Supersymmetric Wilson Loops in $\mathcal{N} = 4$ Super Yang-Mills Theory

Diplomarbeit

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Abstract

We study an extension of classical Wilson loops around lightlike polygons in $\mathcal{N} = 4$ super Yang-Mills theory, which was introduced by Caron-Huot as an observable dual to scattering amplitudes. After a thorough exposition of the underlying quantum field theory, we use symmetry considerations to derive explicit expressions for the edge and vertex operators of the resulting supersymmetric Wilson loops. The contributions to the quantum expectation value coming from the edges and vertices are conveniently summarised in the form of Feynman rules. Furthermore, we yield expressions for tree-level components by explicit calculations. It turns out that there is indeed a striking partial duality with scattering amplitudes which is, however, broken. This should be independent of the regularisation method used. Finally, we repeat our treatment for a natural variant of the edge and vertex operators with an analogous result.

Zusammenfassung

Wir untersuchen eine Erweiterung der klassischen Wilson-Schleifen um lichtartige Polygone in der $\mathcal{N} = 4$ Super-Yang-Mills-Theorie, die von Caron-Huot als zu Streuamplituden duale Observable eingeführt wurde. Nach einer gründlichen Darstellung der zugrunde liegenden Quantenfeldtheorie benutzen wir Symmetrieüberlegungen, um explizite Ausdrücke für die Kanten- und Vertex-Operatoren der resultierenden supersymmetrischen Wilson-Schleifen herzuleiten. Die Beiträge zum Quanten-Erwartungswert, die von den Kanten und Ecken herrühren, lassen sich zweckmäßig in der Form von Feynman-Regeln zusammenfassen. Desweiteren gewinnen wir durch explizite Rechnungen Ausdrücke für Baumgraphen-Komponenten. Es stellt sich heraus, dass tatsächlich eine bemerkenswerte Teil-Dualität mit Streuamplituden existiert, die jedoch gebrochen ist. Dies sollte unabhängig von der benutzten Regularisierungs-Methode sein. Schließlich wiederholen wir unsere Behandlung für eine natürliche Variante der Kanten- und Ecken-Operatoren mit einem analogen Resultat.

Contents

Preface

Gluon scattering amplitudes have been known to be dual to Wilson loops along lightlike polygons. While first shown at strong coupling (cf. [AM07]) through the famous AdS/CFT duality introduced in [Mal98], this result has later been verified at weak coupling (cf. [DKS08] and [BHT08]). For a review, consult [AR08]. Recently, a similar duality (at weak coupling) between the full scattering amplitudes of $\mathcal{N} = 4$ super Yang-Mills theory and a supersymmetric extension of the Wilson loop observable has been claimed (cf. [CH11]). In this thesis, we analyse this conjecture through explicit calculations. It is organised as follows.

In Chp. 1, we provide a thorough introduction to $\mathcal{N} = 4$ super Yang-Mills theory. The purpose is twofold. First, we lay the foudations for later chapters, providing notation and important formulas including the Euler-Lagrange equations, supersymmetry generators as well as the Feynman rules. Second, we aim at a self-contained introduction to this theory, which is also readable for people with a mathematical background.

In Chp. 2, we analyse an ansatz for a super Wilson loop along lightlike polygons which is obtained by symmetry considerations. For the edge and vertex operators, we derive explicit formulas. While the calculations, especially for the vertex operators, turn out to be rather lengthy, the result can be stated in the form of a simple recursion formula. Our treatment remains purely classical.

In Chp. 3, we study the quantum field theory of supersymmetric Wilson loops. After establishing Feynman rules for the contributions to the quantum expectation value coming from the edges and vertices, we calculate explicit examples of tree-level components and compare them with scattering amplitudes. Our main result, the breaking of the proposed duality, can be found here. Finally, we repeat our treatment for a natural variant of the supersymmetric Wilson loop.

Chapter 1

$\mathcal{N} = 4$ Super Yang-Mills Theory

$\mathcal{N} = 4$ super Yang-Mills (SYM) theory plays an important role as a theoretical testing ground for its less symmetric cousin, quantum chromodynamics. In this chapter, we review its definition and properties. The advanced reader may skip the details.

1.1 Algebraic Preliminaries

In this section, we introduce the algebraic background for the fields of the theory. We adopt the conventions of [BDKM04] whenever applicable, with the exception of the charge conjugation matrix (cf. Sec. 1.1.2 below), where we follow [Ton07].

1.1.1 Odd Quantities and Matrices

In general, fermionic fields in a (pseudo-)classical field theory underlying a quantum field theory are assumed to be "odd", i.e. anticommuting, in order to ensure Fermi-Dirac statistics upon quantisation (consult [Nic91] as well as standard text books such as [PS95]). Mathematically, oddness can be modelled by making the fields take values in the odd part of a Grassmann algebra with suitably many generators. By consistency, the bosonic fields should then take values in the even part of the same algebra.

We denote the parity of such a Grassmann "number" a by $|a| \in \mathbb{Z}_2$ which, by definition, is 0 if a is even and 1 if it is odd. The commutation rule can thus be written $a \cdot b = (-1)^{|a||b|} b \cdot a$. This is equivalent to the vanishing of the super commutator

$$[a,\, b] := a \cdot b - (-1)^{|a||b|} b \cdot a \tag{1.1}$$

In the following, we shall implicitly assume that all quantities occurring are Grassmann valued, and as such are either even or odd. As the first example, consider real $n \times n$ matrices A with parity $|A|$ in the sense that their entries all have parity $\left|A^{ij}\right| = |A|$. This is to be distinguished from the parity of matrices in super-linear algebra ([Var04]).

Lemma 1.1.1. Let A, B, C be real $n \times n$ matrices. Then

$$\begin{aligned} \mathrm{tr}(A \cdot B) &= (-1)^{|A||B|} \mathrm{tr}(B \cdot A) \\ \mathrm{tr}(A\,[B,\, C]) &= (-1)^{(|A|+|B|)|C|} \mathrm{tr}(C\,[A,\, B]) \end{aligned}$$

where the bracket denotes the super commutator (1.1).

Proof. The first equation follows immediately.

$$\mathrm{tr}(AB) = A^{ab}B^{ba} = (-1)^{|A||B|}B^{ba}A^{ab} = (-1)^{|A||B|}\mathrm{tr}(BA)$$

For the second, we calculate

$$\begin{aligned}
\mathrm{tr}(A\,[B,\,C]) &= A^{ab}\,[B,\,C]^{ba} \\
&= A^{ab}\left(B^{bc}C^{ca} - (-1)^{|B||C|}C^{bc}B^{ca}\right) \\
&= A^{ab}B^{bc}C^{ca} - (-1)^{|B||C|}A^{ab}C^{bc}B^{ca} \\
&= (-1)^{(|A|+|B|)|C|}C^{ef}\left(A^{fb}B^{be} - A^{ae}B^{fa}\right) \\
&= (-1)^{(|A|+|B|)|C|}C^{ef}\left(A^{fb}B^{be} - (-1)^{|A||B|}B^{fb}A^{be}\right) \\
&= (-1)^{(|A|+|B|)|C|}C^{ef}\,[A,\,B]^{fe} \\
&= (-1)^{(|A|+|B|)|C|}\mathrm{tr}(C\,[A,\,B])
\end{aligned}$$

□

Special Unitary Lie Algebra

The fields of $\mathcal{N} = 4$ SYM theory take value in the (real) Lie algebra $i \cdot \mathfrak{su}(N)$. By definition, $\mathfrak{su}(N)$ is the Lie algebra associated to the special unitary group $SU(N)$ which, by definition, consists of all complex $n \times n$ matrices that are both unitary and have determinant 1. Upon differentiating these defining properties, we see that $T \in \mathfrak{su}(N)$ if and only if T is both antihermitian and traceless. The factor "i" turns antihermitian into hermitian. Therefore

$$T \in i \cdot \mathfrak{su}(N) \iff T = T^\dagger \quad \text{and} \quad \mathrm{tr}(T) = 0$$

where, as usual, "†" denotes transposition followed by complex conjugation. It follows immediately that $\dim(i \cdot \mathfrak{su}(N)) = N^2 - 1$.

Lemma 1.1.2. For every constant $C > 0$ there is a basis $(T^1, \ldots, T^{N^2-1})$ of $i \cdot \mathfrak{su}(N)$ such that $\mathrm{tr}(T^aT^b) = C \cdot \delta^{ab}$.

Proof. By Lem. 1.1.1, the $N^2 - 1 \times N^2 - 1$-matrix $D^{ab} := \mathrm{tr}(T^aT^b)$ is symmetric. Denoting complex conjugation by "$*$", we conclude from

$$(D^{ab})^* = \mathrm{tr}(T^aT^b)^* = \mathrm{tr}((T^aT^b)^T)^* = \mathrm{tr}((T^aT^b)^\dagger) = \mathrm{tr}((T^b)^\dagger(T^a)^\dagger) = \mathrm{tr}(T^bT^a) = D^{ab}$$

that it is also real. We may therefore replace each of the original basis elements by a linear transformation thereof to obtain a new basis, also denoted $(T^1, \ldots, T^{N^2-1})$, such that D^{ab} is a diagonal matrix. By

$$\mathrm{tr}(T^aT^a) = \mathrm{tr}(T^a(T^a)^\dagger) = (T^a)_{ij}(T^a)^*_{ij} = \sum\nolimits_{i,j} |(T^a)_{ij}|^2 > 0$$

it moreover follows that D^{ab} is positive definite, and the statement follows upon further normalisation of the T^a. □

Basis elements T^a of $i \cdot \mathfrak{su}(N)$ are often called generators. In the following, they are implicitly chosen such as to satisfy the statement of Lem. 1.1.2 with $C = \frac{1}{2}$, i.e.

$$\mathrm{tr}(T^a T^b) = \frac{1}{2} \cdot \delta^{ab} \tag{1.2}$$

Since the commutator of two $i \cdot \mathfrak{su}(N)$-matrices is an $\mathfrak{su}(N)$-matrix, there are real numbers f^{abc}, called structure constants, such that

$$\left[T^a,\, T^b\right] = i f^{abc} T^c \tag{1.3}$$

Lemma 1.1.3. f^{abc} is totally antisymmetric in the indices abc. Moreover, it has the explicit expression

$$f^{abc} = -2i\,\mathrm{tr}\left(T^c \left[T^a,\, T^b\right]\right)$$

Proof. Multiplying either side of $\left[T^a,\, T^b\right] = i f^{abd} T^d$ by T^c and taking the trace, we obtain

$$\mathrm{tr}\left(T^c \left[T^a,\, T^b\right]\right) = i f^{abd}\mathrm{tr}(T^c T^d) = \frac{i}{2} f^{abc}$$

and thus the formula stated. By Lem. 1.1.1, this expression is totally antisymmetric. □

Lemma 1.1.4. The generators T^a satisfy the following identity.

$$(T^a)_{ij}(T^a)_{kl} = \frac{1}{2}\delta_{li}\delta_{kj} - \frac{1}{2N}\delta_{ij}\delta_{kl}$$

Proof. The set of $N^2 - 1$ traceless generators T^a is extended by the identity matrix to a basis of $\mathbb{R}^{n \times n}$. Therefore, any such matrix can be written $X = X^0 \cdot \mathrm{id} + X^a T^a$. Taking the trace on either side, we obtain $X^0 = \frac{1}{N}\mathrm{tr}(X)$. On the other hand, contracting first with T^b and then taking the trace, we obtain $\mathrm{tr}(XT^b) = X^a \mathrm{tr}(T^a T^b) = X^a \frac{1}{2}\delta^{ab} = \frac{1}{2}X^b$ or, equivalently, $X^a = 2\mathrm{tr}(XT^a)$. Therefore, we find

$$X_{ij} = X^0 \cdot \mathrm{id}_{ij} + X^a (T^a)_{ij} = \frac{1}{N} X_{mm}\delta_{ij} + 2X_{lm}(T^a)_{ml}(T^a)_{ij}$$

This can be rewritten

$$X_{lm}\left(\delta_{il}\delta_{jm} - \frac{1}{N}\delta_{ml}\delta_{ij} - 2(T^a)_{ml}(T^a)_{ij}\right) = 0$$

which immediately implies the statement. □

1.1.2 Vectors and Spinors in Various Dimensions

As we will see below, $\mathcal{N} = 4$ SYM, which is a theory in Minkowski space, arises from dimensional reduction of a Lagrangian in $1 + 9$ dimensions. We thus need to fix conventions for vectors and spinors in dimension $1 + 3$ (Minkowski space) and dimension $1 + 9$ (ten-dimensional Lorentz space) as well as dimension 6 (six-dimensional Euclidean space). While some of the constructions are specific to the dimensions mentioned, others hold in much more generality. We refer to the standard books on spinors and Dirac operators for the general theory such as [Bau81] and [Fri00].

Minkowski Space I: Spinors

Minkowski space is the vector space $\mathbb{R}^4$ together with the metric (scalar product) of signature $(1,3)$ which, in the convention chosen here, reads $\eta = \mathrm{diag}(+,-,-,-)$. We denote the standard basis by (e_0, e_1, e_2, e_3) and let v^μ denote the corresponding coefficients of a vector $v = v^\mu e_\mu$. We further adopt the standard notation $v_\mu := \eta_{\mu\nu} v^\nu$.

The representation space $\Delta_{1,3} \cong \mathbb{C}^4$ of the spinor representation has dimension four. Elements $\psi \in \Delta_{1,3}$ are often called Dirac spinors. We use the convention

$$\psi = \begin{pmatrix} \lambda_\alpha \\ \tilde{\lambda}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \tilde{\lambda}^1 & \tilde{\lambda}^2 \end{pmatrix}^T$$

with $\lambda, \tilde{\lambda} \in \mathbb{C}^2$ and indices $\alpha, \dot{\alpha} \in \{1,2\}$. It can be shown that, if we let $SL(2,\mathbb{C}) \cong \mathrm{Spin}^+(1,3)$ act on λ with the standard matrix representation, and on $\tilde{\lambda}$ with the complex-conjugate representation, both subspaces $\cong \mathbb{C}^2$ carry a natural symplectic structure, which is invariant under the respective action of $SL(2,\mathbb{C})$. We shall denote these symplectic structures by

$$\langle v,\, w\rangle := v_\alpha w^\alpha \;, \qquad [v,\, w] := v_{\dot{\alpha}} w^{\dot{\alpha}} \tag{1.4}$$

with the convention, following [BDKM04], that indices are raised and lowered via

$$\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta \;, \quad \tilde{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}} \;, \quad \lambda_\alpha = \lambda^\beta \epsilon_{\beta\alpha} \;, \quad \tilde{\lambda}^{\dot{\alpha}} = \tilde{\lambda}_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}$$

where

$$\epsilon^{12} = \epsilon_{12} = 1 \;, \qquad \epsilon_{\dot{1}\dot{2}} = \epsilon^{\dot{1}\dot{2}} = -1 \;, \qquad \text{antisymmetric}$$

Here, $\epsilon_{\alpha\beta}$ may be identified with the matrix $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Lemma 1.1.5 (Schouten Identity)**.**

$$a_\gamma b^\gamma c^\alpha + b_\gamma c^\gamma a^\alpha + c_\gamma a^\gamma b^\alpha = 0 \qquad \text{or, equivalently,} \qquad \langle a,\, b\rangle\, c + \langle b,\, c\rangle\, a + \langle c,\, a\rangle\, b = 0$$

Proof. Using the identity $\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}$, we yield

$$\begin{aligned}
&\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \epsilon_{\gamma\alpha}\epsilon_{\beta\delta} + \epsilon_{\beta\gamma}\epsilon_{\alpha\delta} \\
&\qquad = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\gamma\beta}\delta_{\alpha\delta} - \delta_{\gamma\delta}\delta_{\alpha\beta} + \delta_{\beta\alpha}\delta_{\gamma\delta} - \delta_{\beta\delta}\delta_{\gamma\alpha} \\
&\qquad = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\beta\delta}\delta_{\gamma\alpha} + \delta_{\gamma\beta}\delta_{\alpha\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\beta\alpha}\delta_{\gamma\delta} - \delta_{\gamma\delta}\delta_{\alpha\beta} \\
&\qquad = 0
\end{aligned}$$

and the statement follows upon contracting either side with $a^\alpha b^\beta c^\gamma$. □

The Clifford relation is $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, and the Dirac matrices can be written, in the so called chiral form,

$$\gamma^\mu = \begin{pmatrix} 0_2 & \overline{\sigma}^\mu_{\alpha\dot{\beta}} \\ \sigma^{\mu\dot{\alpha}\beta} & 0_2 \end{pmatrix} \;, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix} \tag{1.5}$$

where $\sigma^{\mu\dot{\alpha}\beta} = (1, \boldsymbol{\sigma})$ and $\overline{\sigma}^{\mu}_{\alpha\dot{\beta}} = (1, -\boldsymbol{\sigma})$ with the vector $\boldsymbol{\sigma}$ of Pauli matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

One easily checks that the chiral Dirac matrices indeed satisfy the Clifford relation. In terms of the σ-matrices, it reads

$$\overline{\sigma}^{\mu}_{\alpha\dot{\gamma}}\sigma^{\nu\dot{\gamma}\beta} + \overline{\sigma}^{\nu}_{\alpha\dot{\gamma}}\sigma^{\mu\dot{\gamma}\beta} = 2\eta^{\mu\nu}\delta^{\beta}_{\alpha} , \qquad \sigma^{\mu\dot{\alpha}\gamma}\overline{\sigma}^{\nu}_{\gamma\dot{\beta}} + \sigma^{\nu\dot{\alpha}\gamma}\overline{\sigma}^{\mu}_{\gamma\dot{\beta}} = 2\eta^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{1.6}$$

As usual, the volume element γ^5 is an involution ($\gamma^5\gamma^5 = 1_4$) and the spinor module decomposes into its eigenspaces $\Delta^{\pm}_{1,3} \cong \mathbb{C}^2$ to the eigenvalues ± 1. Elements of $\Delta^{\pm}_{1,3}$ are called Weyl spinors. The chiral form of the Dirac matrices is constructed such that a Dirac spinor ψ decomposes, using the above notation, into the Weyl spinors λ_α and $\tilde{\lambda}^{\dot{\alpha}}$, on which the standard spinor representation of $\mathrm{Spin}^+(1,3)$ acts in the aforementioned way (standard or conjugate). The indices α and $\dot{\alpha}$ are, therefore, also referred to as Weyl indices. Consult [Str04] for more on the representation theory of spinors in Minkowski space.

Let C be an invertible matrix (which operates on $\Delta_{1,3}$) that satisfies the following property (following the conventions used in [Ton07]).

$$\gamma^\mu C = -C(\gamma^\mu)^* , \qquad \psi^{(c)} := C\psi^* \tag{1.7}$$

C is referred to as charge conjugation. While not being unique, it can be shown that a charge conjugation matrix exists in any Lorentz space $\mathbb{R}^{1,d-1}$ (cf. [Tod11]). The concrete shape of C clearly depends on the representation of the gamma matrices chosen. In Minkowski space with chiral Dirac matrices as stated above, a possible choice is

$$C_{1,3} := i\gamma^2 = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \tag{1.8}$$

A straightforward calculation shows that, in general, the charge conjugated Dirac spinor $\psi^{(c)}$ transforms like ψ under a Lorentz transformation. As for the name, note that ψ satisfies the Dirac equation $\gamma_\mu D^\mu \psi = 0$ if and only if $\psi^{(c)}$ satisfies the same equation with the coupling constant g (which equals the electric charge e in quantum electrodynamics) replaced by $-g$ (for notation cf. Sec. 1.2 below, the calculation can be found in [Ton07]).

Spinors with the property $\psi^{(c)} = \psi$ are called Majorana spinors. Upon quantisation, a Majorana spinor is a fermion which is its own anti-particle. Majorana spinors that live in either space $\Delta^{\pm}_{1,3}$ are, therefore, referred to as Majorana-Weyl spinors. The Majorana property is in fact a reality condition. Indeed, Majorana spinors exist if and only if the corresponding (complex) pinor representation admits a real structure (cf. [FO06]). This is not true for every spacetime signature. Moreover, the Majorana and Weyl conditions can be mutually exclusive, as it is the case for the Minkowski signature.

Minkowski Space II: Vectors and Tensors

It turns out to be convenient to write vectors and tensors in terms of spinor indices. We need the following three lemmas concerning the sigma matrices.

Lemma 1.1.6. The matrices σ and $\overline{\sigma}$ can be identified as follows.

$$\sigma^{\mu\dot{\alpha}\beta} = \epsilon^{\beta\gamma}\overline{\sigma}^{\mu}_{\gamma\dot{\delta}}\epsilon^{\dot{\delta}\dot{\alpha}} = \overline{\sigma}^{\mu\beta\dot{\alpha}}$$

Proof. From the explicit form of the Pauli matrices we see that $\epsilon\boldsymbol{\sigma}^T\epsilon = \boldsymbol{\sigma}$ and, therefore,

$$\epsilon^{\beta\gamma}\overline{\sigma}^{\mu}_{\gamma\dot{\delta}}\epsilon^{\dot{\delta}\dot{\alpha}} = (\epsilon\overline{\sigma}^{\mu}(-\epsilon))^T = -\epsilon(1,-\boldsymbol{\sigma})^T\epsilon = (-\epsilon^2, \epsilon\boldsymbol{\sigma}^T\epsilon) = (1,\boldsymbol{\sigma}) = \sigma^{\mu\dot{\alpha}\beta}$$

□

Lemma 1.1.7. Contraction of the spacetime indices can be written as follows.

$$\overline{\sigma}^{\mu}_{\alpha\dot{\beta}}\overline{\sigma}_{\mu\gamma\dot{\delta}} = -2\epsilon_{\alpha\gamma}\epsilon_{\dot{\beta}\dot{\delta}}$$

Proof. By a direct calculation, using the explicit form of Pauli matrices, we obtain

$$\begin{aligned}\overline{\sigma}^{\mu}_{1\dot{\beta}}\overline{\sigma}_{\mu 1\dot{\delta}} &= \overline{\sigma}^{0}_{1\dot{\beta}}\overline{\sigma}^{0}_{1\dot{\delta}} - \overline{\sigma}^{1}_{1\dot{\beta}}\overline{\sigma}^{1}_{1\dot{\delta}} - \overline{\sigma}^{2}_{1\dot{\beta}}\overline{\sigma}^{2}_{1\dot{\delta}} - \overline{\sigma}^{3}_{1\dot{\beta}}\overline{\sigma}^{3}_{1\dot{\delta}}\\ &= \delta_{1\dot{\beta}}\delta_{1\dot{\delta}} - \delta_{2\dot{\beta}}\delta_{2\dot{\delta}} - (-i)(-i)\delta_{2\dot{\beta}}\delta_{2\dot{\delta}} - \delta_{1\dot{\beta}}\delta_{1\dot{\delta}} = 0\end{aligned}$$

and

$$\begin{aligned}\overline{\sigma}^{\mu}_{2\dot{\beta}}\overline{\sigma}_{\mu 2\dot{\delta}} &= \overline{\sigma}^{0}_{2\dot{\beta}}\overline{\sigma}^{0}_{2\dot{\delta}} - \overline{\sigma}^{1}_{2\dot{\beta}}\overline{\sigma}^{1}_{2\dot{\delta}} - \overline{\sigma}^{2}_{2\dot{\beta}}\overline{\sigma}^{2}_{2\dot{\delta}} - \overline{\sigma}^{3}_{2\dot{\beta}}\overline{\sigma}^{3}_{2\dot{\delta}}\\ &= \delta_{2\dot{\beta}}\delta_{2\dot{\delta}} - \delta_{1\dot{\beta}}\delta_{1\dot{\delta}} - (i)(i)\delta_{1\dot{\beta}}\delta_{1\dot{\delta}} - (-1)(-1)\delta_{2\dot{\beta}}\delta_{2\dot{\delta}} = 0\end{aligned}$$

and

$$\begin{aligned}\overline{\sigma}^{\mu}_{1\dot{\beta}}\overline{\sigma}_{\mu 2\dot{\delta}} &= \overline{\sigma}^{0}_{1\dot{\beta}}\overline{\sigma}^{0}_{2\dot{\delta}} - \overline{\sigma}^{1}_{1\dot{\beta}}\overline{\sigma}^{1}_{2\dot{\delta}} - \overline{\sigma}^{2}_{1\dot{\beta}}\overline{\sigma}^{2}_{2\dot{\delta}} - \overline{\sigma}^{3}_{1\dot{\beta}}\overline{\sigma}^{3}_{2\dot{\delta}}\\ &= \delta_{1\dot{\beta}}\delta_{2\dot{\delta}} - \delta_{2\dot{\beta}}\delta_{1\dot{\delta}} - (-i)(i)\delta_{2\dot{\beta}}\delta_{1\dot{\delta}} - (-1)\delta_{1\dot{\beta}}\delta_{2\dot{\delta}}\\ &= 2\left(\delta_{1\dot{\beta}}\delta_{2\dot{\delta}} - \delta_{2\dot{\beta}}\delta_{1\dot{\delta}}\right) = -2\epsilon_{\dot{\beta}\dot{\delta}}\end{aligned}$$

By these calculations, the statement is immediate. □

Lemma 1.1.8. Contraction of the spinor indices can be written as follows.

$$\overline{\sigma}^{\mu}_{\alpha\dot{\alpha}}\overline{\sigma}^{\nu\alpha\dot{\alpha}} = 2\eta^{\mu\nu}$$

Proof. Using $\epsilon = i\sigma^2$, we calculate

$$\overline{\sigma}^{\mu}_{\alpha\dot{\alpha}}\overline{\sigma}^{\nu\alpha\dot{\alpha}} = \overline{\sigma}^{\mu}_{\alpha\dot{\alpha}}\epsilon^{\alpha\beta}\overline{\sigma}^{\nu}_{\beta\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}} = \epsilon^{\beta\alpha}\overline{\sigma}^{\mu}_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\overline{\sigma}^{\nu}_{\beta\dot{\beta}} = \mathrm{tr}\left(\epsilon\overline{\sigma}^{\mu}(-\epsilon)(\overline{\sigma}^{\nu})^T\right) = \mathrm{tr}\left(\sigma^2\overline{\sigma}^{\mu}\sigma^2(\overline{\sigma}^{\nu})^T\right)$$

In the case $\mu = 2$, we thus obtain

$$\begin{aligned}\overline{\sigma}^{2}_{\alpha\dot{\alpha}}\overline{\sigma}^{\nu\alpha\dot{\alpha}} &= -\mathrm{tr}\left(\sigma^2\sigma^2\sigma^2(\overline{\sigma}^{\nu})^T\right) = -\mathrm{tr}\left(\sigma^2(\overline{\sigma}^{\nu})^T\right) = -(-1)^{\delta_{\nu 2}}\mathrm{tr}\left(\sigma^2\overline{\sigma}^{\nu}\right)\\ &= -(-1)^{\nu\in\{1,3\}}\mathrm{tr}\left(\sigma^2\sigma^{\nu}\right) = -(-1)^{\nu\in\{1,3\}}2\delta^{\nu 2}\\ &= -2\delta^{\nu 2}\end{aligned}$$

while in the case $\mu \neq 2$, we yield

$$\begin{aligned}\overline{\sigma}^{\mu}_{\alpha\dot{\alpha}}\overline{\sigma}^{\nu\alpha\dot{\alpha}} &= (-1)^{\mu\in\{1,3\}}\mathrm{tr}\left(\sigma^2\sigma^2\overline{\sigma}^{\mu}(\overline{\sigma}^{\nu})^T\right)\\ &= (-1)^{\mu\in\{1,3\}}\mathrm{tr}\left(\overline{\sigma}^{\mu}(\overline{\sigma}^{\nu})^T\right)\\ &= (-1)^{\mu\in\{1,3\}}(-1)^{\delta_{\nu 2}}\mathrm{tr}\left(\overline{\sigma}^{\mu}\overline{\sigma}^{\nu}\right)\\ &= (-1)^{\mu\in\{1,3\}}(-1)^{\delta_{\nu 2}}\mathrm{tr}\left(\sigma^{\mu}\sigma^{\nu}\right)\\ &= (-1)^{\mu\in\{1,3\}}(-1)^{\delta_{\nu 2}}2\delta^{\mu\nu}\\ &= (-1)^{\mu\in\{1,3\}}2\delta^{\mu\nu}\end{aligned}$$

Taken together, the statement is proved. □

It will turn out to be convenient to use the sigma matrices to assign a bispinor to a vector p^μ as follows.

$$p^{\alpha\dot\alpha} := \overline{\sigma}^{\mu\alpha\dot\alpha} p_\mu = \sigma^{\mu\dot\alpha\alpha} p_\mu =: p^{\dot\alpha\alpha} \tag{1.9}$$

where the equality in the middle holds true by Lem. 1.1.6. In the following, we shall make implicit use of (1.9) along with the identification $p^{\alpha\dot\alpha} = p^{\dot\alpha\alpha}$. Lem. 1.1.8 implies that

$$p_{\alpha\dot\alpha} k^{\alpha\dot\alpha} = \overline{\sigma}^\mu_{\alpha\dot\alpha} p_\mu \overline{\sigma}^{\nu\alpha\dot\alpha} k_\nu = 2\eta^{\mu\nu} p_\mu k_\nu = 2 p_\mu k^\mu \tag{1.10}$$

It follows that for lightlike p^μ ($p^2 = 0$), the rank of the matrix $p^{\alpha\dot\alpha}$ is at most 1 and, therefore, there are Weyl spinors λ^α and $\tilde\lambda^{\dot\alpha}$ (which are unique up to a scaling invariance) such that

$$p^2 = 0 \qquad \Longrightarrow \qquad p^{\alpha\dot\alpha} = \lambda^\alpha \tilde\lambda^{\dot\alpha} \tag{1.11}$$

Defining

$$\sigma^{\mu\nu}{}_\alpha{}^\beta := \frac{i}{2}\left(\overline{\sigma}^\mu_{\alpha\dot\gamma}\sigma^{\nu\dot\gamma\beta} - \overline{\sigma}^\nu_{\alpha\dot\gamma}\sigma^{\mu\dot\gamma\beta}\right), \qquad \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} := \frac{i}{2}\left(\sigma^{\mu\dot\alpha\gamma}\overline{\sigma}^\nu_{\gamma\dot\beta} - \sigma^{\nu\dot\alpha\gamma}\overline{\sigma}^\mu_{\gamma\dot\beta}\right) \tag{1.12}$$

we may similarly assign two bispinors

$$F^{\alpha\beta} := F_{\mu\nu}\sigma^{\mu\nu\alpha\beta}, \qquad F^{\dot\alpha\dot\beta} := F_{\mu\nu}\overline{\sigma}^{\mu\nu\dot\alpha\dot\beta} \tag{1.13}$$

to an antisymmetric 2-tensor $F^{\mu\nu}$. It follows at once that $F^{\alpha\beta} = F^{\beta\alpha}$.

Lemma 1.1.9. Identifying $F^{\mu\nu}$ with a four-spinor according to (1.9), the following identities hold, provided that $F^{\mu\nu}$ is antisymmetric.

$$F_{\alpha\dot\alpha\,\beta\dot\beta} = -\frac{i}{2}\epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta} + \frac{i}{2}\epsilon_{\alpha\beta}F_{\dot\alpha\dot\beta}, \qquad F_{\mu\nu}F^{\mu\nu} = -\frac{1}{8}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{8}F_{\dot\alpha\dot\beta}F^{\dot\alpha\dot\beta}$$

Proof. We calculate

$$\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}\epsilon^{\beta\alpha} = -\epsilon_{\alpha\beta}\epsilon_{\beta\alpha} = -\mathrm{tr}(\epsilon^2) = -\mathrm{tr}(-\mathrm{id}) = 2$$

and, using Lem. 1.1.6 and Lem. 1.1.8,

$$\begin{aligned}
-2i(\sigma^{\mu\nu})_{\alpha\beta}\epsilon^{\alpha\beta} &= \left(\overline{\sigma}^\mu_{\alpha\dot\gamma}\sigma^{\nu\dot\gamma}{}_\beta - \overline{\sigma}^\nu_{\alpha\dot\gamma}\sigma^{\mu\dot\gamma}{}_\beta\right)\epsilon^{\alpha\beta} \\
&= \left(\overline{\sigma}^\mu_{\alpha\dot\gamma}\sigma^{\nu\dot\gamma\alpha} - \overline{\sigma}^\nu_{\alpha\dot\gamma}\sigma^{\mu\dot\gamma\alpha}\right) \\
&= \left(\overline{\sigma}^\mu_{\alpha\dot\gamma}\overline{\sigma}^{\nu\alpha\dot\gamma} - \overline{\sigma}^\nu_{\alpha\dot\gamma}\overline{\sigma}^{\mu\alpha\dot\gamma}\right) \\
&= 2\eta^{\mu\nu} - 2\eta^{\nu\mu} \\
&= 0
\end{aligned}$$

Therefore, only the second term of the right hand side of the first equation survives upon contraction with $\epsilon^{\alpha\beta}$, and we yield

$$\begin{aligned}\left(-\frac{i}{2}\epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}+\frac{i}{2}\epsilon_{\alpha\beta}F_{\dot\alpha\dot\beta}\right)\epsilon^{\alpha\beta} &= \frac{i}{2}2F_{\dot\alpha\dot\beta}\\ &= iF_{\mu\nu}(\overline{\sigma}^{\mu\nu})_{\dot\alpha\dot\beta}\\ &= -\frac{1}{2}F_{\mu\nu}\left(\sigma^{\mu}_{\dot\alpha\alpha}\overline{\sigma}^{\nu}_{\beta\dot\beta}-\sigma^{\nu}_{\dot\alpha\alpha}\overline{\sigma}^{\mu}_{\beta\dot\beta}\right)\epsilon^{\beta\alpha}\\ &= \frac{1}{2}F_{\mu\nu}\left(\overline{\sigma}^{\mu}_{\alpha\dot\alpha}\overline{\sigma}^{\nu}_{\beta\dot\beta}-\overline{\sigma}^{\nu}_{\alpha\dot\alpha}\overline{\sigma}^{\mu}_{\beta\dot\beta}\right)\epsilon^{\alpha\beta}\\ &= F_{\mu\nu}\overline{\sigma}^{\mu}_{\alpha\dot\alpha}\overline{\sigma}^{\nu}_{\beta\dot\beta}\epsilon^{\alpha\beta}\\ &= F_{\alpha\dot\alpha\,\beta\dot\beta}\epsilon^{\alpha\beta}\end{aligned}$$

and, similarly,

$$\left(-\frac{i}{2}\epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}+\frac{i}{2}\epsilon_{\alpha\beta}F_{\dot\alpha\dot\beta}\right)\epsilon^{\dot\alpha\dot\beta} = F_{\alpha\dot\alpha\,\beta\dot\beta}\epsilon^{\dot\alpha\dot\beta}$$

Taken together, the first statement follows.

To show the second statement, we use the expression derived in the first as well as Lem. 1.1.8 to obtain

$$\begin{aligned}F_{\mu\nu}F^{\mu\nu} &= \frac{1}{4}F_{\alpha\dot\alpha\,\beta\dot\beta}F^{\alpha\dot\alpha\,\beta\dot\beta}\\ &= \frac{1}{4}\left(-\frac{i}{2}\epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}+\frac{i}{2}\epsilon_{\alpha\beta}F_{\dot\alpha\dot\beta}\right)\left(-\frac{i}{2}\epsilon^{\dot\alpha\dot\beta}F^{\alpha\beta}+\frac{i}{2}\epsilon^{\alpha\beta}F^{\dot\alpha\dot\beta}\right)\\ &= \frac{1}{4}\left(-\frac{1}{4}\epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}F^{\alpha\beta}-\frac{1}{4}\epsilon_{\alpha\beta}F_{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}F^{\dot\alpha\dot\beta}\right)\\ &= -\frac{1}{8}F_{\alpha\beta}F^{\alpha\beta}-\frac{1}{8}F_{\dot\alpha\dot\beta}F^{\dot\alpha\dot\beta}\end{aligned}$$

using that the mixed terms vanish by the first calculation in this proof. □

Six-Dimensional Euclidean Space

Consider the vector space $\mathbb{R}^6$ with the metric $\eta = \mathrm{diag}(-,-,-,-,-,-)$. We denote the standard basis by $(\hat{e}_1,\dots,\hat{e}_6)$ and let v^a denote the coefficients of a vector $v = v^a\hat{e}_a$. The representation space $\Delta_{0,6}\cong\mathbb{C}^8$ of the spinor representation has dimension eight. For elements $\psi\in\Delta_{0,6}$, we use the convention

$$\psi = \begin{pmatrix}\lambda^A\\ \lambda_A\end{pmatrix} = \begin{pmatrix}\lambda^1 & \lambda^2 & \lambda^3 & \lambda^4 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\end{pmatrix}^T$$

with $\lambda^A,\lambda_A\in\mathbb{C}^4$ and upper and lower indices $A\in\{1,2,3,4\}$. The Clifford relation is $\hat\gamma^a\hat\gamma^b+\hat\gamma^b\hat\gamma^a = -2\delta^{ab}$, and the Dirac matrices can be written

$$\hat\gamma^a = \begin{pmatrix}0_4 & \Sigma^{aAB}\\ \overline{\Sigma}^a_{AB} & 0_4\end{pmatrix}\,,\qquad \hat\gamma^7 = i\hat\gamma^1\hat\gamma^2\hat\gamma^3\hat\gamma^4\hat\gamma^5\hat\gamma^6 = \begin{pmatrix}1_4 & 0_4\\ 0_4 & -1_4\end{pmatrix}$$

where Σ and $\overline{\Sigma}$ denote the sigma matrices

$$(\Sigma^{1AB}, \dots, \Sigma^{6AB}) = (\eta_{1AB}, \eta_{2AB}, \eta_{3AB}, i\overline{\eta}_{1AB}, i\overline{\eta}_{2AB}, i\overline{\eta}_{3AB})$$
$$(\overline{\Sigma}^1_{AB}, \dots, \overline{\Sigma}^6_{AB}) = (\eta_{1AB}, \eta_{2AB}, \eta_{3AB}, -i\overline{\eta}_{1AB}, -i\overline{\eta}_{2AB}, -i\overline{\eta}_{3AB})$$

which are defined in terms of the 't Hooft symbols

$$\eta_{iAB} := \epsilon_{iAB} + \delta_{iA}\delta_{4B} - \delta_{iB}\delta_{4A} , \qquad \overline{\eta}_{iAB} := \epsilon_{iAB} - \delta_{iA}\delta_{4B} + \delta_{iB}\delta_{4A}$$

One easily checks that the Dirac matrices stated indeed satisfy the Clifford relation. Analogous to the Minkowski case, the spinor module decomposes into the eigenspaces $\Delta^{\pm}_{0,6} \cong \mathbb{C}^4$ of the volume element $\hat{\gamma}^7$ such that, using the above notation, a (Dirac) spinor ψ decomposes into the Weyl spinors λ^A and λ_A. As for any Euclidean signature, it is possible to define charge conjugation by a property similar to (1.7). We omit the details and refer the reader to [Tod11].

We denote by ε_{ABCD} the antisymmetric four-tensor, which is normalised to $\varepsilon_{1234} = 1$. It satisfies the identity

$$\varepsilon_{DABC}\varepsilon_{DKLM} = \delta^{KLM}_{ABC} + \delta^{KLM}_{BCA} + \delta^{KLM}_{CAB} - \delta^{KLM}_{CBA} - \delta^{KLM}_{BAC} - \delta^{KLM}_{ACB} \tag{1.14}$$

where the delta symbols on the right hand side are defined to be 1 if the lower indices coincide with the upper ones, and 0 otherwise. By simple calculations, the 't Hooft symbols are seen to have the following properties.

$$\eta_{iAB} = \frac{1}{2}\varepsilon_{ABCD}\,\eta_{iCD} , \qquad \overline{\eta}_{iAB} = -\frac{1}{2}\varepsilon_{ABCD}\,\overline{\eta}_{iCD}$$
$$\eta_{iAB}\eta_{jAB} = 4\delta_{ij} , \qquad \overline{\eta}_{iAB}\overline{\eta}_{jAB} = 4\delta_{ij} , \qquad \eta_{iAB}\overline{\eta}_{jAB} = 0$$
$$\eta_{iAB}\eta_{iCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + \epsilon_{ABCD}$$
$$\overline{\eta}_{iAB}\overline{\eta}_{iCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} - \epsilon_{ABCD}$$

The next lemma follows as an immediate corollary.

Lemma 1.1.10. The sigma matrices satisfy the following identities.

$$\overline{\Sigma}^a_{AB} = (\Sigma^{aAB})^* = \frac{1}{2}\varepsilon_{ABCD}\,\Sigma^{aCD} , \qquad \Sigma^{aAB} = \frac{1}{2}\varepsilon_{ABCD}\,\overline{\Sigma}^a_{CD}$$
$$\overline{\Sigma}^a_{AB}\overline{\Sigma}^a_{CD} = 2\varepsilon_{ABCD} , \qquad \Sigma^{aAB}\overline{\Sigma}^b_{AB} = 4\delta^{ab}$$

By the first equation $\overline{\Sigma}^a_{AB} = -\overline{\Sigma}^a_{BA}$ is, in particular, antisymmetric.

Analogous to (1.9), we assign a matrix to a vector ϕ^M as follows.

$$\phi^{AB} := \frac{1}{\sqrt{2}}\Sigma^{M\,AB}\phi^M , \qquad \overline{\phi}_{AB} := \frac{1}{\sqrt{2}}\overline{\Sigma}^a_{AB}\phi^a \tag{1.15}$$

By Lem. 1.1.10, they are related via

$$\overline{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2}\varepsilon_{ABCD}\phi^{CD} , \qquad \phi^{AB} = (\overline{\phi}_{AB})^* = \frac{1}{2}\varepsilon_{ABCD}\,\overline{\phi}_{CD} \tag{1.16}$$

and the scalar product can be written as a trace:

$$X^{AB}\overline{Y}_{AB} = \frac{1}{2}\Sigma^{aAB}\overline{\Sigma}^b_{AB}X^aY^b = 2X^aY^a = -2X^aY_a \tag{1.17}$$

In particular, it follows that $X^{AB}\overline{Y}_{AB} = \overline{X}_{AB}Y^{AB}$.

Ten-Dimensional Lorentz Space

Consider the vector space $\mathbb{R}^{10}$ with the metric $\eta = \mathrm{diag}(+,-,\dots,-)$. We denote the standard basis by $(e_1,\dots,e_{10})$ and let v^M denote the coefficients of a vector $v = v^M e_M$ The representation space $\Delta_{1,9} \cong \mathbb{C}^{32}$ of the spinor representation has dimension 32. For elements $\xi \in \Delta_{1,9}$, we use the convention

$$\xi = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\xi_\alpha^A\\ \tilde{\xi}^{\dot\alpha A}\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}\xi_{\alpha A}\\ \tilde{\xi}_A^{\dot\alpha}\end{pmatrix}$$

with $\xi_\alpha^A, \tilde{\xi}^{\dot\alpha A}, \xi_{\alpha A}, \tilde{\xi}_A^{\dot\alpha} \in \mathbb{C}^8$ and indices $\alpha, \dot\alpha \in \{1,2\}$ as well as upper and lower indices $A \in \{1,2,3,4\}$. The Clifford relation is $\Gamma^M\Gamma^N + \Gamma^N\Gamma^M = 2\eta^{MN}$, and the Dirac matrices Γ^M and the volume element Γ^{11} can be constructed from those of $\mathbb{R}^{1,3}$ and $\mathbb{R}^6$ as follows.

$$\Gamma^M = \begin{cases} 1_8 \otimes \gamma^\mu & \text{for } M = \mu \in \{0,1,2,3\} \\ \hat\gamma^a \otimes \gamma^5 & \text{for } M = a+3 \in \{4,5,6,7,8,9\} \end{cases}, \qquad \Gamma^{11} = \hat\gamma^7 \otimes \gamma^5$$

One easily checks that the Dirac matrices stated indeed satisfy the Clifford relation. Weyl spinors are defined as usual. Using the explicit forms for γ^5 and $\hat\gamma^7$ as stated above, the defining equation $\Gamma^{11}\xi = \xi$ for $\xi \in \Delta_{1,9}^+ \cong \mathbb{C}^{16}$ reads

$$\Gamma^{11}\xi = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\xi_\alpha^A\\ -\tilde{\xi}^{\dot\alpha A}\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}\xi_{\alpha A}\\ -\tilde{\xi}_A^{\dot\alpha}\end{pmatrix} \stackrel{!}{=} \xi$$

which is equivalent to the vanishing of $\tilde{\xi}^{\dot\alpha A} = \xi_{\alpha A} = 0$. Therefore,

$$\xi \in \Delta_{1,9}^+ \iff \xi = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\xi_\alpha^A\\ 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\ \tilde{\xi}_A^{\dot\alpha}\end{pmatrix} \tag{1.18}$$

Charge conjugation is defined as in (1.7) with γ^μ replaced by Γ^M, and the subsequent remarks (concerning every Lorentz spacetime signature $(1,d-1)$) apply in particular to the present case of $d=10$. We choose the explicit form

$$C_{1,9} := \begin{pmatrix}0 & 1_4\\ 1_4 & 0\end{pmatrix} \otimes C_{1,3}$$

with $C_{1,3}$ as defined in (1.8). One easily checks that $C_{1,9}$ satisfies the analogon of (1.7). Indeed, for $M \in \{0,1,2,3\}$ this is induced by the properties of $C_{1,3}$ while for $M \geq 4$, this follows from $\gamma^5 C_{1,3} = -C_{1,3}\gamma^5 = -C_{1,3}(\gamma^5)^*$ and $\hat\gamma^a \begin{pmatrix}0 & 1_4\\ 1_4 & 0\end{pmatrix} = \begin{pmatrix}0 & 1_4\\ 1_4 & 0\end{pmatrix} (\hat\gamma^a)^*$ where the latter equation holds by definition of $\hat\gamma^a$ and Lem. 1.1.10.

It turns out that, unlike in the Minkowski case, the Majorana and Weyl conditions $\xi = \xi^{(c)}$ and, respectively, $\Gamma^{11}\xi = \xi$ can be (non-trivially) satisfied at the same time: For a Weyl spinor $\xi \in \Delta_{1,9}^+$, the Majorana condition reads

$$C_{1,9}\xi^* = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\ \epsilon\cdot(\xi_\alpha^A)^*\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}-\epsilon\cdot(\tilde{\xi}_A^{\dot\alpha})^*\\ 0\end{pmatrix} \stackrel{!}{=} \xi$$

which can be written

$$\xi = \xi^{(c)} \in \Delta_{1,9}^+ \iff (\tilde{\xi}_{\dot\alpha A})^* = \xi_\alpha^A \quad\text{and}\quad (\xi^{\alpha A})^* = \tilde{\xi}_A^{\dot\alpha} \tag{1.19}$$

Setting $\overline{\xi} := \xi^\dagger\Gamma^0$, the assertion $(\xi,\Psi) \mapsto \overline{\xi}\Psi$ defines an indefinite Hermitian scalar product which is invariant under the action of $\mathrm{Spin}^+(1,9)$, a construction which generalises to all spacetime signatures (cf. [Bau81]).

Lemma 1.1.11. Let ξ and Ψ be Majorana-Weyl spinors. Then

$$\overline{\xi}\Gamma^\mu\Psi = \tilde{\xi}_{\dot{\alpha}A}\sigma^{\mu\dot{\alpha}\beta}\psi^A_\beta + \xi^{\alpha A}\overline{\sigma}^\mu_{\alpha\dot{\beta}}\tilde{\psi}^{\dot{\beta}}_A\,, \qquad \overline{\xi}\Gamma^{a+3}\Psi = -\Sigma^{aAB}\tilde{\xi}_{\dot{\alpha}A}\tilde{\psi}^{\dot{\alpha}}_B + \overline{\Sigma}^a_{AB}\xi^{\alpha A}\psi^B_\alpha$$

Proof. Using (1.19), we calculate

$$\begin{aligned}\overline{\xi} = \xi^\dagger\Gamma^0 &= \left(\begin{pmatrix}1 & 0\end{pmatrix}\otimes\begin{pmatrix}\tilde{\xi}_{\dot{\alpha}A} & 0\end{pmatrix} + \begin{pmatrix}0 & 1\end{pmatrix}\otimes\begin{pmatrix}0 & \xi^{\alpha A}\end{pmatrix}\right)(1_8\otimes\gamma^0)\\ &= \begin{pmatrix}1 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & \tilde{\xi}_{\dot{\alpha}A}\end{pmatrix} + \begin{pmatrix}0 & 1\end{pmatrix}\otimes\begin{pmatrix}\xi^{\alpha A} & 0\end{pmatrix}\end{aligned}$$

We thus obtain

$$\begin{aligned}\overline{\xi}\Gamma^\mu\Psi = \overline{\xi}(1_8\otimes\gamma^\mu)\Psi &= \begin{pmatrix}0 & \tilde{\xi}_{\dot{\alpha}A}\end{pmatrix}\gamma^\mu\begin{pmatrix}\psi^A_\beta\\ 0\end{pmatrix} + \begin{pmatrix}\xi^{\alpha A} & 0\end{pmatrix}\gamma^\mu\begin{pmatrix}0\\ \tilde{\psi}^{\dot{\beta}}_A\end{pmatrix}\\ &= \tilde{\xi}_{\dot{\alpha}A}\sigma^{\mu\dot{\alpha}\beta}\psi^A_\beta + \xi^{\alpha A}\overline{\sigma}^\mu_{\alpha\dot{\beta}}\tilde{\psi}^{\dot{\beta}}_A\end{aligned}$$

and

$$\begin{aligned}\overline{\xi}\Gamma^{a+3}\Psi = \overline{\xi}(\hat{\gamma}^a\otimes\gamma^5)\Psi &= \begin{pmatrix}1 & 0\end{pmatrix}\hat{\gamma}^a\begin{pmatrix}0\\ 1\end{pmatrix}(-\tilde{\xi}_{\dot{\alpha}A}\tilde{\psi}^{\dot{\alpha}}_B) + \begin{pmatrix}0 & 1\end{pmatrix}\hat{\gamma}^a\begin{pmatrix}1\\ 0\end{pmatrix}(\xi^{\alpha A}\psi^B_\alpha)\\ &= -\Sigma^{aAB}\tilde{\xi}_{\dot{\alpha}A}\tilde{\psi}^{\dot{\alpha}}_B + \overline{\Sigma}^a_{AB}\xi^{\alpha A}\psi^B_\alpha\end{aligned}$$

which concludes the proof of the statement. □

From the explicit formulas in Lem. 1.1.11 and the properties of sigma matrices (Lem. 1.1.6 and Lem. 1.1.10) immediately find, for Majorana-Weyl spinors,

$$\overline{\xi}\Gamma^M\Psi = (-1)^{|\xi||\Psi|}\overline{\Psi}\Gamma^M\xi \tag{1.20}$$

where $|\xi|$ denotes the Grassmann parity of ξ as in Sec. 1.1.1.

1.2 The Fields and the Lagrangian

In this section, we introduce the fields and the Lagrangian of $\mathcal{N}=4$ super Yang-Mills (SYM) theory in Minkowski space as induced by $\mathcal{N}=1$ SYM theory in ten dimensions. The gauge group is $SU(N)$ and the coupling constant $g\in\mathbb{R}$. As usual in particle physics, we adopt units such that $\hbar = c = 1$.

In general, a Lagrangian $\mathcal{L}$ in d dimensions is considered only up to transformations which leave the action $\int d^dx\,\mathcal{L}$ invariant, thus leading to the same Euler-Lagrange equations (cf. Sec. 1.4). We assume that every field $B(x)$ goes sufficiently fast to 0 as $|x|\to\infty$. Then, the addition of any exact term to $\mathcal{L}$ is such an invariance transformation. In particular, we are free to move around the covariant derivative D^M as shown in the next lemma.

Lemma 1.2.1. Let B and C be matrix valued fields (Grassmann even or odd) which go sufficiently fast to 0 as $|x|\to\infty$. Then

$$\int d^dx\,\mathrm{tr}\left(B(D^MC)\right) = -\int d^dx\,\mathrm{tr}\left((D^MB)C\right)$$

Proof. For D^M replaced by ∂^M, this follows simply from the product rule, writing $\int := \int d^d x$ for brevity:

$$0 = \int \partial^M \mathrm{tr}(BC) = \int \mathrm{tr}(\partial^M (BC)) = \int \mathrm{tr}\left((\partial^M B)C + B(\partial^M C)\right)$$

Moreover, using both parts of Lem. 1.1.1, we yield

$$\begin{aligned}
\int \mathrm{tr}\left(B(D^M C)\right) &= \int \mathrm{tr}\left(B\partial^M C - igB\left[A^M, C\right]\right) \\
&= \int \mathrm{tr}\left(B\partial^M C - (-1)^{|B||C|} igC\left[B, A^M\right]\right) \\
&= \int \mathrm{tr}\left(-(\partial^M B)C + (-1)^{|B||C|} igC\left[A^M, B\right]\right) \\
&= \int \mathrm{tr}\left(-(\partial^M B)C + ig\left[A^M, B\right]C\right) \\
&= -\int \mathrm{tr}\left((D^M B)C\right)
\end{aligned}$$

□

The Field Content

We introduce the field content of $\mathcal{N} = 4$ SYM. First, denote the gauge field (gluon) by A. To fix notation, this is supposed to mean that $-igA \in \Omega^1(\mathbb{R}^4, \mathfrak{su}(N))$ is a connection on $\mathbb{R}^4$ with covariant derivative (in the adjoint representation)

$$D_\mu f = \partial_\mu f - ig\left[A_\mu, f\right]$$

for any $\mathfrak{su}(N)$-valued field f (cf. [Bau09] and [Bär09] for the general theory of connections). Let F denote the field strenth, i.e. $-igF \in \Omega^2(\mathbb{R}^4, \mathfrak{su}(N))$ is the curvature form of the connection $-igA$. This implies that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \in C^\infty(\mathbb{R}^4, i \cdot \mathfrak{su}(N))$$

such that, obviously, $F_{\mu\nu} = -F_{\nu\mu}$. Moreover, F satisfies the Bianchi identity

$$D_\mu F_{\nu\kappa} = D_\nu F_{\mu\kappa} + D_\kappa F_{\nu\mu} \tag{1.21}$$

A gauge transformation translates one connection into another. In general, this is a diffeomorphism of the principal fiber bundle (here, the frame bundle of Minkowski space) which is compatible with the action of the Lie group (here $SU(N)$). In our context, there is a bijection with the set of smooth maps $V : \mathbb{R}^4 \to SU(N)$, and the action of such a gauge transformation can be written

$$A_\mu \mapsto V \cdot \left(A_\mu + \frac{i}{g}\partial_\mu\right) \cdot V^\dagger\,, \qquad F_{\mu\nu} \mapsto V \cdot F_{\mu\nu} \cdot V^\dagger \tag{1.22}$$

Yang-Mills theories are constructed such as to be symmetric under gauge transformations (cf. [Ebe89]). In terms of the next section, gauge transformations are thus finite symmetries. This applies, in particular, to $\mathcal{N} = 4$ SYM theory. We leave the proof to the reader as an exercise.

The field content further consists of six "scalar" fields

$$\phi^M \in C^\infty(\mathbb{R}^4, i\cdot\mathfrak{su}(N))\,, \qquad M = 1,\dots,6$$

which are rewritten as fields ϕ^{AB} and $\overline{\phi}_{AB}$ according to (1.15), and fermions

$$\psi^A_\alpha, \tilde{\psi}^{\dot\alpha}_A \in C^\infty(\mathbb{R}^4, i\cdot\mathfrak{su}(N))\,, \quad (\psi^A_\alpha)^* = \tilde{\psi}_{\dot\alpha A}\,, \quad A\in\{1,2,3,4\}\,, \quad \alpha,\dot\alpha\in\{1,2\}$$

It is implicitly understood that the fermions are Grassmann odd, in the sense as explained in Sec. 1.1.1, while the gauge and scalar fields, being bosons, are even.

Lemma 1.2.2. With the definition (1.13), we have

$$F^{\alpha\beta} = i\partial^\alpha{}_{\dot\gamma}A^{\beta\dot\gamma} + i\partial^\beta{}_{\dot\gamma}A^{\alpha\dot\gamma} + g\left[A^\alpha{}_{\dot\gamma},\, A^{\dot\gamma\beta}\right]$$

Proof. This is shown by the following straightforward calculation.

$$\begin{aligned}
F^{\alpha\beta} &= \frac{i}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu,\, A_\nu])\left(\overline{\sigma}^{\mu\alpha}{}_{\dot\gamma}\sigma^{\nu\dot\gamma\beta} - \overline{\sigma}^{\nu\alpha}{}_{\dot\gamma}\sigma^{\mu\dot\gamma\beta}\right)\\
&= i\partial_\mu A_\nu\left(\overline{\sigma}^{\mu\alpha}{}_{\dot\gamma}\sigma^{\nu\dot\gamma\beta} - \overline{\sigma}^{\nu\alpha}{}_{\dot\gamma}\sigma^{\mu\dot\gamma\beta}\right) + i(-ig)\,[A_\mu,\, A_\nu]\,\overline{\sigma}^{\mu\alpha}{}_{\dot\gamma}\sigma^{\nu\dot\gamma\beta}\\
&= i\partial^\alpha{}_{\dot\gamma}A^{\dot\gamma\beta} - i\partial^{\dot\gamma\beta}A^\alpha{}_{\dot\gamma} + g\left[A^\alpha{}_{\dot\gamma},\, A^{\dot\gamma\beta}\right]\\
&= i\partial^\alpha{}_{\dot\gamma}A^{\beta\dot\gamma} + i\partial^\beta{}_{\dot\gamma}A^{\alpha\dot\gamma} + g\left[A^\alpha{}_{\dot\gamma},\, A^{\dot\gamma\beta}\right]
\end{aligned}$$

□

Let $A\in\Omega^1(\mathbb{R}^{10}, i\cdot\mathfrak{su}(N))$ be a connection on $\mathbb{R}^{10}$ and $\Psi\in C^\infty(\mathbb{R}^{10}, \Delta^+_{1,9}\otimes i\mathfrak{su}(N))$ be an (odd) Majorana-Weyl spinor, such that A and Ψ only depend on the coordinates x^μ on $\mathbb{R}^4\subseteq\mathbb{R}^{10}$. Then, prescribing,

(1.23) $$A^\mu := A^M \text{ for } M=\mu\in\{0,1,2,3\}\,, \qquad \phi^M := A^{M+3} \text{ for } M\in\{1,\dots,6\}$$

and using the explicit form (1.19) for Ψ, the fields stated are easily obtained, a technique referred to as dimensional reduction.

The Lagrangian from Dimensional Reduction

Dimensional reduction allows the canonical construction of a Lagrangian in Minkowski space out of a Lagrangian in ten-dimensional Lorentz space. Consider thus the $\mathcal{N}=1$ SYM Lagrangian

(1.24) $$\mathcal{L}_{10} := \operatorname{tr}\left(-\frac{1}{2}F_{MN}F^{MN} + i\overline{\Psi}\Gamma_M D^M\Psi\right)$$

We will see in the next section that it is supersymmetric, from which supersymmetry of the dimensionally reduced Lagrangian $\mathcal{L}_4$ in Minkowski space, to be calculated next, then easily follows.

Lemma 1.2.3. Dimensional reduction of $\mathcal{L}_{10}$ to Minkowski space yields $\mathcal{L}_4$ as follows, which is referred to as the $\mathcal{N}=4$ SYM Lagrangian.

$$\begin{aligned}
\mathcal{L}_4 = \operatorname{tr}\Bigg(&-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi^{AB})(D^\mu\overline{\phi}_{AB}) + \frac{1}{8}g^2\left[\phi^{AB},\,\phi^{CD}\right]\left[\overline{\phi}_{AB},\,\overline{\phi}_{CD}\right]\\
&+2i\,\tilde{\psi}_{\dot\alpha A}\sigma^{\dot\alpha\beta}_\mu D^\mu\psi^A_\beta - \sqrt{2}\,g\psi^{\alpha A}\left[\overline{\phi}_{AB},\,\psi^B_\alpha\right] + \sqrt{2}\,g\tilde{\psi}_{\dot\alpha A}\left[\phi^{AB},\,\tilde{\psi}^{\dot\alpha}_B\right]\Bigg)
\end{aligned}$$

Proof. We calculate

$$-\frac{1}{2}F_{MN}F^{MN} = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{ab}F^{ab} - F_{\mu a}F^{\mu a}$$

thus already obtaining the first term of $\mathcal{L}_4$. Since $\partial^a A^b = 0$ by dimensional reduction, we obtain $F^{ab} = -ig\left[A^a,\, A^b\right]$. Lem. 1.17 then yields, for the second term,

$$\begin{aligned}
-\frac{1}{2}F_{ab}F^{ab} &= -\frac{1}{2}(-ig)(-ig)\left[A_a,\, A_b\right]\left[A^a,\, A^b\right] \\
&= \frac{1}{8}g^2\left[\phi^{AB},\, \phi^{CD}\right]\left[\overline{\phi}_{AB},\, \overline{\phi}_{CD}\right]
\end{aligned}$$

while the third term gives

$$\begin{aligned}
-F_{\mu a}F^{\mu a} &= -(D_\mu A_a)(D^\mu A^a) \\
&= (D_\mu A^a)(D^\mu A^a) \\
&= \frac{1}{2}(D_\mu \phi^{AB})(D^\mu \overline{\phi}_{AB})
\end{aligned}$$

Moreover, by Lem. 1.1.11, Lem. 1.1.6 and Lem. 1.2.1 we yield

$$\begin{aligned}
\overline{\Psi} D_\mu \Gamma^\mu \Psi &= \tilde{\psi}_{\dot{\alpha}A}\sigma^{\mu\dot{\alpha}\beta}D_\mu\psi^A_\beta + \psi^{\alpha A}\overline{\sigma}^\mu_{\alpha\dot{\beta}}D_\mu\tilde{\psi}^{\dot{\beta}}_B \\
&= 2\tilde{\psi}_{\dot{\alpha}A}\sigma^{\mu\dot{\alpha}\beta}D_\mu\psi^A_\beta + \text{exact}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\overline{\Psi} D_a \Gamma^a \Psi &= -\Sigma^{aAB}\tilde{\psi}_{\dot{\alpha}A}D_a\tilde{\psi}^{\dot{\alpha}}_B + \overline{\Sigma}^a_{AB}\psi^{\alpha A}D_a\psi^B_\alpha \\
&= -\Sigma^{aAB}\tilde{\psi}_{\dot{\alpha}A}(-ig)\left[A_a,\, \tilde{\psi}^{\dot{\alpha}}_B\right] + \overline{\Sigma}^a_{AB}\psi^{\alpha A}(-ig)\left[A_a,\, \psi^B_\alpha\right] \\
&= -ig\sqrt{2}\tilde{\psi}_{\dot{\alpha}A}\left[\phi^{AB},\, \tilde{\psi}^{\dot{\alpha}}_B\right] + ig\sqrt{2}\psi^{\alpha A}\left[\overline{\phi}_{AB},\, \psi^B_\alpha\right]
\end{aligned}$$

This concludes the derivation of $\mathcal{L}_4$. □

Writing $A^\mu = A^{a\mu}T^a$ with generators T^a of $i \cdot \mathfrak{su}(N)$, and analogous for the other fields, we obtain the following form of the Lagrangian, which turns out to be useful in the derivation of the Feynman rules in Sec. 1.5.

Lemma 1.2.4. The Lagrangian $\mathcal{L}_4$, written in component form, reads

$$\begin{aligned}
\mathcal{L}_4 = &-\frac{1}{4}(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)^2 + \frac{1}{4}(\partial_\mu\phi^{cAB})(\partial^\mu\overline{\phi}^c_{AB}) + i\tilde{\psi}^c_{\dot{\alpha}A}\sigma^{\dot{\alpha}\beta}_\mu\partial^\mu\psi^{cA}_\beta \\
&- gf^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} - \frac{g^2}{4}f^{abe}f^{cde}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu} \\
&+ \frac{g}{2}f^{abc}A^{a\mu}\overline{\phi}^b_{AB}(\partial_\mu\phi^{cAB}) + \frac{g^2}{4}f^{abe}f^{cde}A^a_\mu\phi^{bAB}A^{c\mu}\overline{\phi}^d_{AB} \\
&+ igf^{abc}\sigma^{\dot{\alpha}\beta}_\mu A^{a\mu}\psi^{bA}_\beta\tilde{\psi}^c_{\dot{\alpha}A} - \frac{\sqrt{2}ig}{2}f^{abc}\overline{\phi}^a_{AB}\psi^{bB}_\alpha\psi^{c\alpha A} + \frac{\sqrt{2}ig}{2}f^{abc}\phi^{aAB}\tilde{\psi}^{b\dot{\alpha}}_B\tilde{\psi}^c_{\dot{\alpha}A} \\
&- \frac{g^2}{16}f^{abe}f^{cde}\phi^{aAB}\phi^{bCD}\overline{\phi}^c_{AB}\overline{\phi}^d_{CD}
\end{aligned}$$

Proof. The first three terms $\sim g^0$ follow directly from replacing D^μ by ∂^μ and using the normalisation (1.2). The gluon terms (second line) are obtained by Lem. 1.1.3 to be

$$\begin{aligned}
\mathcal{L}_4|_{g^1,\text{gluon}} &= -\frac{1}{2}\text{tr}(F_{\mu\nu}F^{\mu\nu})|_{g^1} \\
&= ig\,\text{tr}\left((\partial_\mu A_\nu - \partial_\nu A_\mu)\left[A^\mu,\, A^\nu\right]\right) \\
&= -\frac{g}{2}(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)A^{a\mu}A^{b\nu}f^{abc} \\
&= -g(\partial_\mu A^c_\nu)A^{a\mu}A^{b\nu}f^{abc} \\
&= -gf^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\mathcal{L}_4|_{g^2,\text{gluon}} &= -\frac{1}{2}\text{tr}(F_{\mu\nu}F^{\mu\nu})|_{g^2} \\
&= -\frac{1}{2}\text{tr}\left((-ig\left[A_\mu,\, A_\nu\right])(-ig\left[A^\mu,\, A^\nu\right])\right) \\
&= \frac{g^2}{2}\text{tr}\left(\left[A_\mu,\, A_\nu\right]\left[A^\mu,\, A^\nu\right]\right) \\
&= \frac{g^2}{2}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu}\text{tr}\left(\left[T^a,\, T^b\right]\left[T^c,\, T^d\right]\right) \\
&= \frac{g^2}{2}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu}if^{abe}\text{tr}\left(T^e\left[T^c,\, T^d\right]\right) \\
&= -\frac{g^2}{4}f^{abe}f^{cde}A^a_\mu A^b_\nu A^{c\mu}A^{d\nu}
\end{aligned}$$

We further calculate the gluon-scalar terms (third line)

$$\begin{aligned}
\mathcal{L}_4|_{g^1,\text{gluon}-\text{scalar}} &= -\frac{ig}{2}\text{tr}\left((\partial_\mu\phi^{AB})\left[A^\mu,\, \overline{\phi}_{AB}\right] + \left[A_\mu,\, \phi^{AB}\right](\partial^\mu\overline{\phi}_{AB})\right) \\
&= -ig\,\text{tr}\left((\partial_\mu\phi^{AB})\left[A^\mu,\, \overline{\phi}_{AB}\right]\right) \\
&= \frac{g}{2}f^{abc}A^{a\mu}\overline{\phi}^b_{AB}(\partial_\mu\phi^{cAB})
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_4|_{g^2,\text{gluon}-\text{scalar}} &= -\frac{g^2}{2}\text{tr}\left(\left[A_\mu,\, \phi^{AB}\right]\left[A^\mu,\, \overline{\phi}_{AB}\right]\right) \\
&= \frac{g^2}{4}f^{abe}f^{cde}A^a_\mu\phi^{bAB}A^{c\mu}\overline{\phi}^d_{AB}
\end{aligned}$$

The fourth line with terms containing fermions is immediate, while the four-scalar term (last line) is obtained by

$$\begin{aligned}
\mathcal{L}_4|_{g^2,\text{scalar}} &= \frac{g^2}{8}\text{tr}\left(\left[\phi^{AB},\, \phi^{CD}\right]\left[\overline{\phi}_{AB},\, \overline{\phi}_{CD}\right]\right) \\
&= \frac{g^2}{8}\phi^{aAB}\phi^{bCD}\overline{\phi}^c_{AB}\overline{\phi}^d_{CD}\text{tr}(if^{abe}T^e if^{cdf}T^f) \\
&= -\frac{g^2}{16}f^{abe}f^{cde}\phi^{aAB}\phi^{bCD}\overline{\phi}^c_{AB}\overline{\phi}^d_{CD}
\end{aligned}$$

This concludes the derivation of the component form. □

1.3 Supersymmetry

In general, a (finite) symmetry of a (classical) field theory is a transformation of the fields which leaves the action invariant. This holds in particular for a transformation that alters the Lagrangian at most by an exact term. In good cases, the symmetries of a theory are induced by the action of some Lie group. Differentiation then yields the corresponding Lie algebra action which acts by derivations that cancel the Lagrangian (up to an exact term). This is referred to as an infinitesimal symmetry. In the following, saying "symmetry", we will exclusively mean one or several derivations δ which are, by definition, linear and satisfy the product rule

$$\delta(BC) = \delta(B)C + B\delta(C) \tag{1.25}$$

where B and C are any two fields of the theory. In particular, we do not care about whether such a "symmetry" is indeed an infinitesimal symmetry. Consult [DF99a] for a more in-depth treatment.

A supersymmetry (finite or infinitesimal) is a symmetry that exchanges bosons and fermions. Some supersymmetric field theories admit a so called superspace formulation, in which the fields can all be combined into a single morphism of supermanifolds and such that "supersymmetry" really means infinitesimal supersymmetry (cf. [DF99b] and [Hél09]). For $\mathcal{N} = 4$ SYM theory, such a formulation seems not to be known.

However, $\mathcal{N} = 4$ SYM does have supersymmetry transformations (in the simplified meaning explained). They are most conveniently obtained by dimensional reduction from supersymmetry transformations of the ten-dimensional theory as follows. Let $\xi \in \Delta^+_{1,9}$ be a constant (odd) Majorana-Weyl spinor and consider the following derivations.

$$\delta\Psi = \frac{i}{2}F^{MN}\Gamma_{MN}\xi\,, \qquad \delta A_M = -i\bar{\xi}\Gamma_M\Psi\,, \qquad \Gamma_{MN} := \frac{i}{2}(\Gamma_M\Gamma_N - \Gamma_N\Gamma_M) \tag{1.26}$$

By the next theorem, they are indeed symmetries.

Theorem 1.3.1. $\mathcal{L}_{10}$ is invariant under supersymmetry transformations (1.26), i.e.

$$\delta\mathcal{L}_{10} = 0$$

holds (up to an exact term).

Proof. We proof that the variation of the bosonic part of the Lagrangian cancels with that of the fermionic part, up to exact terms (which are not of interest) and an expression trilinear in the spinor field Ψ which can be shown to vanish.

We calculate the variation of the bosonic part:

$$\begin{aligned}
\delta\,\mathrm{tr}\left(-\frac{1}{2}F_{MN}F^{MN}\right) &= -\mathrm{tr}\left((\delta F_{MN})F^{MN}\right)\\
&= -\mathrm{tr}\left(\delta(\partial_M A_N - \partial_N A_M - ig\,[A_M,\,A_N])F^{MN}\right)\\
&= -2\mathrm{tr}\left(\left(\partial_M \delta A_N - ig\,[A_M,\,\delta A_N]\right)F^{MN}\right)\\
&= 2i\mathrm{tr}\left(\bar{\xi}\Gamma_N\partial_M\Psi - ig\left[A_M,\,\bar{\xi}\Gamma_N\Psi\right]\right)\\
&= 2i\mathrm{tr}\left((\bar{\xi}\Gamma_N D_M\Psi)F^{MN}\right)\\
&= 2ig_{ML}\mathrm{tr}\left(\bar{\xi}\Gamma_N(D^L\Psi)F^{MN}\right)\\
&= -2i\,g_{ML}\mathrm{tr}\left(\bar{\xi}\Gamma_N\Psi D^L F^{MN}\right)(+\,\mathrm{exact})\\
&= 2i\,g_{ML}\mathrm{tr}\left(\overline{\Psi}\Gamma_N D^L F^{MN}\xi\right)(+\,\mathrm{exact})
\end{aligned}$$

where the last two equation are, respectively, due to Lem. 1.2.1 and (1.20).

We calculate the variation of the fermionic part:

$$\begin{aligned}\delta\,\mathrm{tr}\left(i\overline{\Psi}\Gamma_M D^M\Psi\right) &= i\mathrm{tr}\left((\delta\overline{\Psi})\Gamma_M D^M\Psi\right) + i\mathrm{tr}\left(\overline{\Psi}\Gamma_M D^M\delta\Psi\right) + i\mathrm{tr}\left(\overline{\Psi}\Gamma_M(\delta D^M)\Psi\right)\\ &=: (1)+(2)+(3)\end{aligned}$$

As a side calculation using $iA \in \mathfrak{su}(N)$, note that

$$\overline{D^M\Psi} = (D^M\Psi)^\dagger\Gamma^0 = D^M\Psi^\dagger\Gamma^0 = D^M\overline{\Psi}$$

Further using $F^\dagger = F$ as well as (1.20) and 1.2.1, we see that the first two contributions coincide:

$$\begin{aligned}(1) &= -\frac{1}{2}\mathrm{tr}\left(F^{AB}(\overline{\Gamma_{AB}\xi})\Gamma_M(D^M\Psi)\right) = \frac{1}{2}\mathrm{tr}\left(F^{AB}\overline{D^M\Psi}\Gamma_M\Gamma_{AB}\xi\right)\\ &= \frac{1}{2}\mathrm{tr}\left(F^{AB}D^M\overline{\Psi}\Gamma_M\Gamma_{AB}\xi\right) = -\frac{1}{2}\mathrm{tr}\left((D^M F^{AB})\overline{\Psi}\Gamma_M\Gamma_{AB}\xi\right)\\ &= -\frac{1}{2}\mathrm{tr}\left(\overline{\Psi}\Gamma_M D^M F^{AB}\Gamma_{AB}\xi\right) = i\mathrm{tr}\left(\overline{\Psi}\Gamma_M D^M\delta\Psi\right) = (2)\end{aligned}$$

Therefore,

$$\begin{aligned}(1)+(2) &= -\mathrm{tr}\left(\overline{\Psi}\Gamma_M D^M F^{AB}\Gamma_{AB}\xi\right)\\ &= -\mathrm{tr}\left(\frac{i}{2}\overline{\Psi}\Gamma_M D^M F^{AB}(\Gamma_A\Gamma_B-\Gamma_B\Gamma_A)\xi\right)\\ &= -i\mathrm{tr}\left(\overline{\Psi}\Gamma_M D^M F^{AB}\Gamma_A\Gamma_B\xi\right)\\ &= -i\mathrm{tr}\left(\overline{\Psi}\Gamma_L\Gamma_M\Gamma_N D^L F^{MN}\xi\right)\end{aligned}$$

Using the Clifford relation and the Bianchi identity (1.21), we further yield

$$\begin{aligned}&\Gamma_L\Gamma_M\Gamma_N D^L F^{MN}\\ &\quad= \Gamma_L\Gamma_M\Gamma_N\left(\frac{2}{3}D^L F^{MN} + \frac{1}{3}D^L F^{MN}\right)\\ &\quad= \frac{1}{3}\Gamma_L\Gamma_M\Gamma_N\left(2D^L F^{MN} + D^M F^{LN} + D^N F^{ML}\right)\\ &\quad= \frac{1}{3}\Gamma_L\Gamma_M\Gamma_N\left(D^L F^{MN} + D^M F^{LN}\right) + \frac{1}{3}\Gamma_L\Gamma_M\Gamma_N\left(D^L F^{MN} + D^N F^{ML}\right)\\ &\quad= \frac{1}{3}(\Gamma_L\Gamma_M+\Gamma_M\Gamma_L)\Gamma_N D^L F^{MN} + \frac{1}{3}(\Gamma_L\Gamma_M\Gamma_N+\Gamma_N\Gamma_M\Gamma_L)D^L F^{MN}\\ &\quad= \frac{2}{3}g_{ML}\Gamma_N D^L F^{MN} + \frac{1}{3}(-\Gamma_L\Gamma_N\Gamma_M + 2g_{MN}\Gamma_L + \Gamma_N\Gamma_M\Gamma_L)D^L F^{MN}\\ &\quad= \frac{2}{3}g_{ML}\Gamma_N D^L F^{MN} + \frac{1}{3}(-\Gamma_L\Gamma_N\Gamma_M + \Gamma_N\Gamma_M\Gamma_L)D^L F^{MN}\\ &\quad= \frac{1}{3}\left(2g_{ML}\Gamma_N + \Gamma_N\Gamma_L\Gamma_M - 2g_{NL}\Gamma_M + \Gamma_N\Gamma_M\Gamma_L\right)D^L F^{MN}\\ &\quad= \frac{1}{3}\left(2g_{ML}\Gamma_N + 2\Gamma_N g_{LM} - 2g_{NL}\Gamma_M\right)D^L F^{MN}\\ &\quad= 2g_{ML}\Gamma_N D^L F^{MN}\end{aligned}$$

Summarising, we obtain

$$(1)+(2) = -i\mathrm{tr}\left(\overline{\Psi}\Gamma_L\Gamma_M\Gamma_N D^L F^{MN}\xi\right) = -2ig_{ML}\mathrm{tr}\left(\overline{\Psi}\Gamma_N D^L F^{MN}\xi\right)$$

But this is exactly minus the bosonic variation.

Up to exact terms, we finally conclude $\delta\mathcal{L}_{10} = (3) \sim \operatorname{tr}\left(\overline{\Psi}\Gamma_M(\overline{\xi}\Gamma^M\Psi)\Psi\right)$. This expression vanishes by the existence of a normed division algebra in dimension 8. Consult [BH10] for a very general and self-contained (but abstract) treatment. □

$\mathcal{N}=4$ Supersymmetry from Dimensional Reduction

Performing dimensional reduction upon (1.26) yields the following result concerning $\mathcal{N}=4$ SYM theory.

Lemma 1.3.2. $\mathcal{L}_4$ is invariant under the following supersymmetry transformations.

$$\delta A^\mu = -i\xi^{\alpha A}\overline{\sigma}^\mu_{\alpha\dot\beta}\tilde\psi^{\dot\beta}_A - i\tilde\xi_{\dot\alpha A}\sigma^{\mu\dot\alpha\beta}\psi^A_\beta \tag{1.27a}$$

$$\delta\phi^{AB} = -i\sqrt{2}\left(\xi^{\alpha A}\psi^B_\alpha - \xi^{\alpha B}\psi^A_\alpha - \varepsilon^{ABCD}\tilde\xi_{\dot\alpha C}\tilde\psi^{\dot\alpha}_D\right) \tag{1.27b}$$

$$\delta\psi^A_\alpha = \frac{i}{2}F_{\mu\nu}\sigma^{\mu\nu}{}_\alpha{}^\beta\xi^A_\beta - \sqrt{2}(D_\mu\phi^{AB})\overline{\sigma}^\mu_{\alpha\dot\beta}\tilde\xi^{\dot\beta}_B + ig\left[\phi^{AB},\,\overline{\phi}_{BC}\right]\xi^C_\alpha \tag{1.27c}$$

$$\delta\tilde\psi^{\dot\alpha}_A = \frac{i}{2}F_{\mu\nu}\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\tilde\xi^{\dot\beta}_A + \sqrt{2}(D_\mu\overline{\phi}_{AB})\sigma^{\mu\dot\alpha\beta}\xi^B_\beta + ig\left[\overline{\phi}_{AB},\,\phi^{BC}\right]\tilde\xi^{\dot\alpha}_C \tag{1.27d}$$

In other words, $\delta\mathcal{L}_4 = 0$ holds up to an exact term.

Proof. Lem. 1.1.11 yields

$$\begin{aligned}
\delta A^\mu &= -i\overline{\xi}\Gamma^\mu\Psi = -i\left(\tilde\xi_{\dot\alpha A}\sigma^{\mu\dot\alpha\beta}\psi^A_\beta + \xi^{\alpha A}\overline{\sigma}^\mu_{\alpha\dot\beta}\tilde\psi^{\dot\beta}_B\right)\\
\delta A^{a+3} &= -i\overline{\xi}\Gamma^{a+3}\Psi = -i\left(-\Sigma^{aAB}\tilde\xi_{\dot\alpha A}\tilde\psi^{\dot\alpha}_B + \overline{\Sigma}^a_{AB}\xi^{\alpha A}\psi^B_\alpha\right)
\end{aligned}$$

By dimensional reduction (1.23), we thus immediately obtain (1.27a), and (1.27b) follows from

$$\begin{aligned}
\delta\phi^{AB} &= \frac{1}{\sqrt{2}}\Sigma^{aAB}\delta A^a\\
&= -\frac{i}{\sqrt{2}}\Sigma^{aAB}\left(-\Sigma^{aCD}\tilde\xi_{\dot\alpha C}\tilde\psi^{\dot\alpha}_D + \overline{\Sigma}^a_{CD}\xi^{\alpha C}\psi^D_\alpha\right)\\
&= -\frac{i}{\sqrt{2}}\left(-2\varepsilon_{ABCD}\tilde\xi_{\dot\alpha C}\tilde\psi^{\dot\alpha}_D + (2\delta_{AC}\delta_{BD} - 2\delta_{AD}\delta_{BC})\xi^{\alpha C}\psi^D_\alpha\right)\\
&= i\sqrt{2}\left(\varepsilon_{ABCD}\tilde\xi_{\dot\alpha C}\tilde\psi^{\dot\alpha}_D - \xi^{\alpha A}\psi^B_\alpha + \xi^{\alpha B}\psi^A_\alpha\right)
\end{aligned}$$

using (1.14) and Lem. 1.1.10. In the following, we blur the distinction between indices a and $a+3$. The respective use should be clear from the context. The curvature term in the variation of Ψ decomposes into

$$F^{MN}\Gamma_{MN} = F^{\mu\nu}\Gamma_{\mu\nu} + F^{ab}\Gamma_{ab} + 2F^{\mu a}\Gamma_{\mu a}$$

Now, according to definitions (1.5) and (1.12), we obtain

$$\Gamma^{\mu\nu} = \frac{i}{2}(1_8\otimes\gamma^\mu\gamma^\nu - 1_8\otimes\gamma^\nu\gamma^\mu) = 1_8\otimes\begin{pmatrix}\sigma^{\mu\nu}{}_\alpha{}^\beta & 0\\ 0 & \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}\end{pmatrix}$$

such that

$$\frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\xi = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\frac{i}{2}F_{\mu\nu}\sigma^{\mu\nu}{}_{\alpha}{}^{\beta}\xi^{A}_{\beta}\\ 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\ \frac{i}{2}F_{\mu\nu}\overline{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}}\tilde{\xi}^{\dot{\beta}}_{A}\end{pmatrix}$$

Similarly, we find

$$\begin{aligned}\Gamma^{ab} &= \frac{i}{2}(\hat{\gamma}^a\hat{\gamma}^b - \hat{\gamma}^b\hat{\gamma}^a)\otimes(\gamma^5)^2\\ &= \frac{i}{2}\begin{pmatrix}\Sigma^{aAB}\overline{\Sigma}^{b}_{BC} - \Sigma^{bAB}\overline{\Sigma}^{a}_{BC} & 0\\ 0 & \overline{\Sigma}^{a}_{AB}\Sigma^{bBC} - \overline{\Sigma}^{b}_{AB}\Sigma^{aBC}\end{pmatrix}\otimes 1_4\end{aligned}$$

Inserting $F^{ab} = -ig\left[A^a,\, A^b\right]$ (derivatives vanish by dimensional reduction), we thus yield

$$\frac{i}{2}F_{ab}\Gamma^{ab}\xi = ig\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}\left[\phi^{AB},\,\overline{\phi}_{BC}\right]\xi^{C}_{\alpha}\\ 0\end{pmatrix} + ig\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\ \left[\overline{\phi}_{AB},\,\phi^{BC}\right]\tilde{\xi}^{\dot{\alpha}}_{C}\end{pmatrix}$$

Moreover

$$\Gamma^{\mu a} = \frac{i}{2}\hat{\gamma}^a\otimes(\gamma^\mu\gamma^5 - \gamma^5\gamma^\mu) = i\hat{\gamma}^a\otimes\gamma^\mu\gamma^5$$

With $F_{\mu a} = \partial_\mu A_a - ig\left[A_\mu,\, A_a\right] = D_\mu A_a$, we thus find

$$\begin{aligned}2\frac{i}{2}F_{\mu a}\Gamma^{\mu a}\xi &= -D_\mu A_a(\hat{\gamma}^a\otimes\gamma^\mu\gamma^5)\xi\\ &= -D_\mu A_a\begin{pmatrix}0 & \Sigma^{aAB}\\ \overline{\Sigma}^{a}_{AB} & 0\end{pmatrix}\otimes\begin{pmatrix}0 & -\overline{\sigma}^{\mu}_{\alpha\dot{\beta}}\\ \sigma^{\mu\dot{\alpha}\beta} & 0\end{pmatrix}\xi\\ &= \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}D_\mu A_a\overline{\sigma}^{\mu}_{\alpha\dot{\beta}}\Sigma^{aAB}\tilde{\xi}^{\dot{\beta}}_{B}\\ 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\ -D_\mu A_a\sigma^{\mu\dot{\alpha}\beta}\overline{\Sigma}^{a}_{AB}\xi^{B}_{\beta}\end{pmatrix}\\ &= \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}-\sqrt{2}(D_\mu\phi^{AB})\overline{\sigma}^{\mu}_{\alpha\dot{\beta}}\tilde{\xi}^{\dot{\beta}}_{B}\\ 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\ \sqrt{2}(D_\mu\overline{\phi}_{AB})\sigma^{\mu\dot{\alpha}\beta}\xi^{B}_{\beta}\end{pmatrix}\end{aligned}$$

in the last step using that $A_a = -A^a$. Picking up terms in this and the previous calculations, we arrive at (1.27c) and (1.27d). □

Supersymmetry Generators

For any field O (e.g. $O = A^\mu$), we define the supersymmetry generators q^{α}_{A} and $\tilde{q}^{A\dot{\alpha}}$ by

$$\delta O = \xi^{A}_{\alpha}\, q^{\alpha}_{A}(O) + \tilde{\xi}_{A\dot{\alpha}}\, \tilde{q}^{A\dot{\alpha}}(O) \tag{1.28}$$

Note that the supersymmetry generators thus defined are odd superderivations rather than ordinary derivations (1.25).

Corollary 1.3.3. $\mathcal{L}_4$ is invariant under the action of supersymmetry generators, i.e.

$$q^{\alpha}_{A}(\mathcal{L}_4) = 0\,, \qquad \tilde{q}^{A\dot{\alpha}}(\mathcal{L}_4) = 0$$

holds up to exact terms.

Proof. Choosing $\xi^A_\alpha := \delta^{AA_0}\delta_{\alpha\alpha_0}$ fixes $\tilde{\xi}_{A\dot{\alpha}}$ by the Majorana condition (1.19), and the right hand side of (1.28) becomes $(q^{\alpha_0}_{A_0} + \tilde{q}^{A_0\dot{\alpha}_0})(O)$. Similarly, $\xi^A_\alpha := i\delta^{AA_0}\delta_{\alpha\alpha_0}$ leads to $i(q^{\alpha_0}_{A_0} - \tilde{q}^{A_0\dot{\alpha}_0})(O)$. By Lem. 1.3.2, both expressions vanish for O replaced by $\mathcal{L}_4$ (up to exact terms) and, therefore, each generator applied to $\mathcal{L}_4$ vanishes individually. □

Lemma 1.3.4. Thus supersymmetry generators act on the fields as follows.

$$\begin{aligned}
q^\alpha_A(A^{\beta\dot{\beta}}) &= 2i\,\epsilon^{\alpha\beta}\tilde{\psi}^{\dot{\beta}}_A\,, & \tilde{q}^{A\dot{\alpha}}(A^{\beta\dot{\beta}}) &= -2i\,\epsilon^{\dot{\alpha}\dot{\beta}}\psi^{\beta A}\\
q^\alpha_A(\overline{\phi}_{BC}) &= i\sqrt{2}\,\varepsilon_{ABCD}\psi^{\alpha D}\,, & \tilde{q}^{A\dot{\alpha}}(\overline{\phi}_{BC}) &= -i\sqrt{2}\,\tilde{\psi}^{\dot{\alpha}}_{[B}\delta^A_{C]}\\
q^\alpha_A(\psi^{B\beta}) &= \frac{i}{2}F^{\beta\alpha}\delta^B_A + i\epsilon^{\beta\alpha}g\left[\overline{\phi}_{AC},\,\phi^{BC}\right]\,, & \tilde{q}^{A\dot{\alpha}}(\psi^{B\beta}) &= -\sqrt{2}\,D^{\beta\dot{\alpha}}\phi^{AB}\\
q^\alpha_A(\tilde{\psi}^{\dot{\beta}}_B) &= -\sqrt{2}\,D^{\dot{\beta}\alpha}\overline{\phi}_{AB}\,, & \tilde{q}^{A\dot{\alpha}}(\tilde{\psi}^{\dot{\beta}}_B) &= -\frac{i}{2}F^{\dot{\beta}\dot{\alpha}}\delta^A_B + ig\left[\phi^{AC},\,\overline{\phi}_{BC}\right]\epsilon^{\dot{\alpha}\dot{\beta}}
\end{aligned}$$

Proof. From (1.27a) we immediately find $q^\alpha_A(A^\mu) = i\overline{\sigma}^{\mu\alpha}{}_{\dot{\beta}}\tilde{\psi}^{\dot{\beta}}_A$ and $\tilde{q}^{A\dot{\alpha}}(A^\mu) = -i\sigma^{\mu\dot{\alpha}\beta}\psi^A_\beta$. By Lem. 1.1.7 and Lem. 1.1.6, we thus obtain

$$\begin{aligned}
q^\alpha_A(A^{\beta\dot{\beta}}) &= \overline{\sigma}_\mu{}^{\beta\dot{\beta}}q^\alpha_A(A^\mu) = i\overline{\sigma}_\mu{}^{\beta\dot{\beta}}\overline{\sigma}^{\mu\alpha}{}_{\dot{\epsilon}}\tilde{\psi}^{\dot{\epsilon}}_A = i(\epsilon^{\beta\gamma}\overline{\sigma}_{\mu\gamma\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\beta}})(\epsilon^{\alpha\delta}\overline{\sigma}^\mu_{\delta\dot{\epsilon}})\tilde{\psi}^{\dot{\epsilon}}_A\\
&= -2i\epsilon^{\beta\gamma}\epsilon^{\dot{\gamma}\dot{\beta}}\epsilon^{\alpha\delta}(\epsilon_{\gamma\delta}\epsilon_{\dot{\gamma}\dot{\epsilon}})\tilde{\psi}^{\dot{\epsilon}}_A = -2i(\epsilon^{\beta\gamma}\epsilon^{\alpha\delta}\epsilon_{\gamma\delta})(\epsilon^{\dot{\gamma}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\epsilon}}\tilde{\psi}^{\dot{\epsilon}}_A) = -2i\,\epsilon^{\beta\alpha}\tilde{\psi}^{\dot{\beta}}_A
\end{aligned}$$

and

$$\begin{aligned}
\tilde{q}^{A\dot{\alpha}}(A^{\beta\dot{\beta}}) &= \overline{\sigma}_\mu{}^{\beta\dot{\beta}}\tilde{q}^{A\dot{\alpha}}(A^\mu) = -i\overline{\sigma}_\mu{}^{\beta\dot{\beta}}\sigma^{\mu\dot{\alpha}\epsilon}\psi^A_\epsilon = -i(\epsilon^{\beta\lambda}\overline{\sigma}_{\mu\lambda\dot{\lambda}}\epsilon^{\dot{\lambda}\dot{\beta}})(\epsilon^{\epsilon\gamma}\overline{\sigma}^\mu_{\gamma\dot{\delta}}\epsilon^{\dot{\delta}\dot{\alpha}})\psi^A_\epsilon\\
&= 2i\epsilon^{\beta\lambda}\epsilon^{\dot{\lambda}\dot{\beta}}\epsilon^{\epsilon\gamma}\epsilon^{\dot{\delta}\dot{\alpha}}(\epsilon_{\lambda\gamma}\epsilon_{\dot{\lambda}\dot{\delta}})\psi^A_\epsilon = 2i(\epsilon^{\dot{\lambda}\dot{\beta}}\epsilon^{\dot{\delta}\dot{\alpha}}\epsilon_{\dot{\lambda}\dot{\delta}})(\epsilon^{\beta\lambda}\epsilon^{\epsilon\gamma}\epsilon_{\lambda\gamma}\psi^A_\epsilon) = 2i\epsilon^{\dot{\beta}\dot{\alpha}}\psi^{\beta A}
\end{aligned}$$

From (1.27b), we find $q^\alpha_A(\phi^{BC}) = i\sqrt{2}\left(\delta^{AB}\psi^{\alpha C} - \delta^{AC}\psi^{\alpha B}\right)$ and, moreover, $\tilde{q}^{A\dot{\alpha}}(\phi^{BC}) = i\sqrt{2}\varepsilon^{ABCD}\tilde{\psi}^{\dot{\alpha}}_D$. We thus obtain

$$q^\alpha_A(\overline{\phi}_{BC}) = \frac{1}{2}\varepsilon_{BCDE}q^\alpha_A(\phi^{DE}) = \frac{i\sqrt{2}}{2}\varepsilon_{BCAE}\psi^{\alpha E} - \frac{i\sqrt{2}}{2}\varepsilon_{BCDA}\psi^{\alpha D} = i\sqrt{2}\,\varepsilon_{ABCD}\psi^{\alpha D}$$

and

$$\begin{aligned}
\tilde{q}^{A\dot{\alpha}}(\overline{\phi}_{BC}) &= \frac{1}{2}\varepsilon_{BCDE}\tilde{q}^{A\dot{\alpha}}(\phi^{DE}) = i\sqrt{2}\frac{1}{2}\varepsilon_{BCDE}\varepsilon^{ADEF}\tilde{\psi}^{\dot{\alpha}}_F = i\sqrt{2}\frac{1}{2}\varepsilon_{DEBC}\varepsilon^{DEAF}\tilde{\psi}^{\dot{\alpha}}_F\\
&= i\sqrt{2}(\delta_{AB}\delta_{CF} - \delta_{AC}\delta_{BF})\tilde{\psi}^{\dot{\alpha}}_F = i\sqrt{2}(\delta_{AB}\tilde{\psi}^{\dot{\alpha}}_C - \delta_{AC}\tilde{\psi}^{\dot{\alpha}}_B) = -i\sqrt{2}\tilde{\psi}^{\dot{\alpha}}_{[B}\delta^A_{C]}
\end{aligned}$$

From (1.27c) and (1.27d), we find, respectively,

$$\begin{aligned}
\delta\psi^{B\beta} &= \frac{i}{2}F_{\mu\nu}\sigma^{\mu\nu\beta\gamma}\xi^B_\gamma - \sqrt{2}(D_\mu\phi^{BC})\overline{\sigma}^{\mu\beta}{}_{\dot{\gamma}}\tilde{\xi}^{\dot{\gamma}}_C + ig\left[\phi^{BC},\,\overline{\phi}_{CD}\right]\xi^{D\beta}\\
&= \xi^A_\alpha\left(\frac{i}{2}F_{\mu\nu}\sigma^{\mu\nu\beta\alpha}\delta^{AB} + ig\left[\phi^{BC},\,\overline{\phi}_{CA}\right]\epsilon^{\beta\alpha}\right) + \tilde{\xi}_{\dot{\alpha}A}\sqrt{2}(D_\mu\phi^{BA})\overline{\sigma}^{\mu\beta\dot{\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
\delta\tilde{\psi}^{\dot{\beta}}_B &= \frac{i}{2}F_{\mu\nu}\overline{\sigma}^{\mu\nu\dot{\beta}}{}_{\dot{\gamma}}\tilde{\xi}^{\dot{\gamma}}_B + \sqrt{2}(D_\mu\overline{\phi}_{BC})\sigma^{\mu\dot{\beta}\gamma}\xi^C_\gamma + ig\left[\overline{\phi}_{BC},\,\phi^{CD}\right]\tilde{\xi}^{\dot{\beta}}_D\\
&= \xi^A_\alpha\sqrt{2}(D_\mu\overline{\phi}_{BA})\sigma^{\mu\dot{\beta}\alpha} + \tilde{\xi}_{A\dot{\alpha}}\left(-\frac{i}{2}F_{\mu\nu}\overline{\sigma}^{\mu\nu\dot{\beta}\dot{\alpha}}\delta_{AB} + ig\left[\overline{\phi}_{BC},\,\phi^{CA}\right]\epsilon^{\dot{\alpha}\dot{\beta}}\right)
\end{aligned}$$

This yields the remaining equations and thus finishes the proof. □

Corollary 1.3.5. The supersymmetry generators act on derived fields as follows.

$$\begin{aligned} q_A^\alpha(F^{\gamma\xi}) &= -2\epsilon^{\alpha\xi} D^\gamma{}_{\dot\gamma}\tilde\psi_A^{\dot\gamma} - 2\epsilon^{\alpha\gamma} D^\xi{}_{\dot\gamma}\tilde\psi_A^{\dot\gamma} \\ q_A^\alpha(D^{\dot\beta\gamma}\overline{\phi}_{BC}) &= 2g\epsilon^{\alpha\gamma}\left[\tilde\psi_A^{\dot\beta},\, \overline{\phi}_{BC}\right] + i\sqrt{2}\,\varepsilon_{ABCD} D^{\dot\beta\gamma}\psi^{\alpha D} \\ q_A^\alpha(D^{\dot\beta\gamma}\psi^{\xi B}) &= \frac{i}{2} D^{\dot\beta\gamma}F^{\xi\alpha}\delta_A^B + 2g\epsilon^{\alpha\gamma}\left[\tilde\psi_A^{\dot\beta},\, \psi^{\xi B}\right] + 2i\epsilon^{\xi\alpha} g\left[\overline{\phi}_{AC},\, D^{\dot\beta\gamma}\phi^{BC}\right] \\ &\quad + \frac{i}{2}\epsilon^{\xi\alpha} g\left[D^{\dot\beta\gamma}\phi^{DE},\, \overline{\phi}_{DE}\right]\delta_A^B \end{aligned}$$

Proof. We use the supersymmetry generators from Lem. 1.3.4. For the first equation, we use Lem. 1.2.2 to calculate

$$\begin{aligned} q_A^\alpha(F^{\gamma\xi}) &= q_A^\alpha\left(i\partial^\gamma{}_{\dot\gamma}A^{\xi\dot\gamma} + i\partial^\xi{}_{\dot\gamma}A^{\gamma\dot\gamma} + g\left[A^\gamma{}_{\dot\gamma},\, A^{\dot\gamma\xi}\right]\right) \\ &= -2\epsilon^{\alpha\xi}\partial^\gamma{}_{\dot\gamma}\tilde\psi_A^{\dot\gamma} - 2\epsilon^{\alpha\gamma}\partial^\xi{}_{\dot\gamma}\tilde\psi_A^{\dot\gamma} + 2ig\epsilon^{\alpha\gamma}\left[\tilde\psi_{\dot\gamma A},\, A^{\dot\gamma\xi}\right] + 2ig\epsilon^{\alpha\xi}\left[A^\gamma{}_{\dot\gamma},\, \tilde\psi_A^{\dot\gamma}\right] \\ &= -2\epsilon^{\alpha\xi}\left(\partial^\gamma{}_{\dot\gamma} - ig\left[A^\gamma{}_{\dot\gamma},\, \cdot\right]\right)\tilde\psi_A^{\dot\gamma} - 2\epsilon^{\alpha\gamma}\left(\partial^\xi{}_{\dot\gamma} - ig\left[A^\xi{}_{\dot\gamma},\, \cdot\right]\right)\tilde\psi_A^{\dot\gamma} \\ &= -2\epsilon^{\alpha\xi} D^\gamma{}_{\dot\gamma}\tilde\psi_A^{\dot\gamma} - 2\epsilon^{\alpha\gamma} D^\xi{}_{\dot\gamma}\tilde\psi_A^{\dot\gamma} \end{aligned}$$

For the second equation, we calculate

$$\begin{aligned} q_A^\alpha(D^{\dot\beta\gamma}\overline{\phi}_{BC}) &= q_A^\alpha(D^{\dot\beta\gamma})\overline{\phi}_{BC} + D^{\dot\beta\gamma}q_A^\alpha(\overline{\phi}_{BC}) \\ &= -ig\left[q_A^\alpha(A^{\dot\beta\gamma}),\, \overline{\phi}_{BC}\right] + D^{\dot\beta\gamma}q_A^\alpha(\overline{\phi}_{BC}) \\ &= 2g\epsilon^{\alpha\gamma}\left[\tilde\psi_A^{\dot\beta},\, \overline{\phi}_{BC}\right] + i\sqrt{2}\,\varepsilon_{ABCD}D^{\dot\beta\gamma}\psi^{\alpha D} \end{aligned}$$

Finally, the third equation follows from

$$\begin{aligned} q_A^\alpha(D^{\dot\beta\gamma}\psi^{\xi B}) &= q_A^\alpha(D^{\dot\beta\gamma})\psi^{\xi B} + D^{\dot\beta\gamma}q_A^\alpha(\psi^{\xi B}) \\ &= 2g\epsilon^{\alpha\gamma}\left[\tilde\psi_A^{\dot\beta},\, \psi^{\xi B}\right] + \frac{i}{2}D^{\dot\beta\gamma}F^{\xi\alpha}\delta_A^B + i\epsilon^{\xi\alpha} g D^{\dot\beta\gamma}\left[\overline{\phi}_{AC},\, \phi^{BC}\right] \end{aligned}$$

and

$$\begin{aligned} D^{\dot\beta\gamma}\left[\overline{\phi}_{AC},\, \phi^{BC}\right] &= \left[\overline{\phi}_{AC},\, D^{\dot\beta\gamma}\phi^{BC}\right] + \left[D^{\dot\beta\gamma}\overline{\phi}_{AC},\, \phi^{BC}\right] \\ &= \left[\overline{\phi}_{AC},\, D^{\dot\beta\gamma}\phi^{BC}\right] + \frac{1}{4}\varepsilon_{ACDE}\varepsilon_{BCFG}\left[D^{\dot\beta\gamma}\phi^{DE},\, \overline{\phi}_{FG}\right] \\ &= \left[\overline{\phi}_{AC},\, D^{\dot\beta\gamma}\phi^{BC}\right] + \frac{1}{2}\left(\delta^{BFG}_{ADE} + \delta^{BFG}_{DEA} + \delta^{BFG}_{EAD}\right)\left[D^{\dot\beta\gamma}\phi^{DE},\, \overline{\phi}_{FG}\right] \\ &= \left[\overline{\phi}_{AC},\, D^{\dot\beta\gamma}\phi^{BC}\right] + \frac{1}{2}\left[D^{\dot\beta\gamma}\phi^{DE},\, \overline{\phi}_{DE}\right]\delta_A^B \\ &\qquad + \frac{1}{2}\left[D^{\dot\beta\gamma}\phi^{BE},\, \overline{\phi}_{EA}\right] + \frac{1}{2}\left[D^{\dot\beta\gamma}\phi^{DB},\, \overline{\phi}_{AD}\right] \\ &= 2\left[\overline{\phi}_{AC},\, D^{\dot\beta\gamma}\phi^{BC}\right] + \frac{1}{2}\left[D^{\dot\beta\gamma}\phi^{DE},\, \overline{\phi}_{DE}\right]\delta_A^B \end{aligned}$$

where we used (1.14) and antisymmetry of ϕ^{DE} (Lem. 1.1.10). □

1.4 Euler-Lagrange Equations

We calculate the Euler-Lagrange equations next, that is the equations of motion which are the critical points of the action functional $\int d^4x\,\mathcal{L}_4$ corresponding to the $\mathcal{N}=4$ SYM Lagrangian $\mathcal{L}_4$. They are easily obtained from those of $\mathcal{L}_{10}$ by dimensional reduction, analogous to the derivation of the supersymmetry transformations in Lem. 1.3.2. For a change, we perform a direct calculation here.

Lemma 1.4.1. The Euler-Lagrange equations of the $\mathcal{N}=4$ SYM Lagrangian $\mathcal{L}_4$ from Lem. 1.2.3 are as follows.

$$\begin{aligned}
D^\nu F_{\nu\mu} &= -\frac{i}{2}g\left[D_\mu\phi^{AB},\,\overline{\phi}_{AB}\right] + g\sigma_\mu^{\dot\alpha\beta}\left[\tilde\psi_{\dot\alpha A},\,\psi_\beta^A\right]\\
D^\mu D_\mu\phi^{AB} &= -\frac{1}{2}g^2\left[\left[\phi^{AB},\,\phi^{CD}\right],\,\overline{\phi}_{CD}\right] + \sqrt{2}g\left[\psi^{\alpha A},\,\psi_\alpha^B\right] - \frac{\sqrt{2}}{2}g\varepsilon_{ABCD}\left[\tilde\psi_{\dot\alpha C},\,\tilde\psi_D^{\dot\alpha}\right]\\
D^{\dot\alpha\beta}\psi_\beta^A &= i\sqrt{2}\,g\left[\phi^{AB},\,\tilde\psi_B^{\dot\alpha}\right]\\
D^{\dot\alpha\beta}\tilde\psi_{\dot\alpha A} &= i\sqrt{2}\,g\left[\overline{\phi}_{AB},\,\psi^{\beta B}\right]
\end{aligned}$$

where the bracket denotes the supercommutator (1.1).

Proof. Since ψ and $\tilde\psi$ are related to each other by complex conjugation, they may be treated as independent variables. Consider first a variation $\tilde\psi=\tilde\psi_\varepsilon$ with $\frac{d\tilde\psi}{d\varepsilon}=\tilde\gamma$ at $\varepsilon=0$ and fix the other fields (A,ψ,ϕ). Using Lem. 1.1.1, we then yield

$$\begin{aligned}
\frac{d\mathcal{L}_4}{d\varepsilon}|_0 &= \frac{d}{d\varepsilon}|_0\mathrm{tr}\left(2i\,\tilde\psi_{\dot\alpha A}\sigma_\mu^{\dot\alpha\beta}D^\mu\psi_\beta^A + \sqrt{2}\,g\tilde\psi_{\dot\alpha A}\left[\phi^{AB},\,\tilde\psi_B^{\dot\alpha}\right]\right)\\
&= \mathrm{tr}\left(2i\,\tilde\gamma_{\dot\alpha A}\sigma_\mu^{\dot\alpha\beta}D^\mu\psi_\beta^A + \sqrt{2}\,g\tilde\gamma_{\dot\alpha A}\left[\phi^{AB},\,\tilde\psi_B^{\dot\alpha}\right] + \sqrt{2}\,g\tilde\psi_{\dot\alpha A}\left[\phi^{AB},\,\tilde\gamma_B^{\dot\alpha}\right]\right)\\
&= \mathrm{tr}\left(2i\,\tilde\gamma_{\dot\alpha A}\sigma_\mu^{\dot\alpha\beta}D^\mu\psi_\beta^A + \sqrt{2}\,g\tilde\gamma_{\dot\alpha A}\left[\phi^{AB},\,\tilde\psi_B^{\dot\alpha}\right] - \sqrt{2}\,g\tilde\gamma_B^{\dot\alpha}\left[\tilde\psi_{\dot\alpha A},\,\phi^{AB}\right]\right)\\
&= \mathrm{tr}\left(2i\,\tilde\gamma_{\dot\alpha A}\sigma_\mu^{\dot\alpha\beta}D^\mu\psi_\beta^A + 2\sqrt{2}\,g\tilde\gamma_{\dot\alpha A}\left[\phi^{AB},\,\tilde\psi_B^{\dot\alpha}\right]\right)
\end{aligned}$$

The integral thereof is required to vanish for any $\tilde\gamma$, and thus we are lead to the following Euler-Lagrange equation.

$$0 = i\,\sigma_\mu^{\dot\alpha\beta}D^\mu\psi_\beta^A + \sqrt{2}\,g\left[\phi^{AB},\,\tilde\psi_B^{\dot\alpha}\right]$$

Next, we consider a variation $\psi=\psi_\varepsilon$ with $\frac{d\psi}{d\varepsilon}=\gamma$ at $\varepsilon=0$ and fix the other fields $(A,\tilde\psi,\phi)$. By Lem. 1.2.1, we replace the term involving ψ and $\tilde\psi$ to yield

$$\begin{aligned}
\frac{d\mathcal{L}_4}{d\varepsilon}|_0 &= \frac{d}{d\varepsilon}|_0\mathrm{tr}\left(-2i\,\sigma_\mu^{\dot\alpha\beta}(D^\mu\tilde\psi_{\dot\alpha A})\psi_\beta^A - \sqrt{2}\,g\psi^{\alpha A}\left[\overline{\phi}_{AB},\,\psi_\alpha^B\right]\right)\\
&= \mathrm{tr}\left(-2i\,\sigma_\mu^{\dot\alpha\beta}(D^\mu\tilde\psi_{\dot\alpha A})\gamma_\beta^A - \sqrt{2}\,g\gamma^{\alpha A}\left[\overline{\phi}_{AB},\,\psi_\alpha^B\right] - \sqrt{2}\,g\psi^{\alpha A}\left[\overline{\phi}_{AB},\,\gamma_\alpha^B\right]\right)\\
&= \mathrm{tr}\left(2i\,\gamma_\beta^A\sigma_\mu^{\dot\alpha\beta}D^\mu\tilde\psi_{\dot\alpha A} - \sqrt{2}\,g\gamma^{\alpha A}\left[\overline{\phi}_{AB},\,\psi_\alpha^B\right] + \sqrt{2}\,g\gamma_\alpha^B\left[\psi^{\alpha A},\,\overline{\phi}_{AB}\right]\right)\\
&= \mathrm{tr}\left(2i\,\gamma_\beta^A\sigma_\mu^{\dot\alpha\beta}D^\mu\tilde\psi_{\dot\alpha A} + 2\sqrt{2}\,g\gamma_\alpha^A\left[\overline{\phi}_{AB},\,\psi^{\alpha B}\right]\right)
\end{aligned}$$

The integral thereof is required to vanish for any γ, and thus we are lead to the following Euler-Lagrange equation.

$$0 = i\,\sigma_\mu^{\dot\alpha\beta}D^\mu\tilde\psi_{\dot\alpha A} + \sqrt{2}\,g\left[\overline{\phi}_{AB},\,\psi^{\beta B}\right]$$

Next, we consider a variation $A = A_\varepsilon$ with $\frac{dA}{d\varepsilon} = B$ at $\varepsilon = 0$. Then

$$\begin{aligned}\frac{d\mathcal{L}_4}{d\varepsilon}|_0 &= \frac{d}{d\varepsilon}|_0 \mathrm{tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi^{AB})(D^\mu\overline{\phi}_{AB}) + 2i\,\tilde{\psi}_{\dot{\alpha}A}\sigma_\mu^{\dot{\alpha}\beta}D^\mu\psi_\beta^A\right)\\ &=: (1) + (2) + (3)\end{aligned}$$

We further evaluate the first term, using Lem. 1.2.1.

$$\begin{aligned}(1) &= -\mathrm{tr}\left(F_{\mu\nu}\frac{d}{d\varepsilon}|_0\,(\partial^\mu A^\nu - \partial^\nu A^\mu - ig\,[A^\mu,\, A^\nu])\right)\\ &= -\mathrm{tr}\,(F_{\mu\nu}(\partial^\mu B^\nu - \partial^\nu B^\mu - ig\,[B^\mu,\, A^\nu] - ig\,[A^\mu,\, B^\nu]))\\ &= -\mathrm{tr}\,(F_{\mu\nu}(D^\mu B^\nu - D^\nu B^\mu))\\ &= \mathrm{tr}\,((D^\mu F_{\mu\nu})B^\nu - (D^\nu F_{\mu\nu})B^\mu)\\ &= 2\mathrm{tr}\,((D^\nu F_{\nu\mu})B^\mu)\end{aligned}$$

The second term reads

$$\begin{aligned}(2) &= \frac{1}{2}\mathrm{tr}\left(D_\mu\phi^{AB}(-ig)\left[B^\mu,\, \overline{\phi}_{AB}\right] + (-ig)\left[B_\mu,\, \phi^{AB}\right]D^\mu\overline{\phi}_{AB}\right)\\ &= \frac{i}{2}g\mathrm{tr}\left(D_\mu\phi^{AB}\left[\overline{\phi}_{AB},\, B^\mu\right] + D^\mu\overline{\phi}_{AB}\left[\phi^{AB},\, B_\mu\right]\right)\\ &= \frac{i}{2}g\mathrm{tr}\left(B^\mu\left[D_\mu\phi^{AB},\, \overline{\phi}_{AB}\right] + B_\mu\left[D^\mu\overline{\phi}_{AB},\, \phi^{AB}\right]\right)\\ &= \frac{i}{2}g\mathrm{tr}\left(B^\mu\left[D_\mu\phi^{AB},\, \overline{\phi}_{AB}\right] - B_\mu\left[\overline{\phi}_{AB},\, D^\mu\phi^{AB}\right]\right)\\ &= ig\mathrm{tr}\left(B^\mu\left[D_\mu\phi^{AB},\, \overline{\phi}_{AB}\right]\right)\end{aligned}$$

while the third equals

$$\begin{aligned}(3) &= \mathrm{tr}\left(2i\,\tilde{\psi}_{\dot{\alpha}A}\sigma_\mu^{\dot{\alpha}\beta}(-ig)\left[B^\mu,\, \psi_\beta^A\right]\right)\\ &= -2g\mathrm{tr}\left(\tilde{\psi}_{\dot{\alpha}A}\sigma_\mu^{\dot{\alpha}\beta}\left[\psi_\beta^A,\, B^\mu\right]\right)\\ &= -2g\mathrm{tr}\left(B^\mu\sigma_\mu^{\dot{\alpha}\beta}\left[\tilde{\psi}_{\dot{\alpha}A},\, \psi_\beta^A\right]\right)\end{aligned}$$

The integral over (1) + (2) + (3) is required to vanish for any B, and thus we are lead to the following Euler-Lagrange equation.

$$0 = 2D^\nu F_{\nu\mu} + ig\left[D_\mu\phi^{AB},\, \overline{\phi}_{AB}\right] - 2g\sigma_\mu^{\dot{\alpha}\beta}\left[\tilde{\psi}_{\dot{\alpha}A},\, \psi_\beta^A\right]$$

Next, we consider a variation $\phi = \phi_\varepsilon$ with $\frac{d\phi}{d\varepsilon} = \xi$ at $\varepsilon = 0$. Then

$$\begin{aligned}\frac{d\mathcal{L}_4}{d\varepsilon}|_0 = \frac{d}{d\varepsilon}|_0\mathrm{tr}\Bigg(&\frac{1}{2}(D_\mu\phi^{AB})(D^\mu\overline{\phi}_{AB}) + \frac{1}{8}g^2\left[\phi^{AB},\, \phi^{CD}\right]\left[\overline{\phi}_{AB},\, \overline{\phi}_{CD}\right]\\ &-\sqrt{2}\,g\psi^{\alpha A}\left[\overline{\phi}_{AB},\, \psi_\alpha^B\right] + \sqrt{2}\,g\tilde{\psi}_{\dot{\alpha}A}\left[\phi^{AB},\, \tilde{\psi}_B^{\dot{\alpha}}\right]\Bigg)\\ =:&\ (1) + (2) + (3) + (4)\end{aligned}$$

For the first term we obtain, using (1.17),

$$\begin{aligned}(1) &= \frac{1}{2}\mathrm{tr}\left((D_\mu \xi^{AB})(D^\mu \overline{\phi}_{AB}) + (D_\mu \phi^{AB})(D^\mu \overline{\xi}_{AB})\right)\\ &= -\frac{1}{2}\mathrm{tr}\left(\xi^{AB} D_\mu D^\mu \overline{\phi}_{AB} + \overline{\xi}_{AB} D^\mu D_\mu \phi^{AB}\right)\\ &= -\mathrm{tr}\left(\overline{\xi}_{AB} D^\mu D_\mu \phi^{AB}\right)\end{aligned}$$

The second term is

$$\begin{aligned}(2) &= \frac{1}{4}g^2 \mathrm{tr}\left(\left[\phi^{AB},\, \phi^{CD}\right]\left(\frac{\partial \mathcal{L}}{\partial \varepsilon}|_0 \left[\overline{\phi}_{AB},\, \overline{\phi}_{CD}\right]\right)\right)\\ &= \frac{1}{4}g^2 \mathrm{tr}\left(\left[\phi^{AB},\, \phi^{CD}\right]\left(\left[\overline{\xi}_{AB},\, \overline{\phi}_{CD}\right] + \left[\overline{\phi}_{AB},\, \overline{\xi}_{CD}\right]\right)\right)\\ &= -\frac{1}{2}g^2 \mathrm{tr}\left(\left[\phi^{AB},\, \phi^{CD}\right]\left[\overline{\phi}_{CD},\, \overline{\xi}_{AB}\right]\right)\\ &= -\frac{1}{2}g^2 \mathrm{tr}\left(\overline{\xi}_{AB}\left[\left[\phi^{AB},\, \phi^{CD}\right],\, \overline{\phi}_{CD}\right]\right)\end{aligned}$$

while the third term equals

$$(3) = \sqrt{2}\, g \mathrm{tr}\left(\psi^{\alpha A}\left[\psi^B_\alpha,\, \overline{\xi}_{AB}\right]\right) = \sqrt{2}\, g \mathrm{tr}\left(\overline{\xi}_{AB}\left[\psi^{\alpha A},\, \psi^B_\alpha\right]\right)$$

and the fourth term

$$\begin{aligned}(4) &= -\sqrt{2}\, g \mathrm{tr}\left(\tilde{\psi}_{\dot{\alpha}A}\left[\tilde{\psi}^{\dot{\alpha}}_B,\, \xi^{AB}\right]\right) = -\sqrt{2}\, g \mathrm{tr}\left(\xi^{AB}\left[\tilde{\psi}_{\dot{\alpha}A},\, \tilde{\psi}^{\dot{\alpha}}_B\right]\right)\\ &= -\frac{\sqrt{2}}{2}\, g \mathrm{tr}\left(\varepsilon_{ABCD}\, \overline{\xi}_{AB}\left[\tilde{\psi}_{\dot{\alpha}C},\, \tilde{\psi}^{\dot{\alpha}}_D\right]\right)\end{aligned}$$

The integral over $-((1)+(2)+(3)+(4))$ is required to for all ξ, and thus we are lead to the following Euler-Lagrange equation.

$$0 = D^\mu D_\mu \phi^{AB} + \frac{1}{2}g^2\left[\left[\phi^{AB},\, \phi^{CD}\right],\, \overline{\phi}_{CD}\right] - \sqrt{2}g\left[\psi^{\alpha A},\, \psi^B_\alpha\right] + \frac{\sqrt{2}}{2}g\varepsilon_{ABCD}\left[\tilde{\psi}_{\dot{\alpha}C},\, \tilde{\psi}^{\dot{\alpha}}_D\right]$$

This concludes the derivation of the Euler-Lagrange equations as stated. □

Corollary 1.4.2. The first Euler-Lagrange equation of Lem. 1.4.1 can be, equivalently, written

$$D^{\beta\dot{\gamma}} F_\beta{}^\gamma = g\left[D^{\gamma\dot{\gamma}}\phi^{AB},\, \overline{\phi}_{AB}\right] + 4ig\left[\tilde{\psi}^{\dot{\gamma}}_A,\, \psi^{\gamma A}\right]$$

Proof. Supposing that there should be a more elegant proof, we calculate

$$D^\nu F_{\nu\mu}\overline{\sigma}^\mu_{\gamma\dot{\gamma}} = \frac{1}{2}D^{\beta\dot{\beta}} F_{\beta\dot{\beta}\gamma\dot{\gamma}} = \frac{i}{4}(-D^{\beta\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}}F_{\beta\gamma} + D^{\beta\dot{\beta}}\epsilon_{\beta\gamma}F_{\dot{\beta}\dot{\gamma}}) = \frac{i}{4}(D_\gamma{}^{\dot{\beta}}F_{\dot{\beta}\dot{\gamma}} + D^\beta{}_{\dot{\gamma}}F_{\beta\gamma})$$

Moreover, Bianchi's identity (1.21) yields

$$\begin{aligned}D_\gamma{}^{\dot{\beta}}F_{\dot{\beta}\dot{\gamma}} &= \overline{\sigma}^\mu{}_\gamma{}^{\dot{\beta}}\overline{\sigma}^{\nu\kappa}{}_{\dot{\beta}\dot{\gamma}}D_\mu F_{\nu\kappa}\\ &= \frac{i}{2}(\overline{\sigma}^\mu{}_\gamma{}^{\dot{\beta}}\sigma^\nu{}_{\dot{\beta}}{}^\lambda\overline{\sigma}^\kappa_{\lambda\dot{\gamma}} - \overline{\sigma}^\mu{}_\gamma{}^{\dot{\beta}}\sigma^\kappa{}_{\dot{\beta}}{}^\lambda\overline{\sigma}^\nu_{\lambda\dot{\gamma}})(D_\nu F_{\mu\kappa} + D_\kappa F_{\nu\mu})\\ &= \frac{i}{2}(\overline{\sigma}^\nu{}_\gamma{}^{\dot{\beta}}\sigma^\mu{}_{\dot{\beta}}{}^\lambda\overline{\sigma}^\kappa_{\lambda\dot{\gamma}} - \overline{\sigma}^\nu{}_\gamma{}^{\dot{\beta}}\sigma^\kappa{}_{\dot{\beta}}{}^\lambda\overline{\sigma}^\mu_{\lambda\dot{\gamma}} + \overline{\sigma}^\kappa{}_\gamma{}^{\dot{\beta}}\sigma^\nu{}_{\dot{\beta}}{}^\lambda\overline{\sigma}^\mu_{\lambda\dot{\gamma}} - \overline{\sigma}^\kappa{}_\gamma{}^{\dot{\beta}}\sigma^\mu{}_{\dot{\beta}}{}^\lambda\overline{\sigma}^\nu_{\lambda\dot{\gamma}})D_\mu F_{\nu\kappa}\end{aligned}$$

We use the Clifford relation for the respective second and third factors in the first and fourth term to obtain

$$\begin{aligned}
D_{\gamma}{}^{\dot\beta}F_{\dot\beta\dot\gamma} &= \frac{i}{2}(2\overline{\sigma}^{\nu}{}_{\gamma}{}^{\dot\beta}g^{\mu\kappa}\delta_{\dot\beta\dot\gamma} - \overline{\sigma}^{\nu}{}_{\gamma}{}^{\dot\beta}\sigma^{\kappa}{}_{\dot\beta}{}^{\lambda}\overline{\sigma}^{\mu}_{\lambda\dot\gamma} - \overline{\sigma}^{\nu}{}_{\gamma}{}^{\dot\beta}\sigma^{\kappa}{}_{\dot\beta}{}^{\lambda}\overline{\sigma}^{\mu}_{\lambda\dot\gamma} + \overline{\sigma}^{\kappa}{}_{\gamma}{}^{\dot\beta}\sigma^{\nu}{}_{\dot\beta}{}^{\lambda}\overline{\sigma}^{\mu}_{\lambda\dot\gamma} \\
&\quad - 2\overline{\sigma}^{\kappa}{}_{\gamma}{}^{\dot\beta}g^{\mu\nu}\delta_{\dot\beta\dot\gamma} + \overline{\sigma}^{\kappa}{}_{\gamma}{}^{\dot\beta}\sigma^{\nu}{}_{\dot\beta}{}^{\lambda}\overline{\sigma}^{\mu}_{\lambda\dot\gamma})D_{\mu}F_{\nu\kappa} \\
&= i(\overline{\sigma}^{\nu}_{\gamma\dot\gamma}g^{\mu\kappa} - \overline{\sigma}^{\kappa}_{\gamma\dot\gamma}g^{\mu\nu} + \overline{\sigma}^{\kappa}{}_{\gamma}{}^{\dot\beta}\sigma^{\nu}{}_{\dot\beta}{}^{\lambda}\overline{\sigma}^{\mu}_{\lambda\dot\gamma} - \overline{\sigma}^{\nu}{}_{\gamma}{}^{\dot\beta}\sigma^{\kappa}{}_{\dot\beta}{}^{\lambda}\overline{\sigma}^{\mu}_{\lambda\dot\gamma})D_{\mu}F_{\nu\kappa} \\
&= (-2\sigma^{\kappa\nu}{}_{\gamma}{}^{\lambda}\overline{\sigma}^{\mu}_{\lambda\dot\gamma} + i\overline{\sigma}^{\nu}_{\gamma\dot\gamma}g^{\mu\kappa} - i\overline{\sigma}^{\kappa}_{\gamma\dot\gamma}g^{\mu\nu})D_{\mu}F_{\nu\kappa} \\
&= 2D_{\lambda\dot\gamma}F_{\gamma}{}^{\lambda} + i\overline{\sigma}^{\nu}_{\gamma\dot\gamma}D^{\kappa}F_{\nu\kappa} - i\overline{\sigma}^{\kappa}_{\gamma\dot\gamma}D^{\nu}F_{\nu\kappa} \\
&= 2D_{\lambda\dot\gamma}F_{\gamma}{}^{\lambda} - 2iD^{\nu}F_{\nu\mu}\overline{\sigma}^{\mu}_{\gamma\dot\gamma}
\end{aligned}$$

Using the first calculation in this proof, we thus yield

$$D_{\gamma}{}^{\dot\beta}F_{\dot\beta\dot\gamma} = 2D_{\lambda\dot\gamma}F_{\gamma}{}^{\lambda} + \frac{1}{2}(D_{\gamma}{}^{\dot\beta}F_{\dot\beta\dot\gamma} + D^{\beta}{}_{\dot\gamma}F_{\beta\gamma})$$

and

$$\frac{1}{2}D_{\gamma}{}^{\dot\beta}F_{\dot\beta\dot\gamma} = 2D_{\lambda\dot\gamma}F_{\gamma}{}^{\lambda} + \frac{1}{2}D^{\beta}{}_{\dot\gamma}F_{\beta\gamma} = -2D^{\lambda}{}_{\dot\gamma}F_{\gamma\lambda} + \frac{1}{2}D^{\beta}{}_{\dot\gamma}F_{\gamma\beta} = -\frac{3}{2}D^{\beta}{}_{\dot\gamma}F_{\beta\gamma}$$

Now, going back to the first calculation, we obtain

$$D^{\nu}F_{\nu\mu}\overline{\sigma}^{\mu}_{\gamma\dot\gamma} = \frac{i}{4}(D_{\gamma}{}^{\dot\beta}F_{\dot\beta\dot\gamma} + D^{\beta}{}_{\dot\gamma}F_{\beta\gamma}) = \frac{i}{4}(-3D^{\beta}{}_{\dot\gamma}F_{\beta\gamma} + D^{\beta}{}_{\dot\gamma}F_{\beta\gamma}) = -\frac{i}{2}D^{\beta}{}_{\dot\gamma}F_{\beta\gamma}$$

Using Lem. 1.1.7, we finally arrive at

$$\begin{aligned}
-\frac{i}{2}D^{\beta\dot\gamma}F_{\beta}{}^{\gamma} &= D^{\nu}F_{\nu\mu}\overline{\sigma}^{\mu\gamma\dot\gamma} \\
&= \left(-\frac{i}{2}g\left[D_{\mu}\phi^{AB},\, \overline{\phi}_{AB}\right] + g\sigma^{\dot\alpha\beta}_{\mu}\left[\tilde\psi_{\dot\alpha A},\, \psi^{A}_{\beta}\right]\right)\overline{\sigma}^{\mu\gamma\dot\gamma} \\
&= -\frac{i}{2}g\left[D^{\gamma\dot\gamma}\phi^{AB},\, \overline{\phi}_{AB}\right] + g(\overline{\sigma}^{\beta\dot\alpha}_{\mu}\overline{\sigma}^{\mu\gamma\dot\gamma})\left[\tilde\psi_{\dot\alpha A},\, \psi^{A}_{\beta}\right] \\
&= -\frac{i}{2}g\left[D^{\gamma\dot\gamma}\phi^{AB},\, \overline{\phi}_{AB}\right] - 2g\epsilon^{\beta\gamma}\epsilon^{\dot\alpha\dot\gamma}\left[\tilde\psi_{\dot\alpha A},\, \psi^{A}_{\beta}\right] \\
&= -\frac{i}{2}g\left[D^{\gamma\dot\gamma}\phi^{AB},\, \overline{\phi}_{AB}\right] + 2g\left[\tilde\psi^{\dot\gamma}_{A},\, \psi^{\gamma A}\right]
\end{aligned}$$

which concludes the proof of the statement. □

1.5 Feynman Rules

Our treatment of $\mathcal{N} = 4$ SYM theory has been completely classical so far. We will now, for the rest of this chapter, derive the Feynman rules, the diagrammatic building blocks for perturbatively calculating quantum field theoretic expectation values of observables. This derivation is conveniently done via the path integral, for which no mathematically satisfying theory exists. For details, consult the usual books such as [PS95], as well as [Pol05] and [Kle11] which are better suited for mathematicians.

QFT in Zero Dimensions

The usual strategy is to study the zero-dimensional case which is, at least in some cases, sound, and then extend the resulting formulas to the cases of interest. For self-containedness, let us briefly repeat the relevant facts. Consider $\mathbb{R}^0 = \{0\}$ as "spacetime" and fields $x : \mathbb{R}^0 \to X$ that assume values in a parameter space $X \cong \mathbb{R}^d$. The space of fields can thus be identified with X itself. Consider a Lagrangian of the form

$$\mathcal{L}(x) = \frac{1}{2}\langle Ax,\, x\rangle + gU(x)\ , \qquad \mathcal{S}(x) = \mathcal{L}(x) \tag{1.29}$$

where A is some $d \times d$ matrix, U a potential function and g the coupling constant of the theory. In zero dimensions, the Lagrangian coincides with the action $\mathcal{S}$. The basic quantity of quantum field theory is the correlation function

$$\left\langle x^{i_1}, \dots, x^{i_m}\right\rangle_U = \frac{1}{\int d^d x\, e^{-\mathcal{S}(x)}} \int d^d x\, e^{-\mathcal{S}(x)} x^{i_1} \cdots x^{i_m} \tag{1.30}$$

In good cases (most notably, A should be invertible and satisfy some definiteness condition), it can be calculated perturbatively (order by order in g) as summarised next. In bad cases, some integrals in the derivation are ill-defined but we still use the resulting formulas and pretend everything to work, since the infinite-dimensional case, which is of interest, is even less based on solid ground. In all cases, convergence of the g^k series is another subtle question.

First, assume that U can be expanded to a sum $U = \sum U_l$ of multilinear maps of the form $U_l = \sum_{k_1,\dots,k_l} U_{k_1,\dots,k_l} x^{k_1} \dots x^{k_l}$. We think of $g \cdot U_l$ as an l-valent vertex with label $gU_{k_1,\dots,k_l}$. Now consider graphs with n inner vertices (corresponding to g^n), m external legs (univalent vertices) labelled $i_1, \dots i_m$ (corresponding to $x^{i_1} \dots x^{i_m}$) and edges, labelled $\left\langle x^i, x^j\right\rangle_0$, connecting two vertices with labels i and j. We denote the set of all such graphs which do not contain vacuum diagrams, i.e. components without legs, by Γ^n_m. Then

$$\left\langle x^{i_1}, \dots, x^{i_m}\right\rangle_U = \sum\nolimits_{\Gamma\in\Gamma^n_m} \frac{g^n}{|Aut\Gamma|} \sum\nolimits_{\text{labels}} \prod\nolimits_{\text{edges}\,(i,j)} \prod\nolimits_{\text{vertices}\, v} U_{v_1,\dots,v_{l_v}} \left\langle x^i, x^j\right\rangle_0$$

The symmetry factor $|Aut\Gamma|$ occurs since every graph is considered only up to automorphisms (mapping the graph to itself, thereby only exchanging names of the labels), whereas the sum $\sum_{\text{labels}}$ leads to vertices being connected in all possible fashions. This can be rewritten

$$\left\langle x^{i_1}, \dots, x^{i_m}\right\rangle_U = \sum\nolimits_{\Gamma\in\Gamma^n_m} \frac{g^n}{|Aut\Gamma|} \prod_{\substack{\text{edges}\,(i,j)\\ i,j\in\{v_i,i_j\}}} \prod_{\substack{\text{vertices}\\ v=(v_1,\dots,v_l)}} I_v \left\langle x^i, x^j\right\rangle_0 \tag{1.31}$$

where now

$$I_v = I_{v_1,\dots,v_l} = \sum\nolimits_{\text{permutations}\,\sigma} U_{\sigma(v_1),\dots,\sigma(v_l)} \tag{1.32}$$

encodes that edges can end in every possible way (permutation) on the corresponding vertex.

In usual terminology, $I_{v_1,\dots,v_l}$ is called an inner vertex while $\left\langle x^i, x^j\right\rangle_0$ is called a propagator. As the notation suggests, the propagator coincides with the correlation function with respect to the potential $U = 0$. One can show that the propagator

$$\left\langle x^i, x^j\right\rangle_0 = \frac{1}{\int d^d x\, e^{-\frac{1}{2}\langle Ax,\, x\rangle}} \int d^d x\, e^{-\frac{1}{2}\langle Ax,\, x\rangle} x^i x^j = (A^{-1})^{ij} \tag{1.33}$$

equals the (i,j)-th entry of the inverse matrix of A. I_v and $\langle x^i, x^j\rangle_0$ are depicted by an l-valent vertex and an edge, both with labels as explained above. By (1.32) and (1.33), they can be calculated directly from the Lagrangian $\mathcal{L}$. Together with the externel legs, these are collectively refered to as the Feynman rules of the theory.

The formulas stated so far accordingly hold (with the difficulties mentioned) for complex valued fields, i.e. for $x \in \mathbb{C}^d$. Moreover, fermionic (anticommuting) fields are, in the zero dimensional theory, Grassmann generators θ^j, and the correlation function involving such generators is defined as in (1.30) but with the usual integral over $\mathbb{R}^d$ replaced by the super integral over all even coordinates x^i and odd coordinates θ^j. By now, the theory of supermanifolds has become a well-established mathematical area. For a good introduction, which also covers integration, we refer the reader to [Var04]. In the following, let θ^j be a complex generator and θ^{*j} its complex conjugate. One can show that, analogous to (1.33),

$$\left\langle \theta^k,\, \theta^{*l} \right\rangle_0 = \frac{\left(\prod_i \int d\theta^{*i} d\theta^i\right) \theta^k \theta^{*l} e^{-\theta^{*i}B_{ij}\theta^j}}{\left(\prod_i \int d\theta^{*i} d\theta^i\right) e^{-\theta^{*i}B_{ij}\theta^j}} = (B^{-1})^{kl} \tag{1.34}$$

holds for a suitable matrix B (cf. (9.69) and (9.70) in [PS95]). Unlike the bosonic propagator (1.33), the fermionic propagator (1.34) is not symmetric in the arguments but antisymmetric. It is, therefore, depicted by a directed edge (an edge with an arrow).

Gauge Theories and Ghosts

The assumption that the matrix A in (1.29) is invertible was crucial for our previous considerations. In gauge theories, on the other hand, this is not the case: Assume that A has l degenerate directions corresponding to the (free etc.) action of an l-dimensional Lie group G on X, which leaves the Lagrangian $\mathcal{L}$ invariant. Instead of critical points, the classical solutions of the Euler-Lagrange equations now come as critical orbits each of which contains a continuous set of physically equivalent states. Therefore, we should count each such state only once and reduce integration over X in (1.30) to the integral over the quotient space $\tilde{X} = X/G$ of G-orbits. In good cases, this is based on solid mathematical ground and related to the Haar measure.

In "many" cases (with less justification) it seems to work as follows. Let $X_0 \subseteq X$ be a submanifold which intersects every G-orbit exactly once, such that there is a bijection with $\tilde{X}$, and assume that it is defined by l equations $F^1(x) = \ldots = F^l(x) = 0$ for some $F : X \to \mathbb{R}^l$, i.e. $X_0 = \{x_0 \in X \mid F(x_0) = 0\}$. Moreover, the orbits should be intersected transversally such that $\int_X dx = \int_{X_0} dx_0 \int_G dg$ holds, at least upon restriction to some tubular neighbourhood of X_0. In such a neighbourhood, we can then treat F as a local coordinate in the fiber over x_0, and the gauge-fixed partition function (the denominator of the correlation function) becomes, performing a change of coordinates,

$$\begin{aligned}\int_{X_0} d^{d-l}x_0\, e^{-\mathcal{S}(x_0)} &= \int_{X_0} d^{d-l}x_0 \int_{\mathbb{R}^l} d^l F\, \delta(F) e^{-\mathcal{S}(x_0)} \\ &= \int_{X_0} d^{d-l}x_0 \int_G dg\, \delta(F(gx_0)) \left|\det\left(\frac{\partial F(gx_0)}{\partial g}\right)\right| e^{-\mathcal{S}(x_0)}\end{aligned}$$

Setting $x = gx_0$, the delta function restricts integration to $F(x) = F(gx_0) = 0$ where $g = 1$ and $x = x_0$. We thus yield

$$\int_{X_0} d^{d-l}x_0\, e^{-\mathcal{S}(x_0)} = \int_X d^d x\, \delta(F(x))\, |\det(\Lambda)|\, e^{-\mathcal{S}(x_0)}\ , \qquad \Lambda := \frac{\partial F(gx)}{\partial g}|_{g=1}$$

Several difficulties arise with this approach. Firstly, the Fadeev-Popov matrix Λ should be invertible, and each orbit should be intersected exactly once. Even if this can be achieved locally, it might not be possible to ensure for the whole of X_0 due to some complicated topology, which is referred to as the Gribov problem (cf. [EPZ04]).

To continue, let us further assume that the integral over X_0 does in fact only depend on $\tilde{X}$. The exact form of F is then unimportant, and we may replace F by $F-\omega$ for some reference vector $\omega \in \mathbb{R}^l$. Let further $\xi \in \mathbb{R}_+$ and $N(\xi)$ be the constant, depending on ξ, which is defined by $N(\xi)^{-1} := \int d\omega\, e^{-\frac{\omega^2}{2\xi}}$ (this is a standard Gaussian integral). Inserting 1 and further redefining $N(\xi)$ to include the sign of $\det \Lambda$, we thus yield

$$\begin{aligned}\int_{X_0} d^{d-l}x_0\, e^{-\mathcal{S}(x_0)} &= N(\xi)\int d\omega \int_X d^d x\, e^{-\frac{\omega^2}{2\xi}}\delta(F(x)-\omega)\det\Lambda e^{-\mathcal{S}(x)} \\ &= N(\xi)\int_X d^d x\, e^{-\mathcal{S}(x)}e^{-\frac{F(x)^2}{2\xi}}\det\Lambda\end{aligned}$$

To deal with $\det \Lambda$, we introduce new (complex) Grassmann generators $c^1, \ldots, c^l$. Analogous to (1.34) one can show that (for suitable Λ)

$$\left(\prod_i \int dc^{*i}dc^i\right) e^{-c^{*i}\Lambda_{ij}c^j} = \det\Lambda$$

holds. Remembering $\mathcal{S} = \mathcal{L}$, our calculations may be summarised

$$\int_{X_0} dx_0\, e^{-\mathcal{L}(x_0)} = N(\xi)\int_X dxdc^*dc\, e^{-\mathcal{L}_{\mathrm{GF}}(x)}\,, \quad \mathcal{L}_{\mathrm{GF}} = \mathcal{L} + \frac{F(x)^2}{2\xi} + \langle c^*, \Lambda c\rangle \tag{1.35}$$

We are free to choose a convenient value of the parameter ξ (a "gauge"). Constant factors in the correlation function (1.30) cancel out. We have thus solved the gauge ambiguity problem by introducing new fermionic fields c and c^*, which are called ghosts, and replacing the original Lagrangian $\mathcal{L}$ in (1.29) with the gauge fixed Lagrangian $\mathcal{L}_{\mathrm{GF}}$ as in (1.35). The first extra term allows for the calculation of Feynman rules for the gauge field A, while the second introduces new Feynman rules involving the ghosts.

QFT in Finite Dimensions: The Infinite Dimensional Case

Let us now move from zero to four (or any finite number of) spacetime dimensions. The space of fields is then a non-trivial infinite set of functions. Let us pretend that the formulas for the zero-dimensional theory have canonical analoga and consider again $\mathcal{N}=4$ super Yang-Mills theory.

As mentioned above, gauge transformations are finite symmetries, such that the original Lagrangian $\mathcal{L}_4$ need to be replaced by a gauge fixed Lagrangian. We choose the Lorentz gauge $F(A) = i\partial^\mu A_\mu(x)$. The first extra term in (1.35) then becomes $-\frac{1}{2\xi}(\partial_\mu A^{c\mu})^2$. To calculate the ghost contribution, note that the group action, previously denoted gx, is now replaced by gauge transformations (1.22). Let $\alpha : \mathbb{R}^4 \to \mathfrak{su}(N)$. The set of such maps can be identified with the tangent space at a gauge transformation. This is a bit subtle but should be OK (cf. similar situations such as the transversality arguments in Chp. 3 of [MS04]). We thus calculate, using $\alpha^\dagger = -\alpha$,

$$\Lambda[\alpha] = i\partial^\mu \frac{d}{dV}|_{V=1}\left(VA_\mu V^\dagger + \frac{i}{g}V\partial_\mu V^\dagger\right)[\alpha] = i\partial^\mu\left(-[A_\mu, \alpha] - \frac{i}{g}\partial_\mu\alpha\right) = \frac{1}{g}\partial^\mu D_\mu \alpha$$

After having calculated the ghost propagator and ghost-gluon vertex later in Sec. 1.5.2 and Sec. 1.5.3, we will see that every graph in (1.31) contains exactly the same number of propagators and vertices such that absorbing the scaling factor $1/g$ into the normalisation of c and c^* does not change the result and can be omitted. Written as a matrix with respect to the generators T^a of $i \cdot \mathfrak{su}(N)$, we thus arrive at $\Lambda^{ac} = \delta^{ac}\partial^\mu\partial_\mu(\cdot) + gf^{abc}\partial^\mu(A^b_\mu\cdot)$, and the gauge-fixed Lagrangian becomes

$$\mathcal{L}_{\text{GF}} = \left(\mathcal{L}_4 - \frac{1}{2\xi}(\partial_\mu A^{c\mu})^2\right) + c^{*a}(\partial^\mu\partial_\mu c^a) - gf^{abc}\partial^\mu(c^{*a})A^b_\mu c^c \tag{1.36}$$

Upon replacing $\mathcal{L}_4$ by $\mathcal{L}_{\text{GF}}$ we obtain, in particular, the analoga of the formulas (1.33) and (1.34) for bosonic fields ϕ (and now also A) and fermionic fields ψ (and ghosts c) in four dimensions:

$$\langle\phi(x),\, \phi(y)\rangle = \frac{\int \mathcal{D}\phi \exp\left(i\int d^4x\, \mathcal{L}(\phi)|_{g^0}\right)\phi(x)\phi(y)}{\int \mathcal{D}\phi \exp\left(i\int d^4x\, \mathcal{L}\right)} \tag{1.37a}$$

$$\left\langle\psi(x),\, \tilde{\psi}(y)\right\rangle = \frac{\int \mathcal{D}\tilde{\psi}\mathcal{D}\psi \exp\left(i\int d^4x\, \mathcal{L}(\psi)|_{g^0}\right)\psi(x)\tilde{\psi}(y)}{\int \mathcal{D}\tilde{\psi}\mathcal{D}\psi \exp\left(i\int d^4x\, \mathcal{L}\right)} \tag{1.37b}$$

where the (ill-defined) integral (the path-integral) goes over the respective space of all fields. These propagators are calculated analogous to the zero dimensional case as the matrix elements of the "inverse" (Green's function) of the integral kernels which correspond to the previous matrices A and B.

In fact, the analogy between the respective propagators as stated holds only up to a factor of $-i$. This is another subtlety due to the transition from Euclidean to Minkowski space. In between, one performs some Wick rotation to translate the Euclidean into the Minkowski metric. While this is a standard trick that seems to work, it is usually hard to justify mathematically. For simplicity, we shall use (1.37a) and (1.37b) without further ado in the following, pressed into the form (1.33) and (1.34) by

$$\mathcal{L} \to -i\mathcal{L} \tag{1.38}$$

When it comes to Green's function in the next subsection, we will encounter some problems which are inevitable relics of this transformation.

1.5.1 Green's Functions

The infinite-dimensional analogon for the inverse of a matrix A or B, as occurring in (1.33) and (1.34) is the Green's function of some differential operator. It is worthwhile summarising the relevant formulas here before applying the Feynman formalism to $\mathcal{N} = 4$ super Yang-Mills theory in the next two subsections. We need the following lemma.

Lemma 1.5.1. The four-dimensional delta function can be expressed as the Fourier transform of the constant function 1:

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} = \delta^{(4)}(x-y)$$

Proof. It is well-known that the delta "function" is a distribution. Likewise, the left hand side (the Fourier transform of 1) should be regarded as a distribution. For a formal treatment, consult e.g. Chp. VIII of [Wer02]. In this context, the formula appears rather as a definition than a statement.

The left hand side (with $x-y$ replaced by x) also appears as the limit (in a distributional sense) of the sequence

$$\delta_k(x) := \int \frac{d^4p}{(2\pi)^4} e^{-ipx-\delta_k p_E^2} \,, \qquad k \in \mathbb{N} \,, \qquad \delta_k \to 0 \tag{1.39}$$

where $p_E^2 := \sum_\mu p^\mu \cdot p^\mu$ denotes the Euclidean square of p (in contrast to the Minkowski square $p^2 = p_\mu p^\mu$). For this to see, it suffices to show that $\delta_k(x)$ is a Dirac sequence (cf. Chp. 2 of [Alt02]): In (1.39), we perform the transformation $p^0 \to p^0$ and $p^l \to -p^l$ for $l = 1, 2, 3$ to obtain

$$\int_{-\infty}^{\infty} dp^l \, e^{-ip^l \cdot x^l - \delta_k p^l \cdot p^l} = \int_{\infty}^{-\infty} dp^l \, (-1) e^{ip^l \cdot x^l - \delta_k p^l \cdot p^l} = \int_{-\infty}^{\infty} dp^l \, e^{ip^l \cdot x^l - \delta_k p^l \cdot p^l}$$

and, therefore,

$$\delta_k(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot_E x - \delta_k p_E^2}$$

where $p \cdot_E x := \sum_\mu p^\mu \cdot x^\mu$. We now make a shift $p_\mu = q_\mu - \frac{i}{2\delta_k} x_\mu$ such that

$$\begin{aligned} -ip \cdot_E x - \delta_k \cdot p_E^2 &= -i \left(q \cdot_E x - \frac{i}{2\delta_k} x_E^2 \right) - \delta_k \left(q_E^2 - \frac{i}{\delta_k} q \cdot_E x - \frac{1}{4\delta_k^2} x_E^2 \right) \\ &= -\delta_k q_E^2 - \frac{1}{4\delta_k} x_E^2 \end{aligned}$$

and

$$\delta_k(x) = \prod\nolimits_\mu \left(\int_{-\infty + \frac{i}{2\delta_k} x^\mu}^{\infty + \frac{i}{2\delta_k} x^\mu} \frac{dq^\mu}{2\pi} \right) e^{-\delta_k q_E^2} e^{-\frac{1}{4\delta_k} x_E^2}$$

Now, the function $e^{-\delta_k q^\mu \cdot q^\mu}$ is holomorphic (without poles) in q^μ and, therefore, the integral thereof over the boundary of a box $(-R, R) \times (0, \frac{1}{2\delta_k} x^\mu)$ with $R > 0$ in the complex q^μ plane vanishes. For the integral over the right side vertical line, we obtain

$$\left| \int_R^{R + \frac{i}{2\delta_k} x^\mu} dq^\mu \, e^{-\delta_k q^\mu \cdot q^\mu} \right| \leq \frac{x^\mu}{2\delta_k} e^{-\delta_k R^2} e^{\delta_k \left(\frac{x^\mu}{2\delta_k} \right)^2} \xrightarrow{R \to \infty} 0$$

and analogous for the integral over left side vertical line. Therefore, we may replace, for each μ, the integral over the shifted real line by minus the integral over the ordinary real line to obtain the Gaussian integral

$$\delta_k(x) = \int \frac{d^4p}{(2\pi)^4} e^{-\delta_k p_E^2} e^{-\frac{1}{4\delta_k} x_E^2} = \frac{1}{(2\pi)^4} \sqrt{\frac{\pi}{\delta_k}}^4 e^{-\frac{1}{4\delta_k} x_E^2}$$

By this expression, it is clear that $\delta_k(x)$ is indeed a Dirac sequence. In particular, it satisfies the normalisation condition

$$\int d^4x \, \delta_k(x) = \frac{1}{(2\pi)^4} \sqrt{\frac{\pi}{\delta_k}}^4 \sqrt{4\pi\delta_k}^4 = 1$$

which is calculated again by the Gaussian integral. □

Lemma 1.5.2. The function

$$G(x-y) := -\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2+i\varepsilon}$$

is a Green's function for $\partial_\mu \partial^\mu$ (i.e. it satisfies $\partial_\mu \partial^\mu G(x-y) = \delta^{(4)}(x-y)$).

Here, we meet the "$i\varepsilon$-prescription": The p_0-integral $\int dp_0 \frac{e^{-ip(x-y)}}{p^2}$ is not well-defined. This can be remedied by shifting the poles, which are located at $(p_0)_\pm = \pm |\boldsymbol{p}|$, slightly parallel to the imaginary axis towards $(p_0)_\pm \to \pm \left(|\boldsymbol{p}| - \frac{i\varepsilon}{2|\boldsymbol{p}|}\right)$ with some small parameter $\varepsilon > 0$. As a result, the p_0-integral can now be evaluated as the residue at one of the poles, depending on whether x_0 is positive or negative such that the corresponding arc which closes the integration line vanishes at infinity. In the limit $\varepsilon \to 0$, the integral is independent of the convention for the shift, which was chosen such that

$$\begin{aligned} p^2 = (p_0 - |\boldsymbol{p}|)(p_0 + |\boldsymbol{p}|) &\to \left(p_0 - \left(|\boldsymbol{p}| - \frac{i\varepsilon}{2\,|\boldsymbol{p}|}\right)\right)\left(p_0 + \left(|\boldsymbol{p}| - \frac{i\varepsilon}{2\,|\boldsymbol{p}|}\right)\right) \\ &= (p_0)^2 - \left(|\boldsymbol{p}| - \frac{i\varepsilon}{2\,|\boldsymbol{p}|}\right)^2 \\ &= p^2 + i\varepsilon + \frac{\varepsilon^2}{4\,|\boldsymbol{p}|^2} \end{aligned}$$

In the limit $\varepsilon \to 0$, one can further see that the term $\frac{\varepsilon^2}{4|\boldsymbol{p}|^2}$ is irrelevant and, as a consequence, we can write the shift as $p^2 \to p^2 + i\varepsilon$. Although usually omitted, it is implicitly understood that ε is sent to 0 (in front of the integral).

Proof of Lem. 1.5.2. This follows from Lem. 1.5.1 by the following calculation.

$$\begin{aligned} -\partial_\mu \partial^\mu \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2+i\varepsilon} &= -\int \frac{d^4p}{(2\pi)^4} (-p^2) \frac{e^{-ip(x-y)}}{p^2+i\varepsilon} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \\ &= \delta^{(4)}(x-y) \end{aligned}$$

In this calculation, we freely exchanged differentiation, limits and integration without having checked the hypotheses. We believe that this can be done (and has beed done somewhere) rigorously and leave the details to the reader. □

Lemma 1.5.3. The function

$$G(x-y)_{\alpha\dot{\alpha}} := -\overline{\sigma}^\mu_{\alpha\dot{\alpha}} \partial_\mu \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2+i\varepsilon}$$

is a Green's function for $\sigma_\mu^{\dot{\beta}\beta} \partial^\mu$.

Proof. This follows from (1.6) and Lem. 1.5.1 by the following calculation, where the comment in the proof of Lem. 1.5.2 applies accordingly.

$$
\begin{aligned}
-\sigma_{\nu}^{\dot\beta\beta}\overline{\sigma}^{\mu}_{\beta\dot\gamma}\partial^\nu\partial_\mu \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2+i\epsilon} &= \int \frac{d^4p}{(2\pi)^4}\frac{p^\nu\sigma_{\nu}^{\dot\beta\beta}p_\mu\overline{\sigma}^{\mu}_{\beta\dot\gamma}}{p^2+i\epsilon}e^{-ip(x-y)} \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{p_\nu\sigma^{\nu\dot\beta\beta}p_\mu\overline{\sigma}^{\mu}_{\beta\dot\gamma}}{p^2+i\epsilon}e^{-ip(x-y)} \\
&= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{p_\nu p_\mu(\sigma^{\nu\dot\beta\beta}\overline{\sigma}^{\mu}_{\beta\dot\gamma}+\sigma^{\mu\dot\beta\beta}\overline{\sigma}^{\nu}_{\beta\dot\gamma})}{p^2+i\epsilon}e^{-ip(x-y)} \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{p_\nu p_\mu\eta^{\mu\nu}\delta^{\dot\beta}_{\dot\gamma}}{p^2+i\epsilon}e^{-ip(x-y)} \\
&= \delta^{\dot\beta}_{\dot\gamma}\int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)} \\
&= \delta^{\dot\beta}_{\dot\gamma}\delta^{(4)}(x-y)
\end{aligned}
$$

□

For the integral $A_4(x)$ occuring in Lem. 1.5.2 and Lem. 1.5.3, we need an explicit expression. For dimensional regularisation, to be explained below, it will be necessary to consider the analogous integral in d dimensional Lorentz space with signature $(+,-,\dots,-)$ rather than Minkowski space.

Lemma 1.5.4. The integral $A_d(x)$ can be written

$$
A_d(x) := \int \frac{d^dp}{(2\pi)^d}\frac{e^{-ipx}}{p^2+i\varepsilon} = i\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\frac{1}{(x^2)^{\frac{d}{2}-1}}\,, \qquad A_4(x) = \frac{i}{4\pi^2}\frac{1}{x^2}
$$

Proof. Again, the right hand side has to be understood as a distribution (more precisely also with some kind of $i\varepsilon$-prescription). By a simple calculation, one finds that the application of the operator $\partial_\mu\partial^\mu$ to it vanishes for $x \neq 0$. It should be possible to prove that it is indeed a Green's function and, moreover, coincides with the Green's function on the left hand side.

One could also first evaluate the p_0-integral by the residue theorem and then solve the remaining integral in $d-1$ (Euclidean) dimensions by a direct calculation involving spherical coordinates, following [LFQ+10] and (for $d=4$) [GR09].

Instead, we shall be satisfied by a simple calculation using Schwinger parametrisation and Gauss integration. This calculation contains several steps a mathematician would call terribly wrong (the reader is invited to spot all those), yet it seems to provide a reasonable result. Here is how it works. The Schwinger trick is writing

$$
\frac{1}{p^2} = \frac{1}{p^2}\int_0^\infty d\sigma\, e^{-\sigma} = \int_0^\infty d\tau\, e^{-\tau\cdot p^2}
$$

Therefore

$$
A_d(x) = \frac{1}{(2\pi)^d}\int_0^\infty d\tau \int d^dp\, e^{-\tau\cdot p^2 - ip\cdot x}
$$

As in the proof of Lem. 1.5.1, we now make a shift $p_\mu = q_\mu - \frac{i}{2\tau} x_\mu$ such that

$$A_d(x) = \frac{1}{(2\pi)^d} \int_0^\infty d\tau \left(\int d^d q \, e^{-\tau q^2} \right) e^{-\frac{1}{4\tau} x^2} = \frac{\sqrt{\pi}^d}{(2\pi)^d} \int_0^\infty d\tau \, \tau^{-\frac{d}{2}} e^{-\frac{1}{4\tau} x^2}$$

Besides having tacitly moved the integration line back to $\mathbb{R}$, we have neglected that q^2 can be negative. To solve the latter problem (and make the calculation even wronger), we perform a Wick rotation $q_0 \to i q_0$, followed by changing the integration line from $i\mathbb{R}$ back to $\mathbb{R}$. This gives an additional factor i. The substitution $\alpha = \frac{x^2}{4\tau}$ then yields

$$\begin{aligned} A_d(x) &= i \frac{\sqrt{\pi}^d}{(2\pi)^d} \int_\infty^0 d\alpha \left(-\frac{x^2}{4\alpha^2} \right) \left(\frac{x^2}{4\alpha} \right)^{-\frac{d}{2}} e^{-\alpha} \\ &= i \frac{\sqrt{\pi}^d}{(2\pi)^d} \frac{4^{\frac{d}{2}-1}}{(x^2)^{\frac{d}{2}-1}} \int_0^\infty d\alpha \, \alpha^{\frac{d}{2}-2} e^{-\alpha} \\ &= i \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \frac{1}{(x^2)^{\frac{d}{2}-1}} \end{aligned}$$

having introduced the Γ-function. For $d = 4$, this coincides with the expression stated in Chp. 2 of [GR09]. □

1.5.2 Propagators

With this preparation, we are now in a position to derive explicit expressions for the propagators of $\mathcal{N} = 4$ SYM theory. We consider the gauge-fixed Lagrangian $\mathcal{L}_{\mathrm{GF}}$ of $\mathcal{N} = 4$ SYM theory from (1.36), multiplied with a factor $-i$ due to (1.38).

Scalar Propagator

We calculate the scalar propagator. By Lem. 1.2.4 and (1.17), the relevant term can be written as follows.

$$\begin{aligned} -i\mathcal{L}_4(\phi)|_{g^0} &= -\frac{i}{4} (\partial_\mu \phi^{cCD})(\partial^\mu \overline{\phi}^c_{CD}) \\ &= -\frac{i}{2} (\partial_\mu \phi^{cM})(\partial^\mu \phi^{cM}) \\ &= \frac{i}{2} \left(\partial_\mu \partial^\mu \phi^{cM} \right) \phi^{cM} \end{aligned}$$

where the last equality holds upon integration. This is identified with the bilinear form $-\frac{1}{2} \langle A\phi, \phi \rangle$ where A denotes the "matrix" with entries $A^{aM\,bN} = -i\delta^{ab}\delta^{MN}\partial_\mu\partial^\mu$. We use the formula in infinite dimensions which is analogous to (1.33), where as the "inverse" of A we take a suitable Green's function (Green's functions are not unique), chosen according to a standard physical argument involving the causal structure of the theory (cf. Chp. 2 of [PS95]), which is built from the Green's function $G(x-y) =$ "$(\partial_\mu\partial^\mu)^{-1}$"

of $\partial_\mu \partial^\mu$ as stated in Lem. 1.5.2. Further using Lem. 1.1.10, we thus obtain

$$\begin{aligned}
\left\langle \overline{\phi}^a_{AB}(x),\, \overline{\phi}^b_{CD}(y) \right\rangle &= \frac{1}{2} \overline{\Sigma}^M_{AB} \overline{\Sigma}^N_{CD} \left\langle \phi^{aM}(x),\, \phi^{bN}(y) \right\rangle \\
&= \frac{1}{2} \overline{\Sigma}^M_{AB} \overline{\Sigma}^N_{CD} (A^{-1})^{aM\, bN} \\
&= \frac{i}{2} \overline{\Sigma}^M_{AB} \overline{\Sigma}^N_{CD} \delta^{ab} \delta^{MN} (\partial_\mu \partial^\mu)^{-1} \\
&= -i \delta^{ab} \varepsilon_{ABCD} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i\varepsilon}
\end{aligned}$$

Substituting the expression from Lem. 1.5.4 for the integral $A_4(x-y)$, we thus find

$$\bullet\text{-------}\bullet = \left\langle \overline{\phi}^a_{AB}(x),\, \overline{\phi}^b_{CD}(y) \right\rangle = \frac{1}{4\pi^2} \frac{\varepsilon_{ABCD} \delta^{ab}}{(x-y)^2} \tag{1.40}$$

as the scalar propagator.

Quantum field theory is a theory of infinities. Besides the convergence problem of the infinite dimensional analogon of the sum (1.31), already individual graphs can contribute as infinity due to a vanishing denominator in the propagator (1.40). To solve this problem, we use a trick which seems to be standard: We make the propagators depend on a parameter $\varepsilon > 0$ (which has nothing to do with the $i\varepsilon$-prescription) such that, in the limit $\varepsilon \to 0$, the original propagators are obtained. If done neatly, the individual graphs would then have a finite part plus some summand with a pole in ε and, in the sum over all graphs contributing to some order n of the coupling constant g, the latter "infinities" cancel out, leaving a finite result which survives in the limit $\varepsilon \to 0$. This technique is referred to as regularisation.

From a mathematician's point of view, the infinity problem is closely related to the propagator, and a fortiori the individual Feynman graph, being a distribution rather than a function (cf. the discussion in the proof of Lem. 1.5.4). Via regularisation, each graph is approximated by a series of ordinary functions (like a Dirac sequence approximating the delta function). It seems natural but remains unclear (at least to the author) whether this approach can be made rigorous.

Of the several possible ways of regularisation, we choose the so called dimensional regularisation. This works as follows. Calculate the propagators up to the occurrence of the integral $A_4(x-y)$ in the calculation preceeding (1.40); we shall see below that the same integral appears for the fermion and gluon propagators as well. Then replace $A_4(x-y)$ by $A_d(x-y)$ (cf. Lem. 1.5.4) and, finally, set $d = 4 - 2\varepsilon$, except that the Lorentz metric $\eta_{\mu\nu}$ is considered in honest $d = 4$ dimensions (the factor 2 convention turns out to be convenient). Doing so for the scalar propagator, we immediately yield

$$\bullet\text{-------}\bullet = \left\langle \overline{\phi}^a_{AB}(x),\, \overline{\phi}^b_{CD}(y) \right\rangle = \frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}} \frac{\varepsilon_{ABCD} \delta^{ab}}{((x-y)^2)^{1-\varepsilon}} \tag{1.41}$$

as the (dimensionally) regularised scalar propagator. For comparison with the literature, note that the coupling constant g carries a (physical) dimension in dimensions other than 4. This can be seen by the necessity of the action being dimensionless, which appears in the analogon of (1.33) as the argument of the exponential. To make g again dimensionless, it is often rescaled using a mass scale parameter μ which then would appear in (1.41).

In the following lemma, we state derivatives of the regularised propagator expression needed later on, which are easily calculated. Here, the last expression in $\mathcal{O}(\varepsilon)$ does not completely vanish only because $\eta_{\mu\nu}$ is the metric in $d = 4$ rather than in $d = 4 - 2\varepsilon$ dimensions (for which then $\eta^{\mu\nu}\eta_{\mu\nu} = d = 4 - 2\varepsilon$).

Lemma 1.5.5. For $x, y \in \mathbb{R}^4$ with $(x-y)^2 \neq 0$, we have

$$\begin{aligned} \partial_{(x)\mu} \frac{1}{((x-y)^2)^{1-\varepsilon}} &= -2(1-\varepsilon) \frac{(x-y)_\mu}{((x-y)^2)^{2-\varepsilon}} \\ \partial_{(x)\nu}\partial_{(x)\mu} \frac{1}{((x-y)^2)^{1-\varepsilon}} &= -\frac{2(1-\varepsilon)}{((x-y)^2)^{2-\varepsilon}} \left(\eta_{\mu\nu} - 2(2-\varepsilon) \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right) \end{aligned}$$

Replacing the x-derivative $\partial_{(x)\mu}$ by the y-derivative $\partial_{(y)\mu}$ yields a minus sign. Moreover, contraction with $\eta^{\mu\nu}$ yields

$$\partial_{(x)\mu}\partial^\mu_{(x)} \frac{1}{((x-y)^2)^{1-\varepsilon}} = -\frac{4\varepsilon(1-\varepsilon)}{((x-y)^2)^{2-\varepsilon}} \xrightarrow{\varepsilon \to 0} 0$$

Fermion Propagator

We calculate the fermion propagator. By Lem. 1.2.4, the relevant term can be written as follows.

$$-i\mathcal{L}_4(\psi)|_{g^0} = \tilde{\psi}^c_{\dot{\beta}C} \sigma_\mu^{\dot{\beta}\beta} \partial^\mu \psi^{cC}_\beta$$

This is identified with the bilinear form $-\left\langle \tilde{\psi},\, B\psi \right\rangle$ where B denotes the "matrix" with entries $B^{a\dot{\beta}A\,b\beta B} = -\delta^{ab}\delta^{AB}\sigma_\mu^{\dot{\beta}\beta}\partial^\mu$. We use the formula in infinite dimensions which is analogous to (1.34), where as the "inverse" of B we take the Green's function which is built from the Green's function $G(x-y)_{\alpha\dot{\alpha}} = ("(\sigma_\mu\partial^\mu)^{-1}")_{\alpha\dot{\alpha}}$ of $\sigma_\mu^{\dot{\beta}\beta}\partial^\mu$ as stated in Lem. 1.5.3. We thus obtain

$$\begin{aligned} \left\langle \psi^{aA}_\alpha(x),\, \tilde{\psi}^b_{\dot{\alpha}B}(y) \right\rangle &= -\delta^{ab}\delta^{AB} G(x-y)_{\alpha\dot{\alpha}} \\ &= \delta^{ab}\delta^{AB} \overline{\sigma}^\mu_{\alpha\dot{\alpha}} \partial_\mu \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i\varepsilon} \end{aligned}$$

At this point, we perform dimensional regularisation as in the calculation of the scalar propagator to obtain the following expression (using Lem. 1.5.4 and setting $d = 4 - 2\varepsilon$).

$$\left\langle \psi^{aA}_\alpha(x),\, \tilde{\psi}^b_{\dot{\alpha}B}(y) \right\rangle = i\frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}} \partial_\mu \frac{\delta^{ab}\delta^{AB}\overline{\sigma}^\mu_{\alpha\dot{\alpha}}}{((x-y)^2)^{1-\varepsilon}}$$

Calculating the derivative (Lem. 1.5.5), we immediately yield

(1.42)  $= \left\langle \psi^{aA}_\alpha(x),\, \tilde{\psi}^b_{\dot{\alpha}B}(y) \right\rangle = -\frac{i(1-\varepsilon)\Gamma(1-\varepsilon)}{2\pi^{2-\varepsilon}} \delta^{ab}\delta^{AB} \frac{(x-y)_{\alpha\dot{\alpha}}}{((x-y)^2)^{2-\varepsilon}}$

as the (dimensionally) regularised fermion propagator (without having introduced a mass scale). For $\varepsilon = 0$, this propagator reduces to

(1.43)  $= \left\langle \psi^{aA}_\alpha(x),\, \tilde{\psi}^b_{\dot{\alpha}B}(y) \right\rangle = -\frac{i}{2\pi^2} \delta^{ab}\delta^{AB} \frac{(x-y)_{\alpha\dot{\alpha}}}{(x-y)^4}$

Gluon Propagator

We calculate the gluon propagator. Here, we need to consider indeed the gauged fixed Lagrangian (1.36), and the relevant terms can be written

$$\begin{aligned} -i\mathcal{L}_{\text{GF}}(A)|_{g^0} &= \frac{i}{4}(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)^2 + \frac{i}{2\xi}(\partial_\mu A^{c\mu})^2 \\ &= \frac{i}{2}\left((\partial_\mu A^c_\nu)(\partial^\mu A^{c\nu}) - (\partial_\mu A^c_\nu)(\partial^\nu A^{c\mu}) + \frac{1}{\xi}(\partial^\nu A^c_\nu)(\partial^\mu A^c_\mu)\right) \\ &= -\frac{i}{2}A^c_\nu\left(\partial_\rho\partial^\rho\eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)A^c_\mu \end{aligned}$$

where the last equality holds upon integration. From now on choosing Feynman gauge $\xi = 1$, this is identified with the bilinear form $-\frac{1}{2}\langle CA,\, A\rangle$ where C denotes the "matrix" with entries $C^{a\mu\, b\nu} = i\delta^{ab}\eta^{\mu\nu}\partial_\rho\partial^\rho$. We thus obtain

$$\left\langle A^a_\mu(x),\, A^b_\nu(y)\right\rangle = (A^{-1})^{a\,b}_{\mu\nu} = -i\delta^{ab}\eta_{\mu\nu}(\partial_\rho\partial^\rho)^{-1}$$

The further derivation exactly parallels that of the scalar propagator, and we obtain

(1.44)  $$= \left\langle A^a_\mu(x),\, A^b_\nu(y)\right\rangle = -\frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}}\frac{\eta_{\mu\nu}\delta^{ab}}{((x-y)^2)^{1-\varepsilon}}$$

as the regularised gluon propagator in Feynman gauge. For $\varepsilon = 0$, this propagator reduces to

(1.45)  $$= \left\langle A^a_\mu(x),\, A^b_\nu(y)\right\rangle = -\frac{1}{4\pi^2}\frac{\eta_{\mu\nu}\delta^{ab}}{(x-y)^2}$$

Ghost Propagator

We finally calculate the ghost propagator. The relevant term of (1.36) is

$$-i\mathcal{L}_{\text{GF}}(c)|_{g^0} = -ic^{*a}(\partial_\mu\partial^\mu c^a)$$

As in the derivation of the fermion propagator (actually, this is a fermionic propagator), this expression is identified with the bilinear form $-\langle c^*,\, Bc\rangle$ where B denotes the "matrix" with entries $B^{ab} = i\delta^{ab}\partial_\mu\partial^\mu$. The inverse is $(B^{-1})^{ab} = -i\delta^{ab}(\partial_\mu\partial^\mu)^{-1}$, and we immediately obtain

(1.46) $$= \left\langle c^a(x),\, c^{*b}(y)\right\rangle = -\frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}}\frac{\delta^{ab}}{((x-y)^2)^{1-\varepsilon}}$$

as the regularised ghost propagator. For $\varepsilon = 0$, this propagator reduces to

(1.47)  $$= \left\langle c^a(x),\, c^{*b}(y)\right\rangle = -\frac{1}{4\pi^2}\frac{\delta^{ab}}{(x-y)^2}$$

1.5.3 Inner Vertices

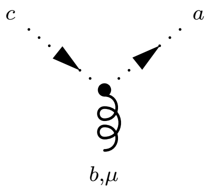
Having calculated the propagators, we now come to the inner vertices. According to (1.32), every vertex is obtained from symmetrisation of the correponding interaction term of the $\mathcal{N} = 4$ SYM Lagrangian from Lem. 1.2.4 with the addition of the gauge-fixing terms in (1.36) and multiplied with a factor $-i$ due to (1.38). For the rest of this chapter, we summarise these calculations and summarise the resulting formulas.

Antighost-Gluon-Ghost (cgc) Vertex

The antighost-gluon-ghost interaction term of the Lagrangian reads

$$-i\mathcal{L}_{c^*gc} = igf^{abc}\partial^\mu(c^{*a})A^b_\mu c^c = \left(igf^{abc}\partial^\mu_{(1)}\right)\left(c^{*a}A^b_\mu c^c\right)$$

Here, we do not need to consider permutations since we have three different fields such that $I = U$. Therefore, we find the following expression for the antighost-gluon-ghost vertex.



$$= (I_{c^*gc})^{a\ b\mu\ c} = igf^{abc}\partial^\mu_{(1)}$$

3-Gluon (ggg) Vertex

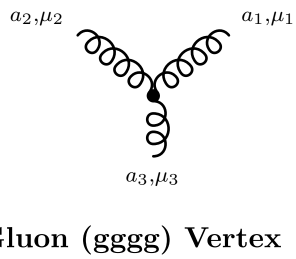
The 3-gluon interaction term of the Lagrangian reads

$$-i\mathcal{L}_{ggg} = \left(igf^{a_1a_2a_3}\eta^{\mu_1\mu_3}\partial^{\mu_2}_{(1)}\right)(A^{a_1}_{\mu_1}A^{a_2}_{\mu_2}A^{a_3}_{\mu_3}) =: (U^{a_1\mu_1\ a_2\mu_2\ a_3\mu_3})(A^{a_1}_{\mu_1}A^{a_2}_{\mu_2}A^{a_3}_{\mu_3})$$

Using Lem. 1.1.3, we calculate

$$\begin{aligned}
&(I_{ggg})^{a_1\mu_1\ a_2\mu_2\ a_3\mu_3} \\
&\quad= \sum\nolimits_{\substack{\text{permutations of}\\ a_1\mu_1\ a_2\mu_2\ a_3\mu_3}} U^{a_1\mu_1\ a_2\mu_2\ a_3\mu_3} \\
&\quad= ig\Big(f^{a_1a_2a_3}\eta^{\mu_1\mu_3}\partial^{\mu_2}_{(1)} + f^{a_2a_3a_1}\eta^{\mu_2\mu_1}\partial^{\mu_3}_{(2)} + f^{a_3a_1a_2}\eta^{\mu_3\mu_2}\partial^{\mu_1}_{(3)} \\
&\qquad\quad + f^{a_1a_3a_2}\eta^{\mu_1\mu_2}\partial^{\mu_3}_{(1)} + f^{a_3a_2a_1}\eta^{\mu_3\mu_1}\partial^{\mu_2}_{(3)} + f^{a_2a_1a_3}\eta^{\mu_2\mu_3}\partial^{\mu_1}_{(2)}\Big) \\
&\quad= igf^{a_1a_2a_3}\Big(\eta^{\mu_1\mu_3}\partial^{\mu_2}_{(1)} + \eta^{\mu_2\mu_1}\partial^{\mu_3}_{(2)} + \eta^{\mu_3\mu_2}\partial^{\mu_1}_{(3)} \\
&\qquad\quad - \eta^{\mu_1\mu_2}\partial^{\mu_3}_{(1)} - \eta^{\mu_3\mu_1}\partial^{\mu_2}_{(3)} - \eta^{\mu_2\mu_3}\partial^{\mu_1}_{(2)}\Big)
\end{aligned}$$

Therefore, we find the following expression for the 3-gluon vertex.



$$\begin{aligned}
&= (I_{ggg})^{a_1\mu_1\ a_2\mu_2\ a_3\mu_3} \\
&= -igf^{a_1a_2a_3}\Big(\eta^{\mu_3\mu_1}(\partial^{\mu_2}_{(3)} - \partial^{\mu_2}_{(1)}) + \eta^{\mu_1\mu_2}(\partial^{\mu_3}_{(1)} - \partial^{\mu_3}_{(2)}) \\
&\qquad + \eta^{\mu_2\mu_3}(\partial^{\mu_1}_{(2)} - \partial^{\mu_1}_{(3)})\Big)
\end{aligned}$$

4-Gluon (gggg) Vertex

The 4-gluon interaction term of the Lagrangian reads

$$\begin{aligned}
-i\mathcal{L}_{gggg} &= \left(\frac{ig^2}{4}f^{a_1a_2b}f^{a_3a_4b}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\right)(A^{a_1}_{\mu_1}A^{a_2}_{\mu_2}A^{a_3}_{\mu_3}A^{a_4}_{\mu_4}) \\
&=: (U^{a_1\mu_1\ a_2\mu_2\ a_3\mu_3\ a_4\mu_4})(A^{a_1}_{\mu_1}A^{a_2}_{\mu_2}A^{a_3}_{\mu_3}A^{a_4}_{\mu_4})
\end{aligned}$$

We abbreviate $U(1,2,3,4) := U^{a_1\mu_1\; a_2\mu_2\; a_3\mu_3\; a_4\mu_4}$, to find

$$\begin{aligned} U(1,2,3,4) &= \left(\frac{ig^2}{4} f^{a_1a_2b} f^{a_3a_4b} \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\right) \\ &= \left(\frac{ig^2}{4} f^{a_2a_1b} f^{a_4a_3b} \eta^{\mu_2\mu_4}\eta^{\mu_1\mu_3}\right) \\ &= U(2,1,4,3) \end{aligned}$$

and

$$\begin{aligned} U(1,2,3,4) &= \left(\frac{ig^2}{4} f^{a_1a_2b} f^{a_3a_4b} \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\right) \\ &= \left(\frac{ig^2}{4} f^{a_3a_4b} f^{a_1a_2b} \eta^{\mu_3\mu_1}\eta^{\mu_4\mu_2}\right) \\ &= U(3,4,1,2) \end{aligned}$$

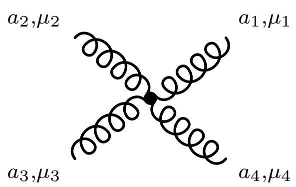
Applying both symmetries after each other, we thus obtain.

$$U(1,2,3,4) = U(3,4,1,2) = U(4,3,2,1) = U(2,1,4,3)$$

Therefore, it suffices to consider permutations of (1234) where 1 is fixed at the first position, weighted with a factor of 4 as follows.

$$\begin{aligned} (I_{gggg})^{a_1\mu_1\; a_2\mu_2\; a_3\mu_3\; a_4\mu_4} &= \sum\nolimits_{\text{permutations}} U(1,2,3,4) \\ &= 4\,(U(1,2,3,4) + U(1,3,4,2) + U(1,4,2,3) \\ &\qquad + U(1,2,4,3) + U(1,4,3,2) + U(1,3,2,4)) \\ &= ig^2 \Big(f^{a_1a_2b} f^{a_3a_4b} \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + f^{a_1a_3b} f^{a_4a_2b} \eta^{\mu_1\mu_4}\eta^{\mu_3\mu_2} \\ &\qquad + f^{a_1a_4b} f^{a_2a_3b} \eta^{\mu_1\mu_2}\eta^{\mu_4\mu_3} + f^{a_1a_2b} f^{a_4a_3b} \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\ &\qquad + f^{a_1a_4b} f^{a_3a_2b} \eta^{\mu_1\mu_3}\eta^{\mu_4\mu_2} + f^{a_1a_3b} f^{a_2a_4b} \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} \Big) \end{aligned}$$

Therefore, we find the following expression for the 4-gluon vertex.



$$\begin{aligned} &= (I_{gggg})^{a_1\mu_1\; a_2\mu_2\; a_3\mu_3\; a_4\mu_4} \\ &= ig^2 \big(f^{a_1a_2b} f^{a_3a_4b} (\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}) \\ &\qquad + f^{a_1a_3b} f^{a_2a_4b} (\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_3\mu_2}) \\ &\qquad + f^{a_1a_4b} f^{a_2a_3b} (\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}) \big) \end{aligned}$$

Gluon-2-Scalar (gss) Vertex

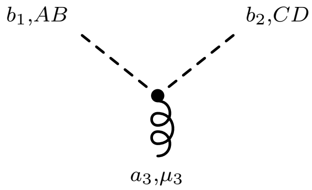
The gluon-2-scalar interaction term of the Lagrangian reads

$$\begin{aligned} -i\mathcal{L}_{gss} &= -\frac{ig}{2} f^{abc} A^{a\mu} \overline{\phi}^{b}_{AB} (\partial_\mu \phi^{cAB}) \\ &= \left(-\frac{ig}{4} f^{ab_2b_3} \varepsilon_{ABCD} \partial^\mu_{(3)}\right) (A^a_\mu \overline{\phi}^{b_2}_{AB} \overline{\phi}^{b_3}_{CD}) \\ &=: (U^{a\mu\; b_2\; b_3}{}_{AB\; CD})(A^a_\mu \overline{\phi}^{b_2}_{AB} \overline{\phi}^{b_3}_{CD}) \end{aligned}$$

such that

$$
\begin{aligned}
(I_{gss})^{a\mu\, b_2\, b_3}_{\, AB\, CD} &= \sum\nolimits_{\text{permutations}} U^{a\mu\, b_2\, b_3}_{\, AB\, CD} \\
&= -\frac{ig}{4}\left(f^{ab_2b_3}\varepsilon_{ABCD}\partial^{\mu}_{(3)} + f^{ab_3b_2}\varepsilon_{CDAB}\partial^{\mu}_{(2)}\right)
\end{aligned}
$$

Therefore, we find the following expression for the gluon-2-scalar vertex.



$$
= (I_{gss})^{a\mu\, b_2\, b_3}_{\, AB\, CD} = -\frac{ig}{4} f^{ab_2b_3}\varepsilon_{ABCD}(\partial^{\mu}_{(3)} - \partial^{\mu}_{(2)})
$$

2-Gluon-2-Scalar (gsgs) Vertex

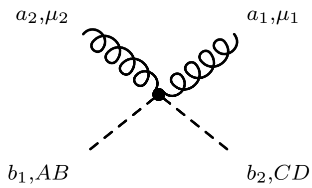
The 2-gluon-2-scalar interaction term of the Lagrangian reads

$$
\begin{aligned}
-i\mathcal{L}_{gsgs} &= -\frac{ig^2}{4} f^{abe} f^{cde} A^a_\mu \phi^{bAB} A^{c\mu}\overline{\phi}^d_{AB} \\
&= \left(-\frac{ig^2}{8}\eta^{\mu_1\mu_2} f^{a_1b_1c} f^{a_2b_2c}\varepsilon_{ABCD}\right)(A^{a_1}_{\mu_1}\overline{\phi}^{b_1}_{AB} A^{a_2}_{\mu_2}\overline{\phi}^{b_2}_{CD}) \\
&= (U^{a_1\mu_1\, b_1\, a_2\mu_2\, b_2}_{\, AB\, \, CD})(A^{a_1}_{\mu_1}\overline{\phi}^{b_1}_{AB} A^{a_2}_{\mu_2}\overline{\phi}^{b_2}_{CD})
\end{aligned}
$$

such that

$$
\begin{aligned}
(I_{gsgs})^{a_1\mu_1\, b_1\, a_2\mu_2\, b_2}_{\, AB\, \, CD} &= \sum\nolimits_{\text{permutations}} U^{a_1\mu_1\, b_1\, a_2\mu_2\, b_2}_{\, AB\, \, CD} \\
&= -\frac{ig^2}{8}\Big(\eta^{\mu_1\mu_2} f^{a_1b_1c} f^{a_2b_2c}\varepsilon_{ABCD} + \eta^{\mu_2\mu_1} f^{a_2b_1c} f^{a_1b_2c}\varepsilon_{ABCD} \\
&\qquad + \eta^{\mu_1\mu_2} f^{a_1b_2c} f^{a_2b_1c}\varepsilon_{CDAB} + \eta^{\mu_2\mu_1} f^{a_2b_2c} f^{a_1b_1c}\varepsilon_{CDAB}\Big)
\end{aligned}
$$

Therefore, we find the following expression for the 2-gluon-2-scalar vertex.



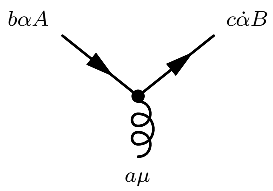
$$
\begin{aligned}
&= (I_{gsgs})^{a_1\mu_1\, b_1\, a_2\mu_2\, b_2}_{\, AB\, \, CD} \\
&= -\tfrac{ig^2}{4}\eta^{\mu_1\mu_2}\varepsilon_{ABCD}(f^{a_1b_1c} f^{a_2b_2c} + f^{a_1b_2c} f^{a_2b_1c})
\end{aligned}
$$

Gluon-Fermion-Antifermion (gfa) Vertex

The gluon-fermion-antifermion interaction term of the Lagrangian reads

$$
-i\mathcal{L}_{gfa} = (gf^{abc}\overline{\sigma}^{\mu\alpha\dot{\alpha}}\delta_{AB})(A^a_\mu \psi^{bA}_\alpha \tilde{\psi}^{cB}_{\dot{\alpha}})
$$

Here, no permutation occurs since there are different types of fields such that $I = U$. Therefore, we find the following expression for the gluon-fermion-antifermion vertex.



$$
= (I_{gfa})^{a\mu\, b\alpha\, c\dot{\alpha}}_{\, A\, B} = gf^{abc}\overline{\sigma}^{\mu\alpha\dot{\alpha}}\delta_{AB}
$$

Scalar-2-Fermion (sff) Vertex

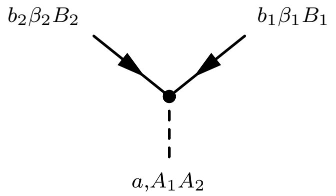
The scalar-2-fermion interaction term of the Lagrangian reads

$$-i\mathcal{L}_{sff} = \left(\frac{\sqrt{2}g}{2} f^{ab_1b_2} \epsilon^{\beta_1\beta_2} \delta_{A_1B_1} \delta_{A_2B_2} \right) (\overline{\phi}^a_{A_1A_2} \psi^{b_1B_1}_{\beta_1} \psi^{b_2B_2}_{\beta_2})$$

Therefore

$$\begin{aligned}(I_{sff})^{a}_{A_1A_2}{}^{b_1\beta_1}_{B_1}{}^{b_2\beta_2}_{B_2} &= \frac{\sqrt{2}g}{2} \left(f^{ab_1b_2} \epsilon^{\beta_1\beta_2} \delta_{A_1B_1} \delta_{A_2B_2} - f^{ab_2b_1} \epsilon^{\beta_2\beta_1} \delta_{A_1B_2} \delta_{A_2B_1} \right) \\ &= \frac{\sqrt{2}g}{2} f^{ab_1b_2} \epsilon^{\beta_1\beta_2} \left(\delta_{A_1B_1} \delta_{A_2B_2} - \delta_{A_1B_2} \delta_{A_2B_1} \right)\end{aligned}$$

where the sign comes from the oddness of ψ (which gives a sign upon permuting). This has to be understood upon contraction with $\overline{\phi}_{A_1A_2}$ (which is antisymmetric in (A_1A_2)). Therefore, we find the following expression for the scalar-2-fermion vertex.



$$= (I_{sff})^{a}_{A_1A_2}{}^{b_1\beta_1}_{B_1}{}^{b_2\beta_2}_{B_2} = \sqrt{2}g f^{ab_1b_2} \epsilon^{\beta_1\beta_2} \delta_{A_1B_1} \delta_{A_2B_2}$$

Scalar-2-Antifermion (saa) Vertex

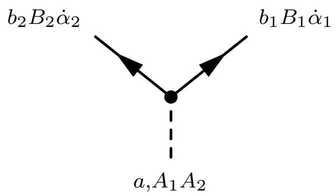
The scalar-2-antifermion interaction term of the Lagrangian reads

$$-i\mathcal{L}_{saa} = \left(-\frac{\sqrt{2}g}{4} f^{ab_1b_2} \varepsilon_{A_1A_2B_1B_2} \epsilon^{\dot{\alpha}_1\dot{\alpha}_2} \right) (\overline{\phi}^a_{A_1A_2} \tilde{\psi}^{b_1}_{\dot{\alpha}_1B_1} \tilde{\psi}^{b_2}_{\dot{\alpha}_2B_2})$$

such that

$$(I_{saa})^{a}_{A_1A_2}{}^{b_1\dot{\alpha}_1}_{B_1}{}^{b_2\dot{\alpha}_2}_{B_2} = -\frac{\sqrt{2}g}{4} \left(f^{ab_1b_2} \varepsilon_{A_1A_2B_1B_2} \epsilon^{\dot{\alpha}_1\dot{\alpha}_2} - f^{ab_2b_1} \varepsilon_{A_1A_2B_2B_1} \epsilon^{\dot{\alpha}_2\dot{\alpha}_1} \right)$$

Therefore, we find the following expression for the scalar-2-antifermion vertex.



$$= (I_{saa})^{a}_{A_1A_2}{}^{b_1\dot{\alpha}_1}_{B_1}{}^{b_2\dot{\alpha}_2}_{B_2} = -\frac{\sqrt{2}g}{2} f^{ab_1b_2} \varepsilon_{A_1A_2B_1B_2} \epsilon^{\dot{\alpha}_1\dot{\alpha}_2}$$

4-Scalar (ssss) Vertex

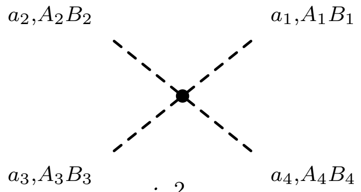
The 4-scalar interaction term of the Lagrangian reads

$$\begin{aligned}-i\mathcal{L}_{ssss} &= \frac{ig^2}{16} f^{abe} f^{cde} \phi^{aAB} \phi^{bCD} \overline{\phi}^c_{AB} \overline{\phi}^d_{CD} \\ &= \left(\frac{ig^2}{64} f^{a_1a_2b} f^{a_3a_4b} \varepsilon_{A_1B_1A_3B_3} \varepsilon_{A_2B_2A_4B_4} \right) (\overline{\phi}^{a_1}_{A_1B_1} \overline{\phi}^{a_2}_{A_2B_2} \overline{\phi}^{a_3}_{A_3B_3} \overline{\phi}^{a_4}_{A_4B_4})\end{aligned}$$

The symmetries $U(1,2,3,4) = U(2,1,4,3) = U(3,4,1,2) = U(4,3,2,1)$ are obtained analogous to the calculation of I_{gggg} above, with analogous notation. Therefore, it suffices to consider permutations of (1234) where 1 is at the first position, weighted with a factor of 4 as follows.

$$
\begin{aligned}
&I_{ssss}(1,2,3,4)\\
&\quad = \sum\nolimits_{\text{permutations}} U(1,2,3,4)\\
&\quad = 4\,(U(1,2,3,4) + U(1,3,4,2) + U(1,4,2,3)\\
&\qquad + U(1,2,4,3) + U(1,4,3,2) + U(1,3,2,4))\\
&\quad = \frac{ig^2}{16}\Big(f^{a_1a_2b}f^{a_3a_4b}\varepsilon_{A_1B_1A_3B_3}\varepsilon_{A_2B_2A_4B_4} + f^{a_1a_3b}f^{a_4a_2b}\varepsilon_{A_1B_1A_4B_4}\varepsilon_{A_3B_3A_2B_2}\\
&\qquad\qquad + f^{a_1a_4b}f^{a_2a_3b}\varepsilon_{A_1B_1A_2B_2}\varepsilon_{A_4B_4A_3B_3} + f^{a_1a_2b}f^{a_4a_3b}\varepsilon_{A_1B_1A_4B_4}\varepsilon_{A_2B_2A_3B_3}\\
&\qquad\qquad + f^{a_1a_4b}f^{a_3a_2b}\varepsilon_{A_1B_1A_3B_3}\varepsilon_{A_4B_4A_2B_2} + f^{a_1a_3b}f^{a_2a_4b}\varepsilon_{A_1B_1A_2B_2}\varepsilon_{A_3B_3A_4B_4}\Big)
\end{aligned}
$$

Therefore, we find the following expression for the 4-scalar vertex.



$$
\begin{aligned}
&= (I_{ssss})^{a_1\ a_2\ a_3\ a_4}_{A_1B_1\,A_2B_2\,A_3B_3\,A_4B_4}\\
&= \frac{ig^2}{16}\Big(f^{a_1a_2b}f^{a_3a_4b}(\varepsilon_{A_1B_1A_3B_3}\varepsilon_{A_2B_2A_4B_4} - \varepsilon_{A_1B_1A_4B_4}\varepsilon_{A_2B_2A_3B_3})\\
&\qquad + f^{a_1a_3b}f^{a_2a_4b}(\varepsilon_{A_1B_1A_2B_2}\varepsilon_{A_3B_3A_4B_4} - \varepsilon_{A_1B_1A_4B_4}\varepsilon_{A_3B_3A_2B_2})\\
&\qquad + f^{a_1a_4b}f^{a_2a_3b}(\varepsilon_{A_1B_1A_2B_2}\varepsilon_{A_4B_4A_3B_3} - \varepsilon_{A_1B_1A_3B_3}\varepsilon_{A_4B_4A_2B_2})\Big)
\end{aligned}
$$

Chapter 2

Supersymmetric Wilson Loops

In this chapter, we derive an extension of the classical Wilson loop which is supersymmetric in a sense to be explained. We will see that this supersymmetry condition ensures the existence of a solution which is not unique but has a determined shape. Our treatment remains purely classical, and we defer the quantum theory to the next chapter.

2.1 The Super Wilson Loop Ansatz and its Symmetries

As in Sec. 1.2, we denote the gauge field by A such that $-igA$ is a connection with gauge group $SU(N)$ on Minkowski space $\mathbb{R}^4$. Let $x : [0,1] \to \mathbb{R}^4$ be a path in Minkowski space connecting $x(0)$ with $x(1)$. Parallel transport, which maps a vector $v_0 \in T_{x(0)}\mathbb{R}^4 \cong \mathbb{R}^4$ to a vector $v(t) \in T_{x(t)}\mathbb{R}^4 \cong \mathbb{R}^4$, is defined by the differential equation $D_t v(t) = 0$. Given the initial condition $v(0) = v_0$, this equation is shown to have a unique solution which can be written in the form $v(t) = B(t) \cdot v_0$ with $B : [0,1] \to \mathfrak{su}(N)$ being the path-ordered exponential of igA. Taking the trace and dividing by N defines the classical Wilson line (for $t = 1$) as follows.

$$W := \frac{1}{N}\mathrm{tr}B(1) = \frac{1}{N}\mathrm{tr}\,\mathcal{P}\exp\left(ig\int_0^1 dt\frac{dx^\mu}{dt}A_\mu(x(t))\right) \tag{2.1}$$

If x is a loop (such that $x(0) = x(1)$), B is also known as the holonomy around x, and W is called a Wilson loop. For the classical theory, consult e.g. [Bau09] or [Bär09].

Recently, two approaches appeared for a supersymmetric version of (2.1), which are both motivated by a proposed extension of the duality of gluon scattering amplitudes with (quantised) Wilson loops to a duality involving scattering amplitudes with any kind of $\mathcal{N} = 4$ SYM particles. We will come back to this duality in Sec. 3.2 and, in this chapter, solely concentrate on the classical theory. The first approach ([MS10]) originates in momentum twistor space and translates into the integral over some kind of superconnection. Without having checked, we assume that this can indeed be identified with the superholonomy of a superconnection in the sense of [Gal09].

The second approach, followed here, is due to Caron-Huot ([CH11]) and works for polygons with lightlike edges only. It was shown in [BKS12] that, in the common domain of definition, both approaches agree modulo the Euler-Lagrange equations of Sec. 1.4. Consider then a polygon with n vertices x_i and vectors $p_i := x_i - x_{i-1}$ such that $x_i(t) := x_{i-1} + tp_i$ connects $x_i(0) = x_{i-1}$ with $x_i(1) = x_i$, and $\frac{dx_i^\mu}{dt} = p_i^\mu$. We shall refer to the p_i as "momenta", a denomination not justified until Sec. 3.2, which we assume to be lightlike such that $p_i = \lambda_i \tilde{\lambda}_i$ decomposes according to (1.11). Since the polygon closes, we trivially observe momentum conservation

$$p_1 + p_2 + \ldots + p_n = 0 \tag{2.2}$$

We endow the spacetime-coordinates of Minkowski space by Grassmann generators

$$\eta_i^A \,, \qquad i \in \{1, \ldots, n\} \,, \qquad A \in \{1, \ldots, 4\} \tag{2.3}$$

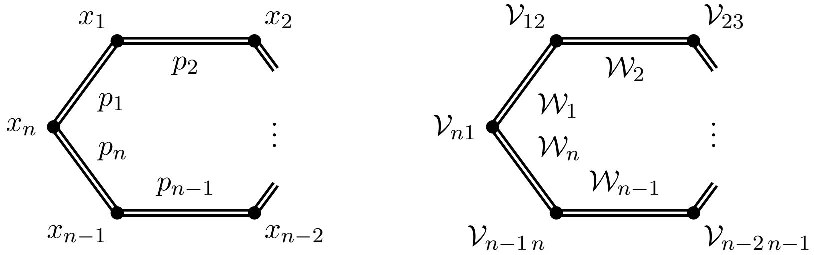
and, for the supersymmetric Wilson loop, make the ansatz

$$W_n = \frac{1}{N} \mathrm{tr}\left(\mathcal{V}_{n1} \mathcal{W}_n \mathcal{V}_{n-1,n} \ldots \mathcal{W}_2 \mathcal{V}_{12} \mathcal{W}_1\right) \tag{2.4}$$

where $\mathcal{V}_{j\,j+1} \in \mathbb{C} \otimes \mathfrak{su}(N)$, which is thought of as a vertex operator located at x_j, takes values in the Grassmann algebra generated by the η_i^A, and where

$$\mathcal{W}_j := \mathcal{P} \exp\left(ig \int_0^1 dt\, \mathcal{E}_j(t)\right) \tag{2.5}$$

is an extension of the parallel transport along p_j with $\mathcal{E}_j = p_j \cdot A + \mathcal{O}(\eta)$.



In the following, we will derive explicit expressions for $\mathcal{V}_{j\,j+1}$ and $\mathcal{E}_j$ by symmetry considerations.

Lemma 2.1.1. Consider the following "super" analogon of a gauge transformation (1.22) with $V : \mathbb{R}^4 \to SU(N)$.

$$\mathcal{E}_i(t) \to V(x_i(t)) \left(\mathcal{E}_i(t) + \frac{i}{g}\partial_t\right) V^\dagger(x_i(t)) \,, \qquad \mathcal{V}_{i,i+1} \to V(x_i) \mathcal{V}_{i,i+1} V^\dagger(x_i) \tag{2.6}$$

Then $W_n \to W_n$ is invariant under (2.6).

Proof. Denote the "super" extension of the covariant derivative D_t along an edge by the same symbol:

$$D_t := \partial_t - ig\,[\mathcal{E}_i(t), \cdot]$$

As for the classical parallel transport, one shows that (2.5) is the unique solution of the equation $D_t \mathcal{W}_i = 0$ for a given initial condition $\mathcal{E}_i(0)$. Under (2.6), D_t transforms by definition as

$$D_t \to \partial_t - igV(x_i(t)) \left(\mathcal{E}_i(t) + \frac{i}{g}\partial_t\right) V^\dagger(x_i(t)) = V(x_i(t)) D_t V^\dagger(x_i(t))$$

and $V(x_i)\mathcal{W}_i V^\dagger(x_{i-1})$ satisfies $(VD_tV^\dagger)(V(x_i(t))\mathcal{W}_iV^\dagger(x_{i-1})) = 0$ if and only if $D_t\mathcal{W}_i = 0$. It follows that $\mathcal{W}_i$ transforms under (2.6) as

$$\mathcal{W}_i \to V(x_i)\mathcal{W}_i V^\dagger(x_{i-1}) \tag{2.7}$$

As a consequence, the Wilson loop ansatz (2.4) transforms as

$$\begin{aligned} W_n &= \frac{1}{N}\mathrm{tr}\left(\mathcal{V}_{n1}\mathcal{W}_n\mathcal{V}_{n-1,n}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1\right) \\ &\to \frac{1}{N}\mathrm{tr}\Big(\Big(V(x_n)\mathcal{V}_{n1}V^\dagger(x_n)\Big)\Big(V(x_n)\mathcal{W}_nV^\dagger(x_{n-1})\Big)\dots \\ &\qquad\qquad \dots\Big(V(x_1)\mathcal{V}_{12}V^\dagger(x_1)\Big)\Big(V(x_1)\mathcal{W}_1V^\dagger(x_n)\Big)\Big) \\ &= \frac{1}{N}\mathrm{tr}\Big(V(x_n)\mathcal{V}_{n1}\mathcal{W}_n\mathcal{V}_{n-1,n}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1V^\dagger(x_n)\Big) \\ &= W_n \end{aligned}$$

where we have used Lem. 1.1.1. □

Lemma 2.1.2. Let $\mathcal{Q}$ be a variation such that

$$\mathcal{Q}\mathcal{E}_i(t) = \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}_i(t),\,\cdot\right]\right)X(x_i(t))\,, \qquad \mathcal{Q}\mathcal{V}_{i,i+1} = iX(x_i)\mathcal{V}_{i,i+1} - i\mathcal{V}_{i,i+1}X(x_i)$$

for some $X : \mathbb{R}^4 \to \mathbb{C}\otimes\mathfrak{su}(N)$. Then $\mathcal{Q}W_n = 0$.

Proof. We denote the gauge transformed edge and vertex operators on the respective right hand side in (2.6) by $\mathcal{E}_i(V)$ and $\mathcal{V}_{i,i+1}(V)$ and consider first $\alpha : \mathbb{R}^4 \to \mathfrak{su}(N)$ and $X := -i\alpha : \mathbb{R}^4 \to i\cdot\mathfrak{su}(N)$. Analogous to the calculation of the gauge fixing term $\Lambda[\alpha]$ in Sec. 1.5, we find that

$$\begin{aligned} d\mathcal{E}_i(V)[\alpha] &= \frac{1}{g}\left(\partial_t X - ig\left[\mathcal{E}_i,\,X\right]\right) = \mathcal{Q}\mathcal{E}_i \\ d\mathcal{V}_{i,i+1}(V)[\alpha] &= iX(x_i)\mathcal{V}_{i,i+1} - i\mathcal{V}_{i,i+1}X(x_i) = \mathcal{Q}\mathcal{V}_{i,i+1} \end{aligned}$$

It follows that

$$d\mathcal{W}_i(V)[\alpha] = d\mathcal{W}_i \circ d\mathcal{E}_i(V)[\alpha] = d\mathcal{W}_i \circ \mathcal{Q}\mathcal{E}_i = \mathcal{Q}\mathcal{W}_i$$

and we obtain

$$\begin{aligned} \mathcal{Q}W_n &= \mathcal{Q}\left(\frac{1}{N}\mathrm{tr}\left(\mathcal{V}_{n1}\mathcal{W}_n\mathcal{V}_{n-1,n}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1\right)\right) \\ &= d\left(\frac{1}{N}\mathrm{tr}\left(\mathcal{V}_{n1}\mathcal{W}_n\mathcal{V}_{n-1,n}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1\right)\right)(V)[\alpha] \\ &= dW_n(V)[\alpha] = 0 \end{aligned}$$

by invariance $W_n = W_n(V)$ from Lem. 2.1.1. The general case (X and α taking values in $\mathbb{C}\otimes\mathfrak{su}(N)$) follows by complex linear extension of the derivatives. □

We have seen that the variation stated in the previous lemma is just the infinitesimal form of a super gauge transformation (2.6). In the following, we shall need a weaker form as treated next.

Lemma 2.1.3. Let $\mathcal{Q}$ be a variation such that

$$\mathcal{Q}\mathcal{E}_i(t) = \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}_i(t),\,\cdot\right]\right)X_i(t)\,, \qquad \mathcal{Q}\mathcal{V}_{i,i+1} = iX_{i+1}(0)\mathcal{V}_{i,i+1} - i\mathcal{V}_{i,i+1}X_i(1)$$

for some $X_i : [0,1] \to \mathbb{C}\otimes\mathfrak{su}(N)$. Then $\mathcal{Q}W_n = 0$.

Proof. If we can find $X : \mathbb{R}^4 \to \mathbb{C}\otimes\mathfrak{su}(N)$ such that $X(x_i(t)) = X_i(t)$ for all i, the statement follows immediately from the previous lemma. Otherwise (if X_i cannot be smoothly connected with X_{i+1}) we proceed as follows. Assuming that X_i has sufficient regularity properties, we at least find $\tilde{X}_i$, which is defined in a neighbourhood of the "interval" $[x_{i-1}, x_i] \subseteq \mathbb{R}^4$, such that $\tilde{X}_i(x_i(t)) = X_i(t)$. Then

$$\mathcal{Q}\mathcal{W}_i = d\mathcal{W}_i[\tilde{\alpha}_i(x_i(t))] = i\tilde{X}_i(x_i)\mathcal{W}_i - i\mathcal{W}_i\tilde{X}_i(x_{i-1}) = iX_i(1)\mathcal{W}_i - i\mathcal{W}_iX_i(0)$$

holds with $\tilde{\alpha}_i = i\tilde{X}_i$, using (2.7). We then calculate

$$\begin{aligned}\mathcal{Q}(\mathcal{V}_{i,i+1}\mathcal{W}_i) &= (\mathcal{Q}\mathcal{V}_{i,i+1})\mathcal{W}_i + \mathcal{V}_{i,i+1}(\mathcal{Q}\mathcal{W}_i)\\ &= \left(iX_{i+1}(0)\mathcal{V}_{i,i+1} - i\mathcal{V}_{i,i+1}X_i(1)\right)\mathcal{W}_i + \mathcal{V}_{i,i+1}\left(iX_i(1)\mathcal{W}_i - i\mathcal{W}_iX_i(0)\right)\\ &= iX_{i+1}(0)\mathcal{V}_{i,i+1}\mathcal{W}_i - i\mathcal{V}_{i,i+1}\mathcal{W}_iX_i(0)\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}(\mathcal{V}_{i,i+1}\mathcal{W}_i\mathcal{V}_{i-1,i}\mathcal{W}_{i-1}) &= \mathcal{Q}(\mathcal{V}_{i,i+1}\mathcal{W}_i)\mathcal{V}_{i-1,i}\mathcal{W}_{i-1} + \mathcal{V}_{i,i+1}\mathcal{W}_i\mathcal{Q}(\mathcal{V}_{i-1,i}\mathcal{W}_{i-1})\\ &= (iX_{i+1}(0)\mathcal{V}_{i,i+1}\mathcal{W}_i - i\mathcal{V}_{i,i+1}\mathcal{W}_iX_i(0))\mathcal{V}_{i-1,i}\mathcal{W}_{i-1}\\ &\quad + \mathcal{V}_{i,i+1}\mathcal{W}_i(iX_i(0)\mathcal{V}_{i-1,i}\mathcal{W}_{i-1} - i\mathcal{V}_{i-1,i}\mathcal{W}_{i-1}X_{i-1}(0))\\ &= iX_{i+1}(0)\mathcal{V}_{i,i+1}\mathcal{W}_i\mathcal{V}_{i-1,i}\mathcal{W}_{i-1} - i\mathcal{V}_{i,i+1}\mathcal{W}_i\mathcal{V}_{i-1,i}\mathcal{W}_{i-1}X_{i-1}(0)\end{aligned}$$

Proceeding by induction, we obtain

$$\begin{aligned}&\mathcal{Q}(\mathcal{V}_{n-1,n}\mathcal{W}_{n-1}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1)\\ &\qquad = iX_n(0)\mathcal{V}_{n-1,n}\mathcal{W}_{n-1}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1 - i\mathcal{V}_{n-1,n}\mathcal{W}_{n-1}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1X_1(0)\end{aligned}$$

and, therefore,

$$\begin{aligned}\mathcal{Q}W_n &= \frac{1}{N}\mathrm{tr}\,\mathcal{Q}\left(\mathcal{V}_{n1}\mathcal{W}_n\mathcal{V}_{n-1,n}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1\right)\\ &= \frac{i}{N}\mathrm{tr}\left(X_1(0)\mathcal{V}_{n1}\mathcal{W}_n\mathcal{V}_{n-1,n}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1 - \mathcal{V}_{n1}\mathcal{W}_n\mathcal{V}_{n-1,n}\dots\mathcal{W}_2\mathcal{V}_{12}\mathcal{W}_1X_1(0)\right)\end{aligned}$$

which vanishes by Lem. 1.1.1. □

Our ansatz (2.4) for the Wilson loop W_n depends on the fields of the theory as well as on the Grassmann generators (2.3), and we endow the supersymmetry generators q_A^α as stated in Lem. 1.3.4 by generators Q_A^α which act on superspace as follows.

$$\mathcal{Q}_A^\alpha := q_A^\alpha + Q_A^\alpha := q_A^\alpha + c_0\sum\nolimits_i \lambda_i^\alpha\frac{\partial}{\partial\eta_i^A} \tag{2.8}$$

Since the two sets of generators act on different spaces, we have added a constant c_0 which remains undetermined for the time being. We aim at constructing $\mathcal{E}_i$ and $\mathcal{V}_{i,i+1}$ such that W_n is supersymmetric in the sense of $\mathcal{Q}_A^\alpha W_n = 0$ (ignoring the second half of supersymmetries $\tilde{q}^{A\dot{\alpha}}$). By Lem. 2.1.3, this is achieved if we further find $X_{iA}^\alpha(t)$ such that

$$\mathcal{Q}_A^\alpha\mathcal{E}_i = \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}_i,\,\cdot\right]\right)X_{iA}^\alpha(t) \tag{2.9a}$$

$$\mathcal{Q}_A^\alpha\mathcal{V}_{i,i+1} = iX_{i+1\,A}^\alpha(0)\mathcal{V}_{i,i+1} - i\mathcal{V}_{i,i+1}X_{iA}^\alpha(1) \tag{2.9b}$$

2.2 Derivation of Edge Operators

In this section, we derive a solution of (2.9a) whose shape is determined by this symmetry. It turns out that, once X^α_{iA} is fixed, (2.9b) has a unique solution, for which an explicit recursion formula exist. This will be the subject matter of Sec. 2.3. For the sake of brevity, we write

$$\langle i,\, j\rangle := \langle \lambda_i,\, \lambda_j\rangle \;, \qquad [i,\, j] := \left[\tilde\lambda_i,\, \tilde\lambda_j\right] \;, \qquad 2p_i\cdot p_j = \langle i,\, j\rangle\, [i,\, j] \tag{2.10}$$

where the brackets denote the symplectic structures (1.4), and the last equation holds by (1.10). Moreover, we denote by $\mathcal{O}(\eta^k)$ any polynomial in the Grassmann generators (2.3) of a degree of at least k.

Theorem 2.2.1. We make the ansatz $\mathcal{E}_i = p_i\cdot A + \mathcal{O}(\eta)$ (such that, at lowest order, $\mathcal{W}_i$ in (2.5) equals $B(1)$ in the classical Wilson line (2.1)). Then (2.9a) is satisfied with

$$\begin{aligned}\mathcal{E}_i = {}& \frac{1}{2}\lambda_{i\beta}\tilde\lambda_{i\dot\beta}A^{\beta\dot\beta} + \frac{i}{c_0}\tilde\lambda_{i\dot\beta}\tilde\psi^{\dot\beta}_A\eta^A_i - \frac{i\sqrt{2}}{2c_0^2}\frac{\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\gamma}D^{\dot\beta\gamma}\overline{\phi}_{AB}}{\langle i,\, i-1\rangle}\eta^A_i\eta^B_i\\ &+\frac{1}{3c_0^3}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\xi}\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\gamma}D^{\dot\beta\gamma}\psi^{\xi A}}{\langle i,\, i-1\rangle^2}\eta^B_i\eta^C_i\eta^D_i\\ &+\frac{i}{24c_0^4}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\gamma}\lambda_{(i-1)\xi}\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\beta}D^{\dot\beta\beta}F^{\gamma\xi}}{\langle i,\, i-1\rangle^3}\eta^A_i\eta^B_i\eta^C_i\eta^D_i\end{aligned}$$

and

$$\begin{aligned}X^\alpha_{iA} := \frac{g\lambda^\alpha_{i-1}}{c_0\,\langle i,\, i-1\rangle}\Bigg(&-2i\sqrt{2}\,\overline{\phi}_{AB}\eta^B_i + \varepsilon_{ABCD}\frac{2\lambda_{(i-1)\gamma}\psi^{\gamma B}}{c_0\,\langle i,\, i-1\rangle}\eta^C_i\eta^D_i\\ &+\frac{i}{3c_0^2}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}F^{\gamma\beta}}{\langle i,\, i-1\rangle^2}\eta^B_i\eta^C_i\eta^D_i\Bigg)\end{aligned}$$

This result holds upon the Euler-Lagrange equations of Sec. 1.4.

Proof. We use the notation

$$\mathcal{E}_i = \mathcal{E}^{(0)}_i + \mathcal{E}^{(1)}_i + \mathcal{E}^{(2)}_i + \mathcal{E}^{(3)}_i + \mathcal{E}^{(4)}_i \;, \qquad \mathcal{E}^k_i := \mathcal{E}^{(0)}_i + \ldots + \mathcal{E}^{(k)}_i$$

and similarly for X^α_{iA}, with $\mathcal{E}^{(k)}_i$ denoting the term of order $(\eta_i)^k$. Starting with $\mathcal{E}^0_i := p_i\cdot A = \frac{1}{2}\lambda_{i\beta}\tilde\lambda_{i\dot\beta}A^{\beta\dot\beta}$, we calculate, order by order in the η terms, the supersymmetry variation $\mathcal{Q}^\alpha_A(\mathcal{E}^k_i)$ (Lem. 1.3.4). Upon multiplication with $1 = \frac{\lambda_{i\gamma}\lambda^\gamma_{i-1}}{\langle i\, i-1\rangle}$ and using the Schouten identity (Lem. 1.1.5), this variation can be written as a sum $\mathcal{Q}^\alpha_A(\mathcal{E}^k_i) = \lambda^\alpha_i E^k_{iA} + \lambda^\alpha_{i-1}R^k_{iA}$ of terms proportional to either λ^α_i or λ^α_{i-1}. Now

$$\lambda^\alpha_i E^k_{iA} = \frac{1}{S}\cdot\left(c_0\sum\nolimits_i \lambda^\alpha_i\frac{\partial}{\partial\eta^A_i}\right)\left(\eta^A_i\frac{E^k_{iA}}{c_0}\right) = Q^\alpha_A\left(\eta^A_i\frac{E^k_{iA}}{S\cdot c_0}\right)$$

where S is a symmetry factor depending on E^k_{iA}. Setting $\mathcal{E}_i^{(k+1)} := -\eta_i^A \frac{E^k_{iA}}{S\cdot c_0}$, we thus obtain

$$\begin{aligned}
\mathcal{Q}^\alpha_A(\mathcal{E}_i^{k+1}) &= \lambda_i^\alpha E^k_{iA} + \lambda^\alpha_{i-1} R^k_{iA} - \mathcal{Q}^\alpha_A\left(\eta_i^A \frac{E^k_{iA}}{S\cdot c_0}\right)\\
&= -q^\alpha_A\left(\eta_i^A \frac{E^k_{iA}}{S\cdot c_0}\right) + \lambda^\alpha_{i-1} R^k_{iA}\\
&= q^\alpha_A\left(\mathcal{E}_i^{(k+1)}\right) + \lambda^\alpha_{i-1} R^k_{iA}\\
&\overset{!}{=} \lambda_i^\alpha E^{k+1}_{iA} + \lambda^\alpha_{i-1} R^{k+1}_{iA}
\end{aligned}$$

By recursion, this fixes the edge operator $\mathcal{E}_i = \mathcal{E}_i^4$. Finally, we find that $\mathcal{Q}^\alpha_A(\mathcal{E}_i) = \lambda^\alpha_{i-1} R^4_{iA}$ (i.e. $E^4_{iA} = 0$). Moreover we find, again order by order, that there is X^α_{iA} such that

$$\lambda^\alpha_{i-1} R^4_{iA} = \frac{1}{g}\left(\partial_t - ig\,[\mathcal{E}_i,\,\cdot]\right) X^\alpha_{iA}(t) \tag{2.11}$$

The details of the calculations occupy the rest of this section. □

2.2.1 Proof: Part 1 (Calculation of $\mathcal{E}_i$ and R_{iA})

First Order

Starting with $\mathcal{E}_i^0 = \frac{1}{2}\lambda_{i\beta}\tilde{\lambda}_{i\dot\beta} A^{\beta\dot\beta}$, we find

$$\mathcal{Q}^\alpha_A(\mathcal{E}_i^0) = \frac{1}{2}\lambda_{i\beta}\tilde{\lambda}_{i\dot\beta} q^\alpha_A(A^{\beta\dot\beta}) = i\lambda_{i\beta}\tilde{\lambda}_{i\dot\beta}\epsilon^{\alpha\beta}\tilde{\psi}^{\dot\beta}_A = \lambda_i^\alpha\cdot i\tilde{\lambda}_{i\dot\beta}\tilde{\psi}^{\dot\beta}_A = \lambda_i^\alpha E^0_{iA}$$

such that we find $\mathcal{E}_i^{(1)} := -\eta_i^A \frac{i}{c_0}\tilde{\lambda}_{i\dot\beta}\tilde{\psi}^{\dot\beta}_A = \frac{i}{c_0}\tilde{\lambda}_{i\dot\beta}\tilde{\psi}^{\dot\beta}_A\eta_i^A$.

Second Order

Furthermore

$$\mathcal{Q}^\alpha_A(\mathcal{E}_i^1) = \frac{i}{c_0}\tilde{\lambda}_{i\dot\beta}\, q^\alpha_A(\tilde{\psi}^{\dot\beta}_B)\,\eta_i^B = -\frac{i\sqrt{2}}{c_0}\tilde{\lambda}_{i\dot\beta} D^{\dot\beta\alpha}\overline{\phi}_{AB}\eta_i^B = -\frac{\lambda_{i\gamma}\lambda^\gamma_{i-1}}{\langle i,\, i-1\rangle}\frac{i\sqrt{2}}{c_0}\tilde{\lambda}_{i\dot\beta} D^{\dot\beta\alpha}\overline{\phi}_{AB}\eta_i^B$$

and the Schouten identity yields

$$\begin{aligned}
\mathcal{Q}^\alpha_A(\mathcal{E}_i^1) &= \frac{i\sqrt{2}\,\lambda_i^\alpha}{c_0}\frac{\tilde{\lambda}_{i\dot\beta}\lambda_{(i-1)\gamma}}{\langle i,\, i-1\rangle} D^{\dot\beta\gamma}\overline{\phi}_{AB}\eta_i^B - \frac{i\sqrt{2}\,\lambda^\alpha_{i-1}}{c_0}\frac{\tilde{\lambda}_{i\dot\beta}\lambda_{i\gamma}}{\langle i,\, i-1\rangle} D^{\dot\beta\gamma}\overline{\phi}_{AB}\eta_i^B\\
&= \lambda_i^\alpha E^1_{iA} + \lambda^\alpha_{i-1} R^1_{iA}
\end{aligned}$$

Now $\frac{\partial}{\partial\eta_i^A}\left(\overline{\phi}_{BC}\eta_i^B\eta_i^C\right) = \overline{\phi}_{AC}\eta_i^C - \overline{\phi}_{BA}\eta_i^B = 2\overline{\phi}_{AB}\eta_i^B$ such that $S = 2$ and we find $\mathcal{E}_i^{(2)} := -\frac{i\sqrt{2}}{2c_0^2}\frac{\tilde{\lambda}_{i\dot\beta}\lambda_{(i-1)\gamma}}{\langle i, i-1\rangle} D^{\dot\beta\gamma}\overline{\phi}_{BC}\eta_i^B\eta_i^C$.

Third Order

We further calculate, using Cor. 1.3.5:

$$
\begin{aligned}
&\mathcal{Q}^{\alpha}_{A}(\mathcal{E}^{2}_{i})\\
&\quad = q^{\alpha}_{A}\left(-\frac{i\sqrt{2}}{c_0^2}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}}{\langle i,\, i-1\rangle}D^{\dot{\beta}\gamma}\overline{\phi}_{BC}\eta^{B}_{i}\eta^{C}_{i}\right) + \lambda^{\alpha}_{i-1}R^{1}_{iA}\\
&\quad = \frac{1}{c_0^2}\varepsilon_{ABCD}\frac{\lambda_{i\xi}\lambda^{\xi}_{(i-1)}}{\langle i,\, i-1\rangle}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\psi^{\alpha D}}{\langle i,\, i-1\rangle}\eta^{B}_{i}\eta^{C}_{i}\\
&\qquad\qquad - \frac{ig\sqrt{2}}{c_0^2}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda^{\alpha}_{(i-1)}\left[\tilde{\psi}^{\dot{\beta}}_{A},\, \overline{\phi}_{BC}\right]}{\langle i,\, i-1\rangle}\eta^{B}_{i}\eta^{C}_{i} + \lambda^{\alpha}_{i-1}R^{1}_{iA}\\
&\quad = -\frac{\lambda^{\alpha}_{i}}{c_0^2}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\psi^{\xi D}}{\langle i,\, i-1\rangle^2}\eta^{B}_{i}\eta^{C}_{i}\\
&\qquad + \lambda^{\alpha}_{i-1}\left(\varepsilon_{ABCD}\frac{\lambda_{i\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\psi^{\xi D}}{c_0^2\,\langle i,\, i-1\rangle^2}\eta^{B}_{i}\eta^{C}_{i} - \frac{ig\sqrt{2}}{c_0^2}\frac{\tilde{\lambda}_{i\dot{\beta}}\left[\tilde{\psi}^{\dot{\beta}}_{A},\, \overline{\phi}_{BC}\right]}{\langle i,\, i-1\rangle}\eta^{B}_{i}\eta^{C}_{i} + R^{1}_{iA}\right)\\
&\quad = \lambda^{\alpha}_{i}E^{2}_{iA} + \lambda^{\alpha}_{i-1}R^{2}_{iA}
\end{aligned}
$$

Now $\frac{\partial}{\partial\eta^{A}_{i}}\left(\varepsilon_{BCDE}\psi^{\xi E}\eta^{B}_{i}\eta^{C}_{i}\eta^{D}_{i}\right) = 3\varepsilon_{ACDE}\psi^{\xi E}\eta^{C}_{i}\eta^{D}_{i}$ holds such that $S = 3$, and we find $\mathcal{E}^{(3)}_{i} = \frac{1}{3c_0^3}\varepsilon_{BCDE}\frac{\lambda_{(i-1)\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\psi^{\xi B}}{\langle i, i-1\rangle^2}\eta^{C}_{i}\eta^{D}_{i}\eta^{E}_{i}$.

Fourth Order

We further calculate, using Cor. 1.3.5,

$$
\begin{aligned}
\mathcal{Q}^{\alpha}_{A}(\mathcal{E}^{3}_{i}) &= q^{\alpha}_{A}\left(\frac{1}{3c_0^3}\varepsilon_{BCDE}\frac{\lambda_{(i-1)\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\psi^{\xi B}}{\langle i,\, i-1\rangle^2}\eta^{C}_{i}\eta^{D}_{i}\eta^{E}_{i}\right) + \lambda^{\alpha}_{i-1}R^{2}_{iA}\\
&= \frac{i}{6c_0^3}\varepsilon_{ABCD}\frac{\lambda_{i\gamma}\lambda^{\gamma}_{i-1}}{\langle i\, i-1\rangle}\frac{\lambda_{(i-1)\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\beta}D^{\dot{\beta}\beta}F^{\xi\alpha}}{\langle i,\, i-1\rangle^2}\eta^{B}_{i}\eta^{C}_{i}\eta^{D}_{i} + \ldots + \lambda^{\alpha}_{i-1}R^{2}_{iA}
\end{aligned}
$$

where "..." refers to the remaining terms of $q_A^\alpha(D^{\dot\beta\gamma}\psi^{\xi B})$. By the Schouten identity, we thus find

$$\begin{aligned}
\mathcal{Q}_A^\alpha(\mathcal{E}_i^3) &= -\frac{i}{6c_0^3}\varepsilon_{ACDE}\frac{\lambda_i^\alpha\lambda_{(i-1)\gamma}\lambda_{(i-1)\xi}\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\beta}D^{\dot\beta\beta}F^{\gamma\xi}}{\langle i,\,i-1\rangle^3}\eta_i^C\eta_i^D\eta_i^E \\
&\quad+\lambda_{i-1}^\alpha\left(\frac{i}{6c_0^3}\varepsilon_{ACDE}\frac{\lambda_{i\gamma}\lambda_{(i-1)\xi}\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\beta}D^{\dot\beta\beta}F^{\gamma\xi}}{\langle i,\,i-1\rangle^3}\eta_i^C\eta_i^D\eta_i^E\right. \\
&\qquad+\frac{2g}{3c_0^3}\varepsilon_{BCDE}\frac{\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\gamma}\left[\tilde\psi_A^{\dot\beta},\,\psi^{\gamma B}\right]}{\langle i,\,i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E \\
&\qquad-\frac{2ig}{3c_0^3}\varepsilon_{BCDE}\frac{\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\gamma}\left[\overline{\phi}_{AF},\,D^{\dot\beta\gamma}\phi^{BF}\right]}{\langle i,\,i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E \\
&\qquad\left.-\frac{ig}{6c_0^3}\varepsilon_{ACDE}\frac{\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\gamma}\left[D^{\dot\beta\gamma}\phi^{FG},\,\overline{\phi}_{FG}\right]}{\langle i,\,i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E+R_{iA}^2\right) \\
&=\lambda_i^\alpha E_{iA}^3+\lambda_{i-1}^\alpha R_{iA}^3
\end{aligned}$$

Now $\frac{\partial}{\partial\eta_i^A}\left(\varepsilon_{BCDE}\eta_i^B\eta_i^C\eta_i^D\eta_i^E\right)=4\varepsilon_{ACDE}\eta_i^C\eta_i^D\eta_i^E$ holds such that $S=4$, and we find $\mathcal{E}_i^{(4)}=\frac{i}{24c_0^4}\varepsilon_{BCDE}\frac{\lambda_{(i-1)\gamma}\lambda_{(i-1)\xi}\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\beta}D^{\dot\beta\beta}F^{\gamma\xi}}{\langle i,\,i-1\rangle^3}\eta_i^B\eta_i^C\eta_i^D\eta_i^E$. This finishes the calculation of $\mathcal{E}_i$.

The Remaining Terms

To finish the calculation of R_{iA}, we calculate

$$\mathcal{Q}_A^\alpha(\mathcal{E}_i^4)=q_A^\alpha\left(\frac{i}{24c_0^4}\varepsilon_{BCDE}\frac{\lambda_{(i-1)\gamma}\lambda_{(i-1)\xi}\tilde\lambda_{i\dot\beta}\lambda_{(i-1)\beta}D^{\dot\beta\beta}F^{\gamma\xi}}{\langle i,\,i-1\rangle^3}\eta_i^B\eta_i^C\eta_i^D\eta_i^E\right)+\lambda_{i-1}^\alpha R_{iA}^3$$

To proceed, we calculate, using Cor. 1.3.5,

$$\begin{aligned}
q_A^\alpha(D^{\dot\beta\beta}F^{\gamma\xi}) &= q_A^\alpha(D^{\dot\beta\beta})F^{\gamma\xi}+D^{\dot\beta\beta}q_A^\alpha(F^{\gamma\xi}) \\
&= -ig\left[q_A^\alpha(A^{\dot\beta\beta}),\,F^{\gamma\xi}\right]+D^{\dot\beta\beta}\left(-2\epsilon^{\alpha\xi}D^\gamma{}_{\dot\gamma}\tilde\psi_A^{\dot\gamma}-2\epsilon^{\alpha\gamma}D^\xi{}_{\dot\gamma}\tilde\psi_A^{\dot\gamma}\right) \\
&\cong 2g\epsilon^{\alpha\beta}\left[\tilde\psi_A^{\dot\beta},\,F^{\gamma\xi}\right]+4\epsilon^{\alpha\xi}D^{\dot\beta\beta}D^{\gamma\dot\gamma}\tilde\psi_{\dot\gamma A}
\end{aligned}$$

where the last equation holds upon contraction with $\lambda_{(i-1)\gamma}\lambda_{(i-1)\xi}$. Using the equations of motion (Lem. 1.4.1), we thus yield

$$\begin{aligned}
&q_A^\alpha(D^{\dot\beta\beta}F^{\gamma\xi}) \\
&\quad\cong 2g\epsilon^{\alpha\beta}\left[\tilde\psi_A^{\dot\beta},\,F^{\gamma\xi}\right]+4i\sqrt2\,g\epsilon^{\alpha\xi}D^{\dot\beta\beta}\left[\overline{\phi}_{AB},\,\psi^{\gamma B}\right] \\
&\quad= 2g\epsilon^{\alpha\beta}\left[\tilde\psi_A^{\dot\beta},\,F^{\gamma\xi}\right]+4i\sqrt2\,g\epsilon^{\alpha\xi}\left[D^{\dot\beta\beta}\overline{\phi}_{AB},\,\psi^{\gamma B}\right]-4i\sqrt2\,g\epsilon^{\alpha\xi}\left[D^{\dot\beta\beta}\psi^{\gamma B},\,\overline{\phi}_{AB}\right]
\end{aligned}$$

and

$$\mathcal{Q}^{\alpha}_{A}(\mathcal{E}^4_i) = \lambda^{\alpha}_{i-1} R^3_{iA} + \frac{i\lambda^{\alpha}_{i-1}\varepsilon_{BCDE}\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}\tilde{\lambda}_{i\dot{\beta}}}{24c_0^4\,\langle i,\, i-1\rangle^3}\Big(2g\left[\tilde{\psi}^{\dot{\beta}}_{A},\, F^{\gamma\beta}\right]$$
$$+4i\sqrt{2}\,g\left[D^{\dot{\beta}\beta}\overline{\phi}_{AK},\, \psi^{\gamma K}\right] - 4i\sqrt{2}\,g\left[D^{\dot{\beta}\beta}\psi^{\gamma K},\, \overline{\phi}_{AK}\right]\Big)\eta^B_i\eta^C_i\eta^D_i\eta^E_i$$
$$= \lambda^{\alpha}_{i-1} R^4_{iA}$$

This finishes the calculation of R_{iA}.

2.2.2 Proof: Part 2 (Calculation of X^{α}_{iA})

We need to find X^{α}_{iA} such that (2.11) holds. Sorting order by order, this is equivalent to

$$\lambda^{\alpha}_{i-1}R^{(1)}_{iA} = \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}^0_i,\,\cdot\right]\right)X^{\alpha(1)}_{iA}$$
$$\lambda^{\alpha}_{i-1}R^{(2)}_{iA} = \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}^0_i,\,\cdot\right]\right)X^{\alpha(2)}_{iA} - i\left[\mathcal{E}^{(1)}_i,\, X^{\alpha(1)}_{iA}\right]$$
$$\lambda^{\alpha}_{i-1}R^{(3)}_{iA} = \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}^0_i,\,\cdot\right]\right)X^{\alpha(3)}_{iA} - i\left[\mathcal{E}^{(1)}_i,\, X^{\alpha(2)}_{iA}\right] - i\left[\mathcal{E}^{(2)}_i,\, X^{\alpha(1)}_{iA}\right]$$
$$\lambda^{\alpha}_{i-1}R^{(4)}_{iA} = \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}^0_i,\,\cdot\right]\right)X^{\alpha(4)}_{iA} - i\left[\mathcal{E}^{(1)}_i,\, X^{\alpha(3)}_{iA}\right] - i\left[\mathcal{E}^{(2)}_i,\, X^{\alpha(2)}_{iA}\right] - i\left[\mathcal{E}^{(3)}_i,\, X^{\alpha(1)}_{iA}\right]$$

First Order

From the above calculations, we collect

$$\lambda^{\alpha}_{i-1}R^{(1)}_{iA} = -\frac{i\sqrt{2}\,\lambda^{\alpha}_{i-1}}{c_0}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{i\gamma}}{\langle i,\, i-1\rangle}D^{\dot{\beta}\gamma}\overline{\phi}_{AB}\eta^B_i$$
$$= -\frac{2i\sqrt{2}\,\lambda^{\alpha}_{i-1}}{c_0\,\langle i,\, i-1\rangle}\frac{1}{2}\tilde{\lambda}_{i\dot{\beta}}\lambda_{i\beta}\left(\partial^{\dot{\beta}\beta} - ig\left[A^{\dot{\beta}\beta},\,\cdot\right]\right)\overline{\phi}_{AB}\,\eta^B_i$$
$$= \left(\partial_t - ig\left[\frac{1}{2}\tilde{\lambda}_{i\dot{\beta}}\lambda_{i\beta}A^{\beta\dot{\beta}},\,\cdot\right]\right)\left(-\frac{2i\sqrt{2}\,\lambda^{\alpha}_{i-1}}{c_0\,\langle i,\, i-1\rangle}\overline{\phi}_{AB}\eta^B_i\right)$$
$$= \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}^0_i,\,\cdot\right]\right)\left(-\frac{2ig\sqrt{2}\,\lambda^{\alpha}_{i-1}}{c_0\,\langle i,\, i-1\rangle}\overline{\phi}_{AB}\eta^B_i\right)$$
$$=: \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}^0_i,\,\cdot\right]\right)X^{\alpha(1)}_{iA}$$

where we used $\frac{1}{2}\tilde{\lambda}_i\lambda_i D = (\partial_t - ig[\mathcal{E}^0_i,\cdot])$, thus fixing $X^{\alpha(1)}_{iA}$.

Second Order

From the above calculation, we deduce

$$\lambda^{\alpha}_{i-1}R^{(2)}_{iA} = -\frac{ig\sqrt{2}}{c_0^2}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda^{\alpha}_{(i-1)}\left[\tilde{\psi}^{\dot{\beta}}_{A},\,\overline{\phi}_{BC}\right]}{\langle i,\, i-1\rangle}\eta^B_i\eta^C_i$$
$$+ \frac{\lambda^{\alpha}_{(i-1)}}{c_0^2}\varepsilon_{ABCD}\frac{\lambda_{i\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\psi^{\xi D}}{\langle i,\, i-1\rangle^2}\eta^B_i\eta^C_i$$

We transform the second term by the Schouten identity and the equations of motion (Lem. 1.4.1):

$$\begin{aligned}\lambda_{i\xi}\tilde{\lambda}_{i\dot{\beta}}(\lambda_{(i-1)\gamma}D^{\dot{\beta}\gamma}\psi^{\xi D}) &= \lambda_{i\xi}\tilde{\lambda}_{i\dot{\beta}}(\lambda^{\xi}_{(i-1)}D^{\dot{\beta}\gamma}\psi^{D}_{\gamma}) + \lambda_{i\xi}\tilde{\lambda}_{i\dot{\beta}}(\lambda_{(i-1)\gamma}D^{\dot{\beta}\xi}\psi^{\gamma D}) \\ &= i\sqrt{2}\,g\,\langle i,\, i-1\rangle\,\tilde{\lambda}_{i\dot{\beta}}\left[\phi^{DK},\,\tilde{\psi}^{\dot{\beta}}_{K}\right] + \lambda_{i\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\xi}\psi^{\gamma D}\end{aligned}$$

In the next step, we combine the two terms with a commutator to obtain

$$\begin{aligned}&\frac{ig\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left(\left[\overline{\phi}_{BC},\,\tilde{\psi}^{\dot{\beta}}_{A}\right] + \varepsilon_{ABCD}\left[\phi^{DK},\,\tilde{\psi}^{\dot{\beta}}_{K}\right]\right)\eta_i^B\eta_i^C \\ &\quad= \frac{ig\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left(\left[\overline{\phi}_{BC},\,\tilde{\psi}^{\dot{\beta}}_{A}\right] + \frac{1}{2}\varepsilon_{ABCD}\varepsilon_{DKLM}\left[\overline{\phi}_{LM},\,\tilde{\psi}^{\dot{\beta}}_{K}\right]\right)\eta_i^B\eta_i^C \\ &\quad= \frac{ig\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left[\overline{\phi}_{LM},\,\tilde{\psi}^{\dot{\beta}}_{K}\right]\left(\delta^{KLM}_{ABC} - \frac{1}{2}\varepsilon_{DABC}\varepsilon_{DKLM}\right)\eta_i^B\eta_i^C\end{aligned}$$

Further using (1.14), we obtain for the commutator terms

$$\begin{aligned}&\frac{ig\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left[\overline{\phi}_{LM},\,\tilde{\psi}^{\dot{\beta}}_{K}\right]\left(\delta^{KLM}_{ABC} - \delta^{KLM}_{ABC} - \delta^{KLM}_{BCA} - \delta^{KLM}_{CAB}\right)\eta_i^B\eta_i^C \\ &\quad= \frac{ig\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left[\tilde{\psi}^{\dot{\beta}}_{K},\,\overline{\phi}_{LM}\right]\left(\delta^{KLM}_{BCA} + \delta^{KLM}_{CAB}\right)\eta_i^B\eta_i^C \\ &\quad= \frac{ig\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}(\left[\tilde{\psi}^{\dot{\beta}}_{B},\,\overline{\phi}_{CA}\right] + \left[\tilde{\psi}^{\dot{\beta}}_{C},\,\overline{\phi}_{AB}\right])\eta_i^B\eta_i^C \\ &\quad= -\frac{2ig\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left[\tilde{\psi}^{\dot{\beta}}_{B},\,\overline{\phi}_{AC}\right]\eta_i^B\eta_i^C\end{aligned}$$

Therefore

$$\begin{aligned}\lambda^{\alpha}_{i-1}R^{(2)}_{iA} &= -\frac{2ig\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left[\tilde{\psi}^{\dot{\beta}}_{B},\,\overline{\phi}_{AC}\right]\eta_i^B\eta_i^C \\ &\qquad + \frac{\lambda^{\alpha}_{(i-1)}}{c_0^2}\varepsilon_{ABCD}\frac{\lambda_{i\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}D^{\dot{\beta}\xi}\psi^{\gamma D}}{\langle i,\, i-1\rangle^2}\eta_i^B\eta_i^C \\ &= \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}_i^0,\,\cdot\right]\right)\left(\varepsilon_{ABCD}\frac{2g\lambda^{\alpha}_{(i-1)}\lambda_{(i-1)\gamma}\psi^{\gamma B}}{c_0^2\,\langle i,\, i-1\rangle^2}\eta_i^C\eta_i^D\right) \\ &\qquad - i\left(\frac{2g\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left[\tilde{\psi}^{\dot{\beta}}_{B},\,\overline{\phi}_{AC}\right]\eta_i^B\eta_i^C\right) \\ &=: \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}_i^0,\,\cdot\right]\right)(X^{\alpha(2)}_{iA}) - i\left(\frac{2g\sqrt{2}\lambda^{\alpha}_{(i-1)}\tilde{\lambda}_{i\dot{\beta}}}{c_0^2\,\langle i,\, i-1\rangle}\left[\tilde{\psi}^{\dot{\beta}}_{B},\,\overline{\phi}_{AC}\right]\eta_i^B\eta_i^C\right) \\ &= \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}_i^0,\,\cdot\right]\right)(X^{\alpha(2)}_{iA}) - i\left(\left[\frac{i}{c_0}\tilde{\lambda}_{i\dot{\beta}}\tilde{\psi}^{\dot{\beta}}_{B}\eta_i^B,\, -\frac{2ig\sqrt{2}\,\lambda^{\alpha}_{i-1}}{c_0\,\langle i,\, i-1\rangle}\overline{\phi}_{AC}\eta_i^C\right]\right) \\ &= \frac{1}{g}\left(\partial_t - ig\left[\mathcal{E}_i^0,\,\cdot\right]\right)(X^{\alpha(2)}_{iA}) - i\left(\left[\mathcal{E}_i^{(1)},\,X^{\alpha(1)}_{iA}\right]\right)\end{aligned}$$

This fixes $X^{\alpha(2)}_{iA}$, and at the same time provides a consistency check for $X^{\alpha(1)}_{iA}$.

Third Order

From the above calculation, we deduce

$$\begin{aligned}
\lambda^{\alpha}_{i-1}R^{(3)}_{iA} = \lambda^{\alpha}_{i-1}\Bigg(& \frac{i}{6c_0^3}\varepsilon_{ACDE}\frac{\lambda_{i\gamma}\lambda_{(i-1)\xi}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\beta}D^{\dot{\beta}\beta}F^{\gamma\xi}}{\langle i,\, i-1\rangle^3}\eta_i^C\eta_i^D\eta_i^E \\
& + \frac{2g}{3c_0^3}\varepsilon_{BCDE}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}\left[\tilde{\psi}^{\dot{\beta}}_{A},\, \psi^{\gamma B}\right]}{\langle i,\, i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E \\
& - \frac{2ig}{3c_0^3}\varepsilon_{BCDE}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}\left[\overline{\phi}_{AF},\, D^{\dot{\beta}\gamma}\phi^{BF}\right]}{\langle i,\, i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E \\
& - \frac{ig}{6c_0^3}\varepsilon_{ACDE}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}\left[D^{\dot{\beta}\gamma}\phi^{FG},\, \overline{\phi}_{FG}\right]}{\langle i,\, i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E\Bigg) \\
=: (1) + (2) + (3) + (4) &
\end{aligned}$$

Consider the first term. We use the Schouten identity and Cor. 1.4.2 to obtain

$$\begin{aligned}
&\lambda_{i\gamma}\lambda_{(i-1)\beta}D^{\dot{\beta}\beta}F^{\gamma\xi} \\
&\quad = \lambda_{i\beta}\lambda_{(i-1)\gamma}D^{\dot{\beta}\beta}F^{\gamma\xi} - \lambda_{i\beta}\lambda^{\beta}_{i-1}D^{\dot{\beta}}{}_{\gamma}F^{\gamma\xi} \\
&\quad = \lambda_{i\beta}\lambda_{(i-1)\gamma}D^{\dot{\beta}\beta}F^{\gamma\xi} + \lambda_{i\beta}\lambda^{\beta}_{i-1}D^{\dot{\beta}\gamma}F_{\gamma}{}^{\xi} \\
&\quad = \lambda_{i\beta}\lambda_{(i-1)\gamma}D^{\dot{\beta}\beta}F^{\gamma\xi} + \langle i,\, i-1\rangle\, g\left[D^{\xi\dot{\beta}}\phi^{AB},\, \overline{\phi}_{AB}\right] + 4i\,\langle i,\, i-1\rangle\, g\left[\tilde{\psi}^{\dot{\beta}}_{A},\, \psi^{\xi A}\right]
\end{aligned}$$

and thus

$$\begin{aligned}
(1) = & \frac{i}{6c_0^3}\varepsilon_{ACDE}\frac{\lambda^{\alpha}_{i-1}\tilde{\lambda}_{i\dot{\beta}}\lambda_{i\beta}\lambda_{(i-1)\xi}\lambda_{(i-1)\gamma}D^{\dot{\beta}\beta}F^{\gamma\xi}}{\langle i,\, i-1\rangle^3}\eta_i^C\eta_i^D\eta_i^E \\
& + \frac{ig}{6c_0^3}\varepsilon_{ACDE}\frac{\lambda^{\alpha}_{i-1}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\xi}\left[D^{\xi\dot{\beta}}\phi^{FG},\, \overline{\phi}_{FG}\right]}{\langle i,\, i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E \\
& - \frac{2g}{3c_0^3}\varepsilon_{ACDE}\frac{\lambda^{\alpha}_{i-1}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\xi}\left[\tilde{\psi}^{\dot{\beta}}_{F},\, \psi^{\xi F}\right]}{\langle i,\, i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E \\
=: & (1a) + (1b) + (1c)
\end{aligned}$$

We identify the first term thereof with

$$\begin{aligned}
(1a) &= \frac{i}{3c_0^3}\varepsilon_{ACDE}\frac{\lambda^{\alpha}_{i-1}(\partial_t - ig\left[\mathcal{E}^0_i,\, \cdot\right])\lambda_{(i-1)\xi}\lambda_{(i-1)\gamma}F^{\gamma\xi}}{\langle i,\, i-1\rangle^3}\eta_i^C\eta_i^D\eta_i^E \\
&= \frac{1}{g}(\partial_t - ig\left[\mathcal{E}^0_i,\, \cdot\right])(X^{\alpha(3)}_{iA})
\end{aligned}$$

This fixes $X_{iA}^{\alpha(3)}$. Now (4) cancels with (1b), and we arrive at

$$\begin{aligned}\lambda_{i-1}^{\alpha}R_{iA}^{(3)}-(1a)=&-\frac{2g}{3c_0^3}\varepsilon_{ACDE}\frac{\lambda_{i-1}^{\alpha}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\xi}\left[\tilde{\psi}_F^{\dot{\beta}},\,\psi^{\xi F}\right]}{\langle i,\,i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E\\&+\frac{2g}{3c_0^3}\varepsilon_{BCDE}\frac{\lambda_{i-1}^{\alpha}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}\left[\tilde{\psi}_A^{\dot{\beta}},\,\psi^{\gamma B}\right]}{\langle i,\,i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E\\&-\frac{2ig}{3c_0^3}\varepsilon_{BCDE}\frac{\lambda_{i-1}^{\alpha}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}\left[\overline{\phi}_{AF},\,D^{\dot{\beta}\gamma}\phi^{BF}\right]}{\langle i,\,i-1\rangle^2}\eta_i^C\eta_i^D\eta_i^E\\=:&\,(i)+(ii)+(iii)\end{aligned}$$

It remains to show that this equals the sum of the commutators $\sim[\eta,\eta\eta]+[\eta\eta,\eta]$ of $-i[\mathcal{E},X]$. We show that the first two terms can be combined into the form required. In the first equation, we use that (with $ACDE$ fixed) F can obtain exactly the values A and C because otherwise η's cancel.

$$\begin{aligned}&(\tilde{\psi}_F\eta^F)(\varepsilon_{ACDE}\psi^C\eta^D\eta^E)\\&=\varepsilon_{ACDE}\tilde{\psi}_A\eta^A\psi^C\eta^D\eta^E+\varepsilon_{ACDE}\tilde{\psi}_C\eta^C\psi^C\eta^D\eta^E\\&=(\tilde{\psi}_A\psi^C)\varepsilon_{CADE}\eta^A\eta^D\eta^E-(\tilde{\psi}_C\psi^C)\varepsilon_{ACDE}\eta^C\eta^D\eta^E\\&=\frac{1}{3}\sum\nolimits_{C\neq A}(\tilde{\psi}_A\psi^C)\varepsilon_{CKDE}\eta^K\eta^D\eta^E-\frac{1}{3}\sum\nolimits_{F\neq A}(\tilde{\psi}_F\psi^F)\varepsilon_{ACDE}\eta^C\eta^D\eta^E\\&=\frac{1}{3}\sum\nolimits_{C\neq A}(\tilde{\psi}_A\psi^C)\varepsilon_{CKDE}\eta^K\eta^D\eta^E\\&\qquad+\frac{1}{3}(\tilde{\psi}_A\psi^A)\varepsilon_{ACDE}\eta^C\eta^D\eta^E-\frac{1}{3}(\tilde{\psi}_F\psi^F)\varepsilon_{ACDE}\eta^C\eta^D\eta^E\\&=\frac{1}{3}(\tilde{\psi}_A\psi^C)\varepsilon_{CKDE}\eta^K\eta^D\eta^E-\frac{1}{3}(\tilde{\psi}_F\psi^F)\varepsilon_{ACDE}\eta^C\eta^D\eta^E\end{aligned}$$

We thus conclude that

$$\begin{aligned}(i)+(ii)&=\frac{2g}{c_0^3}\frac{\lambda_{i-1}^{\alpha}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\xi}}{\langle i,\,i-1\rangle^2}\left[\tilde{\psi}_F^{\dot{\beta}}\eta_i^F,\,\varepsilon_{ACDE}\psi^{\xi C}\eta_i^D\eta_i^E\right]\\&=-i\left[\frac{i}{c_0}\tilde{\lambda}_{i\dot{\beta}}\tilde{\psi}_F^{\dot{\beta}}\eta_i^F,\,\frac{2g\lambda_{i-1}^{\alpha}\lambda_{(i-1)\xi}}{c_0^2\,\langle i,\,i-1\rangle^2}\varepsilon_{ACDE}\psi^{\xi C}\eta_i^D\eta_i^E\right]\\&=-i\left[\mathcal{E}_i^{(1)},\,X_{iA}^{\alpha(2)}\right]\end{aligned}$$

as it should be. We finally show that the last term has the form required.

$$\begin{aligned}\varepsilon_{BCDE}\overline{\phi}_{AF}D\phi^{BF}\eta^C\eta^D\eta^E&=\frac{1}{2}\varepsilon_{BCDE}\varepsilon_{BFKL}\overline{\phi}_{AF}D\overline{\phi}_{KL}\eta^C\eta^D\eta^E\\&=3\delta_{FKL}^{CDE}\overline{\phi}_{AF}D\overline{\phi}_{KL}\eta^C\eta^D\eta^E\\&=3\overline{\phi}_{AF}D\overline{\phi}_{KL}\eta^F\eta^K\eta^L\\&=3(\overline{\phi}_{AF}\eta^F)(D\overline{\phi}_{KL}\eta^K\eta^L)\end{aligned}$$

Therefore

$$\begin{aligned}(iii) &= -\frac{2ig}{c_0^3}\frac{\lambda^\alpha_{i-1}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}}{\langle i,\, i-1\rangle^2}\left[\overline{\phi}_{AF}\eta_i^F,\, D^{\dot{\beta}\gamma}\overline{\phi}_{KL}\eta_i^K\eta_i^L\right]\\ &= -i\left[-\frac{i\sqrt{2}}{2c_0^2}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}}{\langle i,\, i-1\rangle}D^{\dot{\beta}\gamma}\overline{\phi}_{KL}\eta_i^K\eta_i^L,\, -\frac{2i\sqrt{2}\,g\lambda^\alpha_{i-1}\overline{\phi}_{AF}\eta_i^F}{c_0\,\langle i,\, i-1\rangle}\right]\\ &= -i\left[\mathcal{E}_i^{(2)},\, X_{iA}^{\alpha(1)}\right]\end{aligned}$$

This concludes the verification to third order, thus providing a consistency check for $X_{iA}^{\alpha(1)}$ and $X_{iA}^{\alpha(2)}$.

Fourth Order

From the above calculation, we deduce

$$\begin{aligned}\lambda^\alpha_{i-1}R_{iA}^{(4)} &= \frac{i\lambda^\alpha_{i-1}\varepsilon_{BCDE}\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}\tilde{\lambda}_{i\dot{\beta}}}{24c_0^4\,\langle i,\, i-1\rangle^3}\Big(2g\left[\tilde{\psi}_A^{\dot{\beta}},\, F^{\gamma\beta}\right]\\ &\quad +4i\sqrt{2}\,g\left[D^{\dot{\beta}\beta}\overline{\phi}_{AK},\, \psi^{\gamma K}\right]-4i\sqrt{2}\,g\left[D^{\dot{\beta}\beta}\psi^{\gamma K},\, \overline{\phi}_{AK}\right]\Big)\,\eta_i^B\eta_i^C\eta_i^D\eta_i^E\\ &=: (1)+(2)+(3)\end{aligned}$$

In the following, we show that each of these three terms equals one of the three terms in $-i\left[\mathcal{E}_i,\, X^\alpha_{iA}\right]|_{\eta^4}$. This fixes $X_{iA}^{\alpha(4)}=0$. We calculate (no sum over A!):

$$\begin{aligned}\tilde{\psi}_A\,\varepsilon_{BCDE}\,\eta^B\eta^C\eta^D\eta^E &= 4\tilde{\psi}_A\,\varepsilon_{ACDE}\,\eta^A\eta^C\eta^D\eta^E\\ &= 4(\tilde{\psi}_A\eta^A)(\varepsilon_{ACDE}\,\eta^C\eta^D\eta^E)\\ &= 4(\tilde{\psi}_F\eta^F)(\varepsilon_{ACDE}\,\eta^C\eta^D\eta^E)\end{aligned}$$

where the last equation holds since, for CDE fixed, the value of F is required to be A since otherwise the η terms vanish. Therefore

$$\begin{aligned}(1) &= \frac{i\lambda^\alpha_{i-1}\varepsilon_{BCDE}\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}\tilde{\lambda}_{i\dot{\beta}}}{24c_0^4\,\langle i,\, i-1\rangle^3}2g\left[\tilde{\psi}_A^{\dot{\beta}},\, F^{\gamma\beta}\right]\eta_i^B\eta_i^C\eta_i^D\eta_i^E\\ &= \frac{ig\lambda^\alpha_{i-1}\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}\tilde{\lambda}_{i\dot{\beta}}}{3c_0^4\,\langle i,\, i-1\rangle^3}\left[\tilde{\psi}_F^{\dot{\beta}}\eta_i^F,\, F^{\gamma\beta}\varepsilon_{ACDE}\,\eta_i^C\eta_i^D\eta_i^E\right]\\ &= -i\left[\frac{i}{c_0}\tilde{\lambda}_{i\dot{\beta}}\tilde{\psi}_F^{\dot{\beta}}\eta_i^F,\, \frac{ig\lambda^\alpha_{i-1}}{3c_0^3}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}F^{\gamma\beta}}{\langle i,\, i-1\rangle^3}\eta_i^B\eta_i^C\eta_i^D\right]\\ &= -i\left[\mathcal{E}_i^{(1)},\, X_{iA}^{\alpha(3)}\right]\end{aligned}$$

As for the second term, we similarly calculate (no sum over A and K):

$$\begin{aligned}\overline{\phi}_{AK}\varepsilon_{BCDE}\eta^B\eta^C\eta^D\eta^E &= 4\overline{\phi}_{AK}\varepsilon_{ACDE}\eta^A\eta^C\eta^D\eta^E\\ &= 12\overline{\phi}_{AK}\varepsilon_{AKDE}\eta^A\eta^K\eta^D\eta^E\\ &= 12(\overline{\phi}_{AK}\eta^A\eta^K)(\varepsilon_{AKDE}\eta^D\eta^E)\\ &= 6(\overline{\phi}_{LM}\eta^L\eta^M)(\varepsilon_{AKDE}\eta^D\eta^E)\end{aligned}$$

Therefore

$$
\begin{aligned}
(2) &= \frac{i\lambda^{\alpha}_{i-1}\varepsilon_{BCDE}\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}\tilde{\lambda}_{i\dot{\beta}}}{24c_0^4\left\langle i,\, i-1\right\rangle^3} 4i\sqrt{2}\, g \left[D^{\dot{\beta}\beta}\overline{\phi}_{AK},\, \psi^{\gamma K}\right]\eta_i^B\eta_i^C\eta_i^D\eta_i^E \\
&= -\frac{\sqrt{2}\, g\lambda^{\alpha}_{i-1}\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}\tilde{\lambda}_{i\dot{\beta}}}{c_0^4\left\langle i,\, i-1\right\rangle^3}\left[D^{\dot{\beta}\beta}\overline{\phi}_{LM}\eta^L\eta^M,\, \psi^{\gamma K}\varepsilon_{AKDE}\eta^D\eta^E\right] \\
&= -i\left[-\frac{i\sqrt{2}}{2c_0^2}\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\beta}D^{\dot{\beta}\beta}\phi_{LM}}{\left\langle i,\, i-1\right\rangle}\eta_i^L\eta_i^M,\, \frac{2g\lambda^{\alpha}_{i-1}\lambda_{(i-1)\gamma}\psi^{\gamma K}}{c_0^2\left\langle i,\, i-1\right\rangle^2}\varepsilon_{AKDE}\eta^D\eta^E\right] \\
&= -i\left[\mathcal{E}_i^{(2)},\, X_{iA}^{\alpha(2)}\right]
\end{aligned}
$$

Moreover (no sum over A and K):

$$
\begin{aligned}
\psi^K\overline{\phi}_{AK}\varepsilon_{BCDE}\eta^B\eta^C\eta^D\eta^E &= 4\psi^K\overline{\phi}_{AK}\varepsilon_{KCDE}\eta^K\eta^C\eta^D\eta^E \\
&= -4(\varepsilon_{KCDE}\psi^K\eta^C\eta^D\eta^E)(\overline{\phi}_{AK}\eta^K) \\
&= -4(\varepsilon_{KCDE}\psi^K\eta^C\eta^D\eta^E)(\overline{\phi}_{AL}\eta^L)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(3) &= -\frac{i\lambda^{\alpha}_{i-1}\varepsilon_{BCDE}\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}\tilde{\lambda}_{i\dot{\beta}}}{24c_0^4\left\langle i,\, i-1\right\rangle^3} 4i\sqrt{2}\, g \left[D^{\dot{\beta}\beta}\psi^{\gamma K},\, \overline{\phi}_{AK}\right]\eta_i^B\eta_i^C\eta_i^D\eta_i^E \\
&= -\frac{2\sqrt{2}\, g\lambda^{\alpha}_{i-1}\lambda_{(i-1)\gamma}\lambda_{(i-1)\beta}\tilde{\lambda}_{i\dot{\beta}}}{3c_0^4\left\langle i,\, i-1\right\rangle^3}\left[\varepsilon_{KCDE}D^{\dot{\beta}\beta}\psi^{\gamma K}\eta_i^C\eta_i^D\eta_i^E,\, \overline{\phi}_{AL}\eta_i^L\right] \\
&= -i\left[\frac{1}{3c_0^3}\varepsilon_{ABCD}\frac{\lambda_{(i-1)\gamma}\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\beta}D^{\dot{\beta}\beta}\psi^{\gamma A}}{\left\langle i,\, i-1\right\rangle^2}\eta_i^B\eta_i^C\eta_i^D,\, -\frac{2i\sqrt{2}\, g\lambda^{\alpha}_{i-1}}{c_0\left\langle i,\, i-1\right\rangle}\overline{\phi}_{AL}\eta_i^L\right] \\
&= -i\left[\mathcal{E}_i^{(3)},\, X_{iA}^{\alpha(1)}\right]
\end{aligned}
$$

This concludes the verification to fourth order, thus providing a consistency check for $X_{iA}^{\alpha(1)}$ and $X_{iA}^{\alpha(2)}$ and $X_{iA}^{\alpha(3)}$ and fixing $X_{iA}^{\alpha(4)} = 0$.

2.3 Derivation of Vertex Operators

For the vertex operators, we make the ansatz $\mathcal{V}_{i,i+1} = 1 + \mathcal{O}(\eta)$ and require that it only depends on the generators η_i and η_{i+1}. By (requested) construction, $\mathcal{V}_{i,i+1}$ then consists of terms with maximal η order $\sim \eta_i^4\eta_{i+1}^4$. We can thus expand

$$
\mathcal{V}_{i,i+1} = \sum\nolimits_{k=0}^{4}\sum\nolimits_{l=0}^{4} V_{A_1\ldots A_k\; B_1\ldots B_l}\,\eta_i^{A_1}\ldots\eta_i^{A_k}\eta_{i+1}^{B_1}\ldots\eta_{i+1}^{B_l} \tag{2.12}
$$

and aim at a result in this form. Similarly, we denote the coefficients of X_{iA}^{α} by

$$
X_{iA}^{\alpha} = X_{iA}^{\alpha(1)} + X_{iA}^{\alpha(2)} + X_{iA}^{\alpha(3)} = X_{iAA_1}^{\alpha(1)}\eta_i^{A_1} + X_{iAA_1A_2}^{\alpha(2)}\eta_i^{A_1}\eta_i^{A_2} + X_{iAA_1A_2A_3}^{\alpha(3)}\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A_3}
$$

Proposition 2.3.1. Let $V_0 = 1$ (i.e. $\mathcal{V}_{i,i+1} = 1 + \mathcal{O}(\eta)$) and require that $\mathcal{V}_{i\,i+1}$ only depends on the generators η_i and η_{i+1}. Then (2.9b) with X_{iA}^{α} as in Thm. 2.2.1 has the following unique solution: All coefficients $V_{B_1,\dots,B_d} = 0$ for $d > 0$ (i.e. all "pure η_{i+1}-terms") vanish and the remaining coefficients are determined by the following recursion formula.

$$\begin{aligned}
&V_{A\,A_1\dots A_k\,B_1\dots B_l} \\
&\quad= \frac{(-1)^{d+1}\lambda_{(i+1)\alpha}}{(k+1)c_0\,\langle i+1,\,i\rangle}\Big(-q_A^\alpha(V_{A_1\dots A_k\,B_1\dots B_l}) + iX^{\alpha(1)}_{(i+1)AB_l}V_{A_1\dots A_k\,B_1\dots B_{l-1}} \\
&\qquad +iX^{\alpha(2)}_{(i+1)AB_{l-1}B_l}V_{A_1\dots A_k\,B_1\dots B_{l-2}} + iX^{\alpha(3)}_{(i+1)AB_{l-2}B_{l-1}B_l}V_{A_1\dots A_k\,B_1\dots B_{l-3}} \\
&\qquad -i(-1)^l V_{A_1\dots A_{k-1}\,B_1\dots B_l}X^{\alpha(1)}_{iAA_k} - i(-1)^d V_{A_1\dots A_{k-2}\,B_1\dots B_l}X^{\alpha(2)}_{iAA_{k-1}A_k} \\
&\qquad -i(-1)^l V_{A_1\dots A_{k-3}\,B_1\dots B_l}X^{\alpha(3)}_{iAA_{k-2}A_{k-1}A_k}\Big)
\end{aligned}$$

where $d = k + l$, and we use the notation (2.10).

Proof. For calculations, it is easier to work with an expansion where the generators η_i and η_{i+1} can stand in any order:

$$\begin{aligned}
\mathcal{V}_{i,i+1} &= \sum\nolimits_{d=0}^{8} C^{j_1\dots j_d}_{B_1\dots B_d}\eta^{B_1}_{j_1}\dots\eta^{B_d}_{j_d} \\
&= 1 + C^j_B\eta^B_j + C^{jk}_{BC}\eta^B_j\eta^C_k + C^{jkl}_{BCD}\eta^B_j\eta^C_k\eta^D_l + C^{jklm}_{BCDE}\eta^B_j\eta^C_k\eta^D_l\eta^E_m + \dots
\end{aligned}$$

with $j_i \in \{i, i+1\}$ and $B_i \in \{1,2,3,4\}$. By construction, the coefficient $C^{j_1\dots j_d}_{B_1\dots B_d}$ is totally antisymmetric with respect to index pairs ${}^{j_i}_{B_i}$. Now consider terms with k times an η_i and l times an η_{i+1}. There are $\binom{k+l}{k}$ possibilities to have k η_i-terms within a set of $k+l$ η-terms, and thus

$$\begin{aligned}
V_{A_1\dots A_k\;B_1\dots B_l}\,\eta^{A_1}_i\dots\eta^{A_k}_i\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1} &= \sum\nolimits_{\substack{\#\{j_m=i\}=k \\ \#\{j_m=i+1\}=l}} C^{j_1\dots j_{k+l}}_{B_1\dots B_{k+l}}\eta^{B_1}_{j_1}\dots\eta^{B_{k+l}}_{j_{k+l}} \\
&= \binom{k+l}{k} C^{i\;\;\dots i\;\;i+1\dots i+1}_{A_1\dots A_k B_1\;\dots B_l}\,\eta^{A_1}_i\dots\eta^{A_k}_i\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1}
\end{aligned}$$

or, equivalently,

$$V_{A_1\dots A_k\;B_1\dots B_l} = \binom{k+l}{k} C^{i\;\;\dots i\;\;i+1\dots i+1}_{A_1\dots A_k B_1\;\dots B_l} \tag{2.13}$$

Now, applying from the left a fixed $\frac{\partial}{\partial\eta^A_k}$ in the C-expansion kills the corresponding η terms which can occur at every position, thus giving a symmetry factor of d and a sign. Therefore

$$\begin{aligned}
\Big(c_0\sum\nolimits_k \lambda^\alpha_k\frac{\partial}{\partial\eta^A_k}\Big)(\mathcal{V}_{i,i+1}) &= c_0\sum\nolimits_{d=1}^{8} d\,(-1)^{\left|C^{kj_1\dots j_{d-1}}_{AB_1\dots B_{d-1}}\right|}\lambda^\alpha_k\,C^{kj_1\dots j_{d-1}}_{AB_1\dots B_{d-1}}\,\eta^{B_1}_{j_1}\dots\eta^{B_{d-1}}_{j_{d-1}} \\
&= c_0\sum\nolimits_{d=0}^{7}(d+1)\,(-1)^{\left|C^{kj_1\dots j_d}_{AB_1\dots B_d}\right|}\lambda^\alpha_k\,C^{kj_1\dots j_d}_{AB_1\dots B_d}\,\eta^{B_1}_{j_1}\dots\eta^{B_d}_{j_d}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}^\alpha_A(\mathcal{V}_{i,i+1}) &= \sum\nolimits_{d=0}^{8}\Big(q^\alpha_A(C^{j_1\dots j_d}_{B_1\dots B_d}) + c_0(d+1)\,(-1)^{\left|C^{kj_1\dots j_d}_{AB_1\dots B_d}\right|}\lambda^\alpha_k\,C^{kj_1\dots j_d}_{AB_1\dots B_d}\Big)\,\eta^{B_1}_{j_1}\dots\eta^{B_d}_{j_d} \\
&= c_0(-1)^{|C^k_A|}\lambda^\alpha_k C^k_A + \Big(q^\alpha_A(C^j_B) + 2c_0(-1)^{|C^{kj}_{AB}|}\lambda^\alpha_k C^{kj}_{AB}\Big)\,\eta^B_j + \mathcal{O}(\eta^2)
\end{aligned}$$

with the implicit understanding that C "with too many indices" vanishes.

We now show by induction that the coefficients are of parity $\left|C^{j_1\ldots j_d}_{B_1\ldots B_d}\right| \equiv_2 d$: The base case $d=0$ is already established. Restricting to $\sim \eta^d$, the previous calculation shows that (2.9b) is equivalent to the recursion formula

$$\begin{aligned}
&c_0(d+1)\,(-1)^{\left|C^{kj_1\ldots j_d}_{AB_1\ldots B_d}\right|}\lambda^\alpha_k\,C^{kj_1\ldots j_d}_{AB_1\ldots B_d}\eta^{B_1}_{j_1}\ldots\eta^{B_d}_{j_d}\\
&\quad= -q^\alpha_A(C^{j_1\ldots j_d}_{B_1\ldots B_d})\eta^{B_1}_{j_1}\ldots\eta^{B_d}_{j_d} + (iX^\alpha_{i+1\,A}\mathcal{V}_{i,i+1} - i\mathcal{V}_{i,i+1}X^\alpha_{iA})|_{\eta^d}\\
&\quad= -q^\alpha_A(C^{j_1\ldots j_d}_{B_1\ldots B_d})\eta^{B_1}_{j_1}\ldots\eta^{B_d}_{j_d} + i\sum\nolimits_{k+l=d}(X^\alpha_{i+1\,A}|_{\eta^k}\mathcal{V}_{i,i+1}|_{\eta^l} - \mathcal{V}_{i,i+1}|_{\eta^k}X^\alpha_{iA}|_{\eta^l})
\end{aligned}$$

By induction hypothesis, $C^{j_1\ldots j_d}_{B_1\ldots B_d}$ has parity d and, therefore, q^α_A applied to it has parity $d+1$ as it should be. We further note that the η-coefficient of $X^\alpha_{i+1\,A}|_{\eta^k}$ has parity $k+1$ and the coefficient of $\mathcal{V}_{i,i+1}|_{\eta^l}$ has parity l (again by induction hypothesis), and analogous for the last summand, such that the sum term on the right hand side also has parity $k+1+l=d+1$. Therefore, the left hand side has parity $d+1$ which was to be shown.

Also by induction, we see that that all coefficients $C^{i+1\ldots i+1}_{B_1\ldots B_d} = 0$ vanish: In the recursion formula so far established, we consider the case $j_1 = \ldots j_d = i+1$ and multiply both sides with $\lambda_{i\alpha}$. Then only the left hand side with $k = i+1$ remains and

$$C^{i+1,i+1\ldots i+1}_{AB_1\ldots B_d}\eta^{B_1}_{i+1}\ldots\eta^{B_d}_{i+1} = \frac{(-1)^{d+1}\lambda_{i\alpha}}{\langle i,\, i+1\rangle\, c_0(d+1)}\left(-q^\alpha_A(C^{i+1\ldots i+1}_{B_1\ldots B_d})\eta^{B_1}_{i+1}\ldots\eta^{B_d}_{i+1}\right)$$

since $\lambda_{i\alpha}X^\alpha_{i+1\,A} = 0$ and $X^\alpha_{iA} = \mathcal{O}(\eta_i)$. For $d=0$, the right hand side $\sim q^\alpha_A(1) = 0$ vanishes and thus $C^{i+1}_B = 0$. Take this as induction basis and assume that $C^{i+1\ldots i+1}_{B_1\ldots B_d} = 0$. The same recursion formula then implies that $C^{i+1,i+1\ldots i+1}_{AB_1\ldots B_d} = 0$, thus proving the claim.

Now, by multiplying both sides of the recursion formula with $\lambda_{(i+1)\alpha}$, only the left hand side with $k = i$ remains and

$$\begin{aligned}
&C^{ij_1\ldots j_d}_{AB_1\ldots B_d}\eta^{B_1}_{j_1}\ldots\eta^{B_d}_{j_d}\\
&\quad= \frac{(-1)^{d+1}\lambda_{(i+1)\alpha}}{\langle i+1,\, i\rangle\, c_0(d+1)}\left(-q^\alpha_A(C^{j_1\ldots j_d}_{B_1\ldots B_d})\eta^{B_1}_{j_1}\ldots\eta^{B_d}_{j_d} + (iX^\alpha_{i+1\,A}\mathcal{V}_{i,i+1} - i\mathcal{V}_{i,i+1}X^\alpha_{iA})|_{\eta^d}\right)
\end{aligned}$$

where, with the parities already established,

$$\begin{aligned}
(X^\alpha_{i+1\,A}\mathcal{V}_{i,i+1})|_{\eta^d} &= (X^{\alpha(1)}_{(i+1)AC}\eta^C_{i+1})(C^{j_1\ldots j_{d-1}}_{B_1\ldots B_{d-1}}\eta^{B_1}_{j_1}\ldots\eta^{B_{d-1}}_{j_{d-1}})\\
&\quad+ (X^{\alpha(2)}_{(i+1)ACD}\eta^C_{i+1}\eta^D_{i+1})(C^{j_1\ldots j_{d-2}}_{B_1\ldots B_{d-2}}\eta^{B_1}_{j_1}\ldots\eta^{B_{d-2}}_{j_{d-2}})\\
&\quad+ (X^{\alpha(e)}_{(i+1)ACDE}\eta^C_{i+1}\eta^D_{i+1}\eta^E_{i+1})(C^{j_1\ldots j_{d-3}}_{B_1\ldots B_{d-3}}\eta^{B_1}_{j_1}\ldots\eta^{B_{d-3}}_{j_{d-3}})\\
&= (X^{\alpha(1)}_{(i+1)AC}C^{j_1\ldots j_{d-1}}_{B_1\ldots B_{d-1}})\,\eta^{B_1}_{j_1}\ldots\eta^{B_{d-1}}_{j_{d-1}}\,\eta^C_{i+1}\\
&\quad+ (X^{\alpha(2)}_{(i+1)ACD}C^{j_1\ldots j_{d-2}}_{B_1\ldots B_{d-2}})\,\eta^{B_1}_{j_1}\ldots\eta^{B_{d-2}}_{j_{d-2}}\,\eta^C_{i+1}\eta^D_{i+1}\\
&\quad+ (X^{\alpha(3)}_{(i+1)ACDE}C^{j_1\ldots j_{d-3}}_{B_1\ldots B_{d-3}})\,\eta^{B_1}_{j_1}\ldots\eta^{B_{d-3}}_{j_{d-3}}\,\eta^C_{i+1}\eta^D_{i+1}\eta^E_{i+1}
\end{aligned}$$

and

$$
\begin{aligned}
(\mathcal{V}_{i,i+1}X^\alpha_{iA})|_{\eta^d} &= (C^{j_1\dots j_{d-1}}_{B_1\dots B_{d-1}}\eta^{B_1}_{j_1}\dots\eta^{B_{d-1}}_{j_{d-1}})(X^{\alpha(1)}_{iAC}\eta^C_i) \\
&\quad + (C^{j_1\dots j_{d-2}}_{B_1\dots B_{d-2}}\eta^{B_1}_{j_1}\dots\eta^{B_{d-2}}_{j_{d-2}})(X^{\alpha(2)}_{iACD}\eta^C_i\eta^D_i) \\
&\quad + (C^{j_1\dots j_{d-3}}_{B_1\dots B_{d-3}}\eta^{B_1}_{j_1}\dots\eta^{B_{d-3}}_{j_{d-3}})(X^{\alpha(3)}_{iACDE}\eta^C_i\eta^D_i\eta^E_i) \\
&= (-1)^{d+1}(C^{j_1\dots j_{d-1}}_{B_1\dots B_{d-1}}X^{\alpha(1)}_{iAC})\,\eta^C_i\,\eta^{B_1}_{j_1}\dots\eta^{B_{d-1}}_{j_{d-1}} \\
&\quad + (-1)^d(C^{j_1\dots j_{d-2}}_{B_1\dots B_{d-2}}X^{\alpha(2)}_{iACD})\,\eta^C_i\eta^D_i\,\eta^{B_1}_{j_1}\dots\eta^{B_{d-2}}_{j_{d-2}} \\
&\quad + (-1)^{d+1}(C^{j_1\dots j_{d-3}}_{B_1\dots B_{d-3}}X^{\alpha(3)}_{iACDE})\,\eta^C_i\eta^D_i\eta^E_i\,\eta^{B_1}_{j_1}\dots\eta^{B_{d-3}}_{j_{d-3}}
\end{aligned}
$$

Let $k+l=d$ and consider only the terms such that $\#\{j_m=i\}=k$ and $\#\{j_m=i+1\}=l$ in the recursion formula. For the left hand side, we then obtain, using (2.13)

$$
\begin{aligned}
C^{ij_1\dots j_d}_{AB_1\dots B_d}\eta^{B_1}_{j_1}\dots\eta^{B_d}_{j_d}|_{\eta^k_i\eta^l_{i+1}} &= \binom{k+l}{k}C^{i\;i\dots i\quad\;\; i+1\dots i+1}_{A\,A_1\dots A_k\,B_1\dots B_l}\,\eta^{A_1}_i\dots\eta^{A_k}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1} \\
&= \frac{\binom{k+l}{k}}{\binom{k+l+1}{k+1}}V_{A\,A_1\dots A_k\,B_1\dots B_l}\eta^{A_1}_i\dots\eta^{A_k}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1} \\
&= \frac{k+1}{d+1}V_{A\,A_1\dots A_k\,B_1\dots B_l}\eta^{A_1}_i\dots\eta^{A_k}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1}
\end{aligned}
$$

For the first term on the right side, we simply obtain

$$
q^\alpha_A(C^{j_1\dots j_d}_{B_1\dots B_d})\eta^{B_1}_{j_1}\dots\eta^{B_d}_{j_d}|_{\eta^k_i\eta^l_{i+1}} = q^\alpha_A(V_{A_1\dots A_k\,B_1\dots B_l})\,\eta^{A_1}_i\dots\eta^{A_k}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1}
$$

Similarly, the second term yields

$$
\begin{aligned}
&(X^\alpha_{i+1\,A}\mathcal{V}_{i,i+1})|_{\eta^k_i\eta^l_{i+1}} \\
&\quad = (X^{\alpha(1)}_{(i+1)AC}V_{A_1\dots A_k\,B_1\dots B_{l-1}})\,\eta^{A_1}_i\dots\eta^{A_k}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_{l-1}}_{i+1}\,\eta^C_{i+1} \\
&\qquad + (X^{\alpha(2)}_{(i+1)ACD}V_{A_1\dots A_k\,B_1\dots B_{l-2}})\,\eta^{A_1}_i\dots\eta^{A_k}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_{l-2}}_{i+1}\,\eta^C_{i+1}\eta^D_{i+1} \\
&\qquad + (X^{\alpha(3)}_{(i+1)ACDE}V_{A_1\dots A_k\,B_1\dots B_{l-3}})\,\eta^{A_1}_i\dots\eta^{A_k}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_{l-3}}_{i+1}\,\eta^C_{i+1}\eta^D_{i+1}\eta^E_{i+1}
\end{aligned}
$$

and the third

$$
\begin{aligned}
&(\mathcal{V}_{i,i+1}X^\alpha_{iA})|_{\eta^k_i\eta^l_{i+1}} \\
&\quad = (-1)^{d+1}(V_{A_1\dots A_{k-1}\,B_1\dots B_l}X^{\alpha(1)}_{iAC})\,\eta^C_i\,\eta^{A_1}_i\dots\eta^{A_{k-1}}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1} \\
&\qquad + (-1)^d(V_{A_1\dots A_{k-2}\,B_1\dots B_l}X^{\alpha(2)}_{iACD})\,\eta^C_i\eta^D_i\,\eta^{A_1}_i\dots\eta^{A_{k-2}}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1} \\
&\qquad + (-1)^{d+1}(V_{A_1\dots A_{k-3}\,B_1\dots B_l}X^{\alpha(3)}_{iACDE})\,\eta^C_i\eta^D_i\eta^E_i\,\eta^{A_1}_i\dots\eta^{A_{k-3}}_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1} \\
&\quad = (-1)^l(V_{A_1\dots A_{k-1}\,B_1\dots B_l}X^{\alpha(1)}_{iAC})\,\eta^{A_1}_i\dots\eta^{A_{k-1}}_i\,\eta^C_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1} \\
&\qquad + (-1)^d(V_{A_1\dots A_{k-2}\,B_1\dots B_l}X^{\alpha(2)}_{iACD})\,\eta^{A_1}_i\dots\eta^{A_{k-2}}_i\,\eta^C_i\eta^D_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1} \\
&\qquad + (-1)^l(V_{A_1\dots A_{k-3}\,B_1\dots B_l}X^{\alpha(3)}_{iACDE})\,\eta^{A_1}_i\dots\eta^{A_{k-3}}_i\,\eta^C_i\eta^D_i\eta^E_i\,\eta^{B_1}_{i+1}\dots\eta^{B_l}_{i+1}
\end{aligned}
$$

Putting everything together, the statement is proved. □

By the recursion formula of Prp. 2.3.1, the coefficients of higher order Grassmann monomials are uniquely determined by those of lower order ones (and X) and can be explicitly calculated. The following brackets will be needed throughout the calculations.

$$i_- := \langle i,\, i-1\rangle\ , \qquad i_+ := \langle i+1,\, i\rangle\ , \qquad i_\pm := \langle i+1,\, i-1\rangle$$

such that

$$X^{\alpha(1)}_{iAA_1} = -\frac{2i\sqrt{2}\,g\lambda^\alpha_{i-1}}{c_0 i_-}\overline{\phi}_{AA_1}\ , \qquad X^{\alpha(2)}_{iAA_1A_2} = \varepsilon_{AA_1A_2C}\frac{2g\lambda^\alpha_{i-1}\lambda_{(i-1)\gamma}\psi^{\gamma C}}{c_0^2 i_-^2}$$

$$X^{\alpha(3)}_{iAA_1A_2A_3} = \varepsilon_{AA_1A_2A_3}\frac{ig\lambda^\alpha_{i-1}\lambda_{(i-1)\beta}\lambda_{(i-1)\gamma}F^{\beta\gamma}}{3c_0^3 i_-^3}$$

and

$$X^{\alpha(1)}_{(i+1)AB_1} = -\frac{2i\sqrt{2}\,g\lambda^\alpha_{i}}{c_0 i_+}\overline{\phi}_{AB_1}\ , \qquad X^{\alpha(2)}_{(i+1)AB_1B_2} = \varepsilon_{AB_1B_2C}\frac{2g\lambda^\alpha_{i}\lambda_{i\gamma}\psi^{\gamma C}}{c_0^2 i_+^2}$$

$$X^{\alpha(3)}_{(i+1)AB_1B_2B_3} = \varepsilon_{AB_1B_2B_3}\frac{ig\lambda^\alpha_{i}\lambda_{i\beta}\lambda_{i\gamma}F^{\beta\gamma}}{3c_0^3 i_+^3}$$

Theorem 2.3.2. Make the ansatz $\mathcal{V}_{i,i+1} = 1+\mathcal{O}(\eta)$ and require that it only depends on the generators η_i and η_{i+1}. Then (2.9b) with X^α_{iA} as in Thm. 2.2.1 is uniquely satisfied by

$$\begin{aligned}
\mathcal{V}_{i,i+1} = 1 &- \frac{\sqrt{2}\,g i_\pm}{c_0^2 i_- i_+}\overline{\phi}_{A_1A_2}\,\eta_i^{A_1}\eta_i^{A_2} + \frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{A_1B_1}\,\eta_i^{A_1}\eta_{i+1}^{B_1} \\
&+ \frac{2ig}{3c_0^3}\frac{i_\pm\left(-i_-\lambda_{(i+1)\gamma} + i_+\lambda_{(i-1)\gamma}\right)\psi^{\gamma C}}{i_-^2 i_+^2}\varepsilon_{A_1A_2A_3C}\,\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A_3} \\
&+ \frac{2ig\lambda_{(i+1)\gamma}\psi^{\gamma C}}{c_0^3 i_+^2}\varepsilon_{A_1A_2B_1C}\,\eta_i^{A_1}\eta_i^{A_2}\eta_{i+1}^{B_1} - \frac{2ig\lambda_{i\gamma}\psi^{\gamma C}}{c_0^3 i_+^2}\varepsilon_{A_1B_1B_2C}\,\eta_i^{A_1}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2} \\
&+ \mathcal{O}(\eta^4)
\end{aligned}$$

along with the higher order terms listed next.

Fourth Order

The (non-vanishing) fourth order coefficients (2.12) of $\mathcal{V}_{i,i+1}$ in Thm. 2.3.2 are as follows.

$$\begin{aligned}
V_{A_1A_2A_3A_4} &= \Bigg(\frac{g i_\pm\left(i_-^2\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma} - i_-i_+\lambda_{(i-1)\beta}\lambda_{(i+1)\gamma} + i_+^2\lambda_{(i-1)\beta}\lambda_{(i-1)\gamma}\right)F^{\beta\gamma}}{12c_0^4 i_-^3 i_+^3} \\
&\qquad + \frac{g^2 i_\pm^2\,\overline{\phi}_{CD}\phi^{CD}}{12c_0^4 i_-^2 i_+^2}\Bigg)\varepsilon_{A_1A_2A_3A_4} \\
V_{A_1A_2A_3B_1} &= -\frac{g\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}}{3c_0^4 i_+^3}\varepsilon_{A_1A_2A_3B_1} - \frac{4g^2 i_\pm}{c_0^4 i_- i_+^2}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2A_3} \\
V_{A_1A_2B_1B_2} &= \frac{g\lambda_{i\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}}{2c_0^4 i_+^3}\varepsilon_{A_1A_2B_1B_2} - \frac{g^2}{c_0^4 i_+^2}\left[\overline{\phi}_{A_1A_2},\,\overline{\phi}_{B_1B_2}\right] - \frac{4g^2}{c_0^4 i_+^2}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2} \\
V_{A_1B_1B_2B_3} &= -\frac{g\lambda_{i\beta}\lambda_{i\gamma}F^{\beta\gamma}}{3c_0^4 i_+^3}\varepsilon_{A_1B_1B_2B_3}
\end{aligned}$$

Fifth Order

The fifth order coefficients of $\mathcal{V}_{i,i+1}$ in Thm. 2.3.2 are as follows.

$$
\begin{aligned}
V_{A_1A_2A_3A_4B_1} = {} & \frac{i\sqrt{2}\,g^2 i_\pm}{3c_0^5 i_-^2 i_+^3}\big(4(i_-\lambda_{(i+1)\gamma} - i_+\lambda_{(i-1)\gamma})\varepsilon_{A_2A_3A_4C}\overline{\phi}_{A_1B_1}\psi^{\gamma C} \\
& \qquad -6i_-\lambda_{(i+1)\gamma}\psi^{\gamma C}\varepsilon_{A_1A_2B_1C}\overline{\phi}_{A_3A_4}\big) \\
V_{A_1A_2A_3B_1B_2} = {} & \frac{i\sqrt{2}\,g^2\lambda_{(i+1)\beta}}{3c_0^5 i_+^3}\Big(\varepsilon_{A_2A_3B_1B_2}\left[\overline{\phi}_{A_1C},\,\psi^{\beta C}\right] + \varepsilon_{A_1A_2A_3C}\left[\overline{\phi}_{B_1B_2},\,\psi^{\beta C}\right] \\
& \qquad - \varepsilon_{A_1B_1B_2C}\left[\overline{\phi}_{A_2A_3},\,\psi^{\beta C}\right] - 4\varepsilon_{A_1A_2B_1C}\psi^{\beta C}\overline{\phi}_{A_3B_2} \\
& \qquad -8\varepsilon_{A_2A_3B_1C}\overline{\phi}_{A_1B_2}\psi^{\beta C}\Big) \\
& + \frac{2i\sqrt{2}\,g^2 i_\pm\lambda_{i\gamma}}{c_0^5 i_- i_+^3}\varepsilon_{A_1B_1B_2C}\psi^{\gamma C}\overline{\phi}_{A_2A_3} \\
V_{A_1A_2B_1B_2B_3} = {} & \frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}}{3c_0^5 i_+^3}\Big(-\left[\overline{\phi}_{A_1C},\,\psi^{\gamma C}\right]\varepsilon_{A_2B_1B_2B_3} + 3\left[\overline{\phi}_{A_1B_1},\,\psi^{\gamma C}\right]_+\varepsilon_{A_2B_2B_3C}\Big)
\end{aligned}
$$

where $[X,\,Y]_+ := XY + YX$ (regardless of the Grassmann parity of X and Y) denotes the anticommutator.

Sixth Order

The sixth order coefficients of $\mathcal{V}_{i,i+1}$ in Thm. 2.3.2 are as follows.

$$
\begin{aligned}
V_{A_1A_2A_3A_4B_1B_2} = {} & -\frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{24c_0^6 i_+^4}\varepsilon_{A_1A_2A_3A_4}\left[\overline{\phi}_{B_1B_2},\,F^{\beta\alpha}\right] \\
& + \frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{6c_0^6 i_+^4}\varepsilon_{A_1A_2A_3B_1}\Big(F^{\beta\alpha}\overline{\phi}_{A_4B_2} + 3\overline{\phi}_{A_4B_2}F^{\beta\alpha}\Big) \\
& - \frac{\sqrt{2}g^2 i_\pm}{2c_0^6 i_- i_+^4}\lambda_{i\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}\overline{\phi}_{A_1A_4}\varepsilon_{A_2A_3B_1B_2} \\
& + \frac{\sqrt{2}\,g^3 i_\pm}{2c_0^6 i_- i_+^3}\left(2\left[\overline{\phi}_{A_1A_2},\,\overline{\phi}_{B_1B_2}\right]\overline{\phi}_{A_3A_4} + 8\overline{\phi}_{A_2B_1}\overline{\phi}_{A_3B_2}\overline{\phi}_{A_1A_4}\right) \\
& + \frac{g^2}{3c_0^6 i_-^2 i_+^4}\big(\varepsilon_{A_2A_3A_4C}\varepsilon_{A_1B_1B_2D}(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta}) \\
& \qquad + \varepsilon_{A_1B_1B_2C}\varepsilon_{A_2A_3A_4D}(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta} + 4i_-i_\pm\lambda_{i\gamma}\lambda_{(i+1)\delta} \\
& \qquad\qquad - 4i_+i_\pm\lambda_{i\gamma}\lambda_{(i-1)\delta}) \\
& \qquad + \varepsilon_{A_2A_3B_1C}\varepsilon_{A_1A_4B_2D}(6i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta})\big)\,\psi^{\gamma C}\psi^{\delta D}
\end{aligned}
$$

and

$$
\begin{aligned}
V_{A_1A_2A_3B_1B_2B_3} = {} & \frac{\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4}\big(\big[\overline{\phi}_{A_1A_2},\,F^{\gamma\alpha}\big]\,\varepsilon_{A_3B_1B_2B_3} \\
& \qquad +3\,\big[\overline{\phi}_{A_2B_1},\,F^{\gamma\alpha}\big]_+\varepsilon_{A_3B_2B_3A_1} + 3\overline{\phi}_{A_1B_3}F^{\gamma\alpha}\varepsilon_{A_2A_3B_1B_2}\Big) \\
& + \frac{\sqrt{2}\,g^2 i_\pm\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}\overline{\phi}_{A_1A_2}}{3c_0^6 i_- i_+^4}\varepsilon_{A_3B_1B_2B_3} \\
& + \frac{2\sqrt{2}\,g^3}{18c_0^6 i_+^3}\big(\big[\overline{\phi}_{A_2C},\,\big[\overline{\phi}_{A_1D},\,\overline{\phi}_{EF}\big]\big]\,\varepsilon_{CDEF}\varepsilon_{A_3B_1B_2B_3} \\
& \qquad + 6\,\big[\overline{\phi}_{A_2B_1},\,\big[\overline{\phi}_{A_1A_3},\,\phi_{B_2B_3}\big]\big]_+ \\
& \qquad -6\overline{\phi}_{A_1B_3}\big[\overline{\phi}_{A_2A_3},\,\overline{\phi}_{B_1B_2}\big] - 24\overline{\phi}_{A_1B_3}\overline{\phi}_{A_2B_1}\overline{\phi}_{A_3B_2}\big) \\
& + \frac{4g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4}\Big(-\varepsilon_{A_1A_2CD}\varepsilon_{A_3B_1B_2B_3}\big[\psi^{\alpha D},\,\psi^{\gamma C}\big]_+ \\
& \qquad + 3\varepsilon_{A_3B_2B_3C}\varepsilon_{A_1A_2B_1D}\big[\psi^{\alpha D},\,\psi^{\gamma C}\big] \\
& \qquad -3\varepsilon_{A_1B_2B_3C}\varepsilon_{A_2A_3B_1D}\psi^{\gamma C}\psi^{\alpha D}\big) \\
V_{A_1A_2B_1B_2B_3B_4} = {} & \frac{\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{i\beta}\big[F^{\gamma\beta},\,\overline{\phi}_{A_1B_1}\big]_+\varepsilon_{A_2B_2B_3B_4}}{3c_0^6 i_+^4} \\
& + \frac{2g^2\lambda_{i\gamma}\lambda_{i\delta}\psi^{\gamma C}\psi^{\delta D}}{c_0^6 i_+^4}\varepsilon_{A_1B_3B_4C}\varepsilon_{A_2B_1B_2D}
\end{aligned}
$$

Seventh and Eighth Order

The seventh and eighth order coefficients of $\mathcal{V}_{i,i+1}$ in Thm. 2.3.2 have the following structure.

$$
\begin{aligned}
V_{A_1A_2A_3B_1B_2B_3B_4} &\sim \frac{g^2}{c_0^7}F\psi + \frac{g^3}{c_0^7}\overline{\phi\phi}\psi \\
V_{A_1A_2A_3A_4B_1B_2B_3} &\sim \frac{g^3}{c_0^7}\overline{\phi\phi}\psi + \frac{g^2}{c_0^7}F\psi \\
V_{A_1A_2A_3A_4B_1B_2B_3B_4} &\sim \frac{g^3}{c_0^8}\overline{\phi}\psi\psi + \frac{g^2}{c_0^8}FF + \frac{g^3}{c_0^8}F\overline{\phi\phi} + \frac{g^4}{c_0^8}\overline{\phi}^4
\end{aligned}
$$

Here, we (schematically) denote terms such as $\big[\overline{\phi},\,F\big]$ simply by $\overline{\phi}F \sim F\overline{\phi}$. We decided in favour of schematic expressions since the exact formulas (and their calculation) turn out to be very long and are not needed in the following.

2.3.1 Calculation up to Fourth Order

To proof Thm. 2.3.2, we use the recursion formula in Prp. 2.3.1 to calculate the coefficients (2.12) of $\mathcal{V}_{i,i+1}$. Consider first $d = 0$, i.e. the coefficients $\sim \eta^1$ of order 1. We already know that $V_B = 0$. Considering V_A, we observe that the right hand side of the formula only consists of the (vanishing) variation of $V_0 = 1$ such that also $V_A = 0$ vanishes. The calculation of the higher order coefficients is analogous and occupies the rest of this chapter.

Second Order ($d = 1$, $k = 1$ and $l = 0$)

$$\begin{aligned}V_{AA_1} &= \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+}\left(-q^\alpha_A(V_{A_1}) + 0 + 0 + 0 - iX^{\alpha(1)}_{iAA_1} + 0 + 0\right)\\ &= -\frac{i\lambda_{(i+1)\alpha}}{2c_0 i_+}X^{\alpha(1)}_{iAA_1} = -\frac{\sqrt{2}\,gi_\pm}{c_0^2 i_- i_+}\overline{\phi}_{AA_1}\end{aligned}$$

Second Order ($d = 1$, $k = 0$ and $l = 1$)

$$V_{AB_1} = \frac{\lambda_{(i+1)\alpha}}{c_0 i_+}\left(iX^{\alpha(1)}_{(i+1)AB_1}\right) = \frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{AB_1}$$

Third Order ($d = 2$, $k = 2$ and $l = 0$)

Using Lem. 1.3.4, we calculate

$$\begin{aligned}V_{AA_1A_2} &= \frac{-\lambda_{(i+1)\alpha}}{3c_0 i_+}\left(-q^\alpha_A(V_{A_1A_2}) - iX^{\alpha(2)}_{iAA_1A_2}\right)\\ &= \frac{-\lambda_{(i+1)\alpha}}{3c_0 i_+}\left(\frac{\sqrt{2}\,gi_\pm}{c_0^2 i_- i_+}(i\sqrt{2}\,\varepsilon_{AA_1A_2C}\psi^{\alpha C}) - \varepsilon_{AA_1A_2C}\frac{2ig\lambda^\alpha_{i-1}\lambda_{(i-1)\gamma}\psi^{\gamma C}}{c_0^2 i_-^2}\right)\\ &= \frac{2ig}{3c_0^3}\frac{i_\pm\left(-i_-\lambda_{(i+1)\gamma} + i_+\lambda_{(i-1)\gamma}\right)\psi^{\gamma C}}{i_-^2 i_+^2}\varepsilon_{AA_1A_2C}\end{aligned}$$

Third Order ($d = 2$, $k = 1$ and $l = 1$)

$$\begin{aligned}V_{AA_1B_1} &= \frac{-\lambda_{(i+1)\alpha}}{2c_0 i_+}\left(-q^\alpha_A(V_{A_1B_1})\right) = \frac{\sqrt{2}\,g\lambda_{(i+1)\alpha}}{c_0^3 i_+^2}(i\sqrt{2}\,\varepsilon_{AA_1B_1C}\psi^{\alpha C})\\ &= \frac{2ig\lambda_{(i+1)\gamma}\psi^{\gamma C}}{c_0^3 i_+^2}\varepsilon_{AA_1B_1C}\end{aligned}$$

Third Order ($d = 2$, $k = 0$ and $l = 2$)

Now, for $k = 0$ and $l = 2$:

$$V_{AB_1B_2} = \frac{-\lambda_{(i+1)\alpha}}{c_0 i_+}\left(iX^{\alpha(2)}_{(i+1)AB_1B_2}\right) = \frac{-2ig\lambda_{i\gamma}\psi^{\gamma C}}{c_0^3 i_+^2}\varepsilon_{AB_1B_2C}$$

Fourth Order ($d = 3$, $k = 3$ and $l = 0$)

$$\begin{aligned} V_{AA_1A_2A_3} &= \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\left(-q^\alpha_A(V_{A_1A_2A_3}) - iV_{A_1A_2}X^{\alpha(1)}_{iAA_3} - iX^{\alpha(3)}_{iAA_1A_2A_3}\right)\\ &= \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\left(\frac{2ig}{3c_0^3}\frac{i_\pm\left(i_-\lambda_{(i+1)\gamma} - i_+\lambda_{(i-1)\gamma}\right)q^\alpha_A(\psi^{\gamma C})}{i_-^2 i_+^2}\varepsilon_{A_1A_2A_3C}\right.\\ &\qquad -i\left(-\frac{\sqrt{2}\,gi_\pm}{c_0^2 i_- i_+}\overline{\phi}_{A_1A_2}\right)\left(-\frac{2i\sqrt{2}\,g\lambda^\alpha_{i-1}}{c_0 i_-}\overline{\phi}_{AA_3}\right)\\ &\qquad \left. -i\left(\varepsilon_{AA_1A_2A_3}\frac{ig\lambda^\alpha_{i-1}\lambda_{(i-1)\beta}\lambda_{(i-1)\gamma}F^{\beta\gamma}}{3c_0^3 i_-^3}\right)\right)\\ &=: (1) + (2) + (3) \end{aligned}$$

We calculate (1). The term corresponding to the second summand $\sim \left[\overline{\phi}_{AD}, \phi^{CD}\right]$ of the supersymmetry transformation $q^\alpha_A(\psi^{\gamma C})$ in Lem. 1.3.4 vanishes upon pairing with the η terms. This follows from

$$\begin{aligned} \left[\overline{\phi}_{AD}, \phi^{CD}\right]\varepsilon_{CA_1A_2A_3}\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3} &= \frac{1}{2}\varepsilon_{CDEF}\varepsilon_{CA_1A_2A_3}\left[\overline{\phi}_{AD}, \overline{\phi}_{EF}\right]\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3}\\ &= \frac{1}{2}(\delta^{A_1A_2A_3}_{DEF} + \ldots - \ldots)\left[\overline{\phi}_{AD}, \overline{\phi}_{EF}\right]\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3}\\ &= 3\left[\overline{\phi}_{AD}, \overline{\phi}_{EF}\right]\eta_i^A\eta_i^D\eta_i^E\eta_i^F \end{aligned}$$

and

$$\begin{aligned} \left[\overline{\phi}_{AD}, \overline{\phi}_{EF}\right]\eta_i^A\eta_i^D\eta_i^E\eta_i^F &= -\left[\overline{\phi}_{EF}, \overline{\phi}_{AD}\right]\eta_i^A\eta_i^D\eta_i^E\eta_i^F\\ &= -\left[\overline{\phi}_{EF}, \overline{\phi}_{AD}\right]\eta_i^E\eta_i^F\eta_i^A\eta_i^D\\ &= -\left[\overline{\phi}_{AD}, \overline{\phi}_{EF}\right]\eta_i^A\eta_i^D\eta_i^E\eta_i^F \end{aligned}$$

such that

$$\begin{aligned} (1) &= \frac{2ig\lambda_{(i+1)\alpha}}{12c_0^4 i_+}\frac{i_\pm\left(i_-\lambda_{(i+1)\gamma} - i_+\lambda_{(i-1)\gamma}\right)}{i_-^2 i_+^2}\left(\frac{i}{2}F^{\gamma\alpha}\delta^C_A\right)\varepsilon_{A_1A_2A_3C}\\ &= \frac{gi_\pm\left(i_-\lambda_{(i+1)\gamma} - i_+\lambda_{(i-1)\gamma}\right)\lambda_{(i+1)\alpha}F^{\gamma\alpha}}{12c_0^4 i_-^2 i_+^3}\varepsilon_{AA_1A_2A_3} \end{aligned}$$

Moreover

$$\begin{aligned} (3) &= \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\varepsilon_{AA_1A_2A_3}\frac{g\lambda^\alpha_{i-1}\lambda_{(i-1)\beta}\lambda_{(i-1)\gamma}F^{\beta\gamma}}{3c_0^3 i_-^3}\\ &= \frac{gi_\pm\lambda_{(i-1)\beta}\lambda_{(i-1)\gamma}F^{\beta\gamma}}{12c_0^4 i_-^3 i_+}\varepsilon_{AA_1A_2A_3} \end{aligned}$$

such that

$$(1) + (3) = \frac{gi_\pm\left(i_-^2\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma} - i_-i_+\lambda_{(i-1)\beta}\lambda_{(i+1)\gamma} + i_+^2\lambda_{(i-1)\beta}\lambda_{(i-1)\gamma}\right)F^{\beta\gamma}}{12c_0^4 i_-^3 i_+^3}\varepsilon_{AA_1A_2A_3}$$

Moreover

$$(2) = \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\left(\frac{\sqrt{2}\,g i_\pm}{c_0^2 i_- i_+}\overline{\phi}_{A_1A_2}\right)\left(\frac{2\sqrt{2}\,g\lambda^\alpha_{i-1}}{c_0 i_-}\overline{\phi}_{AA_3}\right)$$
$$= \frac{g^2 i_\pm^2}{c_0^4 i_-^2 i_+^2}\overline{\phi}_{A_1A_2}\overline{\phi}_{AA_3}$$

We calculate (using $\{CD\} = \{A_1A_2\}$ in the second equation):

$$\begin{aligned}
&\overline{\phi}_{A_1A_2}\overline{\phi}_{AA_3}\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3}\\
&\quad= \frac{1}{2}\varepsilon_{AA_3CD}\overline{\phi}_{A_1A_2}\phi^{CD}\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3}\\
&\quad= \frac{1}{2}(\varepsilon_{AA_3A_1A_2}\overline{\phi}_{A_1A_2}\phi^{A_1A_2} + \varepsilon_{AA_3A_2A_1}\overline{\phi}_{A_1A_2}\phi^{A_2A_1})\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3}\\
&\quad= (\overline{\phi}_{A_1A_2}\phi^{A_1A_2})(\varepsilon_{AA_3A_1A_2}\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3})
\end{aligned}$$

Here, the sum runs over all $AA_1A_2A_3$. For A_1A_2 fixed, the epsilon tensor allows for one of the two permutation of the remaining indices AA_3 in $1\ldots4$, and the second factor in parentheses is always the same. Therefore

$$\begin{aligned}
\overline{\phi}_{A_1A_2}\overline{\phi}_{AA_3}\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3} &= 2\overline{\phi}_{CD}\phi^{CD}\eta_i^1\eta_i^2\eta_i^3\eta_i^4\\
&= \frac{1}{12}\overline{\phi}_{CD}\phi^{CD}\varepsilon_{AA_1A_2A_3}\eta_i^A\eta_i^{A_1}\eta_i^{A_2}\eta_i^{A^3}
\end{aligned}$$

such that

$$(2) = \frac{g^2 i_\pm^2\overline{\phi}_{CD}\phi^{CD}}{12c_0^4 i_-^2 i_+^2}\varepsilon_{AA_1A_2A_3}$$

thus determining $V_{AA_1A_2A_3}$.

Fourth Order ($d = 3$, $k = 2$ and $l = 1$)

$$\begin{aligned}
V_{AA_1A_2B_1} &= \frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\left(-q_A^\alpha(V_{A_1A_2B_1}) + iX^{\alpha(1)}_{(i+1)AB_1}V_{A_1A_2} + iV_{A_1B_1}X^{\alpha(1)}_{iAA_2}\right)\\
&= \frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\Bigg(-\frac{2ig\lambda_{(i+1)\gamma}q_A^\alpha(\psi^{\gamma C})}{c_0^3 i_+^2}\varepsilon_{A_1A_2B_1C}\\
&\qquad + i\left(-\frac{2i\sqrt{2}\,g\lambda_i^\alpha}{c_0 i_+}\overline{\phi}_{AB_1}\right)\left(-\frac{\sqrt{2}\,g i_\pm}{c_0^2 i_- i_+}\overline{\phi}_{A_1A_2}\right)\\
&\qquad + i\left(\frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{A_1B_1}\right)\left(-\frac{2i\sqrt{2}\,g\lambda^\alpha_{i-1}}{c_0 i_-}\overline{\phi}_{AA_2}\right)\Bigg)\\
&=: (1) + (2) + (3)
\end{aligned}$$

We calculate (1). As before, the term corresponding to the second summand in the supersymmetry variation $q_A^\alpha(\psi^{\gamma C})$ vanishes, but now because

$$\lambda_{(i+1)\alpha}\lambda_{(i+1)\gamma}q_A^\alpha(\psi^{\gamma C})|_2 \sim \lambda_{(i+1)\alpha}\lambda_{(i+1)\gamma}\epsilon^{\gamma\alpha} = 0$$

and we are left with

$$\begin{aligned}(1) &= -\frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\frac{2ig\lambda_{(i+1)\gamma}}{c_0^3 i_+^2}\frac{i}{2}F^{\gamma\alpha}\delta_A^C\varepsilon_{A_1A_2B_1C}\\ &= -\frac{g\lambda_{(i+1)\alpha}\lambda_{(i+1)\gamma}F^{\gamma\alpha}}{3c_0^4 i_+^3}\varepsilon_{AA_1A_2B_1}\end{aligned}$$

Moreover

$$\begin{aligned}(2)+(3) &= \frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\left(-\frac{4g^2 i_\pm\lambda_i^\alpha}{c_0^3 i_- i_+^2}\overline{\phi}_{AB_1}\overline{\phi}_{A_1A_2}+\frac{8g^2\lambda_{i-1}^\alpha}{c_0^3 i_- i_+}\overline{\phi}_{A_1B_1}\overline{\phi}_{AA_2}\right)\\ &= -\frac{4g^2 i_\pm}{3c_0^4 i_- i_+^2}\overline{\phi}_{AB_1}\overline{\phi}_{A_1A_2}+\frac{8g^2 i_\pm}{3c_0^4 i_- i_+^2}\overline{\phi}_{A_1B_1}\overline{\phi}_{AA_2}\\ &= \frac{4g^2 i_\pm}{3c_0^4 i_- i_+^2}\left(-\overline{\phi}_{AB_1}\overline{\phi}_{A_1A_2}+2\overline{\phi}_{A_1B_1}\overline{\phi}_{AA_2}\right)\\ &\cong -\frac{4g^2 i_\pm}{c_0^4 i_- i_+^2}\overline{\phi}_{AB_1}\overline{\phi}_{A_1A_2}\end{aligned}$$

where, as before, "$\cong$" denotes equality upon pairing with the corresponding η terms.

Fourth Order ($d=3$, $k=1$ and $l=2$)

$$\begin{aligned}&V_{AA_1B_1B_2}\\ &\quad= \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+}\left(-q_A^\alpha(V_{A_1B_1B_2})+iX^{\alpha(1)}_{(i+1)AB_2}V_{A_1B_1}\right)\\ &\quad= \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+}\left(\frac{2ig\lambda_{i\gamma}q_A^\alpha(\psi^{\gamma C})}{c_0^3 i_+^2}\varepsilon_{A_1B_1B_2C}+i\left(-\frac{2i\sqrt{2}\,g\lambda_i^\alpha}{c_0 i_+}\overline{\phi}_{AB_2}\right)\left(\frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{A_1B_1}\right)\right)\\ &\quad=: (1)+(2)\end{aligned}$$

To calculate (1), we make the following side calculation.

$$\begin{aligned}&\left[\overline{\phi}_{AD},\,\phi^{CD}\right]\varepsilon_{CA_1B_1B_2}\eta_i^A\eta_i^{A_1}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2}=\dots\\ &\quad= \left(\delta_{DEF}^{A_1B_1B_2}+\delta_{DEF}^{B_1B_2A_1}+\delta_{DEF}^{B_2A_1B_1}\right)\left[\overline{\phi}_{AD},\,\overline{\phi}_{EF}\right]\eta_i^A\eta_i^{A_1}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2}\\ &\quad= \left(\left[\overline{\phi}_{AA_1},\,\overline{\phi}_{B_1B_2}\right]+\left[\overline{\phi}_{AB_1},\,\overline{\phi}_{B_2A_1}\right]+\left[\overline{\phi}_{AB_2},\,\overline{\phi}_{A_1B_1}\right]\right)\eta_i^A\eta_i^{A_1}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2}\\ &\quad= \left[\overline{\phi}_{AA_1},\,\overline{\phi}_{B_1B_2}\right]\eta_i^A\eta_i^{A_1}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2}\end{aligned}$$

since the second and third terms vanish by symmetry considerations: For example, for the third term:

$$\begin{aligned}\left[\overline{\phi}_{AB_2},\,\overline{\phi}_{A_1B_1}\right]\eta_i^A\eta_i^{A_1}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2} &= \left[\overline{\phi}_{AB_2},\,\overline{\phi}_{A_1B_1}\right]\eta_i^{A_1}\eta_i^{A}\eta_{i+1}^{B_2}\eta_{i+1}^{B_1}\\ &= \left[\overline{\phi}_{A_1B_1},\,\overline{\phi}_{AB_2}\right]\eta_i^A\eta_i^{A_1}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2}\\ &= -\left[\overline{\phi}_{AB_2},\,\overline{\phi}_{A_1B_1}\right]\eta_i^A\eta_i^{A_1}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2}\end{aligned}$$

where the second equality follows from renaming the summation indices. Therefore

$$\begin{aligned}(1) &= \frac{ig\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{c_0^4 i_+^3} q_A^\alpha(\psi^{\gamma C})\varepsilon_{A_1B_1B_2C}\\ &= -\frac{ig\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{c_0^4 i_+^3}\left(\frac{i}{2}F^{\gamma\alpha}\delta_A^C + i\epsilon^{\gamma\alpha}g\left[\overline{\phi}_{AD},\, \phi^{CD}\right]\right)\varepsilon_{CA_1B_1B_2}\\ &\cong \frac{g\lambda_{i\gamma}\lambda_{(i+1)\alpha}F^{\gamma\alpha}}{2c_0^4 i_+^3}\varepsilon_{AA_1B_1B_2} - \frac{g^2}{c_0^4 i_+^2}\left[\overline{\phi}_{AA_1},\, \overline{\phi}_{B_1B_2}\right]\end{aligned}$$

For the second term, we calculate

$$(2) = \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+}\frac{8g^2\lambda_i^\alpha}{c_0^3 i_+^2}\overline{\phi}_{AB_2}\overline{\phi}_{A_1B_1} = \frac{4g^2}{c_0^4 i_+^2}\overline{\phi}_{AB_2}\overline{\phi}_{A_1B_1} \cong -\frac{4g^2}{c_0^4 i_+^2}\overline{\phi}_{AB_1}\overline{\phi}_{A_1B_2}$$

Fourth Order ($d = 3$, $k = 0$ and $l = 3$)

$$V_{AB_1B_2B_3} = \frac{\lambda_{(i+1)\alpha}}{c_0 i_+} iX^{\alpha(3)}_{(i+1)AB_1B_2B_3} = -\frac{g\lambda_{i\beta}\lambda_{i\gamma}F^{\beta\gamma}}{3c_0^4 i_+^3}\varepsilon_{AB_1B_2B_3}$$

2.3.2 Fifth Order

Fifth Order ($d = 4$, $k = 4$ and $l = 0$)

It is clear that $V_{AA_1A_2A_3A_4} \sim \eta_i^5 = 0$.

Fifth Order ($d = 4$, $k = 3$ and $l = 1$)

Now, consider $k = 3$ and $l = 1$.

$$\begin{aligned}&V_{AA_1A_2A_3B_1}\\ &= -\frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\Big(-q_A^\alpha(V_{A_1A_2A_3B_1}) + iX^{\alpha(1)}_{(i+1)AB_1}V_{A_1A_2A_3}\\ &\qquad + iV_{A_1A_2B_1}X^{\alpha(1)}_{iAA_3} - iV_{A_1B_1}X^{\alpha(2)}_{iAA_2A_3}\Big)\\ &= -\frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\left(\frac{g\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma}q_A^\alpha(F^{\gamma\beta})}{3c_0^4 i_+^3}\varepsilon_{A_1A_2A_3B_1} + \frac{4g^2 i_\pm}{c_0^4 i_- i_+^2}q_A^\alpha(\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2A_3})\right.\\ &\qquad + i\left(-\frac{2i\sqrt{2}\,g\lambda_i^\alpha}{c_0 i_+}\overline{\phi}_{AB_1}\right)\left(\frac{2ig}{3c_0^3}\frac{i_\pm\left(-i_-\lambda_{(i+1)\gamma} + i_+\lambda_{(i-1)\gamma}\right)\psi^{\gamma C}}{i_-^2 i_+^2}\varepsilon_{A_1A_2A_3C}\right)\\ &\qquad + i\left(\frac{2ig\lambda_{(i+1)\beta}\psi^{\beta C}}{c_0^3 i_+^2}\varepsilon_{A_1A_2B_1C}\right)\left(-\frac{2i\sqrt{2}\,g\lambda_{i-1}^\alpha}{c_0 i_-}\overline{\phi}_{AA_3}\right)\\ &\qquad \left. - i\left(\frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{A_1B_1}\right)\left(\varepsilon_{AA_2A_3C}\frac{2g\lambda_{i-1}^\alpha\lambda_{(i-1)\gamma}\psi^{\gamma C}}{c_0^2 i_-^2}\right)\right)\\ &=: (1) + (2) + (3) + (4) + (5)\end{aligned}$$

Using Cor. 1.3.5, we calculate

$$\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma}q_A^\alpha(F^{\gamma\beta}) \sim \lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma}(\epsilon^{\alpha\beta} + \epsilon^{\alpha\gamma}) = 0$$

and, therefore, (1) = 0. Moreover, by Lem. 1.3.4,

$$(2) = -\frac{g^2 i_\pm \lambda_{(i+1)\alpha}}{c_0^5 i_- i_+^3}\left(q_A^\alpha(\overline{\phi}_{A_1B_1})\overline{\phi}_{A_2A_3} + \overline{\phi}_{A_1B_1} q_A^\alpha(\overline{\phi}_{A_2A_3})\right)$$
$$= -\frac{i\sqrt{2}\,g^2 i_\pm \lambda_{(i+1)\alpha}}{c_0^5 i_- i_+^3}\left(\varepsilon_{AA_1B_1C}\psi^{\alpha C}\overline{\phi}_{A_2A_3} + \varepsilon_{AA_2A_3C}\overline{\phi}_{A_1B_1}\psi^{\alpha C}\right)$$

Moreover

$$(3) = \frac{1}{4c_0} i\left(\frac{2i\sqrt{2}\,g}{c_0 i_+}\overline{\phi}_{AB_1}\right)\left(\frac{2ig}{3c_0^3}\frac{i_\pm\left(-i_-\lambda_{(i+1)\gamma} + i_+\lambda_{(i-1)\gamma}\right)\psi^{\gamma C}}{i_-^2 i_+^2}\varepsilon_{A_1A_2A_3C}\right)$$
$$= \frac{i\sqrt{2}\,g^2}{3c_0^5}\frac{i_\pm\left(i_-\lambda_{(i+1)\gamma} - i_+\lambda_{(i-1)\gamma}\right)}{i_-^2 i_+^3}\varepsilon_{A_1A_2A_3C}\overline{\phi}_{AB_1}\psi^{\gamma C}$$

and

$$(4) = \frac{i_\pm}{4c_0 i_+} i\left(\frac{2ig\lambda_{(i+1)\beta}\psi^{\beta C}}{c_0^3 i_+^2}\varepsilon_{A_1A_2B_1C}\right)\left(\frac{2i\sqrt{2}\,g}{c_0 i_-}\overline{\phi}_{AA_3}\right)$$
$$= -\frac{i\sqrt{2}\,g^2 i_\pm \lambda_{(i+1)\beta}}{c_0^5 i_- i_+^3}\varepsilon_{A_1A_2B_1C}\psi^{\beta C}\overline{\phi}_{AA_3}$$

and

$$(5) = \frac{i_\pm}{4c_0 i_+} i\left(\frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{A_1B_1}\right)\left(\varepsilon_{AA_2A_3C}\frac{2g\lambda_{(i-1)\gamma}\psi^{\gamma C}}{c_0^2 i_-^2}\right)$$
$$= \frac{i\sqrt{2}\,g^2 i_\pm \lambda_{(i-1)\gamma}}{c_0^5 i_-^2 i_+^2}\varepsilon_{AA_2A_3C}\overline{\phi}_{A_1B_1}\psi^{\gamma C}$$

Now, combining (2) to (5), we find

$$\begin{aligned}
V_{AA_1A_2A_3B_1} &= (2) + (3) + (4) + (5)\\
&= \frac{i\sqrt{2}\,g^2 i_\pm}{3c_0^5 i_-^2 i_+^3}\big(-3i_-\lambda_{(i+1)\alpha}\varepsilon_{AA_1B_1C}\psi^{\alpha C}\overline{\phi}_{A_2A_3} - 3i_-\lambda_{(i+1)\alpha}\varepsilon_{AA_2A_3C}\overline{\phi}_{A_1B_1}\psi^{\alpha C}\\
&\qquad + i_-\lambda_{(i+1)\alpha}\varepsilon_{A_1A_2A_3C}\overline{\phi}_{AB_1}\psi^{\alpha C} - i_+\lambda_{(i-1)\alpha}\varepsilon_{A_1A_2A_3C}\overline{\phi}_{AB_1}\psi^{\alpha C}\\
&\qquad -3i_-\lambda_{(i+1)\alpha}\varepsilon_{A_1A_2B_1C}\psi^{\alpha C}\overline{\phi}_{AA_3} + 3i_+\lambda_{(i-1)\alpha}\varepsilon_{AA_2A_3C}\overline{\phi}_{A_1B_1}\psi^{\alpha C}\big)\\
&= \frac{i\sqrt{2}\,g^2 i_\pm}{3c_0^5 i_-^2 i_+^3}\big(i_+\lambda_{(i-1)\alpha}(-\varepsilon_{A_1A_2A_3C}\overline{\phi}_{AB_1} + 3\varepsilon_{AA_2A_3C}\overline{\phi}_{A_1B_1})\psi^{\alpha C}\\
&\qquad + i_-\lambda_{(i+1)\alpha}(-3\varepsilon_{AA_2A_3C}\overline{\phi}_{A_1B_1} + \varepsilon_{A_1A_2A_3C}\overline{\phi}_{AB_1})\psi^{\alpha C}\\
&\qquad -3i_-\lambda_{(i+1)\alpha}\psi^{\alpha C}(\varepsilon_{AA_1B_1C}\overline{\phi}_{A_2A_3} + \varepsilon_{A_1A_2B_1C}\overline{\phi}_{AA_3})\big)\\
&\cong \frac{i\sqrt{2}\,g^2 i_\pm}{3c_0^5 i_-^2 i_+^3}\big(-4i_+\lambda_{(i-1)\alpha}\varepsilon_{A_1A_2A_3C}\overline{\phi}_{AB_1}\psi^{\alpha C}\\
&\qquad + 4i_-\lambda_{(i+1)\alpha}\varepsilon_{A_1A_2A_3C}\overline{\phi}_{AB_1}\psi^{\alpha C}\\
&\qquad -6i_-\lambda_{(i+1)\alpha}\psi^{\alpha C}\varepsilon_{AA_1B_1C}\overline{\phi}_{A_2A_3}\big)\\
&= \frac{i\sqrt{2}\,g^2 i_\pm}{3c_0^5 i_-^2 i_+^3}\big(4(i_-\lambda_{(i+1)\gamma} - i_+\lambda_{(i-1)\gamma})\varepsilon_{A_1A_2A_3C}\overline{\phi}_{AB_1}\psi^{\gamma C}\\
&\qquad\quad -6i_-\lambda_{(i+1)\gamma}\psi^{\gamma C}\varepsilon_{AA_1B_1C}\overline{\phi}_{A_2A_3}\big)
\end{aligned}$$

where the fourth equation holds upon contracting with the η terms.

Fifth Order ($d = 4$, $k = 2$ and $l = 2$)

$$\begin{aligned}
V_{AA_1A_2B_1B_2} &= -\frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\Big(-q^\alpha_A(V_{A_1A_2B_1B_2}) + iX^{\alpha(1)}_{(i+1)AB_2}V_{A_1A_2B_1} \\
&\qquad + iX^{\alpha(2)}_{(i+1)AB_1B_2}V_{A_1A_2} - iV_{A_1B_1B_2}X^{\alpha(1)}_{iAA_2}\Big) \\
&= -\frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\Bigg(-\frac{g\lambda_{i\gamma}\lambda_{(i+1)\beta}q^\alpha_A(F^{\gamma\beta})}{2c_0^4 i_+^3}\varepsilon_{A_1A_2B_1B_2} + \frac{g^2}{c_0^4 i_+^2}q^\alpha_A\left[\overline{\phi}_{A_1A_2},\, \overline{\phi}_{B_1B_2}\right] \\
&\qquad + \frac{4g^2}{c_0^4 i_+^2}q^\alpha_A(\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}) \\
&\qquad + i\left(-\frac{2i\sqrt{2}\,g\lambda^\alpha_i}{c_0 i_+}\overline{\phi}_{AB_2}\right)\left(\frac{2ig\lambda_{(i+1)\beta}\psi^{\beta C}}{c_0^3 i_+^2}\varepsilon_{A_1A_2B_1C}\right) \\
&\qquad + i\left(\varepsilon_{AB_1B_2C}\frac{2g\lambda^\alpha_i\lambda_{i\gamma}\psi^{\gamma C}}{c_0^2 i_+^2}\right)\left(-\frac{\sqrt{2}\,g i_\pm}{c_0^2 i_- i_+}\overline{\phi}_{A_1A_2}\right) \\
&\qquad - i\left(-\frac{2ig\lambda_{i\gamma}\psi^{\gamma C}}{c_0^3 i_+^2}\varepsilon_{A_1B_1B_2C}\right)\left(-\frac{2i\sqrt{2}\,g\lambda^\alpha_{i-1}}{c_0 i_-}\overline{\phi}_{AA_2}\right)\Bigg) \\
&=: (1) + (2) + (3) + (4) + (5) + (6)
\end{aligned}$$

Using Cor. 1.3.5 and the last Euler-Lagrange equation in Lem. 1.4.1, we calculate

$$\begin{aligned}
(1) &= -\frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\frac{g\lambda_{i\gamma}\lambda_{(i+1)\beta}2\epsilon^{\alpha\gamma}D^\beta{}_{\dot\gamma}\tilde{\psi}^{\dot\gamma}_A}{2c_0^4 i_+^3}\varepsilon_{A_1A_2B_1B_2} \\
&= -\frac{g\lambda_{(i+1)\beta}D^\beta{}_{\dot\gamma}\tilde{\psi}^{\dot\gamma}_A}{3c_0^5 i_+^3}\varepsilon_{A_1A_2B_1B_2} \\
&= \frac{i\sqrt{2}\,g^2\lambda_{(i+1)\beta}}{3c_0^5 i_+^3}\left[\overline{\phi}_{AC},\, \psi^{\beta C}\right]\varepsilon_{A_1A_2B_1B_2}
\end{aligned}$$

Moreover

$$\begin{aligned}
(2) &= -\frac{g^2\lambda_{(i+1)\alpha}}{3c_0^5 i_+^3}\left(\left[q^\alpha_A(\overline{\phi}_{A_1A_2}),\, \overline{\phi}_{B_1B_2}\right] + \left[\overline{\phi}_{A_1A_2},\, q^\alpha_A(\overline{\phi}_{B_1B_2})\right]\right) \\
&= -\frac{i\sqrt{2}\,g^2\lambda_{(i+1)\alpha}}{3c_0^5 i_+^3}\left(\varepsilon_{AA_1A_2C}\left[\psi^{\alpha C},\, \overline{\phi}_{B_1B_2}\right] + \varepsilon_{AB_1B_2C}\left[\overline{\phi}_{A_1A_2},\, \psi^{\alpha C}\right]\right)
\end{aligned}$$

and

$$\begin{aligned}
(3) &= -\frac{4g^2\lambda_{(i+1)\alpha}}{3c_0^5 i_+^3}\left(q^\alpha_A(\overline{\phi}_{A_1B_1})\overline{\phi}_{A_2B_2} + \overline{\phi}_{A_1B_1}q^\alpha_A(\overline{\phi}_{A_2B_2})\right) \\
&= -\frac{4i\sqrt{2}\,g^2\lambda_{(i+1)\alpha}}{3c_0^5 i_+^3}\left(\varepsilon_{AA_1B_1C}\psi^{\alpha C}\overline{\phi}_{A_2B_2} + \varepsilon_{AA_2B_2C}\overline{\phi}_{A_1B_1}\psi^{\alpha C}\right)
\end{aligned}$$

and

$$\begin{aligned}
(4) &= \frac{1}{3c_0}i\left(\frac{2i\sqrt{2}\,g}{c_0 i_+}\overline{\phi}_{AB_2}\right)\left(\frac{2ig\lambda_{(i+1)\beta}\psi^{\beta C}}{c_0^3 i_+^2}\varepsilon_{A_1A_2B_1C}\right) \\
&= -\frac{4i\sqrt{2}\,g^2\lambda_{(i+1)\beta}}{3c_0^5 i_+^3}\varepsilon_{A_1A_2B_1C}\overline{\phi}_{AB_2}\psi^{\beta C}
\end{aligned}$$

and

$$(5) = \frac{1}{3c_0} i \left(\varepsilon_{AB_1B_2C} \frac{2g\lambda_{i\gamma}\psi^{\gamma C}}{c_0^2 i_+^2} \right) \left(\frac{\sqrt{2}\, g i_\pm}{c_0^2 i_- i_+} \overline{\phi}_{A_1A_2} \right)$$
$$= \frac{2i\sqrt{2}\, g^2 i_\pm \lambda_{i\gamma}}{3c_0^5 i_- i_+^3} \varepsilon_{AB_1B_2C} \psi^{\gamma C} \overline{\phi}_{A_1A_2}$$

and

$$(6) = \frac{i_\pm}{3c_0 i_+} i \left(\frac{2ig\lambda_{i\gamma}\psi^{\gamma C}}{c_0^3 i_+^2} \varepsilon_{A_1B_1B_2C} \right) \left(\frac{2i\sqrt{2}\, g}{c_0 i_-} \overline{\phi}_{AA_2} \right)$$
$$= -\frac{4i\sqrt{2}\, g^2 i_\pm \lambda_{i\gamma}}{3c_0^5 i_- i_+^3} \varepsilon_{A_1B_1B_2C} \psi^{\gamma C} \overline{\phi}_{AA_2}$$

We further calculate

$$(3) + (4) = -\frac{4i\sqrt{2}\, g^2 \lambda_{(i+1)\beta}}{3c_0^5 i_+^3} \Big(\varepsilon_{AA_1B_1C} \psi^{\beta C} \overline{\phi}_{A_2B_2} + \varepsilon_{AA_2B_2C} \overline{\phi}_{A_1B_1} \psi^{\beta C}$$
$$+ \varepsilon_{A_1A_2B_1C} \overline{\phi}_{AB_2} \psi^{\beta C} \Big)$$
$$\cong -\frac{4i\sqrt{2}\, g^2 \lambda_{(i+1)\beta}}{3c_0^5 i_+^3} \left(\varepsilon_{AA_1B_1C} \psi^{\beta C} \overline{\phi}_{A_2B_2} + 2\varepsilon_{A_1A_2B_1C} \overline{\phi}_{AB_2} \psi^{\beta C} \right)$$

and

$$(5) + (6) = \frac{2i\sqrt{2}\, g^2 i_\pm \lambda_{i\gamma}}{3c_0^5 i_- i_+^3} \left(\varepsilon_{AB_1B_2C} \psi^{\gamma C} \overline{\phi}_{A_1A_2} - 2\varepsilon_{A_1B_1B_2C} \psi^{\gamma C} \overline{\phi}_{AA_2} \right)$$
$$\cong \frac{2i\sqrt{2}\, g^2 i_\pm \lambda_{i\gamma}}{c_0^5 i_- i_+^3} \varepsilon_{AB_1B_2C} \psi^{\gamma C} \overline{\phi}_{A_1A_2}$$

and, together,

$$V_{AA_1A_2B_1B_2} = \frac{i\sqrt{2}\, g^2 \lambda_{(i+1)\beta}}{3c_0^5 i_+^3} \Big(\left[\overline{\phi}_{AC},\, \psi^{\beta C} \right] \varepsilon_{A_1A_2B_1B_2} + \varepsilon_{AA_1A_2C} \left[\overline{\phi}_{B_1B_2},\, \psi^{\beta C} \right]$$
$$- \varepsilon_{AB_1B_2C} \left[\overline{\phi}_{A_1A_2},\, \psi^{\beta C} \right] - 4\varepsilon_{AA_1B_1C} \psi^{\beta C} \overline{\phi}_{A_2B_2} - 8\varepsilon_{A_1A_2B_1C} \overline{\phi}_{AB_2} \psi^{\beta C} \Big)$$
$$+ \frac{2i\sqrt{2}\, g^2 i_\pm \lambda_{i\gamma}}{c_0^5 i_- i_+^3} \varepsilon_{AB_1B_2C} \psi^{\gamma C} \overline{\phi}_{A_1A_2}$$

Fifth Order ($d = 4$, $k = 1$ and $l = 3$)

Now consider $d = 4$ and $k = 1$ and $l = 3$.

$$\begin{aligned}
V_{AA_1B_1B_2B_3} &= -\frac{\lambda_{(i+1)\alpha}}{2c_0 i_+}\left(-q^\alpha_A(V_{A_1B_1B_2B_3}) + iX^{\alpha(1)}_{(i+1)AB_3}V_{A_1B_1B_2} + iX^{\alpha(2)}_{(i+1)AB_2B_3}V_{A_1B_1}\right)\\
&= -\frac{\lambda_{(i+1)\alpha}}{2c_0 i_+}\left(\frac{g\lambda_{i\gamma}\lambda_{i\beta}q^\alpha_A(F^{\gamma\beta})}{3c_0^4 i_+^3}\varepsilon_{A_1B_1B_2B_3}\right.\\
&\qquad + i\left(-\frac{2i\sqrt{2}\,g\lambda^\alpha_i}{c_0 i_+}\overline{\phi}_{AB_3}\right)\left(-\frac{2ig\lambda_{i\gamma}\psi^{\gamma C}}{c_0^3 i_+^2}\varepsilon_{A_1B_1B_2C}\right)\\
&\qquad \left. + i\left(\varepsilon_{AB_2B_3C}\frac{2g\lambda^\alpha_i\lambda_{i\gamma}\psi^{\gamma C}}{c_0^2 i_+^2}\right)\left(\frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{A_1B_1}\right)\right)\\
&=: (1) + (2) + (3)
\end{aligned}$$

We calculate, using Cor. 1.3.5 and Lem. 1.4.1,

$$\begin{aligned}
(1) &= -\frac{g\lambda_{(i+1)\alpha}\lambda_{i\gamma}\lambda_{i\beta}}{6c_0^5 i_+^4}q^\alpha_A(F^{\gamma\beta})\varepsilon_{A_1B_1B_2B_3}\\
&= \frac{g\lambda_{(i+1)\alpha}\lambda_{i\gamma}\lambda_{i\beta}}{3c_0^5 i_+^4}\left(\epsilon^{\alpha\beta}D^\gamma{}_{\dot\gamma}\tilde\psi^{\dot\gamma}_A + \epsilon^{\alpha\gamma}D^\beta{}_{\dot\gamma}\tilde\psi^{\dot\gamma}_A\right)\varepsilon_{A_1B_1B_2B_3}\\
&= \frac{g}{3c_0^5 i_+^3}\left(\lambda_{i\gamma}D^\gamma{}_{\dot\gamma}\tilde\psi^{\dot\gamma}_A + \lambda_{i\beta}D^\beta{}_{\dot\gamma}\tilde\psi^{\dot\gamma}_A\right)\varepsilon_{A_1B_1B_2B_3}\\
&= \frac{2g\lambda_{i\gamma}D^\gamma{}_{\dot\gamma}\tilde\psi^{\dot\gamma}_A}{3c_0^5 i_+^3}\varepsilon_{A_1B_1B_2B_3}\\
&= -\frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}\left[\overline{\phi}_{AC},\,\psi^{\gamma C}\right]}{3c_0^5 i_+^3}\varepsilon_{A_1B_1B_2B_3}
\end{aligned}$$

Moreover

$$\begin{aligned}
(2) &= -\frac{1}{2c_0}i\left(\frac{2i\sqrt{2}\,g}{c_0 i_+}\overline{\phi}_{AB_3}\right)\left(\frac{2ig\lambda_{i\gamma}\psi^{\gamma C}}{c_0^3 i_+^2}\varepsilon_{A_1B_1B_2C}\right)\\
&= \frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}}{c_0^5 i_+^3}\overline{\phi}_{AB_3}\psi^{\gamma C}\varepsilon_{A_1B_1B_2C}
\end{aligned}$$

and

$$\begin{aligned}
(3) &= -\frac{i}{2c_0}\left(\varepsilon_{AB_2B_3C}\frac{2g\lambda_{i\gamma}\psi^{\gamma C}}{c_0^2 i_+^2}\right)\left(\frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{A_1B_1}\right)\\
&= -\frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}}{c_0^5 i_+^3}\psi^{\gamma C}\overline{\phi}_{A_1B_1}\varepsilon_{AB_2B_3C}
\end{aligned}$$

such that

$$(2) + (3) \cong \frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}}{c_0^5 i_+^3}\left[\overline{\phi}_{AB_1},\,\psi^{\gamma C}\right]_+\varepsilon_{A_1B_2B_3C}$$

and

$$V_{AA_1B_1B_2B_3} = \frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}}{3c_0^5 i_+^3}\left(-\left[\overline{\phi}_{AC},\,\psi^{\gamma C}\right]\varepsilon_{A_1B_1B_2B_3} + 3\left[\overline{\phi}_{AB_1},\,\psi^{\gamma C}\right]_+\varepsilon_{A_1B_2B_3C}\right)$$

Fifth Order ($d = 4$, $k = 0$ and $l = 4$)

It is easy to see that $V_{AB_1B_2B_3B_4} = 0$.

2.3.3 Sixth Order

Sixth Order ($d = 5$, $k = 5$ and $l = 0$)

Obviously $V_{AA_1A_2A_3A_4A_5} = 0$.

Sixth Order ($d = 5$, $k = 4$ and $l = 1$)

Obviously $V_{AA_1A_2A_3A_4B_1} = 0$.

Sixth Order ($d = 5$, $k = 3$ and $l = 2$)

Now consider $d = 5$ and $k = 3$ and $l = 2$.

$$\begin{aligned} V_{AA_1A_2A_3B_1B_2} = \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+} \Big(& -q^\alpha_A(V_{A_1A_2A_3B_1B_2}) + iX^{\alpha(1)}_{(i+1)AB_2} V_{A_1A_2A_3B_1} \\ & + iX^{\alpha(2)}_{(i+1)AB_1B_2} V_{A_1A_2A_3} - iV_{A_1A_2B_1B_2} X^{\alpha(1)}_{iAA_3} \\ & + iV_{A_1B_1B_2} X^{\alpha(2)}_{iAA_2A_3} \Big) \\ =: & \ (1) + (2) + (3) + (4) + (5) \end{aligned}$$

We denote the terms of (1) corresponding to the six terms of $V_{A_1A_2A_3B_1B_2}$ by

$$(1) =: (1a) + (1b) + (1c) + (1d) + (1e) + (1f)$$

We calculate $(1a)$. It consists of three terms, one corresponding to the variation of $\overline{\phi}_{A_1C}$ and two correponding to the variation of $\psi^{\beta C}$. The first term vanishes since it is proportional to

$$(1a1) \sim \lambda_{(i+1)\alpha}\lambda_{(i+1)\beta} \left[q^\alpha_A(\overline{\phi}_{A_1C}),\, \psi^{\beta C} \right] \sim \lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}\varepsilon_{AA_1CD} \left[\psi^{\alpha D},\, \psi^{\beta C} \right]_+ = 0$$

and vanishes since it is symmetric $\alpha\beta$ but antisymmetric in CD. The third term is propertional to

$$(1a3) \sim \lambda_{(i+1)\alpha}\lambda_{(i+1)\beta} \left[\overline{\phi}_{A_1C},\, \epsilon^{\beta\alpha} \dots \right] = 0$$

and obviously also vanishes. Therefore

$$\begin{aligned} (1a) &= -\frac{\lambda_{(i+1)\alpha}}{4c_0 i_+} \frac{i\sqrt{2}\, g^2 \lambda_{(i+1)\beta}}{3c_0^5 i_+^3} \varepsilon_{A_2A_3B_1B_2} \left[\overline{\phi}_{A_1C},\, \frac{i}{2} F^{\beta\alpha} \delta^C_A \right] \\ &= \frac{\sqrt{2} g^2 \lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{24 c_0^6 i_+^4} \varepsilon_{A_2A_3B_1B_2} \left[\overline{\phi}_{A_1A},\, F^{\beta\alpha} \right] \end{aligned}$$

Now consider $(1bc) := (1b) + (1c)$.

$$\begin{aligned}
(1bc1) &= -\frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\frac{i\sqrt{2}\,g^2\lambda_{(i+1)\beta}}{3c_0^5 i_+^3}\Big(\varepsilon_{A_1A_2A_3C}\left[q_A^\alpha(\overline{\phi}_{B_1B_2}),\,\psi^{\beta C}\right] \\
&\qquad -\varepsilon_{A_1B_1B_2C}\left[q_A^\alpha(\overline{\phi}_{A_2A_3}),\,\psi^{\beta C}\right]\Big) \\
&= \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+}\frac{g^2\lambda_{(i+1)\beta}}{3c_0^5 i_+^3}\left(\varepsilon_{A_1A_2A_3C}\varepsilon_{AB_1B_2D} - \varepsilon_{AA_2A_3D}\varepsilon_{A_1B_1B_2C}\right)\left[\psi^{\alpha D},\,\psi^{\beta C}\right]_+ \\
&= \frac{g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{6c_0^6 i_+^4}\left(\varepsilon_{A_1A_2A_3C}\varepsilon_{AB_1B_2D} - \varepsilon_{AA_2A_3C}\varepsilon_{A_1B_1B_2D}\right)\left[\psi^{\alpha D},\,\psi^{\beta C}\right]_+ \\
&\cong \frac{g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{3c_0^6 i_+^4}\varepsilon_{A_1A_2A_3C}\varepsilon_{AB_1B_2D}\left[\psi^{\alpha D},\,\psi^{\beta C}\right]_+
\end{aligned}$$

As before, $(1bc3) \sim \lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}\epsilon^{\beta\alpha} = 0$ and

$$\begin{aligned}
(1bc2) &= -\frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\frac{i\sqrt{2}\,g^2\lambda_{(i+1)\beta}}{3c_0^5 i_+^3}\Big(\varepsilon_{A_1A_2A_3C}\left[\overline{\phi}_{B_1B_2},\,\frac{i}{2}F^{\beta\alpha}\delta_A^C\right] \\
&\qquad -\varepsilon_{A_1B_1B_2C}\left[\overline{\phi}_{A_2A_3},\,\frac{i}{2}F^{\beta\alpha}\delta_A^C\right]\Big) \\
&= \frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{24c_0^6 i_+^4}\Big(\varepsilon_{A_1A_2A_3A}\left[\overline{\phi}_{B_1B_2},\,F^{\beta\alpha}\right] - \varepsilon_{A_1B_1B_2A}\left[\overline{\phi}_{A_2A_3},\,F^{\beta\alpha}\right]\Big)
\end{aligned}$$

We calculate $(1de) := (1d) + (1e)$.

$$\begin{aligned}
&(1de) \\
&= -\frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\frac{i\sqrt{2}\,g^2\lambda_{(i+1)\beta}}{3c_0^5 i_+^3}\Big(-4\varepsilon_{A_1A_2B_1C}q_A^\alpha(\psi^{\beta C}\overline{\phi}_{A_3B_2}) - 8\varepsilon_{A_2A_3B_1C}q_A^\alpha(\overline{\phi}_{A_1B_2}\psi^{\beta C})\Big) \\
&= \frac{i\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{3c_0^6 i_+^4}\Big(\varepsilon_{A_1A_2B_1C}\frac{i}{2}F^{\beta\alpha}\delta_A^C\overline{\phi}_{A_3B_2} - \varepsilon_{A_1A_2B_1C}\psi^{\beta C}i\sqrt{2}\varepsilon_{AA_3B_2D}\psi^{\alpha D} \\
&\qquad +2\varepsilon_{A_2A_3B_1C}i\sqrt{2}\varepsilon_{AA_1B_2D}\psi^{\alpha D}\psi^{\beta C} + 2\varepsilon_{A_2A_3B_1C}\frac{i}{2}\overline{\phi}_{A_1B_2}F^{\beta\alpha}\delta_A^C\Big) \\
&= -\frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{6c_0^6 i_+^4}\Big(\varepsilon_{A_1A_2B_1A}F^{\beta\alpha}\overline{\phi}_{A_3B_2} + 2\varepsilon_{A_2A_3B_1A}\overline{\phi}_{A_1B_2}F^{\beta\alpha}\Big) \\
&\quad + \frac{2g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{3c_0^6 i_+^4}\Big(\varepsilon_{A_1A_2B_1C}\varepsilon_{AA_3B_2D}\psi^{\beta C}\psi^{\alpha D} - 2\varepsilon_{A_2A_3B_1C}\varepsilon_{AA_1B_2D}\psi^{\alpha D}\psi^{\beta C}\Big) \\
&\cong -\frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{6c_0^6 i_+^4}\varepsilon_{A_1A_2B_1A}\Big(F^{\beta\alpha}\overline{\phi}_{A_3B_2} + 2\overline{\phi}_{A_3B_2}F^{\beta\alpha}\Big) \\
&\quad + \frac{2g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{3c_0^6 i_+^4}\left(\varepsilon_{A_1A_2B_1C}\varepsilon_{AA_3B_2D} - 2\varepsilon_{AA_1B_2C}\varepsilon_{A_2A_3B_1D}\right)\psi^{\beta C}\psi^{\alpha D} \\
&\cong -\frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{6c_0^6 i_+^4}\varepsilon_{A_1A_2B_1A}\Big(F^{\beta\alpha}\overline{\phi}_{A_3B_2} + 2\overline{\phi}_{A_3B_2}F^{\beta\alpha}\Big) \\
&\quad + \frac{2g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{c_0^6 i_+^4}\varepsilon_{A_1A_2B_1C}\varepsilon_{AA_3B_2D}\psi^{\beta C}\psi^{\alpha D}
\end{aligned}$$

We calculate $(1f)$.

$$\begin{aligned}(1f) &= -\frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\frac{2i\sqrt{2}\,g^2 i_\pm \lambda_{i\gamma}}{c_0^5 i_- i_+^3}\varepsilon_{A_1B_1B_2C}q_A^\alpha(\psi^{\gamma C}\overline{\phi}_{A_2A_3})\\ &= -\frac{2i\sqrt{2}\,g^2 i_\pm \lambda_{(i+1)\alpha}\lambda_{i\gamma}}{4c_0^6 i_- i_+^4}\varepsilon_{A_1B_1B_2C}\left(\frac{i}{2}F^{\gamma\alpha}\delta_A^C\overline{\phi}_{A_2A_3} + i\epsilon^{\gamma\alpha}g\left[\overline{\phi}_{AD},\, \phi^{CD}\right]\overline{\phi}_{A_2A_3}\right.\\ &\qquad\qquad \left. -\psi^{\gamma C} i\sqrt{2}\varepsilon_{AA_2A_3D}\psi^{\alpha D}\right)\\ &=: (1f1) + (1f2) + (1f3)\end{aligned}$$

with

$$\begin{aligned}(1f1) &= \frac{\sqrt{2}\,g^2 i_\pm \lambda_{(i+1)\alpha}\lambda_{i\gamma}}{4c_0^6 i_- i_+^4}\varepsilon_{A_1B_1B_2A}F^{\gamma\alpha}\overline{\phi}_{A_2A_3}\\ (1f3) &= -\frac{g^2 i_\pm \lambda_{(i+1)\alpha}\lambda_{i\gamma}}{c_0^6 i_- i_+^4}\varepsilon_{A_1B_1B_2C}\varepsilon_{AA_2A_3D}\psi^{\gamma C}\psi^{\alpha D}\end{aligned}$$

and

$$\begin{aligned}(1f2) &= \frac{2\sqrt{2}\,g^3 i_\pm}{4c_0^6 i_- i_+^3}\varepsilon_{CA_1B_1B_2}\left[\overline{\phi}_{AD},\, \phi^{CD}\right]\overline{\phi}_{A_2A_3}\\ &= \frac{\sqrt{2}\,g^3 i_\pm}{4c_0^6 i_- i_+^3}\varepsilon_{CA_1B_1B_2}\varepsilon_{CDEF}\left[\overline{\phi}_{AD},\, \overline{\phi}_{EF}\right]\overline{\phi}_{A_2A_3}\\ &= \frac{\sqrt{2}\,g^3 i_\pm}{2c_0^6 i_- i_+^3}(\delta^{DEF}_{A_1B_1B_2} + \text{cyclic})\left[\overline{\phi}_{AD},\, \overline{\phi}_{EF}\right]\overline{\phi}_{A_2A_3}\\ &= \frac{\sqrt{2}\,g^3 i_\pm}{2c_0^6 i_- i_+^3}(\left[\overline{\phi}_{AA_1},\, \overline{\phi}_{B_1B_2}\right] + \left[\overline{\phi}_{AB_1},\, \overline{\phi}_{B_2A_1}\right] + \left[\overline{\phi}_{AB_2},\, \overline{\phi}_{A_1B_1}\right])\overline{\phi}_{A_2A_3}\\ &\cong \frac{\sqrt{2}\,g^3 i_\pm}{2c_0^6 i_- i_+^3}\left(\left[\overline{\phi}_{AA_1},\, \overline{\phi}_{B_1B_2}\right] + 2\left[\overline{\phi}_{AB_1},\, \overline{\phi}_{B_2A_1}\right]\right)\overline{\phi}_{A_2A_3}\end{aligned}$$

Moreover

$$\begin{aligned}(2) &= \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}i\left(-\frac{2i\sqrt{2}\,g\lambda_i^\alpha}{c_0 i_+}\overline{\phi}_{AB_2}\right)\\ &\qquad\qquad \cdot\left(-\frac{g\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}}{3c_0^4 i_+^3}\varepsilon_{A_1A_2A_3B_1} - \frac{4g^2 i_\pm}{c_0^4 i_- i_+^2}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2A_3}\right)\\ &= \left(\frac{\sqrt{2}g}{2c_0^2 i_+}\overline{\phi}_{AB_2}\right)\left(-\frac{g\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}}{3c_0^4 i_+^3}\varepsilon_{A_1A_2A_3B_1} - \frac{4g^2 i_\pm}{c_0^4 i_- i_+^2}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2A_3}\right)\\ &= -\frac{\sqrt{2}g^2}{6c_0^6}\frac{\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma}\overline{\phi}_{AB_2}F^{\beta\gamma}}{i_+^4}\varepsilon_{A_1A_2A_3B_1} - \frac{2\sqrt{2}g^3 i_\pm}{c_0^6 i_- i_+^3}\overline{\phi}_{AB_2}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2A_3}\end{aligned}$$

and

$$
\begin{aligned}
(3) &= \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+} i \left(\varepsilon_{AB_1B_2D} \frac{2g\lambda_i^\alpha \lambda_{i\beta}\psi^{\beta D}}{c_0^2 i_+^2} \right) \\
&\qquad \cdot \left(\frac{2ig}{3c_0^3} \frac{i_\pm \left(-i_-\lambda_{(i+1)\gamma} + i_+\lambda_{(i-1)\gamma}\right)\psi^{\gamma C}}{i_-^2 i_+^2} \varepsilon_{A_1A_2A_3C} \right) \\
&= \left(\frac{ig\lambda_{i\beta}\psi^{\beta D}}{2c_0^3 i_+^2} \varepsilon_{AB_1B_2D} \right) \left(\frac{2ig}{3c_0^3} \frac{i_\pm \left(-i_-\lambda_{(i+1)\gamma} + i_+\lambda_{(i-1)\gamma}\right)\psi^{\gamma C}}{i_-^2 i_+^2} \varepsilon_{A_1A_2A_3C} \right) \\
&= \frac{g^2}{3c_0^6} \frac{i_\pm \lambda_{i\beta} \left(i_-\lambda_{(i+1)\gamma} - i_+\lambda_{(i-1)\gamma}\right)\psi^{\beta D}\psi^{\gamma C}}{i_-^2 i_+^4} \varepsilon_{AB_1B_2D}\varepsilon_{A_1A_2A_3C}
\end{aligned}
$$

and

$$
\begin{aligned}
(4) &= \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}(-i) \left(\frac{g\lambda_{i\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}}{2c_0^4 i_+^3} \varepsilon_{A_1A_2B_1B_2} - \frac{g^2}{c_0^4 i_+^2} \left[\overline{\phi}_{A_1A_2}, \overline{\phi}_{B_1B_2}\right] \right. \\
&\qquad \left. - \frac{4g^2}{c_0^4 i_+^2} \overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2} \right) \cdot \left(- \frac{2i\sqrt{2}\, g\lambda_{i-1}^\alpha}{c_0 i_-} \overline{\phi}_{AA_3} \right) \\
&= - \left(\frac{g\lambda_{i\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}}{2c_0^4 i_+^3} \varepsilon_{A_1A_2B_1B_2} - \frac{g^2}{c_0^4 i_+^2} \left[\overline{\phi}_{A_1A_2}, \overline{\phi}_{B_1B_2}\right] \right. \\
&\qquad \left. - \frac{4g^2}{c_0^4 i_+^2} \overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2} \right) \cdot \left(\frac{\sqrt{2} g i_\pm}{2c_0^2 i_- i_+} \overline{\phi}_{AA_3} \right) \\
&= - \frac{\sqrt{2} g^2 i_\pm}{4c_0^6 i_- i_+^4} \lambda_{i\beta}\lambda_{(i+1)\gamma} F^{\beta\gamma} \overline{\phi}_{AA_3} \varepsilon_{A_1A_2B_1B_2} + \frac{\sqrt{2} g^3 i_\pm}{2c_0^6 i_- i_+^3} \left[\overline{\phi}_{A_1A_2}, \overline{\phi}_{B_1B_2}\right] \overline{\phi}_{AA_3} \\
&\quad + \frac{2\sqrt{2} g^3 i_\pm}{c_0^6 i_- i_+^3} \overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\overline{\phi}_{AA_3}
\end{aligned}
$$

and

$$
\begin{aligned}
(5) &= \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+} i \left(- \frac{2ig\lambda_{i\beta}\psi^{\beta D}}{c_0^3 i_+^2} \varepsilon_{A_1B_1B_2D} \right) \left(\varepsilon_{AA_2A_3C} \frac{2g\lambda_{i-1}^\alpha \lambda_{(i-1)\gamma}\psi^{\gamma C}}{c_0^2 i_-^2} \right) \\
&= \left(\frac{2g\lambda_{i\beta}\psi^{\beta D}}{c_0^3 i_+^2} \varepsilon_{A_1B_1B_2D} \right) \left(\varepsilon_{AA_2A_3C} \frac{g i_\pm \lambda_{(i-1)\gamma}\psi^{\gamma C}}{2c_0^3 i_-^2 i_+} \right) \\
&= \frac{g^2 i_\pm \lambda_{i\beta}\lambda_{(i-1)\gamma}\psi^{\beta D}\psi^{\gamma C}}{c_0^6 i_-^2 i_+^3} \varepsilon_{A_1B_1B_2D}\varepsilon_{AA_2A_3C}
\end{aligned}
$$

Collecting terms, we thus obtain

$$
V_{AA_1A_2A_3B_1B_2} = V_{AA_1A_2A_3B_1B_2}|_{\phi F} + V_{AA_1A_2A_3B_1B_2}|_{\phi^3} + V_{AA_1A_2A_3B_1B_2}|_{\psi^2}
$$

where

$$
\begin{aligned}
&V_{AA_1A_2A_3B_1B_2}|_{\phi F}\\
&\quad= \frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{24c_0^6 i_+^4}\varepsilon_{A_2A_3B_1B_2}\left[\overline{\phi}_{A_1A},\, F^{\beta\alpha}\right]\\
&\qquad+\frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{24c_0^6 i_+^4}\left(\varepsilon_{A_1A_2A_3A}\left[\overline{\phi}_{B_1B_2},\, F^{\beta\alpha}\right]-\varepsilon_{A_1B_1B_2A}\left[\overline{\phi}_{A_2A_3},\, F^{\beta\alpha}\right]\right)\\
&\qquad-\frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{6c_0^6 i_+^4}\varepsilon_{A_1A_2B_1A}\left(F^{\beta\alpha}\overline{\phi}_{A_3B_2}+2\overline{\phi}_{A_3B_2}F^{\beta\alpha}\right)\\
&\qquad+\frac{\sqrt{2}\,g^2 i_\pm\lambda_{(i+1)\alpha}\lambda_{i\gamma}}{4c_0^6 i_- i_+^4}\varepsilon_{A_1B_1B_2A}F^{\gamma\alpha}\overline{\phi}_{A_2A_3}\\
&\qquad-\frac{\sqrt{2}g^2}{6c_0^6}\frac{\lambda_{(i+1)\beta}\lambda_{(i+1)\gamma}\overline{\phi}_{AB_2}F^{\beta\gamma}}{i_+^4}\varepsilon_{A_1A_2A_3B_1}\\
&\qquad-\frac{\sqrt{2}g^2 i_\pm}{4c_0^6 i_- i_+^4}\lambda_{i\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}\overline{\phi}_{AA_3}\varepsilon_{A_1A_2B_1B_2}
\end{aligned}
$$

Here, we see that the first and third terms cancel upon contraction with the η terms. Moreover, the last and third-to-last terms are equal. Moreover, the fifth term equals (up to factor 2) the second-to-last. Therefore

$$
\begin{aligned}
V_{AA_1A_2A_3B_1B_2}|_{\phi F} &= \frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{24c_0^6 i_+^4}\varepsilon_{A_1A_2A_3A}\left[\overline{\phi}_{B_1B_2},\, F^{\beta\alpha}\right]\\
&\quad-\frac{\sqrt{2}g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{6c_0^6 i_+^4}\varepsilon_{A_1A_2B_1A}\left(F^{\beta\alpha}\overline{\phi}_{A_3B_2}+3\overline{\phi}_{A_3B_2}F^{\beta\alpha}\right)\\
&\quad-\frac{\sqrt{2}g^2 i_\pm}{2c_0^6 i_- i_+^4}\lambda_{i\beta}\lambda_{(i+1)\gamma}F^{\beta\gamma}\overline{\phi}_{AA_3}\varepsilon_{A_1A_2B_1B_2}
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&V_{AA_1A_2A_3B_1B_2}|_{\phi^3}\\
&\quad=\frac{\sqrt{2}\,g^3 i_\pm}{2c_0^6 i_- i_+^3}\left(\left[\overline{\phi}_{AA_1},\,\overline{\phi}_{B_1B_2}\right]+2\left[\overline{\phi}_{AB_1},\,\overline{\phi}_{B_2A_1}\right]\right)\overline{\phi}_{A_2A_3}\\
&\qquad-\frac{2\sqrt{2}g^3 i_\pm}{c_0^6 i_- i_+^3}\overline{\phi}_{AB_2}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2A_3}+\frac{\sqrt{2}g^3 i_\pm}{2c_0^6 i_- i_+^3}\left[\overline{\phi}_{A_1A_2},\,\overline{\phi}_{B_1B_2}\right]\overline{\phi}_{AA_3}\\
&\qquad+\frac{2\sqrt{2}g^3 i_\pm}{c_0^6 i_- i_+^3}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\overline{\phi}_{AA_3}\\
&\quad=\frac{\sqrt{2}\,g^3 i_\pm}{2c_0^6 i_- i_+^3}\left(\left[\overline{\phi}_{AA_1},\,\overline{\phi}_{B_1B_2}\right]\overline{\phi}_{A_2A_3}+2\left[\overline{\phi}_{AB_1},\,\overline{\phi}_{B_2A_1}\right]\overline{\phi}_{A_2A_3}\right.\\
&\qquad\quad\left.-4\overline{\phi}_{AB_2}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2A_3}+\left[\overline{\phi}_{A_1A_2},\,\overline{\phi}_{B_1B_2}\right]\overline{\phi}_{AA_3}+4\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\overline{\phi}_{AA_3}\right)\\
&\quad\cong\frac{\sqrt{2}\,g^3 i_\pm}{2c_0^6 i_- i_+^3}\left(2\left[\overline{\phi}_{AA_1},\,\overline{\phi}_{B_1B_2}\right]\overline{\phi}_{A_2A_3}+2\left[\overline{\phi}_{AB_1},\,\overline{\phi}_{B_2A_1}\right]\overline{\phi}_{A_2A_3}+8\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\overline{\phi}_{AA_3}\right)\\
&\quad\cong\frac{\sqrt{2}\,g^3 i_\pm}{2c_0^6 i_- i_+^3}\left(2\left[\overline{\phi}_{AA_1},\,\overline{\phi}_{B_1B_2}\right]\overline{\phi}_{A_2A_3}+8\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\overline{\phi}_{AA_3}\right)
\end{aligned}
$$

and

$$\begin{aligned}
V_{AA_1A_2A_3B_1B_2}|_{\psi^2} &= \frac{g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{3c_0^6 i_+^4}\varepsilon_{A_1A_2A_3C}\varepsilon_{AB_1B_2D}\left[\psi^{\alpha D},\,\psi^{\beta C}\right]_+ \\
&\quad + \frac{2g^2\lambda_{(i+1)\alpha}\lambda_{(i+1)\beta}}{c_0^6 i_+^4}\varepsilon_{A_1A_2B_1C}\varepsilon_{AA_3B_2D}\psi^{\beta C}\psi^{\alpha D} \\
&\quad - \frac{g^2 i_\pm\lambda_{(i+1)\alpha}\lambda_{i\gamma}}{c_0^6 i_- i_+^4}\varepsilon_{A_1B_1B_2C}\varepsilon_{AA_2A_3D}\psi^{\gamma C}\psi^{\alpha D} \\
&\quad + \frac{g^2}{3c_0^6}\frac{i_\pm\lambda_{i\beta}\left(i_-\lambda_{(i+1)\gamma} - i_+\lambda_{(i-1)\gamma}\right)\psi^{\beta D}\psi^{\gamma C}}{i_-^2 i_+^4}\varepsilon_{AB_1B_2D}\varepsilon_{A_1A_2A_3C} \\
&\quad + \frac{g^2 i_\pm\lambda_{i\beta}\lambda_{(i-1)\gamma}\psi^{\beta D}\psi^{\gamma C}}{c_0^6 i_-^2 i_+^3}\varepsilon_{A_1B_1B_2D}\varepsilon_{AA_2A_3C} \\
&= \frac{g^2}{3c_0^6 i_-^2 i_+^4}\big(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta}\varepsilon_{A_1A_2A_3D}\varepsilon_{AB_1B_2C} \\
&\qquad + i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta}\varepsilon_{A_1A_2A_3C}\varepsilon_{AB_1B_2D} \\
&\qquad + 6i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta}\varepsilon_{A_1A_2B_1C}\varepsilon_{AA_3B_2D} \\
&\qquad - 3i_-i_\pm\lambda_{i\gamma}\lambda_{(i+1)\delta}\varepsilon_{A_1B_1B_2C}\varepsilon_{AA_2A_3D} \\
&\qquad + i_\pm\lambda_{i\gamma}\left(i_-\lambda_{(i+1)\delta} - i_+\lambda_{(i-1)\delta}\right)\varepsilon_{AB_1B_2C}\varepsilon_{A_1A_2A_3D} \\
&\qquad + 3i_+i_\pm\lambda_{i\gamma}\lambda_{(i-1)\delta}\varepsilon_{A_1B_1B_2C}\varepsilon_{AA_2A_3D}\big)\,\psi^{\gamma C}\psi^{\delta D}
\end{aligned}$$

Therefore

$$\begin{aligned}
&V_{AA_1A_2A_3B_1B_2}|_{\psi^2} \\
&\cong \frac{g^2}{3c_0^6 i_-^2 i_+^4}\big(\varepsilon_{A_1A_2A_3C}\varepsilon_{AB_1B_2D}(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta}) \\
&\qquad + \varepsilon_{AB_1B_2C}\varepsilon_{A_1A_2A_3D}(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta} + 3i_-i_\pm\lambda_{i\gamma}\lambda_{(i+1)\delta} \\
&\qquad\qquad + i_\pm\lambda_{i\gamma}\left(i_-\lambda_{(i+1)\delta} - i_+\lambda_{(i-1)\delta}\right) - 3i_+i_\pm\lambda_{i\gamma}\lambda_{(i-1)\delta}) \\
&\qquad + \varepsilon_{A_1A_2B_1C}\varepsilon_{AA_3B_2D}(6i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta})\big)\,\psi^{\gamma C}\psi^{\delta D} \\
&= \frac{g^2}{3c_0^6 i_-^2 i_+^4}\big(\varepsilon_{A_1A_2A_3C}\varepsilon_{AB_1B_2D}(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta}) \\
&\qquad + \varepsilon_{AB_1B_2C}\varepsilon_{A_1A_2A_3D}(i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta} + 4i_-i_\pm\lambda_{i\gamma}\lambda_{(i+1)\delta} - 4i_+i_\pm\lambda_{i\gamma}\lambda_{(i-1)\delta}) \\
&\qquad + \varepsilon_{A_1A_2B_1C}\varepsilon_{AA_3B_2D}(6i_-^2\lambda_{(i+1)\gamma}\lambda_{(i+1)\delta})\big)\,\psi^{\gamma C}\psi^{\delta D}
\end{aligned}$$

This finishes the calculation.

Sixth Order ($d = 5$, $k = 2$ and $l = 3$)

Now consider $d = 5$ and $k = 2$ and $l = 3$.

$$\begin{aligned}
V_{AA_1A_2B_1B_2B_3} &= \frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\Big(-q_A^\alpha(V_{A_1A_2B_1B_2B_3}) + iX^{\alpha(1)}_{(i+1)AB_3}V_{A_1A_2B_1B_2} \\
&\qquad + iX^{\alpha(2)}_{(i+1)AB_2B_3}V_{A_1A_2B_1} + iX^{\alpha(3)}_{(i+1)AB_1B_2B_3}V_{A_1A_2} \\
&\qquad + iV_{A_1B_1B_2B_3}X^{\alpha(1)}_{iAA_2}\Big) \\
&=: (1) + (2) + (3) + (4) + (5)
\end{aligned}$$

We calculate

$$
\begin{aligned}
(1) &= -\frac{\lambda_{(i+1)\alpha}}{3c_0 i_+} q_A^\alpha (V_{A_1A_2B_1B_2B_3}) \\
&= -\frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4} \left(-q_A^\alpha \left[\overline{\phi}_{A_1C},\, \psi^{\gamma C}\right] \varepsilon_{A_2B_1B_2B_3} \right.\\
&\qquad \left. +3q_A^\alpha \left[\overline{\phi}_{A_1B_1},\, \psi^{\gamma C}\right]_+ \varepsilon_{A_2B_2B_3C}\right) \\
&= -\frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4} \left(-i\sqrt{2}\,\varepsilon_{AA_1CD} \left[\psi^{\alpha D},\, \psi^{\gamma C}\right]_+ \varepsilon_{A_2B_1B_2B_3} \right.\\
&\qquad + 3i\sqrt{2}\,\varepsilon_{AA_1B_1D} \left[\psi^{\alpha D},\, \psi^{\gamma C}\right] \varepsilon_{A_2B_2B_3C} \\
&\qquad - \left[\overline{\phi}_{A_1C},\, \frac{i}{2}F^{\gamma\alpha}\delta_A^C + i\epsilon^{\gamma\alpha} g \left[\overline{\phi}_{AD},\, \phi^{CD}\right]\right] \varepsilon_{A_2B_1B_2B_3} \\
&\qquad \left. +3\left[\overline{\phi}_{A_1B_1},\, \frac{i}{2}F^{\gamma\alpha}\delta_A^C + i\epsilon^{\gamma\alpha} g \left[\overline{\phi}_{AD},\, \phi^{CD}\right]\right]_+ \varepsilon_{A_2B_2B_3C}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(2) &= \frac{\lambda_{(i+1)\alpha}}{3c_0 i_+} i \left(-\frac{2i\sqrt{2}\,g\lambda_i^\alpha}{c_0 i_+}\overline{\phi}_{AB_3}\right) V_{A_1A_2B_1B_2} \\
&= \frac{2\sqrt{2}\,g}{3c_0^2 i_+}\overline{\phi}_{AB_3} V_{A_1A_2B_1B_2} \\
&= (2a) + (2b) + (2c)
\end{aligned}
$$

corresponding to the three terms of $V_{A_1A_2B_1B_2}$. Now

$$
\begin{aligned}
(2a) &= \frac{2\sqrt{2}\,g}{3c_0^2 i_+}\overline{\phi}_{AB_3} \left(\frac{g\lambda_{i\gamma}\lambda_{(i+1)\alpha}F^{\gamma\alpha}}{2c_0^4 i_+^3}\varepsilon_{A_1A_2B_1B_2}\right) \\
&= \frac{\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}\overline{\phi}_{AB_3}F^{\gamma\alpha}}{3c_0^6 i_+^4}\varepsilon_{A_1A_2B_1B_2}
\end{aligned}
$$

and

$$
\begin{aligned}
(2b) &= \frac{2\sqrt{2}\,g}{3c_0^2 i_+}\overline{\phi}_{AB_3} \left(-\frac{g^2}{c_0^4 i_+^2}\left[\overline{\phi}_{A_1A_2},\, \overline{\phi}_{B_1B_2}\right]\right) \\
&= -\frac{2\sqrt{2}\,g^3}{3c_0^6 i_+^3}\overline{\phi}_{AB_3}\left[\overline{\phi}_{A_1A_2},\, \overline{\phi}_{B_1B_2}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(2c) &= \frac{2\sqrt{2}\,g}{3c_0^2 i_+}\overline{\phi}_{AB_3} \left(-\frac{4g^2}{c_0^4 i_+^2}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\right) \\
&= -\frac{8\sqrt{2}\,g^3}{3c_0^6 i_+^3}\overline{\phi}_{AB_3}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}
\end{aligned}
$$

Moreover

$$\begin{aligned}(3) &= \frac{\lambda_{(i+1)\alpha}}{3c_0 i_+} i \left(\varepsilon_{AB_2B_3C} \frac{2g\lambda_i^\alpha \lambda_{i\gamma}\psi^{\gamma C}}{c_0^2 i_+^2}\right)\left(\frac{2ig\lambda_{(i+1)\beta}\psi^{\beta D}}{c_0^3 i_+^2}\varepsilon_{A_1A_2B_1D}\right)\\ &= -\frac{1}{3c_0}\left(\varepsilon_{AB_2B_3C}\frac{2g\lambda_{i\gamma}\psi^{\gamma C}}{c_0^2 i_+^2}\right)\left(\frac{2g\lambda_{(i+1)\beta}\psi^{\beta D}}{c_0^3 i_+^2}\varepsilon_{A_1A_2B_1D}\right)\\ &= -\frac{4g^2\lambda_{i\gamma}\lambda_{(i+1)\beta}\psi^{\gamma C}\psi^{\beta D}}{3c_0^6 i_+^4}\varepsilon_{AB_2B_3C}\varepsilon_{A_1A_2B_1D}\end{aligned}$$

and

$$\begin{aligned}(4) &= \frac{\lambda_{(i+1)\alpha}}{3c_0 i_+} i \left(\varepsilon_{AB_1B_2B_3}\frac{ig\lambda_i^\alpha\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}}{3c_0^3 i_+^3}\right)\left(-\frac{\sqrt{2}\,gi_\pm}{c_0^2 i_- i_+}\overline{\phi}_{A_1A_2}\right)\\ &= \frac{1}{3c_0}\left(\varepsilon_{AB_1B_2B_3}\frac{g\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}}{3c_0^3 i_+^3}\right)\left(\frac{\sqrt{2}\,gi_\pm}{c_0^2 i_- i_+}\overline{\phi}_{A_1A_2}\right)\\ &= \frac{\sqrt{2}\,g^2 i_\pm\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}\overline{\phi}_{A_1A_2}}{9c_0^6 i_- i_+^4}\varepsilon_{AB_1B_2B_3}\end{aligned}$$

and

$$\begin{aligned}(5) &= \frac{\lambda_{(i+1)\alpha}}{3c_0 i_+} i \left(-\frac{g\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}}{3c_0^4 i_+^3}\varepsilon_{A_1B_1B_2B_3}\right)\left(-\frac{2i\sqrt{2}\,g\lambda_{i-1}^\alpha}{c_0 i_-}\overline{\phi}_{AA_2}\right)\\ &= -\frac{i_\pm}{3c_0 i_+}\left(\frac{g\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}}{3c_0^4 i_+^3}\varepsilon_{A_1B_1B_2B_3}\right)\left(\frac{2\sqrt{2}\,g}{c_0 i_-}\overline{\phi}_{AA_2}\right)\\ &= -\frac{2\sqrt{2}\,g^2 i_\pm\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}\overline{\phi}_{AA_2}}{9c_0^6 i_- i_+^4}\varepsilon_{A_1B_1B_2B_3}\end{aligned}$$

such that

$$(4)+(5) \cong \frac{\sqrt{2}\,g^2 i_\pm\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}\overline{\phi}_{AA_1}}{3c_0^6 i_- i_+^4}\varepsilon_{A_2B_1B_2B_3}$$

Collecting terms, we thus obtain

$$V_{AA_1A_2B_1B_2B_3} = V_{AA_1A_2B_1B_2B_3}|_{\phi F} + V_{AA_1A_2B_1B_2B_3}|_{\phi^3} + V_{AA_1A_2B_1B_2B_3}|_{\psi^2}$$

where

$$\begin{aligned}
V_{AA_1A_2B_1B_2B_3}|_{\phi F} = &-\frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4}\left(-\left[\overline{\phi}_{A_1C},\,\frac{i}{2}F^{\gamma\alpha}\delta_A^C\right]\varepsilon_{A_2B_1B_2B_3}\right.\\
&\left.+3\left[\overline{\phi}_{A_1B_1},\,\frac{i}{2}F^{\gamma\alpha}\delta_A^C\right]_+\varepsilon_{A_2B_2B_3C}\right)\\
&+\frac{\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}\overline{\phi}_{AB_3}F^{\gamma\alpha}}{3c_0^6 i_+^4}\varepsilon_{A_1A_2B_1B_2}\\
&+\frac{\sqrt{2}\,g^2 i_\pm\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}\overline{\phi}_{AA_1}}{3c_0^6 i_- i_+^4}\varepsilon_{A_2B_1B_2B_3}\\
= &\frac{\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4}\left(\left[\overline{\phi}_{AA_1},\,F^{\gamma\alpha}\right]\varepsilon_{A_2B_1B_2B_3}\right.\\
&\left.+3\left[\overline{\phi}_{A_1B_1},\,F^{\gamma\alpha}\right]_+\varepsilon_{A_2B_2B_3A}+3\overline{\phi}_{AB_3}F^{\gamma\alpha}\varepsilon_{A_1A_2B_1B_2}\right)\\
&+\frac{\sqrt{2}\,g^2 i_\pm\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}\overline{\phi}_{AA_1}}{3c_0^6 i_- i_+^4}\varepsilon_{A_2B_1B_2B_3}
\end{aligned}$$

and

$$\begin{aligned}
V_{AA_1A_2B_1B_2B_3}|_{\phi^3} = &-\frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4}\left(-\left[\overline{\phi}_{A_1C},\,i\epsilon^{\gamma\alpha}g\left[\overline{\phi}_{AD},\,\phi^{CD}\right]\right]\varepsilon_{A_2B_1B_2B_3}\right.\\
&\left.+3\left[\overline{\phi}_{A_1B_1},\,i\epsilon^{\gamma\alpha}g\left[\overline{\phi}_{AD},\,\phi^{CD}\right]\right]_+\varepsilon_{A_2B_2B_3C}\right)\\
&-\frac{2\sqrt{2}\,g^3}{3c_0^6 i_+^3}\overline{\phi}_{AB_3}\left[\overline{\phi}_{A_1A_2},\,\overline{\phi}_{B_1B_2}\right]-\frac{8\sqrt{2}\,g^3}{3c_0^6 i_+^3}\overline{\phi}_{AB_3}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\\
= &\frac{2\sqrt{2}\,g^3}{9c_0^6 i_+^3}\left(\left[\overline{\phi}_{A_1C},\,\left[\overline{\phi}_{AD},\,\phi^{CD}\right]\right]\varepsilon_{A_2B_1B_2B_3}\right.\\
&-3\left[\overline{\phi}_{A_1B_1},\,\left[\overline{\phi}_{AD},\,\phi^{CD}\right]\right]_+\varepsilon_{A_2B_2B_3C}\\
&\left.-3\overline{\phi}_{AB_3}\left[\overline{\phi}_{A_1A_2},\,\overline{\phi}_{B_1B_2}\right]-12\overline{\phi}_{AB_3}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\right)\\
= &\frac{2\sqrt{2}\,g^3}{18c_0^6 i_+^3}\left(\left[\overline{\phi}_{A_1C},\,\left[\overline{\phi}_{AD},\,\overline{\phi}_{EF}\right]\right]\varepsilon_{CDEF}\varepsilon_{A_2B_1B_2B_3}\right.\\
&+3\left[\overline{\phi}_{A_1B_1},\,\left[\overline{\phi}_{AD},\,\phi_{EF}\right]\right]_+\varepsilon_{CDEF}\varepsilon_{CA_2B_2B_3}\\
&\left.-6\overline{\phi}_{AB_3}\left[\overline{\phi}_{A_1A_2},\,\overline{\phi}_{B_1B_2}\right]-24\overline{\phi}_{AB_3}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\right)
\end{aligned}$$

In the last expression, we calculate the second term as follows.

$$\begin{aligned}
&3\left[\overline{\phi}_{A_1B_1},\,\left[\overline{\phi}_{AD},\,\phi_{EF}\right]\right]_+\varepsilon_{CDEF}\varepsilon_{CA_2B_2B_3}\\
&\quad=6\left[\overline{\phi}_{A_1B_1},\,\left[\overline{\phi}_{AA_2},\,\phi_{B_2B_3}\right]+\left[\overline{\phi}_{AB_2},\,\phi_{B_3A_2}\right]+\left[\overline{\phi}_{AB_3},\,\phi_{A_2B_2}\right]\right]_+\\
&\quad=6\left[\overline{\phi}_{A_1B_1},\,\left[\overline{\phi}_{AA_2},\,\phi_{B_2B_3}\right]\right]_+-6\overline{\phi}_{A_1B_1}\left[\overline{\phi}_{AB_2},\,\phi_{A_2B_3}\right]-6\left[\overline{\phi}_{AB_2},\,\phi_{A_2B_3}\right]\overline{\phi}_{A_1B_1}\\
&\qquad\quad+6\overline{\phi}_{A_1B_1}\left[\overline{\phi}_{AB_3},\,\phi_{A_2B_2}\right]+6\left[\overline{\phi}_{AB_3},\,\phi_{A_2B_2}\right]\overline{\phi}_{A_1B_1}\\
&=6\left[\overline{\phi}_{A_1B_1},\,\left[\overline{\phi}_{AA_2},\,\phi_{B_2B_3}\right]\right]_+-12\overline{\phi}_{A_1B_1}\left[\overline{\phi}_{AB_2},\,\phi_{A_2B_3}\right]-12\left[\overline{\phi}_{AB_2},\,\phi_{A_2B_3}\right]\overline{\phi}_{A_1B_1}\\
&=6\left[\overline{\phi}_{A_1B_1},\,\left[\overline{\phi}_{AA_2},\,\phi_{B_2B_3}\right]\right]_+-12\overline{\phi}_{A_1B_1}\overline{\phi}_{AB_2}\phi_{A_2B_3}+12\overline{\phi}_{A_1B_1}\phi_{A_2B_3}\overline{\phi}_{AB_2}\\
&\qquad-12\overline{\phi}_{AB_2}\phi_{A_2B_3}\overline{\phi}_{A_1B_1}+12\phi_{A_2B_3}\overline{\phi}_{AB_2}\overline{\phi}_{A_1B_1}\\
&\cong 6\left[\overline{\phi}_{A_1B_1},\,\left[\overline{\phi}_{AA_2},\,\phi_{B_2B_3}\right]\right]_+
\end{aligned}$$

such that

$$
\begin{aligned}
V_{AA_1A_2B_1B_2B_3}|_{\phi^3} = \frac{2\sqrt{2}\,g^3}{18c_0^6 i_+^3} \big(& \left[\overline{\phi}_{A_1C},\, \left[\overline{\phi}_{AD},\, \overline{\phi}_{EF}\right]\right] \varepsilon_{CDEF}\varepsilon_{A_2B_1B_2B_3} \\
& + 6\left[\overline{\phi}_{A_1B_1},\, \left[\overline{\phi}_{AA_2},\, \phi_{B_2B_3}\right]\right]_+ \\
& -6\overline{\phi}_{AB_3}\left[\overline{\phi}_{A_1A_2},\, \overline{\phi}_{B_1B_2}\right] - 24\overline{\phi}_{AB_3}\overline{\phi}_{A_1B_1}\overline{\phi}_{A_2B_2}\big)
\end{aligned}
$$

Moreover

$$
\begin{aligned}
V_{AA_1A_2B_1B_2B_3}|_{\psi^2} = & -\frac{2i\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4}\Big(-i\sqrt{2}\,\varepsilon_{AA_1CD}\left[\psi^{\alpha D},\, \psi^{\gamma C}\right]_+ \varepsilon_{A_2B_1B_2B_3} \\
& \qquad +3i\sqrt{2}\,\varepsilon_{AA_1B_1D}\left[\psi^{\alpha D},\, \psi^{\gamma C}\right]\varepsilon_{A_2B_2B_3C}\Big) \\
& -\frac{4g^2\lambda_{i\gamma}\lambda_{(i+1)\beta}\psi^{\gamma C}\psi^{\beta D}}{3c_0^6 i_+^4}\varepsilon_{AB_2B_3C}\varepsilon_{A_1A_2B_1D} \\
= & \frac{4g^2\lambda_{i\gamma}\lambda_{(i+1)\alpha}}{9c_0^6 i_+^4}\Big(-\varepsilon_{AA_1CD}\varepsilon_{A_2B_1B_2B_3}\left[\psi^{\alpha D},\, \psi^{\gamma C}\right]_+ \\
& \qquad + 3\varepsilon_{A_2B_2B_3C}\varepsilon_{AA_1B_1D}\left[\psi^{\alpha D},\, \psi^{\gamma C}\right] \\
& \qquad -3\varepsilon_{AB_2B_3C}\varepsilon_{A_1A_2B_1D}\psi^{\gamma C}\psi^{\alpha D}\Big)
\end{aligned}
$$

This finishes the calculation.

Sixth Order ($d = 5$, $k = 1$ and $l = 4$)

Now consider $d = 5$ and $k = 1$ and $l = 4$.

$$
\begin{aligned}
V_{AA_1B_1B_2B_3B_4} &= \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+}\Big(iX^{\alpha(1)}_{i+1AB_4}V_{A_1B_1B_2B_3} + iX^{\alpha(2)}_{(i+1)AB_3B_4}V_{A_1B_1B_2} \\
&\qquad + iX^{\alpha(3)}_{(i+1)AB_2B_3B_4}V_{A_1B_1}\Big) \\
&=: (1) + (2) + (3)
\end{aligned}
$$

We calculate

$$
\begin{aligned}
(1) &= \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+} i \left(-\frac{2i\sqrt{2}\,g\lambda_i^\alpha}{c_0 i_+}\overline{\phi}_{AB_4}\right)\left(-\frac{g\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}}{3c_0^4 i_+^3}\varepsilon_{A_1B_1B_2B_3}\right) \\
&= -\frac{\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{i\beta}\overline{\phi}_{AB_4}F^{\gamma\beta}\varepsilon_{A_1B_1B_2B_3}}{3c_0^6 i_+^4}
\end{aligned}
$$

and

$$
\begin{aligned}
(2) &= \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+} i \left(\varepsilon_{AB_3B_4C}\frac{2g\lambda_i^\alpha\lambda_{i\gamma}\psi^{\gamma C}}{c_0^2 i_+^2}\right)\left(-\frac{2ig\lambda_{i\gamma}\psi^{\gamma D}}{c_0^3 i_+^2}\varepsilon_{A_1B_1B_2D}\right) \\
&= \frac{2g^2\lambda_{i\gamma}\lambda_{i\delta}\psi^{\gamma C}\psi^{\delta D}}{c_0^6 i_+^4}\varepsilon_{AB_3B_4C}\varepsilon_{A_1B_1B_2D}
\end{aligned}
$$

and

$$
\begin{aligned}
(3) &= \frac{\lambda_{(i+1)\alpha}}{2c_0 i_+} i \left(\varepsilon_{AB_2B_3B_4}\frac{ig\lambda_i^\alpha\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}}{3c_0^3 i_+^3}\right)\left(\frac{2\sqrt{2}\,g}{c_0^2 i_+}\overline{\phi}_{A_1B_1}\right) \\
&= -\frac{\sqrt{2}\,g^2\lambda_{i\gamma}\lambda_{i\beta}F^{\gamma\beta}\overline{\phi}_{A_1B_1}\varepsilon_{AB_2B_3B_4}}{3c_0^6 i_+^4}
\end{aligned}
$$

such that

$$(1)+(3) \cong -\frac{\sqrt{2}\, g^2 \lambda_{i\gamma}\lambda_{i\beta}\left[F^{\gamma\beta},\, \overline{\phi}_{A_1B_1}\right]_+ \varepsilon_{AB_2B_3B_4}}{3c_0^6 i_+^4}$$

and thus

$$V_{AA_1B_1B_2B_3B_4} = \frac{\sqrt{2}\, g^2 \lambda_{i\gamma}\lambda_{i\beta}\left[F^{\gamma\beta},\, \overline{\phi}_{AB_1}\right]_+ \varepsilon_{A_1B_2B_3B_4}}{3c_0^6 i_+^4} + \frac{2g^2\lambda_{i\gamma}\lambda_{i\delta}\psi^{\gamma C}\psi^{\delta D}}{c_0^6 i_+^4}\varepsilon_{AB_3B_4C}\varepsilon_{A_1B_1B_2D}$$

Sixth Order ($d = 5$, $k = 0$ and $l = 5$)

Obviously $V_{AB_1B_2B_3B_4B_5} = 0$.

2.3.4 Seventh and Eighth Order

In order 7, there are only two nonvanishing coefficients corresponding to $\sim \eta_i^3\eta_{i+1}^4$ and $\sim \eta_i^4\eta_{i+1}^3$.

Seventh Order ($d = 6$, $k = 2$ and $l = 4$)

We consider $d = 6$ and $k = 2$ and $l = 4$.

$$\begin{aligned} V_{AA_1A_2B_1B_2B_3B_4} &= -\frac{\lambda_{(i+1)\alpha}}{3c_0 i_+}\Big(-q_A^{\alpha}(V_{A_1A_2B_1B_2B_3B_4}) + iX^{\alpha(1)}_{(i+1)AB_4}V_{A_1A_2B_1B_2B_3} \\ &\qquad + iX^{\alpha(2)}_{(i+1)AB_3B_4}V_{A_1A_2B_1B_2} + iX^{\alpha(3)}_{(i+1)AB_2B_3B_4}V_{A_1A_2B_1}\Big) \\ &=: (1)+(2)+(3)+(4) \end{aligned}$$

where, using $q(F) \sim D_{\dot\gamma}\tilde{\psi}^{\dot\gamma} \sim g\overline{\phi}\psi$,

$$(1) \sim \frac{g^2}{c_0^7}(q(F\overline{\phi}) + q(\psi\psi)) \sim \frac{g^2}{c_0^7}(D_{\dot\gamma}\tilde{\psi}^{\dot\gamma}\overline{\phi} + F\psi + g\overline{\phi\phi}\psi) \sim \frac{g^2}{c_0^7}(F\psi + g\overline{\phi\phi}\psi)$$

and

$$\begin{aligned} (2) &\sim \frac{1}{c_0}\frac{g\overline{\phi}}{c_0}\frac{g^2}{c_0^5}\overline{\phi}\psi \sim \frac{g^3}{c_0^7}\overline{\phi\phi}\psi \\ (3) &\sim \frac{1}{c_0}\frac{g\psi}{c_0^2}\Big(\frac{gF}{c_0^4} + \frac{g^2\overline{\phi\phi}}{c_0^4}\Big) \sim \frac{g^2}{c_0^7}\psi F + \frac{g^3}{c_0^7}\psi\overline{\phi\phi} \\ (4) &\sim \frac{1}{c_0}\frac{gF}{c_0^3}\frac{g\psi}{c_0^3} \sim \frac{g^2}{c_0^7}F\psi \end{aligned}$$

such that

$$V_{AA_1A_2B_1B_2B_3B_4} \sim \frac{g^2}{c_0^7}F\psi + \frac{g^3}{c_0^7}\overline{\phi\phi}\psi$$

Seventh Order ($d = 6$, $k = 3$ and $l = 3$)

We consider $d = 6$ and $k = 3$ and $l = 3$.

$$\begin{aligned} V_{AA_1A_2A_3B_1B_2B_3} = & -\frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\Big(-q_A^\alpha(V_{A_1A_2A_3B_1B_2B_3}) + iX^{\alpha(1)}_{(i+1)AB_3}V_{A_1A_2A_3B_1B_2} \\ & + iX^{\alpha(2)}_{(i+1)AB_2B_3}V_{A_1A_2A_3B_1} + iX^{\alpha(3)}_{(i+1)AB_1B_2B_3}V_{A_1A_2A_3} \\ & + iV_{A_1A_2B_1B_2B_3}X^{\alpha(1)}_{iAA_3} - iV_{A_1B_1B_2B_3}X^{\alpha(2)}_{AA_2A_3}\Big) \\ =: & (1) + (2) + (3) + (4) + (5) + (6) \end{aligned}$$

where

$$\begin{aligned} (1) & \sim \frac{1}{c_0}q\left(\frac{g^2}{c_0^6}F\overline{\phi} + \frac{g^3}{c_0^6}\overline{\phi\phi\phi} + \frac{g^2}{c_0^6}\psi\psi\right) \\ & \sim \frac{g^3}{c_0^7}\overline{\phi\phi}\psi + \frac{g^2}{c_0^7}F\psi + \frac{g^3}{c_0^7}\overline{\phi\phi}\psi + \frac{g^2}{c_0^7}F\psi + \frac{g^3}{c_0^7}\overline{\phi\phi}\psi \\ & \sim \frac{g^3}{c_0^7}\overline{\phi\phi}\psi + \frac{g^2}{c_0^7}F\psi \end{aligned}$$

and

$$\begin{aligned} (2) & \sim \frac{1}{c_0}\frac{g\overline{\phi}}{c_0}\frac{g^2\overline{\phi}\psi}{c_0^5} \sim \frac{g^3}{c_0^7}\overline{\phi\phi}\psi \\ (3) & \sim \frac{1}{c_0}\frac{g\psi}{c_0^2}\left(\frac{gF}{c_0^4} + \frac{g^2\overline{\phi\phi}}{c_0^4}\right) \sim \frac{g^2}{c_0^7}F\psi + \frac{g^3}{c_0^7}\overline{\phi\phi}\psi \\ (4) & \sim \frac{1}{c_0}\frac{gF}{c_0^3}\frac{g\psi}{c_0^3} \sim \frac{g^2}{c_0^7}F\psi \\ (5) & \sim \frac{1}{c_0}\frac{g^2\overline{\phi}\psi}{c_0^5}\frac{g\overline{\phi}}{c_0} \sim \frac{g^3}{c_0^7}\overline{\phi\phi}\psi \\ (6) & \sim \frac{1}{c_0}\frac{gF}{c_0^4}\frac{g\psi}{c_0^2} \sim \frac{g^2}{c_0^7}F\psi \end{aligned}$$

such that

$$V_{AA_1A_2A_3B_1B_2B_3} \sim \frac{g^3}{c_0^7}\overline{\phi\phi}\psi + \frac{g^2}{c_0^7}F\psi$$

Eighth Order ($d = 7$, $k = 3$ and $l = 4$)

In order 8, there is only one nonvanishing coefficient corresponding to $\sim \eta_i^4\eta_{i+1}^4$. Therefore, we consider $d = 7$ and $k = 3$ and $l = 4$.

$$\begin{aligned} V_{AA_1A_2A_3B_1B_2B_3B_4} = & \frac{\lambda_{(i+1)\alpha}}{4c_0 i_+}\Big(-q_A^\alpha(V_{A_1A_2A_3B_1B_2B_3B_4}) + iX^{\alpha(1)}_{(i+1)AB_4}V_{A_1A_2A_3B_1B_2B_3} \\ & + iX^{\alpha(2)}_{(i+1)AB_3B_4}V_{A_1A_2A_3B_1B_2} + iX^{\alpha(3)}_{(i+1)AB_2B_3B_4}V_{A_1A_2A_3B_1} \\ & - iV_{A_1A_2B_1B_2B_3B_4}X^{\alpha(1)}_{iAA_3}\Big) \\ =: & (1) + (2) + (3) + (4) + (5) \end{aligned}$$

where

$$
\begin{aligned}
(1) &\sim \frac{1}{c_0} q \left(\frac{g^2}{c_0^7} F\psi + \frac{g^3}{c_0^7} \overline{\phi\phi}\psi \right) \sim \frac{g^3}{c_0^8} \overline{\phi}\psi\psi + \frac{g^2}{c_0^8} FF + \frac{g^3}{c_0^8} F\overline{\phi\phi} + \frac{g^4}{c_0^8} \overline{\phi}^4 \\
(2) &\sim \frac{1}{c_0} \frac{g\overline{\phi}}{c_0} \left(\frac{g^2 \overline{\phi} F}{c_0^6} + \frac{g^3 \overline{\phi\phi\phi}}{c_0^6} + \frac{g^2 \psi\psi}{c_0^6} \right) \sim \frac{g^3}{c_0^8} \overline{\phi}\psi\psi + \frac{g^3}{c_0^8} F\overline{\phi\phi} + \frac{g^4}{c_0^8} \overline{\phi}^4 \\
(3) &\sim \frac{1}{c_0} \frac{g\psi}{c_0^2} \frac{g^2 \overline{\phi}\psi}{c_0^5} \sim \frac{g^3}{c_0^8} \overline{\phi}\psi\psi \\
(4) &\sim \frac{1}{c_0} \frac{gF}{c_0^3} \left(\frac{gF}{c_0^4} + \frac{g^2 \overline{\phi\phi}}{c_0^4} \right) \sim \frac{g^2}{c_0^8} FF + \frac{g^3}{c_0^8} F\overline{\phi\phi} \\
(5) &\sim \frac{1}{c_0} \left(\frac{g^2 \overline{\phi} F}{c_0^6} + \frac{g^2 \psi\psi}{c_0^6} \right) \frac{g\overline{\phi}}{c_0} \sim \frac{g^3}{c_0^8} F\overline{\phi\phi} + \frac{g^3}{c_0^8} \overline{\phi}\psi\psi
\end{aligned}
$$

such that

$$
V_{AA_1A_2A_3B_1B_2B_3B_4} \sim \frac{g^3}{c_0^8} \overline{\phi}\psi\psi + \frac{g^2}{c_0^8} FF + \frac{g^3}{c_0^8} F\overline{\phi\phi} + \frac{g^4}{c_0^8} \overline{\phi}^4
$$

Chapter 3

Wilson Loops and Scattering Amplitudes

In this chapter, we study the quantum theory field theory of the supersymmetric Wilson loops established in Chp. 2. In particular, we compare components of the expectation value with scattering amplitudes. This matching will provide us with the yet undetermined constant c_0, which then depends on the coupling constant g. We will explicitly see that Wilson loops are not dual to scattering amplitudes at tree level.

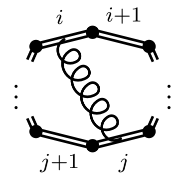
3.1 The Quantum Theory of Super Wilson Loops

Consider a super Wilson loop W_n as in (2.4). We are interested in the expectation value

$$\langle W_n \rangle = \frac{\int \mathcal{D}\Phi \exp\left(i \int d^4x\, \mathcal{L}\right) W_n}{\int \mathcal{D}\Phi \exp\left(i \int d^4x\, \mathcal{L}\right)} \tag{3.1}$$

where by $\int \mathcal{D}\Phi$ we denote the path integral over all fields $\Phi \in \{\phi, \tilde{\psi}, \psi, A\}$ of $\mathcal{N} = 4$ SYM theory. Up to a certain order g^k in the coupling constant, W_n is the sum over finitely many integrals over products of fields multiplied with Grassmann monomials. Exchanging these integrals with the path integral, we can thus apply the machinery developped in Sec. 1.5.

A typical diagram that contributes to (3.1) is e.g. given by connecting two edges with a gluon propagator (and integrating over the edges):



This diagram, in turn, has several contributions according to the different gluon terms in the edge operators $\mathcal{E}_i$ and $\mathcal{E}_j$. Similarly, a propagator coming from an edge may end in a vertex or an inner vertex. This can be depicted by a new set of Feynman rules in addition to the usual ones derived in Sec. 1.5.

3.1.1 Feynman Rules

We summarise, schematically, the edge and vertex terms $ig\mathcal{E}_i$ and $\mathcal{V}_{i,i+1}$ which enter the Wilson loop W_n according to (2.4). From Thm. 2.2.1 we obtain

$$\begin{aligned} ig\mathcal{E}_i \sim gA \cdot 1 + \frac{g}{c_0}\tilde{\psi} \cdot \eta_i + \frac{g}{c_0^2}\overline{\phi} \cdot (\eta_i)^2 + \frac{g^3}{c_0^2}A\overline{\phi} \cdot (\eta_i)^2 \\ + \frac{g}{c_0^3}\psi \cdot (\eta_i)^3 + \frac{g^2}{c_0^3}A\psi \cdot (\eta_i)^3 + \frac{g}{c_0^4} \cdot F(\eta_i)^4 + \frac{g^2}{c_0^4}AF \cdot (\eta_i)^4 \end{aligned}$$

where $F \sim A + gAA$. Similarly, Thm. 2.3.2 yields

$$\begin{aligned} \mathcal{V}_{i,i+1} \sim 1 + \frac{g}{c_0^2}\overline{\phi} \cdot \eta^2 + \frac{g}{c_0^3}\psi \cdot \eta^3 + \frac{g}{c_0^4}F \cdot \eta^4 + \frac{g^2}{c_0^4}\overline{\phi}^2 \cdot \eta^4 + \frac{g^2}{c_0^5}\overline{\phi}\psi \cdot \eta^5 \\ + \frac{g^2}{c_0^6}\overline{\phi}F \cdot \eta^6 + \frac{g^3}{c_0^6} \cdot \overline{\phi}^3 \cdot \eta^6 + \frac{g^2}{c_0^6}\psi^2 \cdot \eta^6 + \frac{g^2}{c_0^7}F\psi \cdot \eta^7 + \frac{g^3}{c_0^7}\overline{\phi\phi}\psi \cdot \eta^7 \\ + \frac{g^3}{c_0^8}\overline{\phi}\psi\psi \cdot \eta^8 + \frac{g^2}{c_0^8}FF \cdot \eta^8 + \frac{g^3}{c_0^8}F\overline{\phi\phi} \cdot \eta^8 + \frac{g^4}{c_0^8}\overline{\phi}^4 \cdot \eta^8 \end{aligned}$$

where the k-fold product of generators η_i^A is denoted η^k. Here, the constant c_0 is yet undetermined. It would seem natural to have it independent of the coupling constant, i.e. $c_0 \sim g^0 = 1$, resulting in $\langle W_n \rangle = 1 + \mathcal{O}(g^2)$ as for the usual Wilson loop. However, since we want to compare Wilson loops with scattering amplitudes, for which tree level expectation values do exist, we must choose a different dependence. It turns out (cf. Sec. 3.2 below) that

$$c_0^2 \sim g \tag{3.2}$$

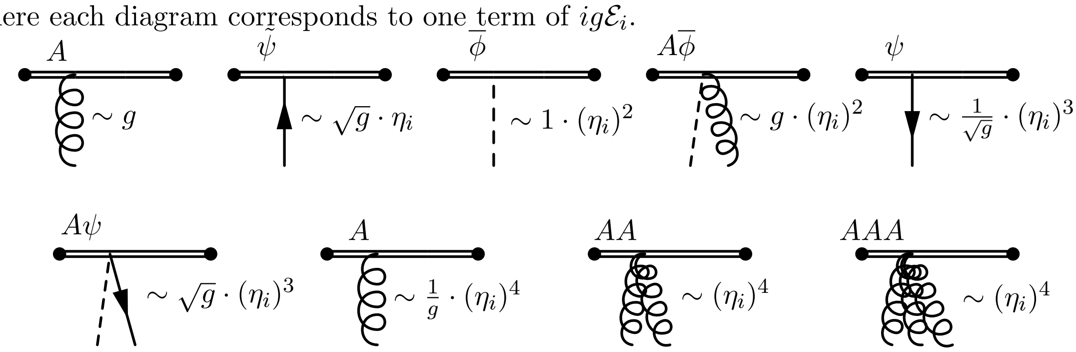
is a good choice, that we shall consider from now on.

Edge Contributions

From the above formula, we immediately find

$$\begin{aligned} ig\mathcal{E}_i \sim gA \cdot 1 + \sqrt{g}\tilde{\psi} \cdot \eta_i + \overline{\phi} \cdot (\eta_i)^2 + gA\overline{\phi} \cdot (\eta_i)^2 \\ + \frac{1}{\sqrt{g}}\psi \cdot (\eta_i)^3 + \sqrt{g}A\psi \cdot (\eta_i)^3 + \frac{1}{g} \cdot A(\eta_i)^4 + AA(\eta_i)^4 + gAAA \cdot (\eta_i)^4 \end{aligned}$$

We illustrate the possible edge contributions by the following additional Feynman rules, where each diagram corresponds to one term of $ig\mathcal{E}_i$.



Vertex Contributions

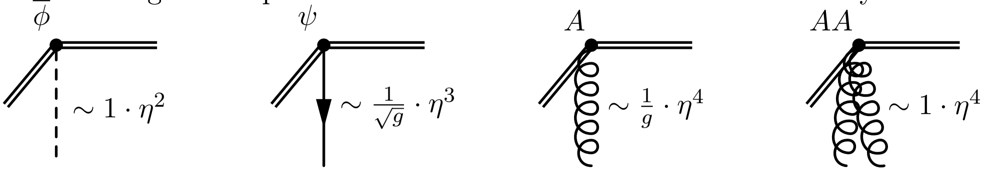
From the above formula for $\mathcal{V}_{i\,i+1}$, we see that higher order terms factor into terms with the structure of lower order terms as follows.

$$\begin{aligned}\mathcal{V}_{i,i+1} \sim\; & 1+\overline{\phi}\cdot\eta^2+\frac{1}{\sqrt{g}}\psi\cdot\eta^3+\frac{1}{g}F\cdot\eta^4+\left(\overline{\phi}\eta^2\right)^2+\left(\overline{\phi}\eta^2\right)\left(\frac{1}{\sqrt{g}}\psi\eta^3\right)\\ &+\left(\overline{\phi}\eta^2\right)\left(\frac{1}{g}F\eta^4\right)+\left(\overline{\phi}\eta^2\right)^3+\left(\frac{1}{\sqrt{g}}\psi\eta^3\right)^2+\left(\frac{1}{\sqrt{g}}\psi\eta^3\right)\left(\frac{1}{g}F\eta^4\right)\\ &+\left(\overline{\phi}\eta^2\right)^2\left(\frac{1}{\sqrt{g}}\psi\eta^3\right)+\left(\overline{\phi}\eta^2\right)\left(\frac{1}{\sqrt{g}}\psi\eta^3\right)^2+\left(\frac{1}{g}F\eta^4\right)^2\\ &+\left(\overline{\phi}\eta^2\right)^2\left(\frac{1}{g}F\eta^4\right)+\left(\overline{\phi}\eta^2\right)^4\end{aligned}$$

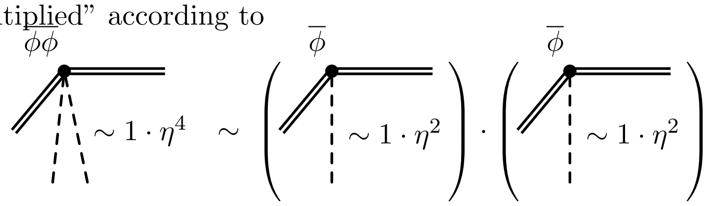
such that

$$\begin{aligned}\mathcal{V}_{i,i+1} &\sim \sum\prod\left(1+\overline{\phi}\cdot\eta^2+\frac{1}{\sqrt{g}}\psi\cdot\eta^3+\frac{1}{g}F\cdot\eta^4\right)\\ &\sim \sum\prod\left(1+\overline{\phi}\cdot\eta^2+\frac{1}{\sqrt{g}}\psi\cdot\eta^3+\frac{1}{g}A\cdot\eta^4+AA\cdot\eta^4\right)\end{aligned}$$

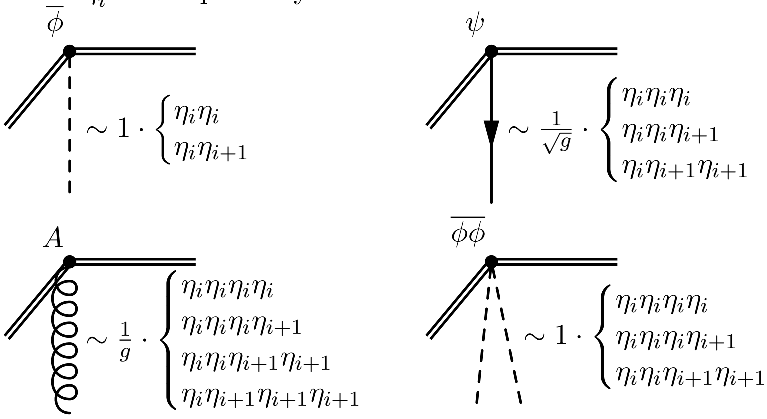
It is understood that this is not an equation but only a similarity to memorise the types of terms occurring. We depict the relevant contributions as additional Feynman rules



which are "multiplied" according to



and analogous for the other contributions. We summarise these Feynman rules up to order 4 in the generators η_i^A more precisely as follows.

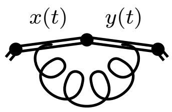


3.1.2 Discussion

Having derived the Feynman rules for the super Wilson loop with (3.2), let us draw some consequences.

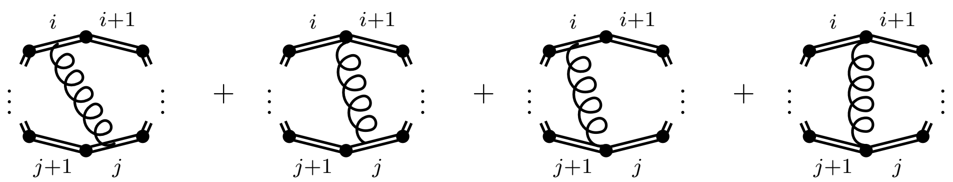
As our first observation, note that $\langle W_n \rangle$ is the sum over Grassmann monomials whose degree is a multiple of 4. Indeed the structure is such that, connecting any two edges/vertices with a propagator, we pick up a factor η^4, and the reasoning remains analogous if inner vertices are present (cf. the Feynman rules derived in Sec. 1.5).

Secondly, we run into divergences due to the vanishing of denominators of the propagators. Consider e.g. the following diagram.



As $x(t)$ and $y(t)$ approach the vertex in between the respective edges, the gluon propagator (1.45) becomes infinite. This problem, which is already present for the case of classical Wilson loops, requires regularisation as explained in Sec. 1.5.2. In particular, one has to take the regularised gluon propagator (1.44). Regularisation splits a single diagram into a finite plus a diverging part. However, the sum of all diagrams contributing to (3.1) should add up to a finite expression without divergences. It remains unclear whether this is indeed the case here.

There is another possible problem due to (3.2): Our Feynman rules allow diagrams which are $\sim g^k$ for $k < 1$. However, perturbative quantum field theory (weak coupling) makes sense only if these diagrams, if not vanishing individually, at least add up to zero. To make an example, consider the term $\sim \frac{1}{g^2} \cdot (\eta_i)^4 (\eta_j)^4$ where i and j are not neighbours. According to the Feynman rules, this contribution comes from the following four diagrams.



In formulas, we have

$$\langle W_n \rangle |_{\frac{1}{g^2}\cdot(\eta_i)^4(\eta_j)^4} = \Big((ig)^2 \int\int \Big\langle \mathcal{E}_i|_{(\eta_i)^4}\, \mathcal{E}_j|_{(\eta_j)^4} \Big\rangle + ig \int \Big\langle \mathcal{V}_{i\,i+1}|_{(\eta_i)^4}\, \mathcal{E}_j|_{(\eta_j)^4} \Big\rangle$$
$$+ ig \int \Big\langle \mathcal{E}_i|_{(\eta_i)^4}\, \mathcal{V}_{j\,j+1}|_{(\eta_j)^4} \Big\rangle + \Big\langle \mathcal{V}_{i\,i+1}|_{(\eta_i)^4}\, \mathcal{V}_{j\,j+1}|_{(\eta_j)^4} \Big\rangle \Big) |_{\frac{1}{g^2}}$$

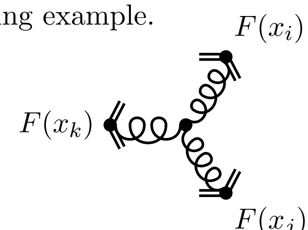
where the integrals refer to the edge integrations. By the explicit formulas for the edge and vertex operators stated in Thms. 2.2.1 and 2.3.2, each of the four brackets is proportional to $\langle F^{a\alpha\beta}|_{g^0}(x)\, F^{b\gamma\delta}|_{g^0}(y) \rangle$ or some derivatives thereof. By Lem. 1.2.2, this bracket consists of four terms, each of which has the following form.

$$
\begin{aligned}
\left\langle i\partial^{\alpha}{}_{\dot\beta} A^{a\beta\dot\beta}(x)\; i\partial^{\gamma}{}_{\dot\delta} A^{b\delta\dot\delta}(y)\right\rangle &= -\overline{\sigma}^{\mu\alpha}{}_{\dot\beta}\overline{\sigma}^{\nu\gamma}{}_{\dot\delta}\overline{\sigma}^{\kappa\beta\dot\beta}\overline{\sigma}^{\lambda\delta\dot\delta}\partial_{(x)\mu}\partial_{(y)\nu}\left\langle A^a_\kappa(x)\; A^b_\lambda(y)\right\rangle\\
&= \frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}}\overline{\sigma}^{\mu\alpha}{}_{\dot\beta}\overline{\sigma}^{\nu\gamma}{}_{\dot\delta}\overline{\sigma}^{\kappa\beta\dot\beta}\overline{\sigma}^{\lambda\delta\dot\delta}\partial_{(x)\mu}\partial_{(y)\nu}\frac{\eta_{\kappa\lambda}\delta^{ab}}{((x-y)^2)^{1-\varepsilon}}\\
&= \frac{\Gamma(1-\varepsilon)}{2\pi^{2-\varepsilon}}\overline{\sigma}^{\mu\alpha}{}_{\dot\beta}\overline{\sigma}^{\nu\gamma\dot\beta}\epsilon^{\beta\delta}\partial_{(x)\mu}\partial_{(y)\nu}\frac{\delta^{ab}}{((x-y)^2)^{1-\varepsilon}}\\
&= \frac{\Gamma(1-\varepsilon)}{2\pi^{2-\varepsilon}}\eta^{\mu\nu}\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\delta^{ab}\partial_{(x)\mu}\partial_{(y)\nu}\frac{1}{\left((x-y)^2\right)^{1-\varepsilon}}
\end{aligned}
$$

In this calculation, we used the regularised gluon propagator (1.44), Lem. 1.1.7 and (1.6) together with the fact that the derivative term is symmetric in μ and ν. By Lem. 1.5.5, the last expression goes to zero as $\varepsilon \to 0$ and, therefore, we arrive at $\langle W_n\rangle|_{\frac{1}{g^2}\cdot(\eta_i)^4(\eta_j)^4} = 0$. We also managed to show that problematic terms involving the three-gluon inner vertex vanish. It remains unclear, however, whether all problematic terms vanish.

3.1.3 Tree-Level Calculations

Having established the Feynman rules, we calculate some tree-level ($\sim g^0 = 1$) contributions. The prescription (3.2) is such that "tree-level" diagrams may contain inner vertices. Consider the following example.



It contains (vanishing) problematic as well as both tree-level and one-loop ($\sim g^1$) contributions.

To make life simple, we restrict our study to tree-level components proportional to the product of four Grassmann generators, where the contributing diagrams contain propagators but no inner vertices. For the rest of this section, we calculate some of such contributions. As explained above, we actually need to consider regularised propagators due to divergences. However, it is also possible to start with un-regularised calculations and only later argue at which places in the calculation regularisation plays a role. We will encounter integrals as in the next lemma.

Lemma 3.1.1. Setting $x_{i\,j} := x_i - x_j$ and assuming that $p_i = x_{i\,i-1}$ is lightlike, the following formula holds.

$$\int_0^1 dt\,\frac{1}{(x_j - x_{i-1} - tp_i)^4} = \frac{1}{x^2_{j\,i-1}x^2_{j\,i}}$$

Proof. With $\overline{t} := (1-t)$, we transform

$$
\begin{aligned}
x_j - x_{i-1} - tp_i &= x_j - x_{i-1} - tx_i + tx_{i-1}\\
&= (x_j - x_{i-1})(1-t) + (x_j - x_i)t\\
&= (x_j - x_{i-1})\overline{t} + (x_j - x_i)t\\
&= x_{j\,i-1}\overline{t} + x_{j\,i}t
\end{aligned}
$$

For computing the square thereof, we need the following side calculation.

$$\begin{aligned}
-(x_{ij}^2 - x_{ik}^2 - x_{jk}^2) &= -(x_i^2 + x_j^2 - 2x_ix_j - x_i^2 - x_k^2 + 2x_ix_k - x_j^2 - x_k^2 + 2x_jx_k) \\
&= 2x_ix_j + 2x_k^2 - 2x_ix_k - 2x_jx_k \\
&= 2(x_i(x_j - x_k) - x_k(x_j - x_k)) \\
&= 2(x_i - x_k)(x_j - x_k) \\
&= 2x_{ik}x_{jk}
\end{aligned}$$

We thus find

$$\begin{aligned}
(x_j - x_{i-1} - tp_i)^2 &= x_{j\,i-1}^2\bar{t}^2 + x_{j\,i}^2t^2 + 2t\bar{t}x_{j\,i-1}x_{j\,i} \\
&= x_{j\,i-1}^2\bar{t}^2 + x_{j\,i}^2t^2 - t\bar{t}(x_{i-1\,i}^2 - x_{j\,i-1}^2 - x_{j\,i}^2) \\
&= x_{j\,i-1}^2(\bar{t}^2 + t\bar{t}) + x_{j\,i}^2(t^2 + t\bar{t}) \\
&= x_{j\,i-1}^2\bar{t} + x_{j\,i}^2t
\end{aligned}$$

Now, by the substitution rule with $s(t) = x_{j\,i-1}^2\bar{t} + x_{j\,i}^2t$ and $s^{'}(t) = -x_{j\,i-1}^2 + x_{j\,i}^2$, we yield

$$\begin{aligned}
\int_0^1 dt\,\frac{1}{(x_j - x_{i-1} - tp_i)^4} &= \int_0^1 dt\,\frac{1}{(x_{j\,i-1}^2\bar{t} + x_{j\,i}^2t)^2} \\
&= \frac{1}{-x_{j\,i-1}^2 + x_{j\,i}^2}\int_{x_{j\,i-1}^2}^{x_{j\,i}^2} ds\,\frac{1}{s^2} \\
&= \frac{1}{-x_{j\,i-1}^2 + x_{j\,i}^2}\left(\frac{1}{x_{j\,i-1}^2} - \frac{1}{x_{j\,i}^2}\right) \\
&= \frac{1}{x_{j\,i-1}^2x_{j\,i}^2}
\end{aligned}$$

□

Lemma 3.1.2. The following expressions hold for tree-level contributions with two vertices being connected by a scalar propagator.

$$\begin{aligned}
&\left\langle \mathcal{V}_{i,i+1}(x_i)|_{\eta_i\eta_{i+1}}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}}\right\rangle \\
&\quad = C_0\cdot\eta_i^A\eta_{i+1}^B\eta_j^C\eta_{j+1}^D\frac{\varepsilon_{ABCD}}{\langle i\,i+1\rangle\langle j\,j+1\rangle(x_i - x_j)^2}\,, \qquad C_0 := \left(\frac{g^2(N^2-1)}{N\pi^2c_0^4}\right) \\
&\left\langle \mathcal{V}_{i,i+1}(x_i)|_{\eta_i\eta_i}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}}\right\rangle \\
&\quad = C_0\cdot\eta_i^A\eta_i^B\eta_j^C\eta_{j+1}^D\frac{\varepsilon_{ABCD}\langle i+1\,i-1\rangle}{2\langle i-1\,i\rangle\langle i\,i+1\rangle\langle j\,j+1\rangle(x_i - x_j)^2}
\end{aligned}$$

The constant C_0 is independent of the coupling constant g through (3.2).

Proof. We calculate the first bracket, using $\delta^{ab}\delta^{ab} = \delta^{aa} = \dim(\mathfrak{su}(N)) = N^2 - 1$.

$$\begin{aligned}
&\left\langle \mathcal{V}_{i,i+1}(x_i)|_{\eta_i\eta_{i+1}}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}} \right\rangle \\
&= \frac{1}{N}\mathrm{tr}\left\langle \frac{2\sqrt{2}\,g}{c_0^2\,\langle i+1\,i\rangle}\overline{\phi}_{AB}(x_i)\eta_i^A\eta_{i+1}^B \quad \frac{2\sqrt{2}\,g}{c_0^2\,\langle j+1\,j\rangle}\overline{\phi}_{CD}(x_j)\eta_j^C\eta_{j+1}^D \right\rangle \\
&= \eta_i^A\eta_{i+1}^B\eta_j^C\eta_{j+1}^D \frac{8g^2}{Nc_0^4\,\langle i\,i+1\rangle\,\langle j\,j+1\rangle}\mathrm{tr}\left\langle \overline{\phi}^a_{AB}(x_i)T^a\ \overline{\phi}^b_{CD}(x_j)T^b \right\rangle \\
&= \eta_i^A\eta_{i+1}^B\eta_j^C\eta_{j+1}^D \frac{8g^2\mathrm{tr}(T^aT^b)}{Nc_0^4\,\langle i\,i+1\rangle\,\langle j\,j+1\rangle}\left\langle \overline{\phi}^a_{AB}(x_i)\ \overline{\phi}^b_{CD}(x_j) \right\rangle \\
&= \eta_i^A\eta_{i+1}^B\eta_j^C\eta_{j+1}^D \frac{4g^2\delta^{ab}}{Nc_0^4\,\langle i\,i+1\rangle\,\langle j\,j+1\rangle}\left(\frac{1}{4\pi^2}\frac{\varepsilon_{ABCD}\delta^{ab}}{(x_i-x_j)^2} \right) \\
&= \left(\frac{g^2(N^2-1)}{N\pi^2c_0^4} \right)\eta_i^A\eta_{i+1}^B\eta_j^C\eta_{j+1}^D \frac{\varepsilon_{ABCD}}{\langle i\,i+1\rangle\,\langle j\,j+1\rangle\,(x_i-x_j)^2}
\end{aligned}$$

Similarly, we obtain the second equality as follows.

$$\begin{aligned}
&\left\langle \mathcal{V}_{i,i+1}(x_i)|_{\eta_i\eta_i}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}} \right\rangle \\
&= \frac{1}{N}\mathrm{tr}\left\langle -\frac{\sqrt{2}\,g\,\langle i+1\,i-1\rangle}{c_0^2\,\langle i\,i-1\rangle\,\langle i+1\,i\rangle}\overline{\phi}_{AB}(x_i)\eta_i^A\eta_i^B \quad \frac{2\sqrt{2}\,g}{c_0^2\,\langle j+1\,j\rangle}\overline{\phi}_{CD}(x_j)\eta_j^C\eta_{j+1}^D \right\rangle \\
&= -\frac{\langle i+1\,i-1\rangle}{2\,\langle i\,i-1\rangle}\frac{1}{N}\mathrm{tr}\left\langle \frac{2\sqrt{2}\,g}{c_0^2\,\langle i+1\,i\rangle}\overline{\phi}_{AB}(x_i)\eta_i^A\eta_i^B \quad \frac{2\sqrt{2}\,g}{c_0^2\,\langle j+1\,j\rangle}\overline{\phi}_{CD}(x_j)\eta_j^C\eta_{j+1}^D \right\rangle \\
&= -\frac{\langle i+1\,i-1\rangle}{2\,\langle i\,i-1\rangle}C_0\eta_i^A\eta_i^B\eta_j^C\eta_{j+1}^D \frac{\varepsilon_{ABCD}}{\langle i\,i+1\rangle\,\langle j\,j+1\rangle\,(x_i-x_j)^2} \\
&= C_0\cdot\eta_i^A\eta_i^B\eta_j^C\eta_{j+1}^D \frac{\varepsilon_{ABCD}\,\langle i+1\,i-1\rangle}{2\,\langle i-1\,i\rangle\,\langle i\,i+1\rangle\,\langle j\,j+1\rangle\,(x_i-x_j)^2}
\end{aligned}$$

□

Lemma 3.1.3. The following expressions hold for tree-level contributions with a vertex being connected with an edge (and integrated over) by a scalar propagator, where C_0 is the constant as defined in the previous lemma.

$$\begin{aligned}
ig\int \left\langle \mathcal{E}_i|_{\eta_i\eta_i}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}} \right\rangle &= C_0\cdot\frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}x_{j\,i-1}^{\dot{\beta}\gamma}\,\varepsilon_{ABCD}}{2\,\langle i\,i-1\rangle\,\langle j+1\,j\rangle\,x_{j\,i-1}^2x_{j\,i}^2}\eta_i^A\eta_i^B\eta_j^C\eta_{j+1}^D \\
ig\int \left\langle \mathcal{V}_{j\,j+1}(x_j)|_{\eta_j\eta_{j+1}\eta_{j+1}}\ \mathcal{E}_i|_{\eta_i} \right\rangle &= C_0\cdot\frac{\lambda_{j\gamma}\tilde{\lambda}_{i\dot{\beta}}x_{j\,i-1}^{\gamma\dot{\beta}}\varepsilon_{ABCD}}{2\,\langle j+1\,j\rangle^2\,x_{j\,i-1}^2x_{j\,i}^2}\eta_i^A\eta_j^B\eta_{j+1}^C\eta_{j+1}^D \\
ig\int \left\langle \mathcal{V}_{j\,j+1}(x_j)|_{\eta_j\eta_j\eta_{j+1}}\ \mathcal{E}_i|_{\eta_i} \right\rangle &= -C_0\cdot\frac{\lambda_{(j+1)\gamma}\tilde{\lambda}_{i\dot{\beta}}x_{j\,i-1}^{\gamma\dot{\beta}}\varepsilon_{ABCD}}{2\,\langle j+1\,j\rangle^2\,x_{j\,i-1}^2x_{j\,i}^2}\eta_i^A\eta_j^B\eta_j^C\eta_{j+1}^D
\end{aligned}$$

Here, and in the following, it is implicitly understood that the respective left hand side is restricted to the tree-level ($\sim g^0 = 1$) part.

Proof. We calculate the first term.

$$
\begin{aligned}
&ig \int \left\langle \mathcal{E}_i|_{\eta_i\eta_i}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}} \right\rangle \\
&= \frac{ig}{N} \int_{x=x_{i-1}+tp_i} \mathrm{tr} \left\langle -\frac{i\sqrt{2}}{2c_0^2} \frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}\partial^{\dot{\beta}\gamma}\overline{\phi}_{AB}(x)}{\langle i\, i-1\rangle} \eta_i^A \eta_i^B\ \frac{2\sqrt{2}\,g}{c_0^2\,\langle j+1\, j\rangle} \overline{\phi}_{CD}(x_j)\, \eta_j^C\, \eta_{j+1}^D \right\rangle \\
&= \frac{2g^2 \tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}}{Nc_0^4\,\langle i\, i-1\rangle\,\langle j+1\, j\rangle} \eta_i^A \eta_i^B \eta_j^C\, \eta_{j+1}^D \int_{x=x_{i-1}+tp_i} \frac{1}{2}\delta^{ab} \left\langle \partial^{\dot{\beta}\gamma}\overline{\phi}^a_{AB}(x)\ \overline{\phi}^b_{CD}(x_j) \right\rangle \\
&= \frac{g^2 \tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}\varepsilon_{ABCD}\delta^{ab}\delta^{ab}}{4\pi^2 N c_0^4\,\langle i\, i-1\rangle\,\langle j+1\, j\rangle} \eta_i^A \eta_i^B \eta_j^C\, \eta_{j+1}^D \int_{x=x_{i-1}+tp_i} \partial^{\dot{\beta}\gamma} \frac{1}{(x-x_j)^2} \\
&= \left(\frac{g^2(N^2-1)}{c_0^4\pi^2 N}\right) \cdot \frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}\varepsilon_{ABCD}}{2\,\langle i\, i-1\rangle\,\langle j+1\, j\rangle} \eta_i^A \eta_i^B \eta_j^C\, \eta_{j+1}^D \int_{x=x_{i-1}+tp_i} \frac{(x_j-x)^{\dot{\beta}\gamma}}{(x_j-x)^4}
\end{aligned}
$$

The numerator is independent of t,

$$
\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}(x_j-x)^{\dot{\beta}\gamma} = \tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}(x_j-x_{i-1}-tp_i)^{\dot{\beta}\gamma} = \tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}x_{j\,i-1}^{\dot{\beta}\gamma}
$$

and the first equality follows by Lem. 3.1.1.

$$
\begin{aligned}
&ig \int \left\langle \mathcal{E}_i|_{\eta_i\eta_i}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}} \right\rangle \\
&\qquad = C_0 \cdot \frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}x_{j\,i-1}^{\dot{\beta}\gamma}\,\varepsilon_{ABCD}}{2\,\langle i\, i-1\rangle\,\langle j+1\, j\rangle} \eta_i^A \eta_i^B \eta_j^C \eta_{j+1}^D \int_0^1 dt\, \frac{1}{(x_j-x_{i-1}-tp_i)^4} \\
&\qquad = C_0 \cdot \frac{\tilde{\lambda}_{i\dot{\beta}}\lambda_{(i-1)\gamma}x_{j\,i-1}^{\dot{\beta}\gamma}\,\varepsilon_{ABCD}}{2\,\langle i\, i-1\rangle\,\langle j+1\, j\rangle\, x_{j\,i-1}^2 x_{j\,i}^2} \eta_i^A \eta_i^B \eta_j^C \eta_{j+1}^D
\end{aligned}
$$

Similarly, we obtain the second equality as follows.

$$
\begin{aligned}
&ig \int \left\langle \mathcal{V}_{j\,j+1}(x_j)|_{\eta_j\eta_{j+1}\eta_{j+1}}\ \mathcal{E}_i|_{\eta_i} \right\rangle \\
&\quad = \frac{ig}{N} \int_{x=x_{i-1}+tp_i} \mathrm{tr} \left\langle -\frac{2ig\lambda_{j\gamma}\psi^{\gamma C}(x_j)}{c_0^3\,\langle j+1\, j\rangle^2} \varepsilon_{AB_1B_2C}\, \eta_j^A \eta_{j+1}^{B_1}\eta_{j+1}^{B_2}\ \frac{i}{c_0}\tilde{\lambda}_{i\dot{\beta}}\tilde{\psi}_F^{\dot{\beta}}(x)\eta_i^F \right\rangle \\
&\quad = \frac{ig^2\lambda_{j\gamma}\tilde{\lambda}_{i\dot{\beta}}\varepsilon_{AB_1B_2C}\delta^{ab}}{Nc_0^4\,\langle j+1\, j\rangle^2} \eta_i^F \eta_j^A \eta_{j+1}^{B_1}\eta_{j+1}^{B_2} \int_{x=x_{i-1}+tp_i} \left\langle \psi^{\gamma Ca}(x_j)\ \tilde{\psi}_F^{\dot{\beta}b}(x) \right\rangle \\
&\quad = \left(\frac{g^2(N^2-1)}{c_0^4\pi^2 N}\right) \frac{\lambda_{j\gamma}\tilde{\lambda}_{i\dot{\beta}}\varepsilon_{AB_1B_2C}}{2\,\langle j+1\, j\rangle^2} \eta_i^C \eta_j^A \eta_{j+1}^{B_1}\eta_{j+1}^{B_2} \int_{x=x_{i-1}+tp_i} \frac{(x_j-x)^{\gamma\dot{\beta}}}{(x_j-x)^4} \\
&\quad = C_0 \cdot \frac{\lambda_{j\gamma}\tilde{\lambda}_{i\dot{\beta}}x_{j\,i-1}^{\gamma\dot{\beta}}\varepsilon_{ABCD}}{2\,\langle j+1\, j\rangle^2} \eta_i^A \eta_j^B \eta_{j+1}^C \eta_{j+1}^D \int_0^1 dt\, \frac{1}{(x_j-x_{i-1}-tp_i)^4} \\
&\quad = C_0 \cdot \frac{\lambda_{j\gamma}\tilde{\lambda}_{i\dot{\beta}}x_{j\,i-1}^{\gamma\dot{\beta}}\varepsilon_{ABCD}}{2\,\langle j+1\, j\rangle^2\, x_{j\,i-1}^2 x_{j\,i}^2} \eta_i^A \eta_j^B \eta_{j+1}^C \eta_{j+1}^D
\end{aligned}
$$

Similarly, we obtain the third equality as follows.

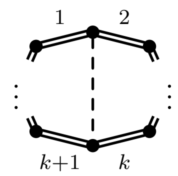
$$
\begin{aligned}
ig \int \langle \mathcal{V}_{j\,j+1}(x_j)|_{\eta_j\eta_j\eta_{j+1}}\, \mathcal{E}_i|_{\eta_i} \rangle
&= \frac{ig}{N} \int_{x=x_{i-1}+tp_i} \operatorname{tr} \left\langle \frac{2ig\lambda_{(j+1)\gamma}\psi^{\gamma C}(x_j)}{c_0^3 \langle j+1\, j\rangle^2} \varepsilon_{A_1A_2BC}\, \eta_j^{A_1}\eta_j^{A_2}\eta_{j+1}^B\, \frac{i}{c_0} \tilde{\lambda}_{i\dot{\beta}}\tilde{\psi}_F^{\dot{\beta}}(x)\eta_i^F \right\rangle \\
&= -\frac{ig^2\lambda_{(j+1)\gamma}\tilde{\lambda}_{i\dot{\beta}}\varepsilon_{A_1A_2BC}\delta^{ab}}{Nc_0^4 \langle j+1\, j\rangle^2} \eta_i^F\eta_j^{A_1}\eta_j^{A_2}\eta_{j+1}^B \int_{x=x_{i-1}+tp_i} \left\langle \psi^{\gamma Ca}(x_j)\ \tilde{\psi}_F^{\dot{\beta}b}(x) \right\rangle \\
&= -C_0 \cdot \frac{\lambda_{(j+1)\gamma}\tilde{\lambda}_{i\dot{\beta}}\varepsilon_{ABCD}}{2\langle j+1\, j\rangle^2} \eta_i^A\eta_j^B\eta_j^C\eta_{j+1}^D \int_{x=x_{i-1}+tp_i} \frac{(x_j-x)^{\gamma\dot{\beta}}}{(x_j-x)^4} \\
&= -C_0 \cdot \frac{\lambda_{(j+1)\gamma}\tilde{\lambda}_{i\dot{\beta}}x_{j\,i-1}^{\gamma\dot{\beta}}\varepsilon_{ABCD}}{2\langle j+1\, j\rangle^2 x_{j\,i-1}^2 x_{j\,i}^2} \eta_i^A\eta_j^B\eta_j^C\eta_{j+1}^D
\end{aligned}
$$

□

Having calculated some building blocks of tree-level components in Lem. 3.1.2 and Lem. 3.1.3, we now study some examples. Consider $\langle W_n\rangle|_{\eta_a\eta_b\eta_c\eta_d}$, which is the sum of all components of $\langle W_n\rangle$ proportional to $\eta_a^A\eta_b^B\eta_c^C\eta_d^D$ for any values of A, B, C, D.

Example 1 ($n > 4$ and $(a, b, c, d) = (1, 2, k, k+1)$ with $k > 2$)

Consider the component $\sim \eta_1\eta_2\eta_k\eta_{k+1}$ of $\langle W_n\rangle$ with $k > 2$ and $n > 4$. It comes from only one diagram involving two vertex terms.

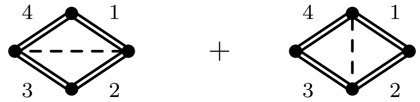


By Lem. 3.1.2, we immediately find

$$
\begin{aligned}
\langle W_n\rangle|_{\eta_1\eta_2\eta_k\eta_{k+1}} &= \left\langle \mathcal{V}_{1,2}(x_1)|_{\eta_1\eta_2}\ \mathcal{V}_{k,k+1}(x_k)|_{\eta_k\eta_{k+1}} \right\rangle \\
&= C_0 \cdot \eta_1^A\eta_2^B\eta_k^C\eta_{k+1}^D \frac{\varepsilon_{ABCD}}{\langle 1\,2\rangle \langle k\, k+1\rangle (x_1-x_k)^2}
\end{aligned}
$$

Example 2 ($n = 4$ and $(a, b, c, d) = (1, 2, 3, 4)$)

Consider the component $\sim \eta_1\eta_2\eta_3\eta_4$ of $\langle W_4\rangle$. This time, there are two graphs involved.



By Lem. 3.1.2, we calculate

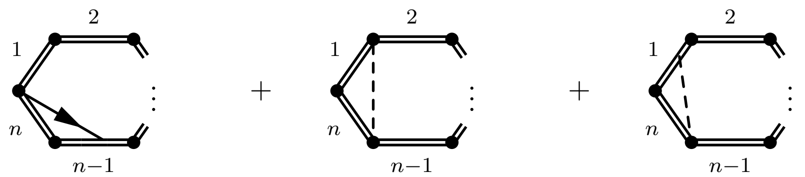
$$
\begin{aligned}
&\langle W_4\rangle\,|_{\eta_1\eta_2\eta_3\eta_4}\\
&\quad= \langle \mathcal{V}_{12}(x_1)|_{\eta_1\eta_2}\ \mathcal{V}_{34}(x_3)|_{\eta_3\eta_4}\rangle + \langle \mathcal{V}_{23}(x_2)|_{\eta_2\eta_3}\ \mathcal{V}_{41}(x_4)|_{\eta_4\eta_1}\rangle\\
&\quad= C_0\cdot \eta_1^A\eta_2^B\eta_3^C\eta_4^D\frac{\varepsilon_{ABCD}}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,(x_1-x_3)^2} + C_0\cdot \eta_2^B\eta_3^C\eta_4^D\eta_1^A\frac{\varepsilon_{BCDA}}{\langle 2\,3\rangle\,\langle 4\,1\rangle\,(x_2-x_4)^2}\\
&\quad= C_0\cdot \eta_1^A\eta_2^B\eta_3^C\eta_4^D\varepsilon_{ABCD}\left(\frac{1}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,(x_1-x_3)^2}+\frac{1}{\langle 2\,3\rangle\,\langle 4\,1\rangle\,(x_2-x_4)^2}\right)\\
&\quad= C_0\cdot \eta_1^A\eta_2^B\eta_3^C\eta_4^D\varepsilon_{ABCD}\left(\frac{1}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 2\,3\rangle\,[2\,3]}+\frac{1}{\langle 2\,3\rangle\,\langle 4\,1\rangle\,\langle 3\,4\rangle\,[3\,4]}\right)\\
&\quad= C_0\cdot \eta_1^A\eta_2^B\eta_3^C\eta_4^D\varepsilon_{ABCD}\frac{\langle 1\,4\rangle\,[3\,4]-\langle 1\,2\rangle\,[2\,3]}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 1\,2\rangle\,\langle 1\,4\rangle\,[2\,3]\,[3\,4]}\\
&\quad= 0
\end{aligned}
$$

using (2.10) and momentum conservation (2.2) in the form

$$
\langle 1\,4\rangle\,[3\,4] = \lambda_{1\alpha}\tilde\lambda_{3\dot\alpha}p_4^{\alpha\dot\alpha} = -\lambda_{1\alpha}\tilde\lambda_{3\dot\alpha}(p_1^{\alpha\dot\alpha}+p_2^{\alpha\dot\alpha}+p_3^{\alpha\dot\alpha}) = \langle 1\,2\rangle\,[2\,3]
$$

Example 3 ($n\geq 4$ and $(a,b,c,d)=(1,1,n-1,n)$)

Consider the component $\sim \eta_1\eta_1\eta_{n-1}\eta_n$ of $\langle W_n\rangle$ with $n\geq 4$, which is the sum of three graphs.



In formulas, we yield

$$
\begin{aligned}
\langle W_n\rangle\,|_{\eta_1\eta_1\eta_{n-1}\eta_n} &= ig\int \big\langle \mathcal{V}_{n1}(x_n)|_{\eta_n\eta_1\eta_1}\ \mathcal{E}_{n-1}|_{\eta_{n-1}}\big\rangle + \big\langle \mathcal{V}_{12}(x_1)|_{\eta_1\eta_1}\ \mathcal{V}_{n-1,n}(x_{n-1})|_{\eta_{n-1}\eta_n}\big\rangle\\
&\quad + ig\int \big\langle \mathcal{E}_1|_{\eta_1\eta_1}\ \mathcal{V}_{n-1,n}(x_{n-1})|_{\eta_{n-1}\eta_n}\big\rangle\\
&=: (1)+(2)+(3)
\end{aligned}
$$

Consider the terms (1) and (2). In each case, the numerator in the expression stated in Lem. 3.1.3 vanishes due to

$$
\begin{aligned}
&(1)\sim \lambda_{n\gamma}\tilde\lambda_{(n-1)\dot\beta}(x_n-x_{n-2})^{\dot\beta\gamma} = \lambda_{n\gamma}\tilde\lambda_{(n-1)\dot\beta}(p_n+p_{n-1})^{\dot\beta\gamma} = 0\\
&(3)\sim \tilde\lambda_{1\dot\beta}\lambda_{n\gamma}(x_n-x_{n-1})^{\dot\beta\gamma} = \tilde\lambda_{1\dot\beta}\lambda_{n\gamma}p_n^{\dot\beta\gamma} = 0
\end{aligned}
$$

However, the denominators also vanish, and we conclude that we should have performed regularisation. Going back to the proof of Lem. 3.1.3, we observe that regularisation replaces the integrands by terms depending on ε which, for $\varepsilon\to 0$, go to the original integrands, leading to a finite plus a diverging contribution. Yet the numerator outside the integral remains unchanged and, therefore, our original reasoning remains valid such that both the finite and the diverging part (for both graphs individually) vanish.

Therefore, only (2) remains which, by Lem. 3.1.2 reads

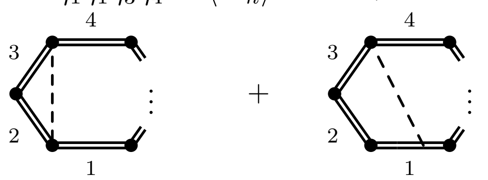
$$\begin{aligned}\langle W_n\rangle\,|_{\eta_1\eta_1\eta_{n-1}\eta_n} = (2) &= C_0\cdot\eta_1^A\eta_1^B\eta_{n-1}^C\eta_n^D\frac{\varepsilon_{ABCD}\,\langle 2\,n\rangle}{2\,\langle n\,1\rangle\,\langle 1\,2\rangle\,\langle n-1\,n\rangle\,(x_1-x_{n-1})^2}\\ &= C_0\cdot\eta_1^A\eta_1^B\eta_{n-1}^C\eta_n^D\frac{\varepsilon_{ABCD}\,\langle 2\,n\rangle}{2\,\langle n\,1\rangle^2\,\langle 1\,2\rangle\,\langle n-1\,n\rangle\,[n\,1]}\end{aligned}$$

For the cases $n=4$ and $n=5$, we thus obtain

$$\begin{aligned}\langle W_4\rangle\,|_{\eta_1\eta_1\eta_3\eta_4} &= C_0\cdot\eta_1^A\eta_1^B\eta_3^C\eta_4^D\frac{\varepsilon_{ABCD}\,\langle 2\,4\rangle}{2\,\langle 1\,4\rangle^2\,\langle 1\,2\rangle\,\langle 3\,4\rangle\,[4\,1]}\\ \langle W_5\rangle\,|_{\eta_1\eta_1\eta_4\eta_5} &= C_0\cdot\eta_1^A\eta_1^B\eta_4^C\eta_5^D\frac{\varepsilon_{ABCD}\,\langle 2\,5\rangle}{2\,\langle 1\,5\rangle^2\,\langle 1\,2\rangle\,\langle 4\,5\rangle\,[5\,1]}\end{aligned}$$

Example 4 ($n=5$ and $(a,b,c,d)=(1,1,3,4)$)

Consider the component $\sim\eta_1\eta_1\eta_3\eta_4$ of $\langle W_n\rangle$ with $n>4$.



Compared to the previous example, there are two simplifications: Here, there are only two diagrams contributing and, as we will see in a minute, there is no need for regularisation. Moreover, it is an easy exercise to generalise this example towards $n\geq 5$ and $(a,b,c,d)=(1,1,k,k+1)$ with $2<k<n-1$. Using Lem. 3.1.2 and Lem. 3.1.3, we calculate

$$\begin{aligned}\langle W_5\rangle\,|_{\eta_1\eta_1\eta_3\eta_4} &= \langle\mathcal{V}_{12}(x_1)|_{\eta_1\eta_1}\ \mathcal{V}_{3,4}(x_3)|_{\eta_3\eta_4}\rangle + ig\int\langle\mathcal{E}_1|_{\eta_1\eta_1}\ \mathcal{V}_{3,4}(x_3)|_{\eta_3\eta_4}\rangle\\ &= C_0\cdot\eta_1^A\eta_1^B\eta_3^C\eta_4^D\frac{\varepsilon_{ABCD}\,\langle 2\,5\rangle}{2\,\langle 5\,1\rangle\,\langle 1\,2\rangle\,\langle 3\,4\rangle\,(x_1-x_3)^2}\\ &\quad + C_0\cdot\frac{\tilde{\lambda}_{1\dot{\beta}}\lambda_{5\gamma}x_{3\,5}^{\dot{\beta}\gamma}\,\varepsilon_{ABCD}}{2\,\langle 1\,5\rangle\,\langle 4\,3\rangle\,x_{3\,5}^2x_{3\,1}^2}\eta_1^A\eta_1^B\eta_3^C\eta_4^D\\ &= C_0\cdot\eta_1^A\eta_1^B\eta_3^C\eta_4^D\varepsilon_{ABCD}\frac{\langle 2\,5\rangle\,(x_5-x_3)^2+\langle 1\,2\rangle\,\tilde{\lambda}_{1\dot{\beta}}\lambda_{5\gamma}(x_3-x_5)^{\dot{\beta}\gamma}}{2\,\langle 5\,1\rangle\,\langle 1\,2\rangle\,\langle 3\,4\rangle\,(x_1-x_3)^2(x_5-x_3)^2}\\ &= C_0\cdot\eta_1^A\eta_1^B\eta_3^C\eta_4^D\varepsilon_{ABCD}\frac{\langle 2\,5\rangle\,\langle 5\,4\rangle\,[5\,4]-\langle 1\,2\rangle\,[1\,4]\,\langle 5\,4\rangle}{2\,\langle 5\,1\rangle\,\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 2\,3\rangle\,[2\,3]\,\langle 5\,4\rangle\,[5\,4]}\\ &= -C_0\cdot\eta_1^A\eta_1^B\eta_3^C\eta_4^D\varepsilon_{ABCD}\frac{\langle 2\,5\rangle\,[4\,5]+\langle 2\,1\rangle\,[4\,1]}{2\,\langle 5\,1\rangle\,\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 2\,3\rangle\,[2\,3]\,[5\,4]}\end{aligned}$$

By momentum conservation (2.2), we further transform

$$\langle 2\,5\rangle\,[4\,5]+\langle 2\,1\rangle\,[4\,1]=\lambda_2\tilde{\lambda}_4(p_5+p_1)=-\lambda_2\tilde{\lambda}_4(p_2+p_3+p_4)=-\langle 2\,3\rangle\,[4\,3]$$

such that we arrive at

$$\langle W_5\rangle\,|_{\eta_1\eta_1\eta_3\eta_4}=C_0\cdot\eta_1^A\eta_1^B\eta_3^C\eta_4^D\frac{\varepsilon_{ABCD}\,\langle 2\,3\rangle\,[3\,4]}{2\,\langle 5\,1\rangle\,\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 2\,3\rangle\,[2\,3]\,[4\,5]}$$

3.2 Scattering Amplitudes in $\mathcal{N} = 4$ SYM

Supersymmetric Wilson loops as studied so far have been introduced in [CH11] with the aim of obtaining a duality with scattering amplitudes. After summarising the relevant background we will, in this section, explicitly compare components of the two observables.

Consider n particles $\{|i\rangle\}_{i=1}^{n}$, each of which with a fixed momentum p_i (such that $|i\rangle = |i(p_i)\rangle$) and of one of the following types corresponding to the field content of $\mathcal{N} = 4$ SYM theory: A gluon $|i\rangle = |g_i^+\rangle$ of positive helicity, a fermion $|i\rangle = |\psi_i^A\rangle$, a scalar $|i\rangle = |\overline{\phi}_{iAB}\rangle$, an anti-fermion $|i\rangle = \left|\tilde{\psi}_{iA}\right\rangle$ or a gluon $|i\rangle = |g_i^-\rangle$ of negative helicity. The scattering amplitude $\mathcal{A}_n(a_1, |1\rangle, \ldots, a_n, |n\rangle)$ further depends on the colours a_i and, as in any quantum field theory, is calculated as a sum over graphs resulting from the Feynman rules as derived in Sec. 1.5, however Fourier transformed to momentum space.

While in principle possible, actual calculations, even at lowest order $\sim g^0$ (tree level), turn out to be nearly impossible, containing too many and complicated diagrams and variables. In the last two decades, there has been much progress in the calculation of scattering amplitudes by means of more sophisticated methods, whose proof of validity, of course, still relies on Feynman rules (cf. [Dix96] and [Dru10]). As a first step, it is possible to separate the colour structure from the rest by eliminating the structure constants f^{abc} in favour of generators T^a (Lem. 1.1.3) and then using the identity of Lem. 1.1.4. At tree level, one arrives at the colour decomposition

$$\mathcal{A}_n^{(0)}(a_1, |i\rangle, \ldots, a_n, |n\rangle) = \sum\nolimits_{\sigma \in S_n/\mathbb{Z}_n} \operatorname{tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}\right) A_n^{(0)}(|\sigma(1)\rangle, \ldots, |\sigma(n)\rangle)$$

where $A_n^{(0)}$ is referred to as the colour ordered (tree-level) amplitude. At higher orders in g, one obtains a similar formula, however containing multi-trace terms which vanish only in the planar limit ($N \to \infty$). A convenient way to calculate the colour-ordered amplitudes is to use BCFW recursion (cf. [BCFW05]).

Following [Nai88] and using the notation of [CH11] we introduce, for each $i \in \{1, \ldots, n\}$, four Grassmann generators $\tilde{\eta}_i^A$ and put the different on-shell state into a single on-shell superstate as follows.

$$|\Phi_i\rangle = |g_i^+\rangle + \tilde{\eta}_i^A |\psi_i^A\rangle + \frac{1}{2}\tilde{\eta}_i^A\tilde{\eta}_i^B |\overline{\phi}_{iAB}\rangle + \frac{1}{3!}\varepsilon_{ABCD}\tilde{\eta}_i^A\tilde{\eta}_i^B\tilde{\eta}_i^C \left|\tilde{\psi}_{iD}\right\rangle + \frac{1}{4!}(\tilde{\eta}_i)^4 |g_i^-\rangle$$

where we set $(\tilde{\eta}_i)^4 := \varepsilon_{ABCD}\tilde{\eta}_i^A\tilde{\eta}_i^B\tilde{\eta}_i^C\tilde{\eta}_i^D$. Extending the colour ordered amplitudes $A_n^{(0)}$ by (super-)linearity over the generators $\tilde{\eta}_i^A$, one can thus define the super-amplitude $A_n^{(0)}(|\Phi_1\rangle, \ldots, |\Phi_n\rangle)$ which, in turn, by definition contains the original amplitudes as the coefficients of the products of generators in the Grassmann expansion. It turns out (cf. [DHKS10]) that the super amplitude can be factorised

$$A_n^{(0)}(|\Phi_1\rangle, \ldots, |\Phi_n\rangle) = A_{n\,\mathrm{MHV}}^{(0)} \cdot \left(1 + M_{n\,\mathrm{NMHV}}^{(0)} + M_{n\,\mathrm{NNMHV}}^{(0)} + \ldots + M_{n\,\overline{\mathrm{MHV}}}^{(0)}\right)$$

where the subscripts MHV="maximally helicity violating" etc. correspond to the type of amplitudes and, as a consequence, the terms in parantheses on the right hand side have respective Grassmann degrees $0, 4, 8, \ldots, (4n-16)$. Moreover, all terms on the right hand side now depend only on the momenta λ_i and $\tilde{\lambda}_i$ (the theory is massless such that $p_i = \lambda_i\tilde{\lambda}_i$ as in (1.11)) and the generators $\tilde{\eta}_i^A$.

In the following, we shall focus on the next-to-MHV subamplitude $M^{(0)}_{n\,\mathrm{NMHV}}$ of Grassmann degree 4, which can be written as a compact and explicit expression in terms of momentum super-twistors as summarised next. Consult [Hod09] and [MS09] for momentum twistors as well as [WW90] for a general introduction to twistor theory. Let η_i^A (untilded) denote the Grassmann odd coordinates of the momentum super-twistor associated to the i-th particle, which turns out to be a linear combination of the tilded generators with coefficients depending on the (half-)momenta λ_i. In terms of these momentum super-twistors, the next-to-MHV subamplitude can be written as follows ([DDH12], cf. also [MS09] and [BMS10]).

$$M^{(0)}_{n\,\mathrm{NMHV}} = \sum\nolimits_{1<i<j<n} [1\, i\, i+1\, j\, j+1] \tag{3.3a}$$

$$[a\,b\,c\,d\,e] := \frac{\delta^{0|4}\left(\eta_a \langle b\,c\,d\,e\rangle + \text{cyclic}\right)}{\langle a\,b\,c\,d\rangle \langle b\,c\,d\,e\rangle \langle c\,d\,e\,a\rangle \langle d\,e\,a\,b\rangle \langle e\,a\,b\,c\rangle} \tag{3.3b}$$

where $\langle a\,b\,c\,d\rangle$ is the totally antisymmetric contraction of the momentum twistor coordinates, which satisfies

$$\langle i\, i+1\, k\, k+1\rangle = (x_i - x_k)^2 \cdot \langle i,\, i+1\rangle \langle k,\, k+1\rangle \tag{3.4}$$

The Grassmann delta function is defined as

$$\delta^{0|4}\left(\eta_a \langle b\,c\,d\,e\rangle + \text{cyclic}\right) := \prod\nolimits_{A=1}^{4} \left(\eta_a^A \langle b\,c\,d\,e\rangle + \text{cyclic}\right) \tag{3.5}$$

In the following lemma, we calculate some examples using (3.3a).

Lemma 3.2.1. For brevity, we set $M_n := M^{(0)}_{n\,\mathrm{NMHV}}$. The four-point amplitude $M_4 = 0$ vanishes. For $n > 4$, we obtain

$$M_n|_{\eta_1\eta_2\eta_k\eta_{k+1}} = \eta_1^A \eta_2^B \eta_k^C \eta_{k+1}^D \frac{\varepsilon_{ABCD}}{\langle 1\,2\rangle \langle k\, k+1\rangle (x_1 - x_k)^2}\,, \qquad k > 2$$

$$M_n|_{\eta_1\eta_2\eta_3\eta_4} = \eta_1^A \eta_2^B \eta_3^C \eta_4^D \frac{\varepsilon_{ABCD}}{\langle 1\,2\rangle \langle 3\,4\rangle \langle 2\,3\rangle [2\,3]}$$

while the five-point amplitude M_5 has components

$$M_5|_{\eta_1\eta_1\eta_2\eta_3} = \eta_1^A \eta_1^B \eta_2^C \eta_3^D \frac{\varepsilon_{ABCD}}{2} \frac{\langle 4\,5\rangle [3\,4]}{\langle 1\,2\rangle^2 [1\,2] \langle 2\,3\rangle [2\,3] \langle 5\,1\rangle}$$

$$M_5|_{\eta_1\eta_1\eta_3\eta_4} = \eta_1^A \eta_1^B \eta_3^C \eta_4^D \frac{\varepsilon_{ABCD}}{2} \frac{\langle 2\,3\rangle [3\,4]}{\langle 5\,1\rangle \langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle [2\,3] [4\,5]}$$

$$M_5|_{\eta_1\eta_1\eta_4\eta_5} = \eta_1^A \eta_1^B \eta_4^C \eta_5^D \frac{\varepsilon_{ABCD}}{2} \frac{\langle 2\,3\rangle [3\,4]}{\langle 5\,1\rangle^2 [4\,5] \langle 4\,5\rangle \langle 1\,2\rangle [5\,1]}$$

Proof. It is well-known that, for $n = 4$, the only non-vanishing amplitudes are MHV such that, in particular, $M_4 = M^{(0)}_{4\,\mathrm{NMHV}} = 0$. This can also be seen directly from (3.3a) as follows. The only summand is $[1\,2\,3\,3\,4]$. However, the five-bracket vanishes if two entries are the same: For simplicity, assume $a = b$. Then, in each of the 4 factors on the right hand side of (3.5), antisymmetry of the four-bracket is survived only by the sum $\eta_a^A \langle a\,c\,d\,e\rangle + \eta_a^A \langle c\,d\,e\,a\rangle$, which vanishes for the same reason.

For the calculation of the amplitudes with $n > 4$, observe the following remark. The delta function (3.5) is a sum of terms of Grassmann order 4. Consider, in the corresponding such term, the coefficient of $\eta_a^1 \eta_b^2 \eta_c^3 \eta_d^4$. By construction, it remains unchanged under any permutation of the indices *abcd*. The coefficient in front of $\eta_a^A\eta_b^B\eta_c^C\eta_d^D$ thus equals the coefficient of $\eta_a^1\eta_b^2\eta_c^3\eta_d^4$ multiplied by ε_{ABCD}, which arises from permuting the η variables such that the upper indices $ABCD$ are translated into 1234. As usual, the product $\varepsilon_{ABCD}\,\eta_a^A\eta_b^B\eta_c^C\eta_d^D$ must be multiplied by a symmetry factor S if any of the lower indices *abcd* are equal. If only $a=b$, we obtain $S=1/2$. It is clear that the same remarks apply to $M^{(0)}_{n\,\mathrm{NMHV}}$.

Consider the component $\sim \eta_1\eta_2\eta_k\eta_{k+1}$ with $k>2$ of the n-point NMHV subamplitude. Only one term contributes to the sum (3.3a), and we calculate

$$\begin{aligned}
M_n|_{\eta_1\eta_2\eta_k\eta_{k+1}} &= [1\,2\,3\,k\,k+1]\,|_{\eta_1\eta_2\eta_k\eta_{k+1}}\\
&= \varepsilon_{ABCD}\frac{\eta_1^A\,\langle 2\,3\,k\,k+1\rangle\,\eta_2^B\,\langle 3\,k\,k+1\,1\rangle\,\eta_k^C\,\langle k+1\,1\,2\,3\rangle\,\eta_{k+1}^D\,\langle 1\,2\,3\,k\rangle}{\langle 1\,2\,3\,k\rangle\,\langle 2\,3\,k\,k+1\rangle\,\langle 3\,k\,k+1\,1\rangle\,\langle k\,k+1\,1\,2\rangle\,\langle k+1\,1\,2\,3\rangle}\\
&= \eta_1^A\eta_2^B\eta_k^C\eta_{k+1}^D\frac{\varepsilon_{ABCD}}{\langle k\,k+1\,1\,2\rangle}
\end{aligned}$$

The first formula stated now follows immediately from (3.4), and the second is the case $k=3$, using $x_1-x_3=x_1-x_2+x_2-x_3=-p_2-p_3$.

Now consider the amplitude M_5. In this case, (3.3a) has only one summand such that $M_5=[1\,2\,3\,4\,5]$. For the component $\sim \eta_1\eta_1\eta_2\eta_3$ of M_5, we thus obtain

$$\begin{aligned}
M_5|_{\eta_1\eta_1\eta_2\eta_3} &= [1\,2\,3\,4\,5]\,|_{\eta_1\eta_1\eta_2\eta_3}\\
&= \eta_1^A\eta_1^B\eta_2^C\eta_3^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\,4\,5\rangle}{\langle 1\,2\,3\,4\rangle\,\langle 5\,1\,2\,3\rangle}\\
&= \eta_1^A\eta_1^B\eta_2^C\eta_3^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\rangle\,\langle 4\,5\rangle\,(x_2-x_4)^2}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,(x_1-x_3)^2\,\langle 5\,1\rangle\,\langle 2\,3\rangle\,(x_5-x_2)^2}\\
&= \eta_1^A\eta_1^B\eta_2^C\eta_3^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 4\,3\rangle\,[4\,3]}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 3\,2\rangle\,[3\,2]\,\langle 5\,1\rangle\,\langle 2\,3\rangle\,\langle 2\,1\rangle\,[2\,1]}
\end{aligned}$$

which equals the expression stated. Similarly, the component $\sim \eta_1\eta_1\eta_3\eta_4$ reads

$$\begin{aligned}
M_5|_{\eta_1\eta_1\eta_3\eta_4} &= \eta_1^A\eta_1^B\eta_3^C\eta_4^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\,4\,5\rangle}{\langle 1\,2\,3\,4\rangle\,\langle 3\,4\,5\,1\rangle}\\
&= \eta_1^A\eta_1^B\eta_3^C\eta_4^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\rangle\,\langle 4\,5\rangle\,(x_2-x_4)^2}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,(x_1-x_3)^2\,\langle 3\,4\rangle\,\langle 5\,1\rangle\,(x_3-x_5)^2}\\
&= \eta_1^A\eta_1^B\eta_3^C\eta_4^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 3\,4\rangle\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle\,\langle 2\,3\rangle\,[2\,3]\,\langle 3\,4\rangle\,\langle 5\,1\rangle\,\langle 4\,5\rangle\,[4\,5]}
\end{aligned}$$

while the component $\sim \eta_1\eta_1\eta_4\eta_5$ evaluates to

$$\begin{aligned}
M_5|_{\eta_1\eta_1\eta_4\eta_5} &= \eta_1^A\eta_1^B\eta_4^C\eta_5^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\,4\,5\rangle}{\langle 3\,4\,5\,1\rangle\,\langle 4\,5\,1\,2\rangle}\\
&= \eta_1^A\eta_1^B\eta_4^C\eta_5^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\rangle\,\langle 4\,5\rangle\,(x_2-x_4)^2}{\langle 3\,4\rangle\,\langle 5\,1\rangle\,(x_3-x_5)^2\,\langle 4\,5\rangle\,\langle 1\,2\rangle\,(x_4-x_1)^2}\\
&= \eta_1^A\eta_1^B\eta_4^C\eta_5^D\frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,3\rangle\,\langle 4\,5\rangle\,\langle 3\,4\rangle\,[3\,4]}{\langle 3\,4\rangle\,\langle 5\,1\rangle\,\langle 4\,5\rangle\,[4\,5]\,\langle 4\,5\rangle\,\langle 1\,2\rangle\,\langle 5\,1\rangle\,[5\,1]}
\end{aligned}$$

This finishes the calculation. □

3.2.1 Comparison with Supersymmetric Wilson Loops

Gluon scattering amplitudes have been known to be dual to Wilson loops along lightlike polygons. In [CH11], a similar duality (at weak coupling) between the full (super) scattering amplitudes of $\mathcal{N} = 4$ SYM theory and super Wilson loops has been claimed with the following identification of parameters. The number n of particles corresponds to the number n of polygon vertices while the particle momenta p_i are translated into the differences $x_i - x_{i-1}$. By construction, the latter are lightlike which matches with the massless theory. Moreover, the (odd) momemtum supertwistors η_i^A are identified with the Grassmann generators (2.3) which occur in the construction of the super Wilson loop.

We will now check this proposal at tree-level through the examples worked out in this and in the previous section. First, observe that both observables are the sum over monomials of Grassmann degree $0, 4, 8, \ldots$. As our first test, we try to match the components $\sim \eta_1\eta_2\eta_k\eta_{k+1}$ (with $n > 4$ and $k > 2$) with the result

$$\langle W_n \rangle |_{\eta_1\eta_2\eta_k\eta_{k+1}} = C_0 \cdot M_n|_{\eta_1\eta_2\eta_k\eta_{k+1}}$$

which perfectly agrees upon setting $C_0 \stackrel{!}{=} 1$ which fixes the so far undetermined constant c_0 in (2.8) to be

$$c_0^4 = \frac{g^2(N^2-1)}{N\pi^2} \tag{3.6}$$

In particular, we see that $c_0^2 \sim g$ in agreement with (3.2). Similarly, we obtain an agreement for the components $\sim \eta_1\eta_2\eta_3\eta_4$ in the case $n = 4$, where both vanish.

$$\langle W_4 \rangle |_{\eta_1\eta_2\eta_3\eta_4} = M_4|_{\eta_1\eta_2\eta_3\eta_4} = 0$$

For $n = 5$ and the components $\sim \eta_1\eta_1\eta_3\eta_4$, we yield

$$\langle W_5 \rangle |_{\eta_1\eta_1\eta_3\eta_4} = M_5|_{\eta_1\eta_1\eta_3\eta_4}$$

with the implicit use of the normalisation (3.6).

There exist, however, components for which the agreement is explicitly broken. Consider, for example, the components $\sim \eta_1\eta_1\eta_3\eta_4$ with $n = 4$. While the scatting amplitude vanishes, the super Wilson loop does not.

$$\langle W_4 \rangle |_{\eta_1\eta_1\eta_3\eta_4} \neq M_4|_{\eta_1\eta_1\eta_3\eta_4} = 0$$

The mismatch is also obtained for the components $\sim \eta_1\eta_1\eta_4\eta_5$ with $n = 5$. We calculate

$$\begin{aligned} &M_5|_{\eta_1\eta_1\eta_4\eta_5} - \langle W_5 \rangle |_{\eta_1\eta_1\eta_4\eta_5} \\ &\quad = \eta_1^A\eta_1^B\eta_4^C\eta_5^D \frac{\varepsilon_{ABCD}}{2}\left(\frac{\langle 2\,5\rangle}{\langle 1\,5\rangle^2\langle 1\,2\rangle\langle 4\,5\rangle[5\,1]} - \frac{\langle 2\,3\rangle[3\,4]}{\langle 5\,1\rangle^2[4\,5]\langle 4\,5\rangle\langle 1\,2\rangle[5\,1]}\right) \\ &\quad = \eta_1^A\eta_1^B\eta_4^C\eta_5^D \frac{\varepsilon_{ABCD}}{2}\frac{\langle 2\,5\rangle[4\,5] + \langle 2\,3\rangle[4\,3]}{\langle 5\,1\rangle^2[4\,5]\langle 4\,5\rangle\langle 1\,2\rangle[5\,1]} \end{aligned}$$

By momentum conservation (2.2), we further transform

$$\langle 2\,5\rangle[4\,5] + \langle 2\,3\rangle[4\,3] = \lambda_2\tilde{\lambda}_4(p_3 + p_5) = -\lambda_2\tilde{\lambda}_4(p_1 + p_2 + p_4) = -\langle 2\,1\rangle[4\,1]$$

and thus

$$M_5|_{\eta_1\eta_1\eta_4\eta_5} - \langle W_5\rangle|_{\eta_1\eta_1\eta_4\eta_5} = \eta_1^A\eta_1^B\eta_4^C\eta_5^D\frac{\varepsilon_{ABCD}}{2}\frac{[4\,1]}{\langle 5\,1\rangle^2[4\,5]\langle 4\,5\rangle[5\,1]} \neq 0$$

We expect an analogous behaviour for the components $\sim \eta_1\eta_1\eta_{n-1}\eta_n$ with arbitrary $n \geq 4$.

To summarise, we have shown that scattering amplitudes are not dual to super Wilson loops. On the other hand, the remaining partial duality is still striking. We wonder whether there is a way to repair the mismatching problem. We have seen that the problematic terms are connected with diverging diagrams which require regularisation. We have also seen that, then, both the finite and the diverging parts of these diagrams vanish individually. Therefore, the use of a different regularisation method should not make a difference. In the next section, we consider a natural variant of the super Wilson loop which, however, turns out not to solve the problem.

3.3 A Natural Variant

The solution for the edge and vertex operators found in Thm. 2.2.1 and Thm. 2.3.2 is not uniquely determined by the supersymmetry conditions (2.9a) and (2.9b). More precisely, the edge operators are not unique while the vertex operators are determined by the edge operators, as we have seen.

By construction, $\mathcal{E}_i$ and X^α_{iA} depend on p_i and p_{i-1}. By using the Schouten identity in the proof of Thm. 2.2.1 with $1 = \frac{\lambda_{i\gamma}\lambda^\gamma_{i+1}}{\langle i\, i{+}1\rangle}$ instead, one obtains a different solution depending on p_i and p_{i+1}. The proof goes through verbatim with λ_{i-1} and $\langle i\, i-1\rangle$ replaced by λ_{i+1} and $\langle i\, i+1\rangle$, respectively. For future reference, we state the first terms next. In fact, there are many more solutions which, however, seem unnatural.

Proposition 3.3.1. The following edge operator satisfies the ansatz $\mathcal{E}_i = p_i \cdot A + \mathcal{O}(\eta)$ as well as (2.9a):

$$\mathcal{E}_i = \frac{1}{2}\lambda_{i\beta}\tilde{\lambda}_{i\dot\beta}A^{\beta\dot\beta} + \frac{i}{c_0}\tilde{\lambda}_{i\dot\beta}\tilde{\psi}^{\dot\beta}_A\eta_i^A - \frac{i\sqrt{2}}{2c_0^2}\frac{\tilde{\lambda}_{i\dot\beta}\lambda_{(i+1)\gamma}D^{\dot\beta\gamma}\overline{\phi}_{AB}}{\langle i\, i+1\rangle}\eta_i^A\eta_i^B + \mathcal{O}((\eta_i)^3)$$

with

$$X^\alpha_{iA} = -\frac{2i\sqrt{2}\,g\lambda^\alpha_{i+1}}{c_0\langle i\, i+1\rangle}\overline{\phi}_{AB}\eta_i^B + \varepsilon_{ABCD}\frac{2g\lambda^\alpha_{i+1}\lambda_{(i+1)\gamma}\psi^{\gamma B}}{c_0^2\langle i\, i+1\rangle^2}\eta_i^C\eta_i^D + \mathcal{O}((\eta_i)^3)$$

As for the vertex terms, it would be no difficulty to establish a recursion formula along the lines of Prp. 2.3.1. For the few terms needed below, we provide a direct proof instead.

Proposition 3.3.2. The following vertex operator satisfies the ansatz $\mathcal{V}_{i,i+1} = 1+\mathcal{O}(\eta)$, only depends on the generators η_i and η_{i+1} and satisfies (2.9b) with X^α_{iA} as in Prp. 3.3.1.

$$\begin{aligned}\mathcal{V}_{i,i+1} = 1 &- \frac{2\sqrt{2}g}{c_0^2\langle i\, i+1\rangle}\overline{\phi}_{AB}\eta_i^A\eta_{i+1}^B + \frac{\sqrt{2}g\langle i\, i+2\rangle}{c_0^2\langle i\, i+1\rangle\langle i+1\, i+2\rangle}\overline{\phi}_{AB}\eta_{i+1}^{B_1}\eta_{i+1}^{B_2}\\ &+ \frac{2ig}{c_0^3}\frac{\left(\langle i\, i+1\rangle\lambda_{(i+2)\gamma} - \langle i\, i+2\rangle\lambda_{(i+1)\gamma}\right)\psi^{\gamma C}}{\langle i\, i+1\rangle^2\langle i+1\, i+2\rangle}\varepsilon_{AB_1B_2C}\eta_i^A\eta_{i+1}^{B_1}\eta_{i+1}^{B_2}\\ &+ \frac{2ig}{c_0^3}\frac{\lambda_{(i+1)\gamma}\psi^{\gamma C}}{\langle i\, i+1\rangle^2}\varepsilon_{A_1A_2BC}\eta_i^{A_1}\eta_i^{A_2}\eta_{i+1}^B + \mathcal{O}((\eta_{i+1})^3) + \mathcal{O}(\eta^4)\end{aligned}$$

Proof. It is clear that the first order terms vanish. As in the proof of Prp. 2.3.1, we find

$$\mathcal{Q}^\alpha_A(\mathcal{V}_{i,i+1})|_{\eta^1} = 2c_0\lambda^\alpha_k C^{kj}_{AB}\eta^B_j$$
$$\left(iX^\alpha_{(i+1)A}\mathcal{V}_{i,i+1} - iV_{i,i+1}X^\alpha_{iA}\right)|_{(\eta_i)^1} = -iX^\alpha_{iA}|_{(\eta_i)^1}$$

such that (2.9b) implies

$$2c_0\lambda^\alpha_k C^{ki}_{AB} = -\frac{2\sqrt{2}g\lambda^\alpha_{i+1}}{c_0\left\langle i\, i+1\right\rangle}\overline{\phi}_{AB}$$

Multiplying both sides with, respectively, $\lambda_{(i+1)\alpha}$ and $\lambda_{i\alpha}$, we obtain

$$C^{ii}_{AB} = 0\ , \qquad C^{i,i+1}_{AB} = -\frac{\sqrt{2}g}{c_0^2\left\langle i\, i+1\right\rangle}\overline{\phi}_{AB}$$

Similarly, we find

$$2c_0\lambda^\alpha_k C^{k,i+1}_{AB} = iX^\alpha_{(i+1)iA}|_{(\eta_{i+1})^1} = \frac{2\sqrt{2}g\lambda^\alpha_{i+2}}{c_0\left\langle i+1\, i+2\right\rangle}\overline{\phi}_{AB}$$

such that

$$C^{i+1,i+1}_{AB} = \frac{\sqrt{2}g\left\langle i\, i+2\right\rangle}{c_0^2\left\langle i\, i+1\right\rangle\left\langle i+1\, i+2\right\rangle}\overline{\phi}_{AB}$$

thus determining the terms of second order. As for the third order, we find

$$\mathcal{Q}^\alpha_A(\mathcal{V}_{i,i+1})|_{(\eta_{i+1})^2} = \left(q^\alpha_A(C^{i+1,i+1}_{B_1B_2}) - 3c_0\lambda^\alpha_k C^{k,i+1,i+1}_{AB_1B_2}\right)\eta^{B_1}_{i+1}\eta^{B_2}_{i+1}$$
$$\left(iX^\alpha_{(i+1)A}\mathcal{V}_{i,i+1} - iV_{i,i+1}X^\alpha_{iA}\right)|_{(\eta_{i+1})^2} = iX^\alpha_{(i+1)A}|_{(\eta_{i+1})^2}$$

such that (2.9b) implies

$$3c_0\lambda^\alpha_k C^{k,i+1,i+1}_{AB_1B_2} = q^\alpha_A(C^{i+1,i+1}_{B_1B_2}) - iX^\alpha_{(i+1)A}|_{(\eta_{i+1})^2}$$

Therefore

$$\begin{aligned}C^{i,i+1,i+1}_{AB_1B_2} &= \frac{\lambda_{(i+1)\alpha}}{3c_0\left\langle i+1\, i\right\rangle}\left(q^\alpha_A(C^{i+1,i+1}_{B_1B_2}) - iX^\alpha_{(i+1)A}|_{(\eta_{i+1})^2}\right)\\ &= \frac{2ig}{3c_0^3}\frac{\left(\left\langle i\, i+1\right\rangle\lambda_{(i+2)\gamma} - \left\langle i\, i+2\right\rangle\lambda_{(i+1)\gamma}\right)\psi^{\gamma C}}{\left\langle i\, i+1\right\rangle^2\left\langle i+1\, i+2\right\rangle}\varepsilon_{AB_1B_2C}\end{aligned}$$

Similarly, we find

$$\mathcal{Q}^\alpha_A(\mathcal{V}_{i,i+1})|_{\eta_i\eta_{i+1}} = 2\left(q^\alpha_A(C^{i,i+1}_{B_1B_2}) - 3c_0\lambda^\alpha_k C^{k,i,i+1}_{AB_1B_2}\right)\eta^{B_1}_i\eta^{B_2}_{i+1}$$
$$\left(iX^\alpha_{(i+1)A}\mathcal{V}_{i,i+1} - iV_{i,i+1}X^\alpha_{iA}\right)|_{\eta_i\eta_{i+1}} = 0$$

such that (2.9b) implies

$$C^{i,i,i+1}_{AB_1B_2} = \frac{\lambda_{(i+1)\alpha}}{3c_0\left\langle i+1\, i\right\rangle}q^\alpha_A(C^{i,i+1}_{B_1B_2}) = \frac{2ig}{3c_0^3}\frac{\lambda_{(i+1)\gamma}\psi^{\gamma C}}{\left\langle i\, i+1\right\rangle^2}\varepsilon_{AB_1B_2C}$$

Moreover, we find $C^{iii}_{ABC} = 0$, thus determining the terms of order three stated. □

Tree-Level Calculations

We shall now calculate some tree-level components of the expectation value (3.1) with respect to the variant Wilson loop with edges and vertices as in Prp. 3.3.1 and Prp. 3.3.2. Comparison with scattering amplitudes will then give a result analogous to the original case in the previous section.

The $\sim \eta_i \eta_{i+1}$ component of $\mathcal{V}_{i,i+1}$ remains unchanged. Therefore, the connector

$$\left\langle \mathcal{V}_{i,i+1}(x_i)|_{\eta_i\eta_{i+1}}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}} \right\rangle = C_0 \cdot \eta_i^A \eta_{i+1}^B \eta_j^C \eta_{j+1}^D \frac{\varepsilon_{ABCD}}{\langle i\, i+1\rangle \langle j\, j+1\rangle (x_i - x_j)^2}$$

between two vertices remains as in Lem. 3.1.2. Moreover, $\mathcal{V}_{i,i+1}|_{\eta_{i+1}\eta_{i+1}}$ of the variant equals the original $\mathcal{V}_{i,i+1}|_{\eta_i\eta_i}$ upon changing the index i to $i+1$ and, therefore,

$$\begin{aligned} &\left\langle \mathcal{V}_{i,i+1}(x_i)|_{\eta_{i+1}\eta_{i+1}}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}} \right\rangle \\ &\quad = C_0 \cdot \eta_{i+1}^A \eta_{i+1}^B \eta_j^C \eta_{j+1}^D \frac{\varepsilon_{ABCD}\, \langle i+2\, i\rangle}{2\, \langle i\, i+1\rangle \langle i+1\, i+2\rangle \langle j\, j+1\rangle (x_i - x_j)^2} \end{aligned}$$

Similarly, $\mathcal{E}_i|_{\eta_i\eta_i}$ is changed by $i-1 \to i+1$, while $\mathcal{V}_{j,j+1}|_{\eta_j\eta_j\eta_{j+1}}$ and $\mathcal{E}_i|_{\eta_i}$ both remain unchanged. As in Lem. 3.1.3, we thus find

$$\begin{aligned} ig \int \left\langle \mathcal{E}_i|_{\eta_i\eta_i}\ \mathcal{V}_{j,j+1}(x_j)|_{\eta_j\eta_{j+1}} \right\rangle &= C_0 \cdot \frac{\tilde\lambda_{i\dot\beta} \lambda_{(i+1)\gamma} x_{j\,i-1}^{\dot\beta\gamma}\, \varepsilon_{ABCD}}{2\, \langle i\, i+1\rangle \langle j+1\, j\rangle\, x_{j\,i-1}^2 x_{j\,i}^2} \eta_i^A \eta_i^B \eta_j^C \eta_{j+1}^D \\ ig \int \left\langle \mathcal{V}_{j\,j+1}(x_j)|_{\eta_j\eta_j\eta_{j+1}}\ \mathcal{E}_i|_{\eta_i} \right\rangle &= -C_0 \cdot \frac{\lambda_{(j+1)\gamma} \tilde\lambda_{i\dot\beta} x_{j\,i-1}^{\gamma\dot\beta} \varepsilon_{ABCD}}{2\, \langle j+1\, j\rangle^2\, x_{j\,i-1}^2 x_{j\,i}^2} \eta_i^A \eta_j^B \eta_j^C \eta_{j+1}^D \end{aligned}$$

Having derived the expressions for some connectors, we are now in a position to calculate tree-level components and compare them with scattering amplitudes. The first observation is that $\langle W_n\rangle|_{\eta_1\eta_2\eta_k\eta_{k+1}}$ (with $n > 4$ and $k > 2$) remains unchanged and continues to match with the amplitude

$$\langle W_n\rangle|_{\eta_1\eta_2\eta_k\eta_{k+1}} = C_0 \cdot M_n|_{\eta_1\eta_2\eta_k\eta_{k+1}}$$

thus leading to the same constant fixing (3.6) as before. The component $\sim \eta_1\eta_1\eta_3\eta_4$ with $n = 5$ is the sum of two diagrams

$$\begin{aligned} &\langle W_5\rangle|_{\eta_1\eta_1\eta_3\eta_4} \\ &\quad = \langle \mathcal{V}_{51}(x_5)|_{\eta_1\eta_1}\ \mathcal{V}_{3,4}(x_3)|_{\eta_3\eta_4}\rangle + ig \int \langle \mathcal{E}_1|_{\eta_1\eta_1}\ \mathcal{V}_{3,4}(x_3)|_{\eta_3\eta_4}\rangle \\ &\quad = \left(\frac{\langle 2\,5\rangle}{2\,\langle 5\,1\rangle \langle 1\,2\rangle \langle 3\,4\rangle (x_5 - x_3)^2} + \frac{\tilde\lambda_{1\dot\beta} \lambda_{2\gamma} x_{3\,5}^{\dot\beta\gamma}}{2\,\langle 1\,2\rangle \langle 4\,3\rangle\, x_{3\,5}^2 x_{3\,1}^2} \right) \varepsilon_{ABCD} \eta_1^A \eta_1^B \eta_3^C \eta_4^D \\ &\quad = -\frac{\langle 2\,5\rangle\,[2\,3] + \langle 1\,5\rangle\,[1\,3]}{2\,\langle 1\,2\rangle \langle 3\,4\rangle \langle 5\,1\rangle \langle 4\,5\rangle\,[4\,5]\,[3\,2]} \varepsilon_{ABCD} \eta_1^A \eta_1^B \eta_3^C \eta_4^D \\ &\quad = \frac{\langle 4\,5\rangle\,[4\,3]}{2\,\langle 1\,2\rangle \langle 3\,4\rangle \langle 5\,1\rangle \langle 4\,5\rangle\,[4\,5]\,[3\,2]} \varepsilon_{ABCD} \eta_1^A \eta_1^B \eta_3^C \eta_4^D \\ &\quad = M_5|_{\eta_1\eta_1\eta_3\eta_4} \end{aligned}$$

and, therefore, also continues to match with the amplitude.

Consider next the component $\sim \eta_1\eta_1\eta_2\eta_3$ with $n \geq 4$. It consists of three diagrams

$$\begin{aligned}\langle W_n \rangle |_{\eta_1\eta_1\eta_2\eta_3} &= \langle \mathcal{V}_{n,1}(x_n)|_{\eta_1\eta_1}\ \mathcal{V}_{2,3}(x_2)|_{\eta_2\eta_3}\rangle + \langle \mathcal{E}_1|_{\eta_1\eta_1}\ \mathcal{V}_{2,3}(x_2)|_{\eta_2\eta_3}\rangle \\ &\quad + \langle \mathcal{V}_{1\,2}(x_1)|_{\eta_1\eta_1\eta_2}\ \mathcal{E}_3|_{\eta_3}\rangle \\ &=: (1) + (2) + (3)\end{aligned}$$

of which the second and third

$$(2) \sim \tilde{\lambda}_{1\dot{\beta}}\lambda_{2\gamma}x_{2\,n}^{\dot{\beta}\gamma} = 0\ , \qquad (3) \sim \lambda_{2\gamma}\tilde{\lambda}_{3\dot{\beta}}x_{1\,2}^{\gamma\dot{\beta}} = 0$$

vanish. Here, the denominators also vanish, but the regularisation argument of Exp. 3 in Sec. 3.1.3 above goes through verbatim. We thus arrive at

$$\langle W_n \rangle |_{\eta_1\eta_1\eta_2\eta_3} = (1) = \varepsilon_{ABCD}\,\eta_1^A\eta_1^B\eta_2^C\eta_3^D\,\frac{\langle 2\,n\rangle}{2\,\langle n\,1\rangle\,\langle 1\,2\rangle^2\,[1\,2]\,\langle 2\,3\rangle}$$

For $n = 4$, this expression does not vanish, and we conclude that

$$\langle W_4 \rangle |_{\eta_1\eta_1\eta_2\eta_3} \neq M_4|_{\eta_1\eta_1\eta_2\eta_3} = 0$$

Similarly, for $n = 5$, we obtain

$$\begin{aligned}&M_5|_{\eta_1\eta_1\eta_2\eta_3} - \langle W_5 \rangle |_{\eta_1\eta_1\eta_2\eta_3} \\ &\quad = \eta_1^A\eta_1^B\eta_2^C\eta_3^D\,\frac{\varepsilon_{ABCD}}{2}\left(\frac{\langle 4\,5\rangle\,[3\,4]}{\langle 1\,2\rangle^2\,[1\,2]\,\langle 2\,3\rangle\,[2\,3]\,\langle 5\,1\rangle} - \frac{\langle 2\,5\rangle}{\langle 5\,1\rangle\,\langle 1\,2\rangle^2\,[1\,2]\,\langle 2\,3\rangle}\right) \\ &\quad = \eta_1^A\eta_1^B\eta_2^C\eta_3^D\,\frac{\varepsilon_{ABCD}}{2}\,\frac{[3\,1]}{\langle 1\,2\rangle^2\,[1\,2]\,\langle 2\,3\rangle\,[2\,3]} \\ &\quad \neq 0\end{aligned}$$

We expect an analogous result for arbitrary $n \geq 4$.

To summarise, the variant of the super Wilson loop considered has an analogous behaviour as the original. In particular, it is partially, but not completely, dual to scattering amplitudes.

References

[Alt02] H. Alt. *Lineare Funktionalanalysis.* Springer, 2002.

[AM07] L. Alday and J. Maldacena. Gluon scattering amplitudes at strong coupling. *JHEP*, 0706, 2007.

[AR08] L. Alday and R. Roiban. Scattering Amplitudes, Wilson Loops and the String/Gauge Theory Correspondence. *Phys.Rept.*, 468:153–211, 2008.

[Bär09] C. Bär. Gauge theory. Lecture notes, Universität Potsdam, 2009.

[Bau81] H. Baum. *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten.* Teubner, 1981.

[Bau09] H. Baum. *Eichfeldtheorie.* Springer, 2009.

[BCFW05] R. Britto, F. Cachazo, B. Feng, and E. Witten. Direct proof of tree-level recursion relation in Yang-Mills theory. *Phys.Rev.Lett.*, 94, 2005.

[BDKM04] A. V. Belitsky, S. É. Derkachov, G. P. Korchemsky, and A. N. Manashov. Superconformal operators in $N = 4$ super-Yang-Mills theory. *Phys. Rev. D*, 70(4):045021, Aug 2004.

[BH10] J. Baez and J. Huerta. Division Algebras and Supersymmetry I. In R. Doran, G. Friedman, and J. Rosenberg, editors, *Superstrings, Geometry, Topology, and C*-algebras*, pages 65–80. AMS, 2010.

[BHT08] A. Brandhuber, P. Heslop, and G. Travaglini. MHV amplitudes in N=4 super Yang-Mills and Wilson loops. *Nucl.Phys.*, B794:231–243, 2008.

[BKS12] A. Belitsky, G. Korchemsky, and E. Sokatchev. Are scattering amplitudes dual to super Wilson loops? *Nucl.Phys.*, B855:333–360, 2012.

[BMS10] M. Bullimore, L. Mason, and D. Skinner. MHV Diagrams in Momentum Twistor Space. *JHEP*, 1012, 2010.

[CH11] Simon Caron-Huot. Notes on the scattering amplitude / Wilson loop duality. *JHEP*, 07:058, 2011.

[DDH12] L. Dixon, J. Drummond, and J. Henn. Analytic result for the two-loop six-point NMHV amplitude in N=4 super Yang-Mills theory. *JHEP*, 1201, 2012.

[DF99a] P. Deligne and D. Freed. Classical field theory. In P. Deligne et al., editor, *Quantum Fields and Strings: A Course for Mathematicians.* American Mathematical Society, 1999.

[DF99b] P. Deligne and D. Freed. Supersolutions. In P. Deligne et al., editor, *Quantum Fields and Strings: A Course for Mathematicians.* American Mathematical Society, 1999.

[DHKS10] J. Drummond, J. Henn, G. Korchemsky, and E. Sokatchev. Dual superconformal symmetry of scattering amplitudes in super Yang–Mills theory. *Nuclear Physics B*, 828(1–2):317 – 374, 2010.

[Dix96] L. Dixon. Calculating scattering amplitudes efficiently. Lecture notes, TASI, 1996.

[DKS08] J. Drummond, G. Korchemsky, and E. Sokatchev. Conformal properties of four-gluon planar amplitudes and Wilson loops. *Nucl.Phys.*, B795:385–408, 2008.

[Dru10] J. Drummond. Hidden Simplicity of Gauge Theory Amplitudes. *Class.Quant.Grav.*, 27, 2010.

[Ebe89] D. Ebert. *Eichtheorien. Grundlagen der Elementarteilchenphysik.* Wiley, 1989.

[EPZ04] G. Esposito, D. Pelliccia, and F. Zaccaria. Gribov problem for gauge theories: A pedagogical introduction. *Int.J.Geom.Meth.Mod.Phys.*, 1:423–441, 2004.

[FO06] J. Figueroa-O'Farrill. Majorana spinors. Lecture notes, University of Edinburgh, 2006.

[Fri00] Thomas Friedrich. *Dirac Operators in Riemannian Geometry.* American Mathematical Society, 2000.

[Gal09] Anton Galaev. Holonomy of supermanifolds. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 79:47–78, 2009.

[GR09] W. Greiner and J. Reinhardt. *Quantum Electrodynamics.* Springer, 2009.

[Hél09] F. Hélein. An introduction to supermanifolds and supersymmetry. In P. Baird, F. Hélein, J. Kouneiher, F. Pedit, and V. Roubtsov, editors, *Systèmes intégrables et théorie des champs quantiques*, pages 103–157. Hermann, 2009.

[Hod09] A. Hodges. Eliminating spurious poles from gauge-theoretic amplitudes. Preprint, Oxford University, 2009.

[Kle11] R. Kleiss. Pictures, paths, particles and processes. Lecture notes, Radboud University, 2011.

[LFQ+10] X. Li, K. Feng, S. Qiu, H. Zhang, and A. Zhao. On analytic formulas of Feynman propagators in position space. *Chinese Phys. C*, 34(10):1576–1582, 2010.

[Mal98] J. Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv.Theor.Math.Phys.*, 2:231–252, 1998.

[MS04] D. McDuff and D. Salamon. *J-holomorphic Curves and Symplectic Topology.* American Mathematical Society, 2004.

[MS09] L. Mason and D. Skinner. Dual superconformal invariance, momentum twistors and grassmannians. *Journal of High Energy Physics*, 2009(11):045, 2009.

[MS10] L. Mason and D. Skinner. The Complete Planar S-matrix of N=4 SYM as a Wilson Loop in Twistor Space. *JHEP*, 1012:018, 2010.

[Nai88] V. Nair. A current algebra for some gauge theory amplitudes. *Physics Letters B*, 214(2):215 – 218, 1988.

[Nic91] H. Nicolai. Supersymmetrische Quantenmechanik. *Phys. Blätter*, 47:387–392, 1991.

[Pol05] M. Polyak. Feynman diagrams for pedestrians and mathematicians. *Proc. Symp. Pure Math.*, 73:15–42, 2005.

[PS95] M. Peskin and D. Schroeder. *An Introduction to Quantum Field Theory.* Perseus Books, 1995.

[Str04] N. Straumann. *Relativistische Quantentheorie.* Springer, 2004.

[Tod11] I. Todorov. Clifford algebras and spinors. *Bulg. J. Phys.*, 38:3–28, 2011.

[Ton07] D. Tong. Quantum field theory. Lecture notes, University of Cambridge, 2007.

[Var04] V. Varadarajan. *Supersymmetry for Mathematicians: An Introduction.* Providence, R.I., 2004.

[Wer02] D. Werner. *Funktionalanalysis.* Springer, 2002.

[WW90] R. Ward and R. Wells. *Twistor Geometry and Field Theory.* Cambridge Monographs, 1990.

Addendum

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