}\left(\phi^i\phi^j\right) \qquad (i<j) \tag{6.60}$$

$$\mathcal{O}_{2B,i} = \mathrm{Tr}\left(\phi^i\phi^i\right) - \frac{1}{\sqrt{3}}\,\mathcal{K} \qquad (i = 2\ldots 6) \tag{6.61}$$

Length Three

$$\mathcal{O}_{3A,i[jk]} = \mathrm{Tr}\left(\phi^i \phi^{[j} \phi^{k]}\right) \qquad (i < j < k) \quad (6.62)$$

$$\mathcal{O}_{3B,i} = \sum_{j=1}^{6} \mathrm{Tr}\left(\phi^i \phi^j \phi^j\right) \quad (6.63)$$

$$\mathcal{O}_{3C,i(jk)} = \mathrm{Tr}\left(\phi^i \phi^{(j} \phi^{k)}\right) \qquad (i < j < k) \quad (6.64)$$

$$\mathcal{O}_{3C,ij} = 8\ \mathrm{Tr}\left(\phi^i \phi^j \phi^j\right) - \sum_{k=1}^{6} \mathrm{Tr}\left(\phi^i \phi^k \phi^k\right) \qquad (i \neq j, j = 2 \dots 6) \quad (6.65)$$

$$\mathcal{O}_{3C,i} = 8\ \mathrm{Tr}\left(\phi^i \phi^i \phi^i\right) - 3 \sum_{j=1}^{6} \mathrm{Tr}\left(\phi^i \phi^j \phi^j\right) \qquad (i = 2 \dots 6) \quad (6.66)$$

Note that these operators are neither normalised nor diagonal at tree-level within the eigenspaces. As described in section 6.1.3, the choice of the basis within the eigenspaces has no influence on the quotients $C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$ as long as these are constant within the eigenspaces.

Length Four Singlets The length four singlets can be easily found by hand. They are

$$\mathcal{O}_{4A} = \sum_{i=1}^{6} \sum_{j=1}^{6} \left[4\ \mathrm{Tr}\left(\phi^i \phi^i \phi^j \phi^j\right) + \left(5 - \sqrt{41}\right) \mathrm{Tr}\left(\phi^i \phi^j \phi^i \phi^j\right)\right] \quad (6.67)$$

$$\mathcal{O}_{4E} = \sum_{i=1}^{6} \sum_{j=1}^{6} \left[4\ \mathrm{Tr}\left(\phi^i \phi^i \phi^j \phi^j\right) + \left(5 + \sqrt{41}\right) \mathrm{Tr}\left(\phi^i \phi^j \phi^i \phi^j\right)\right]. \quad (6.68)$$

We can therefore calculate the structure constants for these operators manually, and obtain the following non-zero results:

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
3A	3A	4A	$-\frac{21+\sqrt{41}}{32\,\pi^2}$	$-\left(\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\right)$,
3A	3A	4E	$-\frac{21-\sqrt{41}}{32\,\pi^2}$	$-\left(\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\right)$,
3B	3B	4A	$-\frac{261+9\sqrt{41}}{800\,\pi^2}$	$-\frac{9}{25}\left(3\,\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\right)$,
3B	3B	4E	$-\frac{261-9\sqrt{41}}{800\,\pi^2}$	$-\frac{9}{25}\left(3\,\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\right)$,

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
3C	3C	4A	$-\frac{13+\sqrt{41}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \qquad (\widetilde{\gamma}_\alpha=\widetilde{\gamma}_\beta=0)$
3C	3C	4E	$-\frac{13-\sqrt{41}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \qquad (\widetilde{\gamma}_\alpha=\widetilde{\gamma}_\beta=0)$
4A	4A	4A	$-\frac{7185+309\sqrt{41}}{11728\,\pi^2}$	not possible
4A	4A	4E	$-\frac{21-\sqrt{41}}{160\,\pi^2}$	$-\frac{2}{65}\left(4\,\widetilde{\gamma}_{\alpha=\beta}+17\,\widetilde{\gamma}_\gamma\right)$
4A	4E	4E	$-\frac{21+\sqrt{41}}{160\,\pi^2}$	$-\frac{2}{65}\left(17\,\widetilde{\gamma}_\alpha+4\,\widetilde{\gamma}_{\beta=\gamma}\right)$
4E	4E	4E	$-\frac{7185-309\sqrt{41}}{11728\,\pi^2}$	not possible

For the correlators of three 4A operators or three 4E operators there is no possibility to write the structure constants as a function of the type (6.57).

6.6.4 Operators up to Length Four

More results can be obtained by numerical calculations. The exact results stated above are confirmed by the numerical results. Up to length four operators we obtain the following further results:

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
2B	4A	4B	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \qquad (\widetilde{\gamma}_\alpha=0)$
2B	4A	4F	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \qquad (\widetilde{\gamma}_\alpha=0)$
2B	4A	4G	0	$(\widetilde{\gamma}_\alpha=\widetilde{\gamma}_\gamma=0)$
2B	4B	4B	$-\frac{115+14\sqrt{5}}{632\,\pi^2}$	not possible
2B	4B	4C	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta, \qquad (\widetilde{\gamma}_\alpha=0)$
2B	4B	4D	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta, \qquad (\widetilde{\gamma}_\alpha=0)$
2B	4B	4E	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta, \qquad (\widetilde{\gamma}_\alpha=0)$
2B	4B	4F	0	$(\widetilde{\gamma}_\alpha=0)$
2B	4B	4G	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta, \qquad (\widetilde{\gamma}_\alpha=\widetilde{\gamma}_\gamma=0)$
2B	4C	4C	$-\frac{3}{8\,\pi^2}$	$-2\,\widetilde{\gamma}_{\beta=\gamma}, \qquad (\widetilde{\gamma}_\alpha=0)$
2B	4C	4D	$-\frac{1}{4\,\pi^2}$	$-\left(\widetilde{\gamma}_\beta+\frac{1}{2}\,\widetilde{\gamma}_\gamma\right), \qquad (\widetilde{\gamma}_\alpha=0)$
2B	4C	4F	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \qquad (\widetilde{\gamma}_\alpha=0)$
2B	4C	4G	0	$(\widetilde{\gamma}_\alpha=\widetilde{\gamma}_\gamma=0)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
2B	4D	4F	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
2B	4D	4G	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
2B	4E	4F	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
2B	4E	4G	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
2B	4F	4F	$-\frac{115-14\sqrt{5}}{632\,\pi^2}$	not possible
2B	4F	4G	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta, \quad (\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
2B	4G	4G	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = \widetilde{\gamma}_\gamma = 0)$
3A	3A	4B	$-\frac{9+\sqrt{5}}{16\,\pi^2}$	$-\big(\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\big)$
3A	3A	4C	$-\frac{5}{8\,\pi^2}$	$-\big(\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\big)$
3A	3A	4D	$-\frac{3}{8\,\pi^2}$	$-\big(\widetilde{\gamma}_{\alpha=\beta} + \widetilde{\gamma}_\gamma\big)$
3A	3A	4F	$-\frac{9-\sqrt{5}}{16\,\pi^2}$	$-\big(\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\big)$
3A	3A	4G	$-\frac{5}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_{\alpha=\beta}, \quad (\widetilde{\gamma}_\gamma = 0)$
3A	3B	4C	$-\frac{1}{2\,\pi^2}$	ambiguous
3A	3B	4D	$-\frac{5}{8\,\pi^2}$	ambiguous
3A	3C	4D	$-\frac{3}{8\,\pi^2}$	$-\big(\widetilde{\gamma}_\alpha + \widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\beta = 0)$
3B	3B	4A	$-\frac{261+9\sqrt{41}}{800\,\pi^2}$	$-\frac{18}{25}\,\big(3\,\widetilde{\gamma}_{\alpha=\beta} + \widetilde{\gamma}_\gamma\big)$
3B	3B	4B	$-\frac{87+3\sqrt{5}}{176\,\pi^2}$	$-\frac{3}{11}\,\big(9\,\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\big)$
3B	3B	4E	$-\frac{261-9\sqrt{41}}{800\,\pi^2}$	$-\frac{18}{25}\,\big(3\,\widetilde{\gamma}_{\alpha=\beta} + \widetilde{\gamma}_\gamma\big)$
3B	3B	4F	$-\frac{87-3\sqrt{5}}{176\,\pi^2}$	$-\frac{3}{11}\,\big(9\,\widetilde{\gamma}_{\alpha=\beta} + 2\,\widetilde{\gamma}_\gamma\big)$
3B	3C	4B	$-\frac{39+7\sqrt{5}}{176\,\pi^2}$	$-\frac{1}{11}\,\big(\frac{3}{2}\,\widetilde{\gamma}_\alpha + 14\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\beta = 0)$
3B	3C	4C	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha + \frac{4}{3}\,(1-\xi)\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\beta = 0)$
3B	3C	4D	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha + 2\,(1-\xi)\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\beta = 0)$
3B	3C	4G	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha, \quad (\widetilde{\gamma}_\beta = \widetilde{\gamma}_\gamma = 0)$
3C	3C	4B	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$
3C	3C	4C	$-\frac{3}{8\,\pi^2}$	$-2\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$
3C	3C	4D	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
3C	3C	4G	0	$(\widetilde{\gamma}_\alpha=\widetilde{\gamma}_\beta=\widetilde{\gamma}_\gamma=0)$
4A	4A	4G	$-\frac{13+\sqrt{41}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_{\alpha=\beta}$
4A	4B	4F	$-\frac{9+\sqrt{41}}{32\,\pi^2}$	$-2\widetilde{\gamma}_\alpha+\frac{2}{5}\,(\widetilde{\gamma}_\beta+\widetilde{\gamma}_\gamma)$
4A	4D	4D	$-\frac{27-\sqrt{41}}{32\,\pi^2}$	$+2\,\widetilde{\gamma}_\alpha-10\,\widetilde{\gamma}_{\beta=\gamma}$
4A	4D	4G	$-\frac{13+\sqrt{41}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_\alpha,\quad(\widetilde{\gamma}_\gamma=0)$
4A	4G	4G	$-\frac{13+\sqrt{41}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_\alpha,\quad(\widetilde{\gamma}_\beta=\widetilde{\gamma}_\gamma=0)$
4B	4B	4C	$-\frac{1+4\sqrt{5}}{16\,\pi^2}$	$-8\,\widetilde{\gamma}_{\alpha=\beta}+\frac{19}{3}\,\widetilde{\gamma}_\gamma$
4B	4B	4D	$-\frac{25+7\sqrt{5}}{76\,\pi^2}$	$-\frac{4}{19}\,\big(14\,\widetilde{\gamma}_{\alpha=\beta}-5\,\widetilde{\gamma}_\gamma\big)$
4B	4B	4G	$-\frac{25+7\sqrt{5}}{76\,\pi^2}$	not possible
4B	4C	4D	$-\frac{13+\sqrt{5}}{32\,\pi^2}$	$-\big(\widetilde{\gamma}_\alpha+\frac{4}{3}\,\xi\,\widetilde{\gamma}_\beta+2\,(1-\xi)\,\widetilde{\gamma}_\gamma\big)$
4B	4D	4F	$-\frac{5}{24\,\pi^2}$	$-\frac{1}{3}\,\big(2\,\xi\,(\widetilde{\gamma}_\alpha+\widetilde{\gamma}_\gamma)+5\,(1-\xi)\,\widetilde{\gamma}_\beta\big)$
4B	4D	4G	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\alpha,\quad(\widetilde{\gamma}_\gamma=0)$
4B	4E	4F	$-\frac{9-\sqrt{41}}{32\,\pi^2}$	$\frac{2}{5}\,(\widetilde{\gamma}_\alpha+\widetilde{\gamma}_\gamma)-2\,\widetilde{\gamma}_\beta$
4B	4F	4G	$-\frac{5}{24\,\pi^2}$	$-\frac{2}{3}\,\big(\widetilde{\gamma}_\alpha+\widetilde{\gamma}_\beta\big),\quad(\widetilde{\gamma}_\gamma=0)$
4B	4G	4G	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\alpha,\quad(\widetilde{\gamma}_\beta=\widetilde{\gamma}_\gamma=0)$
4C	4C	4G	$-\frac{3}{4\,\pi^2}$	$-4\,\widetilde{\gamma}_{\alpha=\beta},\quad(\widetilde{\gamma}_\gamma=0)$
4C	4D	4F	$-\frac{13-\sqrt{5}}{32\,\pi^2}$	$-\big(\frac{4}{3}\,\xi\,\widetilde{\gamma}_\alpha+2\,(1-\xi)\,\widetilde{\gamma}_\beta+\widetilde{\gamma}_\gamma\big)$
4C	4D	4G	$-\frac{3}{8\,\pi^2}$	$-\big(2\,\xi\,\widetilde{\gamma}_\alpha+3\,(1-\xi)\,\widetilde{\gamma}_\beta\big),\quad(\widetilde{\gamma}_\gamma=0)$
4C	4F	4F	$-\frac{1-4\sqrt{5}}{16\,\pi^2}$	$\frac{19}{3}\,\widetilde{\gamma}_\alpha-8\,\widetilde{\gamma}_{\beta=\gamma}$
4C	4G	4G	$-\frac{3}{8\,\pi^2}$	$-2\,\widetilde{\gamma}_\alpha,\quad(\widetilde{\gamma}_\beta=\widetilde{\gamma}_\gamma=0)$
4D	4D	4D	$-\frac{1}{4\,\pi^2}$	$-2\,\widetilde{\gamma}_{\alpha=\beta=\gamma}$
4D	4D	4E	$-\frac{27+\sqrt{41}}{32\,\pi^2}$	$-10\,\widetilde{\gamma}_{\alpha=\beta}+2\,\widetilde{\gamma}_\gamma$
4D	4D	4G	$-\frac{1}{4\,\pi^2}$	$-2\,\widetilde{\gamma}_{\alpha=\beta},\quad(\widetilde{\gamma}_\gamma=0)$
4D	4E	4G	$-\frac{13-\sqrt{41}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta,\quad(\widetilde{\gamma}_\gamma=0)$
4D	4F	4F	$-\frac{25-7\sqrt{5}}{76\,\pi^2}$	$-\frac{4}{19}\,\big(-5\,\widetilde{\gamma}_\alpha+14\,\widetilde{\gamma}_{\beta=\gamma}\big)$
4D	4F	4G	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta,\quad(\widetilde{\gamma}_\gamma=0)$
4D	4G	4G	0	$(\widetilde{\gamma}_\beta=\widetilde{\gamma}_\gamma=0)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
4E	4E	4G	$-\frac{13-\sqrt{41}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_{\alpha=\beta}, \quad (\widetilde{\gamma}_\gamma = 0)$
4E	4G	4G	$-\frac{13-\sqrt{41}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_{\alpha}, \quad (\widetilde{\gamma}_\beta = \widetilde{\gamma}_\gamma = 0)$
4F	4F	4G	$-\frac{25-7\sqrt{5}}{76\,\pi^2}$	not possible
4F	4G	4G	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_{\alpha}, \quad (\widetilde{\gamma}_\beta = \widetilde{\gamma}_\gamma = 0)$
4G	4G	4G	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = \widetilde{\gamma}_\gamma = 0)$

6.6.5 Operators up to Length Five

For the correlation functions that contain length five operators the structure constants are:

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
2B	5A	5C	$-\frac{9}{32\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5A	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5A	5K	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
2B	5B	5K	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
2B	5C	5G	$-\frac{5}{32\,\pi^2}$	$-\frac{5}{2}\big(\frac{1}{3}\,\xi\,\widetilde{\gamma}_\beta + \frac{5}{8}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5C	5J	$-\frac{25}{32\,\pi^2}$	$-\frac{5}{2}\big(\frac{1}{3}\,\xi\,\widetilde{\gamma}_\beta + \frac{5}{4}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5D	5E	$+\frac{1-3\sqrt{5}}{16\,\pi^2}$	$-5\,\big(\frac{3}{2}\,\widetilde{\gamma}_\beta - \frac{4}{3}\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5D	5J	$-\frac{95-\sqrt{5}}{656\,\pi^2}$	$-\frac{5}{41}\,\big(-\frac{1}{2}\,\widetilde{\gamma}_\beta + 25\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5E	5I	$+\frac{1+3\sqrt{5}}{16\,\pi^2}$	$-5\,\big(-\frac{4}{3}\,\widetilde{\gamma}_\beta + \frac{3}{2}\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5E	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\big(\frac{1}{3}\,\xi\,\widetilde{\gamma}_\beta + (1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5E	5K	$-\frac{3}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\beta, \quad (\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
2B	5F	5G	$-\frac{5}{16\,\pi^2}$	$-\frac{5}{2}\big(\xi\,\widetilde{\gamma}_\beta + \frac{5}{4}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5F	5J	$+\frac{5}{16\,\pi^2}$	$+\frac{5}{2}\big(\xi\,\widetilde{\gamma}_\beta + \frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5G	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\big(\frac{1}{2}\,\xi\,\widetilde{\gamma}_\beta + (1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
2B	5H	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
2B	5I	5J	$-\frac{95+\sqrt{5}}{656\,\pi^2}$	$-\frac{5}{41}\left(-\frac{1}{2}\widetilde{\gamma}_\beta+25\,\widetilde{\gamma}_\gamma\right),\quad(\widetilde{\gamma}_\alpha=0)$
2B	5J	5J	$-\frac{5}{56\,\pi^2}$	$-\frac{25}{14}\,\widetilde{\gamma}_{\beta=\gamma},\quad(\widetilde{\gamma}_\alpha=0)$
2B	5J	5K	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\beta,\quad(\widetilde{\gamma}_\alpha=\widetilde{\gamma}_\gamma=0)$
2B	5K	5K	0	$(\widetilde{\gamma}_\alpha=\widetilde{\gamma}_\beta=\widetilde{\gamma}_\gamma=0)$
3A	4A	5A	$-\frac{9+\sqrt{13}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4A	5D	$-\frac{7+\sqrt{5}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4A	5H	$-\frac{9-\sqrt{13}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4A	5I	$-\frac{7-\sqrt{5}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4B	5C	$+\frac{11+18\sqrt{5}}{32\,\pi^2}$	$-\left(\frac{79}{8}\,\xi\,\widetilde{\gamma}_\alpha-18\,\widetilde{\gamma}_\beta+\frac{79}{6}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3A	4B	5D	$-\frac{9+\sqrt{5}}{16\,\pi^2}$	$-\left(\widetilde{\gamma}_\alpha+2\,\xi\,\widetilde{\gamma}_\beta+\frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3A	4B	5F	$-\frac{7}{16\,\pi^2}$	$-\frac{7}{2}\left(\frac{1}{2}\,\xi\,\widetilde{\gamma}_\alpha+(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3A	4C	5A	$-\frac{9+\sqrt{13}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4C	5E	$-\frac{5}{8\,\pi^2}$	ambiguous
3A	4C	5H	$-\frac{9-\sqrt{13}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4C	5F	$-\frac{1}{2\,\pi^2}$	ambiguous
3A	4D	5A	$-\frac{1}{2\,\pi^2}$	$-2\left(\xi\,\widetilde{\gamma}_\alpha+2\,(1-\xi)\,\widetilde{\gamma}_\beta\right)$
3A	4D	5D	$-\frac{10+\sqrt{5}}{4\,\pi^2}$	$-5\left(\xi\,\widetilde{\gamma}_\alpha+2\,(1-\xi)\,\widetilde{\gamma}_\beta+2\,\widetilde{\gamma}_\gamma\right)$
3A	4D	5G	$-\frac{1}{2\,\pi^2}$	ambiguous
3A	4D	5H	$-\frac{1}{2\,\pi^2}$	$-2\left(\xi\,\widetilde{\gamma}_\alpha+2\,(1-\xi)\,\widetilde{\gamma}_\beta\right)$
3A	4D	5I	$-\frac{10-\sqrt{5}}{4\,\pi^2}$	$-5\left(\xi\,\widetilde{\gamma}_\alpha+2\,(1-\xi)\,\widetilde{\gamma}_\beta+2\,\widetilde{\gamma}_\gamma\right)$
3A	4D	5J	$-\frac{3}{8\,\pi^2}$	ambiguous
3A	4D	5K	$-\frac{1}{2\,\pi^2}$	$-2\left(\xi\,\widetilde{\gamma}_\alpha+2\,(1-\xi)\,\widetilde{\gamma}_\beta\right),\quad(\widetilde{\gamma}_\gamma=0)$
3A	4E	5A	$-\frac{9+\sqrt{13}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4E	5D	$-\frac{7+\sqrt{5}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4E	5H	$-\frac{9-\sqrt{13}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4E	5I	$-\frac{7-\sqrt{5}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right)$
3A	4F	5C	$+\frac{11-18\sqrt{5}}{32\,\pi^2}$	$-\left(\frac{79}{8}\,\xi\,\widetilde{\gamma}_\alpha-18\,\widetilde{\gamma}_\beta+\frac{79}{6}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
3A	4F	5F	$-\frac{7}{16\,\pi^2}$	$-\frac{7}{2}\left(\frac{1}{2}\,\xi\,\widetilde{\gamma}_\alpha+(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3A	4F	5I	$-\frac{9-\sqrt{5}}{16\,\pi^2}$	$-\left(\widetilde{\gamma}_\alpha+2\,\xi\,\widetilde{\gamma}_\beta+\frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3A	4G	5D	$-\frac{7+\sqrt{5}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right),\qquad(\widetilde{\gamma}_\beta=0)$
3A	4G	5I	$-\frac{7-\sqrt{5}}{16\,\pi^2}$	$-\frac{1}{2}\left(\widetilde{\gamma}_\alpha+5\,\widetilde{\gamma}_\gamma\right),\qquad(\widetilde{\gamma}_\beta=0)$
3B	4A	5B	$-\frac{5}{8\,\pi^2}$	$-\frac{5}{2}\left(\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha+(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3B	4A	5E	$-\frac{1}{4\,\pi^2}$	$-\left(\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha+\frac{5}{3}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3B	4A	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{1}{2}\left(\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha+5\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3B	4A	5K	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\qquad(\widetilde{\gamma}_\gamma=0)$
3B	4B	5C	$-\frac{1+6\,\sqrt{5}}{32\,\pi^2}$	$-\left(-\frac{87}{16}\,\xi\,\widetilde{\gamma}_\alpha+6\,\widetilde{\gamma}_\beta-\frac{29}{6}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3B	4B	5D	$+\frac{7+21\,\sqrt{5}}{176\,\pi^2}$	$-\frac{21}{11}\left(\frac{7}{4}\,\widetilde{\gamma}_\alpha-2\,\xi\,\widetilde{\gamma}_\beta+\frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3B	4B	5E	$-\frac{73+6\,\sqrt{5}}{152\,\pi^2}$	$-\frac{1}{19}\left(\frac{129}{4}\,\xi\,\widetilde{\gamma}_\alpha+24\,\widetilde{\gamma}_\beta+\frac{215}{6}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3B	4B	5F	$-\frac{19+4\,\sqrt{5}}{48\,\pi^2}$	$\frac{1}{8}\,\xi\,\widetilde{\gamma}_\alpha-\frac{8}{3}\,\widetilde{\gamma}_\beta+\frac{1}{6}\,(1-\xi)\,\widetilde{\gamma}_\gamma$
3B	4B	5I	$-\frac{27+3\,\sqrt{5}}{64\,\pi^2}$	$-\frac{3}{4}\left(\frac{21}{4}\xi\widetilde{\gamma}_\alpha+\frac{14}{5}(1-\xi)\widetilde{\gamma}_\beta+(1-\frac{7}{2}\xi)\widetilde{\gamma}_\gamma\right)$
3B	4B	5J	$-\frac{13+2\,\sqrt{5}}{24\,\pi^2}$	$-\left(\frac{3}{4}\,\xi\,\widetilde{\gamma}_\alpha+\frac{8}{3}\,\widetilde{\gamma}_\beta+\frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\right)$
3B	4B	5K	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\qquad(\widetilde{\gamma}_\gamma=0)$
3B	4C	5A	$-\frac{1}{2\,\pi^2}$	$-\left(3\,\xi\,\widetilde{\gamma}_\alpha+\frac{8}{3}\,(1-\xi)\,\widetilde{\gamma}_\beta\right)$
3B	4C	5B	$-\frac{1}{2\,\pi^2}$	ambiguous
3B	4C	5D	$-\frac{1}{2\,\pi^2}$	$-\left(3\,\xi\,\widetilde{\gamma}_\alpha+\frac{8}{3}\,(1-\xi)\,\widetilde{\gamma}_\beta\right)$
3B	4C	5E	$-\frac{5}{8\,\pi^2}$	ambiguous
3B	4C	5G	$-\frac{1}{2\,\pi^2}$	ambiguous
3B	4C	5H	$-\frac{1}{2\,\pi^2}$	$-\left(3\,\xi\,\widetilde{\gamma}_\alpha+\frac{8}{3}\,(1-\xi)\,\widetilde{\gamma}_\beta\right)$
3B	4C	5I	$-\frac{1}{2\,\pi^2}$	$-\left(3\,\xi\,\widetilde{\gamma}_\alpha+\frac{8}{3}\,(1-\xi)\,\widetilde{\gamma}_\beta\right)$
3B	4D	5A	$-\frac{77+\sqrt{13}}{272\,\pi^2}$	$-\frac{5}{17}\left(\frac{21}{4}\,\xi\,\widetilde{\gamma}_\alpha+7\,(1-\xi)\,\widetilde{\gamma}_\beta+\frac{1}{2}\,\widetilde{\gamma}_\gamma\right)$
3B	4D	5C	$-\frac{19}{32\,\pi^2}$	ambiguous
3B	4D	5D	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$
3B	4D	5F	$-\frac{3}{8\,\pi^2}$	ambiguous
3B	4D	5H	$-\frac{77-\sqrt{13}}{272\,\pi^2}$	$-\frac{5}{17}\left(\frac{21}{4}\,\xi\,\widetilde{\gamma}_\alpha+7\,(1-\xi)\,\widetilde{\gamma}_\beta+\frac{1}{2}\,\widetilde{\gamma}_\gamma\right)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
3B	4D	5I	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$
3B	4D	5K	$-\frac{1}{4\,\pi^2}$	$-\big(\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha + 2\,(1-\xi)\,\widetilde{\gamma}_\beta\big),\quad (\widetilde{\gamma}_\gamma = 0)$
3B	4E	5E	$-\frac{1}{4\,\pi^2}$	$-\big(\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha + \frac{5}{3}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big)$
3B	4E	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{1}{2}\,\big(\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha + 5\,(1-\xi)\,\widetilde{\gamma}_\gamma\big)$
3B	4E	5K	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\quad (\widetilde{\gamma}_\gamma = 0)$
3B	4F	5C	$-\frac{1-6\,\sqrt{5}}{32\,\pi^2}$	$-\big(-\frac{87}{16}\,\xi\,\widetilde{\gamma}_\alpha + 6\,\widetilde{\gamma}_\beta - \frac{29}{6}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big)$
3B	4F	5D	$-\frac{27-3\,\sqrt{5}}{64\,\pi^2}$	$-\frac{3}{4}\big(\frac{21}{4}\xi\widetilde{\gamma}_\alpha + \frac{14}{5}(1-\xi)\widetilde{\gamma}_\beta + (1-\frac{7}{2}\xi)\widetilde{\gamma}_\gamma\big)$
3B	4F	5E	$-\frac{73-6\,\sqrt{5}}{152\,\pi^2}$	$-\frac{1}{19}\,\big(\frac{129}{4}\,\xi\,\widetilde{\gamma}_\alpha + 24\,\widetilde{\gamma}_\beta + \frac{215}{6}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big)$
3B	4F	5F	$-\frac{19-4\,\sqrt{5}}{48\,\pi^2}$	$\frac{1}{8}\,\xi\,\widetilde{\gamma}_\alpha - \frac{8}{3}\,\widetilde{\gamma}_\beta + \frac{1}{6}\,(1-\xi)\,\widetilde{\gamma}_\gamma$
3B	4F	5I	$+\frac{7-21\,\sqrt{5}}{176\,\pi^2}$	$-\frac{21}{11}\,\big(\frac{7}{4}\,\widetilde{\gamma}_\alpha - 2\,\xi\,\widetilde{\gamma}_\beta + \frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big)$
3B	4F	5J	$-\frac{13-2\,\sqrt{5}}{24\,\pi^2}$	$-\big(\frac{3}{4}\,\xi\,\widetilde{\gamma}_\alpha + \frac{8}{3}\,\widetilde{\gamma}_\beta + \frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big)$
3B	4F	5K	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\quad (\widetilde{\gamma}_\gamma = 0)$
3B	4G	5A	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5B	$-\frac{1}{4\,\pi^2}$	$-\frac{1}{2}\,\big(3\,\xi\,\widetilde{\gamma}_\alpha + 2\,(1-\xi)\,\widetilde{\gamma}_\gamma\big),\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5C	$-\frac{1}{4\,\pi^2}$	$-\frac{1}{2}\,\big(3\,\xi\,\widetilde{\gamma}_\alpha + \frac{8}{3}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big),\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5D	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5E	$-\frac{1}{4\,\pi^2}$	$-\frac{1}{2}\,\big(3\,\xi\,\widetilde{\gamma}_\alpha + \frac{10}{3}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big),\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5F	$-\frac{1}{4\,\pi^2}$	$-\frac{1}{2}\,\big(3\,\xi\,\widetilde{\gamma}_\alpha + 4\,(1-\xi)\,\widetilde{\gamma}_\gamma\big),\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5G	$-\frac{1}{4\,\pi^2}$	$-\frac{1}{2}\,\big(3\,\xi\,\widetilde{\gamma}_\alpha + 5\,(1-\xi)\,\widetilde{\gamma}_\gamma\big),\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5H	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5I	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{1}{2}\,\big(\frac{3}{2}\,\xi\,\widetilde{\gamma}_\alpha + 5\,(1-\xi)\,\widetilde{\gamma}_\gamma\big),\quad (\widetilde{\gamma}_\beta = 0)$
3B	4G	5K	$-\frac{1}{4\,\pi^2}$	$-\frac{3}{2}\,\widetilde{\gamma}_\alpha,\quad (\widetilde{\gamma}_\beta = \widetilde{\gamma}_\gamma = 0)$
3C	4A	5E	$-\frac{3}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma,\quad (\widetilde{\gamma}_\alpha = 0)$
3C	4A	5F	$-\frac{5}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma,\quad (\widetilde{\gamma}_\alpha = 0)$
3C	4A	5G	$-\frac{1}{4\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma,\quad (\widetilde{\gamma}_\alpha = 0)$
3C	4A	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma,\quad (\widetilde{\gamma}_\alpha = 0)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
3C	4A	5K	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
3C	4B	5C	$-\frac{435-130\sqrt{5}}{1888\,\pi^2}$	$-\frac{5}{59}\big(-26\,\widetilde{\gamma}_\beta + \frac{217}{6}\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4B	5E	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{6}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4B	5F	$-\frac{5}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4B	5G	$+\frac{1-3\sqrt{5}}{16\,\pi^2}$	$-2\,\big(3\,\widetilde{\gamma}_\beta - 5\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4B	5I	$-\frac{5+3\sqrt{5}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta + \frac{5}{4}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4B	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4B	5K	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta, \quad (\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
3C	4C	5A	$-\frac{1}{4\,\pi^2}$	$-\frac{4}{3}\,\widetilde{\gamma}_\beta, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4C	5B	$-\frac{1}{4\,\pi^2}$	$-\big(\frac{4}{3}\,\xi\,\widetilde{\gamma}_\beta + (1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4C	5D	$-\frac{1}{4\,\pi^2}$	$-\frac{4}{3}\,\widetilde{\gamma}_\beta, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4C	5E	$-\frac{3}{8\,\pi^2}$	$-\big(2\,\xi\,\widetilde{\gamma}_\beta + \frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4C	5G	$-\frac{1}{4\,\pi^2}$	$-\big(\frac{4}{3}\,\xi\,\widetilde{\gamma}_\beta + \frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4C	5H	$-\frac{1}{4\,\pi^2}$	$-\frac{4}{3}\,\widetilde{\gamma}_\beta, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4C	5I	$-\frac{1}{4\,\pi^2}$	$-\frac{4}{3}\,\widetilde{\gamma}_\beta, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4C	5J	$-\frac{1}{8\,\pi^2}$	$-\big(\frac{2}{3}\,\xi\,\widetilde{\gamma}_\beta + \frac{5}{2}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4D	5A	$-\frac{17+\sqrt{13}}{48\,\pi^2}$	$-\frac{5}{6}\,\big(2\,\widetilde{\gamma}_\beta + \widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4D	5C	$-\frac{17}{96\,\pi^2}$	$-\frac{17}{6}\,\big(\frac{1}{2}\,\xi\,\widetilde{\gamma}_\beta + \frac{1}{3}\,(1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4D	5F	$-\frac{1}{8\,\pi^2}$	$-\big(\xi\,\widetilde{\gamma}_\beta + (1-\xi)\,\widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4E	5C	$-\frac{15}{32\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4E	5D	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4E	5E	$-\frac{3}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4E	5F	$-\frac{5}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4E	5G	$-\frac{1}{4\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4D	5H	$-\frac{17-\sqrt{13}}{48\,\pi^2}$	$-\frac{5}{6}\,\big(2\,\widetilde{\gamma}_\beta + \widetilde{\gamma}_\gamma\big), \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4E	5I	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$
3C	4E	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma, \quad (\widetilde{\gamma}_\alpha = 0)$

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$	As function of scaling dim.
3C	4E	5K	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
3C	4F	5C	$-\frac{435+130\sqrt{5}}{1888\,\pi^2}$	$-\frac{5}{59}\big(-26\,\widetilde{\gamma}_\beta + \frac{217}{6}\,\widetilde{\gamma}_\gamma\big)$, $(\widetilde{\gamma}_\alpha = 0)$
3C	4F	5D	$-\frac{5-3\sqrt{5}}{32\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta + \frac{5}{4}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = 0)$
3C	4F	5E	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{6}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = 0)$
3C	4F	5F	$-\frac{5}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = 0)$
3C	4F	5G	$+\frac{1+3\sqrt{5}}{16\,\pi^2}$	$-2\,\big(3\,\widetilde{\gamma}_\beta - 5\,\widetilde{\gamma}_\gamma\big)$, $(\widetilde{\gamma}_\alpha = 0)$
3C	4F	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = 0)$
3C	4F	5K	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-2\,\widetilde{\gamma}_\beta$, $(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\gamma = 0)$
3C	4G	5C	$-\frac{15}{32\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$
3C	4G	5D	$-\frac{5+\sqrt{5}}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$
3C	4G	5F	$-\frac{5}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$
3C	4G	5G	$-\frac{1}{4\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$
3C	4G	5I	$-\frac{5-\sqrt{5}}{16\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$
3C	4G	5J	$-\frac{1}{8\,\pi^2}$	$-\frac{5}{2}\,\widetilde{\gamma}_\gamma$, $(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = 0)$
3C	4G	5K	0	$(\widetilde{\gamma}_\alpha = \widetilde{\gamma}_\beta = \widetilde{\gamma}_\gamma = 0)$

We can state a function of the form (6.57) for most of these operators but not for all of them.

In order to obtain the analytic results given above, we assumed the structure constants to take the form

$$C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma} = \frac{a + b\,\sqrt{\{5,\,13,\,41\}}}{c\,\pi^2} \tag{6.69}$$

with integers a, b and c and sought for values of these integers reproducing the numerical results. Regarding the anomalous dimensions, we see that for each of these values there should also appear the value where $-b$ is substituted for b. The appearance of this value can serve as a consistency check.

For some of the numerical results listed in appendix A.2 we could not determine the analytic form. For these the integers a, b and c respectively may be too large, or these results do not hold the structure (6.69) at all.

6.7 Three-Point Functions for BMN Operators

Let us now consider the two impurity BMN operators

$$\mathcal{B}^{J,n}_{(ij)} = N(J,n) \sum_{k=0}^{J} \cos \frac{\pi n(2k+1)}{J+1} \Big[\mathrm{Tr}\left(\phi^i Z^k \phi^j Z^{J-k}\right) - \frac{\delta^{ij}}{4} \sum_{l=3}^{6} \mathrm{Tr}\left(\phi^l Z^k \phi^l Z^{J-k}\right) \Big] \quad (6.70)$$

$$\mathcal{B}^{J,n}_{[ij]} = N(J,n) \sum_{k=0}^{J} \sin \frac{\pi n(2k+2)}{J+2} \mathrm{Tr}\left(\phi^i Z^k \phi^j Z^{J-k}\right) \quad (6.71)$$

$$\mathcal{B}^{J,n}_{\mathrm{tr}} = N(J,n) \sum_{k=0}^{J} \sum_{i=1}^{6} \cos \frac{\pi n(2k+3)}{J+3} \mathrm{Tr}\left(\phi^i Z^k \phi^i Z^{J-k}\right) \quad (6.72)$$

introduced in [7], keeping the convention $Z = \phi_1 + \mathrm{i}\phi_2$. In the table of section 6.6.1 we denoted to which anomalous dimension eigenspace these operators belong. They have anomalous dimensions

$$\gamma_{\mathcal{B}^{J,n}_{(ij)}} = \frac{1}{\pi^2} \sin^2 \frac{\pi n}{J+1} \quad (6.73)$$

$$\gamma_{\mathcal{B}^{J,n}_{[ij]}} = \frac{1}{\pi^2} \sin^2 \frac{\pi n}{J+2} \quad (6.74)$$

$$\gamma_{\mathcal{B}^{J,n}_{\mathrm{tr}}} = \frac{1}{\pi^2} \sin^2 \frac{\pi n}{J+3}, \quad (6.75)$$

and are already part of a diagonal basis. Thus, we can calculate their structure constants by hand.

6.7.1 Correlators with Length Two Operators

First, we consider three-point functions of two BMN operators of equal length with a length two operator. In order to obtain these, we have to find a diagonal basis for the length two operators. According to [7] the length two operators split in the Konishi operator

$$\mathcal{K} = \mathrm{Tr}\left(Z\overline{Z}\right) + \sum_{i=3}^{6} \mathrm{Tr}\left(\phi^i \phi^i\right), \qquad \gamma_{\mathcal{K}} = \frac{3}{4\pi^2} \quad (6.76)$$

and four classes of protected operators that we denote by

$$\mathcal{O}_a = \operatorname{Tr}(ZZ) \tag{6.77}$$

$$\mathcal{O}_b = \sum_{i=3}^{6} \operatorname{Tr}\left(\phi^i\phi^i\right) - 2\operatorname{Tr}\left(Z\overline{Z}\right) \tag{6.78}$$

$$\mathcal{O}_c^i = \operatorname{Tr}\left(Z\phi^i\right) \qquad (i = 3\ldots 6) \tag{6.79}$$

$$\mathcal{O}_d^{(ij)} = \operatorname{Tr}\left(\phi^i\phi^j\right) - \frac{\delta^{ij}}{4}\sum_{k=3}^{6}\operatorname{Tr}\left(\phi^k\phi^k\right) \qquad (i = 3\ldots 6). \tag{6.80}$$

Together with their complex conjugates $\overline{\mathcal{O}}_a$ and $\overline{\mathcal{O}}_c^i$, these are the twenty operators with $\gamma = 0$, i. e. the operators of the class 2B.

One Konishi and Two BMN Operators Let us first calculate some correlators with the Konishi operator:

$$C^{(1)}_{\mathcal{K}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{2,0}_{(ij)}} = -\frac{3}{8\pi^2}\, C^{(0)}_{\mathcal{K}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{2,0}_{(ij)}} \tag{6.81}$$

$$C^{(1)}_{\mathcal{K}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,1}_{(ij)}} = -\frac{3}{4\pi^2}\, C^{(0)}_{\mathcal{K}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,1}_{(ij)}} \tag{6.82}$$

$$C^{(1)}_{\mathcal{K}\mathcal{B}^{3,0}_{(ij)}\overline{\mathcal{B}}^{3,0}_{(ij)}} = -\frac{3}{8\pi^2}\, C^{(0)}_{\mathcal{K}\mathcal{B}^{3,0}_{(ij)}\overline{\mathcal{B}}^{3,0}_{(ij)}} \tag{6.83}$$

$$C^{(1)}_{\mathcal{K}\mathcal{B}^{3,1}_{(ij)}\overline{\mathcal{B}}^{3,1}_{(ij)}} = -\frac{23}{40\pi^2}\, C^{(0)}_{\mathcal{K}\mathcal{B}^{3,1}_{(ij)}\overline{\mathcal{B}}^{3,1}_{(ij)}} \tag{6.84}$$

$$C^{(1)}_{\mathcal{K}\mathcal{B}^{4,0}_{(ij)}\overline{\mathcal{B}}^{4,0}_{(ij)}} = -\frac{3}{8\pi^2}\, C^{(0)}_{\mathcal{K}\mathcal{B}^{4,0}_{(ij)}\overline{\mathcal{B}}^{4,0}_{(ij)}} \tag{6.85}$$

$$C^{(1)}_{\mathcal{K}\mathcal{B}^{4,1}_{(ij)}\overline{\mathcal{B}}^{4,1}_{(ij)}} = -\frac{14-\sqrt{5}}{24\pi^2}\, C^{(0)}_{\mathcal{K}\mathcal{B}^{4,1}_{(ij)}\overline{\mathcal{B}}^{4,1}_{(ij)}} \tag{6.86}$$

$$C^{(1)}_{\mathcal{K}\mathcal{B}^{4,2}_{(ij)}\overline{\mathcal{B}}^{4,2}_{(ij)}} = -\frac{14+\sqrt{5}}{24\pi^2}\, C^{(0)}_{\mathcal{K}\mathcal{B}^{4,2}_{(ij)}\overline{\mathcal{B}}^{4,2}_{(ij)}} \tag{6.87}$$

These results are in complete accordance with our general result (6.39).[2]

[2] Note that Okuyama and Tseng come to a different result in equation (5.26) of [40]. While our results for $n = 0$ are in accordance with theirs all others are not.

One Protected and Two BMN Operators For the protected length two operators, only the operators $\mathcal{O}_b$ and $\mathcal{O}_d^{(ij)}$ give non-zero results in correlators $\langle \mathcal{O}_{a,b,c,d}\,\mathrm{BMN}\,\overline{\mathrm{BMN}}\rangle$. These are the structure constants for all non-vanishing three-point functions that can be built from the $J=1$ and $J=2$ operators:

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{1,0}_{(ij)}\overline{\mathcal{B}}^{1,0}_{(ij)}} = 0 \tag{6.88}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{1,0}_{(ij)}\overline{\mathcal{B}}^{1,1}_{\mathrm{tr}}} = -\frac{1}{4\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{1,0}_{(ij)}\overline{\mathcal{B}}^{1,1}_{\mathrm{tr}}} \tag{6.89}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{1,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{1,1}_{\mathrm{tr}}} = -\frac{1}{6\pi^2}\,C^{(0)}_{\mathcal{O}_b\mathcal{B}^{1,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{1,1}_{\mathrm{tr}}} \tag{6.90}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{2,0}_{(ij)}} = 0 \tag{6.91}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{2,0}_{(ij)}} = 0 \tag{6.92}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{2,1}_{\mathrm{tr}}} = -\frac{5-\sqrt{5}}{16\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{2,1}_{\mathrm{tr}}} \tag{6.93}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{2,2}_{\mathrm{tr}}} = -\frac{5+\sqrt{5}}{16\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{2,2}_{\mathrm{tr}}} \tag{6.94}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,1}_{(ij)}} = -\frac{3}{8\pi^2}\,C^{(0)}_{\mathcal{O}_b\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,1}_{(ij)}} \tag{6.95}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,1}_{(ij)}} = -\frac{3}{8\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,1}_{(ij)}} \tag{6.96}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,1}_{\mathrm{tr}}} = -\frac{5-\sqrt{5}}{16\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,1}_{\mathrm{tr}}} \tag{6.97}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,2}_{\mathrm{tr}}} = -\frac{5+\sqrt{5}}{16\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{2,2}_{\mathrm{tr}}} \tag{6.98}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{2,1}_{[ij]}\overline{\mathcal{B}}^{2,1}_{[ij]}} = +\frac{1}{16\pi^2}\,C^{(0)}_{\mathcal{O}_b\mathcal{B}^{2,1}_{[ij]}\overline{\mathcal{B}}^{2,1}_{[ij]}} \tag{6.99}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,1}_{[ij]}\overline{\mathcal{B}}^{2,1}_{[ij]}} = -\frac{3}{8\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{2,1}_{[ij]}\overline{\mathcal{B}}^{2,1}_{[ij]}} \tag{6.100}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{2,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{2,1}_{\mathrm{tr}}} = -\frac{115-14\sqrt{5}}{632\pi^2}\,C^{(0)}_{\mathcal{O}_b\mathcal{B}^{2,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{2,1}_{\mathrm{tr}}} \tag{6.101}$$

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{2,1}_{\text{tr}} \overline{\mathcal{B}}^{2,2}_{\text{tr}}} = 0 \tag{6.102}$$

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{2,2}_{\text{tr}} \overline{\mathcal{B}}^{2,2}_{\text{tr}}} = -\frac{115 + 14\sqrt{5}}{632\pi^2} \, C^{(0)}_{\mathcal{O}_b \mathcal{B}^{2,2}_{\text{tr}} \overline{\mathcal{B}}^{2,2}_{\text{tr}}} \tag{6.103}$$

For $J = 3$ and $J = 4$ we calculate only some correlation functions with two equal BMN operators. We obtain:

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{3,0}_{(ij)} \overline{\mathcal{B}}^{3,0}_{(ij)}} = 0 \tag{6.104}$$

$$C^{(1)}_{\mathcal{O}^{(ij)}_d \mathcal{B}^{3,0}_{(ij)} \overline{\mathcal{B}}^{3,0}_{(ij)}} = 0 \tag{6.105}$$

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{3,1}_{(ij)} \overline{\mathcal{B}}^{3,1}_{(ij)}} = -\frac{1}{8\pi^2} \, C^{(0)}_{\mathcal{O}_b \mathcal{B}^{3,1}_{(ij)} \overline{\mathcal{B}}^{3,1}_{(ij)}} \tag{6.106}$$

$$C^{(1)}_{\mathcal{O}^{(ij)}_d \mathcal{B}^{3,1}_{(ij)} \overline{\mathcal{B}}^{3,1}_{(ij)}} = -\frac{1}{4\pi^2} \, C^{(0)}_{\mathcal{O}^{(ij)}_d \mathcal{B}^{3,1}_{(ij)} \overline{\mathcal{B}}^{3,1}_{(ij)}} \tag{6.107}$$

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{3,1}_{[ij]} \overline{\mathcal{B}}^{3,1}_{[ij]}} = -\frac{7 - 3\sqrt{5} - 2\sqrt{2(3-\sqrt{5})}}{16\pi^2\,(5-\sqrt{5})} \, C^{(0)}_{\mathcal{O}_b \mathcal{B}^{3,1}_{[ij]} \overline{\mathcal{B}}^{3,1}_{[ij]}} \tag{6.108}$$

$$C^{(1)}_{\mathcal{O}^{(ij)}_d \mathcal{B}^{3,1}_{[ij]} \overline{\mathcal{B}}^{3,1}_{[ij]}} = -\frac{3(5-\sqrt{5}) - 2\sqrt{2(3-\sqrt{5})}}{8\pi^2\,(5-\sqrt{5})} \, C^{(0)}_{\mathcal{O}^{(ij)}_d \mathcal{B}^{3,1}_{[ij]} \overline{\mathcal{B}}^{3,1}_{[ij]}} \tag{6.109}$$

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{3,2}_{[ij]} \overline{\mathcal{B}}^{3,2}_{[ij]}} = +\frac{67 - 31\sqrt{5} - 16\sqrt{2(3-\sqrt{5})}}{128\pi^2\,(5-\sqrt{5})} \, C^{(0)}_{\mathcal{O}_b \mathcal{B}^{3,2}_{[ij]} \overline{\mathcal{B}}^{3,2}_{[ij]}} \tag{6.110}$$

$$C^{(1)}_{\mathcal{O}^{(ij)}_d \mathcal{B}^{3,2}_{[ij]} \overline{\mathcal{B}}^{3,2}_{[ij]}} = -\frac{3(5-\sqrt{5}) + 2\sqrt{2(3-\sqrt{5})}}{8\pi^2\,(5-\sqrt{5})} \, C^{(0)}_{\mathcal{O}^{(ij)}_d \mathcal{B}^{3,2}_{[ij]} \overline{\mathcal{B}}^{3,2}_{[ij]}} \tag{6.111}$$

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{3,1}_{\text{tr}} \overline{\mathcal{B}}^{3,1}_{\text{tr}}} = -\frac{5}{56\pi^2} \, C^{(0)}_{\mathcal{O}_b \mathcal{B}^{3,1}_{\text{tr}} \overline{\mathcal{B}}^{3,1}_{\text{tr}}} \tag{6.112}$$

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{3,2}_{\text{tr}} \overline{\mathcal{B}}^{3,2}_{\text{tr}}} = -\frac{3}{40\pi^2} \, C^{(0)}_{\mathcal{O}_b \mathcal{B}^{3,2}_{\text{tr}} \overline{\mathcal{B}}^{3,2}_{\text{tr}}} \tag{6.113}$$

$$C^{(1)}_{\mathcal{O}_b \mathcal{B}^{4,0}_{(ij)} \overline{\mathcal{B}}^{4,0}_{(ij)}} = 0 \tag{6.114}$$

$$C^{(1)}_{\mathcal{O}^{(ij)}_d \mathcal{B}^{4,0}_{(ij)} \overline{\mathcal{B}}^{4,0}_{(ij)}} = 0 \tag{6.115}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{4,1}_{(ij)}\overline{\mathcal{B}}^{4,1}_{(ij)}} = -\frac{5-\sqrt{5}}{48\pi^2}\,C^{(0)}_{\mathcal{O}_b\mathcal{B}^{4,1}_{(ij)}\overline{\mathcal{B}}^{4,1}_{(ij)}} \tag{6.116}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{4,1}_{(ij)}\overline{\mathcal{B}}^{4,1}_{(ij)}} = -\frac{5-\sqrt{5}}{16\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{4,1}_{(ij)}\overline{\mathcal{B}}^{4,1}_{(ij)}} \tag{6.117}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{4,2}_{(ij)}\overline{\mathcal{B}}^{4,2}_{(ij)}} = -\frac{5+\sqrt{5}}{48\pi^2}\,C^{(0)}_{\mathcal{O}_b\mathcal{B}^{4,2}_{(ij)}\overline{\mathcal{B}}^{4,2}_{(ij)}} \tag{6.118}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{4,2}_{(ij)}\overline{\mathcal{B}}^{4,2}_{(ij)}} = -\frac{5+\sqrt{5}}{16\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{4,2}_{(ij)}\overline{\mathcal{B}}^{4,2}_{(ij)}} \tag{6.119}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{4,1}_{[ij]}\overline{\mathcal{B}}^{4,1}_{[ij]}} = +\frac{1}{48\pi^2}\,C^{(0)}_{\mathcal{O}_b\mathcal{B}^{4,1}_{[ij]}\overline{\mathcal{B}}^{4,1}_{[ij]}} \tag{6.120}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{4,1}_{[ij]}\overline{\mathcal{B}}^{4,1}_{[ij]}} = -\frac{3}{16\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{4,1}_{[ij]}\overline{\mathcal{B}}^{4,1}_{[ij]}} \tag{6.121}$$

$$C^{(1)}_{\mathcal{O}_b\mathcal{B}^{4,2}_{[ij]}\overline{\mathcal{B}}^{4,2}_{[ij]}} = -\frac{1}{16\pi^2}\,C^{(0)}_{\mathcal{O}_b\mathcal{B}^{4,2}_{[ij]}\overline{\mathcal{B}}^{4,2}_{[ij]}} \tag{6.122}$$

$$C^{(1)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{4,2}_{[ij]}\overline{\mathcal{B}}^{4,2}_{[ij]}} = -\frac{7}{16\pi^2}\,C^{(0)}_{\mathcal{O}_d^{(ij)}\mathcal{B}^{4,2}_{[ij]}\overline{\mathcal{B}}^{4,2}_{[ij]}} \tag{6.123}$$

As we see, we get different results for three-point functions of $\mathcal{O}_b$ and $\mathcal{O}_d^{(ij)}$ respectively with the same BMN operators. The protected length two operators, i.e. the operators of class 2B, are thus an example of a class of operators of the same anomalous dimension that hold an additional degeneracy.

6.7.2 Correlators of Three BMN Operators

In addition to the three-point functions with length two operators, the correlators

$$\left\langle \mathcal{B}^{J_1,n_1}_{\dots}(x_1)\mathcal{B}^{J_2,n_2}_{\dots}(x_2)\overline{\mathcal{B}}^{J_3,n_3}_{\dots}(x_3)\right\rangle \tag{6.124}$$

with $J_1+J_2=J_3$ are generally non-zero. As we have seen in the previous section, the results obtained for the BMN operators are in accordance with those that we calculated numerically. We therefore focus on three-point functions of operators for which the numerical calculations gave no definite results, as well as those that we needed to interpret the numerical results.

These are the results:

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{1,0}_{(ij)}\overline{\mathcal{B}}^{2,1}_{\mathrm{tr}}} = -\frac{5-\sqrt{5}}{16\pi^2}\, C^{(0)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{1,0}_{(ij)}\overline{\mathcal{B}}^{2,1}_{\mathrm{tr}}} \tag{6.125}$$

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{3,0}_{(ij)}} = 0 \tag{6.126}$$

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{3,1}_{\mathrm{tr}}} = -\frac{1}{8\pi^2}\, C^{(0)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,0}_{(ij)}\overline{\mathcal{B}}^{3,1}_{\mathrm{tr}}} \tag{6.127}$$

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{[ij]}\overline{\mathcal{B}}^{3,1}_{[ij]}} = -\frac{7-\sqrt{5}}{16\pi^2}\, C^{(0)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{[ij]}\overline{\mathcal{B}}^{3,1}_{[ij]}} \tag{6.128}$$

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{[ij]}\overline{\mathcal{B}}^{3,2}_{[ij]}} = -\frac{7+\sqrt{5}}{16\pi^2}\, C^{(0)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{[ij]}\overline{\mathcal{B}}^{3,2}_{[ij]}} \tag{6.129}$$

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,0}_{(ij)}} = -\frac{5-\sqrt{5}}{16\pi^2}\, C^{(0)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,0}_{(ij)}} \tag{6.130}$$

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,1}_{(ij)}} = +\frac{1+3\sqrt{5}}{16\pi^2}\, C^{(0)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,1}_{(ij)}} \tag{6.131}$$

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,2}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,0}_{(ij)}} = -\frac{5+\sqrt{5}}{16\pi^2}\, C^{(0)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,2}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,0}_{(ij)}} \tag{6.132}$$

$$C^{(1)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,2}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,1}_{(ij)}} = +\frac{1-3\sqrt{5}}{16\pi^2}\, C^{(0)}_{\mathcal{B}^{1,0}_{(ij)}\mathcal{B}^{2,2}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,1}_{(ij)}} \tag{6.133}$$

$$C^{(1)}_{\mathcal{B}^{1,1}_{\mathrm{tr}}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{3,0}_{(ij)}} = -\frac{1}{4\pi^2}\, C^{(0)}_{\mathcal{B}^{1,1}_{\mathrm{tr}}\mathcal{B}^{2,1}_{(ij)}\overline{\mathcal{B}}^{3,0}_{(ij)}} \tag{6.134}$$

$$C^{(1)}_{\mathcal{B}^{1,1}_{\mathrm{tr}}\mathcal{B}^{2,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,2}_{\mathrm{tr}}} = -\frac{73-6\sqrt{5}}{152\pi^2}\, C^{(0)}_{\mathcal{B}^{1,1}_{\mathrm{tr}}\mathcal{B}^{2,1}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,2}_{\mathrm{tr}}} \tag{6.135}$$

$$C^{(1)}_{\mathcal{B}^{1,1}_{\mathrm{tr}}\mathcal{B}^{2,2}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,2}_{\mathrm{tr}}} = -\frac{73+6\sqrt{5}}{152\pi^2}\, C^{(0)}_{\mathcal{B}^{1,1}_{\mathrm{tr}}\mathcal{B}^{2,2}_{\mathrm{tr}}\overline{\mathcal{B}}^{3,2}_{\mathrm{tr}}} \tag{6.136}$$

These results add to the numerical results obtained in the previous section. In principle, we could calculate three-point functions for BMN operators of arbitrary length, as long as $J_1 + J_2 = J_3$, but for longer operators the calculations get more time-consuming.

Chapter 7

All's Well That Ends Well

7.1 Summary and Conclusion

Let us begin our conclusion with what we did *not* find. We were not able to state a general form of the one-loop structure constants for arbitrary operators, for example in terms of the anomalous dimensions. Nevertheless we identified such a structure for at least some special classes of three-point functions.

As we know from [40] and explained in section 4.3.1, the structure constants for extremal correlators take the simple form

$$C^{(1)}_{\alpha\beta\gamma} = \frac{1}{2}\left(\gamma_\alpha + \gamma_\beta - \gamma_\gamma\right) C^{(0)}_{\alpha\beta\gamma}. \tag{7.1}$$

The $SU(2)$ operators in section 6.4 yielded the structure constants

$$C^{(1)}_{\alpha\beta\gamma} = -\frac{1}{2}\left(\gamma_\alpha + \gamma_\beta + \gamma_\gamma\right) C^{(0)}_{\alpha\beta\gamma}. \tag{7.2}$$

For three-point functions with a Konishi operator we found that

$$C^{(1)}_{\alpha\beta\mathcal{K}} = -\left(\frac{\gamma_\alpha}{{\Delta^{(0)}}_\alpha} + \frac{\gamma_\beta}{{\Delta^{(0)}}_\beta} + \frac{\gamma_\mathcal{K}}{{\Delta^{(0)}}_\mathcal{K}}\right) C^{(0)}_{\alpha\beta\mathcal{K}}. \tag{7.3}$$

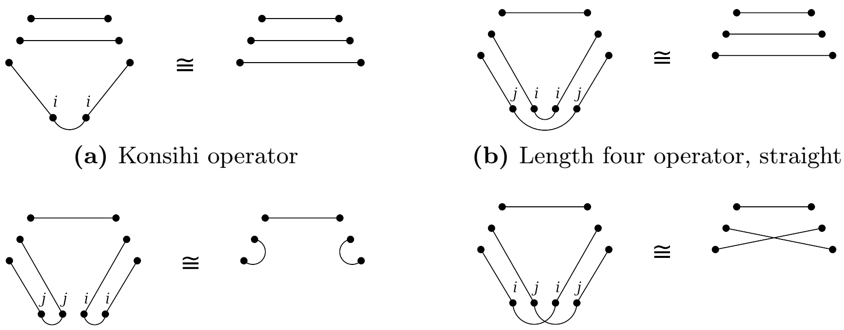
According to equation (6.22), the same relation holds for correlators of only length two operators of a diagonal basis.

Most of the results we obtained by the numerical calculations in section 6.6 seem to point in a similar direction, as for example the three-point function of two operators of the class 3A with a 4A operator whose structure constant could be written as

$$C^{(1)}_{3\mathrm{A},\,3\mathrm{A},\,4\mathrm{A}} = -\left(\frac{1}{2}\,\frac{\gamma_{3\mathrm{A}}}{{\Delta^{(0)}}_{3\mathrm{A}}} + \frac{1}{2}\,\frac{\gamma_{3\mathrm{A}}}{{\Delta^{(0)}}_{3\mathrm{A}}} + 2\,\frac{\gamma_{4\mathrm{A}}}{{\Delta^{(0)}}_{4\mathrm{A}}}\right) C^{(0)}_{3\mathrm{A},\,3\mathrm{A},\,4\mathrm{A}}. \tag{7.4}$$

Therefore, we can write all these structure constants as functions of only the tree-level and one-loop scaling dimensions, as well as the tree-level structure constants of the corresponding operators.

On the other hand, the anomalous dimension and structure constant of



(a) Konsihi operator

(b) Length four operator, straight

(c) Length four operator, self-contraction

(d) Length four operator, crossed

Figure 7.1: While the index contractions of the Konishi operator always yield straight 2-gon contractions, they yield also self-contractions and crossed contractions for the length four singlets.

three operators of the class 4A are

$$\gamma_{4\text{A}} = \frac{13+\sqrt{41}}{16\,\pi^2} \qquad \text{and}$$

$$C^{(1)}_{4\text{A},\,4\text{A},\,4\text{A}} = -\frac{7185+309\,\sqrt{41}}{11728\,\pi^2}\,C^{(0)}_{4\text{A},\,4\text{A},\,4\text{A}}, \tag{7.5}$$

which cannot be written as any function of $\gamma_{4\text{A}}$ holding a structure like (7.4). Results like this as well as the fact that for some operators there seems to be an additional degeneracy of the one-loop scaling dimensions that is broken by the three-point functions, suggest that there are contributions to the one-loop structure constants that do not only depend on the anomalous dimensions.

7.2 Outlook

In the long term one would of course like to find an easy way of determining the three-point functions of $\mathcal{N}=4$ super Yang-Mills and—going even further—also regard n-point functions with $n>3$.

It might be an interesting step into this direction to try to generalise the proof from section 6.3 for the Konishi operator to longer singlet operators, starting with the length four singlets 4A and 4E. The indices of these operators have to be fully contracted. The crucial point for safeguarding that the structure constants simplify for the Konishi operator is that the index contraction corresponds to a straight contraction of the vectors attached to the Konishi operator as depicted in figure 7.1 (a).

For the length four operators that we wrote down explicitly in equation (6.68), we obtain two contributions, proportional to the traces $\mathrm{Tr}\left(\phi^i\phi^i\phi^j\phi^j\right)$ and $\mathrm{Tr}\left(\phi^i\phi^j\phi^i\phi^j\right)$ respectively. If we take all permutations into account, we are left with three different ways of contracting the indices that are depicted in figures 7.1 (b)–(d). These correspond to straight contractions as well as crossed contractions and self-contractions. Applying the 2-gon dressing to obtain the one-loop structure constants then, on one hand, yields terms which hold two crossings or self-contractions. These look like two-loop order contributions, but there are no next-to-nearest-neighbour contractions in this picture.

On the other hand, as described in section 4.3.2, we could also omit the 2-gons and consider only 3-gon dressings. We would then obtain *only* next-to-nearest-neighbour contractions and no diagrams with two separate crossings or self-contractions.

Therefore, it may be possible to express the one-loop structure constants for length $2s$ singlets by the scaling dimensions up to s loops. This—if it should turn out to be correct—could also explain why the additional degeneracy of the two-point functions at one-loop level is broken by the one-loop three-point functions.

It might also be instructive considering three-point functions similar to those that reveal a simple structure. Among these are for example near-extremal three-point functions, although, unfortunately, our calculations for some of these structure constants seem to show no simple structure at all.

In order to proceed to four-point functions, one could try to extend the dressing formulae from section 4.2 and in this way obtain a diagrammatic description as for the three-point functions.

Anyway, we notice that there remains a lot of work to be done.

Appendix A

Results

A.1 Calculations for Twisted Operators

We list the results of the calculations for the twisted operators from section 5.2 here.

A.1.1 Length Two Operators

Two-Point Functions

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(B^2\right)(x_2)\ \right\rangle = 0 \tag{A.1}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC\right)(x_2)\ \right\rangle = 0 \tag{A.2}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BV_a\right)(x_2)\ \right\rangle = 0 \tag{A.3}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \right\rangle = \frac{1}{2\pi^4 x_{12}^4} \tag{A.4}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(CV_a\right)(x_2)\ \right\rangle = 0 \tag{A.5}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(V_aV_b\right)(x_2)\ \right\rangle = 0 \tag{A.6}$$

$$\left\langle \mathrm{Tr}\left(BC\right)(x_1)\ \mathrm{Tr}\left(BC\right)(x_2)\ \right\rangle = \frac{1}{4\pi^4 x_{12}^4}\left(1+\frac{\lambda}{4\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right) \tag{A.7}$$

$$\left\langle \mathrm{Tr}\left(BC\right)(x_1)\ \mathrm{Tr}\left(BV_a\right)(x_2)\ \right\rangle = 0 \tag{A.8}$$

$$\left\langle \mathrm{Tr}\left(BC\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \right\rangle = -\frac{1}{2\pi^4 x_{12}^2} \tag{A.9}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\,(CV_a)\,(x_2)\ \right\rangle = \frac{x_{12}\cdot a}{4\pi^4 x_{12}^4} \tag{A.10}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\,(V_aV_b)\,(x_2)\ \right\rangle = \frac{\lambda\, a\cdot b}{32\pi^6 x_{12}^4}\Big(\ln\frac{\varepsilon^2}{x_{12}^2}-1\Big) \tag{A.11}$$

$$\left\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(BV_b)\,(x_2)\ \right\rangle = 0 \tag{A.12}$$

$$\left\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \right\rangle = -\frac{x_{12}\cdot a}{2\pi^4 x_{12}^4} \tag{A.13}$$

$$\left\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(CV_b)\,(x_2)\ \right\rangle = \frac{a\cdot b}{8\pi^4 x_{12}^4} \tag{A.14}$$

$$\left\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(V_bV_c)\,(x_2)\ \right\rangle = 0 \tag{A.15}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \right\rangle = \frac{1}{2\pi^4} \tag{A.16}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\,(CV_a)\,(x_2)\ \right\rangle = -\frac{x_{12}\cdot a}{2\pi^4 x_{12}^2} \tag{A.17}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\,(V_aV_b)\,(x_2)\ \right\rangle = \frac{x_{12}\cdot a\, x_{12}\cdot b}{2\pi^4 x_{12}^4} \tag{A.18}$$

$$\left\langle \mathrm{Tr}\,(CV_a)\,(x_1)\ \mathrm{Tr}\,(CV_b)\,(x_2)\ \right\rangle = -\frac{a\cdot b\, x_{12}^2 + 2x_{12}\cdot a\, x_{12}\cdot b}{8\pi^4 x_{12}^4} \tag{A.19}$$

$$\left\langle \mathrm{Tr}\,(CV_a)\,(x_1)\ \mathrm{Tr}\,(V_bV_c)\,(x_2)\ \right\rangle = \frac{x_{12}\cdot b\, a\cdot c + x_{12}\cdot c\, a\cdot b}{8\pi^4 x_{12}^4} \tag{A.20}$$

$$\begin{aligned}\left\langle \mathrm{Tr}\,(V_aV_b)\,(x_1)\ \mathrm{Tr}\,(V_cV_d)\,(x_2)\ \right\rangle &= \frac{1}{16\pi^4 x_{12}^4}\Big(a\cdot c\, b\cdot d + a\cdot d\, b\cdot c \\ &+ \frac{\lambda\, a\cdot b\, c\cdot d}{4\pi^2}\Big(\ln\frac{\varepsilon^2}{x_{12}^2}-1\Big)\Big)\end{aligned} \tag{A.21}$$

Three-Point Functions

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(B^2\right)(x_2)\ \mathrm{anything}(x_3)\right\rangle = 0 \tag{A.22}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC\right)(x_2)\ \mathrm{Tr}\left(BC\right)(x_3)\right\rangle = 0 \tag{A.23}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC\right)(x_2)\ \mathrm{Tr}\left(BV_a\right)(x_3)\right\rangle = 0 \tag{A.24}$$

$$\begin{aligned}\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC\right)(x_2)\ \mathrm{Tr}\left(C^2\right)(x_3)\right\rangle &= \frac{1}{2\pi^6 x_{12}^2 x_{13}^2 x_{23}^2}\Big(1 + \frac{\lambda}{8\pi^2} \\ &\times \left(\ln \frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 2\right)\Big)\end{aligned} \tag{A.25}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC\right)(x_2)\ \mathrm{Tr}\left(CV_a\right)(x_3)\right\rangle = 0 \tag{A.26}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC\right)(x_2)\ \mathrm{Tr}\left(V_aV_b\right)(x_3)\right\rangle = 0 \tag{A.27}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BV_a\right)(x_2)\ \mathrm{anything}(x_3)\right\rangle = 0 \tag{A.28}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\left(C^2\right)(x_3)\right\rangle = -\frac{1}{\pi^6\, x_{12}^2 x_{13}^2} \tag{A.29}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\left(CV_a\right)(x_3)\right\rangle = \frac{x_{23}\cdot a}{2\pi^6 x_{12}^2 x_{13}^2 x_{23}^2} \tag{A.30}$$

$$\begin{aligned}\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\left(V_aV_b\right)(x_3)\right\rangle &= 0 + \frac{\lambda}{32\pi^8}\,\frac{a\cdot b}{x_{12}^2 x_{13}^2 x_{23}^2} \\ &\times \left(\ln \frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2\right)\end{aligned} \tag{A.31}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(CV_a\right)(x_2)\ \mathrm{Tr}\left(CV_b\right)(x_3)\right\rangle = \frac{1}{8\pi^6}\,\frac{a\cdot b}{x_{12}^2 x_{13}^2 x_{23}^2} \tag{A.32}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(CV_a\right)(x_2)\ \mathrm{Tr}\left(V_bV_c\right)(x_3)\right\rangle = 0 \tag{A.33}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(V_aV_b\right)(x_2)\ \mathrm{Tr}\left(V_cV_d\right)(x_3)\right\rangle = 0 \tag{A.34}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BC\right)\left(x_2\right)\,\mathrm{Tr}\left(BC\right)\left(x_3\right)\right\rangle = \frac{1}{4\pi^6\,x_{12}^2x_{13}^2x_{23}^2}\Big(1+\frac{\lambda}{8\pi^2}\Big(\ln\frac{\varepsilon^2}{x_{12}^2}$$

$$+\ln\frac{\varepsilon^2}{x_{13}^2}+\ln\frac{\varepsilon^2}{x_{23}^2}-6\Big)\Big) \tag{A.35}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BC\right)\left(x_2\right)\,\mathrm{Tr}\left(BV_a\right)\left(x_3\right)\right\rangle = 0 \tag{A.36}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BC\right)\left(x_2\right)\,\mathrm{Tr}\left(C^2\right)\left(x_3\right)\right\rangle = -\frac{1}{4\pi^6}\frac{x_{12}^2+x_{13}^2+x_{23}^2}{x_{12}^2x_{13}^2x_{23}^2}\Big(1+\frac{\lambda}{4\pi^2}$$

$$\times\frac{1}{x_{12}^2+x_{13}^2+x_{23}^2}\Big[x_{13}^2\Big(\ln\frac{\varepsilon^2x_{13}^2}{x_{12}^2x_{23}^2}-2\Big)+x_{23}^2\Big(\ln\frac{\varepsilon^2x_{23}^2}{x_{12}^2x_{13}^2}-2\Big)\Big]\Big) \tag{A.37}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BC\right)\left(x_2\right)\,\mathrm{Tr}\left(CV_a\right)\left(x_3\right)\right\rangle = \frac{1}{8\pi^6\,x_{12}^2x_{13}^2x_{23}^2}\Big((x_{13}+x_{23})\cdot a$$

$$+\frac{\lambda}{4\pi^2}\Big[x_{23}\cdot a\Big(\ln\frac{\varepsilon^2x_{23}^2}{x_{12}^2x_{13}^2}-2\Big)+x_{13}\cdot a\Big(\ln\frac{\varepsilon^2x_{13}^2}{x_{12}^2x_{23}^2}-2\Big)\Big]\Big) \tag{A.38}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BC\right)\left(x_2\right)\,\mathrm{Tr}\left(V_aV_b\right)\left(x_3\right)\right\rangle = 0+\frac{\lambda}{64\pi^8}\,\frac{a\cdot b}{x_{12}^2x_{13}^2x_{23}^2}$$

$$\times\Big(\ln\frac{\varepsilon^2x_{12}^2}{x_{13}^2x_{23}^2}-2\Big) \tag{A.39}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BV_a\right)\left(x_2\right)\,\mathrm{Tr}\left(BV_b\right)\left(x_3\right)\right\rangle = 0 \tag{A.40}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BV_a\right)\left(x_2\right)\,\mathrm{Tr}\left(C^2\right)\left(x_3\right)\right\rangle = \frac{1}{4\pi^6x_{12}^2x_{13}^2x_{23}^2}\Big((x_{12}-x_{23})\cdot a$$

$$-\frac{\lambda}{4\pi^2}x_{23}\cdot a\Big(\ln\frac{\varepsilon^2x_{23}^2}{x_{12}^2x_{13}^2}-2\Big)\Big) \tag{A.41}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BV_a\right)\left(x_2\right)\,\mathrm{Tr}\left(CV_b\right)\left(x_3\right)\right\rangle = \frac{a\cdot b}{16\pi^6\,x_{12}^2x_{13}^2x_{23}^2}\Big(1+\frac{\lambda}{4\pi^2}$$

$$\times\Big(\ln\frac{\varepsilon^2x_{23}^2}{x_{12}^2x_{13}^2}-2\Big)\Big) \tag{A.42}$$

$$\left\langle \mathrm{Tr}\left(BC\right)\left(x_1\right)\,\mathrm{Tr}\left(BV_a\right)\left(x_2\right)\,\mathrm{Tr}\left(V_bV_c\right)\left(x_3\right)\right\rangle = 0 \tag{A.43}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\;\mathrm{Tr}\left(C^2\right)(x_2)\;\mathrm{Tr}\left(C^2\right)(x_3)\right\rangle = \frac{1}{2\pi^6\,x_{12}^2 x_{13}^2}\Big(x_{12}^2 + x_{13}^2 + \frac{\lambda}{8\pi^2}\,x_{23}^2 \times \Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big)\Big) \tag{A.44}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\;\mathrm{Tr}\left(C^2\right)(x_2)\;\mathrm{Tr}\,(CV_a)\,(x_3)\right\rangle = -\frac{1}{4\pi^6\,x_{12}^2 x_{13}^2 x_{23}^2}\Big[\Big(x_{12}^2 + x_{13}^2\Big)x_{23} + x_{23}^2 x_{13} + \frac{\lambda}{4\pi^2}x_{23}^2 x_{23}\Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big)\Big]\cdot a \tag{A.45}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\;\mathrm{Tr}\left(C^2\right)(x_2)\;\mathrm{Tr}\,(V_aV_b)\,(x_3)\right\rangle = \frac{1}{4\pi^6\,x_{12}^2 x_{13}^2 x_{23}^2}\Big(\Big(x_{13}\cdot a\,x_{23}\cdot b + x_{13}\cdot b\,x_{23}\cdot a\Big) + \frac{\lambda}{4\pi^2}\Big[x_{23}\cdot a\,x_{23}\cdot b\,\Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big) - \frac{a\cdot b}{2}\,x_{12}^2\,\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2\Big)\Big]\Big) \tag{A.46}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\;\mathrm{Tr}\,(CV_a)\,(x_2)\;\mathrm{Tr}\,(CV_b)\,(x_3)\right\rangle = -\frac{1}{16\pi^6\,x_{12}^2 x_{13}^2 x_{23}^2}\Big((x_{12}^2 + x_{13}^2) \times a\cdot b + 2x_{23}\cdot a\,x_{13}\cdot b - 2x_{12}\cdot a\,x_{23}\cdot b + \frac{\lambda}{4\pi^2}\Big(a\cdot b\,x_{23}^2 + 2x_{23}\cdot a\,x_{23}\cdot b \times \Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big) \tag{A.47}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\;\mathrm{Tr}\,(CV_a)\,(x_2)\;\mathrm{Tr}\,(V_bV_c)\,(x_3)\right\rangle = \frac{1}{16\pi^2\,x_{12}^2 x_{13}^2 x_{23}^2}\Big((a\cdot b\,x_{13}\cdot c + a\cdot c\,x_{13}\cdot b) + \frac{\lambda}{4\pi^2}\Big[x_{12}\cdot a\,b\cdot c\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2\Big) + \Big(x_{23}\cdot b\,a\cdot c + x_{23}\cdot c\,a\cdot b\Big) \times \Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big)\Big]\Big) \tag{A.48}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\;\mathrm{Tr}\,(V_aV_b)\,(x_2)\;\mathrm{Tr}\,(V_cV_d)\,(x_3)\right\rangle = 0 + \frac{\lambda}{128\pi^8}\,\frac{a\cdot c\,b\cdot d + a\cdot d\,b\cdot c}{x_{12}^2 x_{13}^2 x_{23}^2} \times \Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big) \tag{A.49}$$

$$\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(BV_b)\,(x_2)\ \mathrm{Tr}\,(BV_c)\,(x_3)\rangle = 0 \tag{A.50}$$

$$\left\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(BV_b)\,(x_2)\ \mathrm{Tr}\left(C^2\right)(x_3)\right\rangle = \frac{a\cdot b}{8\pi^6\, x_{12}^2 x_{13}^2 x_{23}^2} \tag{A.51}$$

$$\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(BV_b)\,(x_2)\ \mathrm{Tr}\,(CV_c)\,(x_3)\rangle = 0 \tag{A.52}$$

$$\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(BV_b)\,(x_2)\ \mathrm{Tr}\,(V_cV_d)\,(x_3)\rangle = 0 \tag{A.53}$$

$$\left\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\left(C^2\right)(x_3)\right\rangle = -\frac{1}{2\pi^6}\frac{x_{12}+x_{13}}{x_{12}^2 x_{13}^2}\cdot a \tag{A.54}$$

$$\left\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\,(CV_b)\,(x_3)\right\rangle = -\frac{2(x_{12}+x_{13})\cdot a x_{23}\cdot b + x_{23}^2 a\cdot b}{8\pi^6\, x_{12}^2 x_{13}^2 x_{23}^2} \tag{A.55}$$

$$\left\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\,(V_bV_c)\,(x_3)\right\rangle = \frac{1}{8\pi^6\, x_{12}^2 x_{13}^2 x_{23}^2}\Big(a\cdot b\, x_{23}\cdot c$$
$$+ a\cdot c\, x_{23}\cdot b - \frac{\lambda}{4\pi^2} b\cdot c\, x_{12}\cdot a\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2\Big)\Big) \tag{A.56}$$

$$\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(CV_b)\,(x_2)\ \mathrm{Tr}\,(CV_c)\,(x_3)\rangle = -\frac{1}{16\pi^6\, x_{12}^2 x_{13}^2 x_{23}^2}\Big((x_{12}+x_{13})\cdot a$$
$$\times\, b\cdot c + x_{23}\cdot b\, a\cdot c - x_{23}\cdot c\, a\cdot b\Big) \tag{A.57}$$

$$\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(CV_b)\,(x_2)\ \mathrm{Tr}\,(V_cV_d)\,(x_3)\rangle = \frac{1}{32\pi^6\, x_{12}^2 x_{13}^2 x_{23}^2}\Big(a\cdot c\, b\cdot d$$
$$+ a\cdot d\, b\cdot c + \frac{\lambda}{4\pi^2} a\cdot b\, c\cdot d\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2\Big)\Big) \tag{A.58}$$

$$\langle \mathrm{Tr}\,(BV_a)\,(x_1)\ \mathrm{Tr}\,(V_bV_c)\,(x_2)\ \mathrm{Tr}\,(V_dV_e)\,(x_3)\rangle = 0 \tag{A.59}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\left(C^2\right)(x_3)\right\rangle = -\frac{1}{\pi^6} \tag{A.60}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\left(CV_a\right)(x_3)\right\rangle = \frac{1}{2\pi^6}\left(\frac{x_{13}}{x_{13}^2}+\frac{x_{23}}{x_{23}^2}\right)\cdot a \tag{A.61}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\left(C^2\right)(x_2)\ \mathrm{Tr}\left(V_aV_b\right)(x_3)\right\rangle = -\frac{1}{2\pi^6\, x_{13}^2x_{23}^2}\Big(x_{13}\cdot a\, x_{23}\cdot b + x_{23}\cdot a\, x_{13}\cdot b - \frac{\lambda}{16\pi^2}x_{12}^2 a\cdot b\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2x_{23}^2}-2\Big)\Big) \tag{A.62}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\left(CV_a\right)(x_2)\ \mathrm{Tr}\left(CV_b\right)(x_3)\right\rangle = \frac{1}{4\pi^6}\Big(\frac{a\cdot b}{2x_{23}^2}+\frac{x_{23}\cdot a\, x_{13}\cdot b}{x_{13}^2x_{23}^2} - \frac{x_{12}\cdot a\, x_{23}\cdot b}{x_{12}^2x_{23}^2}-\frac{x_{12}\cdot a\, x_{13}\cdot b}{x_{12}^2x_{13}^2}\Big) \tag{A.63}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\left(CV_a\right)(x_2)\ \mathrm{Tr}\left(V_bV_c\right)(x_3)\right\rangle = -\frac{1}{8\pi^6 x_{13}^2x_{23}^2}\Big(x_{13}\cdot b\, a\cdot c + x_{13}\cdot c\, a\cdot b - \frac{2x_{12}\cdot a}{x_{12}^2}(x_{13}\cdot b\, x_{23}\cdot c + x_{13}\cdot c\, x_{23}\cdot b) + \frac{\lambda}{4\pi^2}\Big(\ln\frac{\varepsilon^2x_{12}^2}{x_{13}^2x_{23}^2}-2\Big)x_{12}\cdot a\, b\cdot c\Big) \tag{A.64}$$

$$\left\langle \mathrm{Tr}\left(C^2\right)(x_1)\ \mathrm{Tr}\left(V_aV_b\right)(x_2)\ \mathrm{Tr}\left(V_cV_d\right)(x_3)\right\rangle = \frac{1}{8\pi^6 x_{12}^2x_{13}^2x_{23}^2}\Big(x_{12}\cdot a\, x_{13}\cdot c\, b\cdot d + x_{12}\cdot b\, x_{13}\cdot c\, a\cdot d + x_{12}\cdot a\, x_{13}\cdot d\, b\cdot c + x_{12}\cdot b\, x_{13}\cdot d\, a\cdot c + \frac{\lambda}{4\pi^2} \times\Big[\Big(\ln\frac{\varepsilon^2x_{13}^2}{x_{12}^2x_{23}^2}-2\Big)x_{13}\cdot c\, x_{13}\cdot d\, a\cdot b + \Big(\ln\frac{\varepsilon^2x_{12}^2}{x_{13}^2x_{23}^2}-2\Big)x_{12}\cdot a\, x_{12}\cdot b\, c\cdot d\Big]\Big) \tag{A.65}$$

$$\left\langle \mathrm{Tr}\left(CV_a\right)(x_1)\ \mathrm{Tr}\left(CV_b\right)(x_2)\ \mathrm{Tr}\left(CV_c\right)(x_3)\right\rangle = -\frac{1}{16\pi^6x_{12}^2x_{13}^2x_{23}^2}\Big[a\cdot b\,(x_{13}^2x_{23} + x_{23}^2x_{13})\cdot c + a\cdot c\,(x_{12}^2x_{23}+x_{23}^2x_{12})\cdot b - b\cdot c\,(x_{12}^2x_{13}+x_{13}^2x_{12})\cdot a + 2x_{13}\cdot a\, x_{13}\cdot b\, x_{23}\cdot c - 2x_{12}\cdot a\, x_{23}\cdot b\, x_{13}\cdot c\Big] \tag{A.66}$$

$$\langle \mathrm{Tr}\,(CV_a)\,(x_1)\ \mathrm{Tr}\,(CV_b)\,(x_2)\ \mathrm{Tr}\,(V_cV_d)\,(x_3)\rangle = -\frac{1}{32\pi^6 x_{12}^2 x_{13}^2 x_{23}^2}\Big(-2a\cdot c\, x_{12}\cdot b \times x_{23}\cdot d + a\cdot c\, b\cdot d\, x_{12}^2 - 2a\cdot b\, x_{13}\cdot c\, x_{23}\cdot d + 2b\cdot d\, x_{12}\cdot a\, x_{13}\cdot c - 2a\cdot d \times x_{12}\cdot b\, x_{23}\cdot c + a\cdot d\, b\cdot c\, x_{12}^2 - 2a\cdot b\, x_{13}\cdot d\, x_{23}\cdot c + 2b\cdot c\, x_{12}\cdot a\, x_{13}\cdot d + \frac{\lambda}{4\pi^2}\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2\Big) c\cdot d(2x_{12}\cdot a x_{12\cdot b} + a\cdot b x_{12}^2)\Big) \tag{A.67}$$

$$\langle \mathrm{Tr}\,(CV_a)\,(x_1)\ \mathrm{Tr}\,(V_bV_c)\,(x_2)\ \mathrm{Tr}\,(V_dV_e)\,(x_3)\rangle = \frac{1}{32\pi^6 x_{12}^2 x_{13}^2 x_{23}^2}\Big((x_{12}\cdot b\, a\cdot d\, c\cdot e + x_{13}\cdot d\, a\cdot b\, c\cdot e + x_{12}\cdot c\, a\cdot d\, b\cdot e + x_{13}\cdot d\, a\cdot c\, b\cdot e + x_{12}\cdot b\, a\cdot e\, c\cdot d + x_{13}\cdot e \times a\cdot b\, c\cdot d + x_{12}\cdot c\, a\cdot e\, b\cdot d + x_{13}\cdot e\, a\cdot c\, b\cdot d) + \frac{\lambda}{2\pi^2}\Big[(x_{13}\cdot e\, a\cdot d + x_{13}\cdot d \times a\cdot e) b\cdot c\Big(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 2\Big) + (x_{12}\cdot b\, a\cdot c + x_{12}\cdot c\, a\cdot b) d\cdot e\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2\Big)\Big]\Big) \tag{A.68}$$

$$\langle \mathrm{Tr}\,(V_aV_b)\,(x_1)\ \mathrm{Tr}\,(V_cV_d)\,(x_2)\ \mathrm{Tr}\,(V_eV_f)\,(x_3)\rangle = \frac{1}{64\pi^6 x_{12}^2 x_{13}^2 x_{23}^2}\Big((a\cdot e\, b\cdot d\, c\cdot f + a\cdot f\, b\cdot d\, c\cdot e + a\cdot e\, b\cdot c\, d\cdot f + a\cdot f\, b\cdot c\, d\cdot e + b\cdot e\, a\cdot d\, c\cdot f + b\cdot f\, a\cdot d\, c\cdot e + b\cdot e\, a\cdot c\, d\cdot f + b\cdot f\, a\cdot c\, d\cdot e) + \frac{\lambda}{2\pi^2}\Big[a\cdot b(c\cdot e\, d\cdot f + c\cdot f\, d\cdot e)\Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big) + c\cdot d(a\cdot f\, b\cdot e + a\cdot e\, b\cdot f)\Big(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 2\Big) + e\cdot f(a\cdot d\, b\cdot e + a\cdot e\, b\cdot d) \times \Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 2\Big)\Big]\Big) \tag{A.69}$$

A.1.2 Length Three Operators

Two-Point Functions

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(B^3\right)(x_2)\right\rangle = 0 \tag{A.70}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(B^2C\right)(x_2)\right\rangle = 0 \tag{A.71}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(B^2V_a\right)(x_2)\right\rangle = 0 \tag{A.72}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(BC^2\right)(x_2)\right\rangle = 0 \tag{A.73}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(BCV_a\right)(x_2)\right\rangle = 0 \tag{A.74}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(BV_aV_b\right)(x_2)\right\rangle = 0 \tag{A.75}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\right\rangle = \frac{3}{8\pi^6 x_{12}^6} \tag{A.76}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(C^2V_a\right)(x_2)\right\rangle = 0 \tag{A.77}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(CV_aV_b\right)(x_2)\right\rangle = 0 \tag{A.78}$$

$$\left\langle \mathrm{Tr}\left(B^3\right)(x_1)\ \mathrm{Tr}\left(V_aV_bV_c\right)(x_2)\right\rangle = 0 \tag{A.79}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(B^2C\right)(x_2)\right\rangle = 0 \tag{A.80}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(B^2V_a\right)(x_2)\right\rangle = 0 \tag{A.81}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(BC^2\right)(x_2)\right\rangle = \frac{1}{8\pi^6 x_{12}^6}\left(1+\frac{\lambda}{4\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right) \tag{A.82}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(BCV_a\right)(x_2)\right\rangle = 0 \tag{A.83}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(BV_aV_b\right)(x_2)\right\rangle = 0 \tag{A.84}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\right\rangle = -\frac{3}{8\pi^6 x_{12}^4} \tag{A.85}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(C^2V_a\right)(x_2)\right\rangle = \frac{x_{12}\cdot a}{8\pi^6 x_{12}^6} \tag{A.86}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(CV_aV_b\right)(x_2)\right\rangle = \frac{\lambda\, a\cdot b}{128\pi^8 x_{12}^6}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right) \tag{A.87}$$

$$\left\langle \mathrm{Tr}\left(B^2C\right)(x_1)\ \mathrm{Tr}\left(V_aV_bV_c\right)(x_2)\right\rangle = 0 \tag{A.88}$$

$$\left\langle \mathrm{Tr}\left(B^2V_a\right)(x_1)\ \mathrm{Tr}\left(B^2V_b\right)(x_2)\right\rangle = 0 \tag{A.89}$$

$$\left\langle \mathrm{Tr}\left(B^2V_a\right)(x_1)\ \mathrm{Tr}\left(BC^2\right)(x_2)\right\rangle = 0 \tag{A.90}$$

$$\left\langle \mathrm{Tr}\left(B^2V_a\right)(x_1)\ \mathrm{Tr}\left(BCV_b\right)(x_2)\right\rangle = 0 \tag{A.91}$$

$$\left\langle \mathrm{Tr}\left(B^2V_a\right)(x_1)\ \mathrm{Tr}\left(BV_bV_c\right)(x_2)\right\rangle = 0 \tag{A.92}$$

$$\left\langle \mathrm{Tr}\left(B^2V_a\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\right\rangle = -\frac{3x_{12}\cdot a}{8\pi^6 x_{12}^6} \tag{A.93}$$

$$\left\langle \mathrm{Tr}\left(B^2V_a\right)(x_1)\ \mathrm{Tr}\left(C^2V_b\right)(x_2)\right\rangle = \frac{a\cdot b}{16\pi^6 x_{12}^6} \tag{A.94}$$

$$\left\langle \mathrm{Tr}\left(B^2V_a\right)(x_1)\ \mathrm{Tr}\left(CV_bV_c\right)(x_2)\right\rangle = 0 \tag{A.95}$$

$$\left\langle \mathrm{Tr}\left(B^2V_a\right)(x_1)\ \mathrm{Tr}\left(V_bV_cV_d\right)(x_2)\right\rangle = 0 \tag{A.96}$$

$$\left\langle \mathrm{Tr}\left(BC^2\right)(x_1)\ \mathrm{Tr}\left(BC^2\right)(x_2)\right\rangle = -\frac{1}{4\pi^6 x_{12}^4}\left(1+\frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right) \tag{A.97}$$

$$\left\langle \mathrm{Tr}\left(BC^2\right)(x_1)\ \mathrm{Tr}\left(BCV_a\right)(x_2)\right\rangle = \frac{x_{12}\cdot a}{8\pi^6 x_{12}^6}\left(1+\frac{\lambda}{8\pi^2}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right)\right) \tag{A.98}$$

$$\left\langle \mathrm{Tr}\left(BC^2\right)(x_1)\ \mathrm{Tr}\left(BV_aV_b\right)(x_2)\right\rangle = \frac{\lambda\, a\cdot b}{128\pi^8 x_{12}^6}\left(\ln\frac{\varepsilon^2}{x_{12}^2}-1\right) \tag{A.99}$$

$$\left\langle \mathrm{Tr}\left(BC^2\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\right\rangle = \frac{3}{8\pi^6 x_{12}^2} \tag{A.100}$$

$$\left\langle \mathrm{Tr}\left(BC^2\right)(x_1)\ \mathrm{Tr}\left(C^2V_a\right)(x_2)\right\rangle = -\frac{x_{12}\cdot a}{4\pi^6 x_{12}^4} \tag{A.101}$$

$$\left\langle \mathrm{Tr}\left(BC^2\right)(x_1)\ \mathrm{Tr}\left(CV_aV_b\right)(x_2)\right\rangle = \frac{1}{8\pi^6 x_{12}^6}\Big(x_{12}\cdot a\, x_{12}\cdot b - \frac{\lambda}{16\pi^2}x_{12}^2\, a\cdot b \times \Big(\ln\frac{\varepsilon^2}{x_{12}^2}-1\Big)\Big) \tag{A.102}$$

$$\left\langle \mathrm{Tr}\left(BC^2\right)(x_1)\ \mathrm{Tr}\left(V_aV_bV_c\right)(x_2)\right\rangle = \frac{\lambda}{128\pi^8 x_{12}^6}\Big(\ln\frac{\varepsilon^2}{x_{12}^2}-1\Big)\Big(a\cdot b\, x_{12}\cdot c + a\cdot c\, x_{12}\cdot b + b\cdot c\, x_{12}\cdot a\Big) \tag{A.103}$$

$$\left\langle \mathrm{Tr}\left(BCV_a\right)(x_1)\ \mathrm{Tr}\left(BCV_b\right)(x_2)\right\rangle = -\frac{5\lambda\, a\cdot b}{256\pi^8 x_{12}^6}\Big(\ln\frac{\varepsilon^2}{x_{12}^2}-1\Big) \tag{A.104}$$

$$\left\langle \mathrm{Tr}\left(BCV_a\right)(x_1)\ \mathrm{Tr}\left(BV_bV_c\right)(x_2)\right\rangle = 0 \tag{A.105}$$

$$\left\langle \mathrm{Tr}\left(BCV_a\right)(x_1)\ \mathrm{Tr}\left(CBV_b\right)(x_2)\right\rangle = \frac{a\cdot b}{16\pi^6 x_{12}^6}\Big(1+\frac{7\lambda}{16\pi^2}\Big(\ln\frac{\varepsilon^2}{x_{12}^2}-1\Big)\Big) \tag{A.106}$$

$$\left\langle \mathrm{Tr}\left(BCV_a\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\right\rangle = \frac{3x_{12}\cdot a}{8\pi^6 x_{12}^4} \tag{A.107}$$

$$\left\langle \mathrm{Tr}\left(BCV_a\right)(x_1)\ \mathrm{Tr}\left(C^2V_b\right)(x_2)\right\rangle = -\frac{x_{12}^2\, a\cdot b + 2x_{12}\cdot a\, x_{12}\cdot b}{16\pi^6 x_{12}^6} \tag{A.108}$$

$$\left\langle \mathrm{Tr}\left(BCV_a\right)(x_1)\ \mathrm{Tr}\left(CV_bV_c\right)(x_2)\right\rangle = \frac{1}{16\pi^6 x_{12}^6}\Big(x_{12}\cdot b\, a\cdot c + \frac{\lambda}{16\pi^2}\Big(6x_{12}\cdot b\, a\cdot c - 6x_{12}\cdot c\, a\cdot b - x_{12}\cdot a\, b\cdot c\Big)\Big(\ln\frac{\varepsilon^2}{x_{12}^2}-1\Big)\Big) \tag{A.109}$$

$$\left\langle \mathrm{Tr}\left(BCV_a\right)(x_1)\ \mathrm{Tr}\left(V_bV_cV_d\right)(x_2)\right\rangle = \frac{\lambda}{512\pi^8 x_{12}^6}(a\cdot b\, c\cdot d + a\cdot c\, b\cdot d + a\cdot d\, b\cdot c) \times \Big(\ln\frac{\varepsilon^2}{x_{12}^2}-1\Big) \tag{A.110}$$

$$\left\langle \mathrm{Tr}\left(BV_aV_b\right)(x_1)\ \mathrm{Tr}\left(BV_cV_d\right)(x_2)\right\rangle = 0 \tag{A.111}$$

$$\left\langle \mathrm{Tr}\left(BV_aV_b\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\right\rangle = \frac{3x_{12}\cdot a\, x_{12}\cdot b}{8\pi^6 x_{12}^6} \tag{A.112}$$

$$\left\langle \mathrm{Tr}\left(BV_aV_b\right)(x_1)\ \mathrm{Tr}\left(C^2V_c\right)(x_2)\right\rangle = -\frac{x_{12}\cdot a\, b\cdot c + x_{12}\cdot b\, a\cdot c}{16\pi^6 x_{12}^6} \tag{A.113}$$

$$\left\langle \mathrm{Tr}\left(BV_aV_b\right)(x_1)\ \mathrm{Tr}\left(CV_cV_d\right)(x_2)\right\rangle = \frac{1}{32\pi^6 x_{12}^6}\Big(a\cdot c\, b\cdot d + \frac{\lambda}{16\pi^2}\Big(\ln\frac{\varepsilon^2}{x_{12}^2} - 1\Big)$$
$$\times\Big(6a\cdot c\, b\cdot d - 6a\cdot d\, b\cdot c + a\cdot b\, c\cdot d\Big)\Big) \tag{A.114}$$

$$\left\langle \mathrm{Tr}\left(BV_aV_b\right)(x_1)\ \mathrm{Tr}\left(V_cV_dV_e\right)(x_2)\right\rangle = 0 \tag{A.115}$$

$$\left\langle \mathrm{Tr}\left(C^3\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\right\rangle = -\frac{3}{8\pi^6} \tag{A.116}$$

$$\left\langle \mathrm{Tr}\left(C^3\right)(x_1)\ \mathrm{Tr}\left(C^2V_a\right)(x_2)\right\rangle = \frac{3x_{12}\cdot a}{8\pi^6 x_{12}^2} \tag{A.117}$$

$$\left\langle \mathrm{Tr}\left(C^3\right)(x_1)\ \mathrm{Tr}\left(CV_aV_b\right)(x_2)\right\rangle = -\frac{3x_{12}\cdot a\, x_{12}\cdot b}{8\pi^6 x_{12}^4} \tag{A.118}$$

$$\left\langle \mathrm{Tr}\left(C^3\right)(x_1)\ \mathrm{Tr}\left(V_aV_bV_c\right)(x_2)\right\rangle = \frac{3x_{12}\cdot a\, x_{12}\cdot b\, x_{12}\cdot c}{8\pi^6 x_{12}^6} \tag{A.119}$$

$$\left\langle \mathrm{Tr}\left(C^2V_a\right)(x_1)\ \mathrm{Tr}\left(C^2V_b\right)(x_2)\right\rangle = \frac{x_{12}^2\, a\cdot b + 4x_{12}\cdot a\, x_{12}\cdot b}{16\pi^6 x_{12}^4} \tag{A.120}$$

$$\left\langle \mathrm{Tr}\left(C^2V_a\right)(x_1)\ \mathrm{Tr}\left(CV_bV_c\right)(x_2)\right\rangle = -\frac{1}{16\pi^6 x_{12}^6}\Big(x_{12}^2\, x_{12}\cdot b\, a\cdot c + x_{12}^2\, x_{12}\cdot c\, a\cdot b$$
$$+ 2x_{12}\cdot a\, x_{12}\cdot b\, x_{12}\cdot c\Big) \tag{A.121}$$

$$\left\langle \mathrm{Tr}\left(C^2V_a\right)(x_1)\ \mathrm{Tr}\left(V_bV_cV_d\right)(x_2)\right\rangle = \frac{1}{16\pi^6 x_{12}^6}\Big(x_{12}\cdot c\, x_{12}\cdot d\, a\cdot b + x_{12}\cdot b\, x_{12}\cdot d$$
$$\times a\cdot c + x_{12}\cdot b\, x_{12}\cdot c\, a\cdot d\Big) \tag{A.122}$$

$$\langle \mathrm{Tr}\,(CV_aV_b)\,(x_1)\ \mathrm{Tr}\,(CV_cV_d)\,(x_2)\rangle = -\frac{1}{32\pi^6 x_{12}^6}\Big(x_{12}^2\,a\cdot c\,b\cdot d + 2x_{12}\cdot b\,x_{12}\cdot c \times a\cdot d + 2x_{12}\cdot a\,x_{12}\cdot d\,b\cdot c + \frac{\lambda}{16\pi^2}\Big(\ln\frac{\varepsilon^2}{x_{12}^2} - 1\Big)(6x_{12}^2\,a\cdot c\,b\cdot d + 12x_{12}\cdot b \times x_{12}\cdot c\,a\cdot d + 12x_{12}\cdot a\,x_{12}\cdot d\,b\cdot c - 6x_{12}^2\,a\cdot d\,b\cdot c - 12x_{12}\cdot a\,x_{12}\cdot c\,b\cdot d - 12x_{12}\cdot b\,x_{12}\cdot d\,a\cdot c + x_{12}^2\,a\cdot b\,c\cdot d)\Big) \tag{A.123}$$

$$\langle \mathrm{Tr}\,(CV_aV_b)\,(x_1)\ \mathrm{Tr}\,(V_cV_dV_e)\,(x_2)\rangle = \frac{1}{32\pi^6 x_{12}^6}\Big((x_{12}\cdot c\,a\cdot d\,b\cdot e + x_{12}\cdot d\,a\cdot e\,b\cdot c + x_{12}\cdot e\,a\cdot c\,b\cdot d + \frac{\lambda}{16\pi^2}\Big(\ln\frac{\varepsilon^2}{x_{12}^2} - 1\Big)\Big[x_{12}\cdot c(6a\cdot d\,b\cdot e - 6a\cdot e\,b\cdot d + a\cdot b\,d\cdot e) + x_{12}\cdot d(6a\cdot e\,b\cdot c - 6a\cdot c\,b\cdot e + a\cdot b\,c\cdot e) + x_{12}\cdot e(6a\cdot c\,b\cdot d - 6a\cdot d\,b\cdot c + a\cdot b\,c\cdot d)\Big]\Big) \tag{A.124}$$

$$\langle \mathrm{Tr}\,(V_aV_bV_c)\,(x_1)\ \mathrm{Tr}\,(V_dV_eV_f)\,(x_2)\rangle = \frac{1}{64\pi^6 x_{12}^6}\Big(a\cdot d\,b\cdot e\,c\cdot f + a\cdot e\,b\cdot f\,c\cdot d + a\cdot f\,b\cdot d\,c\cdot e + \frac{\lambda}{16\pi^2}\Big(\ln\frac{\varepsilon^2}{x_{12}^2} - 1\Big)\Big[6a\cdot d\,b\cdot e\,c\cdot f + 6a\cdot e\,b\cdot f\,c\cdot d + 6a\cdot f \times b\cdot d\,c\cdot e - 6a\cdot e\,b\cdot d\,c\cdot f - 6a\cdot d\,b\cdot f\,c\cdot e - 6a\cdot f\,b\cdot e\,c\cdot d + a\cdot b(c\cdot f\,d\cdot e + c\cdot d\,e\cdot f + c\cdot e\,d\cdot f) + a\cdot c(b\cdot e\,d\cdot f + b\cdot d\,e\cdot f + b\cdot f\,d\cdot e) + b\cdot c(a\cdot d\,e\cdot f + b\cdot d\,e\cdot f + b\cdot f\,d\cdot e)\Big]\Big) \tag{A.125}$$

Three-Point Functions

$$\big\langle \mathrm{Tr}\,\big(B^2\big)\,(x_1)\ \mathrm{Tr}\,\big(B^2C\big)\,(x_2)\ \mathrm{Tr}\,\big(C^3\big)\,(x_3)\big\rangle = \frac{3}{8\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(1 + \frac{\lambda}{8\pi^2} \times \Big(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 2\Big)\Big) \tag{A.126}$$

$$\big\langle \mathrm{Tr}\,\big(B^2\big)\,(x_1)\ \mathrm{Tr}\,\big(BC^2\big)\,(x_2)\ \mathrm{Tr}\,\big(BC^2\big)\,(x_3)\big\rangle = \frac{1}{4\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(1 + \frac{\lambda}{8\pi^2} \times \Big(2\ln\frac{\varepsilon^2}{x_{23}^2} - 3\Big)\Big) \tag{A.127}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC^2\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle = -\frac{3}{4\pi^8 x_{12}^2 x_{13}^2 x_{23}^2}\Big(1+\frac{\lambda}{16\pi^2}$$
$$\times\left(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2}-2\right)\Big) \tag{A.128}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC^2\right)(x_2)\ \mathrm{Tr}\left(C^2V_a\right)(x_3)\right\rangle = \frac{x_{23}\cdot a}{4\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(1+\frac{\lambda}{16\pi^2}$$
$$\times\left(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2}-2\right)\Big) \tag{A.129}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BC^2\right)(x_2)\ \mathrm{Tr}\left(CV_aV_b\right)(x_3)\right\rangle = \frac{\lambda\, a\cdot b}{256\pi^{10} x_{12}^2 x_{13}^2 x_{23}^4}\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2}$$
$$+2\ln\frac{\varepsilon^2}{x_{23}^2}-4\Big) \tag{A.130}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BCV_a\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle = -\frac{3x_{23}\cdot a}{8\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(1+\frac{\lambda}{16\pi^2}$$
$$\times\left(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2}-2\right)\Big) \tag{A.131}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BCV_a\right)(x_2)\ \mathrm{Tr}\left(C^2V_b\right)(x_3)\right\rangle = \frac{a\cdot b}{16\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(1+\frac{\lambda}{16\pi^2}$$
$$\times\left(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2}-2\right)\Big) \tag{A.132}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BV_aV_b\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle = \frac{3\lambda\, a\cdot b}{256\pi^{10} x_{12}^2 x_{13}^2 x_{23}^4}$$
$$\times\left(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2}-2\right) \tag{A.133}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(CBV_a\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle$$
$$=\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BCV_a\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle \tag{A.134}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(CBV_a\right)(x_2)\ \mathrm{Tr}\left(C^2V_b\right)(x_3)\right\rangle$$
$$=\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(BCV_a\right)(x_2)\ \mathrm{Tr}\left(C^2V_b\right)(x_3)\right\rangle \tag{A.135}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle = \frac{9}{8\pi^8 x_{12}^2 x_{13}^2} \tag{A.136}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\ \mathrm{Tr}\left(C^2V_a\right)(x_3)\right\rangle = -\frac{3x_{23}\cdot a}{4\pi^8 x_{12}^2 x_{13}^2 x_{23}^2} \tag{A.137}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\ \mathrm{Tr}\left(CV_aV_b\right)(x_3)\right\rangle = \frac{3}{8\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(x_{23}\cdot a x_{23}\cdot b$$
$$+\frac{\lambda}{32\pi^2}x_{23}^2\, a\cdot b\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2}-2\Big)\Big) \tag{A.138}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^3\right)(x_2)\ \mathrm{Tr}\left(V_aV_bV_c\right)(x_3)\right\rangle = -\frac{3\lambda}{256\pi^{10} x_{12}^2 x_{13}^2 x_{23}^4}(x_{23}\cdot a\, b\cdot c$$
$$+x_{23}\cdot b\, a\cdot c+x_{23}\cdot c\, a\cdot b)\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2}-2\Big) \tag{A.139}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^2V_a\right)(x_2)\ \mathrm{Tr}\left(C^2V_b\right)(x_3)\right\rangle = -\frac{x_{23}^2\, a\cdot b+2x_{23}\cdot a\, x_{23}\cdot b}{8\pi^8 x_{12}^2 x_{13}^2 x_{23}^4} \tag{A.140}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^2V_a\right)(x_2)\ \mathrm{Tr}\left(CV_bV_c\right)(x_3)\right\rangle = \frac{1}{16\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(x_{23}\cdot b\, a\cdot c$$
$$+x_{23}\cdot c\, a\cdot b-\frac{\lambda}{16\pi^2}x_{23}\cdot a\, b\cdot c\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2}-2\Big)\Big) \tag{A.141}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(C^2V_a\right)(x_2)\ \mathrm{Tr}\left(V_bV_cV_d\right)(x_3)\right\rangle = \frac{\lambda}{512\pi^{10} x_{12}^2 x_{13}^2 x_{23}^4}\Big(a\cdot b\, c\cdot d$$
$$+a\cdot c\, b\cdot d+a\cdot d\, b\cdot c\Big)\Big(\ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2}-2\Big) \tag{A.142}$$

$$\left\langle \mathrm{Tr}\left(B^2\right)(x_1)\ \mathrm{Tr}\left(CV_aV_b\right)(x_2)\ \mathrm{Tr}\left(CV_cV_d\right)(x_3)\right\rangle = \frac{1}{32\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(a\cdot d\, b\cdot c$$
$$+\frac{\lambda}{16\pi^2}\Big[(a\cdot d\, b\cdot c-a\cdot c\, b\cdot d)\Big(6\ln\frac{\varepsilon^2}{x_{23}^2}-10\Big)+a\cdot b\, c\cdot d\Big(\ln\frac{\varepsilon^2}{x_{23}^2}-1\Big)\Big]\Big) \tag{A.143}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\left(B^3\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle = \frac{9}{16\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(1+\frac{\lambda}{8\pi^2} \times \Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big)\Big) \tag{A.144}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\left(B^2C\right)(x_2)\ \mathrm{Tr}\left(BC^2\right)(x_3)\right\rangle = \frac{3}{16\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(1+\frac{\lambda}{24\pi^2} \times \Big(3\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} + 4\ln\frac{\varepsilon^2}{x_{23}^2} - 12\Big)\Big) \tag{A.145}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\left(B^2C\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle = -\frac{3}{16\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(x_{12}^2 + x_{13}^2 + 2x_{23}^2 + \frac{\lambda}{8\pi^2}\Big[3x_{23}^2\Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big) + x_{13}^2\Big(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 2\Big)\Big]\Big) \tag{A.146}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\left(B^2C\right)(x_2)\ \mathrm{Tr}\left(C^2V_a\right)(x_3)\right\rangle = \frac{1}{16\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(x_{13} + 2x_{23} + \frac{\lambda}{8\pi^2}\Big[3x_{23}\Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big) + x_{13}\Big(\ln\frac{\varepsilon^2 x_{13}^2}{x_{12}^2 x_{23}^2} - 2\Big)\Big]\Big)\cdot a \tag{A.147}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\left(B^2C\right)(x_2)\ \mathrm{Tr}\,(CV_aV_b)\,(x_3)\right\rangle = \frac{\lambda\, a\cdot b}{512\pi^{10} x_{12}^2 x_{13}^2 x_{23}^4}\Big(2\ln\frac{\varepsilon^2}{x_{23}^2} + \ln\frac{\varepsilon^2 x_{12}^2}{x_{13}^2 x_{23}^2} - 4\Big) \tag{A.148}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\left(B^2V_a\right)(x_2)\ \mathrm{Tr}\left(C^3\right)(x_3)\right\rangle = -\frac{3}{16\pi^8 x_{12}^2 x_{13}^2 x_{23}^2}\Big(2x_{23} - x_{12} + \frac{3\lambda}{8\pi^2}x_{23}\Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big)\Big)\cdot a \tag{A.149}$$

$$\left\langle \mathrm{Tr}\,(BC)\,(x_1)\ \mathrm{Tr}\left(B^2V_a\right)(x_2)\ \mathrm{Tr}\left(C^2V_b\right)(x_3)\right\rangle = \frac{a\cdot b}{16\pi^8 x_{12}^2 x_{13}^2 x_{23}^4}\Big(1+\frac{3\lambda}{16\pi^2} \times \Big(\ln\frac{\varepsilon^2 x_{23}^2}{x_{12}^2 x_{13}^2} - 2\Big)\Big) \tag{A.150}$$

A.2 Numerical Results for Short $SO(6)$ Operators

All definite numerical results of the calculations from section 6.6 are listed below. We also specify the margin of deviation of the results. The definition of the classes of operators can be found in section 6.6.1. For classes of large dimension we calculate only about 250 random samples of structure constants. The column titled *"# values"* contains the number of non-zero values obtained. We quote the values with the smallest numerical error, i. e. the ones with the largest values for $C^{(1)}_{\alpha\beta\gamma}$ and $C^{(0)}_{\alpha\beta\gamma}$. The specified "errors" are the differences to the largest and smallest value respectively.

A.2.1 Operators up to Length Four

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	# values	$-32\pi^2\, C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$
2B	4A	4B	380	14.4721359550 (0)
2B	4A	4F	250	5.5278640450 (0)
2B	4A	4G	250	0.0000 (106)
2B	4B	4B	380	7.407845655 (185)
2B	4B	4C	250	14.4721359550 (0)
2B	4B	4D	248	14.472 (201)
2B	4B	4E	250	14.4721359550 (0)
2B	4B	4F	8000	0.000000000 (623)
2B	4B	4G	250	14.472135955 (553)
2B	4C	4C	15234	12.0000000000 (0)
2B	4C	4D	250	8.0000000000 (0)
2B	4C	4F	250	5.5278640450 (1)
2B	4C	4G	250	0.0000 (119)
2B	4D	4F	247	5.5279 (245)
2B	4D	4G	250	0.00000 (183)
2B	4E	4F	250	5.5278640450 (0)
2B	4E	4G	250	0.00000 (141)
2B	4F	4F	380	4.2377239654 (127)
2B	4F	4G	250	5.527864045 (294)
2B	4G	4G	250	0.0000000000 (0)
3A	3A	4A	49	27.40312 (0)
3A	3A	4B	3620	22.47214 (0)
3A	3A	4C	20790	20.0000000000 (0)
3A	3A	4D	4	12.0 (4)

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	# values	$-32\pi^2\, C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$
3A	3A	4E	49	14.59688 (0)
3A	3A	4F	3620	13.52786 (0)
3A	3A	4G	9961	20.0 (11)
3A	3B	4C	10494	16.00000 (0)
3A	3B	4D	10800	20.0000000000 (1)
3A	3C	4D	250	12.0000000000 (0)
3B	3B	4A	16	12.74512 (0)
3B	3B	4B	401	17.03786 (0)
3B	3B	4E	16	8.13488 (0)
3B	3B	4F	401	14.59851 (0)
3B	3C	4B	248	9.93681379 (13)
3B	3C	4C	248	8.0000 (400)
3B	3C	4D	250	8.00000 (243)
3B	3C	4G	249	8.0000000000 (0)
3C	3C	4B	250	14.472135963 (455)
3C	3C	4C	250	12.000000000 (473)
3C	3C	4D	4995	0.0000 (344)
3C	3C	4G	250	0.0000000000 (0)
4A	4A	4G	250	19.4028 (10)
4A	4B	4B	210	24.6559311634 (899)
4A	4B	4F	400	15.403124237 (210)
4A	4D	4D	325	20.59687576 (117)
4A	4D	4G	250	19.4031018 (79316)
4A	4F	4F	210	9.731571887 (171)
4A	4G	4G	250	19.4031242 (0)
4B	4B	4B	1540	13.545909392 (459)
4B	4B	4C	250	19.8885438198 (490)
4B	4B	4D	232	17.12 (2)
4B	4B	4E	210	76.040506 (363)
4B	4B	4F	4200	52.153275 (158)
4B	4B	4G	250	17.1168319337 (76)
4B	4C	4D	250	15.2360679775 (0)
4B	4D	4F	232	6.66 (1)
4B	4D	4G	250	14.472 (39)
4B	4E	4F	232	2.596875763 (346)
4B	4F	4F	4200	10.74995130 (709)
4B	4F	4G	250	6.6666666697 (411)

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	# values	$-32\pi^2\, C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$
4B	4G	4G	250	14.47213595 (24)
4C	4C	4G	250	24.000000000 (258)
4C	4D	4F	250	10.7639320225 (0)
4C	4D	4G	250	12.0000 (138)
4C	4E	4E	2	-14.1573750398 (0)
4C	4F	4F	250	-15.88854381 (231)
4C	4G	4G	250	12.0000000000 (653)
4D	4D	4D	156	8.000 (10)
4D	4D	4E	623	33.4031243 (489)
4D	4D	4G	530	8.0000000000 (0)
4D	4E	4G	250	6.5969 (18)
4D	4F	4F	250	3.93 (2)
4D	4F	4G	250	5.5278 (145)
4D	4G	4G	250	0.0000 (970)
4E	4E	4G	250	6.596879 (651)
4E	4G	4G	250	6.59687576 (13)
4E	4F	4F	210	9.571990923 (871)
4F	4F	4F	1540	8.92997485 (742)
4F	4F	4G	250	3.9357996451 (387)
4F	4G	4G	250	5.527864044 (43)
4G	4G	4G	250	0.0000000000 (0)

A.2.2 Operators up to Length Five

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	# values	$-32\pi^2\, C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$
2B	5A	5A	3869	15.2947444 (1562948)
2B	5A	5C	248	9.000000000 (986)
2B	5A	5D	250	13.8416192530 (28)
2B	5A	5I	250	9.3694832980 (13)
2B	5A	5J	229	4.000 (690)
2B	5A	5K	248	0.000 (663)
2B	5B	5K	238	0.000 (223)
2B	5C	5G	250	5.0000000000 (151)
2B	5C	5J	250	25.00000000 (214)
2B	5D	5E	250	11.4164078649 (222)

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	# values	$-32\pi^2\ C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$
2B	5D	5H	250	6.6305167020 (13)
2B	5D	5J	250	4.5250698548 (231)
2B	5E	5I	250	-15.4164079 (39)
2B	5E	5J	250	4.0000000000 (449)
2B	5E	5K	250	12.000000000 (226)
2B	5F	5G	250	10.0000000000 (259)
2B	5F	5J	250	-10.000000000 (102)
2B	5G	5J	250	4.0000000000 (258)
2B	5H	5H	238	6.4195412873 (1416831)
2B	5H	5I	250	2.1583807470 (48)
2B	5H	5J	223	4.0 (8)
2B	5I	5J	250	4.7432228282 (70)
2B	5J	5J	250	2.8571428571 (78)
2B	5J	5K	250	4.0000000000 (142)
2B	5K	5K	45	0.0000000000 (0)
3A	4A	5A	190	25.2111025509 (97)
3A	4A	5D	250	18.47 (11)
3A	4A	5H	197	10.7888974491 (408)
3A	4A	5I	250	9.52 (1)
3A	4B	5A	240	22.4171957150 (1417)
3A	4B	5C	50	-51.2492235950 (532)
3A	4B	5D	50	22.4721359550 (37)
3A	4B	5F	50	14.0000000000 (12)
3A	4B	5H	222	13.9742897905 (4035)
3A	4C	5A	250	25.2111025509 (79)
3A	4C	5E	50	20.0000000000 (3)
3A	4C	5F	50	16.0000000000 (1)
3A	4C	5H	236	10.7888974491 (2)
3A	4D	5A	242	16.0 (6)
3A	4D	5D	50	19.57770876 (2)
3A	4D	5G	50	16.0000000000 (0)
3A	4D	5H	235	16.00 (72)
3A	4D	5I	50	12.4222912 (3)
3A	4D	5J	250	12.0000000000 (452)
3A	4D	5K	50	16.0 (2)
3A	4E	5A	181	25.2111025509 (536950)
3A	4E	5D	50	18.472 (4)

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	# values	$-32\pi^2\, C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$
3A	4E	5H	193	10.7888974491 (1)
3A	4E	5I	50	9.527 (2)
3A	4F	5C	50	29.2492235950 (3)
3A	4F	5F	50	14.0000000000 (1)
3A	4F	5I	50	13.527864045 (107)
3A	4G	5D	50	18.4721359550 (8)
3A	4G	5I	50	9.5278640450 (0)
3B	4A	5B	31	20.0000000000 (1)
3B	4A	5D	43	21.0936915713 (0)
3B	4A	5E	50	8.00 (13)
3B	4A	5I	5	-1.0936915713 (0)
3B	4A	5J	299	4.0000 (1647)
3B	4A	5K	50	8.000 (43)
3B	4B	5B	118	12.9660490573 (0)
3B	4B	5C	50	14.4164078650 (0)
3B	4B	5D	50	-9.8104413686 (45)
3B	4B	5E	50	18.19292797 (11)
3B	4B	5F	50	18.6295146067 (0)
3B	4B	5I	50	16.8541019662 (1)
3B	4B	5J	250	23.29618127 (858)
3B	4B	5K	50	8.000 (19)
3B	4C	5A	250	16.00000 (7969)
3B	4C	5B	247	16.00000 (10960)
3B	4C	5D	50	16.000 (124)
3B	4C	5E	50	20.000000000 (514)
3B	4C	5G	50	16.0000000000 (0)
3B	4C	5H	236	16.00000 (509)
3B	4C	5I	50	16.00 (28)
3B	4D	5A	250	9.4830060324 (20)
3B	4D	5C	50	19.0000000000 (12)
3B	4D	5D	50	5.5278640450 (16)
3B	4D	5F	50	12.0000000000 (13)
3B	4D	5H	241	8.6346410264 (25)
3B	4D	5I	50	14.4721359550 (0)
3B	4D	5K	50	8.00 (38)
3B	4E	5A	58	10.34 (10)
3B	4E	5D	41	9.6394347207 (0)

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	# values	$-32\pi^2\, C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$
3B	4E	5E	50	8.000 (59)
3B	4E	5I	44	10.3605652793 (0)
3B	4E	5J	298	4.0000 (5322)
3B	4E	5K	50	8.000 (34)
3B	4F	5B	127	7.5085272139 (2533852496)
3B	4F	5C	50	-12.416407865 (2)
3B	4F	5D	50	10.1458980338 (1)
3B	4F	5E	50	12.543914133 (24)
3B	4F	5F	50	6.7038187267 (1)
3B	4F	5I	50	7.2649868232 (0)
3B	4F	5J	250	11.370485393 (306)
3B	4F	5K	50	8.000 (12)
3B	4G	5A	50	8.0000 (38)
3B	4G	5B	50	8.0000 (40)
3B	4G	5C	50	8.000 (35)
3B	4G	5D	50	8.0000 (27)
3B	4G	5E	50	8.0000000000 (0)
3B	4G	5F	50	8.000 (17)
3B	4G	5G	50	8.000 (42)
3B	4G	5H	50	8.000 (13)
3B	4G	5I	50	8.0000 (89)
3B	4G	5J	50	4.0000000000 (7)
3B	4G	5K	50	8.0000000000 (0)
3C	4A	5E	48	12.0000000000 (0)
3C	4A	5F	50	10.000 (35)
3C	4A	5G	49	8.000 (11)
3C	4A	5J	250	4.0000000 (1721)
3C	4A	5K	50	0.000 (32)
3C	4B	5C	50	12.2998107979 (40)
3C	4B	5D	50	10.322465743 (1)
3C	4B	5E	50	4.0000000000 (27)
3C	4B	5F	50	10.0000000000 (20)
3C	4B	5G	50	11.41640786 (1)
3C	4B	5I	50	11.70820393 (8)
3C	4B	5J	250	4.0000000000 (44)
3C	4B	5K	50	14.4721359 (1)
3C	4C	5A	247	8.000 (303)

$\mathcal{O}_\alpha$	$\mathcal{O}_\beta$	$\mathcal{O}_\gamma$	# values	$-32\pi^2\, C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$
3C	4C	5B	250	8.000 (169)
3C	4C	5D	50	8.000000000 (382)
3C	4C	5E	50	12.0000000000 (7)
3C	4C	5G	50	8.0000000000 (0)
3C	4C	5H	248	8.000 (123)
3C	4C	5I	50	8.0000000000 (6)
3C	4C	5J	250	4.0000000000 (44)
3C	4D	5A	250	13.7370341836 (29)
3C	4D	5C	50	5.6666666667 (691)
3C	4D	5F	50	4.0000000000 (52)
3C	4D	5H	238	8.9296324830 (1)
3C	4E	5C	50	15.00 (39)
3C	4E	5D	49	14.47 (50)
3C	4E	5E	50	12.0000000000 (0)
3C	4E	5F	50	10.0000 (28)
3C	4E	5G	50	8.0000 (12)
3C	4E	5I	50	5.52 (1)
3C	4E	5J	250	4.0000000 (7218)
3C	4E	5K	50	0.00000 (76)
3C	4F	5C	50	2.44595191 (9)
3C	4F	5D	50	-1.7082039335 (337)
3C	4F	5E	50	4.0000000000 (4)
3C	4F	5F	50	10.0000000000 (31)
3C	4F	5G	50	-15.41640785 (10)
3C	4F	5I	50	4.7920380737 (10)
3C	4F	5J	250	4.0000000000 (86)
3C	4F	5K	50	5.52786404 (1)
3C	4G	5C	50	15.0000000 (305)
3C	4G	5D	50	14.47213595 (3)
3C	4G	5F	50	10.0000000000 (22)
3C	4G	5G	50	8.0000000000 (10)
3C	4G	5I	50	5.5278640450 (27)
3C	4G	5J	50	4.0000000000 (2)
3C	4G	5K	50	0.0000000000 (2)

A.2.3 Qualitative Structure

Beside the results given above, there are classes for which all three-point functions vanish identically and therefore no results are obtained at all. For some classes of operators we obtain ambiguous results, these are divided into three types:

- If $C^{(0)}_{\alpha\beta\gamma}$ is finite, ambiguous results suggest that there is an additional degeneracy that is hidden at one-loop level. We would have to choose the right diagonal basis to obtain proper results.
- If $C^{(0)}_{\alpha\beta\gamma} \approx 0$ but all values for $C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$ hint at a definite result, this result is probably right but has a large deviation.
- If on the other hand $C^{(0)}_{\alpha\beta\gamma} \approx 0$ and $C^{(1)}_{\alpha\beta\gamma}/C^{(0)}_{\alpha\beta\gamma}$ varies considerably, there is no clear interpretation of these results.

The qualitative structure of the three-point functions is listed in the tables below. We use the following symbolic notations:

Symbol	Description
✓	results are available
(✓)	results with large deviation are available
0	zero one-loop but non-zero tree-level structure constants
×	vanishing three-point function (also on tree-level)
D	results hint at an additional degeneracy
?	there is no clear interpretation of the results

Class 2B with Two Operators of Length Three

2B	3A	3B	3C
3A	✓	×	×
3B		✓	✓
3C			0

Class 2B with Two Operators of Length Four

2B	4A	4B	4C	4D	4E	4F	4G
4A	×	✓	?	×	×	✓	0
4B		✓	✓	(✓)	✓	0	✓
4C			✓	✓	×	✓	0
4D				D	×	(✓)	0
4E					×	✓	0
4F						✓	✓
4G							0

Class 3A with Operators of Length Three and Four

3A	4A	4B	4C	4D	4E	4F	4G
3A	✓	✓	✓	(✓)	✓	✓	(✓)
3B	×	×	✓	✓	×	×	?
3C	×	?	?	✓	×	?	?

Class 3B with Operators of Length Three and Four

3B	4A	4B	4C	4D	4E	4F	4G
3B	✓	✓	×	×	✓	✓	?
3C	×	✓	✓	✓	×	?	✓

Class 3C with Operators of Length Three and Four

3C	4A	4B	4C	4D	4E	4F	4G
3C	✓	✓	✓	0	✓	?	0

Class 4A with Two Operators of Length Four

4A	4A	4B	4C	4D	4E	4F	4G
4A	✓	×	?	×	✓	×	(✓)
4B		✓	?	×	×	✓	?
4C			D	✓	?	?	?
4D				✓	×	×	(✓)
4E					✓	×	?
4F						✓	?
4G							✓

Class 4B with Two Operators of Length Four

4B	4B	4C	4D	4E	4F	4G
4B	✓	✓	(✓)	✓	✓	✓
4C		D	✓	?	?	?
4D			D	×	(✓)	(✓)
4E				×	✓	?
4F					✓	✓
4G						✓

Class 4C and 4D with Two Operators of Length Four

4C	4C	4D	4E	4F	4G
4C	D	?	D	D	✓
4D		D	?	✓	✓
4E			(✓)	?	D
4F				✓	D
4G					✓

4D	4D	4E	4F	4G
4D	✓	✓	D	✓
4E		×	×	(✓)
4F			(✓)	(✓)
4G				0

Class 4E, 4F and 4G with Two Operators of Length Four

4E	4E	4F	4G
4E	✓	×	✓
4F		✓	?
4G			✓

4F	4F	4G
4F	✓	✓
4G		✓

4G	4G
4G	0

Class 2B with Two Operators of Length Five

2B	5A	5B	5C	5D	5E	5F	5G	5H	5I	5J	5K
5A	✓	?	✓	✓	?	?	?	D	✓	(✓)	0
5B		D	D	?	D	D	?	?	D	D	0
5C			D	D	D	D	✓	?	D	✓	?
5D				D	✓	D	?	✓	D	✓	?
5E					D	D	D	?	✓	✓	✓
5F						D	✓	?	D	✓	?
5G							D	?	?	✓	?
5H								✓	✓	(✓)	?
5I									D	✓	?
5J										✓	✓
5K											0

Class 3A with Operators of Length Four and Five

3A	5A	5B	5C	5D	5E	5F	5G	5H	5I	5J	5K
4A	✓	×	×	(✓)	×	×	?	✓	(✓)	×	?
4B	✓	?	✓	✓	?	✓	?	✓	D	?	?
4C	✓	D	D	D	✓	✓	?	✓	D	?	?
4D	(✓)	D	D	✓	D	D	✓	(✓)	✓	✓	(✓)
4E	✓	×	×	(✓)	×	×	?	✓	(✓)	×	?
4F	D	?	✓	D	?	✓	?	?	✓	?	?
4G	?	?	?	✓	?	?	?	?	✓	?	?

Class 3B with Operators of Length Four and Five

3B	5A	5B	5C	5D	5E	5F	5G	5H	5I	5J	5K
4A	?	✓	×	✓	(✓)	×	?	×	✓	✓	✓
4B	×	✓	✓	✓	✓	✓	?	×	✓	✓	✓
4C	✓	✓	D	(✓)	✓	D	✓	✓	(✓)	?	?
4D	✓	?	✓	✓	?	✓	?	✓	✓	?	(✓)
4E	(✓)	D	?	✓	✓	?	?	✓	✓	✓	✓
4F	?	✓	✓	✓	✓	✓	?	?	✓	✓	✓
4G	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Class 3C with Operators of Length Four and Five

3C	5A	5B	5C	5D	5E	5F	5G	5H	5I	5J	5K
4A	?	?	?	?	✓	✓	✓	?	?	✓	0
4B	?	?	✓	✓	✓	✓	✓	?	✓	✓	✓
4C	✓	✓	D	✓	✓	D	✓	✓	✓	✓	?
4D	✓	?	✓	D	?	✓	?	✓	D	?	?
4E	?	?	(✓)	(✓)	✓	✓	✓	?	(✓)	✓	0
4F	?	?	✓	✓	✓	✓	✓	?	✓	✓	✓
4G	?	?	✓	✓	?	✓	✓	?	✓	✓	0

Appendix B

Program Code Listings

B.1 Summation of Permutations

As described in section 6.5 we can handle the summation of index contractions for all permutations computationally. The code of the corresponding Matlab® programs is listed here. These program codes, as well as those in the next section, are by no means as beautiful or simple as possible but they serve their purpose.

B.1.1 Anomalous Dimensions

This program sums all tree-level and one-loop contributions to the two-point functions.

Listing B.1: zpf.m

```
function [tree,loop] = zpf(b1,b2a)
tree = 0;
loop = 0;
l = size(b1,1);
if size(b2a,1) == l
 for r = 0:l-1
  b2 = circshift(flipud(b2a),r);
  tree = tree + prod(diag(b1*b2.'));
  if l > 2
   for s = 0:l-1
    if s == 0
     triv1 = b1(2:l-1,:);
     triv2 = b2(2:l-1,:);
     u1 = b1(1,:);
     u2 = b1(l,:);
     v1 = b2(1,:);
     v2 = b2(l,:);
    else
     triv1 = b1;
     triv1(s:s+1,:) = [];
     triv2 = b2;
     triv2(s:s+1,:) = [];
```

```
      u1 = b1(s,:);
      u2 = b1(s+1,:);
      v1 = b2(s,:);
      v2 = b2(s+1,:);
     end
     trivprod = prod(diag(triv1*triv2.'));
     if trivprod ~= 0
      loop = loop + trivprod*( 2*(u1*v1.')*(u2*v2.')
           - 2*(u1*v2.')*(u2*v1.') + (u1*u2.')*(v1*v2.') );
     end
    end
   else
    loop = loop + 2*(b1(1,:)*b1(2,:).')*(b2(1,:)*b2(2,:).');
   end
  end
 end
end
```

B.1.2 Structure Constants

This program sums the 3-gons. The renormalisation scheme independent structure constants can be calculated with these results.

Listing B.2: dpf.m

```
function [c0,c1] = dpf(b1,b2,b3)
l = size(b1,1);
m = size(b2,1);
n = size(b3,1);
c0 = 0;
c1 = 0;
for r = 0:l-1
 for s = 0:m-1
  for t = 0:n-1
   p1 = circshift(b1,r);
   p2 = circshift(b2,s);
   p3 = circshift(b3,t);
   c0 = c0 + c0komp(p1,p2,p3,l,m,n);
   c1 = c1 + c1komp(p1,p2,p3,l,m,n);
  end
 end
end
end
```

```
function c = c0komp(p1,p2,p3,l,m,n)
 r = (l + m - n) / 2;
 s = (m + n - l) / 2;
 t = (l + n - m) / 2;
 c = (r>0)*prod(diag(p1(1:r,:)*flipud(p2(s+1:m,:)).'))
   * (s>0)*prod(diag(p2(1:s,:)*flipud(p3(t+1:n,:)).'))
   * (t>0)*prod(diag(p3(1:t,:)*flipud(p1(r+1:l,:)).'));
end

function c = c1komp(p1,p2,p3,l,m,n)
 if l == 2
  if m == 2
   if n == 2
    c = 2*((p1(1,:)*p1(2,:).')*prod(diag(p2*flipud(p3).'))
      + (p2(1,:)*p2(2,:).')*prod(diag(p1*flipud(p3).'))
      + (p3(1,:)*p3(2,:).')*prod(diag(p2*flipud(p1).')));
   else
    c = 0;
   end
  else
   c = 2*(p1(1,:)*p1(2,:).')*prod(diag(p2*flipud(p3).'))
     + prod(diag(p2(1:m-2,:)*flipud(p3(3:n,:)).'))
       * ( 4 * (p1(1,:)*p2(m,:).') * (p1(2,:)*p3(1,:).')
             * (p2(m-1,:)*p3(2,:).')
         - 2 * (p1(1,:)*p2(m-1,:).') * (p1(2,:)*p3(1,:).')
             * (p2(m,:)*p3(2,:).')
         - 2 * (p1(1,:)*p2(m,:).') * (p1(2,:)*p3(2,:).')
             * (p2(m-1,:)*p3(1,:).')
         +     (p2(m-1,:)*p2(m,:).') * (p1(2,:)*p3(1,:).')
             * (p1(1,:)*p3(2,:).')
         +     (p1(1,:)*p2(m,:).') * (p3(2,:)*p3(1,:).')
             * (p2(m-1,:)*p1(2,:).') )
     + prod(diag(p2(2:m-1,:)*flipud(p3(2:n-1,:)).'))
       * ( 4 * (p1(1,:)*p2(m,:).') * (p1(2,:)*p3(1,:).')
             * (p2(1,:)*p3(n,:).')
         - 2 * (p1(1,:)*p2(1,:).') * (p1(2,:)*p3(1,:).')
             * (p2(m,:)*p3(n,:).')
         - 2 * (p1(1,:)*p2(m,:).') * (p1(2,:)*p3(n,:).')
             * (p2(1,:)*p3(1,:).')
         +     (p2(1,:)*p2(m,:).') * (p1(2,:)*p3(1,:).')
             * (p1(1,:)*p3(n,:).')
         +     (p1(1,:)*p2(m,:).') * (p3(n,:)*p3(1,:).')
             * (p2(1,:)*p1(2,:).') );
  end
 else
```

```
  r = (l + m - n) / 2;
  s = (m + n - l) / 2;
  t = (l + n - m) / 2;
  finnen  = prod(diag(p1(1:r-1,:)*flipud(p2(s+2:m,:)).'))
          * prod(diag(p2(1:s-1,:)*flipud(p3(t+2:n,:)).'))
          * prod(diag(p3(1:t-1,:)*flipud(p1(r+2:l,:)).'));
  faussen = prod(diag(p1(2:r,:)*flipud(p2(s+1:m-1,:)).'))
          * prod(diag(p2(2:s,:)*flipud(p3(t+1:n-1,:)).'))
          * prod(diag(p3(2:t,:)*flipud(p1(r+1:l-1,:)).'));
  c = 0;
  if finnen
   c = finnen * dreigons(p1(r,:), p1(r+1,:), p2(s,:),
                         p2(s+1,:), p3(t,:), p3(t+1,:));
  end
  if faussen
   c = c + faussen * dreigons(p1(1,:), p1(l,:), p2(1,:),
                              p2(m,:), p3(1,:), p3(n,:));
  end
 end
end

function d = dreigons(u1,u2,v1,v2,w1,w2)
 d = 6 * (u1*v2.') * (v1*w2.') * (w1*u2.')
   - 2 * (u2*v2.') * (v1*w2.') * (w1*u1.')
   - 2 * (u1*v1.') * (v2*w2.') * (w1*u2.')
   - 2 * (u1*v2.') * (v1*w1.') * (w2*u2.')
   +     (u1*u2.') * (v1*w2.') * (w1*v2.')
   +     (u1*w2.') * (v1*v2.') * (w1*u2.')
   +     (u1*v2.') * (v1*u2.') * (w1*w2.');
end
```

B.2 Diagonal Structure Constants from Non-Diagonal Bases

The Mathematica® and Matlab® routines to calculate the whole spectrum of structure constants for given lengths by brute force are listed below.

B.2.1 Calculation with Mathematica®

The $SO(6)$ standard bases for length two and three operators are defined in `basen.m`. The vectors `basis2` and `basis3` defined here are taken as arguments by the following programs.

The program file `mmatrizen.m` listed below can be used to calculate all two-point functions (`gMatrix`) and diagonalise them. The change-of-basis matrix and anomalous dimensions are written in text files as output by the functions `WriteLx`.

The program `strukkonst.m` calculates all diagonal structure constants. The `WriteCxxx` functions are the main functions. They load the change-of-basis matrix stored from the above program and calculate the diagonal structure constants by summation with help from the change-of-basis matrix.

Listing B.3: basen.m

```
CanonicalOrder[p_] := Sort[Table[RotateLeft[p,k],
                          {k,Length[p]}]][[1]]
basis2 = DeleteDuplicates[Flatten[Table[Sort[{i,j}],
           {i,6},{j,6}],1]];
basis3 = DeleteDuplicates[Flatten[Table[CanonicalOrder[
           {i,j,k}],{i,6},{j,6},{k,6}],2]];
```

Listing B.4: mmatrizen.m

```
zpfvorfaktor[a_,b_,l_] := Module[ {erg,i,n1,n2},
 Catch[
  n1 = 1;
  n2 = 1;
  For[ i=1, i<l, i++,
   If[ a==RotateLeft[a,i], n1++];
   If[ b==RotateLeft[b,i], n2++]
  ];
  erg = 1/Sqrt[n1*n2];
 Throw[erg]] ]

gElem[a_,b_,l_] := Module[ {i,j,erg},
 Catch[
  erg=0;
  If[l>2,
   For[i=0,i<l,i++,
   For[j=0,j<l,j++,
    If[RotateLeft[a,i+j][[3;;l]]==RotateLeft[b,j][[3;;l]],
     If[RotateLeft[a,i+j][[1;;2]]==RotateLeft[b,j][[1;;2]],
         erg+=2];
     If[RotateLeft[a,i+j][[1;;2]]==Reverse[RotateLeft[b,j
         ][[1;;2]]], erg-=2];
     If[RotateLeft[a,i+j][[1]]==RotateLeft[a,i+j][[2]] &&
         RotateLeft[b,j][[1]]==RotateLeft[b,j][[2]], erg
         ++]]]],
```

```
    If[a[[1]]==a[[2]] && b[[1]]==b[[2]], erg+=4]];
  Throw[erg]]  ]

gMatrix[basis_,l_] := Table[zpfvorfaktor[basis[[i]],basis[[
      j]],l]*gElem[basis[[i]],basis[[j]],l],{i,Length[basis
      ]},{j,Length[basis]}]

MMgm[basis_,l_] := Module[ {g,es,gm,mm},
  Catch[
    g  = gMatrix[basis,l];
    es = Eigensystem[g];
    gm = es[[1]];
    mm = Inverse[Transpose[es[[2]]]];
  Throw[{gm,mm}]]  ]

WriteL2[] := Module[ {erg},
  Get["basen.m"];
  erg = MMgm[basis2,2];
  Put[erg[[1]],"data/gm2.txt"];
  Put[erg[[2]],"data/mm2.txt"];  ]

WriteL3[] := Module[ {erg},
  Get["basen.m"];
  erg = MMgm[basis3,3];
  Put[erg[[1]],"data/gm3.txt"];
  Put[erg[[2]],"data/mm3.txt"];  ]
```

Listing B.5: strukkonst.m

```
vorfaktor[a_,b_,c_,l_,m_,n_] := Module[ {erg,i,n1,n2,n3},
  Catch[
    n1 = 1;
    For[i=1,i<l,i++,
      If[a==RotateLeft[a,i],n1++]];
    n2 = 1;
    For[i=1,i<m,i++,
      If[b==RotateLeft[b,i],n2++]];
    n3 = 1;
    For[i=1,i<n,i++,
      If[c==RotateLeft[c,i],n3++]];
    erg = 1/Sqrt[n1*n2*n3];
  Throw[erg]]  ]

C0[a_,b_,c_,l_,m_,n_,r_,s_,t_] := Module[ {i,j,k,erg},
  Catch[
    erg = 0;
```

```
  For[i=0,i<l,i++,For[j=0,j<m,j++,For[k=0,k<n,k++,
   If[RotateLeft[a,i][[1;;r]] == Reverse[RotateLeft[b,j
       ]][[1;;r]]
    && Reverse[RotateLeft[a,i]][[1;;t]] == RotateLeft[c,k
       ][[1;;t]]
    && RotateLeft[b,j][[1;;s]] == Reverse[RotateLeft[c,k
       ]][[1;;s]],erg++]
  ]]];
  erg *= vorfaktor[a,b,c,l,m,n];
 Throw[erg]]  ]

dreigon[a_,b_,c_,l_,m_,n_,r_,s_,t_] := Module[ {erg,i,j,k},
 Catch[
  erg = 0;
  For[i=0,i<l,i++,For[j=0,j<m,j++,For[k=0,k<n,k++,
   If[RotateLeft[b,j][[1;;s]]
      ==Reverse[RotateLeft[c,k]][[1;;s]],
    If[RotateLeft[a,i][[1;;r-1]]
       ==Reverse[RotateLeft[b,j]][[1;;r-1]]
     && Reverse[RotateLeft[a,i]][[1;;t-1]]
       ==RotateLeft[c,k][[1;;t-1]],
      If[RotateLeft[a,i][[r]]==RotateLeft[b,j][[s+1]]
      && RotateLeft[a,i][[r+1]]==RotateLeft[c,k][[t]],
         erg += 2];
      If[RotateLeft[a,i][[r]]==RotateLeft[c,k][[t]]
      && RotateLeft[a,i][[r+1]]==RotateLeft[b,j][[s+1]],
         erg -= 2];
      If[RotateLeft[a,i][[r]]==RotateLeft[a,i][[r+1]]
      && RotateLeft[b,j][[s+1]]==RotateLeft[c,k][[t]],
         erg++]];
    If[RotateLeft[a,i][[2;;r]]
       ==Reverse[RotateLeft[b,j]][[2;;r]]
     && Reverse[RotateLeft[a,i]][[2;;t]]
       ==RotateLeft[c,k][[2;;t]],
      If[RotateLeft[a,i][[1]]==RotateLeft[b,j][[m]]
      && RotateLeft[a,i][[l]]==RotateLeft[c,k][[1]],
         erg += 2];
      If[RotateLeft[a,i][[1]]==RotateLeft[c,k][[1]]
      && RotateLeft[a,i][[l]]==RotateLeft[b,j][[m]],
         erg -= 2];
      If[RotateLeft[a,i][[1]]==RotateLeft[a,i][[l]]
      && RotateLeft[b,j][[m]]==RotateLeft[c,k][[1]],
         erg++]]]  ]]];
 Throw[erg]]  ]
```

```
C1222[a_,b_,c_] := Module[ {erg},
 Catch[
  erg = 0;
  If[a[[1]]==a[[2]],
   If[b==c,erg+=8];
   If[b==Reverse[c],erg+=8]];
  If[b[[1]]==b[[2]],
   If[a==c,erg+=8];
   If[a==Reverse[c],erg+=8]];
  If[c[[1]]==c[[2]],
   If[b==a,erg+=8];
   If[b==Reverse[a],erg+=8]];
  erg *= vorfaktor[a,b,c,2,2,2];
 Throw[erg]]  ]

C1233[a_,b_,c_] := Module[ {erg},
 Catch[
  erg = dreigon[b,c,a,3,3,2,2,1,1]
      + dreigon[c,a,b,3,2,3,1,1,2];
  If[a[[1]]==a[[2]],
   If[b==Reverse[c],erg+=12];
   If[b==Reverse[RotateLeft[c,1]],erg+=12];
   If[b==Reverse[RotateLeft[c,2]],erg+=12]];
  erg *= vorfaktor[a,b,c,2,3,3];
 Throw[erg]]  ]

WriteC222[] := Module[ {i,j,k,c0,c1,M2,r,s,t,cache0,cache1,
                        l2,quot,leer,print},
 Get["basen.m"];
 l2 = Length[basis2];
 M2 = SparseArray[Get["data/mm2.txt"]];
 cache0 = SparseArray[{},{l2,l2,l2},leer];
 cache1 = SparseArray[{},{l2,l2,l2},leer];
 For[i=1,i<=l2,i++,For[j=i,j<=l2,j++,For[k=j,k<=l2,k++,
  print = False;
  c0 = 0;
  c1 = 0;
  For[r=1,r<=l2,r++,For[s=1,s<=l2,s++,For[t=1,t<=l2,t++,
   If[M2[[i,r]]!=0 && M2[[j,s]]!=0 && M2[[k,t]]!=0,
    If[cache0[[r,s,t]]==leer,cache0[[r,s,t]]=C0[basis2[[r
        ]],basis2[[s]],basis2[[t]],2,2,2,1,1,1]];
    If[cache1[[r,s,t]]==leer,cache1[[r,s,t]]=C1222[basis2[[
        r]],basis2[[s]],basis2[[t]]]];
    c0 += M2[[i,r]]*M2[[j,s]]*M2[[k,t]]*cache0[[r,s,t]];
    c1 += M2[[i,r]]*M2[[j,s]]*M2[[k,t]]*cache1[[r,s,t]]]
```

```
    ]]];
   If[c0==0,If[c1!=0,print=True;quot="Inf"],print=True;quot=
      c1/c0];
   If[print,Print[{i,j,k,TeXForm[c0],TeXForm[c1],TeXForm[
      quot]}]]
  ]]];
Exit[]

WriteC233[] := Module[ {i,j,k,c0,c1,M2,M3,r,s,t,cache0,
                          cache1,l2,quot,leer,print},
 Get["basen.m"];
 l2 = Length[basis2];
 l3 = Length[basis3];
 M2 = SparseArray[Get["data/mm2.txt"]];
 M3 = SparseArray[Get["data/mm3.txt"]];
 cache0 = SparseArray[{},{l2,l3,l3},leer];
 cache1 = SparseArray[{},{l2,l3,l3},leer];
 For[i=1,i<=l2,i++,For[j=1,j<=l3,j++,For[k=j,k<=l3,k++,
   print=False;
   c0 = 0;
   c1 = 0;
   For[r=1,r<=l2,r++,For[s=1,s<=l3,s++,For[t=1,t<=l3,t++,
   If[M2[[i,r]]!=0 && M3[[j,s]]!=0 && M3[[k,t]]!=0,
    If[cache0[[r,s,t]]==leer,cache0[[r,s,t]]=C0[basis2[[r]],
       basis3[[s]],basis3[[t]],2,3,3,1,2,1]];
    If[cache1[[r,s,t]]==leer,cache1[[r,s,t]]=C1233[basis2[[r
       ]],basis3[[s]],basis3[[t]]]]];
    c0 += M2[[i,r]]*M3[[j,s]]*M3[[k,t]]*cache0[[r,s,t]];
    c1 += M2[[i,r]]*M3[[j,s]]*M3[[k,t]]*cache1[[r,s,t]]]
   ]]];
   If[c0==0,If[c1!=0,print=True;quot="Inf"],print=True;quot=
      c1/c0];
   If[print,Print[{i,j,k,TeXForm[c0],TeXForm[c1],TeXForm[
      quot]}]]
  ]]];
Exit[]
```

B.2.2 Calculation with Matlab®

The programs listed below mainly serve the same purpose as the Mathematica® programs listed above. Nevertheless there are some differences owed to the fact that the language of Matlab® is fundamentally different from that of Mathematica®.

The standard bases are defined by the `basis` function and the change-of-basis matrices are calculated in `mmgm`. The function `writec` stores a rank-three tensor of the non-diagonal structure constants calculated by `ctensor`. The non-diagonal constants are diagonalised by the `diagc` function.

Listing B.6: canonicalorder.m

```
function c = canonicalorder(parr)
c = zeros(size(parr));
for row = 1:size(parr,1)
  p = parr(row,:);
  table = zeros(length(p));
  for i = 1:length(p)
   table(i,:) = circshift(p',i-1)';
  end
  sorted = sortrows(table);
  c(row,:) = sorted(1,:);
end
end
```

Listing B.7: basis.m

```
function b = basis(l)
E          = eye(6);
count      = zeros(1,6);
count(1)   = l;
numbers    = ones(1,l);
while count(6) < l
 if count(6) == 0
  for i = 0:4
   if count(5-i) > 0
    count(5-i) = count(5-i) - 1;
    count(6-i) = count(6-i) + 1;
    break
   end
  end
 else
  for i = 0:4
   if count(5-i) > 0
    count(5-i) = count(5-i) - 1;
    c = count(6);
    count(6) = 0;
    count(6-i) = count(6-i) + 1 + c;
    break
   end
  end
 end
```

```
 numbers = [numbers ; unique(canonicalorder(perms([ones(1,
     count(1)) 2*ones(1,count(2)) 3*ones(1,count(3)) 4*ones
     (1,count(4)) 5*ones(1,count(5)) 6*ones(1,count(6))])),
     'rows')];
end
b = zeros(l,6,size(numbers,1));
for i = 1:size(numbers,1)
 for j = 1:l
  b(j,:,i) = E(numbers(i,j),:);
 end
end
end
```

Listing B.8: tgmatrix.m

```
function [t,g] = tgmatrix(l)
b   = basis(l);
dim = size(b,3);
t   = zeros(dim);
g   = zeros(dim);
for i = 1:dim
 for j = 1:dim
  sprintf('i,j:␣%d,␣%d\n',i,j)
  for r=0:l-1
   b1 = b(:,:,i);
   b2 = circshift(flipud(b(:,:,j)),r);
   t(i,j) = t(i,j) + prod(diag(b1*b2'));
   if l > 2
    for s=0:l-1
     if s == 0
      triv1 = b1(2:l-1,:);
      triv2 = b2(2:l-1,:);
      u1    = b1(1,:);
      u2    = b1(l,:);
      v1    = b2(1,:);
      v2    = b2(l,:);
     else
      triv1 = b1;
      triv1(s:s+1,:) = [];
      triv2 = b2;
      triv2(s:s+1,:) = [];
      u1 = b1(s,:);
      u2 = b1(s+1,:);
      v1 = b2(s,:);
      v2 = b2(s+1,:);
     end
```

```
      trivprod = prod(diag(triv1*triv2'));
      if trivprod ~= 0
       g(i,j) = g(i,j) + trivprod*( 2*(u1*v1')*(u2*v2') -
           2*(u1*v2')*(u2*v1') + (u1*u2')*(v1*v2') );
      end
     end
    else
     g(i,j) = g(i,j) + 2*(b1(1,:)*b1(2,:)')*(b2(1,:)*b2(2,:)
         ');
    end
   end
  end
 end
end
```

Listing B.9: mmgm.m

```
function mmgm(l)
 [t g] = tgmatrix(l);
 [V gm] = eig(g*inv(t));
 gmvsorted = sortrows([diag(gm) inv(V)]);
 gm = gmvsorted(:,1)';
 M = gmvsorted(:,2:end);
 datei = strcat('data/mmgm',int2str(l),'.mat');
 save(datei,'gm','M');
end
```

Listing B.10: ctensor.m

```
function [c0,c1] = ctensor(l,m,n)
lmn = sort([l m n]);
l   = lmn(1);
m   = lmn(2);
n   = lmn(3);
clear lmn;
basis1 = basis(l);
if m == l
 basis2 = basis1;
else
 basis2 = basis(m);
end
switch n
 case l
  basis3 = basis1;
 case m
  basis3 = basis2;
```

```
  otherwise
   basis3 = basis(n);
end
dim1 = size(basis1,3);
dim2 = size(basis2,3);
dim3 = size(basis3,3);
c0 = zeros(dim1,dim2,dim3);
c1 = zeros(dim1,dim2,dim3);
for i = 1:dim1
 for j = 1:dim2
  for k = 1:dim3
   sprintf('%d %d %d',i,j,k)
   b1 = basis1(:,:,i);
   b2 = basis2(:,:,j);
   b3 = basis3(:,:,k);
   for r = 0:l-1
    for s = 0:m-1
     for t = 0:n-1
      p1 = circshift(b1,r);
      p2 = circshift(b2,s);
      p3 = circshift(b3,t);
      c0(i,j,k) = c0(i,j,k) + c0komp(p1,p2,p3,l,m,n);
      c1(i,j,k) = c1(i,j,k) + c1komp(p1,p2,p3,l,m,n);
     end
    end
   end
  end
 end
end
end

function c = c0komp(p1,p2,p3,l,m,n)
 r = (l + m - n) / 2;
 s = (m + n - l) / 2;
 t = (l + n - m) / 2;
 c = (r>0)*prod(diag(p1(1:r,:)*flipud(p2(s+1:m,:))'))
   * (s>0)*prod(diag(p2(1:s,:)*flipud(p3(t+1:n,:))'))
   * (t>0)*prod(diag(p3(1:t,:)*flipud(p1(r+1:l,:))'));
end

function c = c1komp(p1,p2,p3,l,m,n)
 if l == 2
  if m == 2
   if n == 2
    c = 2*((p1(1,:)*p1(2,:)')*prod(diag(p2*flipud(p3)'))
```

```
          + (p2(1,:)*p2(2,:)')*prod(diag(p1*flipud(p3)'))
          + (p3(1,:)*p3(2,:)')*prod(diag(p2*flipud(p1)')));
    else
      c = 0;
    end
  else
    c = 2*(p1(1,:)*p1(2,:)')*prod(diag(p2*flipud(p3)'))
      + prod(diag(p2(1:m-2,:)*flipud(p3(3:n,:))'))
        * ( 4*(p1(1,:)*p2(m,:)') * (p1(2,:)*p3(1,:)')
        * (p2(m-1,:)*p3(2,:)')
      - 2 * (p1(1,:)*p2(m-1,:)') * (p1(2,:)*p3(1,:)')
          * (p2(m,:)*p3(2,:)')
      - 2 * (p1(1,:)*p2(m,:)') * (p1(2,:)*p3(2,:)')
          * (p2(m-1,:)*p3(1,:)')
      + (p2(m-1,:)*p2(m,:)') * (p1(2,:)*p3(1,:)')
        * (p1(1,:)*p3(2,:)')
      + (p1(1,:)*p2(m,:)') * (p3(2,:)*p3(1,:)')
        * (p2(m-1,:)*p1(2,:)') )
      + prod(diag(p2(2:m-1,:)*flipud(p3(2:n-1,:))'))
        * ( 4*(p1(1,:)*p2(m,:)') * (p1(2,:)*p3(1,:)')
        * (p2(1,:)*p3(n,:)')
      - 2 * (p1(1,:)*p2(1,:)') * (p1(2,:)*p3(1,:)')
          * (p2(m,:)*p3(n,:)')
      - 2 * (p1(1,:)*p2(m,:)') * (p1(2,:)*p3(n,:)')
          * (p2(1,:)*p3(1,:)')
      + (p2(1,:)*p2(m,:)') * (p1(2,:)*p3(1,:)')
        * (p1(1,:)*p3(n,:)')
      + (p1(1,:)*p2(m,:)') * (p3(n,:)*p3(1,:)')
        * (p2(1,:)*p1(2,:)') );
  end
else
  r = (l + m - n) / 2;
  s = (m + n - l) / 2;
  t = (l + n - m) / 2;
  finnen = prod(diag(p1(1:r-1,:)*flipud(p2(s+2:m,:))'))
         * prod(diag(p2(1:s-1,:)*flipud(p3(t+2:n,:))'))
         * prod(diag(p3(1:t-1,:)*flipud(p1(r+2:l,:))'));
  faussen = prod(diag(p1(2:r,:)*flipud(p2(s+1:m-1,:))'))
          * prod(diag(p2(2:s,:)*flipud(p3(t+1:n-1,:))'))
          * prod(diag(p3(2:t,:)*flipud(p1(r+1:l-1,:))'));
  c = 0;
  if finnen
    c = finnen * dreigons(p1(r,:),p1(r+1,:),p2(s,:),
                          p2(s+1,:),p3(t,:),p3(t+1,:));
  end
```

```
    if faussen
     c = c + faussen * dreigons(p1(1,:),p1(l,:),p2(1,:),
                                 p2(m,:),p3(1,:),p3(n,:));
    end
  end
end

function d = dreigons(u1,u2,v1,v2,w1,w2)
  d = 6 * (u1*v2') * (v1*w2') * (w1*u2')
    - 2 * (u2*v2') * (v1*w2') * (w1*u1')
    - 2 * (u1*v1') * (v2*w2') * (w1*u2')
    - 2 * (u1*v2') * (v1*w1') * (w2*u2')
    + (u1*u2') * (v1*w2') * (w1*v2')
    + (u1*w2') * (v1*v2') * (w1*u2')
    + (u1*v2') * (v1*u2') * (w1*w2');
end
```

Listing B.11: writec.m

```
function writec(l,m,n)
  [c0,c1] = ctensor(l,m,n);
  datei = strcat('data/nondiagc',int2str(l),int2str(m),
      int2str(n),'.mat');
  save(datei,'c0','c1');
end
```

Listing B.12: diagc.m

```
function d = diagc(l,m,n)
limit = 1.e-12;
if [l m n] == [3 3 4]
   load data/nondiagc334.mat
   fid=fopen('data/c334.txt','w');
   load data/mmgm3.mat
   M1 = M;
   M2 = M;
   clear M;
   load data/mmgm4.mat
   M3 = M;
end
% [...] %
clear M;
clear gm;

j0 = 1;
k0 = 1;
```

```
for i=1:length(M1)
 if length(M2) == length(M1)
  j0 = i;
 end
 for j=j0:length(M2)
  if length(M3) == length(M2)
   k0 = j;
  end
  for k=k0:length(M3)
   sprintf('%d %d %d',i,j,k)
   dc0 = 0;
   dc1 = 0;
   for r=1:length(M1)
    for s=1:length(M2)
     for t=1:length(M3)
      ms = M1(i,r)*M2(j,s)*M3(k,t);
      dc0 = dc0 + ms*c0(r,s,t);
      dc1 = dc1 + ms*c1(r,s,t);
     end
    end
   end
   if abs(dc0) > limit || abs(dc1) > limit
    quot = NaN;
    if dc0 ~= 0
     quot = dc1/dc0;
    end
    fprintf(fid,'%d %d %d: c0= %10.5f ; c1= %10.5f = %10.5f
       c0\n',i,j,k,dc0,dc1,quot);
   end
  end
 end
end
end
```

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