Integrability of the $AdS_5 \times S^5$ Superstring in Uniform Light-Cone Gauge

Diplomarbeit
zur Erlangung des akademischen Grades
des Diplom Physikers (Dipl. Phys.)

Humboldt-Universität zu Berlin
Mathematisch-Naturwissenschaftliche Fakultät I

Institut für Physik
Lehrstuhl für Quantenfeldtheorie und Stringtheorie

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Berlin, 31. Juli 2007

Hilfsmittel

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Acknowledgements

First of all, at this opportunity I would like to thank my mother Cornelia Hentschel for always having supported me throughout my life, unremittingly assisting me with her experience of life and helping me keeping up my studies. She has a great stake in the success of my work.

From the Quantum Field Theory and String Theory Group of the Humboldt-University of Berlin I wish to devote many thanks to my supervisor Prof. Jan Plefka for his excellent supervision, encouragement and patience as well as supporting my future life in many cases. I would also like to thank Prof. Ulrich Wolff for appraising this diploma thesis even though string theory is not his specific field of research. For many illuminating discussions, dedicating plenty of time for this work as well as for his friendship I thank Per Sundin. I enjoyed very much working in the QFT and String Theory Group and would like to thank all group members for the friendly, communicative and personal atmosphere, especially Silvia Richter for her administrative assistance and her dedication for social concerns and Dr. Hans-Jörg Otto for maintaining our computers.

I would like to express my gratitude and appreciation to Prof. Dietmar Ebert for being devotedly active in teaching and reminding students to keep in mind regarding the overall picture and developments in physics, life and society. Prof. Ebert has continuously supported me as a mentor for many years with personal and well-considered advice. In this respect I am also especially thankfull to Dr. Alejandro Saenz for dedicating plenty of time in supporting my future life with advice and assistance.

I would like to devote special thanks to my colleague Andreas Rodigast for many hours of discussions and help. Finally I would like to thank my colleagues Max Dohse, Jens Grieger, Volker Branding, Nicolai Beck, Hai Ngo Than, Ralf Sattler and Johannes Vetter for the friendly and personal atmosphere and the many discussions, some about physical and lots about other topics in life.

Warm thanks are also devoted to my best friend Felix Hermann who is sedulously assisting me with advice and his support in all possible situations of live. Last but certainly not least I would thank Anke Schneider for her moral support, understanding and for being always a good friend to me.

Inhaltsangabe

In der vorliegenden Diplomarbeit wird im nahen Plane-wave Limes ein detaillierter Test der Quantenintegrabilität des $AdS_5 \times S^5-$Superstrings in uniformer Lichtkegeleichung durchgeführt. Einleitend wird die perturbative Herleitung des Superstringhamiltonians zusammengefasst. Auf dieser Grundlage wird eine Methode zur systematischen Berechnung des Energiespektrums einer allgemeinen Stringkonfiguration entwickelt, die ich in der sogenannten ABAKUS -Software implementiert habe.
Der zweite Teil der Diplomarbeit behandelt den Betheansatz und die Ableitung der $\mathfrak{psu}(2,2|4)$ Bethegleichungen. Die Lösungen dieser Gleichungen liefern die Skalendimensionen eichinvarianter zusammengesetzter Operatoren der $\mathcal{N} = 4$ Super-Yang-Mills-Theorie, die gemäß der AdS/CFT-Korrespondenz dem Stringenergiespektrum entsprechen.
Die durch Diagonalisierung des Lichtkegelhamiltonians berechneten Energiespektren werden mit den Lösungen der Bethegleichungen verglichen, wobei die Untersuchung sowohl analytische als auch numerische Ergebnisse von Zuständen mit maximal sechs Anregungen umfasst. In allen untersuchten Fällen wurde exakte Übereinstimmung der Spektren gefunden, was die vermutete Eigenschaft der Quantenintegrabilität des $AdS_5 \times S^5-$Superstrings stark untermauert.

Abstract

In the present diploma thesis a detailed test of the quantum integrability of the $AdS_5 \times S^5$ superstring in uniform light cone-gauge is performed in the near plane-wave limit. Preliminary the perturbative derivation of the superstring Hamiltonian in $AdS_5 \times S^5$ is reviewed. Based thereon a method is developed to systematically compute the energy spectrum of generic string configurations, which I have implemented in a software system called ABAKUS.
In the second part the Bethe ansatz is introduced and the derivation of the $\mathfrak{psu}(2,2|4)$ Bethe equations is reviewed, yielding the scaling dimension of composite gauge invariant operators of $\mathcal{N} = 4$ super Yang-Mills theory, which is according to the AdS/CFT correspondence equal to the string energy spectrum.
The energy spectra obtained by diagonalization of the light-cone Hamiltonian are thereupon confronted with the solutions of the Bethe equations. The analysis is performed both analytically and numerically up to the level of six impurity states, where perfect agreement is found lending strong support to the quantum integrability of the $AdS_5 \times S^5$ superstring.

Contents

1 Introduction **5**

2 AdS/CFT correspondence and integrability **7**
2.1 Integrability in Gauge Theory and String Theory 9

3 The Superstring on $AdS_5 \times S^5$ **11**
3.1 The $AdS_5 \times S^5$ space 11
3.2 The bosonic Superstring on $AdS_5 \times S^5$ 11
3.2.1 The uniform light-cone gauge 12
3.2.2 Near plane-wave expansion 13
3.2.3 The bosonic $AdS_5 \times S^5$ string Hamiltonian 14
3.2.4 Quantization 15
3.3 The full superstring Hamiltonian on $AdS_5 \times S^5$ 15
3.3.1 Hamiltonian in uniform light-cone gauge 16
3.3.2 $U(1)$ Field Charges 18
3.4 notation of generic string states 19
3.5 Eigenvalues of the Hamiltonian 20
3.5.1 Eigenvalues of $\mathcal{H}_4$ 21
3.6 The $\mathfrak{su}(2)$ sector 21
3.6.1 Solving for the space-time Energy 22

4 Computer-algebraic calculation: the ABAKUS-system **23**
4.1 Physical Fundamentals of the Software 23
4.2 Software Requirements Specification 24
4.3 Algorithmic complexity of the problems 26
4.3.1 Complexity of state generation 26
4.3.2 Complexity of computing the effective Hamiltonian $\mathcal{H}_{\text{eff}}$ 27
4.3.3 Complexity of computing the Hamiltonian matrix representation 27
4.4 Software layout of the ABAKUS-System 28
4.4.1 State generation 31
4.4.2 computing the effective Hamiltonian operator 36
4.4.3 input form of the Hamiltonian 36
4.4.4 algorithmic calculation of $\mathcal{H}_{\text{eff}}$ 36
4.4.5 Hamiltonian matrix representation and its eigenvalues 38
4.4.6 Installation 40

5 String computations with ABAKUS **41**
5.1 The $\mathfrak{su}(2)$ sector 41
5.2 The $\mathfrak{sl}(2)$ sector 41
5.3 The $\mathfrak{su}(1|1)$ sector 41
5.4 The $\mathfrak{su}(1|2)$ sector 42
5.4.1 Two impurities 42
5.4.2 Three impurities with distinct modes 42
5.4.3 Three impurities with confluent modes 43
5.5 The $\mathfrak{su}(1,1|2)$ sector 43
5.5.1 Two impurities 43
5.5.2 Three impurities with confluent modes 43
5.6 The $\mathfrak{su}(2|3)$ sector 44

5.6.1 Two impurities . . . 45
5.7 Numerical results . . . 45

6 String energy spectrum **47**

7 Spin chains and the Bethe ansatz **48**
7.1 Review of the Asymptotic Bethe ansatz . . . 49
7.1.1 Higher loop asymptotic Bethe equations for gauge theory . . . 51
7.1.2 Dressing factor and string Bethe equations . . . 53
7.2 The nested Bethe ansatz . . . 54
7.2.1 Generalization to higher loops . . . 56
7.2.2 Diagonalization of matrix Bethe equations . . . 57
7.3 All loop $\mathfrak{psu}(2,2|4)$ Bethe equations . . . 59
7.4 The light-cone Bethe equations for string theory . . . 60
7.5 Dressing factor S_0 . . . 62
7.5.1 general structure . . . 62
7.5.2 Cancelation of the dressing factor . . . 64
7.6 solutions of the string Bethe equations . . . 65
7.7 Non-confluent mode numbers . . . 65
7.8 Confluent mode numbers . . . 66
7.9 Bethe equations for the smaller spin chains . . . 67

8 Comparing the Bethe equations with string theory **69**
8.1 The rank one sectors $\mathfrak{su}(2)$, $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ . . . 69
8.2 The $\mathfrak{su}(1|2)$ sector . . . 69
8.2.1 Two impurities . . . 70
8.2.2 Three impurities, distinct mode numbers . . . 70
8.2.3 Three impurities, confluent mode numbers . . . 71
8.3 The $\mathfrak{su}(1,1|2)$ sector . . . 71
8.3.1 Two impurities . . . 72
8.3.2 Three impurities, distinct mode numbers . . . 72
8.3.3 Three impurities, confluent mode numbers . . . 73
8.3.4 Higher impurities . . . 74
8.4 The $\mathfrak{su}(2|3)$ sector . . . 75
8.4.1 Two impurities . . . 75
8.4.2 Higher impurities . . . 76

9 Conclusions and outlook **77**
9.1 Comparison of Hamiltonian method and Bethe Ansatz . . . 77
9.2 Summary . . . 78
9.3 Outlook . . . 78

Appendix **79**

A Further details on the $AdS_5 \times S^5$ string Hamiltonian **79**
A.1 Solving the Virasoro constraint for the Hamiltonian 79
A.2 $\Gamma-$matrices 80
A.3 properties of f_n and g_n 81

B Systematic listing of numerical eigenvalues **82**
B.1 Two impurities 82
B.2 Three impurities 82
B.3 Four impurities 84
B.4 Comments on higher impurities 84

C CD containing the ABAKUS Software **85**

1 Introduction

In nature one observes four fundamental forces, which are strong, weak and electromagnetic interaction as well as gravity. At energy scales accessible nowadays the first three interactions are preeminently described by quantum field theories and are combined to a uniform theory by the Standard Model of particle physics. All quantum field theories of the Standard Model are gauge field theories, wherein spin-1 particles are responsible for transmitting the interaction. Gauge theories contain more degrees of freedom than the original physical system. The gauge transformations relate physically equivalent field configurations and form a group. In contrast to the gauge group $U(1)$ of quantum electrodynamics, the gauge group $SU(3)$ of quantum chromodynamics (QCD) and $SU(2)$ of weak interaction are non-Abelian. This property reflects the fact that the gauge particles are self-interacting. At energy scales accessible nowadays the electroweak coupling is small so perturbation theory is applicable. In QCD the situation is quite different: at low energy the coupling constant for the interaction is large, which leads to confinement, while it is small for high energies, resulting in asymptotic freedom of quarks. In the latter perturbation theoretical methods are not applicable, since a power series in the coupling constant does not converge. So far quantum field theories work well in the regime of small couplings, while the strong coupling behavior is understood less well, accessible today only via numerical computations on a discretized spacetime lattice.

For the remaining force of gravity there is currently only a classical theory available, which is the theory of General Relativity. It works well at large length scales corresponding to low energies. Yet a microscopical description of spacetime at lengths near the Planck scale or energies near the Planck energy requires a quantum theory of gravity. The attempt to quantize gravity according to the known procedures leads to a non-renormalizable field theory. Despite non-renormalizability it is nevertheless useful as an effective quantum theory [1] in the low energy limit, including the massless spin-2 graviton as exchange particle of gravitation. But a fully consistent theory of quantum gravity has still not been constructed.
Also a unified quantum theory of gravitation and the Standard Model is needed to describe physics in highly curved backgrounds, like near the horizons of black holes. In such an environment one needs a generalization of the Standard Model including a microscopic theory of gravity. One of the most promising candidates for such a theory is string theory. While in the Standard Model elementary particles are considered to be pointlike and to interact locally, string theory drops this notion and assumes that the fundamental objects are one-dimensional strings.

Even though the motivation for string theory given nowadays is quite different, it was originally developed in an attempt to describe the large number of mesons and hadrons that were experimentally discovered. One surprising issue of the hadronic spectrum is, the hadrons can be sorted into groups in such way that, within every group, mass m and Spin J obey a relation like $m^2 \sim \alpha_0 + TJ$, where only the intercept α_0 differs for each group. This property is well explained by assuming the particles to be different oscillation modes of a rotating, relativistic string with tension T. Unfortunately string theory in this context leads to some properties which drastically disagree with experimental findings. Due to the use of extended objects, string theory predicts an exponential falloff of scattering amplitudes but only powerlike behaviors (possibly deformed by structure functions) have been observed. Later it was discovered that hadrons and mesons are actually built of quarks and the appropriate theoretical description is a non-Abelian $SU(3)$ gauge field theory.
But replacing the picture of pointlike particles by using one-dimensional extended objects of a very small size is actually a quite natural generalization, because by viewing the system on much larger scales, the strings reduce to an almost pointlike structure and therefore string

theory is expected to reproduce many features of conventional gauge theories on larger scales. To the best of our knowledge the only consistent interaction for massless spin-2 particles is that of gravity and since all string theories include such a particle, which is identified with the desired graviton, string theories could represent a unified description of quantized gravity as well as quantum field theories.
One fascinating aspect of string theory is that quantum consistency demands that the theory occupies ten spacetime dimensions[1]. However, we observe only four spacetime dimensions, so theorists are charged with the task of understanding the role of the six extra spatial dimensions, but since it is not known how spacetime looks like at short distances comparable to Planck length, the extra dimensions could simply be highly curved and thus so tiny that it is impossible to detect them at energy scales accessible today.

The course of studying gauge theories and string theories has led to the discovery of a dramatically new class of fundamental symmetries known as dualities. These symmetries stand apart from traditional ones in the sense that dualities connect physical theories which, at least superficially, appear to be entirely distinct in their formulation. In particular two seemingly different theories are considered to be dual, if both models describe equivalent physical systems.
A well known example is T-duality: type IIA string theory with one spatial dimension compactified on a circle of radius R can be translated to type IIB string theory with compactification radius R^{-1}. The usefulness of duality derives in part from the fact that dual descriptions are typically complementary, insofar as information that is inaccessible in one physical theory may often be extracted from a straightforward calculation in the theory's dual description.

In this work we will primarily be concerned with an other famous duality, which is the so called AdS/CFT correspondence. One specific property of the AdS/CFT correspondence is that it claims a strongly coupled super Yang-Mills theory to be dual to a weakly coupled string theory. Provided this duality holds, string theory allows us to access the non-pertubative regime of strongly interacting non-Abelian gauge theories without being restricted to numerical computations on a discrete space-time lattice. It is therefore promising to study string theory, irrespective of whether it will succeed to provide a unified quantum theory of all fundamental forces.
Another very important question to address is, how string theory behaves in a highly curved background, where the extension of single strings is of magnitude of the curvature radius of spacetime. In respect thereof very little is known, since perturbative string theory is not applicable anymore. But using the AdS/CFT correspondence the other way around it gives rise to this regime by working in the dual weakly coupled super Yang-Mills theory.

Altogether the AdS/CFT correspondence could provides us with a powerful tool to explore previously almost unaccessible regimes of different theories. Even though AdS/CFT correspondence has passed several nontrivial test it has not been proven yet.

[1] M-theory is provided with an extra 11^{th} dimension. By different compactifications of this extra dimension M-theory can be reduced to every type of 10-dimensional string theory.

2 AdS/CFT correspondence and integrability

The duality of compactifications of M/sting theory on various Anti-deSitter (AdS) spacetimes and various conformal field theories was conjectured by Maldacena 1998 [2], known as AdS/CFT correspondence. Maldacena's conjecture is based on an idea by 't Hooft [3]: starting with an $SU(N)$ Yang-Mills theory with coupling g_{YM} and N colours one can classify Feynmann graphs according to the their genus H, i.e. the minimum number of handles that must be added to a plane to embed the graph without any crossings of lines. The crucial fact observed by 't Hooft is, that each Feynmann diagram is associated with a factor $r = \lambda^l N^{2-2H-L}$, depending on its number of loops l and genus H. The quantity L enumerates the number of fermionic loops[2] and the 't Hooft coupling is defined as $\lambda = g_{YM}^2 N$. In the 't Hooft limit $N \to \infty$, $g_{YM} \to 0$ with λ fixed, the free energy F of $SU(N)$ gauge theory takes the pictorial form:

$$F = N^2 \quad + 1 \quad + \frac{1}{N^2} \quad + \ \ldots = N^2 \sum_{H=0}^{\infty} \frac{1}{N^{2H}} \sum_{l=0}^{\infty} c_{g,l}\ \lambda^l \tag{2.1}$$

Obviously this genus expansion resembles the pertubative expansion of a string theory in the string couplin constant $g_s \propto N^{-1}$. For large N the string theory becomes free and thus only planar diagramms contribute in the corresponding gauge theory.

The presented argument suggests that different kinds of gauge theories will correspond to differents sting theories but according to experience it is extremely difficult to prove such equivalences. In its purest form, the conjectured AdS/CFT correspondece identifies the type IIB supersting in a ten dimensional anti-de-Sitter cross sphere ($AdS_5 \times S^5$) background with the maximally supersymmetric Yang-Mills theory[3] with gauge group $SU(N)$ and $\mathcal{N} = 4$ spinor supercharges in four dimensions ($\mathcal{N} = 4$ SYM). The gauge theory's Langrangian is completely determined by supersymmetry which has a global $SU(4)_R$ R-symmetry that rotates the six scalar fields and four fermions. Furthermore it is invariant with respect to the conformal group $SO(4,2)$ in four dimensions, including the usual Poincaré transformations as well as scale transformations and special conformal transformations.
These symmetries have to be reflected by the dual string theory description. In fact the five dimensional Anti-de-Sitter space is the only space with local $SO(4,2)$ isometry. It is the maximally symmetric solution of Einstein's euations with negative cosmological constant. At the border of the AdS_5 the remaining five dimensions of the 10 dimensional target space, the type IIB superstring is moving in, are compactified on a five sphere S_5. Thus the $SU(4)$ symmetry of the SYM theory is reproduced by the locally isomorphic $SO(6)$ symmetry of S_5 on the string side.

To establish a connection between the two theories, one has to relate the free parameters of the different models to each other. $\mathcal{N} = 4$ SYM theory is controlled by the rank N of the gauge group and the coupling constant g_{YM} or eqivalently $\lambda = g_{YM}^2 N$ while string theory is paremetrized by the effective string tension R^2/α' and the string coupling g_s, where R is the common radius of the AdS_5 and S^5 geometries and $1/\alpha'$ denotes the string tension. According to the AdS/CFT proposal, these two sets of parameters are identified as

$$g_s = \frac{4\pi\lambda}{N}\ , \qquad\qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}\ . \tag{2.2}$$

[2] In an $SU(N)$ gauge theory there are fermionic particles and gauge bosons transmitting the interaction. To use a uniform notation for differen values of N, we denote the fermions as "quarks" and the $SU(N)$ charge as "colour" (in this convention we also refer to the weak isospin as $SU(2)$ colour). In this notation l is associated with the number of closed colour loops, while L counts only closed quark loops.

[3] Due to it's vanishing β-function, $\mathcal{N} = 4$ SYM is a conformal field theory (CFT).

Low-energy gravitational description in terms of classical type IIB supergravity is perturbatively valid in weakly curved geometries (compared to string units), i.e. for $\sqrt{\lambda} \gg 1$, while perturbative field theory is applicable only in the domain of $g_{YM} \ll 1$, viz small λ. Hence one is facing a strong/weak coupling duality, as stated in chapter 1.

By the physical equivalence of both theories it is required, that for each field $\widehat{O}(x)$ on the boundary Minkowski theory there is a field $\phi(x,z)$ in the bulk string theory with the property that at least the corresponding correlators of both theories agree on the boundary. It is conjectured that the partition function of string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ coincides with the generating function of $\mathcal{N} = 4$ super Yang-Mills on the boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$ [4]:

$$\langle e^{\int d^4x \ \phi_0(\vec{x})\widehat{O}(\vec{x})} \rangle_{CFT} = \mathcal{Z}_{String}\left[\phi_0(\vec{x},z)\big|_{z=0}\right] \tag{2.3}$$

where the left hand side is the generating function of correlation functions in the field theory, i.e. ϕ_0 is an arbitrary function and we can calculate correlation functions of $\widehat{O}$ by taking functional derivatives with respect to ϕ_0 and than setting $\phi_0 = 0$. The right hand side denotes the full partition function of string theory with the boundary condition that the field $\phi(x,z)$ has the value $\phi(x,z)\big|_{z=0} = \phi_0(x)$ on the boundary of AdS. Thus each field propagating on AdS space is in a one-to-one correspondence with an operator of SYM. For the $D = 10$ supergravity multiplet the explicit mapping is given in [5].

Based on (2.3), the energy eigenvalue E of a string state $|\phi_A\rangle$ has been identified with the scaling dimension Δ of the dual gauge theory operator $\widehat{O}_A(x)$ [4], which in turn is determined from the two point function of the conformal field theory by

$$\langle \widehat{O}_A(x)\widehat{O}_B(y)\rangle = \frac{\delta_{\Delta_{\widehat{O}_A},\Delta_{\widehat{O}_B}}}{(x-y)^{2\Delta_{\widehat{O}_A}}} \ . \tag{2.4}$$

The $\widehat{O}$ denote composite gauge invariant operators of the form $\widehat{O}(x) = \mathrm{Tr}\,(\phi_{i_1}(x)\ldots\phi_{i_n}(x))$, where the ϕ_j are elementary fields of $\mathcal{N} = 4$ SYM and their covariant derivatives in the adjoint representation of $SU(N)$.
This remarkable result

$$\Delta(\lambda, \frac{1}{N}) \equiv E(\frac{R^2}{\alpha'}, g_s) \tag{2.5}$$

allows for determination of the all loop scaling dimension in planar gauge theory by calculating the Energy in free string theory in the large N limit. Nevertheless string quantization and determination of its quantum spectrum in curved backgrounds, even in the highly symmetric $\mathrm{AdS}_5 \times \mathrm{S}^5$ space, remains unknown. 2002 Berenstein, Maldacena and Nastase considered a novel limit of a degenerated pointlike string configuration, corresponding to a particle rotating with large angular momentum J on a great circle of the S^5 space [6]. In this so called 'BMN limit' of $J \to \infty$ with λ/J^2 fix, from the fastly moving particles point of view, the geometry of $\mathrm{AdS}_5 \times \mathrm{S}^5$ limits to a gravitational plane wave background

In 'uniform light-cone gauge' an *exact* world-sheet Hamiltonian has been established in [7] and pertubatively quantized in the near-plane wave limit using $J \to \infty$. We are going to review this derivation in section 3. Thus the spectrum of total arbitrary string states can in principle be computed in order $1/J$ and this in turn leads to various predictions for the anomal scaling dimensions of corresponding gauge theory operators.

2.1 Integrability in Gauge Theory and String Theory

In testing the conjectured AdS/CFT correspondece very important progress has been made during recent years building on the concept on integrability [8]. In classical mechanics there is a well-known definition of integrability due to Liouville: a finite-dimensional system is called integrable if it possesses a set of independently conserved charges $\mathcal{Q}_i$ commuting with respect to the Poisson bracket

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$$

and the total number of conserved charges including the Hamiltonian is half of the dimension of the phase space. For quantum theories there is no such strict definition of integrability known, however, it is expected that a quantum system is integrable if the number of conserved charges equals the number of degrees of freedom in the system.
From the most pragmatical point of view one might call a system quantum integrable [9] if provides the opportunity to "exactly" determine the quantities of physical interest. In the given context "exact" means that one can state a fixed set of equations which determine these quantities exactly, however, about the solvability of these equations one does not care at this point.

It has emerged that in planar gauge theory the dilaton operator, whose spectrum yields the desired scaling dimension of composite, gauge invariant operators, is isomorphic to the Hamiltonian of an integrable quantum spin chain [10],[11]. The property of integrability guarantees the existence of a Bethe ansatz which in principle allows for reformulating the quantum spectral problem into the solution of a set of non-linear algebraic equations, the Bethe equations. In other words, the Bethe equations diagonalize the planar gauge theory dilatation operator in the sense that its solutions, the Bethe roots, are eigenvalues of the dilatation operator.

With the AdS/CFT conjecture in mind, immediately a question arises: is the type IIB superstring, propagating in $AdS_5 \times S^5$, a quantum integrable model and is its energy spectrum indeed described by this set of Bethe equations?
Addressing these questions is important, since it will lead to a highly nontrivial test of the AdS/CFT duality conjecture. Moreover the Bethe equations are *all*-loop equations[4] and therefore yield all loop predictions of the quantum string spectrum or the dual scaling dimension of composite, gauge invariant operators, if we manage to solve them non-perturbatively.

In order to investigate the integrability of the $AdS_5 \times S^5$ superstring the perturbative derivation the Hamiltonian is reviewed in section 3. Based thereon in chapter 4 a computer algebraic method is described, which makes it possible to systematically compute the energy spectrum of generic string configurations. In section 5 the superstring spectra are derived analytically and numerically in all closed subsectors for up to six string excitations. Since the light-cone energy is only determined implicitly by the derived string spectra of section 5 the explicit solution for the energy is presented in chapter 6.
The Bethe ansatz leading to $\mathfrak{psu}(2,2|4)$ Bethe equations is reviewed in chapter 7 followed by a detailed discussion in section 8 how to solve these equations in the various sectors. The results are compared to the string results obtaied in section 4.

[4] For a spin chain of length L, by construction [12] the Bethe equations are exact only up to order $\ell < L$ with respect to the expansion in $g \sim \sqrt{\lambda} \ll 1$ for gauge theory and $1/J \ll 1$ in string theory. Consequently the Bethe roots yield all-loop predictions only in the case of an infinite long chain.

3 The Superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$

This section will start with some general remarks on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space. The derivation of the string Hamiltonian is exemplified for the bosonic case in paragraph 3.2 followed by a discussion of the full Hamiltonian including fermions. Subsequently a notation for generic string states is introduced and some general remarks on the Hamiltonian eigenvalues are presented. Section 3 is concluded with the derivation of the spectrum for a generic $\mathfrak{su}(2)$ string states.

3.1 The $\mathrm{AdS}_5 \times \mathrm{S}^5$ space

Embedding the five-dimensional anti-de-Sitter space and the five sphere in $\mathbb{R}^6$ the S^5 is described by $s_1^2 + s_2^2 + \ldots + s_6^2 = R$ and AdS_5 by $-a_{-1}^2 - a_0^2 + a_1^2 + a_2^2 + \ldots + a_4^2 = -R$. R denotes the common radius of AdS_5 and S^5. By parametrization through

$$\begin{aligned} &s_1 + i\, s_2 = \sin\xi\, \cos\psi\, e^{i\,\phi_1}\ , \qquad && s_3 + i\, s_4 = \sin\xi\, \sin\psi\, e^{i\,\phi_2}\ , \qquad && s_5 + i\, s_6 = \cos\xi\ e^{i\,\phi}\ , \\ &a_1 + i\, a_2 = \sinh\rho\, \cos\bar{\psi}\, e^{i\,\varphi_1}\ , \qquad && a_3 + i\, a_4 = \sinh\rho\, \sin\bar{\psi}\, e^{i\,\varphi_2}\ , \qquad && a_{-1} + i\, a_0 = \cosh\rho\ e^{i\,t} \end{aligned}$$

one obtains the metric

$$\begin{aligned} ds^2_{AdS_5} &= d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, (\, d\bar{\psi}^2 + \cos^2\bar{\psi}\, d{\varphi_1}^2 + \sin^2\bar{\psi}\, d{\varphi_2}^2\,) \\ ds^2_{S^5} &= d\xi^2 + \cos^2\xi\, d\phi^2 + \sin^2\xi\, (\, d\psi^2 + \cos^2\psi\, d{\phi_1}^2 + \sin^2\psi\, d{\phi_2}^2\,) \quad . \end{aligned} \tag{3.1}$$

Performing a suitable reparametrization $\{t, \rho, \bar{\psi}, \varphi_1, \varphi_2\} \to \{t, z_1, \ldots, z_4\},\ \{\phi, \xi, \psi, \phi_1, \phi_2\} \to \{\phi, y_1, \ldots, y_4\}$ of the form

$$\cosh\rho = \frac{(1 + z^2/4)}{(1 - z^2/4)}, \quad \cos\xi = \frac{(1 - y^2/4)}{(1 + y^2/4)} \qquad \text{with } z^2 = z_k z_k,\ y^2 = y_k y_k,\ k = 1, \ldots, 4$$

one can cast the metric for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ product space into

$$ds^2_{\mathrm{AdS}_5 \times \mathrm{S}^5} = \underbrace{-\left(\frac{1 + z^2/4}{1 - z^2/4}\right)^2 dt^2 + \frac{dz_k dz_k}{(1 - z^2/4)^2}}_{\text{metric of } \mathrm{AdS}_5} + \underbrace{\left(\frac{1 - y^2/4}{1 + y^2/4}\right)^2 d\phi^2 + \frac{dy_k dy_k}{(1 + y^2/4)^2}}_{\text{metric of } \mathrm{S}^5} \quad . \tag{3.2}$$

By construction, the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space has the symmetry $SO(2,4) \times SO(6)$, but only translation invariance in t and ϕ and the $SO(4) \times SO(4)$ symmetry of the coordiantes $z_k,\ y_k$ remain manifest in this form.

- Introduction to $\mathrm{AdS}_5 \times \mathrm{S}^5$: picture, time winding, Metric

For the sake of simplicity the following notations are used in the next chapter

$t =: z_0$	time on AdS_5	$\{t, z_a\} \equiv z_\mu,\ a = 1, ..., 4,\ \mu = 0, ..., 4$	coordinates on AdS_5
$\phi =: y_0$	angle on S^5	$\{t, y_s\} \equiv y_\nu,\ s = 1, ..., 4,\ \nu = 0, ..., 4$	coordinates on S^5

$x^\mu \equiv \{t, z_1, \ldots, z_4, \phi, y_1, \ldots, z_4\}$ coordinates on $\mathrm{AdS}_5 \times \mathrm{S}^5$

$x^M \equiv \{z^a, y^s\},\quad a, s = 1, ..., 4$ remaining coordinates on $\mathrm{AdS}_5 \times \mathrm{S}^5$ excluding t, ϕ

3.2 The bosonic Superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$

In this chapter the bosonic part of the superstring Hamiltonian will be derived in order to demonstrate the basic procedure to quantize the $\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring perturbatively. We start with the well known Polyakov action where the fermionic contribution is omitted

$$I = \int d\tau d\sigma\ \mathcal{L} \qquad \text{with} \qquad \mathcal{L} = -\frac{\sqrt{\lambda}}{4\pi}\, \gamma^{\alpha\beta} \Big(G^{(AdS_5)}_{\mu\nu}\, \partial_\alpha z^\mu\ \partial_\beta z^\nu + G^{(S^5)}_{\mu\nu}\, \partial_\alpha y^\mu\ \partial_\beta y^\nu \Big)\ . \tag{3.3}$$

Here we use the normalized string world-sheet metric $\gamma^{\alpha\beta}$ with $\det\gamma = -1$ ($\alpha, \beta \in \{\tau, \sigma\}$) and $G^{(AdS_5)}_{\mu\nu}$, $G^{(S^5)}_{\mu\nu}$ denote the target space metrics of AdS_5 and S^5 according to (3.2). $\frac{\sqrt{\lambda}}{2\pi}$ is the effective string tension and the coordinates σ and τ parametrize the string world-sheet.
A closer look at equation (3.1) and (3.3) reveals that the cyclic coordinates of the action I are $(t, \varphi_1, \varphi_2; \phi_1, \phi_2, \phi_3)$ leading to the conserved charges

$$(E, S_1, S_2; J, J_1, J_2) \ , \tag{3.4}$$

where E is the space-time energy, (S_1, S_2) are corresponding to two spins on AdS_5 and (J, J_1, J_2) to three angular momenta on the five sphere respectively.

Using the canonical conjugated momenta p_μ

$$p_\mu = \frac{\delta\mathcal{L}}{\delta\dot{x}^\mu} = -\sqrt{\lambda}\,\gamma^{\tau\tau}\dot{x}_\mu - \sqrt{\lambda}\,\gamma^{\tau\sigma}x'_\mu \qquad \text{with} \quad \dot{x}^\mu \equiv \partial_\tau x^\mu,\ x'^\mu \equiv \partial_\sigma x^\mu \tag{3.5}$$

one can cast the Lagrangian into the form

$$\mathcal{L} = p_\mu\,\dot{x}^\mu + \frac{1}{\sqrt{2}}\frac{1}{\gamma^{\tau\tau}}\Big[p_\mu\,p^\mu + \lambda x'_\mu\,x'^\mu\Big] + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\Big[p_\mu\,x'^\mu\Big] \quad . \tag{3.6}$$

This is easily checked by plugging (3.5) into (3.6) and using the property $-1 = \det\gamma$ of the world sheet metric. The last two terms in (3.6) yield the Virasoro constraints, which arise as equations of motion for the world sheet metric:

$$0 = p_\mu\,x'^\mu \quad , \qquad\qquad 0 = p_\mu\,p^\mu + \lambda x'_\mu\,x'^\mu \ . \tag{3.7}$$

3.2.1 The uniform light-cone gauge

To impose the uniform light-cone gauge we make use of the AdS time t and the angle ϕ on S^5. They parametrize two $U(1)$ isometries of the $AdS_5 \times S^5$ space and the corresponding conserved charges, the space-time energy E and the angular momentum J, are related to the momenta conjugated to t and ϕ by

$$E = -\int_0^{2\pi}\frac{d\sigma}{2\pi}\,p_t \qquad \text{and} \qquad J = \int_0^{2\pi}\frac{d\sigma}{2\pi}\,p_\phi \quad . \tag{3.8}$$

We introduce light cone coordinates $x_\pm$ and the corresponding canonical momenta[5] $p_\mp$

$$x^\pm = \phi \pm t \ , \qquad p_+ = p_\phi - p_t \ , \quad p_- = p_\phi + p_t \ . \tag{3.9}$$

For light-cone coordinates $x^\pm$ the metric G takes the form

$$\begin{aligned} G_{++} = G_{--} &= \frac{1}{4}\left(G_{\phi\phi} + G_{tt}\right) = \frac{1}{4}\left(\tfrac{1-y^2/4}{1+y^2/4}\right)^2 - \frac{1}{4}\left(\tfrac{1+z^2/4}{1-z^2/4}\right)^2 \ , \\ G_{-+} = G_{+-} &= \frac{1}{4}\left(G_{\phi\phi} - G_{tt}\right) = \frac{1}{4}\left(\tfrac{1-y^2/4}{1+y^2/4}\right)^2 + \frac{1}{4}\left(\tfrac{1+z^2/4}{1-z^2/4}\right)^2 \ . \end{aligned} \tag{3.10}$$

and one defines $x_\pm := G_{\pm n}\,x^n$, $p^\pm := G^{\pm n}\,p_n$, $n = +, -$.
The uniform light-cone gauge is imposed by setting

$$x^+ = \tau + \frac{m}{2}\sigma \ , \qquad\qquad p_+ = P_+ = J + E = const, \tag{3.11}$$

[5] Please note that the canonical conjugated momentum corresponding to x_+ is $p_\phi + p_t = p_-$.

The string winding number m appears because ϕ is an angle variable. But in what follows we will use the decompactifying plane-wave limit with $P_+ \to \infty$ and, therefore, we set $m = 0$.

The advantage of this particular gauge choice is that combined with an appropriate κ-symmetry gauge the Poisson structure of fermions simplifies drastically, which is of great advantage for calculating the global symmetry charges and quantization of the theory.

Omitting the Virasoro constraints (3.7) in the Lagrangian (3.6) it acquires in uniform light-cone gauge the form

$$\mathcal{L} = p_M \dot{x}^M + p_+ \dot{x}^- + p_- \ .$$

The second term is a total derivative and thus may be dropped. The upshot is a gauge fixed Lagrangian $\mathcal{L}_{\mathrm{gf}}$ which can be written in the standard form as the difference of a kinetic term $\mathcal{L}_{kin} = p_M \dot{x}^M$ and the Hamiltonian density $\mathcal{H}$

$$\mathcal{L}_{\mathrm{gf}} = \mathcal{L}_{kin} - \mathcal{H} \qquad \text{with} \quad \mathcal{L}_{kin} = p_M \dot{x}^M \ , \quad \mathcal{H} = -p_- \ . \tag{3.12}$$

In light-cone gauge the first Virasoro constraint (3.7) takes the form

$$0 = p_M x'^M + p_+ \partial_\sigma x^- \quad , \tag{3.13}$$

which yields the level matching condition by integration over the closed string

$$0 = \int_0^{2\pi} d\sigma \left(p_M x'^M \right) \qquad\qquad M = 1, \ldots, 8 \quad . \tag{3.14}$$

The second Virasoro constraint determines the Hamiltonian $\mathcal{H} = -p_-$ as a solution of

$$0 = p_M p^M + p_+ p^+ + p_- p^- + \lambda x'_M x'^M + \lambda x'_- x'^- \ , \qquad M = 1, \ldots, 8 \tag{3.15}$$

Up to this point the gauge fixed Lagrangian $\mathcal{L}_{\mathrm{gf}}$ is an exact function of the light-cone momentum P_+ and the string tension $\sqrt{\lambda}$.

3.2.2 Near plane-wave expansion

In order to solve equation (3.15) one needs to consider a simplifying limit.

BNM-limit:
A key idea of Berenstein, Maldacena and Nastase for perturbative quantization of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring was to consider a string circling on S^5 with an infinite large angular momentum J [6]. Reducing the string to a point particle, the energy is classically given by $E = J$. In the so called BMN limit with $J \to \infty$ and $\lambda' := \lambda / J^2$ held fix, all higher string corrections $\mathcal{O}(1/\sqrt{\lambda})$ to the Energy $E = J + E_2(\lambda') + \mathcal{O}(1/\sqrt{\lambda})$ are suppressed, so the approximation of the finite energy contribution $E - J = E_2(\lambda')$ becomes exact. From the perspective of the fast moving string, the space transforms to a plane wave geometry in the BMN limit.

Plane-wave limit:
In the case of the uniform light cone gauge the BMN equivalent choice is

$$P_+ \to \infty \qquad\qquad \text{with} \quad \widetilde{\lambda} := \frac{4\lambda}{P_+^2} \text{ fix} \ . \tag{3.16}$$

Denoting $P_\pm = J \pm E$, we have the identity $E = J - P_-$ and as we will see P_- represents the *finite* correction to the space-time energy E. The BMN effective coupling $\lambda' := \lambda / J^2$ is not equal to the coupling constant $\widetilde{\lambda}$ but reduces to it in the strict $J \to \infty$ limit, since

$$\widetilde{\lambda} = \frac{4\lambda}{P_+^2} = \lambda' \frac{1}{\left(1 - \frac{P_-}{2J}\right)^2} \ . \tag{3.17}$$

3.2.3 The bosonic $AdS_5 \times S^5$ string Hamiltonian

In the near plane-wave limit it is now possible to perturbatively solve equation (3.15) for the Hamiltonian $\mathcal{H} = -p_-$. In order to acquire a canonical Poisson structure and a standard Hamiltonian of the form $\frac{1}{2}(p_M\, p^M + x_M\, x^M)$ we perform a rescaling of the fields and momenta

$$x_M \to \sqrt{\tfrac{2}{P_+}}\, x_M \ , \quad p_M \to \sqrt{\tfrac{P_+}{2}}\, p_M \ . \tag{3.18}$$

Furthermore it is convenient to perform a canonical transformation which simplifies the Hamiltonian again. For details consult the appendix A.1.
One finally obtains the Hamiltonian density in terms of the remaining four bosonic coordinates z_a $(a = 1, \dots, 4)$ of AdS^5, its canonical conjugated Momenta $p_a^{(z)}$ and the four coordinates y_s $(s = 1, \dots, 4)$ with momenta $p_s^{(y)}$, respectively

$$\begin{aligned} \mathcal{H} = \quad & \frac{1}{2}\left(p_a^{(z)} p_a^{(z)} + p_s^{(y)} p_s^{(y)} + z_a\, z_a + y_s\, y_s + \widetilde{\lambda}(z'_a\, z'_a + y'_s\, y'_s)\right) \\ & + \frac{\widetilde{\lambda}}{P_+}\left(y'_s\, y'_s\ z_a\, z_a - z'_a\, z'_a\ y_s\, y_s + z'_a\, z'_a\ z_b\, z_b - y'_s\, y'_s\ y_u\, y_u\right) \quad . \end{aligned} \tag{3.19}$$

Please note that due to the expansion process, the $AdS_5 \times S^5$ metric is not present anymore in equation (3.19), but the indices are contraced using the Kronecker delta. In order to obtain well defined charges $\{S_1, S_2, J_1, J_2\}$ of the bosonic fields it is convenient to express the Hamiltonian density in terms of complex bosonic fields

$$\begin{aligned} &Z_1 = z_2 + i\, z_1\,, \quad Z_2 = z_4 + i\, z_3\,, \quad Z_3 = Z_2^\dagger = z_4 - i\, z_3\,, \quad Z_4 = Z_1^\dagger = z_2 - i\, z_1\,, \\ &Y_1 = y_2 + i\, y_1\,, \quad Y_2 = y_4 + i\, y_3\,, \quad Y_3 = Y_2^\dagger = y_4 - i\, y_3\,, \quad Y_4 = Y_1^\dagger = y_2 - i\, y_1\,, \end{aligned} \tag{3.20}$$

and their canonical P_a^z, P_s^y momenta associated to Z_a, Y_s depending on either $p^{(z)}$ or $p^{(y)}$

$$P_1 = \tfrac{1}{2}(p_2 + i\, p_1)\,, \quad P_2 = \tfrac{1}{2}(p_4 + i\, p_3)\,, \quad P_3 = P_2^\dagger = \tfrac{1}{2}(p_4 - i\, p_3)\,, \quad P_4 = P_1^\dagger = \tfrac{1}{2}(p_2 - i\, p_1)\ .$$

The advantage of the new coordinates is a simple mode expansion (3.31) and standard commutation relation in quantum theory. In terms of the complex fields the kinetic Lagrangian takes the form

$$\mathcal{L}_{kin} = P^z_{5-a}\dot{Z}_a + P^y_{5-s}\dot{Y}_s \qquad \text{with} \quad a, s = 1, \dots, 4 \tag{3.21}$$

and the bosonic Hamiltonian density in uniform light-cone gauge acquires the from

$$\mathcal{H} = \mathcal{H}_2 + \frac{1}{P_+}\mathcal{H}_4 + \mathcal{O}(\tfrac{1}{P_+^2}) \tag{3.22}$$

$$\begin{aligned} \mathcal{H}_2 &= P^z_{5-a}P^z_a + P^y_{5-s}P^y_s + \frac{1}{4}\left(Z_{5-a}Z_a + Y_{5-s}Y_s\right) + \frac{\widetilde{\lambda}}{4}\left(Z'_{5-a}Z'_a + Y'_{5-s}Y'_s\right) \\ \mathcal{H}_4 &= \frac{\widetilde{\lambda}}{4}\left(Y'_{5-s}Y'_s\ Z_{5-a}Z_a - Y_{5-s}Y_s\ Z'_{5-a}Z'_a + Z'_{5-a}Z'_a\ Z_{5-b}Z_b - Y'_{5-s}Y'_s\ Y_{5-u}Y_u\right) \ . \end{aligned} \tag{3.23}$$

3.2.4 Quantization

From (3.21) one reads off the commutator relations

$$\left[Z_a, P^z_{5-b}\right] = i\,\delta_{a,b} \qquad \text{and} \qquad \left[Y_s, P^y_{5-u}\right] = i\,\delta_{s,u} \quad . \tag{3.24}$$

Now one establishs a mode decomposition of the bosonic fields which renders the quadratic terms of the Hamiltonian $\mathcal{H}_2$ in a diagonal form. Here we state only the decomposition for the fields $Z_a,\ P^z_a$

$$Z_a(\tau,\sigma) = \sum_n e^{in\sigma} \frac{1}{i\sqrt{\omega_n}} (\beta^+_{a,n} - \beta^-_{5-a,-n}) \ , \qquad P^z_a(\tau,\sigma) = \sum_n e^{in\sigma} \frac{\sqrt{\omega_n}}{2} (\beta^+_{a,n} + \beta^-_{5-a,-n}) \ ,$$

where the frequency ω_n is defined as $\omega_n := \sqrt{1 + \widetilde{\lambda}\, n}$. The mode decompositions of all fields, including also fermions, are stated together with the full $\mathrm{AdS}_5 \times \mathrm{S}^5$ Hamiltonian in chapter 3.3.1. The creation operators $\alpha^+_{a,n}$ and corresponding annihilation operators $\alpha^-_{a,n}$ carry two indices. The first index $a = 1, \ldots, 4$ denotes the flavor, while the second index n represents a vibrational mode number on the string. Requiring (3.24) to hold, one finds commutation relations for the bosonic creations and annihilation operators

$$[\alpha^-_{a,n}, \alpha^+_{b,m}] = \delta_{a,b}\,\delta_{n,m} \ .$$

In terms of creation and annihilation operators the bosonic Hamiltonian $\mathcal{H}_2$ takes the form

$$\mathcal{H}_2 = \sum_n \omega_n (\beta^+_{a,n}\beta^-_{a,n} + \alpha^+_{a,n}\alpha^-_{a,n}) \, . \tag{3.25}$$

The expression for the next to leading order Hamiltonian $\mathcal{H}_4$ is much longer so we do not write it out explicitly in terms of the creation and annihilation operastors.

3.3 The full superstring Hamiltonian on $\mathrm{AdS}_5 \times \mathrm{S}^5$

The AdS_5 space can also be defined as quotient of $SO(4,2)/SO(4,1)$ while the S^5 manifold is given by $SO(6)/SO(5)$. Furthermore there exits the isomorphism $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \cong \mathfrak{so}(4,2) \oplus \mathfrak{so}(6)$ and the bosonic subalgebra of the superalgebra $\mathfrak{su}(2,2|4)$ admits the following decomposition

$$\mathfrak{su}(2,2|4) \cong \mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1) \ .$$

The superalgebra $\mathfrak{psu}(2,2|4)$ is defined as the quotient algebra of $\mathfrak{su}(2,2|4)$ over the $\mathfrak{u}(1)$ factor. Thus the $\mathrm{AdS}_5 \times \mathrm{S}^5$ target space of the superstring is given by the coset manifold

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1) \times \mathrm{SO}(5)} \tag{3.26}$$

There exists a representation of $\mathfrak{su}(2,2|4)$ in terms of 8×8 matrices but $\mathfrak{psu}(2,2|4)$ has no realization in terms of supermatrices. The construction of the superstring action including fermions uses the $\mathbb{Z}_4-$grading of the superalgebra $\mathfrak{su}(2,2|4)$. Any matrix M from $\mathfrak{su}(2,2|4)$ can then be decomposed into elements $M^{(i)}$ of the four additive groups of the $\mathbb{Z}_4-$grading

$$M = M^{(0)} + M^{(1)} + M^{(2)} + M^{(3)} \ .$$

Using a representative g of the coset space (3.26) and constructing the following current

$$A = -g^{-1}\mathrm{d}g = A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)} \ . \tag{3.27}$$

the Langrangian density [13] for the superstring in $AdS_5 \times S^5$ is given by the sum of kinetic term and the topological Wess-Zumino term:

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{WZ} = -\frac{\sqrt{\lambda}}{2} \operatorname{Str}\left(\gamma^{\alpha\beta} A^{(2)}_{\alpha} A^{(2)}_{\beta} + \kappa \epsilon^{\alpha\beta} A^{(1)}_{\alpha} A^{(3)}_{\beta}\right) \ , \tag{3.28}$$

where we use $\epsilon^{01} \equiv \epsilon^{\tau\sigma} = 1$ and $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$ denotes the Weyl-invariant world-sheet metric with $\det \gamma = -1$. The parameter κ is determined by κ-symmetry to $\kappa = \pm 1$.
Unfortunately the Lagrangian (3.28) suffers from the presence of non-physical degrees of freedom, related to reparametrization invariance and κ-symmetry, which are removed by fixing a gauge for the κ-symmetry and imposing the uniform light-cone gauge (3.11).

In priciple the derivation of the quantized Hamiltonian [7] for the full $AdS_5 \times S^5$ superstring including fermions follows the basic steps performed in capter 3.2 even though it is much more involved. Nevertheless one finds the same relations

$$-p = \mathcal{H} \qquad \text{and} \quad E - J = -P_- = -\int_0^{2\pi} \mathrm{d}\sigma p_- \quad .$$

as in the bosonic case. From now on we will absorb the integration over σ into the Hamiltonian $\mathcal{H}$ even though we will omit writig out the integral explicitly in most of the formulas, i.e. we have the relation $-P_- = \mathcal{H}$ where the integraion over σ is implicit.

Based on the underlying symmetry structure of $SO(4,2) \times SO(6)$ any state operator of the quantized theory can be labeled by the eigenvalues of the six Cartan generators

$$(E, S_1, S_2; J, J_1, J_2) \ , \tag{3.29}$$

where E is corresponding to the energy, (S_1, S_2) correspond to two spins on AdS_5 and (J, J_1, J_2) to the three angular momenta on the five sphere respectively.

3.3.1 Hamiltonian in uniform light-cone gauge

In an impressive computation [7] the quantized Hamiltonian has perturbatively been computed up to next-to-leading order in a $1/P_+$ expansion

$$\mathcal{H} = \mathcal{H}_2 + \frac{1}{P_+}\mathcal{H}_4 + \mathcal{O}(P_+^{-2}) \tag{3.30}$$

in the near plane wave limit and uniform light-cone gauge. The dynamical fields are given by the transverse eight fermionic and eight bosonic fields. We will use the following decomposition of the eight complex bosonic fields Z_a, Y_a and their corresponding canonical momenta P^z_a, P^y_a following the conventions in [7]

$$\begin{aligned}
Z_a(\tau,\sigma) &= \sum_n e^{in\sigma} Z_{a,n}(\tau) & P^z_a(\tau,\sigma) &= \sum_n e^{in\sigma} P^z_{a,n}(\tau) \\
Z_{a,n} &= \frac{1}{i\sqrt{\omega_n}}(\beta^+_{a,n} - \beta^-_{5-a,-n}) & P^z_{a,n} &= \frac{\sqrt{\omega_n}}{2}(\beta^+_{a,n} + \beta^-_{5-a,-n}) \\
Y_a(\tau,\sigma) &= \sum_n e^{in\sigma} Y_{a,n}(\tau) & P^y_a(\tau,\sigma) &= \sum_n e^{in\sigma} P^y_{a,n}(\tau) \\
Y_{a,n} &= \frac{1}{i\sqrt{\omega_n}}(\alpha^+_{a,n} - \alpha^-_{5-a,-n}) & P^y_{a,n} &= \frac{\sqrt{\omega_n}}{2}(\alpha^+_{a,n} + \alpha^-_{5-a,-n}) \ ,
\end{aligned} \tag{3.31}$$

where the frequency ω_n is defined as

$$\omega_n = \sqrt{1 + \widetilde{\lambda}\, n^2}\,. \tag{3.32}$$

The decomposition has been chosen in such a way that the creation and annihilation operators obey canonical commutation relations

$$[\alpha^-_{a,n}, \alpha^+_{b,m}] = \delta_{a,b}\,\delta_{n,m} = [\beta^-_{a,n}, \beta^+_{b,m}]\ . \tag{3.33}$$

The index $a \in \{1,2,3,4\}$ denotes the flavor and n, m are the mode numbers which are subject to the level matching condition

$$\sum_{j=1}^{K_4} m_j = 0\ . \tag{3.34}$$

To use a notation compatible to the Bethe equations of chapter 7.3, the number of string excitations, also called impurities, is denoted by K_4. The index is of no special meaning in the context of string theory.
The mode decompositions for the fermions[6] are:

$$\eta(\tau,\sigma) = \sum_n e^{in\sigma}\eta_n(\tau) \qquad \theta(\tau,\sigma) = \sum_n e^{in\sigma}\theta_n(\tau) \tag{3.35}$$

$$\eta_n = f_n\eta^-_{-n} + ig_n\eta^+_n \qquad \theta_n = f_n\theta^-_{-n} + ig_n\theta^+_n$$

$$\text{with} \qquad \eta^-_n = \eta^-_{a,n}\Gamma_{5-a}\ , \quad \eta^+_n = \eta^+_{a,n}\Gamma_a\ , \qquad \theta^-_n = \theta^-_{a,n}\Gamma_{5-a}\ , \quad \theta^+_n = \theta^+_{a,n}\Gamma_a\,. \tag{3.36}$$

The functions f_m and g_m above are defined as

$$f_m = \sqrt{\frac{1}{2}\Big(1 + \frac{1}{\omega_m}\Big)}, \quad g_m = \frac{\kappa\sqrt{\widetilde{\lambda}}m}{1+\omega_m} f_m\ . \tag{3.37}$$

Here $\kappa = \pm 1$ is the arbitrary relative sign between kinetic and Wess-Zumino term in the worldsheet action. The explicit representation of the Dirac matrices Γ_a is given in the Appendix A.2.
The anti-commutators between the fermionic mode operators are then

$$\{\eta^-_{a,n}, \eta^+_{b,m}\} = \delta_{a,b}\,\delta_{n,m} = \{\theta^-_{a,n}, \theta^+_{b,m}\}\,. \tag{3.38}$$

Using this oscillator representation, the leading order Hamiltonian becomes

$$\mathcal{H}_2 = \sum_n \omega_n(\theta^+_{a,n}\theta^-_{a,n} + \eta^+_{a,n}\eta^-_{a,n} + \beta^+_{a,n}\beta^-_{a,n} + \alpha^+_{a,n}\alpha^-_{a,n})\,. \tag{3.39}$$

[6] In the present context η denotes a fermionic excitation living on the string. It is not to be confused with the grading η_1, η_2, which are used in section 7 to describe different choices of Dynkin diagrams for $\mathfrak{psu}(2,2|4)$

The first order correction to this Hamiltonian is given by [7]

$$\mathcal{H}_4 = \mathcal{H}_{bb} + \mathcal{H}_{bf} + \mathcal{H}_{ff}(\theta) - \mathcal{H}_{ff}(\eta) \tag{3.40}$$

with
$$\mathcal{H}_{bb} = \frac{\widetilde{\lambda}}{4}(Y'_{5-a}Y'_a Z_{5-b}Z_b - Y_{5-a}Y_a Z'_{5-b}Z'_b + Z'_{5-a}Z'_a Z_{5-b}Z_b - Y'_{5-a}Y'_a Y_{5-b}Y_b) \tag{3.41}$$

$$\begin{aligned}
\mathcal{H}_{bf} = \frac{\widetilde{\lambda}}{4}\mathrm{tr}\Big[\quad & (Z_{5-a}Z_a - Y_{5-a}Y_a)(\eta'^\dagger\eta' + \theta'^\dagger\theta') \\
& - Z'_a Z_b[\Gamma_a, \Gamma_b]\left(\mathcal{P}_+(\eta\eta'^\dagger - \eta'\eta^\dagger) - \mathcal{P}_-(\theta^\dagger\theta' - \theta'^\dagger\theta)\right) \\
& + Y'_a Y'_b[\Gamma_a, \Gamma_b]\left(-\mathcal{P}_-(\eta^\dagger\eta' - \eta'^\dagger\eta) - \mathcal{P}_+(\theta\theta'^\dagger - \theta'\theta^\dagger)\right) \\
& - \frac{i\kappa}{\sqrt{\widetilde{\lambda}}}(Z_a P^z_b)'[\Gamma_a, \Gamma_b]\left(\mathcal{P}_+(\eta^\dagger\eta^\dagger + \eta\eta) + \mathcal{P}_-(\theta^\dagger\theta^\dagger + \theta\theta)\right) \\
& + \frac{i\kappa}{\sqrt{\widetilde{\lambda}}}(Y_a P^y_b)'[\Gamma_a, \Gamma_b]\left(\mathcal{P}_-(\eta^\dagger\eta^\dagger + \eta\eta) + \mathcal{P}_+(\theta^\dagger\theta^\dagger + \theta\theta)\right) \\
& + 8i Z_a Y_b\left(-\mathcal{P}_-\Gamma_a\eta'\Gamma_b\theta' + \mathcal{P}_+\Gamma_a\theta'^\dagger\Gamma_b\eta'^\dagger\right) \quad \Big]
\end{aligned} \tag{3.42}$$

$$\mathcal{H}_{ff}(\eta) = \frac{\widetilde{\lambda}}{4}\mathrm{tr}\Big[\Gamma_5\left(\eta'^\dagger\eta\eta'^\dagger\eta + \eta^\dagger\eta'\eta^\dagger\eta' + \eta'^\dagger\eta^\dagger\eta'^\dagger\eta^\dagger + \eta'\eta\eta'\eta\right)\Big]. \tag{3.43}$$

The Hamiltonian (3.40) will serve as input for the ABAKUS software, which will compute its eigenvalues $-\delta P_-$.

3.3.2 $U(1)$ Field Charges

As stated in the beginning of this chapter string excitations are characterized by the values of four $U(1)$ charges: two spins $\{S_1, S_2\}$ on AdS_5 and two angular momenta $\{J_1, J_2\}$ on S^5. In this work the charges (S_+, S_-, J_+, J_-) introduced in [7, 14] are used, which are related to the former quantities via $S_\pm = S_1 \pm S_2$ and $J_\pm = J_1 \pm J_2$. Since the string excitations are represented by creation operators in quantum theory the operators carry the definite charges spelled out in table 1.
The charge pattern of a string state is just the sum of the charges of all creation operators assembling the state. It will turn out to be the appropriate quantity to classify the Hamiltonian eigenvalues.

	S_+	S_-	J_+	J_-
$Y_1,\ P^y_1,\ \alpha^+_{1,m},\ \alpha^-_{4,m}$	0	0	1	1
$Y_2,\ P^y_2,\ \alpha^+_{2,m},\ \alpha^-_{3,m}$	0	0	1	-1
$Y_3,\ P^y_3,\ \alpha^+_{3,m},\ \alpha^-_{2,m}$	0	0	-1	1
$Y_4,\ P^y_4,\ \alpha^+_{4,m},\ \alpha^-_{1,m}$	0	0	-1	-1

	S_+	S_-	J_+	J_-
$Z_1,\ P^z_1,\ \beta^+_{1,m},\ \beta^-_{4,m}$	1	1	0	0
$Z_2,\ P^z_2,\ \beta^+_{2,m},\ \beta^-_{3,m}$	1	-1	0	0
$Z_3,\ P^z_3,\ \beta^+_{3,m},\ \beta^-_{2,m}$	-1	1	0	0
$Z_4,\ P^z_4,\ \beta^+_{4,m},\ \beta^-_{1,m}$	-1	-1	0	0

	S_+	S_-	J_+	J_-
$\theta_1,\ \theta^\dagger_4,\ \theta^+_{1,m},\ \theta^-_{4,m}$	0	1	1	0
$\theta_2,\ \theta^\dagger_3,\ \theta^+_{2,m},\ \theta^-_{3,m}$	0	-1	1	0
$\theta_3,\ \theta^\dagger_2,\ \theta^+_{3,m},\ \theta^-_{2,m}$	0	1	-1	0
$\theta_4,\ \theta^\dagger_1,\ \theta^+_{4,m},\ \theta^-_{1,m}$	0	-1	-1	0

	S_+	S_-	J_+	J_-
$\eta_1,\ \eta^\dagger_4,\ \eta^+_{1,m},\ \eta^-_{4,m}$	1	0	0	1
$\eta_2,\ \eta^\dagger_3,\ \eta^+_{2,m},\ \eta^-_{3,m}$	1	0	0	-1
$\eta_3,\ \eta^\dagger_2,\ \eta^+_{3,m},\ \eta^-_{2,m}$	-1	0	0	1
$\eta_4,\ \eta^\dagger_1,\ \eta^+_{4,m},\ \eta^-_{1,m}$	-1	0	0	-1

Table 1: Charges of annihilation and creation operators of the $AdS_5 \times S^5$ string in uniform light-cone gauge.

3.4 notation of generic string states

In this section a convenient notation for a generic string eigenstate of the leading Hamiltonian $\mathcal{H}_2$ is introduced, generalizing the discussion of [7]. We start with a generic $\mathfrak{su}(2)$ string $|\psi\rangle_{\alpha_1}$ state with K_4 excitations which is composed of creation operators α_1^+ generating modes $n_{K_4}, n_{K_4-1}, \ldots, n_1$.
Note that in general coinciding mode numbers are possible. In the following we distinguish between states where all modes are pairwise unequal, referred to as states with *non-confluent mode numbers*, and states with some coinciding mode numbers, denominated as states with *confluent mode numbers*.
Introducing the multiplicity ν_{n_k} of a mode n_k with respect to a given state $|\psi\rangle_{\alpha_1}$, we denote the number of different modes in $|\psi\rangle_{\alpha_1}$ by K_4'. Since the subscript already indicates whether we are working with the set of only distinct mode numbers or the set of all excited string modes, we allow for a slight abuse of notation by defining

$$\begin{array}{ll} \text{the list of all } K_4 \text{ excited string modes,} & \text{the set of all } K_4' \text{ pairwise unequal string modes.} \\ \{n_{K_4}, n_{K_4-1}, \ldots, n_1\} & \{n_{K_4'}, n_{K_4'-1}, \ldots, n_1\} \end{array} \tag{3.44}$$

It is important to point out that for a certain i the n_i's in both notations do not necessarily refer to the same mode number.
A generic $\mathfrak{su}(2)$ state $|\psi_{\alpha_1}\rangle$ is encoded as

$$|\psi\rangle_{\alpha_1} = c\ \alpha^+_{1,n_{K_4}} \alpha^+_{1,n_{K_4-1}} \ldots \alpha^+_{1,n_1}|0\rangle = c\ (\alpha^+_{1,n_{K_4'}})^{\nu_{K_4'}} (\alpha^+_{1,n_{K_4'-1}})^{\nu_{K_4'-1}} \ldots (\alpha^+_{1,n_1})^{\nu_1}|0\rangle\ , \tag{3.45}$$

where c is the normalization constant. Finally we introduce the notation for a normalized $\mathfrak{su}(2)$ state

$$|\psi\rangle_{\alpha_1} = |G_{\alpha_1};\ n_{K_4'}^{\nu_{K_4'}}, n_{K_4'-1}^{\nu_{K_4'-1}}, \ldots, n_1^{\nu_1}\rangle_{\alpha_1} := \frac{(\alpha^+_{1,n_{K_4'}})^{\nu_{K_4'}}}{\sqrt{\nu_{K_4'}!}} \frac{(\alpha^+_{1,n_{K_4'-1}})^{\nu_{K_4'-1}}}{\sqrt{\nu_{K_4'-1}!}} \ldots \frac{(\alpha^+_{1,n_1})^{\nu_1}}{\sqrt{\nu_1!}}|0\rangle. \tag{3.46}$$

The quantity $G_{\mathcal{O}}$ represents a counter for the number of fermionic creation operators in the particular substate, so in the bosonic case it is simply zero. Having a computer software in mind which is dealing with a large set of states, all carrying the same modes, it is convenient to save the mode numbers separately and to omit the mode numbers in the states:

$$|\psi\rangle_{\alpha_1} = |G_{\alpha_1};\ \nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1\rangle_{\alpha_1}\ . \tag{3.47}$$

Notation (3.47) has been chosen in such a way, it reflects the internal representation of a generic string state in the ABAKUS-software explained in chapter 4. For the sake of simplicity the internal representation should be unique at the software level. That is why for fermionic states the same notation is chosen

$$|\psi\rangle_{\theta,\eta} = |G_{\theta,\eta};\ \nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1\rangle_{\theta,\eta}\ , \tag{3.48}$$

but now the order of the single operators is of course important and in general $G_{\theta,\eta} \neq 0$. Therefore we require operators of the same substate to form a decreasing series with respect to the mode numbers, i.e.

$$n_{K_4'} > n_{K_4'-1} > \ldots > n_1\ . \tag{3.49}$$

We now define a uniform notation for a generic string state using:

$\{ n_{K_4'}, \ldots, n_1 \}$ set of different modes excited on the string
$\{ \nu_{K_4'}, \ldots, \nu_1 \}$ multiplicities, counting the number of excitations of the corresponding modes $\{n_{K_4'}, \ldots, n_1\}$
$\{ \nu_{K_4'}^{(\mathcal{O})}, \ldots, \nu_1^{(\mathcal{O})} \}$ multiplicities, counting the number of excitations with flavor $\mathcal{O}$

A defining property of the mode specific multiplicities $\nu_k^{(\mathcal{O})}$ is, that the sum over all operators yields ν_k

$$\sum_{\substack{\text{flavor} \\ c=1,\ldots,4}} \sum_{\substack{\mathcal{O}\in \\ \{\theta_c,\eta_c,\beta_c,\alpha_c\}}} \nu_k^{(\mathcal{O})} = \nu_k \qquad \text{for all} \quad k = K_4', \ldots, 1 \ .$$

A generic string eigenstate $|\Psi\rangle$ of the quadratic Hamiltonian $\mathcal{H}_2$ can now be written in the form

$$\begin{aligned} |\Psi\rangle &= \prod_{c=1}^{4} |\psi\rangle_{\theta_c} \prod_{c=1}^{4} |\psi\rangle_{\eta_c} \prod_{c=1}^{4} |\psi\rangle_{\beta_c} \prod_{c=1}^{4} |\psi\rangle_{\alpha_c} \\ \text{with} \qquad |\psi\rangle_{\mathcal{O}} &= |G_{\mathcal{O}};\ \nu_{K_4'}^{(\mathcal{O})}, \nu_{K_4'-1}^{(\mathcal{O})}, \ldots, \nu_1^{(\mathcal{O})}\rangle_{\mathcal{O}} \ , \end{aligned} \tag{3.50}$$

where we assume the products to be in decreasing order $\prod_{c=1}^{4} f_c \equiv f_4 f_3 f_2 f_1$. Here also multiplicities $\nu_k^{(\mathcal{O})} = 0$ are allowed, which is important for a software in order to save all substates $|\psi\rangle_{\mathcal{O}}$ of $|\Psi\rangle$ in a structural identical representation. In this notation the level matching condition (3.34) becomes

$$0 = \sum_{i=1}^{K_4} n_i = \sum_{i=1}^{K_4'} \nu_i n_i \ . \tag{3.51}$$

3.5 Eigenvalues of the Hamiltonian

Obviously $\theta_{c,n_i}^{+}\theta_{c,n_i}^{-}$, $\eta_{c,n_i}^{+}\eta_{c,n_i}^{-}$, $\beta_{c,n_i}^{+}\beta_{c,n_i}^{-}$, $\alpha_{c,n_i}^{+}\alpha_{c,n_i}^{-}$ act as mode number operators. Thus the eigenvalues of the leading order Hamiltonian $\mathcal{H}_2$, given in (3.39), are

$$\mathcal{H}_2|\psi\rangle = E_2|\psi\rangle \qquad \text{with} \quad E_2 = \sum_{n=-\infty}^{\infty} \nu_n \omega_n = \sum_{i=1}^{K_4'} \nu_i \omega_i \ . \tag{3.52}$$

In (3.52) two different notations for ν and ω have been used:

ν_m multiplicity of mode number m, where m represents the mode number, i.e $m = -\infty, \ldots, \infty$
$\nu_i \equiv \nu_{n_i}$ multiplicity of mode number n_i, where $i = 1, \ldots, K_4'$ is the index in a set $\{n_{K_4'}, \ldots, n_1\}$ of mode numbers (3.53)

Similarly the notation for ω_n is abbreviated:

$\omega_m = \sqrt{1 + \widetilde{\lambda} m^2}$ in case m represents a mode, i.e $m = -\infty, \ldots, \infty$
$\omega_i \equiv \omega_{n_i} = \sqrt{1 + \widetilde{\lambda} n_i^2}$ in case $i = 1, \ldots, K_4$ is the index in a set of mode numbers (3.54)

In uniform light-cone gauge the Hamiltonian eigenvalue $-P_-$ is then given by

$$P_- = -\sum_{i=1}^{K_4} \omega_i + \delta P_- = -\sum_{i=1}^{K_4'} \nu_i\, \omega_i + \delta P_- \ , \tag{3.55}$$

where $-\delta P_-$ represents the eigenvalues of $\frac{1}{P_+}\mathcal{H}_4$.

3.5.1 Eigenvalues of $\mathcal{H}_4$

Equation (3.52) shows that two states with the same energy E_2 have to carry the same excited modes. However, at leading order the energy is independent of the 16 possible flavours $\{\theta_4, \theta_2, \ldots, \alpha_1\}$ of the excitations. Thus we have to use degenerated pertubation theory to obtain the energy correction $-\delta P_-$. Denoting

$$\begin{array}{lll} \boldsymbol{n}_{K_4'} = & \{n_{K_4'}, n_{K_4'-1}, \ldots, n_1\} & \text{set of distinct excited modes on the string} \\ \boldsymbol{\nu}_{K_4'} = & \{\nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1\} & \text{set of multiplicites corresponding to the modes} \\ \Psi \quad = & (|\Psi_\varrho\rangle, \ldots, |\Psi_1\rangle) & \text{vector of all possible states } |\Psi\rangle, \text{ carrying exactly the} \\ & & \text{modes } \boldsymbol{n}_{K_4'} \text{ with corresponding multiplicity } \boldsymbol{\nu}_{K_4'} \end{array} \tag{3.56}$$

one has to compute the matrix representation $\Psi^\dagger \mathcal{H}_4 \Psi$, whose eigenvalues yield $-\delta P_-$. Looking at the structure of (3.40), $\mathcal{H}_4$ consists, among other terms, of operator products with a different number of creation and annihilation operators. For the given purpose one can just drop these terms, since only matrix elements $\langle \Psi_a | \mathcal{H}_4 | \Psi_b \rangle$ have to be calculated where $|\Psi_a\rangle$ as well as $|\Psi_b\rangle$ carry both K_4' excitations. However, it was shown in [7] that there exists a unitary transformation in pertubation theory around the plane-wave, such that the resulting Hamiltonian contains only terms with an equal number of creation and annihilation operators.

Acting with the Hamiltonian does not change the $U(1)$ charges of section 3.3.2, which is obvious for the bosonic part $\mathcal{H}_{bb}$ (3.41). Therefore mixing states need to carry equal charges in terms of $\{S_+, S_-, J_+, J_-\}$. Hence it is sufficient for a given excitation pattern (3.56) to only generate states with equal changes.

3.6 The $\mathfrak{su}(2)$ sector

As an example we will compute the energy spectrum of the rank one $\mathfrak{su}(2)$ sector. For the simple and structurally identical $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors it is possible to derive closed form expressions for the string energy spectrum.
The $\mathfrak{su}(2)$ sector consists of states which are composed only of $\alpha^+_{1,n}$ creation operators, thus the Hamiltonian (3.40) simplifies dramatically to the effective form

$$\mathcal{H}_4^{(\mathfrak{su}(2))} = \widetilde{\lambda} \sum_{\substack{n+m \\ +k+l}=0} \frac{nk}{\sqrt{\omega_n \omega_m \omega_k \omega_l}} \alpha_n^+ \alpha_m^+ \alpha_{-k}^- \alpha_{-l}^- \quad . \tag{3.57}$$

The mode number operator takes the form

$$\alpha^+_{1,n_i} \alpha^-_{1,n_i} \equiv \nu_i = \sum_{k=1}^{K} \delta_{n_i,n_k} \quad . \tag{3.58}$$

In order to calculate $\langle \Psi | \mathcal{H}_4^{(\mathfrak{su}(2))} | \Psi \rangle = -\delta P_-$, there are obviously only three cases to consinder:
a) $n = -k$, $m = -l$ with $n \neq m$ b) $n = -l$, $m = -k$ with $n \neq m$ c) $n = m = -k = -l$.

case a) $n=-k$, $m=-l$ with $n\neq m$

$$\begin{aligned}
&\frac{\widetilde{\lambda}}{P_+}\sum_{\substack{n,m\\ n\neq m}}\frac{-n^2}{\omega_n\omega_m}\langle\Psi|\alpha_n^+\alpha_n^-\alpha_m^+\alpha_m^-|\Psi\rangle\\
&\overset{(3.58)}{=}\frac{\widetilde{\lambda}}{P_+}\sum_{\substack{n,m\\ n\neq m}}\frac{-n^2}{\omega_n\omega_m}\sum_{\substack{i,j=1\\ i\neq j}}^{K}\delta_{n,n_i}\delta_{m,n_j}=\frac{\widetilde{\lambda}}{P_+}\sum_{n,m}\frac{-n^2}{\omega_n\omega_m}\sum_{\substack{i,j=1\\ i\neq j}}^{K}\delta_{n,n_i}\delta_{m,n_j}+\frac{\widetilde{\lambda}}{P_+}\sum_{n}\frac{n^2}{\omega_n^2}\sum_{\substack{i,j=1\\ i\neq j}}^{K}\delta_{n,n_i}\delta_{n,n_j}\\
&=\frac{\widetilde{\lambda}}{P_+}\sum_{\substack{i,j=1\\ i\neq j}}^{K}\frac{-n_i^2}{\omega_{n_i}\omega_{n_j}}+\frac{\widetilde{\lambda}}{P_+}\sum_{i=1}^{K}\frac{n_i^2}{\omega_{n_i}^2}\Big(\sum_{j=1}^{K}\delta_{n_i,n_j}-1\Big)=\frac{\widetilde{\lambda}}{P_+}\sum_{\substack{i,j=1\\ i\neq j}}^{K}\frac{-n_i^2}{\omega_{n_i}\omega_{n_j}}+\frac{\widetilde{\lambda}}{P_+}\sum_{i=1}^{K}\frac{n_i^2}{\omega_{n_i}^2}\left(\nu_i-1\right)
\end{aligned}\tag{3.59}$$

case b) $n=-l$, $m=-k$ with $n\neq m$

$$\frac{\widetilde{\lambda}}{P_+}\sum_{\substack{n,m\\ n\neq m}}\frac{-nm}{\omega_n\omega_m}\langle\Psi|\alpha_n^+\alpha_n^-\alpha_m^+\alpha_m^-|\Psi\rangle\overset{(3.58)}{=}\frac{\widetilde{\lambda}}{P_+}\sum_{\substack{i,j=1\\ i\neq j}}^{K}\frac{-n_in_j}{\omega_{n_i}\omega_{n_j}}+\frac{\widetilde{\lambda}}{P_+}\sum_{i=1}^{K}\frac{n_i^2}{\omega_{n_i}^2}\left(\nu_i-1\right)\tag{3.60}$$

case c) $n=m=-k=-l$

$$\begin{aligned}
&\frac{\widetilde{\lambda}}{P_+}\sum_{n}\frac{-n^2}{\omega_n^2}\langle\Psi|\alpha_n^+\underbrace{\alpha_n^+\alpha_n^-}_{\alpha_n^-\alpha_n^+-1}\alpha_n^-|\Psi\rangle=\frac{\widetilde{\lambda}}{P_+}\sum_{n}\frac{-n^2}{\omega_n^2}\sum_{i=1}^{K}\delta_{n,n_i}\Big(\sum_{j=1}^{K}\delta_{n,n_j}-1\Big)\\
&=\frac{\widetilde{\lambda}}{P_+}\sum_{i=1}^{K}\frac{-n_i^2}{\omega_{n_i}^2}\left(\nu_{n_i}-1\right)
\end{aligned}\tag{3.61}$$

Adding (3.59), (3.60), (3.61) yields $-\delta P_-$, i.e. we find

$$E-J=\sum_{k=1}^{K}\omega_{n_k}-\frac{\widetilde{\lambda}}{2P_+}\sum_{\substack{i,j=1\\ i\neq j}}^{K}\frac{(n_i+n_j)^2}{\omega_{n_i}\omega_{n_j}}+\frac{\widetilde{\lambda}}{P_+}\sum_{i=1}^{K}\frac{n_i^2}{\omega_{n_i}^2}\left(\nu_{n_i}-1\right)\ .\tag{3.62}$$

(3.62) generalizes the result of [7] to the case of confluent mode numbers.

3.6.1 Solving for the space-time Energy

Since $P_\pm=J\pm E$, the energy is only determined implicitly. By rewriting (3.62) in terms of the global energy E and the BMN quantities J with $\lambda'=\lambda/J^2=\text{fix}$ and subsequently solving for E one obtains the $\mathfrak{su}(2)$ global energy

$$\begin{aligned}
E=J+\sum_{k=1}^{K}\bar{\omega}_{n_k}-\frac{\lambda'}{4J}\sum_{k,j=1}^{K}\frac{n_k^2\bar{\omega}_{n_j}^2+n_j^2\bar{\omega}_{n_k}^2}{\bar{\omega}_{n_k}\bar{\omega}_{n_j}}-\frac{\lambda'}{4J}\sum_{\substack{i,j=1\\ i\neq j}}^{K}\frac{(n_i+n_j)^2}{\bar{\omega}_{n_i}\bar{\omega}_{n_j}}+\frac{\lambda'}{2J}\sum_{i=1}^{K}\frac{n_i^2}{\bar{\omega}_{n_i}^2}\left(\nu_{n_i}-1\right)\\
\text{with}\qquad\bar{\omega}_k:=\sqrt{1+\lambda' m_k^2}\quad.
\end{aligned}$$

This result agrees precisely with the one in [15], where a different gauge has been used, and with the formula derived in [16] from a Bethe ansatz.

4 Computer-algebraic calculation: the ABAKUS-system

In a pioneering work Kurt Gödel proved 1931 that in mathematics there are true statements which however can not be proved using the system of axioms provided by the theory. This fact is known as the Incompleteness Theorem (for general reviews see [17, 18]). The only way to make use of such an unprovable but true statement is to add it as an axiom to the theory.

For instance Goldbach's conjecture of 1742 states, that every even number $2 < n \in \mathbb{N}$ can be decomposed into a sum of two prime numbers. Even the validity is confirmed up to 10^{14}, till this day no prove of Goldbach's conjecture is known.
Also of general interest is the conjecture "$P \neq NP$". Here P and NP denote a certain complexity class of problems. P contains all problems, whose solution can be found in polynomial many calculation steps in terms of the input length. In contrast a problem belongs to NP if any suggested solution can be checked for correctness in polynomial many calculation steps. Of course by constructing the valid solution of a given problem the check for correctness is dispensable, so $P \subset NP$. The question is, if there are problems allowing for a fast validation of a solution, but not for a fast construction of a solution, i.e. problems which are in NP and not in P. For example it can be fast verified that »$97, 89$« is a solution to »find the prime factors of 8633«, so the problem »find the prime factors of number $n \in \mathbb{N}$« is in NP. Factorization of numbers is the key point in modern cryptography, and enormous effort has been taken to find computer algorithms for fast prime factorization in order to crack cryptography protocols. Nevertheless constructing such an algorithm has not been successful and it is believed that no such algorithm exists.
The theorem "$P \neq NP$" is one of the seven Millennium Prize Problems, the Clay Mathematics Institute has put a premium of 1 million Dollar on the prove of each problem. In fact many theoretical computer scientists nowadays believe that there is no prove of "$P \neq NP$", even though nobody seriously doubts "$P \neq NP$". Thus this theorem is accepted as an axiom and is a key ingredient in many proves in the field of theoretical computer science.

The above discussion shows that there are statements of practical interest which are likely to be unprovable and that could well be so for the conjectured AdS/CFT correspondence. Thus it is important to develop tools enabling us to systematically test certain conjectures, which will be computer software in most cases.
A significant part of this Diploma thesis has been to develop a suitable software tool enabling us to compute systematically the spectrum of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. This software, the ABAKUS-System is presented in this chapter. At first a specification for the software is given in 4.2 followed by an analyze of the algorithmic complexity classes of the problem. In 4.4 specific software layout is described and the key algorithms are presented.

4.1 Physical Fundamentals of the Software

According to the mode decompositions (3.35) the fermionic fields $\theta,\ \eta$ consist of a matrix and scalar component. In order to compute traces and products of Γ_a-matrices it is convenient to use a slight different decomposition

$$\begin{aligned} \theta(\tau,\sigma) &= \Gamma_{5-a}\,\widetilde{\theta}_a^-(f) + i\,\Gamma_b\,\widetilde{\theta}_b^+(g) \quad , \qquad \eta(\tau,\sigma) = \Gamma_{5-a}\,\widetilde{\eta}_a^-(f) + i\,\Gamma_b\,\widetilde{\eta}_b^+(g) \quad , \\ \theta^\dagger(\tau,\sigma) &= \Gamma_a\,\widetilde{\theta}_a^+(f) + i\,\Gamma_{5-b}\,\widetilde{\theta}_b^-(g) \quad , \qquad \eta^\dagger(\tau,\sigma) = \Gamma_a\,\widetilde{\eta}_a^+(f) + i\,\Gamma_{5-b}\,\widetilde{\eta}_b^-(g) \quad , \end{aligned} \tag{4.1}$$

with the advantage, that the remaining functions

$$\widetilde{\theta}_a^\pm(k) := \sum_{n=-\infty}^{\infty} e^{in\sigma}\,k_n\,\theta_{a,\pm n}^\pm \ , \qquad \widetilde{\eta}_a^\pm(k) := \sum_{n=-\infty}^{\infty} e^{in\sigma}\,k_n\,\eta_{a,\pm n}^\pm \ , \tag{4.2}$$

have no matrix structure anymore and consist only of creation or annihilation operators of one color a. The functions $f,\ g$ abbreviate the former definitions $f_n,\ g_n$ of (3.37) and thus $k_n \in \{f_n, g_n\}$. The bosonic fields (3.31) are decomposed similarly:

$$\begin{aligned} Z_a(\tau,\sigma) &= -i\,\widetilde{\beta}_a^+(-1) + i\,\widetilde{\beta}_{5-a}^-(-1) \quad , & Y_a(\tau,\sigma) &= -i\,\widetilde{\alpha}_a^+(-1) + i\,\widetilde{\alpha}_{5-a}^-(-1) \quad , \\ P_a^z(\tau,\sigma) &= \tfrac{1}{2}\,\widetilde{\beta}_a^+(+1) + \tfrac{1}{2}\,\widetilde{\beta}_{5-a}^-(+1) \quad , & P_a^y(\tau,\sigma) &= \tfrac{1}{2}\,\widetilde{\alpha}_a^+(+1) + \tfrac{1}{2}\,\widetilde{\alpha}_{5-a}^-(+1) \quad , \end{aligned} \tag{4.3}$$

$$\text{with} \qquad \widetilde{\beta}_a^{\pm}(x) := \sum_{n=-\infty}^{\infty} e^{in\sigma}\,\omega_n^{\frac{x}{2}}\,\beta_{a,\pm n}^{\pm} \qquad ,\widetilde{\alpha}_a^{\pm}(x) := \sum_{n=-\infty}^{\infty} e^{in\sigma}\,\omega_n^{\frac{x}{2}}\,\alpha_{a,\pm n}^{\pm}\ . \tag{4.4}$$

For software purposes the normal order of the Hamiltonian is defined with respect to the sequence

$$\theta_4^+\theta_4^-\,\theta_3^+\theta_3^-\,\ldots\theta_1^+\theta_1^-\,\eta_4^+\eta_4^-\,\ldots\eta_1^+\eta_1^-\,\beta_4^+\beta_4^-\,\ldots\beta_1^+\beta_1^-\,\alpha_4^+\alpha_4^-\,\ldots\alpha_1^+\alpha_1^-\ . \tag{4.5}$$

For the normal ordering procedure the individual mode numbers associated to the creation and annihilation operators are not relevant.

4.2 Software Requirements Specification

A well designed software specification is essential for a fast development and accurate working of computer programs. In the following paragraph a condensed form of a requirement specification for ABAKUS is presented.

1. purpose of the software

The purpose of the ABAKUS-software is to compute eigenvalues of the string Hamiltonian (3.40) for an arbitrary string configuration.

2. essential requirements

- Correctness of the software calculations has to be guaranteed.
- Algorithms have to be optimized with respect to run time requirements, so that complex instances are computable.
- It must be possible to perform the calculations analytically except for the matrix diagonalization, which is not generally possible for higher dimensional matrices.

3. target audience

Physicists, familiar with string theory.

4. runtime environment

operating system:	Linux, kernel version 2.4 (or higher)
required software:	gcc version 3.3.5 (or higher)
	Wolfram MATHEMATICA 5.2 (or higher)
	FORM[7] version 3.1 (or higher)
recommended hardware:	2GHz CPU, 1000 MB Ram, 300 MB space on hard disk

5. user interface

All the necessary input is given in a file `define_states.def` together with some control commands, the output will be given in terms of a MATHEMATICA file.

[7] developed by Jos Vermaseren, The National Institute for Nuclear Physics and High Energy Physics, Netherlands, http://www.nikhef.nl/∼t68/

5.1 physical input

The user has to define the excitation pattern on the string. Since the calculation is performed analytically, the numerical values of the mode numbers are not of interest. What is needed, is the number of excitations and the multiplicity for each mode, which is fully encoded in the command

$$\texttt{\#define ExcitationNumbers } \{\nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1\} \ . \tag{4.6}$$

5.1.1 recommended specification for the structure of considered states

Given the values for the charges by the command

$$\texttt{\#define DefineCharges } \{S_+, S_-, J_+, J_-\} \ , \tag{4.7}$$

all possibly mixing states can be generated.

5.1.2 alternative specification for the structure of considered states

Instead of defining charges, there is the possibility of defining the set of operators, which are allowed to carry excitations

$$\texttt{\#define UseOperators } \{\mathcal{O}_1, \mathcal{O}_2, \ldots\} \qquad \text{with } \mathcal{O}_i \in \theta_4, \ldots, \alpha_1. \tag{4.8}$$

In this case the user has to bear responsibility, that there are no mixing states excluded by the given pattern.
In addition the user might define the specific number of excitations a for each operator in (4.8), by

$$\texttt{\#define HamiltonianSector } \{a_1, a_2, \ldots\} \quad \text{with } \sum_i a_i \stackrel{!}{=} \sum_{i=1}^{K_4'} \nu_i \ . \tag{4.9}$$

Again the user has to bear responsibility, that there are no mixing states excluded by the given pattern.

5.1.3 optional commands

For the propose of diagonalization, the final structure of $\mathcal{H}_4$ in matrix representation is exported to MATHEMATICA. For better readability there is the possibility to define a name for each mode by the command

$$\texttt{\#define ModeIndices } \{\text{»name1«,»name2«}, \ldots\} \ , \tag{4.10}$$

where the list of names has to contain K_4' elements, so that for every distinct mode listed in (4.6) there is a synonym given.

5.2 program control commands

The following commands are mandatory:

file name for generated states :	`#define StatesFile` »file«
name for MATHEMATICA output file :	`#define MathematicaFile` »file« .

The optional command `#define VERBOSE` causes an enhanced output during runtime.

4.3 Algorithmic complexity of the problems

The discussion is restricted to the case, where the user specifies the charge $\{S_+, S_-, J_+, J_-\}$, because in this case the software calculates *exactly all* mixing states. Also the main input to the Bethe equations of chapter 7 are the so called Dynkin node excitations, which can be expressed directly in terms of the charges and K_4'.
In order to compute the energy corrections $-\delta P_-$, all necessary input is given by

$$\boldsymbol{\nu}_{K_4'} = \{\nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1\} \,, \quad \{S_+, S_-, J_+, J_-\} \,, \tag{4.11}$$

where the notation follows (3.56).

At first, all mixing states $\Psi = (|\Psi_\varrho\rangle, \ldots, |\Psi_1\rangle)$ compatible with $\boldsymbol{\nu}_{K_4'}$ and $\{S_+, S_-, J_+, J_-\}$ have to be generated, where the key algorithms are discussed in paragraph 4.4.1. The energy corrections $-\delta P_-$ are given by the eigenvalues of $\Psi^\dagger \mathcal{H}_4 \Psi$. To reduce computational costs, an effective Hamiltonian $\mathcal{H}_{\text{eff}}$ is derived from $\mathcal{H}_4$ by dropping terms, which will evaluate to zero for all generated states. The related algorithm is explained in paragraph 4.4.2. In section 4.4.5 particulars of calculating the matrix representation $\Psi^\dagger \mathcal{H}_{\text{eff}} \Psi$ and its eigenvalues are presented.

4.3.1 Complexity of state generation

Given the multiplicities $\{\nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1\} = \boldsymbol{\nu}_{K_4'}$, the simplest approach to generate states is to choose a flavor $\theta_4^+, \theta_3^+, \ldots, \alpha_1^+$ for all of the $\nu_{K_4'} + \nu_{K_4'-1} + \ldots + \nu_1 = K_4$ modes and afterwards to single out the states with proper charge. Thus, considering one mode with multiplicity ν_i, the task is to pick ν_i flavors where the order does not matter and the flavors can be chosen more than once, viz there are $\binom{q+\nu_i-1}{\nu_i}$ possibilities for $q = 16$ flavors. In total one finds

$$\text{number of generated states } = \prod_{i=1}^{K_4'} \binom{15+\nu_i}{\nu_i} \,. \tag{4.12}$$

In case of non-confluent modes, viz $\nu_{i=1,\ldots,K_4'} = 1$, such an algorithm would compute $16^{K_4'}$ states, but for $\{S_+, S_-, J_+, J_-\} = \{K_4', K_4', 0, 0\}$ there is only one state with proper charge, which is the state composed of α_1^+. This example shows, that the pure approach is extremely inefficient, since it produces exponential overhead. A good algorithm, avoiding this disadvantage, will generate only states of proper charge right from start.

In order to determine the number of states that have to be computed in the worst case, consider the setting $\{S_+, S_-, J_+, J_-\} = \{0, 0, 0, 0\}$, $\nu_{i=1,\ldots,K_4'} = 1$ with even $K_4' = 2k, k \in \mathbb{N}$. For each of the first k modes a flavor $\alpha_1^+, \beta_1^+, \eta_1^+, \theta_1^+$ is chosen independently. The charge of this excitations can easily be annihilated by picking the last k modes from $\alpha_4^+, \beta_4^+, \eta_4^+, \theta_4^+$, which are of complementary charge. Since there are $4^k = 2^{K_4'}$ possibilities to choose the first k flavors, for this particular example the

$$\text{number of contributing states } \geq 2^{K_4'} \,.$$

This observation shows, that there are cases where the number of states grows exponentially with the number of impurities K_4'.

The Abakus-software stores the states using the representation (3.50) introduced in chapter 3.4: for every operator flavor $\mathcal{O} = \theta_4, \theta_3, \ldots, \alpha_1$ multiplicities $\nu_{K_4'}^{(\mathcal{O})}, \ldots, \nu_1^{(\mathcal{O})}$ are stored, which leads to a linear memory requirement of $\mathcal{O}(K_4')$ for every state.

4.3.2 Complexity of computing the effective Hamiltonian $\mathcal{H}_{\text{eff}}$

According to the software requirements specification ABAKUS is designed to compute string spectrum on $\mathrm{AdS}_5 \times \mathrm{S}^5$ in next to leading order. Even though the Hamiltonian (3.40) is stored in a separate file and thus may easily be exchanged,. the software is not designed to deal with user customized Hamiltonians. Therefore we consider the Hamiltonian as permanently been given by (3.40).

The computation of the effective Hamiltonian is a problem of constant complexity. Independent of the given excitations, the Hamiltonian (3.40) consists of a constant number of terms (precisely 8192). To process each term takes a fixed amount of steps, thus it exists an upper bound for the computational steps, i.e. the problem is in $\mathcal{O}(1)$. So in principle the efficiency of solving this problem is not of importance.
Nevertheless one should not be too disregardful with the efficiency, because the number of terms to handle is quite large and computing $\mathcal{H}_{\text{eff}}$ could dominate the runtime in case of solving smaller problem instances.

4.3.3 Complexity of computing the Hamiltonian matrix representation

Given the vector of ϱ mixing states $\Psi = (|\Psi_\varrho\rangle, \ldots, |\Psi_1\rangle)$ the matrix representation $\Psi^\dagger \mathcal{H}_{\text{eff}} \Psi$ contains ϱ^2 matrix elements, where Hermiticity halves the number of independent elements. However, there is still the possibility of an exponential growth in the number of states with respect to the length of the input data, which in this case leads to exponential many matrix entries.

In order to compute a single matrix element, all terms of the effective Hamiltonian have to be processed separately. In principle a single creation or annihilation operator of $\mathcal{H}_{\text{eff}}$ acts on every of the K_4' modes by creating or annihilating a string excitation. As obvious from (3.40) every term in $\mathcal{H}_{\text{eff}}$ consists of four operators. Thus one term of $\mathcal{H}_{\text{eff}}$ results in $\mathcal{O}({K_4'}^4)$ scalar products. Denoting the number of terms in $\mathcal{H}_{\text{eff}}$ by $|\mathcal{H}_{\text{eff}}|$, the computation of the matrix representation is of order $\mathcal{O}(\varrho \cdot {K_4'}^4 \cdot |\mathcal{H}_{\text{eff}}|)$. Therefore the computation of the matrix representation of $\mathcal{H}_{\text{eff}}$ is the problem, which will require most of the runtime.

This section is concluded with some general remarks concerning the fact of exponential scaling behaviors:

- Given the task to compute exponential many objects with respect to the length of the input data, consider an algorithm, that needs A steps to compute one of these objects and a second algorithm using $2A$ steps. Therefore it takes quadratic many steps to compute the entire solution containing exponential many objects for the slower algorithm compared to the faster one.
  Therefore it is important to develop highly efficient software, since also inefficiency scales exponentially.

- Available general purpose software like MATHEMATICA or generic search algorithms are tools for a wide range of applications and thus they can not know about the specific structure of the discussed problems. It is very likely, that using these tools, one will only be able to solve small problem instances. In oder to handle even complex instances, one needs software adapted to the particular problems.

4.4 Software layout of the ABAKUS-System

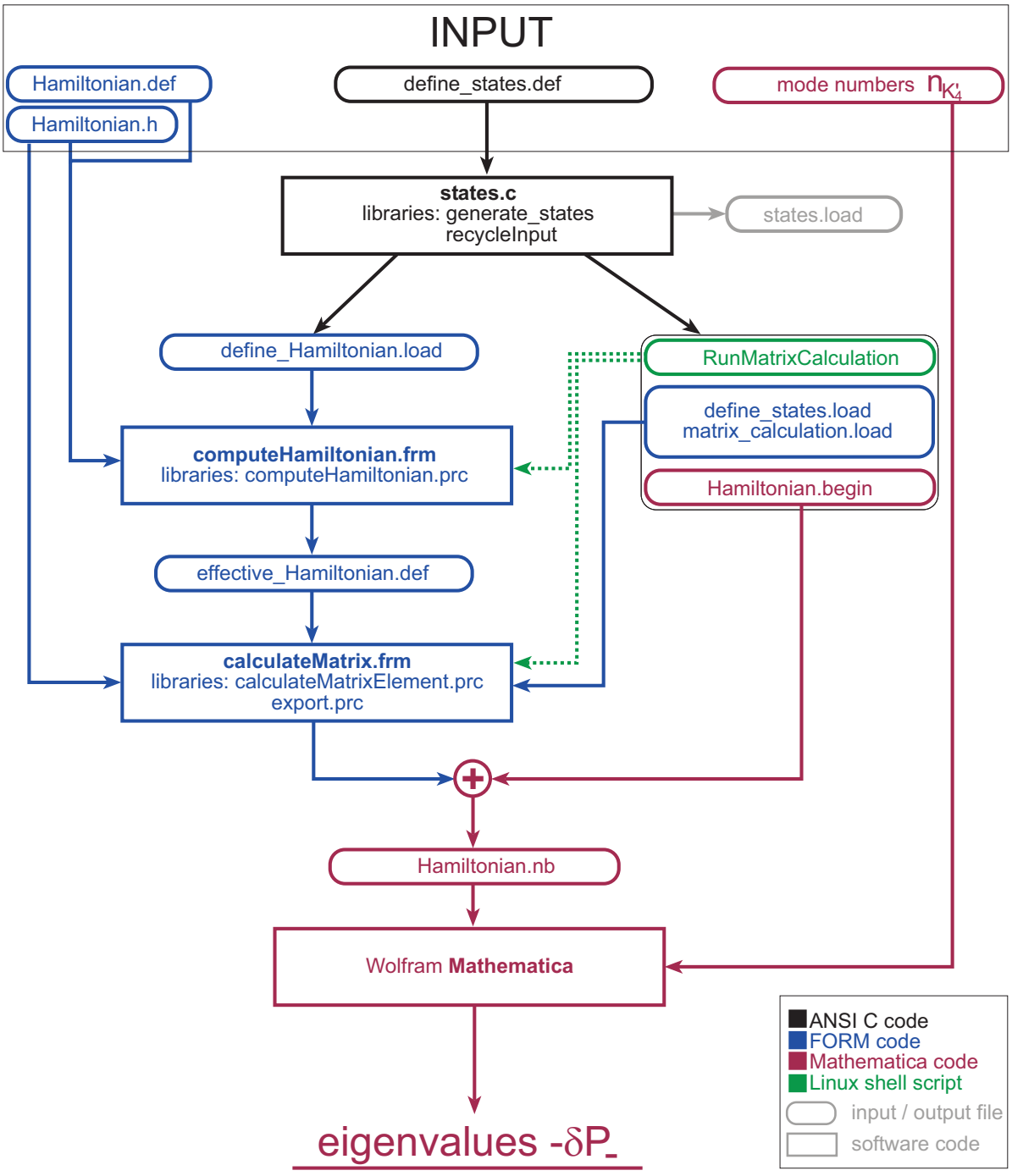


Figure 1: Software layout of the ABAKUS-System

Before the key algorithms for solving the various problems are described, we will sketch the workflow of ABAKUS, which is pictured in figure 1. The ovals represent text files serving as input or output while the boxes stand for software source code.

Programming languages

For the purpose of generating all mixing states, the programming language ANSI C has been chosen, because, due to the pointer concept and the array data structure, it provides highly efficient tools for manipulating composite data structures.

$\mathcal{H}_4$ includes non-commuting objects like matrices and grassmann valued operators. Furthermore great many terms have to be manipulated in every step of the calculation (the fully expanded Hamiltonian $\mathcal{H}_4$ consists of almost 8200 different terms). Thus, FORM has been chosen as an appropriate tool for computing $\mathcal{H}_{\text{eff}}$ and its matrix representation.

General remarks on the software layout

FORM is a script language which is processed by the FORM interpreter at runtime, but also the ANSI C code is not included in a compiled, machine executable form. In fact the ABAKUS-software compiles the ANSI C code everytime the software is started. This approach has two important advantages:

- The user specifies the input as ANSI C preprocessor variables. When the source code is compiled afterwards the preprocessor replaces the input by the user specified values. At the moment when the code is actually been translated to a machine executable form the compiler knows about all the input and thus may substantially optimize the resulting executable program.

- Whole software features are skipped by the preprocessor if permitted by the structure of the input, leading to smaller, more efficient runtime code.

Input

The input consists of three files:

`define_states.def` This is the only file which has to be added by the user of ABAKUS, basically it contains the specification for the multiplicities $\{\nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1\}$ and some general commands. For details consult the software requirements specification in paragraph 4.2. The file is processed by the ANSI C preprocessor.

`Hamiltonian.def` This file is not supposed to be added by the user. It contains the definition of $\mathcal{H}_4$ as FORM code.

`Hamiltonian.h` This file is not supposed to be added by the user. It contains the FORM declarations of all objects needed to define $\mathcal{H}_4$ in `Hamiltonian.def`.

The numerical values of the mode numbers are not of interest during most of the computation. Running ABAKUS will result in a MATHEMATICA file, which contains the mode numbers as unspecified analytical objects.

`mode numbers` $n_{K_4'}$ If numerical eigenvalues of $\mathcal{H}_4$ are to be computed, the user has to plug in numerical values for the mode numbers $n_1, n_2, \ldots$ at the top of the created MATHEMATICAL file.

Workflow

`states.c` ANSI C software, computing all potentially mixing states, creates the files:
`define_Hamiltonian.load`: contains informations about the generated states, needed for the computation of the effective Hamiltonian $\mathcal{H}_{\text{eff}}$;
`define_states.load`: this FORM file contains data of the generated states, needed for the computation of the matrix representation of $\mathcal{H}_{\text{eff}}$;
`matrix_calculation.load`: this FORM file contains adapted FORM code to compute the matrix representation of $\mathcal{H}_{\text{eff}}$;
`Hamiltonian.begin`: is a part of the final MATHEMATICA file;
`RunMatrixCalculation`: Linux shell script controlling the further computation.

`computeHamiltonian.frm` This FORM script computes the effective Hamiltonian based on the input provided by `Hamiltonian.def`, `Hamiltonian.h` and `define_Hamiltonian.load`. The result is stored in the FORM file `effective_Hamiltonian.def`.

`calculateMatrix.frm` is a FORM script for analytic computation of the matrix representation of $\mathcal{H}_{\text{eff}}$. The output is given in terms of MATHEMATICA commands, which are combined with `Hamiltonian.begin` to the final output file[8] `Hamiltonian.nb`.

`Mathematica` After loading the output file `Hamiltonian.nb` into MATHEMATICA one can compute the eigenvalues of $\mathcal{H}_{\text{eff}}$. If the eigenvalues are to be computed numerically one has to specify the values for the mode numbers.

Environment used for testing and computations

The computations have been performed on the hardware system specified below.

CPU	:	AMD Athlon$^{\text{tm}}$ 64 3200+
Memory	:	1GB
Operating system	:	Linux Debian Sarge, stable release

[8] `Hamiltonian.nb` is the default file name for the output file, but it might be changed by the user.

4.4.1 State generation

In this section the algorithm is presented which generates all possible states of a given charge $\{S_+, S_-, J_+, J_-\}$ carrying K_4' distinct modes with the multiplicities $\{\nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1\} = \boldsymbol{\nu}_{K_4'}$. As discussed in chapter 4.3, the number of states might grow exponentially with K_4'. Therefore it is important to generate the states as efficiently as possible.
The charge of an excitation is only depending on its flavor but not on the particular mode number. As an example consider states with $\boldsymbol{\nu}_2 = \{1, 1\}$ and charge $\{S_+, S_-, J_+, J_-\} = \{0, 0, 2, 0\}$. The appropriate input for that example is

```
#define ExcitationNumbers {1,1}
#define DefineCharges {0,0,2,0}
```
(4.13)

The two possible excitation patterns of appropriate charge, called sectors, are

$$\theta_2^+\theta_1^+|0\rangle \ , \qquad \alpha_2^+\alpha_1^+|0\rangle \ . \tag{4.14}$$

In the software context, a *sector* determines how many excitations every flavor carries, irrespective of the singe mode numbers. Consequently a sector specifies for every flavor $\mathcal{O} \in \{\theta_4, \theta_3, \ldots, \alpha_1\}$ the number of excitations $a_\mathcal{O}$

$$\text{sector} \mathrel{\widehat{=}} \{a_{\theta_4}, a_{\theta_3}, \ldots, a_{\alpha_1}\} \ . \tag{4.15}$$

Now let us assign explicit mode numbers m, n to the flavors, which leads to the set of all states with charge $\{0, 0, 2, 0\}$:

sector	$\{a_{\theta_2}=1, a_{\theta_1}=1, a_{\alpha_2}=0, a_{\alpha_1}=0\}$	$\{a_{\theta_2}=0, a_{\theta_1}=0, a_{\alpha_2}=1, a_{\alpha_1}=1\}$
states	$\theta^+_{2,m}\,\theta^+_{1,n}\|0\rangle \ , \quad \theta^+_{2,n}\,\theta^+_{1,m}\|0\rangle$	$\alpha^+_{2,m}\,\alpha^+_{1,n}\|0\rangle \ , \quad \alpha^+_{2,n}\,\alpha^+_{1,m}\|0\rangle$

Since the charge of a single excitation is independent from the mode number, the sector determines the charge of all states belonging to it. According to this description the state generation is split into two parts:

- Part I: calculation of all contributing sectors with charge $\{S_+, S_-, J_+, J_-\}$, i.e. the number of excitations $\{a_{\theta_4}, a_{\theta_3}, \ldots, a_{\alpha_1}\}$ for the 16 operators $\theta_4, \ldots, \alpha_1$.
- Part II: generation of all possible states for every sector of part I.

Part I: contributing sectors

The source code of the key function `genSectors`, calculating the contributing sectors, is given in table 2. It is a depth-first search algorithm recursively writing the number of excitations $a_\mathcal{O}$ for an operator $\mathcal{O}$ into an array `Konf` of 16 integer values. It starts with the total number of impurities `ImpLeft`$= \nu_{K_4'} + \nu_{K_4'-1} + \ldots + \nu_1$ which have to be distributed over the `Op`$=16$ creation in such a way, that the resulting excitation patterns yield the charge configuration $\{S_+, S_-, J_+, J_-\}$. In the beginning the charges $\{S_+, S_-, J_+, J_-\}$ are stored in an array `charges` and for every impurity which is assigned to an operator, the corresponding charge is subtracted from `charges`. Thus the array `charges` denotes the amount of charge, which has to be covered by the remaining unassigned impurities. In every recursion step there are $a_i \leq$ `ImpLeft` excitations assigned to one operator, which leaves the problem to distribute (`ImpLeft`$-a_i$) impurities over (`Op`-1) operators. According to table 1, each operator carries at most 1 unit of a specific charge. The recursive algorithm works as follows:

1. break condition 1: If a charge differs from the target value $S_\pm, J_\pm$ by more units than there are impurities left (encoded in the counter `ImpLeft`) to distribute, it is not possible to generate a sector of appropriate charge by distributing the remaining impurities.

2. break condition 2: If all excitations are set, it is not allowed to have any impurities left, which are not assigned to any mode, i.e. `ImpLeft`$\stackrel{!}{=} 0$.

3. terminating condition: If all impurities are distributed over the operators, i.e. `ImpLeft`$= 0$, a valid configuration with charge $\{S_+, S_-, J_+, J_-\}$ is found[9]. The configuration `Konf` is appended to the list of computed sectors by the function `append_found_Sector(Konf)`.

4. recursion (in all other cases): For any value $A = 0, \ldots,$`ImpLeft`, A modes are assigned to the current operator and the uncovered charges `charges` are calculated. The problem of distributing (`ImpLeft`$-A$) impurities over (`Op`-1) operators remains, which is easily computed by applying the algorithm recursively.

```
1  /* function genSectors(int Op, int ImpLeft)
2  * DESCRIPTION:
3  * the 16 possible creation operators are denoted by the value of Op=15,...,0, where  15~\theta_4, ..., 0~\alpha_1.
4  * This function recursively calculates the sectors with appropriate charge, where it is meant to start with
5  * Op = (NumberOfOperators-1) and decreases the operator-index in each step                                    */
6  void genSectors(int Op, int ImpLeft){
7   int c,A;
8   /* Abbruchbedingungen 1: jeder Operator kann maximal 1 Einheit einer speziellen Ladung tragen, falls fuer eine Ladung gilt:
9                           (Abweichung vom Sollwert) > #(noch zu verteilenden Operatoren) = ImpLeft => dann Abbruch      */
10  for(c=0; c < NumberOfCharges; c++) if(charges[c]*charges[c] > ImpLeft*ImpLeft) return;
11  if (Op == -1) {                                          /* Multiplizitaet aller Operatoren gesetzt              */
12    if (ImpLeft > 0) return;                               /* Abbruchbedingungen 2: Zustand hat nicht genug impurities */
13    append_found_Sector(Konf);                             /* ansonsten: gueltigen Sector gefunden                 */
14    return;
15  }
16  if (ImpLeft == 0) {                                      /* Zustand hat richtige Anzahl von Impurities           */
17    for(c=Op; 0<= c; c--) Konf[c] = 0;                     /* gemaess Abbruchbed. 1 & 2 hat Zustand dann auch geforderte */
18    append_found_Sector(Konf);                             /*         Ladungen => Sector gefunden                  */
19    return;
20  }
21  for(A=0; A<= ImpLeft; A++) {                                    /* Rekursion                                    */
22    Konf[Op] = A;
23    for(c=0; c < NumberOfCharges; c++) charges[c] -= OpCharge[Op*4 + c]*A;    /* berechnen der Ladungen der aktuellen Konf. */
24    genSectors(Op-1,ImpLeft-A);                                   /* rekursiver Abstieg                           */
25    for(c=0; c < NumberOfCharges; c++) charges[c] += OpCharge[Op*4 + c]*A;    /* restore der Werte vor rekursivem Abstieg */
26  }
27 } /* end of function genSectors() */
```

Table 2: ANSI C source code of `function genSectors`

If requested by the user, the calculated sectors are stored in a file. Operators which do not carry excitations in all generated sectors are skipped in the output as well as in the internal representation of the sectors. The software output describing the two sectors of (4.14) is:

```
1   // MESSAGE: order of Operators:
2   //          | Theta2 | Theta1 | Alpha2 | Alpha1 |
3   // MESSAGE: 2 mixing sectors found:
4                | 1 | 1 | 0 | 0 |
5                | 0 | 0 | 1 | 1 |
```

Table 3: Sample output for example (4.14)

For further processing all operators, which carry excitations in any of the sectors, are written

[9] If the algorithm proceeded to step 3, the break condition 1 did not apply. In case `ImpLeft`$= 0$, this ensures that the charge of the generated sector coincides with the target values $\{S_+, S_-, J_+, J_-\}$.

in a set Ω (line 2 in table 3), and for each computed sector the number of excitations are stored for the relevant operators (line 4, 5 in table 3):

$$\begin{array}{ll} \text{set of operators carrying excitations:} & \Omega := \{\mathcal{O}_\varrho, \ldots, \mathcal{O}_1\} \subseteq \{\theta_4, \theta_3, \ldots, \alpha_1\}\ ; \qquad \varrho \leq 16\ , \\ \text{excitation pattern for a sector:} & \{a_\varrho, \ldots, a_1\}\ , \end{array}$$

where the elements of Ω are ordered with respect to the normal order prescription (4.5).

Part II: generating states for a given sector

In order to explain the algorithm we start with some remarks on the internal representation of states. Commensurate to the operators given in $\Omega := \{\mathcal{O}_\varrho, \ldots, \mathcal{O}_1\}$ and the total number of modes K_4' a state is given by[10]

$$\left|\nu_{K_4'}^{(\mathcal{O}_\varrho)}, \nu_{K_4'-1}^{(\mathcal{O}_\varrho)}, \ldots, \nu_1^{(\mathcal{O}_\varrho)}\right\rangle_{\mathcal{O}_\varrho} \ldots \left|\nu_{K_4'}^{(\mathcal{O}_1)}, \nu_{K_4'-1}^{(\mathcal{O}_1)}, \ldots, \nu_1^{(\mathcal{O}_1)}\right\rangle_{\mathcal{O}_1}\ , \tag{4.16}$$

which is conveniently stored into an array of $\varrho\, K_4'$ integer numbers.

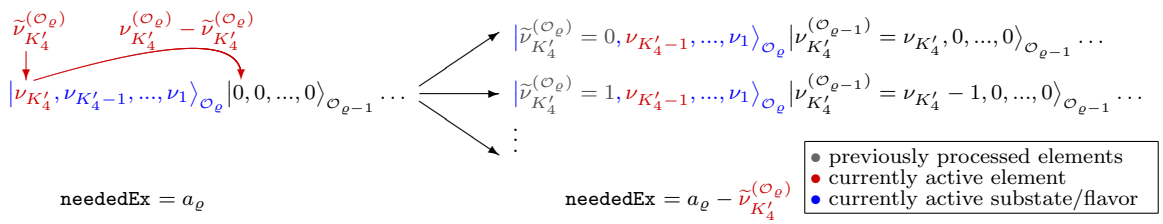
For each of the previously generated sectors all physical states have to be computed. To generate states of a particular sector $\{a_\varrho, \ldots, a_1\}$ the ν_i excitations of every mode need to be distributed over the operator flavors $\mathcal{O}_{i=\varrho,\ldots,1}$ in such a way that the

$$\begin{array}{ll} \text{result} & \left|\widetilde{\nu}_{K_4'}^{(\mathcal{O}_\varrho)}, \widetilde{\nu}_{K_4'-1}^{(\mathcal{O}_\varrho)}, \ldots, \widetilde{\nu}_1^{(\mathcal{O}_\varrho)}\right\rangle_{\mathcal{O}_\varrho} \left|\widetilde{\nu}_{K_4'}^{(\mathcal{O}_{\varrho-1})}, \ldots, \widetilde{\nu}_1^{(\mathcal{O}_{\varrho-1})}\right\rangle_{\mathcal{O}_{\varrho-1}} \ldots \left|\widetilde{\nu}_{K_4'}^{(\mathcal{O}_1)}, \ldots, \widetilde{\nu}_1^{(\mathcal{O}_1)}\right\rangle_{\mathcal{O}_1} \\ \text{satisfies} & \widetilde{\nu}_{K_4'}^{(\mathcal{O}_i)} + \ldots + \widetilde{\nu}_1^{(\mathcal{O}_i)} = a_i \\ & \text{with} \quad \widetilde{\nu}_i^{(\mathcal{O}_\varrho)} + \widetilde{\nu}_i^{(\mathcal{O}_{\varrho-1})} + \ldots + \widetilde{\nu}_i^{(\mathcal{O}_1)} = \nu_i\ . \end{array}$$

The quantities with tilde denote values that are set by the algorithm and are not changed anymore. This is done by basically performing a recursive depth-first search. The algorithm starts with filling the given multiplicities $\nu_{K_4'}, \nu_{K_4'-1}, \ldots, \nu_1$ into the first substate.

$$\text{initialization:} \quad \left|\nu_{K_4'}^{(\mathcal{O}_\varrho)} = \nu_{K_4'}\ ,\ \nu_{K_4'-1}^{(\mathcal{O}_\varrho)} = \nu_{K_4'-1}\ ,\ \ldots,\ \nu_1^{(\mathcal{O}_\varrho)} = \nu_1\right\rangle_{\mathcal{O}_\varrho} |0, 0, \ldots, 0\rangle_{\mathcal{O}_{\varrho-1}} \ldots |0, 0, \ldots, 0\rangle_{\mathcal{O}_1}$$

Then it runs through multiplicities of all substates (or operator flavors $\mathcal{O}_i$) step by step. During each cycle a multiplicity $\nu_k^{(\mathcal{O}_i)}$ is processed: $\widetilde{\nu}_k^{(\mathcal{O}_i)} \leq \nu_k^{(\mathcal{O}_i)}$ modes are assigned to the operator $\mathcal{O}_i$ and the remaining $\nu_k^{(\mathcal{O}_i)} - \widetilde{\nu}_k^{(\mathcal{O}_i)}$ modes are moved to the next subsector for further distribution. The variable `neededEx` counts the number of modes that are missing to achieve the obliged value of a_i assigned modes for the current operator $\mathcal{O}_i$. The first step will clarify the basic principle:



[10] In the present notation $G_\mathcal{O}$ ($\mathcal{O} \in \{\theta, \eta, \beta, \alpha\}$) of (3.46),(3.48) is omitted, since the software adds this counter at the very end of the state generation process.

A general recursion step works out the multiplicity $\nu_k^{(\mathcal{O}_i)}$ belonging to the operator $\mathcal{O}_i$:

$$\ldots|\ldots\rangle_{\mathcal{O}_{i+1}}\,|\tilde{\nu}_{K_4'}^{(\mathcal{O}_i)},\ldots,\tilde{\nu}_{k+1}^{(\mathcal{O}_i)},\nu_k^{(\mathcal{O}_i)},\nu_{k-1}^{(\mathcal{O}_i)},\ldots,\nu_1^{(\mathcal{O}_i)}\rangle_{\mathcal{O}_i}\,|\nu_{K_4'}^{(\mathcal{O}_{i-1})},\ldots,\nu_{k+1}^{(\mathcal{O}_{i-1})},0,0,\ldots,0\rangle_{\mathcal{O}_{i-1}}\ldots$$

$$\ldots|\ldots\rangle_{\mathcal{O}_{i+1}}\,|\tilde{\nu}_{K_4'}^{(\mathcal{O}_i)},\ldots,\tilde{\nu}_{k+1}^{(\mathcal{O}_i)},\tilde{\nu}_k^{(\mathcal{O}_i)},\nu_{k-1}^{(\mathcal{O}_i)},\ldots,\nu_1^{(\mathcal{O}_i)}\rangle_{\mathcal{O}_i}\,|\nu_{K_4'}^{(\mathcal{O}_{i-1})},\ldots,\nu_{k+1}^{(\mathcal{O}_{i-1})},\nu_k^{(\mathcal{O}_i)}-\tilde{\nu}_k^{(\mathcal{O}_i)},0,\ldots,0\rangle_{\mathcal{O}_{i-1}}\ldots$$

There are several break conditions:

1. break condition 0: backtracking in search tree if remaining operators $\mathcal{O}_i,\ldots,\mathcal{O}_1$ are fermionic and $i<\nu_k^{(\mathcal{O}_i)}$, because each of the fermionic operators $\mathcal{O}_i,\ldots,\mathcal{O}_1$ can carry at most one of the $\nu_k^{(\mathcal{O}_i)}$ modes.

2. break condition 1: backtracking in search tree if $\nu_k^{(\mathcal{O}_i)}+\nu_{k-1}^{(\mathcal{O}_i)}+\nu_1^{(\mathcal{O}_i)}<\texttt{neededEx}$, because even by assigning all $\nu_k^{(\mathcal{O}_i)}+\ldots+\nu_1^{(\mathcal{O}_i)}$ modes left to $\mathcal{O}_i$, one will not reach the needed value of a_i impurities.

3. break condition 2: $i=1$ (last operator) and $\mathcal{O}_1$ is bosonic: valid state found;
   $i=1$ and $\mathcal{O}_1$ is fermionic: valid state only if $\nu_k^{(\mathcal{O}_i)},\ldots,\nu_1^{(\mathcal{O}_i)}\leq 1$.
   The valid states are saved into a list.

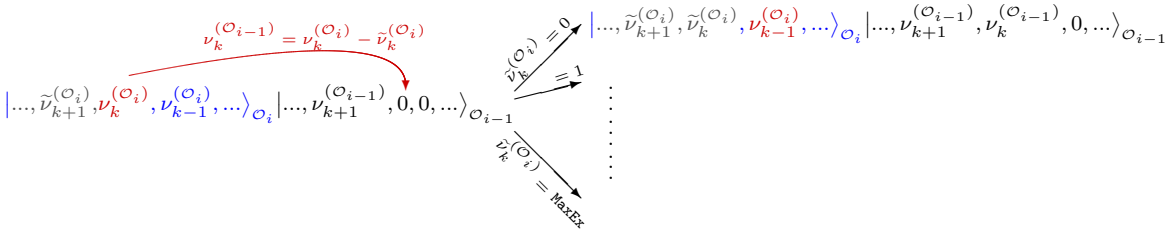
4. recursion possibility 1: $\texttt{neededEx}=0$: all $\nu_k^{(\mathcal{O}_i)},\ldots,\nu_1^{(\mathcal{O}_i)}$ are moved into the next sector, because the needed quantity of a_i impurities has already been assigned to the subsector:

   $$|\underbrace{\tilde{\nu}_{K_4'}^{(\mathcal{O}_i)},\ldots,\tilde{\nu}_{k+1}^{(\mathcal{O}_i)}}_{=a_i},\underbrace{\nu_k^{(\mathcal{O}_i)},\nu_{k-1}^{(\mathcal{O}_i)},\ldots,\nu_1^{(\mathcal{O}_i)}}\rangle_{\mathcal{O}_i}\,|\nu_{K_4'}^{(\mathcal{O}_{i-1})},\ldots,\nu_{k+1}^{(\mathcal{O}_{i-1})},0,0,\ldots,0\rangle_{\mathcal{O}_{i-1}}\ldots$$

   Than the next substate is considered:

   $$\longrightarrow|\tilde{\nu}_{K_4'}^{(\mathcal{O}_i)},\ldots,\tilde{\nu}_{k+1}^{(\mathcal{O}_i)},0,\ldots,0\rangle_{\mathcal{O}_i}\,|\nu_{K_4'}^{(\mathcal{O}_{i-1})},\ldots,\nu_k^{(\mathcal{O}_{i-1})}=\nu_k^{(\mathcal{O}_i)},\ldots,\nu_1^{(\mathcal{O}_{i-1})}=\nu_1^{(\mathcal{O}_i)}\rangle_{\mathcal{O}_{i-1}}|0,\ldots,0\rangle_{\mathcal{O}_{i-2}}\ldots$$

5. recursion possibility 2: if $\mathcal{O}_i$ is bosonic: $\texttt{MaxEx}=\min\{\nu_k^{(\mathcal{O}_i)},\texttt{neededEx}\}$;
   if $\mathcal{O}_i$ is fermionic: $\texttt{MaxEx}=\min\{\nu_k^{(\mathcal{O}_i)},\texttt{neededEx},1\}$
   for every value $\tilde{\nu}_k^{(\mathcal{O}_i)}=\texttt{MaxEx},\texttt{MaxEx}-1,\ldots$ all possibe states are computed recursively:



The source code of the described algorithm `generate_Block_States`, calculating all states contributing to a given sectors, is displayed in table 4.

The discussion of the algorithm is concluded with some remarks concerning runtime: It is important to process the fermionoic substates first, because in this case the algorithm does *not* generate any defective substates.
In every recursion step the algorithm fixes one of the $\varrho K_4'$ multiplicities, i.e. the generation of one state takes $\mathcal{O}(\varrho K_4')$. Also memory requirement for one state is $\mathcal{O}(\varrho K_4')$. Thus the presented algorithm is maximally efficient since the time to generate a state is of the same magnitude as the time to write it into the memory.

```
/* rekursive Funktion generate_Block_States
 * - durchlaeuft in jeder Instanz eine der verschiedenen Moden eines der Operatoren
 * - Anregungszahl der Mode erhaelt Werte 0,...,ExNumber, restliche Anregungen werden zur weiteren Verteilung
 *   in naechsten Operatorsektor verschoben (nextsubstate)
 * - vor Backtracking wird unspruenglicher Anregungswert im aktuellen Sektor (ExNumber) restored
 *   und Anregungswert des naechsten Operatorsektors wieder auf null gesetzt
 * Input:       Mode            - Pointer in Array State auf aktuell zu bearbeitende Mode
 *              neededEx        - Anzahl Anregungen, die noch auf aktuellen Operator zu verteilen sind
 *              ExLeft          - Anzahl aller Anregungen im aktuellen Subsector, die noch verteilt werden koennen
 *              MNextSec        - Anzahl Moden, die in den naechsten Zustandsteil verschoben wurden
 *              ModesLeft       - Anzahl noch nicht abgearbeiteter Moden in aktuellem Operatorsektor (ModesLeft = NumMo,...,1)
 *                                aktuelle Mode wird in ModesLeft mitgezaehlt
 *              OpLeft          - Anzahl noch nicht abgearbeiteter Operatoren (OpLeft = NumOp,...,1)
 *                                aktueller Operatorsektor wird in OpLeft mitgezaehlt
 */
void generate_Block_States(int* Mode, int neededEx, int ExLeft, int MNextSec, int ModesLeft, int OpLeft){
  int *nextsubstate;   /* Pointer in Array State auf Teil des als naechsten aktuell zu bearbeitenden Operators                  */
  int ExNumber;        /* Anzahl der Anregungen der entsprechenden Mode dieser Rekursionsinstanz                               */
  int MaxEx = 1;       /* Anzahl Anregungen (der aktuellen Mode) die (aktueller) Operator tragen kann - default =1 (fermionisch) */
  int MinEx;           /* minimale Anzahl Anregungen (der aktuellen Mode), die die (aktueller) Operator tragen                 */
  int i;
  nextsubstate = Mode + NumMo;
  ExNumber = *Mode;    /* aktuelle Mode */

  /* Abbruchbedingungen 0: nur fermionische Operatoren */       /* jeder der OpLeft fermion. Op. kann maximal eine Anregung     */
  if ((NumOp == NumFermOp) && (OpLeft < ExNumber)) return;      /* der aktuellen Mode tragen - sonst verschwindet Zustand (=0)  */

  /* Abbruchbedingungen 1: nicht mehr genuegend Anregungen uebrig im aktuellen Subsector */
  if(neededEx > ExLeft) return;                                 /* kein zulaessiger Zustand mehr generierbar */

  /* Abbruchbedingungen 2: letztes Operatorsegment erreicht */
  if(OpLeft == 1) {                                             /* aktueller Operatorsektor wird in OpLeft mitgezaehlt          */
     if (NumOp == NumFermOp)                  /* ueberpruefen des letzten Sectors auf Mehrfach-Anregung fermionischer Moden   */
       for(i=0; i< NumMo; i++) if (Mode[i]> 1) return;          /* falls nur fermionische Operatoren beteiligt sind             */
     AgZ++;
     writeStateToFile(State,NumOp,NumMo,Output);
     return;
  } /* Ende Abbruchbed. 2 */

  /* Rekursiver Abstieg 1: Subsector hat geforderte Anzahl an Anregungen (es gilt nun: nicht letztes Operatorsegment)*/
  if(neededEx == 0) {
     if (ModesLeft > 0) {                     /* und noch Moden unbetrachtet                                                  */
        for(i = 0; i < ModesLeft; i++){       /* verschiebe verbleibende Moden-Anregungen in naechsten Operator-Subsector     */
           nextsubstate[i]=Mode[i];
           ExLeft -= Mode[i];
           Mode[i]=0;
           MNextSec += nextsubstate[i];
        }
     }
     /* rekursiver Abstieg in naechsten Operator-subsector*/
     generate_Block_States(State + (NumOp-OpLeft+1)*NumMo,Block[NumOp-OpLeft+1],MNextSec,0,NumMo,OpLeft-1);
     if (ModesLeft > 0) {                                                 /* restore urspruenglichen Anregungswert (ExNumber)     */
        for(i = 0; i < ModesLeft; i++){       /* verschiebe verbleibende Moden-Anregungen in naechsten Operator-Subsector     */
           Mode[i]=nextsubstate[i];
           nextsubstate[i]=0;
        }
     }
     return;
  } /* Ende Rekursiver Abstieg 1 */
  /* es gilt nun: (nicht letztes Operatorsegment) && (ExLeft >= neededEx) && (ModesLeft > 0)                    */
  /* Rekursiver Abstieg 2 */                          /* Index des aktuellen Operators : (NumOp-OpLeft)       */
  MaxEx = (ExNumber < neededEx)? ExNumber : neededEx;  /* maximale Anregung = min(ExNumber,neededEx)           */
  MinEx = neededEx + ExNumber - ExLeft;                /* minimale Anregung = max(0, neededEx+ExNumber-ExLeft) */
  MinEx = (MinEx > 0)? MinEx : 0;
  if ((NumOp-OpLeft)<NumFermOp)                        /* falls aktueller Operator fermionisch ist             */
       MaxEx = (MaxEx<1)? MaxEx : 1;                   /* maximal moegliche Anregung, min(MaxEx,1)             */
  /* rekursiver Abstieg: auf naechste Mode des gleichen Operators nur falls (neededEx > 0) und (ModesLeft > 0)  */
  for(i=MaxEx; i>=0; i--) {
    *Mode = i;
    *nextsubstate = ExNumber -i;
    generate_Block_States(Mode+1, neededEx-i, ExLeft-ExNumber, MNextSec+ExNumber-i, ModesLeft-1, OpLeft);
    *Mode = ExNumber;                                  /* restore urspruenglicher Anregungswert (ExNumber)     */
    *nextsubstate = 0;
  } /* end of for(i=neededEx; i>=0; i--) */
  return;
} /* end of function generate_Block_States */
```

Table 4: Ansi C source code of `function generate_Block_States`

4.4.2 computing the effective Hamiltonian operator

Given the Hamiltonian $\mathcal{H}_4$ in the form of (3.40) and the general structure of the states in terms of the composing creation operators, the task is to compute an effective Hamiltonian $\mathcal{H}_{\text{eff}}$. Consider two states $|\Psi_1\rangle$, $|\Psi_2\rangle$ that do not contain a specific operator flavor $\mathcal{O} \in \{\theta_4, \ldots, \alpha_1\}$. Due to normal ordering, all terms containing the operator $\mathcal{O}$ will drop out of $\mathcal{H}_4$, since $|\Psi_1\rangle$, $|\Psi_2\rangle$ act as the vacuum of $\mathcal{O}$. Given a set of operators $\Omega \subseteq \{\theta_4, \ldots, \alpha_1\}$, the effective Hamiltonian $\mathcal{H}_{\text{eff}}$ is obtained by setting $\mathcal{O} = 0$ for all $\mathcal{O} \notin \Omega$.
Deriving $\mathcal{H}_{\text{eff}}$ is rather straight forward but since the calculation always starts with the full Hamiltonian (3.40) it is important to keep the number of terms small during computation.

4.4.3 input form of the Hamiltonian

The Hamiltonian $\mathcal{H}_4$ is given in `Hamiltonian.def` in terms of FORM statements. The constant factor $\widetilde{\lambda}/(4P_+)$ is not included in the calculation. Einstein's sum convention is applied using the Euclidean metric. The trace is implicit and is taken only for terms, that include matrices.

```
1  Local [Hbb] = Yprime(5-IndexA)*Yprime(IndexA)*Z(5-IndexB)*Z(IndexB)
2              - Y(5-IndexA)*Y(IndexA)*Zprime(5-IndexB)*Zprime(IndexB)
3              + Zprime(5-IndexA)*Zprime(IndexA)*Z(5-IndexB)*Z(IndexB)
4              - Yprime(5-IndexA)*Yprime(IndexA)*Y(5-IndexB)*Y(IndexB);
5
6  Local [Hbf-Part1] = (Z(5-IndexA)*Z(IndexA) - Y(5-IndexA)*Y(IndexA)) * (Etadegprime*Etaprime + Thetadegprime*Thetaprime);
7  Local [Hbf-Part2] = -Zprime(IndexM)*Z(IndexN)*(Gamma(IndexM)*Gamma(IndexN)-Gamma(IndexN)*Gamma(IndexM))
8                      *(MatrixPplus*(Eta*Etadegprime - Etaprime*Etadeg) -MatrixPminus*(Thetadeg*Thetaprime - Thetadegprime*Theta));
9  Local [Hbf-Part3] =  Yprime(IndexM)*Y(IndexN)*(Gamma(IndexM)*Gamma(IndexN)-Gamma(IndexN)*Gamma(IndexM))
10                     *(-MatrixPminus*(Etadeg*Etaprime - Etadegprime*Eta) +MatrixPplus*(Theta*Thetadegprime - Thetaprime*Thetadeg));
11 Local [Hbf-Part4] = -KomplexI*Kappa/Sqrt(Lamdatilde) * (Zprime(IndexN)*PZ(IndexM)+Z(IndexN)*PZprime(IndexM))
12                     *(Gamma(IndexN)*Gamma(IndexM)-Gamma(IndexM)*Gamma(IndexN))
13                     *(MatrixPplus*(Etadeg*Etadeg + Eta*Eta) +MatrixPminus*(Thetadeg*Thetadeg + Theta*Theta));
14 Local [Hbf-Part5] =  KomplexI*Kappa/Sqrt(Lamdatilde) * (Yprime(IndexN)*PY(IndexM)+Y(IndexN)*PYprime(IndexM))
15                     *(Gamma(IndexN)*Gamma(IndexM)-Gamma(IndexM)*Gamma(IndexN))
16                     *(MatrixPminus*(Etadeg*Etadeg + Eta*Eta) +MatrixPplus*(Thetadeg*Thetadeg + Theta*Theta));
17 Local [Hbf-Part6] =  8*KomplexI* Z(IndexM)*Y(IndexN)* ( - MatrixPminus*Gamma(IndexM)*Etaprime*Gamma(IndexN)*Thetaprime
18                                                        + MatrixPplus*Gamma(IndexM)*Thetadegprime*Gamma(IndexN)*Etadegprime );
19
20 Local [Hff-Part1] = - MatrixSigma * (  Etadegprime*Eta*Etadegprime*Eta + Etadeg*Etaprime*Etadeg*Etaprime
21                                       + Etadegprime*Etadeg*Etadegprime*Etadeg + Etaprime*Eta*Etaprime*Eta);
22 Local [Hff-Part2] =   MatrixSigma * (  Thetadegprime*Theta*Thetadegprime*Theta + Thetadeg*Thetaprime*Thetadeg*Thetaprime
23                                       + Thetadegprime*Thetadeg*Thetadegprime*Thetadeg + Thetaprime*Theta*Thetaprime*Theta);
```

Table 5: file `Hamiltonian.def` $-$ $\mathcal{H}_4$ in FORM source code

4.4.4 algorithmic calculation of $\mathcal{H}_{\text{eff}}$

In the following a formal description of the algorithm computing $\mathcal{H}_{\text{eff}}$ is given.

1. According to (3.31), $\{Y_c, P_c^y\}$ will expand to the operator class $\{\alpha_4^{\pm}, \ldots, \alpha_1^{\pm}\}$. If the set $\{\alpha_4^{\pm}, \ldots, \alpha_1^{\pm}\}$ is not included in the set Ω ($\Omega \mathrel{\widehat{=}}$ operators composing the mixing states) the algorithm sets $Y = 0,\ P_c^y = 0$. Appropriate action is done for $\{Z, P^z\}, \{\eta\}, \{\theta\}$ in case the corresponding operator classes are not used.

2. For the products of bosonic fields Z_a, P_a^z, Y_a, P_a^y Einstein's sum convention is applied, so for instance insert $Z_{5-c}Z_c = Z_4Z_1 + \ldots + Z_1Z_4$.

3. The fermionic fields θ, η as well as their derivatives and their adjoint fields are decomposed into the matrix part Γ_c and the scalar part $\widetilde{\theta}_c^{\pm}, \widetilde{\eta}_c^{\pm}$ according to (4.1).
   For all colors $c = 4, \ldots, 1$ the fermionic fields $\widetilde{\theta}_c^{\pm},\ \widetilde{\theta}\,'^{\pm}_c$ are dropped in in case $\theta_c \notin \Omega$. Appropriate action is done for η_c.

4. Order Γ_c-matrices without generating new terms, i.e. use $\{\Gamma_1, \Gamma_2\} = \{\Gamma_1, \Gamma_3\} = 0$ and $\{\Gamma_2, \Gamma_4\} = \{\Gamma_3, \Gamma_4\} = 0$. Whenever possible, apply $\Gamma_c \Gamma_c = 0$ (c fixed).
   Substitute the representation of $\mathcal{P}_\pm$, Σ, Γ_c in terms of Dirac matrices γ_i according to (A.7), (A.8).
   Compute trace, using the built-in trace function of FORM.

5. Rewrite Z_c, P_c^z, Y_c, P_c^y and its derivatives in terms of $\widetilde{\beta}_c^\pm$, $\widetilde{\alpha}_c^\pm$ using (4.3).
   In order to reduce the later costs of normal ordering, the Hamiltonian is pre-ordered in this step with respect to the sequence

$$\widetilde{\theta}^+ \; \widetilde{\theta}^- \; \widetilde{\eta}^+ \; \widetilde{\eta}^- \; \widetilde{\beta}^+ \; \widetilde{\beta}^- \; \widetilde{\alpha}^+ \; \widetilde{\alpha}^- \; ,$$

   but the ordering does not incorporate the colors c of $\widetilde{\alpha}_c^\pm$, $\widetilde{\beta}_c^\pm$, $\widetilde{\eta}_c^\pm$, $\widetilde{\theta}_c^\pm$.
   For all colors $c = 4, \ldots, 1$ drop bosonic fields $\widetilde{\alpha}_c^\pm$, $\widetilde{\alpha}\,'^\pm_c$ in case $\alpha_c \notin \Omega$. Appropriate action is done for β_c.
   Substitute explicit mode expansion for $\widetilde{\alpha}^\pm$, $\widetilde{\beta}^\pm$, $\widetilde{\eta}^\pm$, $\widetilde{\theta}^\pm$ using (4.2), (4.4).

6. Normal order Hamiltonian with respect to (4.5) and create an individual FORM-expression for every unique operator product:
   - $\mathcal{H}_{\text{eff}}$ contains only products of four operators: $\mathcal{O}_{1,a}\,\mathcal{O}_{2,b}\,\mathcal{O}_{3,d}\,\mathcal{O}_{4,l}$. Here $\mathcal{O}_i \in \{\theta_c^\pm, \eta_c^\pm, \beta_c^\pm, \alpha_c^\pm\}$ denotes the operator including the flavour index $c = 4, \ldots, 1$, while the indices a, b, d, l stand for mode numbers. Due to normal ordering the index structure may differ between the operator products, so it has to be changed to a uniform notation, in order that FORM realizes equivalent terms. This is done by applying

$$\mathcal{O}_{1,a}\;\mathcal{O}_{2,b}\,\mathcal{O}_{3,d}\,\mathcal{O}_{4,l} = \;\delta_{a,m_1}\delta_{b,m_2}\delta_{d,m_3}\delta_{l,m_4}\;\mathcal{O}_{1,m_1}\mathcal{O}_{2,m_2}\mathcal{O}_{3,m_3}\mathcal{O}_{4,m_4}\;.$$

   - All summands with exactly equal operators are combined into one term by factoring out the operators; in the simplest case

$$\begin{aligned} &F_1(m_1,...,m_4,\widetilde{\lambda})\;\mathcal{O}_{1,m_1}\mathcal{O}_{2,m_2}\mathcal{O}_{3,m_3}\mathcal{O}_{4,m_4} + F_2(m_1,...,m_4,\widetilde{\lambda})\;\mathcal{O}_{1,m_1}\mathcal{O}_{2,m_2}\mathcal{O}_{3,m_3}\mathcal{O}_{4,m_4} \\ &\quad = \big[\underbrace{F_1(m_1,...,m_4,\widetilde{\lambda}) + F_2(m_1,...,m_4,\widetilde{\lambda})}_{\texttt{HamiltonianFunctionPart}}\big] \cdot \underbrace{\mathcal{O}_{1,m_1}\mathcal{O}_{2,m_2}\mathcal{O}_{3,m_3}\mathcal{O}_{4,m_4}}_{\texttt{HamiltonianOperatorPart}}\;. \end{aligned}$$

     The constituents are written into individual FORM-expressions:
     the operators into $\texttt{HamiltonianOperatorPart}_k$ and the remaining function, depending only on the mode numbers $m_1, \ldots, m_4$ and $\widetilde{\lambda}$, is cast into the FORM-expression $\texttt{HamiltonianFunctionPart}_k$, i.e. the Hamiltonian is decomposed into

$$\mathcal{H}_{\text{eff}} = \sum_k \texttt{HamiltonianFunctionPart}_k \cdot \texttt{HamiltonianOperatorPart}_k\;.$$

     The generated expressions $\texttt{HamiltonianFunctionPart}_\text{i}$ and $\texttt{HamiltonianOperatorPart}_\text{i}$ are stored into the file `effective_Hamiltonian.def`.

In step 5 and 6 operators are ordered. The most efficient way to order functions in FORM is to define them in the appropriate order. In that case FORM orders commuting functions by default where non-commuting objects can be ordered using the `id disorder` command. Therefore the stage-generation algorithm creates the file `define_states.load`. It contains FORM-commands defining all operators $\mathcal{O}^+, \mathcal{O}^-$ for $\mathcal{O} \in \Omega$.

Two properties of the presented algorithm are especially important for runtime:

- By using (4.1), the software computes the trace independently from the explicit mode expansions (4.2) and (4.4). In this way the mode decomposition stays hidden through a large part of the calculation (up to the very end of step 5) and all terms are solely products of four fields (besides a factor of $\kappa\sqrt{\widetilde{\lambda}}$). Inserting the full mode decomposition (3.31),(3.35) at step 3 instead would result in two times longer terms and thus substantially slow down pattern matching and therewith the speed of the program.

- The separation of the operators from the prefactors, performed in step 6, is most important for the subsequent calculation of the matrix representation of $\mathcal{H}_{\text{eff}}$. Computations show that in average 10 terms have identical products of creation and annihilation operators. Skipping the Hamiltonian segmentation of step 6, the software would have to calculate 10 times more expectation values to obtain the matrix representation of $\mathcal{H}_{\text{eff}}$.

4.4.5 Hamiltonian matrix representation and its eigenvalues

Based on the fact that the matrix grows quadratically with the number of states, its computation is unequivocally the most time-consuming problem. On the one hand a numerical calculation of the matrix representation from the outset is therefore expedient. On the other hand it is an immense benefit to compute energy corrections fully analytically and compare the results to analytic solutions of the Bethe equation. I have decided to develop a tool computing the matrix representation analytically and to add a numerical solution later on if required. For this purpose FORM was chosen as programming language to calculate the matrix $\mathcal{H}_{\text{eff}}$ while the result is exported into MATHEMATICA where the numerical or analytical eigenvalues can easily be computed.

The software component computing the matrix representation $\mathcal{H}_{\text{eff}}$ is the FORM program `calculateMatrix.frm`. It loads the file `effective_Hamiltonian.def` containing all terms of $\mathcal{H}_{\text{eff}}$ and calculates an individual matrix for each term. The FORM commands for computing the matrices are encoded in the file `matrix_calculation.load`, which has been generated by `states.c`.

To calculate a matrix element one has to compute an object of the form

$$\begin{aligned}
&\langle\widehat{\Psi}|\mathcal{O}^{\pm}_{1,m_1}\mathcal{O}^{\pm}_{2,m_2}\mathcal{O}^{\pm}_{3,m_3}\mathcal{O}^{\pm}_{4,m_4}|\Psi\rangle\ , && \mathcal{O}_1,\ldots,\mathcal{O}_4\in\Omega\subseteq\{\theta_4,\theta_3,\ldots,\alpha_1\}\ ,\\
\text{with}\quad &|\Psi\rangle = \big|G_{\mathcal{O}_\varrho};\ \nu^{(\mathcal{O}_\varrho)}_{K'_4},\nu^{(\mathcal{O}_\varrho)}_{K'_4-1},\ldots,\nu^{(\mathcal{O}_\varrho)}_{1}\big\rangle_{\mathcal{O}_\varrho}\ldots\big|G_{\mathcal{O}_1};\ \nu^{(\mathcal{O}_1)}_{K'_4},\nu^{(\mathcal{O}_1)}_{K'_4-1},\ldots,\nu^{(\mathcal{O}_1)}_{1}\big\rangle_{\mathcal{O}_1}\\
&\langle\widehat{\Psi}| = {}_{\mathcal{O}_1}\big\langle\widehat{G}_{\mathcal{O}_1};\ \widehat{\nu}^{(\mathcal{O}_1)}_{K'_4},\widehat{\nu}^{(\mathcal{O}_1)}_{K'_4-1},\ldots,\widehat{\nu}^{(\mathcal{O}_1)}_{1}\big|\ldots\ {}_{\mathcal{O}_\varrho}\big\langle\widehat{G}_{\mathcal{O}_\varrho};\ \widehat{\nu}^{(\mathcal{O}_\varrho)}_{K'_4},\widehat{\nu}^{(\mathcal{O}_\varrho)}_{K'_4-1},\ldots,\widehat{\nu}^{(\mathcal{O}_\varrho)}_{1}\big|\ .
\end{aligned}$$

Since in Ω only the operator flavors of the states are stored, the mode-number is represented by the second index of $\mathcal{O}^{\pm}_{i,m_1}$. For example an appropriate operator to $\alpha_3\in\Omega$ is given by $\alpha^{\pm}_{3,m}$, where m denotes a mode number.
As defined in paragraph 3.4, $G_{\mathcal{O}_i}$ denominates the number of Grassmann-valued modes in the appropriate substate:

$$G_{\mathcal{O}_i} := \begin{cases} \nu^{(\mathcal{O}_i)}_{K'_4}+\nu^{(\mathcal{O}_i)}_{K'_4-1}+\ldots+\nu^{(\mathcal{O}_i)}_{1} & \text{if } \mathcal{O}_i \text{ fermionic}\\ 0 & \text{if } \mathcal{O}_i \text{ bosonic}\end{cases} \tag{4.17}$$

In order to compute a matrix element, the creation and annihilation operators of $\mathcal{H}_{\text{eff}}$ have to be moved in front of the corresponding substate. Considering the following example, the importance of the Grassmann counter $G_{\mathcal{O}_i}$ becomes obvious:

$$F(m_4,m_3,m_2,m_1,\widetilde{\lambda})\,\theta^{+}_{2,m_4}\theta^{-}_{2,m_3}\;\theta^{+}_{1,m_2}\underbrace{\theta^{-}_{1,m_1}|G_{\theta_2}\,;\,\nu^{(\theta_2)}_{K'_4},\ldots,\nu^{(\theta_2)}_{1}\rangle_{\theta_2}}_{(-1)^{G_{\theta_2}}}|G_{\theta_1}\,;\,\nu^{(\theta_1)}_{K'_4},\ldots,\nu^{(\theta_1)}_{1}\rangle_{\theta_1}$$

Instead of permuting the grassmann valued operator θ^{-}_{1,m_1} with every mode of a fermionic substate, only one operartion is required to exchange both objects, i.e the operation is in $\mathcal{O}(1)$ instead of $\mathcal{O}(K'_4)$.

The algorithm works as follows:

1. It moves the operators towards the substate of corresponding flavor by applying:

$$\begin{aligned}&\mathcal{O}^{\pm}_{i,m}|G_{\mathcal{O}_\varrho};\,\nu^{(\mathcal{O}_\varrho)}_{K'_4},\nu^{(\mathcal{O}_\varrho)}_{K'_4-1},\ldots,\nu^{(\mathcal{O}_\varrho)}_{1}\rangle_{\mathcal{O}_\varrho}\ldots|G_{\mathcal{O}_1};\,\nu^{(\mathcal{O}_1)}_{K'_4},\nu^{(\mathcal{O}_1)}_{K'_4-1},\ldots,\nu^{(\mathcal{O}_1)}_{1}\rangle_{\mathcal{O}_i}\ldots\\ =(-1)^{G_{\mathcal{O}_\varrho}+G_{\mathcal{O}_{\varrho-1}}+\ldots+G_{\mathcal{O}_{i+1}}}&|G_{\mathcal{O}_\varrho};\,\nu^{(\mathcal{O}_\varrho)}_{K'_4},\nu^{(\mathcal{O}_\varrho)}_{K'_4-1},\ldots,\nu^{(\mathcal{O}_\varrho)}_{1}\rangle\ldots\;\mathcal{O}^{\pm}_{i,m}\,|G_{\mathcal{O}_i};\,\nu^{(\mathcal{O}_1)}_{K'_4},\nu^{(\mathcal{O}_1)}_{K'_4-1},\ldots,\nu^{(\mathcal{O}_1)}_{1}\rangle_{\mathcal{O}_i}\ldots\end{aligned}$$

2. The single substates of $\langle\widehat{\Psi}|$ are moved in front of the corresponding operators or ket-vector with same flavor, so for instance

$$\begin{aligned}&\ldots\;\underbrace{{}_{\mathcal{O}_2}\langle G_{\mathcal{O}_2};\ldots,\nu^{(\mathcal{O}_2)}_{1}|}\;{}_{\mathcal{O}_1}\langle G_{\mathcal{O}_1};\;\ldots,\nu^{(\mathcal{O}_1)}_{1}|\;\mathcal{O}_a|G_{\mathcal{O}_1};\ldots,\nu^{(\mathcal{O}_1)}_{1}\rangle_{\mathcal{O}_1}\mathcal{O}_b\,\mathcal{O}_c|G_{\mathcal{O}_2};\ldots,\nu^{(\mathcal{O}_2)}_{1}\rangle_{\mathcal{O}_2}\ldots\\ =\quad&{}_{\mathcal{O}_1}\langle G_{\mathcal{O}_1};\ldots,\nu^{(\mathcal{O}_1)}_{1}|\;\mathcal{O}_a|G_{\mathcal{O}_1};\ldots,\nu^{(\mathcal{O}_1)}_{1}\rangle_{\mathcal{O}_1}\;\;{}_{\mathcal{O}_2}\langle G_{\mathcal{O}_2};\ldots,\nu^{(\mathcal{O}_2)}_{1}|\mathcal{O}_b\,\mathcal{O}_c|G_{\mathcal{O}_2};\ldots,\nu^{(\mathcal{O}_2)}_{1}\rangle_{\mathcal{O}_2}\;\ldots\end{aligned}$$

   Therewith one is left with a product of independent scalar products, each of an individual flavor.

3. From (4.2), (4.4) it is obvious that all the mode number indices are summed over. Thus the scalar products are computed using

$$\begin{aligned}\sum_m \mathcal{O}^{\pm}_{k,m}\big|\not{G}_{\mathcal{O}_k}\,;\,\nu^{(\mathcal{O}_k)}_{K'_4},\ldots,\nu^{(\mathcal{O}_k)}_{1}\big\rangle_{\mathcal{O}_k}=&\sum_{i=1}^{K'_4}\delta_{m,n_i}\frac{\big|\not{G}_{\mathcal{O}_k}\,;\,\nu^{(\mathcal{O}_k)}_{K'_4},\ldots,\nu^{(\mathcal{O}_k)}_{i}\pm 1,\ldots,\nu^{(\mathcal{O}_k)}_{1}\big\rangle_{\mathcal{O}_k}}{\sqrt{\nu^{(\mathcal{O}_k)}_{i}+\Theta_{\text{h}}(\pm 1)}}\\ &\times\begin{cases}\text{sign}(\nu^{(\mathcal{O}_k)}_{K'_4}+\ldots+\nu^{(\mathcal{O}_k)}_{i+1}) & \text{if } \mathcal{O}_i \text{ fermionic}\\ 1 & \text{if } \mathcal{O}_i \text{ bosonic}\end{cases}\end{aligned}$$

   Θ_{h} denotes the common heavy-side function and according to (3.56) n_i denominates the mode number corresponding to $\nu^{(\mathcal{O}_k)}_{i}$ the operator is acting on. In case $\mathcal{O}^{\pm}_{k,m}$ is fermionic the last term represents just the signature which is produced by permuting $\mathcal{O}^{\pm}_{k,m}$ with Grassmann valued operators assembling the state.
   The counter $G_{\mathcal{O}}$ is crossed out in order to indicate that it is neither used nor changed by the software anymore.

4. In a final step orthogonality of the substates is applied

$${}_{\mathcal{O}_i}\big\langle\not{G}_{\mathcal{O}_i}\,;\,\nu^{(\mathcal{O}_i)}_{K'_4},\ldots,\nu^{(\mathcal{O}_i)}_{1}\big|\big|\not{G}_{\mathcal{O}_i}\,;\,\nu^{(\mathcal{O}_i)}_{K'_4},\ldots,\nu^{(\mathcal{O}_i)}_{1}\big\rangle_{\mathcal{O}_i}=1\;.$$

   Afterwards only vanishing scalar products are left, so the algorithm sets all remaining substates to zero:

$$\big|\not{G}_{\mathcal{O}_i}\,;\,\nu^{(\mathcal{O}_i)}_{K'_4},\ldots,\nu^{(\mathcal{O}_i)}_{1}\big\rangle_{\mathcal{O}_i}=0.$$

Step 1 to 4 computes the contribution of one term of $\mathcal{H}_{\text{eff}}$ to one matrix element $\langle\widehat{\Psi}|\mathcal{H}_{\text{eff}}|\Psi\rangle$. The result is an analytic function in terms of $\omega_{n_i}, f_{n_i}, g_{n_i}$, which is fully determinded by the values of the mode numbers $n_{K_4'}, \ldots, n_1$ and the effective string tension $\widetilde{\lambda}$.
In this way for each term of $\mathcal{H}_{\text{eff}}$ a matrix is generated, whose sum will yield the Hamiltonian matrix representation of $\mathcal{H}_4$. For the sake of simplicity, the FORM program does not sum the matrices but exports them element by element to a MATHEMATICA file.

Eigenvalues of $\mathcal{H}_4$

The computation of the eigenvalues is simple. Provided with the analytic matrix structure of $\mathcal{H}_4$, MATHEMATICA allows for numerical and analytical computation of the eigenvalues as well as analytic simplification of the results in the latter case.
In order to obtain numerical results the user needs to specify values for P_+, $\widetilde{\lambda}$ and the mode numbers, where the latter have to obey the level matching condition (3.51).

4.4.6 Installation

The ABAKUS software project can be downloaded at

http://people.physik.hu-berlin.de/∼hentsche/ABAKUS.tgz .

The version ABAKUS1.02.02.00 can be found on the enclosed CD. In order to run ABAKUS, FORM version 3.1 (or higher) written by Jos Vermaseren and Wolfram MATHEMATICA version 5.2 (or higher) are required. The FORM interpreter has to be available to the Linux operating system as an executable file.
ABAKUS does not require installation, it is sufficient to extract the gzipped tar archive by

```
#⟩ tar -xzvf ABAKUS.tgz
```

In order to run ABAKUS, one needs to execute the shell script `RunCalculation`. All output is placed in the subdirectory `output`.

5 String computations with ABAKUS

To confront the proposed light-cone Bethe equations with the quantum string result extensive computer algebra computations have been performed to diagonalize the worldsheed Hamiltonian perturbatively. For all closed subsectors, i.e. $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$, $\mathfrak{su}(1|1)$, $\mathfrak{su}(1|2)$, $\mathfrak{su}(1,1|2)$ and $\mathfrak{su}(2|3)$, the effective Hamiltonian is stated as well as analytic results for its eigenvalues up to three impurities, whenever available. For higher impurities systematic computations have been performed numerically in all sectors. Unfortunately the feasibility of solving the Bethe equations is quite limited compared to the available string solutions by the ABAKUS software. Therefore in chapter 5.7 only these string eigenvalues are presented for which solutions of the Bethe equations are on-hand, while a full listing of the numerically computed string eigenvalues is given in the appendix B.

As stated in chapter 3.3.2 we will make use of the $U(1)$ charges $\{S_+, S_-, J_+, J_-\}$ in order to classify the Hamiltonian eigenvalues. In order to prevent confusion, recall that the eigenvalues of $\mathcal{H}_4$ are determined by

$$\text{operator } \frac{1}{P_+}\mathcal{H}_4 \quad \longleftrightarrow \quad \text{eigenvalues } -\delta P_- \ . \tag{5.1}$$

5.1 The $\mathfrak{su}(2)$ sector

The $\mathfrak{su}(2)$ sector consists of states, which are composed only of $\alpha^+_{1,n}$ creation operators. For the sake of completeness, the result of section 3.6 is stated again:

$$\mathcal{H}_4^{(\mathfrak{su}(2))} = \widetilde{\lambda} \sum_{\substack{m_1+m_2\\+m_3+m_4=0}} \frac{m_1 m_3}{\sqrt{\omega_{m_1}\omega_{m_2}\omega_{m_3}\omega_{m_4}}}\, \alpha^+_{m_1}\alpha^+_{m_2}\alpha^-_{-m_3}\alpha^-_{-m_4} \tag{5.2}$$

$$\delta P_-^{(\mathfrak{su}(2))} = \frac{\widetilde{\lambda}}{2P_+} \sum_{\substack{i,j=1\\i\neq j}}^{K_4} \frac{(m_i+m_j)^2}{\omega_{m_i}\omega_{m_j}} - \frac{\widetilde{\lambda}}{P_+}\sum_{k=1}^{K_4'} \frac{m_k^2}{\omega_{m_k}^2}\nu_k\left(\nu_k - 1\right) \quad . \tag{5.3}$$

5.2 The $\mathfrak{sl}(2)$ sector

The $\mathfrak{sl}(2)$ states are generated by $\beta^+_{1,n}$ operators. Since the structure of the Hamiltonian (3.41) is identical for $\alpha^\pm_{1,n}$ and $\beta^\pm_{1,n}$ up to a minus sign one finds

$$\mathcal{H}_4^{(\mathfrak{sl}(2))} = -\widetilde{\lambda} \sum_{\substack{m_1+m_2\\+m_3+m_4=0}} \frac{m_2 m_4}{\sqrt{\omega_{m_1}\omega_{m_2}\omega_{m_3}\omega_{m_4}}}\, \beta^+_{1,m_1}\beta^+_{1,m_2}\beta^-_{1,-m_3}\beta^-_{1,-m_4} \tag{5.4}$$

and the global energy shift follows immediately

$$\delta P_-^{(\mathfrak{sl}(2))} = -\delta P_-^{(\mathfrak{su}(2))} \, . \tag{5.5}$$

5.3 The $\mathfrak{su}(1|1)$ sector

States of the $\mathfrak{su}(1|1)$ sector are formed of $\theta^+_{1,n}$ creation operators. As noted in [7] the restriction of the string Hamiltonian $\mathcal{H}_4$ to the pure $\mathfrak{su}(1|1)$ sector vanishes

$$\mathcal{H}_4^{(\mathfrak{su}(1|1))} \equiv 0 \ , \qquad \delta P_-^{(\mathfrak{su}(1|1))} = 0 \ . \tag{5.6}$$

5.4 The $\mathfrak{su}(1|2)$ sector

We now turn to the first larger rank sector $\mathfrak{su}(1|2)$ being spanned by the creation operators $\theta^+_{1,n}$ and $\alpha^+_{1,n}$. The effective Hamiltonian is given by

$$\mathcal{H}_4^{(\mathfrak{su}(1|2))} = \mathcal{H}_4^{(\mathfrak{su}(2))} + \widetilde{\lambda} \sum_{\substack{m_1+m_2\\+m_3+m_4=0}} \frac{X(m_1,m_2,m_3,m_4)}{\sqrt{\omega_{m_3}\omega_{m_4}}}\ \theta^+_{1,m_1}\theta^-_{1,-m_2}\alpha^+_{1,m_3}\alpha^-_{1,-m_4} \quad . \tag{5.7}$$

where $X(m,n,k,l)$ is defined as

$$X(m,n,k,l) := \ \Big[\Big(mn - \tfrac{(m-n)(k-l)}{4}\Big)(f_n f_m + g_n g_m) - \tfrac{\kappa}{4\sqrt{\widetilde{\lambda}}}(k+l)(\omega_k+\omega_l)(f_n g_m + f_m g_n)\Big]\,, \tag{5.8}$$

with $\kappa = \pm 1$.

5.4.1 Two impurities

For two impurity $\mathfrak{su}(1|2)$ states, carrying the modes $m_1 = -m_2$, the Hamiltonian $\mathcal{H}_4$ forms a 4×4 matrix with eigenvalues $-\delta P_-$ where

$$\delta P_- = \Big\{ \pm 2\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1},\ 0,\ 0\Big\}\ . \tag{5.9}$$

5.4.2 Three impurities with distinct modes

Considering the three impurity case with distinct mode numbers m_1, m_2, m_3 the Hamiltonian is represented by an 8×8 matrix which decomposes into 4 non mixing submatrices, where two fall into the rank one sectors $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$. The remaining pieces are two 3×3 matrices.

Since string states only mix if they carry the same charges, we can classify the submatrices and their eigenvalues by the charge of the corresponding states. One finds:

$\{S_+, S_-, J_+, J_-\} = \{0,2,3,1\}_{\theta_1^+\theta_1^+\alpha_1^+|0\rangle}$:

$$\delta P_- = \Big\{ \pm \frac{\widetilde{\lambda}}{P_+}\sum_{j=1}^{3}\frac{m_j^2}{\omega_j},\quad \frac{\widetilde{\lambda}}{P_+\omega_1\omega_2\omega_3}\sum_{j=1}^{3} m_j^2\,\omega_j\Big\} \tag{5.10}$$

$\{S_+, S_-, J_+, J_-\} = \{0,1,3,2\}_{\theta_1^+\alpha_1^+\alpha_1^+|0\rangle}$:

$$\delta P_- = \Big\{0,\quad \frac{\widetilde{\lambda}}{P_+}\frac{m_1^2\omega_{m_1} + m_2^2\omega_{m_2} + m_3^2\omega_{m_3} \pm \Xi_{m_1,m_2,m_3}}{\omega_{m_1}\omega_{m_2}\omega_{m_3}}\Big\} \tag{5.11}$$

$$\begin{aligned}
\text{with}\quad \Xi_{a,b,c} &:= \sqrt{4(\omega_a^2\chi_{b,c}^2 + \omega_b^2\chi_{a,c}^2 + \omega_c^2\chi_{a,b}^2) + (\xi_{a;b,c} - \xi_{b;a,c} + \xi_{c;a,b})^2 - 4\xi_{a;b,c}\xi_{c;a,b}}\\
\xi_{a;b,c} &:= -a(b\omega_b + c\omega_c - a\omega_a)\\
\chi_{a,b} &:= -ab\frac{\widetilde{\lambda}ab - (1+\omega_a)(1+\omega_b)}{\sqrt{(1+\omega_a)(1+\omega_b)}}\,.
\end{aligned}$$

5.4.3 Three impurities with confluent modes

In the case of confluent modes $\{m_1, m_2, m_3\} = \{m, m, -2m\}$ the submatrix with charges $\{0, 2, 3, 1\}$ collapses to a scalar whereas the submatrix of charge $\{0, 1, 3, 2\}$ reduces to a 2×2 matrix. The energy shifts are

$$\{S_+, S_-, J_+, J_-\} = \{0, 2, 3, 1\}_{\theta_1^+\theta_1^+\alpha_1^+|0\rangle}: \qquad \delta P_- = \frac{\widetilde{\lambda}}{P_+}\frac{2m^2}{\omega_m}\Big(\frac{1}{\omega_m} + \frac{1}{\omega_{2m}}\Big) \tag{5.12}$$

$$\{S_+, S_-, J_+, J_-\} = \{0, 1, 3, 2\}_{\theta_1^+\alpha_1^+\alpha_1^+|0\rangle}:$$

$$\delta P_- = 2\frac{\tilde{\lambda}q^2}{P_+\omega_q^2\omega_{2q}}\Big(\omega_q + \omega_{2q} \pm \omega_q\sqrt{3 + 2\omega_{2q}^2 + 4\omega_q\omega_{2q}}\Big) \tag{5.13}$$

5.5 The $\mathfrak{su}(1,1|2)$ sector

States of the $\mathfrak{su}(1,1|2)$ sector are spaned by the set $\{\theta_{1,n}^+, \eta_{1,n}^+, \beta_{1,n}^+, \alpha_{1,n}^+\}$ of creation operators. In this sector the effective Hamiltonian takes the form

$$\begin{aligned}
\mathcal{H}_4^{(\mathfrak{su}(1,1|2))} = \widetilde{\lambda} \sum_{\substack{k+l\\+n+m=0}} & \frac{kl}{\sqrt{\omega_m\omega_n\omega_k\omega_l}}(\alpha_{1,m}^+\alpha_{1,-n}^- - \beta_{1,m}^+\beta_{1,-n}^-)(\alpha_{1,k}^+\alpha_{1,-l}^- + \beta_{1,k}^+\beta_{1,-l}^-) \\
+\widetilde{\lambda} \sum_{\substack{k+l\\+n+m=0}} & 2\,\mathrm{i}\,\frac{f_m f_n - g_m g_n}{\sqrt{\omega_k\omega_l}}(\theta_{1,m}^+\eta_{1,n}^+\beta_{1,-k}^-\alpha_{1,-l}^- + \theta_{1,-m}^-\eta_{1,-n}^-\beta_{1,k}^+\alpha_{1,l}^+) \\
+\widetilde{\lambda} \sum_{\substack{k+l\\+n+m=0}} & \frac{X(m,n,k,l)}{\sqrt{\omega_k\omega_l}}(\theta_{1,m}^+\theta_{1,-n}^- + \eta_{1,m}^+\eta_{1,-n}^-)(\alpha_{1,k}^+\alpha_{1,-l}^- - \beta_{1,k}^+\beta_{1,-l}^-)\,,
\end{aligned} \tag{5.14}$$

where $X(m,n,k,l)$ is given in (5.8).

5.5.1 Two impurities

The Hamiltonian matrix decomposes into several non mixing submatrices. The $\mathfrak{su}(1,1|2)$ sector contains all previous discussed sectors, whose eigenvalues we do not state again. For the two impurity case with mode numbers $m_1 = -m_2$ one obtains the new eigenvalues:

$$\{1,1,1,1\}_{\theta_1^+\eta_1^+|0\rangle,\ \beta_1^+\alpha_1^+|0\rangle}: \qquad \delta P_- = \Big\{\pm 4\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1},\ 0,\ 0\Big\} \tag{5.15}$$

$$\begin{matrix}\{1,2,1,0\}_{\theta_1^+\beta_1^+|0\rangle}, & \{0,1,2,1\}_{\theta_1^+\alpha_1^+|0\rangle} \\ \{2,1,0,1\}_{\eta_1^+\beta_1^+|0\rangle}, & \{1,0,1,2\}_{\eta_1^+\alpha_1^+|0\rangle}\end{matrix} \qquad \delta P_- = \pm 2\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1} \tag{5.16}$$

5.5.2 Three impurities with confluent modes

For higher impurities the situation becomes much more involved. Already the three impurity $\mathfrak{su}(1,1|2)$ Hamiltonian for non-confluent modes becomes a 64×64 matrix with submatrices of rank 9. We will classify the $\mathfrak{su}(1,1|2)$ submatrices with respect to their charges and dimension d. Because $\mathfrak{su}(1,1|2)$ contains previously discussed sectors, we can deduce most of the eigenvalues by using properties of the Hamiltonian $\mathcal{H}_4^{(\mathfrak{su}(1,1|2))}$. The findings are collected in the table 6.

dimension $d = 1$

$\{S_+, S_-, J_+, J_-\}$	State pattern	Property	δP_-
$\{0, 0, 3, 3\}$	$\alpha_1^+\alpha_1^+\alpha_1^+\|0\rangle$	$\mathfrak{su}(2)$ state	(5.3)
$\{3, 3, 0, 0\}$	$\beta_1^+\beta_1^+\beta_1^+\|0\rangle$	$\mathfrak{sl}(2)$ state	(5.5)

dimension $d = 3$

$\{S_+, S_-, J_+, J_-\}$	State pattern	Property	δP_-
$\{0, 2, 3, 1\}$	$\theta_1^+\theta_1^+\alpha_1^+\|0\rangle$	$\mathfrak{su}(1\|2)$ state	$\delta P_-^{\{0,2,3,1\}}$ see (5.10)
$\{2, 0, 1, 3\}$	$\eta_1^+\eta_1^+\alpha_1^+\|0\rangle$	property of (5.14) implies	$\delta P_-^{\{2,1,0,3\}} = +\delta P_-^{\{0,2,3,1\}}$
$\{1, 3, 2, 0\}$	$\theta_1^+\theta_1^+\beta_1^+\|0\rangle$	property of (5.14) implies	$\delta P_-^{\{1,3,2,0\}} = -\delta P_-^{\{0,2,3,1\}}$
$\{3, 1, 0, 2\}$	$\eta_1^+\eta_1^+\beta_1^+\|0\rangle$	property of (5.14) implies	$\delta P_-^{\{3,1,0,2\}} = -\delta P_-^{\{0,2,3,1\}}$
$\{0, 1, 3, 2\}$	$\theta_1^+\alpha_1^+\alpha_1^+\|0\rangle$	$\mathfrak{su}(1\|2)$ state	$\delta P_-^{\{0,1,3,2\}}$ see (5.11)
$\{1, 0, 2, 3\}$	$\eta_1^+\alpha_1^+\alpha_1^+\|0\rangle$	property of (5.14) implies	$\delta P_-^{\{1,0,2,3\}} = +\delta P_-^{\{0,1,3,2\}}$
$\{2, 3, 1, 0\}$	$\theta_1^+\beta_1^+\beta_1^+\|0\rangle$	property of (5.14) implies	$\delta P_-^{\{2,3,1,0\}} = -\delta P_-^{\{0,1,3,2\}}$
$\{3, 2, 0, 1\}$	$\eta_1^+\beta_1^+\beta_1^+\|0\rangle$	property of (5.14) implies	$\delta P_-^{\{3,2,0,1\}} = -\delta P_-^{\{0,1,3,2\}}$

Table 6: Analytically accessible three impurity, distinct $\mathfrak{su}(1,1|2)$ energy shifts.

The structure of the 9×9 submatrices is a bit more involved. Under the oscillator exchange $\theta_{1,m} \leftrightarrow \eta_{1,m}$ and $\alpha_{1,m} \leftrightarrow \beta_{1,m}$ the effective Hamiltonian $\mathcal{H}_4^{(\mathfrak{su}(1,1|2))}$ changes its sign. This exchange translates a state with charge $\{1, 1, 2, 2\}$ into one with $\{2, 2, 1, 1\}$ or a $\{1, 2, 2, 1\}$ charged state into one with $\{2, 1, 1, 2\}$ and vice versa with mutual energy shifts of opposite signs. See table 7 for results.

5.6 The $\mathfrak{su}(2|3)$ sector

Finally the $\mathfrak{su}(2|3)$ sector is spanned by the operators $\theta_{1,n}^+, \theta_{2,n}^+, \alpha_{1,n}^+, \alpha_{2,n}^+$. The effective form of $\mathcal{H}_4$ in this closed subsector reads

$$
\begin{aligned}
\mathcal{H}_4^{(\mathfrak{su}(2|3))} = \\
&\widetilde{\lambda} \sum_{\substack{k+l \\ +n+m}=0} \frac{kl}{\sqrt{\omega_m\omega_n\omega_k\omega_l}}(\alpha_{1,m}^+\alpha_{1,-n}^- + \alpha_{2,m}^+\alpha_{2,-n}^-)(\alpha_{1,k}^+\alpha_{1,-l}^- + \alpha_{2,k}^+\alpha_{2,-l}^-) \\
&+\widetilde{\lambda} \sum_{\substack{k+l \\ +n+m}=0} \frac{X(m,n,k,l)}{\sqrt{\omega_k\omega_l}}(\theta_{1,m}^+\theta_{1,-n}^- + \theta_{2,m}^+\theta_{2,-n}^-)(\alpha_{1,k}^+\alpha_{1,-l}^- + \alpha_{2,k}^+\alpha_{2,-l}^-) \qquad (5.17) \\
&-\frac{\widetilde{\lambda}}{2}\,\mathrm{i} \sum_{\substack{k+l \\ +n+m}=0} \frac{1}{\sqrt{\omega_k\omega_l}}(\theta_{2,m}^+\theta_{1,n}^+\alpha_{2,-k}^-\alpha_{1,-l}^- + \theta_{2,-m}^-\theta_{1,-n}^-\alpha_{2,k}^+\alpha_{1,l}^+) \\
&\qquad\qquad \times \left[(m-n)(k-l)(f_ng_m - f_ng_m) + \frac{\kappa}{\sqrt{\widetilde{\lambda}}}(k+l)(\omega_k - \omega_l)(f_nf_m - g_mg_n)\right] \\
&+\widetilde{\lambda} \sum_{\substack{k+l \\ +n+m}=0} \begin{pmatrix} (f_mg_n + f_ng_m)(f_kg_l + f_lg_k)(mn+kl) \\ +(f_ng_k + f_kg_n)(f_mg_l + f_lg_m)(nk+ml) \\ -(f_nf_l - g_ng_l)(f_mf_k + g_mg_k)(nl+mk) \end{pmatrix} \theta_{2,m}^+\theta_{2,-n}^-\theta_{1,k}^+\theta_{1,-l}^- \,.
\end{aligned}
$$

dimension $d = 9$

$\{S_+, S_-, J_+, J_-\}$	State pattern		δP_-
$\{1,1,2,2\}$	$\beta_1^+\alpha_1^+\alpha_1^+\|0\rangle$,	$\theta_1^+\eta_1^+\alpha_1^+\|0\rangle$	rank 9 matrix, numerical eigenvalues see table 8
$\{2,2,1,1\}$	$\beta_1^+\beta_1^+\alpha_1^+\|0\rangle$,	$\theta_1^+\eta_1^+\beta_1^+\|0\rangle$	$\delta P_-^{\{2,2,1,1\}} = -\delta P_-^{\{1,1,2,2\}}$
$\{1,2,2,1\}$	$\theta_1^+\theta_1^+\eta_1^+\|0\rangle$,	$\theta_1^+\beta_1^+\alpha_1^+\|0\rangle$	rank 6 matrix, numerical eigenvalues see table 8
$\{2,1,1,2\}$	$\theta_1^+\eta_1^+\eta_1^+\|0\rangle$,	$\eta_1^+\beta_1^+\alpha_1^+\|0\rangle$	$\delta P_-^{\{2,1,1,2\}} = -\delta P_-^{\{1,2,2,1\}}$

Table 7: Remaining three impurity, distinct $\mathfrak{su}(1,1|2)$ shifts, which were compared numerically.

5.6.1 Two impurities

For two impurities with mode numbers $m_2 = -m_1$ we find the energy shifts

$$\{0,0,2,0\}_{\theta_2^+\theta_1^+|0\rangle,\ \alpha_2^+\alpha_1^+|0\rangle}: \qquad \delta P_- = \Big\{ \pm 4\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1},\ 0,\ 0\Big\} \tag{5.18}$$

$$\begin{matrix}\{0,1,2,1\}_{\theta_1^+\alpha_1^+|0\rangle}, & \{0,1,2,-1\}_{\theta_1^+\alpha_2^+|0\rangle} \\ \{0,-1,2,1\}_{\theta_2^+\alpha_1^+|0\rangle}, & \{0,-1,2,-1\}_{\theta_2^+\alpha_2^+|0\rangle}\end{matrix}: \qquad \delta P_- = \pm 2\frac{\widetilde{\lambda}}{P_+}\frac{m_1^2}{\omega_1} \tag{5.19}$$

5.7 Numerical results

In order to confront the string results with the predictions of the Bethe equations, we had to retreat to numerical considerations in certain cases. We have considered three impurity excitations in the $\mathfrak{su}(1,1|2)$ subsector with distinct and confluent mode numbers, as well as all three impurity excitations (distinct and confluent) for the $\mathfrak{su}(2|3)$ subsector. In the tables below numerical results for the values $\widetilde{\lambda} = 0.1$ and $P_+ = 100$ are stated. For the case of non-confluent mode numbers $(m_1, m_2, m_3) = (2, 1, -3)$ the string eigenvalues are listed in table 8, while for the confluent modes $(m_1, m_2, m_3) = (3, 3, -6)$ the eigenvalues are given in table 9.

$\mathfrak{su}(2|3)$ sector[11]

$\{S_+, S_-, J_+, J_-\}$	eigenvalues $-\delta P_-$			
{0,0 ,3,±3}	−0.0106324			
{0,±2,3,±1}	±0.0108634	−0.0106324		
{0,±1,3,±2}	−0.0214958	0.000230962	0	
{0,±1,3,0}	0.0217267	3 × −0.0214958	2 × 0.000230962	3 × 0
{0,0,3,±1}	−0.0323591	0.0110943	2 × ±0.0108634	3 × −0.0106324

$\mathfrak{su}(1,1|2)$ sector

$\{S_+, S_-, J_+, J_-\}$	eigenvalues $-\delta P_-$				
{1,1,2,2}	−0.0323591	0.0110943	2 × ±0.0108634	2 × −0.0106324	0.0106324
{1,2,2,1}, {2,1,1,2}	±0.0217267	±0.0214958	±0.000230962	3 × 0	
{2,2,1,1}	0.0323591	−0.0110943	2 × ±0.0108634	2 × 0.0106324	−0.0106324

Table 8: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for three impurity states with distinct mode numbers $m_1 = 2, m_2 = 1, m_3 = -3$. The number in front of some eigenvalues denotes their multiplicity if unequal to one.

[11] The ± signs at some charges are just a shortform of writing several charge combinations all with the same eigenvalues. They are not related to the signatures of the eigenvalues in any sense.

$\mathfrak{su}(2|3)$ sector

$\{S_+, S_-, J_+, J_-\}$	eigenvalues $-\delta P_-$		
{0,±1,3,0}	2×-0.0454059	2×0.0142814	
{0,0,3,±1}	-0.0752496	0.044125	3×-0.0155623
{0,±2,3,±1}, {0,0,3,±3}	-0.0155623		
{0,±1,3,±2}	-0.0454059	0.0142814	

$\mathfrak{su}(1,1|2)$ sector

$\{S_+, S_-, J_+, J_-\}$	eigenvalues $-\delta P_-$			
{1,1,2,2}	-0.0752496	0.044125	0.0155623	2×-0.0155623
{1,2,2,1},{2,1,1,2}	± 0.0454059	± 0.0142814		
{2,2,1,1}	0.0752496	-0.044125	2×0.0155623	-0.0155623

Table 9: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for three impurity states with confluent mode numbers $m_1 = m_2 = 3, m_3 = -6$. The number in front of some eigenvalues denotes their multiplicity if unequal to one.

6 String energy spectrum

As previously stated the Hamiltonian yields the finite part $-P_-$ of the global energy

$$E - J = -P_-(\tfrac{1}{P_+}, \widetilde{\lambda}) \equiv -P_-(\tfrac{1}{E+J}, \tfrac{\lambda}{E+J}) \ . \tag{6.1}$$

Abakus computes $-\delta P_-$ and therefor determines the energy E only implicitly, which is fully sufficient since the Bethe equations are perturbatively expanded in terms of $P_+ \to \infty$ as well. Thus we can directly compare the computational results of the Abakus software with the solutions of the Bethe equations.
Nevertheless Abakus can be used to compute the next to leading order correction of E in both the BMN and the plane-wave limit. A formula for the energy E with precision $\mathcal{O}(\frac{1}{J})$ will be derived.

In general one obtains the expression for E by rewriting P_- in terms of the the BMN quantities J and $\lambda' = \lambda/J^2$ using $P_\pm = J \pm E$, and then subsequently solving for E. The plane-wave quantities are related to the BMN parameters by

$$\frac{1}{P_+} = \frac{1}{2J\big(1 - \frac{P_-}{2J}\big)} = \frac{1}{2J} + \frac{1}{4J^2}P_- + \mathcal{O}(\tfrac{1}{J^2}) \quad \text{and} \quad \widetilde{\lambda} = \frac{4\lambda}{P_+^2} = \lambda' + \frac{\lambda'}{J}P_- + \mathcal{O}(\tfrac{1}{J^2}) \ ,$$

and according to (3.55)

$$P_- = -\sum_{i=1}^{K_4} \omega_i + \delta P_- = -\sum_{i=1}^{K_4} \left(\bar{\omega}_i + \frac{1}{2}\frac{\lambda'}{J}\frac{n_k^2}{\bar{\omega}_{n_k}} P_- \right) + \delta P_- + \mathcal{O}(\tfrac{1}{J^2}) \ , \tag{6.2}$$

where the BMN quantity $\bar{\omega}_i$ is defined as

$$\bar{\omega}_i := \sqrt{1 + \lambda' n_i^2} \ .$$

The eigenvalue $\mathfrak{H}_4$ of the Hamiltonian $\mathcal{H}_4$

$$\delta P_-(n_{K_4'}, \ldots, n_1, \widetilde{\lambda}, P_+) = \frac{1}{P_+}\mathfrak{H}_4(n_{K_4'}, \ldots, n_1, \widetilde{\lambda}) = \frac{1}{2J}\mathfrak{H}_4(n_{K_4'}, \ldots, n_1, \lambda') + \mathcal{O}(\tfrac{1}{J}) \ , \tag{6.3}$$

is computed by the Abakus software and fully determined by the mode numbers $n_{K_4'}, \ldots, n_1$ and the parameter $\widetilde{\lambda}$. The iterative solution of (6.2) yields

$$-P_- = \sum_{i=1}^{K_4} \bar{\omega}_i - \frac{\lambda'}{4J} \sum_{k,j=1}^{K_4} \frac{n_k^2\bar{\omega}_{n_j}^2 + n_j^2\bar{\omega}_{n_k}^2}{\bar{\omega}_{n_k}\bar{\omega}_{n_j}} + \delta P_- + \mathcal{O}(\tfrac{1}{J^2})$$

and with $E - J = -P_-$

$$E = J + \sum_{i=1}^{K_4} \bar{\omega}_i - \frac{\lambda'}{4J} \sum_{k,j=1}^{K_4} \frac{n_k^2\bar{\omega}_{n_j}^2 + n_j^2\bar{\omega}_{n_k}^2}{\bar{\omega}_{n_k}\bar{\omega}_{n_j}} + \delta P_-(n_{K_4'}, \ldots, n_1, \lambda', 2J) + \mathcal{O}(\tfrac{1}{J^2}) \ . \tag{6.4}$$

The term δP_- can easely be computed using the Abakus software and setting $P_+ = 2J$, $\widetilde{\lambda} = \lambda'$ as input which is sufficient for the given precision.

7 Spin chains and the Bethe ansatz

The field content of $\mathcal{N}=4$ super Yang-Mills is given by a gluon field $A_\mu(x)$, six real scalar fields $\phi_i(x)$ $(i=1,\ldots,6)$ and four Weyl gluinos, all in the adjoint representation of $SU(N)$. Adding the scalar fields and the two degrees of freedom of the massless gluon together, there are eight bosonic degrees of freedom in the theory, which can be rearranged into four complex scalars. In the following $\mathcal{Z},\mathcal{X},\mathcal{Y}$ denote three of these complex scalars and $\mathcal{U},\mathcal{W}$ represent two fermions from the field content of $\mathcal{N}=4$ super Yang-Mills. In the planar $N\to\infty$ limit, the relevant operators are single trace operators composed of elementary fields and their derivatives, such as

$$\mathrm{Tr}\left(\mathcal{X}^{K_4}\mathcal{Z}^{J}\right)\ ,$$

where K_4 denotes the number of "impurities" and J denominates an R-charge with respect to $SO(6)$ in gauge theory, which is exactly corresponding to the angular momentum defined in (3.29) in string theory. Classically the scaling dimension D of such an operator is simply the sum of the individual dimensions of the constituent fields. In quantum theory the scaling dimensions receive anomalous corrections, organized in a double expansion in the number of loops ℓ and genus g

$$D=\Delta_0+\sum_{\ell=1}^{\infty}\lambda^\ell\sum_{g=0}^{\infty}\frac{1}{N^{2\,g}}\,\Delta_{\ell,g}=\Delta_0+\delta D\quad .$$

Clearly one is facing a huge operator mixing problem as all operators with arbitrary permutations of $\mathcal{Z}$ and $\mathcal{X}$ are degenerated at tree level, with a conformal dimension $\Delta_0=J+K_4$.
A way to deal with this problem is the dilatation operator $\mathfrak{D}$. It acts on the trace operators at a fixed space time point x and yields the scaling dimension[12] D as its eigenvalues. Thus the diagonalization of $\mathfrak{D}$ solves the mixing problem. Please recall that we are exclusively interested in the planar contribution to $\mathfrak{D}$, as this limit corresponds to the free $\mathrm{AdS}_5\times\mathrm{S}^5$ string theory.
The first crucial hint, that planar $\mathcal{N}=4$ SYM might be integrable was discovered by Minahan and Zarembo in [10], where it was shown, that the dilatation operator could be interpreted as a spin chain Hamiltonian and therefore the dimension of conformal operators may be obtained by diagonalizing an integrable quantum spin chain[13].
In the last few years tremendous process has been made upon exploiting the assumed property of integrability of $\mathcal{N}=4$ super Yang-Mills theory. The discovery of integrability finally led to the construction of a set of nested, asymptotic Bethe equations [19]. The equations proposed by Beisert and Staudacher determine the dilatation operator of planar $\mathcal{N}=4$ SYM in principle, asymptotically[14] to arbitrary loop orders. Due to the AdS/CFT correspondence the type IIB superstring spectrum is expected to match the scaling dimensions of local composite operators in the $\mathcal{N}=4$ super Yang-Mills theory in the 't Hooft limit.

Definition of Integrability

The property of quantum integrability can be defined as

1.) A system is integrable if all interactions of particles can be described as a series of two-body interactions where momenta can be exchanged but not changed in magnitude.

or equivalently

[12] In the literature the scaling dimension is often referred to as Δ, but in our particular case we compute the eigenvalues D of the Dilatation operator $\mathfrak{D}$ and thus stick to the latter notation.
[13] The $\mathfrak{su}(2)$ sector is described by an Heisenberg $\mathrm{XXX}_{1/2}$ quantum spin chain, which is the prototype of an integrable spin chain.
[14] In the given context asymptotically stands for the limit of an infinite large J-charge.

sector	contributing string fields	gauge fields
$\mathfrak{su}(2)$	α_1^+	$\mathcal{Z},\ \mathcal{X}$
$\mathfrak{sl}(2)$	β_1^+	$\mathcal{Z},\ \partial$
$\mathfrak{su}(1\|1)$	θ_1^+	$\mathcal{Z},\ \mathcal{U}$
$\mathfrak{su}(1\|2)$	$\theta_1^+,\ \alpha_1^+$	$\mathcal{Z},\ \mathcal{X},\ \mathcal{U}$
$\mathfrak{su}(2\|3)$	$\theta_2^+,\ \theta_1^+,\ \alpha_2^+,\ \alpha_1^+$	$\mathcal{Z},\ \mathcal{X},\ \mathcal{Y},\ \mathcal{U},\ \mathcal{W}$
$\mathfrak{su}(1,1\|2)$	$\theta_1^+,\ \eta_1^+,\ \beta_1^+,\ \alpha_1^+$	$\mathcal{Z},\ \mathcal{X},\ \partial,\ \mathcal{U},\ \mathcal{W}$

Table 10: All closed subsectors of $\mathfrak{psu}(2,2|4)$ with contributing field content in string and gauge theory. Here ∂ denotes the covariant derivative acting on the contributing fields.
Note that the scalar $\mathcal{Z} \to \mathcal{X}$ is a hard-core excitation, there can be only one such excitation per site. Conversely, the derivative $\mathcal{Z} \to \partial\mathcal{Z}$ is a soft-core excitation, there can be arbitrarily many excitations (derivatives) per site and they also exist on sites which are already occupied by scalars (in the form of $\partial\mathcal{X}, \partial\mathcal{Y}$) or fermions ($\partial\mathcal{U}, \partial\mathcal{W}$).

2.) A system is integrable if there exists a Bethe ansatz, which allows for reformulating the quantum spectral problem into the problem of solving a set of non-linear algebraic equations, the Bethe equations.

In our particular case, the Bethe equations diagonalize the planar dilatation operator in the sense that their roots are eigenvalues of the dilatation operator.

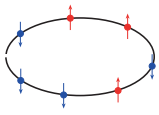
In this section we will review the Bethe ansatz in detail, in order to motivate the $\mathfrak{psu}(2,2|4)$ Bethe equations stated by Beisert and Staudacher [19]. We will perturbatively reformulate these equations in terms of the string quantities P_+ and $\widetilde{\lambda}$ in the near plane-wave limit. Based thereon solutions for all closed subsectors of $\mathfrak{psu}(2,2|4)$ are derived. The corresponding field content of these sectors is displayed in table 10.

7.1 Review of the Asymptotic Bethe ansatz

In this section the perturbative asymptotic Bethe ansatz developed in a very instructive paper of Staudacher [20] will be reviewed. For the sake of simplicity we will restrict the analysis to the simple $\mathfrak{su}(2)$ sector, which is sufficient to explain the basic principle.
The $\mathfrak{su}(2)$ sector consists of operators of the type

$$\mathrm{Tr}\left(\mathcal{X}^{K_4}\mathcal{Z}^{L-K_4}\right) + \ldots = \mathrm{Tr}\left(\mathcal{X}^{K_4}\mathcal{Z}^{J}\right) + \ldots \quad , \tag{7.1}$$

where the dots indicate that we need to include all orderings of the $\mathcal{Z}$ and $\mathcal{X}$ fields. In the spin chain interpretation $L = J + K_4$ is the chain length and M the number of excitations (often also called magnons or impurities). A $\mathcal{Z}$ field is considered as a spin down $|\downarrow\rangle$ and $\mathcal{X}$ as $|\uparrow\rangle$, where we define the vacuum of the spin chain as $|\downarrow\downarrow\downarrow\ldots\downarrow\rangle$. Since the trace links the matrix indices of the first and the last field, the spin chain has periodic boundary conditions: site $x = L + 1$ is to be identified with $x = 1$.

$\mathrm{Tr}\,(\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{Z}\mathcal{X}\mathcal{X}\mathcal{Z}) \Leftrightarrow$  $\Leftrightarrow |\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow\rangle_{\text{cyclic}} \equiv |\mathcal{Z}\mathcal{Z}\mathcal{X}\mathcal{Z}\mathcal{X}\mathcal{X}\mathcal{Z}\rangle_{\text{cyclic}}$

The planar dilatation operator $\mathfrak{D}$ takes the form

$$\mathfrak{D} = \mathfrak{D}^{(0)} + 2g^2\mathfrak{D}^{(1)} + \mathcal{O}(g^4) = \Delta_0 + 2g^2 H_{\mathrm{xxx}_{1/2}} + \mathcal{O}(g^4) \tag{7.2}$$

$$\text{with spin chain Hamiltonian} \quad H_{\mathrm{xxx}_{1/2}} = \sum_{x=1}^{L} (\mathbb{1} - \mathcal{P}_{x,x+1}) = \frac{1}{2}\sum_{x=1}^{L} (\mathbb{1} - \vec{\sigma}_x\vec{\sigma}_{x+1}) \ , \tag{7.3}$$

The permutation operator $\mathcal{P}_{x,x+1}$ exchanges the partons at the lattice sites x and $x+1$. It may alternatively be expressed in terms of the Pauli matrices $\vec{\sigma}_x = (\sigma_x^1, \sigma_x^2, \sigma_x^3)$, when using $|\uparrow\rangle = \binom{1}{0}$ and $|\downarrow\rangle = \binom{0}{1}$. The coupling constant g is defined as

$$g := \frac{g_{YM}\sqrt{N}}{4\pi} = \frac{\sqrt{\lambda}}{4\pi} \overset{(3.16)}{=} \frac{\sqrt{\lambda}P_+}{8\pi} \ . \tag{7.4}$$

A state with two excitations has the form:

$$|\Psi\rangle = \sum_{1\le x_1<x_2\le L} \psi(x_1,x_2) |\ldots \mathcal{Z}\overset{x_1}{\underset{\downarrow}{\mathcal{X}}}\mathcal{Z}\ldots\mathcal{Z}\overset{x_2}{\underset{\downarrow}{\mathcal{X}}}\mathcal{Z}\ldots\rangle \ , \tag{7.5}$$

where $x_{1,2}$ label the position of the two $\mathcal{X}$-excitations in the background of the $\mathcal{Z}$ fields. In position space the Schrödinger equation $H_{\mathrm{xxx}_{1/2}}|\Psi\rangle = \mathcal{E}|\Psi\rangle$ becomes

$$\begin{aligned} \text{for } x_2 > x_1+1: \quad \mathcal{E}\,\psi(x_1,x_2) = \ & 2\,\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1+1,x_2) \\ & + 2\,\psi(x_1,x_2) - \psi(x_1,x_2-1) - \psi(x_1,x_2+1) \ , \\ \text{for } x_2 = x_1+1: \quad \mathcal{E}\,\psi(x_1,x_2) = \ & 2\,\psi(x_1,x_2) - \psi(x_1-1,x_2) - \psi(x_1,x_2+1) \ . \end{aligned} \tag{7.6}$$

The above equation can be fulfilled by a the Bethe ansatz [21]:

$$|\Psi\rangle = e^{ip_1x_1+ip_2x_2} + S(p_2,p_1)e^{ip_2x_1+ip_1x_2} \tag{7.7}$$

The Bethe ansatz is based on the intuition that the excitations, or magnons, (7.7) move freely around the chain until they hit each other at $x_2 = x_1+1$, i.e. we assume a δ-like interaction. Then the magnons can either pass through each other, or exchange momenta with an amplitude given by the S-matrix $S(p_1,p_2)$. This scattering process is non-diffractive, since the momenta p_k are individually conserved.

Plugging (7.7) into the Schrödinger equation (7.6) one finds the energy[15] as the sum of one-particle energies and the S-matrix

$$\mathcal{E} = \sum_{k=1}^{K_4} 4\sin^2\left(\frac{p_k}{2}\right) \ , \tag{7.8}$$

$$S^{(\mathfrak{su}(2))}(p_1,p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_1} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} + 1} \ , \tag{7.9}$$

with $K_4 = 2$. This ansatz solves the infinitely long chain, but for a finite chain one has to impose the periodic boundary condition $\psi(x_1,x_2) = \psi(x_2,x_1+L)$, which leads to the Bethe equations for the two magnon problem:

$$e^{ip_1L} = S^{(\mathfrak{su}(2))}(p_1,p_2) \qquad \text{and} \qquad e^{ip_2L} = S^{(\mathfrak{su}(2))}(p_2,p_1) \ . \tag{7.10}$$

[15] In order to prevent confusion, please note the following important difference: while the string energy E is conjectured to equals the scaling dimension D of the associated conformal operators, the spin chain energy $\mathcal{E}$ correspond to the anomalous dimension δD, i.e.: $E \equiv D = \Delta_0 + \delta D = \Delta_0 + 2g^2\mathcal{E}$.

Equation (7.9) immediately shows $S^{(\mathfrak{su}(2))}(p_1,p_2) = S^{(\mathfrak{su}(2))}(p_2,p_1)^{-1}$ implying $p_1+p_2 = 2\pi m$ with an arbitrary integer m. In order to reinstate the cyclicity of the trace, one needs to further impose the constraint of a total vanishing momentum

$$\sum_{k=1}^{K_4} p_k = 0 \ . \tag{7.11}$$

The key point of integrability is, that the knowledge of the two-particle scattering allows for the immediate solution of the general K_4-particle problem! This phenomenon is known as factorized scattering, viz the multi-body scattering process can be described as a sequence of two-body interactions under which the incoming particles of momenta p_i and p_j scatter elastically and non-diffractively.
The total phase factor acquired by a magnon circling once around the chain should simply be given as a product of the phase factors due to individual collisions with all other $K_4 - 1$ magnons:

$$e^{ip_k L} = \prod_{\substack{i=1\\ i\neq k}}^{K_4} S^{(\mathfrak{su}(2))}(p_k,p_i) \qquad \text{for} \quad k = 1,\dots,K_4 \tag{7.12}$$

The total energy is still given by the sum over all $K_4 \geq 2$ magnons by (7.8). Introducing the *rapidities* $u_k = \frac{1}{2}\cot(\frac{1}{2}p_k)$, (7.12) becomes

$$e^{ip_k L} = \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{i=1\\ i\neq k}}^{K_4} \frac{u_k - u_j + i}{u_k - u_j - i} \qquad k = 1,\dots,K_4 \ . \tag{7.13}$$

while the momentum constraint (7.11) and the energy, yielding the anomalous dimension $\delta D = 2g^2\mathcal{E}$ in first order, turn into

$$\prod_{\substack{i=1\\ i\neq k}}^{K_4} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \qquad \text{and} \qquad \mathcal{E} = \sum_{k=1}^{K_4}\left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}}\right) \ . \tag{7.14}$$

Please note that the anomalous dimension (7.14) does only apply on the gauge theory side with the limit $g \ll 1$ while string theory is working in $g \sim \sqrt{\lambda} \sim P_+ \gg 1$.

7.1.1 Higher loop asymptotic Bethe equations for gauge theory

To diagonalize the higher loop orders of $\mathfrak{D}$, one needs to include interactions between more distant spins on the chain. For a spin chain of length L the known Bethe ansatz works for interactions up to a range of $\ell < L$. For $\ell \geq L$ our methods cease to work: the interactions start to "wrap" around the chain. Inclusion of these *wrapping interactions* in the model will probably require essential modifications of the Bethe ansatz and is an open topic of research at present.
Circumventing this problem by using an asymptotically long spin chain with $L \to \infty$ leads to an infinite charge $L - K_4 = J \to \infty$, which is anyway required by the near-plane-wave limit we performed the string theory computations in. To derive higher loop Bethe equations, Sutherland's *asymptotic* Bethe ansatz is a successful approach. It makes the reasonable assumption that the Bethe ansatz (7.7) is still appropriate as long as the particles are further apart than the range of interaction, which is in our case the considered order ℓ of perturbation theory:

$$\psi(x_1,x_2) \sim e^{ip_1x_1+ip_2x_2} + S(p_2,p_1)e^{ip_2x_1+ip_1x_2} \qquad \text{if} \quad x_2 - x_1 < \ell$$

In order to determine the fine structure of the wave function $\psi(x_1, x_2)$ close to the collision we make the ansatz accurate up to $\mathcal{O}(g^4)$:

$$\begin{aligned}\psi(x_1,x_2) = \quad & \big(1 + C^{(2)}(p_2,p_1)g^{2(x_2-x_1)} + C^{(4)}(p_2,p_1)g^{2+2(x_2-x_1)}\big)e^{ip_1x_1+ip_2x_2} \\ & + \big(1 + C^{(2)\prime}(p_2,p_1)g^{2(x_2-x_1)} + C^{(4)\prime}(p_2,p_1)g^{2+2(x_2-x_1)}\big)S(p_2,p_1)e^{ip_1x_1+ip_2x_2}\end{aligned} \tag{7.15}$$

All coefficient functions including the S-Matrix can be fixed by plugging (7.15) into the Schrödinger equation and one finally obtains 3-loop Bethe equations. The presented technique is referred to as *pertubative asymptotic Bethe ansatz*.

To generalize the Bethe equation (7.12) to all loop orders, we summarize the derivation of [12], where the following analytic all loop expression for the rapidities has been proposed, based on a modification of the Inozemtsev spin chain [22, 23] by demanding BMN-scaling to all loop orders.

$$u_k = \frac{1}{2}\cot\left(\frac{1}{2}p_k\right)\sqrt{1 + 16g^2\sin^2\left(\frac{1}{2}p_k\right)} \quad . \tag{7.16}$$

Based on (7.16) also a generalization of $u_k \pm \frac{i}{2} \to x(u_k \pm \frac{i}{2})$ has to be expected, which in leading order expansion in small g recovers $u_k \pm \frac{i}{2}$. It turns out to be of the remarkably simple form

$$e^{ip_k} = \frac{x(u_k + \frac{i}{2})}{x(u_k - \frac{i}{2})} \qquad \text{with} \quad x(u) := \frac{1}{2}u + \frac{1}{2}\sqrt{u^2 + 4g^2} \quad , \tag{7.17}$$

where $x(u)$ is inverted by $$u(x) = x + \frac{g^2}{x} \quad . \tag{7.18}$$

Using the so called spectral parameters $x_k^\pm \equiv x_k^\pm(p_k)$ first introduced in [24]

$$x_k^\pm(p_k) := x(u_k \pm \tfrac{i}{2}) = \frac{1}{4}(\cot\frac{p_k}{2} \pm i)\left(1 + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p_k}{2}\right)}\right) \tag{7.19}$$

and the identity $u_k - u_j \pm i = (x_k^\pm - x_j^\mp)(1 - g^2/(x_k^\pm x_j^\mp))$ we end up with the *all loop* gauge theory Bethe equations for the $\mathfrak{su}(2)$ sector

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{i=1\\ i\neq k}}^{K_4} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\frac{1 - g^2/(x_k^+ x_j^-)}{1 - g^2/(x_k^- x_j^+)} \qquad \text{for} \quad k = 1, \ldots, K_4 \ . \tag{7.20}$$

In the new notation the additional momentum constraint and the anomalous dimension $\delta D = 2g^2\mathcal{E}$ take the form

$$\prod_{\substack{i=1\\ i\neq k}}^{K_4} \frac{x_k^+}{x_k^-} = 1 \qquad \text{and} \qquad \delta D = 2g^2\sum_{k=1}^{K_4}\left(\frac{i}{x_k^+} - \frac{i}{x_k^-}\right) \ . \tag{7.21}$$

Plane-wave limit: In this limit one takes $L \to \infty$ while keeping K_4 small. This is a dilute gas approximation, where the excitations, in both gauge and string theory, do not feel each

other. In the strict limit, the plane wave approximation is equal to the BNM limit because $L \simeq J = \frac{1}{2}(P_+ - P_-) \simeq \frac{1}{2}P_+$. As argued in [20, 7] the momenta p_k can be expanded as

$$p_k = \frac{p_k^0}{P_+} + \frac{p_k^1}{P_+^2} + \mathcal{O}(\tfrac{1}{P_+^3}) \qquad \text{with } p_k^0 = 4\pi n_k \ , \tag{7.22}$$

which seems quite natural, since in the free limit the momenta of the individual particles on a space of length L with periodic boundary conditions is just given by $2\pi n/L \simeq 4\pi n/P_+$, $n \in \mathbb{Z}$. Expanding δD in terms of P_+ by using (7.19), (7.4) yields

$$\delta D = \sum_{k=1}^{K_4} \frac{\widetilde{\lambda}\, n_k^2}{1+\omega_k} = -K_4 + \sum_{k=1}^{K_4} \omega_k + \mathcal{O}(\tfrac{1}{P_+})$$

$$\text{and thus} \qquad D = \Delta_0 + \delta D = J + \sum_{k=1}^{K_4} \omega_k + \mathcal{O}(\tfrac{1}{P_+}) \ ,$$

which indeed reproduces the $\mathfrak{su}(2)$ string energy (3.62) in leading order.

7.1.2 Dressing factor and string Bethe equations

Even though the Bethe equations (7.20) reproduces the anomalous dimension in the first orders it still does not match the string predictions of order $\mathcal{O}(\frac{1}{P_+})$.
But according to AdS/CFT correspondence quantum strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$ should also be described by an integrable long-range spin chain. String Bethe equations for the $\mathfrak{su}(2)$ sector were first proposed in [16]. Structurally they are identical to the gauge theory apart from an additional scalar factor called *dressing factor*, often denoted as S_0 or σ^2. At first the dressing factor was interpreted as a deformation, which takes us from gauge theory to string theory but later it turned out, that also gauge theory requires a dressing factor at fourth loop order [25].

The difference between gauge theory and string theory is based on slightly different scaling procedures as illustrated in figure 2. The computations in gauge theory are based on perturbation theory around $\lambda = 0$, where in a second step the model is translated into the spin chain picture and the thermodynamic limit $L \to \infty$ is applied. In contrast, thermodynamic limit $L \sim J \to \infty$ is a basic assumption for quantization of our string theory. In order to make contact with gauge theory one might expand arround $\widetilde{\lambda} \simeq 0$ in a second step. If the different expansion patterns for gauge and string theory do not commute, the results for gauge and string theory will disagree.

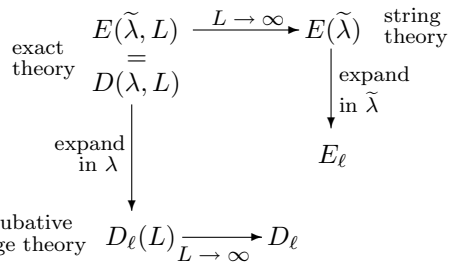


Figure 2: order of limits for gauge and string theory

The important advantage of the all-loop spin chain with its Bethe equations (7.20) is that the pertubative series in λ is summed up before taking the thermodynamic limit and therefore the general structure of the Bethe equations agrees with the string results. But note that the equations have been proposed on three loop computations in gauge theory. Indeed an *all loop dressing factor* S_0 has been proposed in [26] recently, which does not yield a contribution up to fourth order in gauge theory but produces the known string dressing factor in the appropriate limit.

The most general $\mathfrak{su}(2)$ Bethe equations for gauge and string theory acquire the form

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{i=1\\ i\neq k}}^{K_4} \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \frac{1-g^2/(x_k^+ x_j^-)}{1-g^2/(x_k^- x_j^+)} S_0(x_k, x_j) \qquad \text{for } \; k = 1, \ldots, K_4 \; . \tag{7.23}$$

7.2 The nested Bethe ansatz

Derivations [20, 19] similar to the one presented in chapter 7.1 yield the S-matrix for $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$:

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{i=1\\ i\neq k}}^{K_4} \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^\eta \frac{1-g^2/(x_k^+ x_j^-)}{1-g^2/(x_k^- x_j^+)} S_0(x_k, x_j) \qquad \text{with} \quad \begin{array}{ll} \eta = +1 & \text{for } \mathfrak{su}(2) \\ \eta = 0 & \text{for } \mathfrak{su}(1|1) \\ \eta = -1 & \text{for } \mathfrak{sl}(2) \end{array} \tag{7.24}$$

Unfortunately these equations do not allow for computations on larger sectors as for instance the composite $\mathfrak{su}(1|2)$ sectors since (7.24) describes the scattering of either $\mathcal{X}$ and $\mathcal{Z}$ or $\mathcal{X}$ and $\mathcal{U}$ in case of $\mathfrak{su}(1|1)$ but dos not incorporate the interaction of all three fields. In an impressive paper [19] Beisert and Staudacher developed Bethe equations for the full $\mathfrak{psu}(2,2|4)$ using the *nested Bethe ansatz*. This approach is reviewed in this section using the example of $\mathfrak{su}(1|2)$.

In the $\mathcal{N}=4$ gauge theory the planar $\mathfrak{su}(1|2)$ sector consists of operators of the type

$$\mathrm{Tr}\, \mathcal{U}^{K_1} \mathcal{X}^{K_2-K_1} \mathcal{Z}^{L-K_2} + \ldots \quad , \tag{7.25}$$

where $\mathcal{Z}$ and $\mathcal{X}$ are two out of the three complex adjoint scalars of the $\mathcal{N}=4$ model, and $\mathcal{U}$ is an adjoint gaugino. The dots indicate that we need to consider all possible orderings of the fields inside the trace, and diagonalize the set of such operators with respect to the dilatation. As in the previous chapter, this is most easily done when interpreting the dilatation operator as a Hamiltonian acting on a spin chain of length L.
The planar one-loop Hamiltonian in the closed $\mathfrak{su}(1|2)$ sector may be written with the help of the graded permutation operator $\Pi_{x,x+1}$ which exchanges the partons at the lattice sites x and $x+1$, picking up a minus sign if two fermions are involved:

$$\mathrm{H}_0 = \sum_{x=1}^{L} (\mathbb{1} - \Pi_{x,x+1}) \quad .$$

The physics of two scattering fields $\mathcal{X}$ on the background of $\mathcal{Z}$'s has been considered in the last section. The interaction of two fermions $\mathcal{U}$ works similarly and is worked out in [20] in detail. What we are left with is to consider the mixed case of one $\mathcal{X}$ and $\mathcal{U}$.
Choosing the suitable notation for states

$$|\Psi\rangle = \begin{pmatrix} \sum\limits_{1\leq x_1<x_2\leq L} \psi_{\mathcal{U}\mathcal{X}}(x_1,x_2) |\ldots \mathcal{Z}\overset{x_1}{\underset{\downarrow}{\mathcal{U}}}\mathcal{Z}\ldots\mathcal{Z}\overset{x_2}{\underset{\downarrow}{\mathcal{X}}}\mathcal{Z}\ldots\rangle \\ \sum\limits_{1\leq x_1<x_2\leq L} \psi_{\mathcal{X}\mathcal{U}}(x_1,x_2) |\ldots \mathcal{Z}\mathcal{X}\mathcal{Z}\ldots\mathcal{Z}\mathcal{U}\mathcal{Z}\ldots\rangle \end{pmatrix} ,$$

the Schrödinger equation $\mathrm{H}_0|\Psi\rangle = E_0|\Psi\rangle$ is solved by the following Bethe ansatz.

$$\begin{aligned}
\psi_{\mathcal{XU}}(x_1,x_2) &= A_{\mathcal{XU}}\, e^{ip_1x_1+ip_2x_2} + A'_{\mathcal{XU}}\, e^{ip_2x_1+ip_1x_2} \\
\psi_{\mathcal{UX}}(x_1,x_2) &= A_{\mathcal{UX}}\, e^{ip_1x_1+ip_2x_2} + A'_{\mathcal{UX}}\, e^{ip_2x_1+ip_1x_2} \\
\text{with} \qquad \begin{pmatrix} A'_{\mathcal{XU}} \\ A'_{\mathcal{UX}} \end{pmatrix} &= \begin{pmatrix} T^{\mathcal{UX}}_{\mathcal{XU}}(p_2,p_1) & R^{\mathcal{XU}}_{\mathcal{XU}}(p_2,p_1) \\ R^{\mathcal{UX}}_{\mathcal{UX}}(p_2,p_1) & T^{\mathcal{XU}}_{\mathcal{UX}}(p_2,p_1) \end{pmatrix} \begin{pmatrix} A_{\mathcal{XU}} \\ A_{\mathcal{UX}} \end{pmatrix}
\end{aligned} \tag{7.26}$$

The idea behind (7.26) is, that the partons, coming in an arbitrary mixed state with initial amplitudes $A_{\mathcal{UX}}$, $A_{\mathcal{XU}}$, propagate freely along the chain until they scatter at $x_2 = x_1 + 1$. When the particles hit each other they may exchange momenta, and in addition, exchange their flavors. The second terms in (7.26) denominate the outgoing configurations with the amplitudes $A'_{\mathcal{XU}}$, $A'_{\mathcal{UX}}$ and as one easily sees this ansatz assumes the interaction to be non-diffractive. The transmission amplitude for this process is denoted by T and the amplitude for back scattering of the two partons is R. By convention of notation the order of the particles changes if they were transmitted.
The amplitudes may be adjusted by substituting the Bethe ansatz (7.26) into the Schrödinger equation for $x_2 = x_1 + 1$. In the notation of rapidities $u_k = \frac{1}{2}\cot\left(\frac{1}{2}p_k\right)$ one finds:

$$\begin{aligned}
T^{\mathcal{UX}}_{\mathcal{XU}}(p_1,p_2) = T^{\mathcal{XU}}_{\mathcal{UX}}(p_1,p_2) &= \frac{u_k - u_j}{u_k - u_j - i} \\
R^{\mathcal{XU}}_{\mathcal{XU}}(p_1,p_2) = R^{\mathcal{UX}}_{\mathcal{UX}}(p_1,p_2) &= \frac{i}{u_k - u_j - i}
\end{aligned}$$

For the sake of completeness the two-body one-loop S-matrix for the $\mathcal{X}$-$\mathcal{X}$ scattering of section 7.1.1 is listed again and also the $\mathfrak{su}(1|1)$ S-matrix, describing $\mathcal{U}$-$\mathcal{U}$ scattering[16].

$$S^{\mathcal{XX}}_{\mathcal{XX}}(p_1,p_2) = \frac{u_k - u_j + i}{u_k - u_j - i} \quad , \qquad S^{\mathcal{UU}}_{\mathcal{UU}}(p_1,p_2) = 1$$

Using vector notation $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$ for the basis of two-body states $|\mathcal{XX}\rangle, |\mathcal{UX}\rangle, |\mathcal{XU}\rangle, |\mathcal{UU}\rangle$ the one-loop $\mathfrak{su}(1|2)$ S-matrix takes the form

$$S_{k,j} \equiv S_{k,j}(p_k,p_j) = \begin{pmatrix} S^{\mathcal{XX}}_{\mathcal{XX}}(p_k,p_j) & & & \\ & T^{\mathcal{UX}}_{\mathcal{XU}}(p_k,p_j) & R^{\mathcal{XU}}_{\mathcal{XU}}(p_k,p_j) & \\ & R^{\mathcal{UX}}_{\mathcal{UX}}(p_k,p_j) & T^{\mathcal{XU}}_{\mathcal{UX}}(p_k,p_j) & \\ & & & S^{\mathcal{UU}}_{\mathcal{UU}}(p_k,p_j) \end{pmatrix} . \tag{7.27}$$

This result has a remarkable structure: since the vacuum fields $\mathcal{Z}$ dropped out of the picture of this short spin chain, we can interpret the S-matrix (7.27) as the scattering matrix of a very short 'auxiliary' spin two-component spin chain.
The momenta of the excitations living on that short spin chain will be given as functions of the original momenta. Thus one might in principle just solve the short spin chain and than, by inverting the relations between the momenta of the different chains, resolve for the original momenta of the long spin chain.
As one can show (7.27) satisfies the Yang-Baxter equation

$$S_{3,2}S_{3,1}S_{2,1} = S_{2,1}S_{3,1}S_{3,2} \ ,$$

[16] The one loop $\mathfrak{su}(1|1)$ S-matrix may be read of from (7.24) by setting $g = 0$. The dressing factor S_0 has not been considered yet, since it does not contribute on the gauge side in first loop order.

with the important consequence that it is allowed to extend the Bethe ansatz (7.26) to an arbitrary number of particles. Thus let us consider a spin chain with K_4 excitations, where K_3 of these excitations are of flavor $\mathcal{U}$ and the other $K_4 - K_3$ particles are $\mathcal{X}$ fields. We now need to distinguish all possible orderings of the excitations, thus we identify the various configurations with the states $|\ldots\mathcal{X}\mathcal{X}\mathcal{U}\mathcal{X}\ldots\rangle$, where we have left out the vacuum fields $\mathcal{Z}$. As already mentioned, these configurations correspond to a state of a shorter spin chain of length K_4.
If we push any excitation $k \in \{1, \ldots, K_4\}$ once around the chain it collides with all the other $K_4 - 1$ particles, i.e.

$$e^{ip_k L}|\Psi\rangle = \left(\frac{x_k^+}{x_k^-}\right)^L |\Psi\rangle = S_{k,k+1}\ldots S_{k,K_4}S_{k,1}\ldots S_{k,k-1}|\Psi\rangle \ . \tag{7.28}$$

7.2.1 Generalization to higher loops

As discussed in section 7.1.1, the generalization of $(u_k - u_j \pm i)$ is given by $(x_k^\pm - x_j^\mp)(1 - g^2/(x_k^\pm x_j^\mp))$, but it is still unknown what the appropriate all loop expression for $(u_k - u_j)$ is, since $(x_k^\pm - x_j^\pm)(1 - g^2/(x_k^\pm x_j^\pm)) = (u_k - u_j) + \mathcal{O}(g^2)$. This uncertainty can be reduced by an extensive tree-loop computation outlined in [19].
In principle the asymptotic Bethe ansatz (7.15) of section 7.1.1 is applied to the present case

$$\psi_{\mathcal{X}\mathcal{U}}(x_1, x_2) = A_{\mathcal{X}\mathcal{U}}C_{\mathcal{X}\mathcal{U}}(x_2 - x_1)e^{ip_1x_1+ip_2x_2} + A'_{\mathcal{X}\mathcal{U}}C'_{\mathcal{X}\mathcal{U}}(x_2 - x_1)e^{ip_2x_1+ip_1x_2} \tag{7.29}$$

$$\text{with} \quad C_{\mathcal{X}\mathcal{U}}(x_2 - x_1) = 1 + C^{(2)}_{\mathcal{X}\mathcal{U}}(x_2 - x_1)g^{2(x_2-x_1)} + C^{(4)}_{\mathcal{X}\mathcal{U}}(x_2 - x_1)g^{2+2(x_2-x_1)} + \mathcal{O}(g^6) \ , \tag{7.30}$$

and analogous expressions for $C'_{\mathcal{X}\mathcal{U}}(x_2-x_1)$, $\psi_{\mathcal{U}\mathcal{X}}(x_1, x_2)$, $C_{\mathcal{U}\mathcal{X}}(x_2-x_1)$ and $C'_{\mathcal{U}\mathcal{X}}(x_2-x_1)$. All unknowns can be fixed by solving the three-loop Schrödinger equation order by order.
Based on the experience in enlarging simple $\mathfrak{su}(1)$ and $\mathfrak{su}(1|1)$ to all loops, the results yield enough information for an all loop proposal of reflection and transmission amplitudes. For reasons that will become clear shortly, a further index 4 has been added to the spectral parameters $x^\pm$.

$$S_{k,j} = S_0(x_k, x_j)\frac{x^+_{4,k} - x^-_{4,j}}{x^-_{4,k} - x^+_{4,j}}s_{k,j} \tag{7.31}$$

$$\text{with} \quad s_{k,j} = \begin{pmatrix} 1 & & & \\ & \frac{x^+_{4,k}-x^+_{4,j}}{x^+_{4,k}-x^-_{4,j}} & \frac{x^+_{4,j}-x^-_{4,j}}{x^+_{4,k}-x^-_{4,j}} & \\ & \frac{x^+_{4,k}-x^-_{4,k}}{x^+_{4,k}-x^-_{4,j}} & \frac{x^-_{4,k}-x^-_{4,j}}{x^+_{4,k}-x^-_{4,j}} & \\ & & & \frac{x^-_{4,k}-x^+_{4,j}}{x^+_{4,k}-x^-_{4,j}} \end{pmatrix} . \tag{7.32}$$

As in 7.1.2 the dressing factor S_0 has been added to the all loop equations (7.31). This is very natural, as we can think of the $\mathfrak{su}(1|2)$ sector as a "unification" of the $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ sectors.

7.2.2 Diagonalization of matrix Bethe equations

The Bethe equation (7.28) is still a matrix equation, which furthermore has to be simultaneously satisfied for all $k = 1, \ldots, K_4$, i.e an eigenvector $|\Psi\rangle$ is not allowed to depend on k. In order to diagonalize the resulting short spin chain of length K_4, we choose the field $\mathcal{X}$ as a vacuum. Here it is important to note, that individual momenta are assigned to the fields $\mathcal{X}$, and therefore the spin chain is inhomogeneous and thus not translationally invariant.
Defining λ_k as an eigenvalue of the reduced many-body problem, (7.28) can be written as

$$\lambda_k |\Psi\rangle = s_{k,k+1} \ldots s_{k,K_4} s_{k,1} \ldots s_{k,k-1} |\Psi\rangle \tag{7.33}$$

$$\text{with} \qquad \lambda_k = \left(\frac{x^+_{4,k}}{x^-_{4,k}} \right)^L \prod_{\substack{j=1 \\ j \neq k}}^{K_4} S_0^{-1}(x_{4,k}, x_{4,j}) \frac{x^-_{4,k} - x^+_{4,j}}{x^+_{4,k} - x^-_{4,j}} \ . \tag{7.34}$$

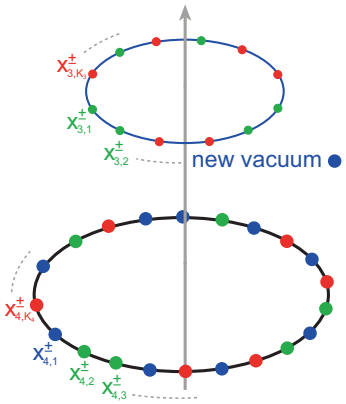


Figure 3: first nesting of spin chain

At first the one magnon problem on the reduced spin chain is solved, i.e. $K_3 = 1$. By applying reduced two-body operators $s_{k,j}$ recursively to the wave function

$$|\Psi\rangle = \sum_{1 \leq k \leq K_4} \psi_k |\mathcal{X} \ldots \overset{k}{\underset{\downarrow}{\mathcal{U}}} \mathcal{X} \ldots \mathcal{X}\rangle \ . \tag{7.35}$$

one obtains a recursion relation for the amplitudes ψ_k, which finally leads to

$$\psi_k(x_1) = \prod_{j=1}^{k-1} \frac{x_3 - x^-_{4,j}}{x_3 - x^+_{4,j+1}} \ , \qquad \lambda_k = \lambda_k(x_1) = \frac{x^-_{4,k} - x_3}{x^+_{4,k} - x_3} \ . \tag{7.36}$$

In (7.36) $x_3 = x_3(x^\pm_{4,1}, \ldots, x^\pm_{4,K_4})$ is a function of all spectral parameters, i.e. of all momenta on the original spin chain. It is found to be independent of the specific choice of k and is interpreted as a new rapidity, parameterizing the momentum of the magnon $\mathcal{U}$ on the nested short spin chain.
The Bethe ansatz for the nested two magnon problem reasonably takes the form

$$\psi_{k_1,k_2}(x_{3,1}, x_{3,2}) = B \psi_{k_1}(x_{3,1}) \psi_{k_2}(x_{3,2}) - B' \psi_{k_1}(x_{3,2}) \psi_{k_2}(x_{3,1}) \ ,$$

where by inserting this ansatz into (7.33) one finds $B = B'$. The general K_1 impurity wave function is thus given as a $K_1 \times K_1$ Slater determinant:

$$\psi_{k_1,\ldots,k_{K_3}}(x_{3,1}, \ldots, x_{3,K_3}) = \det_{\mu,\nu} \ \psi_{k_\mu}(x_{3,\nu})$$

In order to prevent confusion the notation is summarized again:

- μ, ν are labels for the K_3 magnons in the auxiliary, short spin chain of length K_4
- $k_\mu \in \{1, \ldots, K_4\}$ indicate the position of these magnons in the auxiliary chain
- $x_{3,\mu}$ are the rapidities of the K_3 magnons describing their motion in the short chain of length K_4
- $x_{4,k}$ describing the motion of the original K_4 magnons in the long chain of length L

The eigenvalue associated to the wave function (7.37) is clearly given by the product $\lambda_k = \lambda_k(x_{3,1}) \dots \lambda_k(x_{3,K_3})$ of the single eigenvalues. With (7.36), the nested Bethe equations for the $\mathfrak{su}(1|2)$ sector become

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K_4} S_0(x_{4,k}, x_{4,j}) \prod_{\substack{j=1\\ j\neq k}}^{K_4} \frac{x^+_{4,k} - x^-_{4,j}}{x^-_{4,k} - x^+_{4,j}} \prod_{j=1}^{K_3} \frac{x^-_{4,k} - x_3}{x^+_{4,k} - x_3} \tag{7.37}$$

Imposing periodic boundary conditions on the small chain leads to a second set of Bethe equations for the small chain of length K_4:

$$1 = \prod_{j=1}^{K_4} \frac{x_{3,k} - x^+_{4,j}}{x_{3,k} - x^-_{4,j}} \qquad \text{for} \quad k = 1, \dots, K_4. \tag{7.38}$$

7.3 All loop $\mathfrak{psu}(2,2|4)$ Bethe equations

In an inspiring paper [19] the long range gauge and string theory Bethe equations were proposed for the full $\mathfrak{psu}(2,2|4)$ sector. This proposal was based on the nested Bethe ansatz for the smaller $\mathfrak{su}(1,1|2)$ sector by unifying the embedded two component sectors $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$, $\mathfrak{su}(1|1)$ and including an additional fermion $\mathcal{W}$ in the interaction.
Due to further nesting, new spectral parameters $x_2^\pm, x_1^\pm$ and $x_5^\pm, x_6^\pm$ emerge, that correspond to the momenta of excitations living on the reduced spin chains.

The full set of $\mathfrak{psu}(2,2|4)$ Bethe equations proposed in [19] takes the form

$$1 = \prod_{j=1}^{K_4} \frac{x_{4,k}^+}{x_{4,k}^-} \tag{7.39}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k} - u_{2,j} - i\eta_1}{u_{2,k} - u_{2,j} + i\eta_1} \prod_{j=1}^{K_3+K_1} \frac{u_{2,k} - u_{3,j} + \frac{i}{2}\eta_1}{u_{2,k} - u_{3,j} - \frac{i}{2}\eta_1} \tag{7.40}$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}\eta_1}{u_{3,k} - u_{2,j} - \frac{i}{2}\eta_1} \prod_{j=1}^{K_4} \frac{x_{4,j}^{+\eta_1} - x_{3,k}}{x_{4,j}^{-\eta_1} - x_{3,k}} \tag{7.41}$$

$$\begin{aligned} 1 = \Big(\frac{x_{4,k}^-}{x_{4,k}^+}\Big)^{L-\eta_1 K_1 - \eta_2 K_7} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \Big(\frac{x_{4,k}^{+\eta_1} - x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2} - x_{4,j}^{+\eta_2}} \frac{1 - g^2/(x_{4,k}^+ x_{4,j}^-)}{1 - g^2/(x_{4,k}^- x_{4,j}^+)} S_0(x_k, x_j)\Big) \\ \times \prod_{j=1}^{K_3+K_1} \frac{x_{4,k}^{-\eta_1} - x_{3,j}}{x_{4,k}^{+\eta_1} - x_{3,j}} \prod_{j=1}^{K_5+K_7} \frac{x_{4,k}^{-\eta_2} - x_{5,j}}{x_{4,k}^{+\eta_2} - x_{5,j}} \end{aligned} \tag{7.42}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + \frac{i}{2}\eta_2}{u_{5,k} - u_{6,j} - \frac{i}{2}\eta_2} \prod_{j=1}^{K_4} \frac{x_{4,j}^{+\eta_2} - x_{5,k}}{x_{4,j}^{-\eta_2} - x_{5,k}} \tag{7.43}$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k} - u_{6,j} - i\eta_2}{u_{6,k} - u_{6,j} + i\eta_2} \prod_{j=1}^{K_5+K_7} \frac{u_{6,k} - u_{5,j} + \frac{i}{2}\eta_2}{u_{6,k} - u_{5,j} - \frac{i}{2}\eta_2} \,. \tag{7.44}$$

According to (7.18) the variables $u_{i,k}$ are defined by

$$u_{i,k} = x_{i,k} + g^2 \frac{1}{x_{i,k}} \,. \tag{7.45}$$

The Bethe roots $x_{n,k}$ come with the multiplicities equal to the number of excitations $K_{2,\dots,6}$ living on their corresponding (nested) spin chains

$$\begin{array}{lll} x_{2,k} : k = 1, \dots, K_2 & x_{3,k} : k = 1, \dots, (K_1 + K_3) & x_{4,k}^\pm : k = 1, \dots K_4 \\ x_{5,k} : k = 1, \dots, (K_5 + K_7) & x_{6,k} : k = 1, \dots, K_6 & \end{array}$$

The coupling constant g^2 is given by (7.4) and based on (7.19) the spectral parameters $x_{4,k}^\pm$ are related to the magnon momenta p_k via

$$x_{4,k}^\pm = \frac{1}{4}(\cot\frac{p_k}{2} \pm i)\Big(1 + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_k}{2}}\Big) \quad . \tag{7.46}$$

The gradings η_1, η_2 take the values ± 1 and correspond to the four different choices of vacuum fields for the smaller $\mathfrak{su}(1,1|2)$ spin chain. As indicated at the beginning of this chapter, the

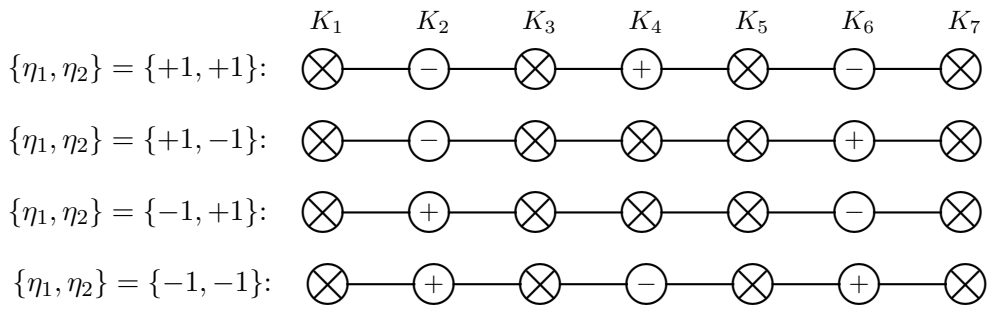


Figure 4: Four different choices of Dynkin diagrams of $\mathfrak{su}(2,2|4)$ specified by the grading η_1 and η_2. The signs in the white nodes indicate the sign of the diagonal elements of the Cartan matrix [19].

full $\mathfrak{psu}(2,2|4)$ Bethe equations are based on examinations in the $\mathfrak{su}(1,1|2)$ sector. In this sector there are four distinct excitations placed on a vacuum of $\mathcal{Z}$ fields. In the nested Bethe ansatz one picks one out of these four excitations as a second effective vacuum of a shorter spin chain, after having eliminated all the sites $\mathcal{Z}$ from the original chain. Referring to table 10, the possible candidates for the second effective vacuum with their associated string oscillators are

$$\mathcal{X} \doteq \alpha_1^+, \quad \partial\mathcal{Z} \doteq \beta_1^+, \quad \mathcal{U} \doteq \theta_1^+, \quad \mathcal{W} \doteq \eta_1^+ \ ,$$

corresponding to four different choices of Dynkin diagrams for $\mathfrak{psu}(2,2|4)$, given in figure 4. The vacuum of $\mathcal{Z}$ fields is associated to the string ground state $|0\rangle$ with charge J.

Note that we have chosen to write down the Bethe equations in a more compact "dynamically" transformed language. In order to convert (7.39)–(7.44) to the form of [19], one introduces the K_1 respectively K_7 roots $x_{1,k}$ and $x_{7,k}$ by splitting off the 'upper' $x_{3,k}$ and $x_{5,k}$ roots via

$$x_{1,k} := g^2/x_{3,K_3+k} \quad k = 1, \dots K_1 \qquad x_{7,k} := g^2/x_{5,K_5+k} \quad k = 1, \dots K_7 \ . \tag{7.47}$$

This coordinate renaming unfolds the equations associated to the fermionic roots (7.40) and (7.43) into two structurally new sets of K_1 and K_7 equations and removes the K_1 and K_7 dependent exponent in the central equation (7.42).
Recall that the first equation (7.39) is the cyclicity constraint on the total momentum of the spin chain. The following $K_2 + (K_1 + K_3) + K_4 + (K_5 + K_7) + K_6$ equations in (7.40)–(7.44) determine the sets of Bethe roots $\{x_{2,k}, x_{3,k}, x^{\pm}_{4,k}, x_{5,k}, x_{6,k}\}$.

7.4 The light-cone Bethe equations for string theory

In [27] an explicit dictionary between the gauge theoretical Bethe equations and perturbative string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ has been established. The string oscillator excitations are characterized by the values of four $U(1)$ charges (S_+, S_-, J_+, J_-) reviewed in chapter 3.3.2. The relationship between the string charges and the excitation numbers $\{K_i\}$ in the Bethe equations is given by

$$\begin{aligned} S_+ &= \ \ \eta_2\,(K_5 + K_7) - (1+\eta_2)\,K_6 + \tfrac{1}{2}(1-\eta_2)\,K_4 \ , \\ S_- &= \ \ \eta_1\,(K_1 + K_3) - (1+\eta_1)\,K_2 + \tfrac{1}{2}(1-\eta_1)\,K_4 \ , \\ J_+ &= -\eta_2\,(K_5 + K_7) - (1-\eta_2)\,K_6 + \tfrac{1}{2}(1+\eta_2)\,K_4 \ , \\ J_- &= -\eta_1\,(K_1 + K_3) - (1-\eta_1)\,K_2 + \tfrac{1}{2}(1+\eta_1)\,K_4 \ . \end{aligned} \tag{7.48}$$

Combining (7.48) with the (S_+, S_-, J_+, J_-) charge values for the string oscillators of table 1 one can construct the excitation pattern for each oscillator, see table 11. The field that is

	$\mathbf{K}_1+\mathbf{K}_3$	$\mathbf{K}_2$	$\mathbf{K}_4$	$\mathbf{K}_6$	$\mathbf{K}_5+\mathbf{K}_7$	S_+	S_-	J_+	J_-
α_1^+	$0+\frac{1}{2}(1-\eta_1)$	0	1	0	$\frac{1}{2}(1-\eta_2)+0$	0	0	1	1
α_2^+	$\frac{1}{2}(1+\eta_1)+1$	1	1	0	$\frac{1}{2}(1-\eta_2)+0$	0	0	1	-1
α_3^+	$0+\frac{1}{2}(1-\eta_1)$	0	1	1	$1+\frac{1}{2}(1+\eta_2)$	0	0	-1	1
α_4^+	$\frac{1}{2}(1+\eta_1)+1$	1	1	1	$1+\frac{1}{2}(1+\eta_2)$	0	0	-1	-1
β_1^+	$0+\frac{1}{2}(1+\eta_1)$	0	1	0	$\frac{1}{2}(1+\eta_2)+0$	1	1	0	0
β_2^+	$\frac{1}{2}(1-\eta_1)+1$	1	1	0	$\frac{1}{2}(1+\eta_2)+0$	1	-1	0	0
β_3^+	$0+\frac{1}{2}(1+\eta_1)$	0	1	1	$1+\frac{1}{2}(1-\eta_2)$	-1	1	0	0
β_4^+	$\frac{1}{2}(1-\eta_1)+1$	1	1	1	$1+\frac{1}{2}(1-\eta_2)$	-1	-1	0	0
θ_1^+	$0+\frac{1}{2}(1+\eta_1)$	0	1	0	$\frac{1}{2}(1-\eta_2)+0$	0	1	1	0
θ_2^+	$\frac{1}{2}(1-\eta_1)+1$	1	1	0	$\frac{1}{2}(1-\eta_2)+0$	0	-1	1	0
θ_3^+	$0+\frac{1}{2}(1+\eta_1)$	0	1	1	$1+\frac{1}{2}(1+\eta_2)$	0	1	-1	0
θ_4^+	$\frac{1}{2}(1-\eta_1)+1$	1	1	1	$1+\frac{1}{2}(1+\eta_2)$	0	-1	-1	0
η_1^+	$0+\frac{1}{2}(1-\eta_1)$	0	1	0	$\frac{1}{2}(1+\eta_2)+0$	1	0	0	1
η_2^+	$\frac{1}{2}(1+\eta_1)+1$	1	1	0	$\frac{1}{2}(1+\eta_2)+0$	1	0	0	-1
η_3^+	$0+\frac{1}{2}(1-\eta_1)$	0	1	1	$1+\frac{1}{2}(1-\eta_2)$	-1	0	0	1
η_4^+	$\frac{1}{2}(1+\eta_1)+1$	1	1	1	$1+\frac{1}{2}(1-\eta_2)$	-1	0	0	-1

Table 11: The translation scheme of string oscillator excitations to the Dynkin node excitation numbers of the Bethe equations. We have also listed the space-time $U(1)$ charges $J_\pm$ and $S_\pm$ of the string oscillators. From this table we easily see which operators represent the middle node for the different choices of gradings. That are, $(\eta_1,\eta_1)=(+,+):\alpha_1^+$, $(-,+):\theta_1^+$, $(+,-):\eta_1^+$ and $(-,-):\beta_1^+$.

picked as the second vacuum in the nested Bethe ansatz only excites the middle node of the Dynkin diagram, so one immediately sees from the table which combinations of the gradings correspond to which choice of vacuum.

Please note, that in the Bethe equations as well as in (7.48) only the combinations (K_1+K_3) and (K_5+K_7) enter. Thus in the dictionary of table 11 a single string oscillator excitation is not described by a single Dynkin node excitation, but rather by a five component excitation vector $(K_1+K_3,K_2,K_4,K_6,K_5+K_7)$, with uniform $K_4=1$ entry. In the string theory the appropriate parametrization is given by the space-time charge vector (S_+,S_-,J_+,J_-). These two labelings are equivalent and the one-to-one map between them is given in (7.48).

There are several things we need to do in order to translate the Bethe equations (7.39)–(7.44) into their light-cone form in order to make a direct comparison to uniform light-cone gauged, near plane-wave string theory. First of all, since the light-cone Hamiltonian is expanded in the large P_+ limit we need to express L in (7.42) in terms of the light-cone momenta. This can be done by using the expression for the eigenvalues of the dilatation operator and the J charge of S^5 [19],

$$\begin{aligned} J &= L+\frac{1}{2}\eta_1(K_3-K_1)-\frac{1}{4}(2+\eta_1+\eta_2)K_4+\frac{1}{2}\eta_2(K_5-K_7),\\ D &= L+\frac{1}{2}\eta_1(K_3-K_1)+\frac{1}{4}(2-\eta_1-\eta_2)K_4+\frac{1}{2}\eta_2(K_5-K_7)+\delta D, \end{aligned} \tag{7.49}$$

where, according to (7.21), the anomalous dimension δD takes the form

$$\delta D = 2g^2\sum_{j=1}^{K_4}\Big(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\Big). \tag{7.50}$$

Using (7.49) and the conjectured equivalence $D \equiv E$ we can write the light-cone momenta P_+ and energy correction P_- as

$$\begin{aligned} P_+ &= D + J = 2L + \eta_1(K_3 - K_1) - \frac{1}{2}(\eta_1+\eta_2)K_4 + \eta_2(K_5 - K_7) + \delta D \\ P_- &= J - D = -K_4 - \delta D \,. \end{aligned} \tag{7.51}$$

Hence it becomes obvious that the large P_+ limit corresponds to an infinitely long chain with a finite number of excitations. Using (7.51), the central K_4 Bethe equations (7.42) become

$$\begin{aligned} \Big(\frac{x^+_{4,k}}{x^-_{4,k}}\Big)^{\frac{1}{2}P_+} = \; & \Big(\frac{x^-_{4,k}}{x^+_{4,k}}\Big)^{\frac{1}{2}(\frac{1}{2}(\eta_1+\eta_2)K_4 - \eta_1(K_1+K_3) - \eta_2(K_5+K_7) - \delta D)} \\ & \times \prod_{\substack{j=1\\ j\neq k}}^{K_4} \Big(\frac{x^{+\eta_1}_{4,k} - x^{-\eta_1}_{4,j}}{x^{-\eta_2}_{4,k} - x^{+\eta_2}_{4,j}} \frac{1 - g^2/(x^+_{4,k}x^-_{4,j})}{1 - g^2/(x^-_{4,k}x^+_{4,j})} S_0^2 \Big) \prod_{j=1}^{K_3+K_1} \frac{x^{-\eta_1}_{4,k} - x_{3,j}}{x^{+\eta_1}_{4,k} - x_{3,j}} \prod_{j=1}^{K_5+K_7} \frac{x^{-\eta_2}_{4,k} - x_{5,j}}{x^{+\eta_2}_{4,k} - x_{5,j}} \,. \end{aligned} \tag{7.52}$$

Anticipanting the result of the following paragraph 7.5, one finds

$$\Big(\frac{x^-_{4,k}}{x^+_{4,k}}\Big)^{-\frac{1}{2}\delta D} \prod_{\substack{j=1\\ j\neq k}}^{K_4} \Big(\frac{1 - g^2/(x^+_{4,k}x^-_{4,j})}{1 - g^2/(x^-_{4,k}x^+_{4,j})} S_0^2 \Big) = 1 + \mathcal{O}(\frac{1}{P_+^3}) \,. \tag{7.53}$$

Curiously enough, not only the $1/P_+$ contribution, but also the $1/P_+^2$ term vanishes in this expansion, in spite of the extremely complicated structure of the involved terms. The $1/P_+^3$ contribution is nonvanishing though. For detailed considerations the reader is referred to the paragraph 7.5. Therefore, to the order we are interested in, the light-cone Bethe equations are given by the previous equations of (7.39)-(7.44) with the central node K_4 Bethe equations (7.42) exchanged by the simpler dressing factor free form

$$\begin{aligned} \Big(\frac{x^+_{4,k}}{x^-_{4,k}}\Big)^{\frac{1}{2}P_+} = \; & \Big(\frac{x^-_{4,k}}{x^+_{4,k}}\Big)^{\frac{1}{2}(\frac{1}{2}(\eta_1+\eta_2)K_4 - \eta_1(K_1+K_3) - \eta_2(K_5+K_7))} \\ & \times \prod_{\substack{j=1\\ j\neq k}}^{K_4} \frac{x^{+\eta_1}_{4,k} - x^{-\eta_1}_{4,j}}{x^{-\eta_2}_{4,k} - x^{+\eta_2}_{4,j}} \prod_{j=1}^{K_3+K_1} \frac{x^{-\eta_1}_{4,k} - x_{3,j}}{x^{+\eta_1}_{4,k} - x_{3,j}} \prod_{j=1}^{K_5+K_7} \frac{x^{-\eta_2}_{4,k} - x_{5,j}}{x^{+\eta_2}_{4,k} - x_{5,j}} + \mathcal{O}(\frac{1}{P_+^2}) \,, \end{aligned} \tag{7.54}$$

Putting all $K_j = 0$, for $j \neq 4$, we indeed reproduce the proposed string Bethe equations for the rank one subsectors presented in [7].

7.5 Dressing factor S_0

7.5.1 general structure

In the limit of large string tension the structure for the dressing factor was introduced on the basis of an $\mathfrak{su}(2)$ string computation in [16]

$$\begin{aligned} & S_0(x_{4,k}, x_{4,j}) = \sigma^2(x_{4,k}, x_{4,j}) = e^{2i\theta(x_{4,k}, x_{4,j})} \\ \text{with} \quad & \theta(x_{4,k}, x_{4,j}) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(g) \Big[q_r(x^\pm_{4,k})\, q_s(x^\pm_{4,j}) - q_r(x^\pm_{4,j})\, q_s(x^\pm_{4,k}) \Big] \,, \end{aligned} \tag{7.55}$$

with the local conserved charge densities

$$q_r(x^\pm) = \frac{i}{r-1}\, g^{r-1} \left[\left(\frac{1}{x^+}\right)^{r-1} - \left(\frac{1}{x^-}\right)^{r-1} \right] \ . \tag{7.56}$$

The all loop expression for $c_{r,s}(g)$ was recently conjectured in [26] which leads to a contribution of the dressing factor also in gauge theory beginning at four loop order.
We will restrict our considerations to the strong coupling limit where $c_{r,s}(g)$ is given [28] by

$$c_{r,s}(g) = g \Big[\delta_{r+1,s} - \frac{(1-(-1)^{r+s})}{g\,\pi} \frac{(r-1)(s-1)}{(r+s-2)(s-r)} + \mathcal{O}(\tfrac{1}{g^2}) \Big] . \tag{7.57}$$

As motivated in (7.22) the momenta are expanded as

$$p_k = \frac{1}{P_+} p_k^0 + \frac{1}{P_+^2} p_k^1 + \frac{1}{P_+^3} p_k^2 + \dots \quad . \tag{7.58}$$

It is convenient to define rescaled spectral parameters $y_k^\pm$

$$y_k^\pm := \frac{1}{g} x_k^\pm = y_k^0 + \frac{1}{P_+} y_k^{1,\pm} + \frac{1}{P_+^2} y_k^{2,\pm} + \frac{1}{P_+^3} y_k^{3,\pm} + \dots \ , \tag{7.59}$$

$$y_k^0 = \frac{1+\omega_k}{\sqrt{\bar{\lambda}}\, m_k} \ , \quad y_{4,k}^{1,\pm} = \frac{2\pi}{\sqrt{\bar{\lambda}}} (1+\omega_k) \Big(\pm i - \frac{2p_k^1}{(p_k^0)^2\, \omega_k} \Big) \ . \tag{7.60}$$

The higher order coefficients $y_k^{\ell,\pm}$ are too long to be displayed here but can simply be obtained by expanding (7.19) using (7.58). In the strong coupling limit the first two orders of the dressing factor can analytically be summed up [28]

$$\theta(x_{4,k}, x_{4,j}) = g\theta^{(0)}(y_k, y_j) + \theta^{(1)}(y_k, y_j) + \mathcal{O}(\tfrac{1}{g}) \tag{7.61}$$

$$\begin{aligned} \theta^{(0)}(y_k, y_j) = \quad & (y_j^+ - y_k^+)\, F(y_k^+ y_j^+) + (y_j^- - y_k^-)\, F(y_k^- y_j^-) \\ & - (y_j^+ - y_k^-)\, F(y_k^- y_j^+) - (y_j^- - y_k^+)\, F(y_k^+ y_j^-) \ , \end{aligned} \tag{7.62}$$

$$\begin{aligned} \theta^{(1)}(y_k, y_j) = \quad & \vartheta(y_j^-, y_k^-) - \vartheta(y_j^-, y_k^+) - \vartheta(y_j^+, y_k^-) + \vartheta(y_j^+, y_k^+) \\ & - \vartheta(y_k^-, y_j^-) + \vartheta(y_k^-, y_j^+) + \vartheta(y_k^+, y_j^-) - \vartheta(y_k^+, y_j^+) \ , \end{aligned} \tag{7.63}$$

where the auxiliary functions F, ϑ are given by

$$F(a) := (1 - \frac{1}{a}) \log(1 - \frac{1}{a}) \ , \tag{7.64}$$

$$\begin{aligned} \vartheta(a,b) := \frac{1}{2\pi} \Bigg[& \log\frac{b-1}{b+1} \log\frac{a-b^{-1}}{a-b} \\ & + \mathrm{Li}\,\frac{\sqrt{b}-\sqrt{b^{-1}}}{\sqrt{b}-\sqrt{a}} - \mathrm{Li}\,\frac{\sqrt{b}+\sqrt{b^{-1}}}{\sqrt{b}-\sqrt{a}} + \mathrm{Li}\,\frac{\sqrt{b}-\sqrt{b^{-1}}}{\sqrt{b}+\sqrt{a}} - \mathrm{Li}\,\frac{\sqrt{b}+\sqrt{b^{-1}}}{\sqrt{b}+\sqrt{a}} \Bigg] \end{aligned} \tag{7.65}$$

With help of the cyclicity constraint (7.11) and defining δd_k

$$\delta D = \sum_{j=1}^{K_4} \delta d_j \qquad \text{with} \quad \delta d_j = g^2 \left(\frac{i}{x_{4,j}^+} - \frac{i}{x_{4,j}^-} \right) \tag{7.66}$$

one can write

$$\left(\frac{x_{4,k}^-}{x_{4,k}^+} \right)^{-\frac{1}{2}\delta D} = \prod_{\substack{j=1 \\ j\neq k}}^{K_4} e^{\frac{i}{2}(p_k \delta d_j - p_j \delta d_k)} \ . \tag{7.67}$$

7.5.2 Cancelation of the dressing factor

With the help of (7.67) equation (7.53) becomes

$$\prod_{\substack{j=1\\ j\neq k}}^{K_4}\left(e^{\frac{i}{2}(p_k\delta d_j - p_j\delta d_k)}\frac{1-1/(y_k^+y_j^-)}{1-1/(y_k^-y_j^+)}\,S_0\right) = 1+\mathcal{O}(\tfrac{1}{P_+^3})\ . \tag{7.68}$$

Expanding the involved terms to first order in $1/P_+$ yields

$$\begin{aligned} g\theta^{(0)}(y_k,y_j) &= \frac{\sqrt{\widetilde{\lambda}}}{8\pi P_+}\,\frac{(y_k^0-y_j^0)(y_k^{1,+}-y_k^{1,-})(y_j^{1,+}-y_j^{1,-})}{(y_j^0)^2(y_k^0)^2(1-y_j^0y_k^0)}+\mathcal{O}(\tfrac{1}{P_+^2})\\ \frac{i}{2}\,(p_k\delta d_j - p_j\delta d_k) &= \frac{\sqrt{\widetilde{\lambda}}}{P_+}\,\frac{m_k(y_k^0)^2(y_j^{1,+}-y_j^{1,-})-m_j(y_j^0)^2(y_k^{1,+}-y_k^{1,-})}{2(y_j^0)^2(y_k^0)^2}+\mathcal{O}(\tfrac{1}{P_+^2})\\ \frac{1-1/(y_k^+y_j^-)}{1-1/(y_k^-y_j^+)} &= 1+\underbrace{\frac{\sqrt{\widetilde{\lambda}}}{P_+}\,\frac{y_k^0(y_j^{1,+}-y_j^{1,-})-y_j^0(y_k^{1,+}-y_k^{1,-})}{y_j^0y_k^0(1-y_j^0y_k^0)}}_{\mathfrak{T}}+\mathcal{O}(\tfrac{1}{P_+^2}) = e^{\mathfrak{T}}+\mathcal{O}(\tfrac{1}{P_+^2})\ . \end{aligned} \tag{7.69}$$

For higher ordes it is usefull to convert the last expansion of (7.69) into an e-function because this slightly simplifies the result. Subsitituting the explicit expressions for y_k^0 and $y_k^{1,\pm}$ from (7.60), the leading order indeed vanishes:

$$2i\,g\theta^{(0)}(y_k,y_j)+\frac{i}{2}\,(p_k\delta d_j - p_j\delta d_k)+\mathfrak{T} = 0+\mathcal{O}(\tfrac{1}{P_+^2})\quad . \tag{7.70}$$

Cancelation of the dressing factor in second order

Computing the leading order of $\theta^{(1)}$ requires quite some effort in using computer algebra. It turns out to be of order

$$\theta^{(1)}(y_k,y_j) = \frac{1}{P_+^3}\mathfrak{I}'_{AFS}+\mathcal{O}(\tfrac{1}{P_+^4}) \tag{7.71}$$

so $\theta^{(1)}$ does not contribute to the next to leading order of (7.68). The second order off the three terms in (7.69) adds to zero

$$e^{\frac{i}{2}(p_k\delta d_j - p_j\delta d_k)}\frac{1-1/(y_k^+y_j^-)}{1-1/(y_k^-y_j^+)}\,e^{2i\,g\theta^{(0)}(y_k,y_j)} = 1+\frac{1}{P_+^3}\mathfrak{I}_{AFS}+\mathcal{O}(\tfrac{1}{P_+^4})\ . \tag{7.72}$$

Now one might ask, if the cancelation of the dressing factor stays valid for higer orders, but this is not the case. One finds $\mathfrak{I}'_{AFS}\neq 0$, $\mathfrak{I}_{AFS}\neq 0$ and also the whole dressing phase $\mathfrak{I}_{AFS}+2i\,\mathfrak{I}'_{AFS}$ does not vansih.
It is however an interesting matter of fact, that the dependences on higher orders of the momenta in $\mathfrak{I}_{AFS}$ and $\mathfrak{I}'_{AFS}$ drop out, consequently the expressions are completely determined by the mode numers $m_{k=1,\dots,K_4}$ and $\widetilde{\lambda}$.

$$\mathfrak{I}_{AFS}(y_k^\pm,y_j^\pm,\widetilde{\lambda}) \equiv \mathfrak{I}_{AFS}(n_k,n_j,\widetilde{\lambda}) \qquad \text{and} \qquad \mathfrak{I}'_{AFS}(y_k^\pm,y_j^\pm,\widetilde{\lambda}) \equiv \mathfrak{I}'_{AFS}(n_k,n_j,\widetilde{\lambda}) \tag{7.73}$$

Due to the complexity of the involved expressions an analytical summation over a valid configuration of momenta and application of the cyclicity constraint (7.39) is extremely difficult. However using an abritrary confluent string configuration such as $m_1=-m_2=2$ and computing $\mathfrak{I}_{AFS}+2i\,\mathfrak{I}'_{AFS}$ numerically, it is proven, that (7.68) does not in general vanish at $\mathcal{O}(\frac{1}{P_+^4})$.

Confluent mode numbers

In order to obtain full consistency, also the expansion

$$p_k = \frac{1}{P_+}p_k^0 + \frac{1}{\sqrt{P_+^3}}p_k^1 + \frac{1}{P_+^2}p_k^2 + \frac{1}{\sqrt{P_+^5}}p_k^3 + \dots \quad .$$

for confluent mode numbers has to be tested. The computations require even more effort since additional terms of order $\mathcal{O}(\frac{1}{P_+^{3/2}})$ and $\mathcal{O}(\frac{1}{P_+^{5/2}})$ arise, but as expected all terms vanish up to $\mathcal{O}(\frac{1}{P_+^3})$.

7.6 solutions of the string Bethe equations

We will now explicitly expand the Bethe equations in the large P_+ limit. The mode numbers of the string oscillators will enter in the equations as the zero mode of the magnon momenta p_k.

7.7 Non-confluent mode numbers

For distinct mode numbers one assumes an expansion of p_k as given in (7.22) determining the analogous expansion of $x_{4,k}^{\pm}$:

$$\begin{aligned} & x_{4,k}^{\pm} = P_+\, x_{4,k}^0 + x_{4,k}^{1,\pm} + \dots \,, \\ \text{with} \quad & x_{4,k}^0 = \frac{1+\omega_k}{2p_k^0}, \qquad x_{4,k}^{1,\pm} = \frac{1}{4}(1+\omega_k)\Big(\pm i - \frac{2p_k^1}{(p_k^0)^2\,\omega_k}\Big)\,, \end{aligned} \tag{7.74}$$

and $\omega_k = \sqrt{1+\widetilde{\lambda}\frac{(p_k^0)^2}{16\pi^2}}$. Consistency then implies that the spectral parameters $x_{3,k}$ and $x_{5,k}$ have the expansion

$$x_{3,k} = P_+\, x_{3,k}^0 + x_{3,k}^1 + \dots, \qquad x_{5,k} = P_+\, x_{5,k}^0 + x_{5,k}^1 + \dots \quad . \tag{7.75}$$

The first order expansion of (7.54) fixes the momentum at leading order p_k^0 to

$$p_k^0 = 4\pi m_k, \qquad m_k \in \mathbb{Z}, \tag{7.76}$$

in agreement with (7.22). The integer here is what will correspond to the mode numbers of the string oscillators. Expanding (7.54) to the next order we find that the p_k^1 satisfies

$$\begin{aligned} p_k^1 = \;& \frac{1}{2}(\eta_1+\eta_2)\sum_{\substack{j=1\\ j\neq k}}^{K_4}\frac{2+\omega_k+\omega_j}{x_{4,k}^0 - x_{4,j}^0} - \eta_1\sum_{j=1}^{K_1+K_3}\frac{1+\omega_k}{x_{4,k}^0 - x_{3,j}^0} \\ & -\eta_2\sum_{j=1}^{K_5+K_7}\frac{1+\omega_k}{x_{4,k}^0 - x_{5,j}^0} - \big(\frac{1}{2}(\eta_1+\eta_2)K_4 - \eta_1(K_1+K_3) - \eta_2(K_5+K_7)\big)p_k^0 \;. \end{aligned} \tag{7.77}$$

Expanding the light-cone energy (7.51), using (7.50) and (7.74) we find

$$P_- = -\sum_{k=1}^{K_4}\omega_k + \delta P_-, \tag{7.78}$$

where the energy shift, δP_-, is given by

$$\delta P_- = -\frac{\widetilde{\lambda}}{P_+}\frac{1}{16\pi^2}\sum_{k=1}^{K_4}\frac{p_k^0 p_k^1}{\omega_k} + \mathcal{O}(\tfrac{1}{P_+^2}) \;. \tag{7.79}$$

7.8 Confluent mode numbers

For the case of confluent mode numbers we run into trouble because of the zero denominator in (7.77), which is caused by the term

$$\prod_{\substack{j=1\\ j\neq k}}^{K_4} \frac{x_{4,k}^{+\eta_1} - x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2} - x_{4,j}^{+\eta_2}} \tag{7.80}$$

of (7.54). In [27] an alternative expansion, first introduced in [16], has successfully been applied

$$p_{k,l_k} = \frac{p_k^0}{P_+} + \frac{p_{k,l_k}^1}{P_+^{3/2}} + \frac{p_{k,l_k}^2}{P_+^2} \qquad l_k \in \{1,2,...,\nu_k\} \ , \quad p_0^k = 4\pi m_k \quad . \tag{7.81}$$

Where we, following section 3.4, denote the multiplicity as ν_k so

$$\sum_{k=1}^{K_4'} \nu_k = K_4 \qquad \text{and} \qquad \sum_{k=1}^{K_4'} \nu_k m_k = 0 \ ,$$

where K_4' is the number of distinct mode numbers. The first order term in (7.81) is degenerate for confluent mode numbers while for the higher order terms the degeneracy might be lifted. The resulting expansion for the spectral parameters $x_{4,k}^{\pm}$ takes the form

$$x_{4,k}^{\pm} = P_+ x_{4,k}^0 + \sqrt{P_+} x_{4,k}^1 + x_{4,k}^{2,\pm} + \dots$$

$$\text{with} \qquad x_{4,k}^0 = \frac{1+\omega_k}{2p_k^0} \ , \quad x_{4,k}^1 = -(1+\omega_k)\frac{p_{k,l_k}^1}{2(p_k^0)^2\omega_k}$$

$$x_{4,k}^{2,\pm} = \frac{1}{4}(1+\omega_k)\Big(\pm i - \frac{2p_k^1}{(p_k^0)^2\,\omega_k}\Big) + \frac{(p_{k,l_k}^1)^2}{4(p_k^0)^3\omega^3}(2\omega_k + 3\omega_k^2 - 1) \quad ,$$

while the expansion (7.75) of $x_{3,k}$ and $x_{5,k}$ is retained unchanged. For consistency reasons it is important notice, that in case of non-coinciding mode numbers (7.54) requires $p_k^1 = 0$.
In the regime of confluent mode numbers the energy shift decomposes as

$$\delta P_- = \sum_{k=1}^{K_4'} \sum_{l_k=1}^{\nu_k} \delta P_{-,k,l_k} \ . \tag{7.82}$$

The contribution from mode numbers m_j with $\nu_j = 1$ look the same as in (7.79) while modes m_k with $\nu_k > 1$ will have contribution from p_{k,l_k}^1. Using (7.81) and expanding (7.80) we find that p_{k,l_k}^1 satisfy a Stieltjes equation [29] of the form

$$p_{k,l_k}^1 = -2(\eta_1+\eta_2)(p_k^0)^2\omega_k \sum_{\substack{\mu_k=1\\ \mu_k\neq l_k}}^{\nu_k} \frac{1}{p_{k,l_k}^1 - p_{k,\mu_k}^1}. \tag{7.83}$$

It is useful to note that $\sum_{l_k=1}^{\nu_k} p_{k,l_k}^1 = 0$. The momenta p_{k,l_k}^1 can be written as

$$(p_{k,l_k}^1)^2 = -2\,(\eta_1+\eta_2)\,(p_k^0)^2\,\omega_k\,h_{\nu_k,l_k}^2 \qquad \text{with} \quad l_k = 1,...,\nu_k \tag{7.84}$$

where h_{ν_k,l_k} are the ν_k roots of Hermite polynomials of degree ν_k. However, the explicit solutions h_{ν_k,l_k} are not needed since when summing over k the following property applies

$$\sum_{l_k=1}^{\nu_k}(h_{\nu_k,l_k})^2 = \frac{\nu_k(\nu_k-1)}{2} \ . \tag{7.85}$$

The expansion for the second order contribution p^2_{k,l_k} in (7.81) is considerably more complicated, we therefore refer only to its general structure

$$p^2_{k,l_k} = \widetilde{p}_k + \sum_{\substack{\mu_k=1\\ \mu_k\neq l_k}}^{\nu_k} f_k(\mu_k,l_k) \ . \tag{7.86}$$

We split p^2_{k,l_k} into a part not depending on l_k, which is equivalent to p^1_k given in (7.77): $\widetilde{p}_k \equiv p^1_k$. The function f_k has the property $f_k(\mu_k,l_k) = -f_k(l_k,\mu_k)$ and thus the second term drops out when summed over l_k. The final expression for the energy shift becomes then

$$\delta P_- = -\frac{1}{P_+}\frac{\widetilde{\lambda}}{32\pi^2}\sum_{k=1}^{K'_4}\nu_k p^0_k\Big(\frac{2\,\widetilde{p}_k\,\omega_k - (\eta_1+\eta_2)p^0_k(\nu_k-1)}{\omega_k^2}\Big). \tag{7.87}$$

7.9 Bethe equations for the smaller spin chains

To be able to solve for p^1_k it is clear from the form of (7.77) that we need the values of the Bethe roots $x_{3,k}$ and $x_{5,k}$ at leading order in P_+. Based on (7.45) the variables u_k scale as $u_k = P_+u^0_k + u^1_k + \ldots$. Expanding (7.40), (7.41), (7.43) and (7.44) yields

$$\begin{aligned}
0 &= \sum_{\substack{j=1\\ j\neq k}}^{K_2}\frac{2}{u^0_{2,j}-u^0_{2,k}} + \sum_{j=1}^{K_1+K_3}\frac{1}{u^0_{2,k}-(x^0_{3,j}+\frac{\widetilde{\lambda}}{64\pi^2}\frac{1}{x^0_{3,j}})}\,,\\
0 &= \eta_1\sum_{j=1}^{K_2}\frac{1}{x^0_{3,k}+\frac{\widetilde{\lambda}}{64\pi^2}\frac{1}{x^0_{3,k}}-u^0_{2,j}} + \frac{1}{2}\sum_{j=1}^{K_4}\frac{1+\omega_j}{x^0_{4,j}-x^0_{3,k}}\,,\\
0 &= \eta_2\sum_{j=1}^{K_6}\frac{1}{x^0_{5,k}+\frac{\widetilde{\lambda}}{64\pi^2}\frac{1}{x^0_{5,k}}-u^0_{6,j}} + \frac{1}{2}\sum_{j=1}^{K_4}\frac{1+\omega_j}{x^0_{4,j}-x^0_{5,k}}\,,\\
0 &= \sum_{\substack{j=1\\ j\neq k}}^{K_6}\frac{2}{u^0_{6,j}-u^0_{6,k}} + \sum_{j=1}^{K_5+K_7}\frac{1}{u^0_{6,k}-(x^0_{5,j}+\frac{\widetilde{\lambda}}{64\pi^2}\frac{1}{x^0_{5,j}})}\,,
\end{aligned} \tag{7.88}$$

which determine the $x^0_{2,k}$, $x^0_{3,k}$, $x^0_{5,k}$ and $x^0_{6,k}$ in terms of $x^0_{4,k}$. Note that the two sets of the first two and the last two equations are decoupled and identical in structure.

Let us briefly discuss how one goes about solving these equations for a given excitation sector. First one needs to commit oneself to a specific grading by specifying the numbers $\eta_{1,2} = \pm 1$. For most of the sectors all choices of gradings will give the same result, however, the calculation will be more or less complicated depending on the choice. Then one reads off the values for $\{K_i\}$ in table 11 corresponding to the string excitation pattern in question. The four different choices of gradings can be grouped into two classes, one with fermionic middle node, $\eta_1 = -\eta_2$, and one with bosonic middle node, $\eta_1 = \eta_2$ in the associated Dynkin diagram. The difference between the two is important in the case of confluent mode numbers. The K_3

and K_5 (and for $\eta_1 = -\eta_2$, also K_4) are fermionic nodes which means that the solutions for $x^0_{3,k}$ and similarly for $x^0_{5,k}$ for different values of k are not allowed to be degenerate by the Pauli principle.

Consider for example the $\mathfrak{su}(1,1|2)$ sector containing only nonvanishing values for $\{K_3, K_4, K_5\}$. Then, due to $K_2 = 0 = K_6$, the equations (7.88) condense to two identical, degree K_4 polynomial equations for $x^0_{3,k}$ and $x^0_{5,k}$ yielding K_4 solutions, including the degenerate solution $\{x^0_{3/5,k} \to \infty\}$. These K_4 solutions are then used once on each node K_3 and K_5, each generating $\frac{K_4\,(K_4-1)\times\ldots\times(K_4-K_j)}{K_j!}$ (with $j = 3, 5$) number of solutions. For a bosonic node, however, we may pick the same solution repeatedly.

Having distributed the solutions for $x^0_{3,k}$ and $x^0_{5,k}$ one then determines p^1_k from (7.77) and finally solves for the energy shift using (7.79) or (7.87). The obtained value is what we then compare with a direct diagonalization of the string Hamiltonian.

8 Comparing the Bethe equations with string theory

After we have transformed the general all loop Bethe equations to suitable variables, we will confront its solutions to the string results of chapter 3. We will present analytical results for all closed subsectors of $\mathfrak{psu}(2,2|4)$ listed in table 10.

8.1 The rank one sectors $\mathfrak{su}(2)$, $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$

The $\mathfrak{su}(2)$ sector is spanned by α_1^+ operators. Choosing the grading $\eta_1 = \eta_2 = 1$ one finds K_4 as the only excited node. Because there is no contribution of $x^0_{3,j}$, $x^0_{3,j}$ equation (7.77) simplifies drastically and we can directly read of the energy shift for abritrary mode numbers from (7.87).
One easily finds

$$(\eta_1+\eta_2)\frac{1}{P_+}\frac{\widetilde{\lambda}}{32\pi^2}\sum_{k=1}^{K_4'}\frac{(p_k^0)^2}{\omega_k^2}\nu_k(\nu_k-1)=\frac{\widetilde{\lambda}}{P_+}\sum_{i=1}^{K_4}\frac{n_i^2}{\omega_{n_i}^2}\left(\nu_{n_i}-1\right)$$

and with a bit more algebra

$$-\frac{1}{P_+}\frac{\widetilde{\lambda}}{16\pi^2}\sum_{k=1}^{K_4'}\nu_k\frac{p_k^0\,\widetilde{p}_k}{\omega_k}=-\frac{\widetilde{\lambda}}{2P_+}\sum_{\substack{i,j=1\\ i\neq j}}^{K_4}\frac{(n_i+n_j)^2}{\omega_{n_i}\omega_{n_j}} \quad .$$

Inserting this results in (7.87) reproduces the $\mathfrak{su}(2)$ string formula (3.62) of section 3.6.

The $\mathfrak{sl}(2)$ sector consists of only β_1^+, thus we choose the grading $\eta_1 = \eta_2 = -1$. One now immediately sees by comparing (7.87) and (7.77) to the $\mathfrak{su}(2)$ sector, that only the signature of δP_- changes.

The $\mathfrak{su}(1|1)$ sector is assembled of the operator θ_1^+. Using the grading $\eta_1 = -\eta_2 = -1$ the vanishing of δP_- is obvious, which is in full agreement with the string result.

8.2 The $\mathfrak{su}(1|2)$ sector

As stated, this sector is spanned by the oscillators α_1^+ and θ_1^+. The contributing parts from the string Hamiltonian are $\mathcal{H}_{bb}$ and $\mathcal{H}_{bf}$. The explicit expression for the effective $\mathfrak{su}(1|2)$ Hamiltonian can be found in (5.7). Let us count the number of solutions for the grading $\eta_1 = \eta_2 = 1$. Then the only excited nodes of the Dynkin diagram in this sector are K_4 and K_3, so the polynomials in (7.88) give $K_4 - \nu$ solutions[17]. Two of these solutions are always 0 and ∞ while the other $K_4 - 2 - \nu$ are non-trivial. Before we perform the actual computation let us count the number of solutions. Say we have a total of K_3 θ_1^+ oscillators and $K_4 - K_3$ α_1^+ oscillators, then this state will yield $\frac{(K_4-\nu)\times(K_4-\nu-1)\times\ldots\times(K_4-\nu-K_3+1)}{K_3!}$ number of solutions. So, for all possible combinations of a general K_4 impurity state the number of solutions are

$$\sum_{K_3=0}^{K_4-\nu}\binom{K_4-\nu}{K_3}=2^{K_4-\nu}. \tag{8.1}$$

Since the worldsheet Hamiltonian is a $2^{K_4-\nu}\times 2^{K_4-\nu}$ matrix, the number of solutions matches.

[16] Mind that $\widetilde{p}$ has defined to equal p_k^1 given in (7.77).

[17] The number of confluent mode numbers must satisfy, $\nu \leq K_4 - K_3 + 1$ since we cannot have fermionic excitations of the same flavor with confluent mode numbers.

8.2.1 Two impurities

For the two impurity sector the perturbative string Hamiltonian is a 4×4 matrix, but we are only interested in a 2×2 submatrix since the other part falls into the rank one sectors $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$. The relevant matrix elements, with mode numbers $\{q, -q\}$, are

$$\left(\begin{array}{c|c|c} & \alpha^+_{1,q}\theta^+_{1,-q}|0\rangle & \alpha^+_{1,-q}\theta^+_{1,q}|0\rangle \\ \hline \langle 0|\alpha^-_{1,q}\theta^-_{1,-q} & \mathcal{H}_{bf} & \mathcal{H}_{bf} \\ \hline \langle 0|\alpha^-_{1,-q}\theta^-_{1,q} & \mathcal{H}_{bf} & \mathcal{H}_{bf} \end{array}\right)$$

The energy shifts are the non-zero values in (5.9). Now, the interesting question is of course if we can reproduce this result from the Bethe equations. For the two impurity state $\alpha^+\theta^+|0\rangle$ it is easiest to work with the gradings $\eta_1 = -1$ and $\eta_2 = 1$ where we have $K_4 = 2$ and $K_3 = 1$. From (7.88) wee see that the only solutions for $x_{3,k}$ are 0 and ∞. Since we have two roots, and one K_3 excitation we get two solutions for p^1_k. Solving (7.77) gives $p^1_k = \pm p^0_k$. Plugging these into (7.79) gives

$$\delta P_- = \pm\frac{\widetilde{\lambda}}{P_+}\sum_{j=1}^{2}\frac{q_j^2}{\omega_{q_j}} = \pm 2\frac{\widetilde{\lambda}}{P_+}\frac{q^2}{\omega_q} =: \kappa_2, \tag{8.2}$$

which equals the non-zero values in (5.9).

8.2.2 Three impurities, distinct mode numbers

The full perturbative string Hamiltonian is a 8×8 matrix but the relevant $\mathfrak{su}(1|2)$ part splits up into two independent submatrices coming from the Fermi-Fermi matrix elements $\langle 0|\alpha_1^-\alpha_1^-\theta_1^-(\mathcal{H}_{bb} + \mathcal{H}_{bf})\theta_1^+\alpha_1^+\alpha_1^+|0\rangle$ and the Bose-Bose elements $\langle 0|\alpha_1^-\theta_1^-\theta_1^-(\mathcal{H}_{bf})\theta_1^+\theta_1^+\alpha_1^+|0\rangle$. Schematically written we have,

$$\left(\begin{array}{c|c|c} & \alpha_1^+\alpha_1^+\theta_1^+|0\rangle & \alpha_1^+\theta_1^+\theta_1^+|0\rangle \\ \hline \langle 0|\theta_1^-\alpha_1^-\alpha_1^- & (\mathcal{H}_{bb}+\mathcal{H}_{bf})^{3\times 3} & 0_{3\times 3} \\ \hline \langle 0|\theta_1^-\theta_1^-\alpha_1^- & 0_{3\times 3} & \mathcal{H}_{bf}^{3\times 3} \end{array}\right) \tag{8.3}$$

The eigenvalues of the Bose-Bose submatrix, the bottom right, is given in (5.10). To reproduce these shifts from the Bethe equations we once again choose $\eta_1 = -1$ and $\eta_2 = 1$ so $K_4 = 3$ and $K_3 = 1$. Solving (7.88) give, as before, $x^0_{3,k} = \{0, \infty\}$ together with a novel third solution

$$y = \frac{(2+\omega_{q_1}+\omega_{q_2})\,x^0_{4,3} + (2+\omega_{q_2}+\omega_{q_3})\,x^0_{4,1} + (2+\omega_{q_1}+\omega_{q_3})\,x^0_{4,2}}{3+\omega_{q_1}+\omega_{q_2}+\omega_{q_3}}\,. \tag{8.4}$$

The first two solutions, 0 and ∞, give as before $p^1_k = \pm p^0_k$. For generic values of K_4, and with $K_3 = 1$, these two solutions will always appear. Using the third solution in (7.77) yields

$$p^1_k = \frac{1+\omega_k}{x^0_{4,k} - y} - p^0_k. \tag{8.5}$$

Plugging this into (7.79), together with some algebra, gives the three solutions

$$\delta P_- = \Big\{ \pm\frac{\widetilde{\lambda}}{P_+}\sum_{j=1}^{3}\frac{q_j^2}{\omega_{q_j}},\ \frac{\widetilde{\lambda}}{P_+\omega_{q_1}\omega_{q_2}\omega_{q_3}}\sum_{j=1}^{3} q_j^2\omega_{q_j}\Big\} =: \Lambda_3\,, \tag{8.6}$$

which agrees with the string result obtained in (5.10).

Let us now focus on the Fermi-Fermi matrix elements, the upper left 3×3 block of (8.3). First, (7.88) give the same three solutions as before, namely $\{0,\infty,y\}$ with the same y as in (8.4). Since $K_3=2$ we now, for each p_k^1, use two of the solutions for $x_{3,k}^0$

$$p_k^1=(1+\omega_{p_k^0})\Big(\frac{1}{x_{4,k}^0-x_{3,1}^0}+\frac{1}{x_{4,k}^0-x_{3,2}^0}\Big)-2p_k^0. \tag{8.7}$$

The three possible distributions of the roots, $\{0,\infty\}$, $\{0,y\}$ and $\{y,\infty\}$, give the three solutions

$$\delta P_-=\Big\{0,\ -\frac{\widetilde{\lambda}}{P_+}\frac{1}{16\pi^2}\sum_{j=1}^{K_4}\frac{p_k^0}{\omega_k}\Big((\frac{1+\omega_k}{x_{4,k}^0-y}-p_k^0)\pm p_k^0\Big)\Big\}=:\Omega_3 \tag{8.8}$$

With a little bit of work one can show that these match the eigenvalues from the string Hamiltonian in (5.11).

8.2.3 Three impurities, confluent mode numbers

For three impurities, with mode numbers $\{q,q,-2q\}$, the only states that do not fall into the previously checked rank one sectors are $\alpha_1^+\alpha_1^+\theta_1^+|0\rangle$ and $\alpha_1^+\theta_1^+\theta_1^+|0\rangle$. For the former, we get from (7.77) (with grading $\eta_1=\eta_2=1$)

$$\widetilde{p}_q=-2p_q^0+\frac{2\omega_q+\omega_{2q}}{x_{4,q}^0-x_{4,2q}^0}-\frac{1+\omega_q}{x_{4,q}^0-x_3^0},\quad \widetilde{p}_{2q}=-2p_{2q}^0+2\frac{2\omega_q+\omega_{2q}}{x_{4,2q}^0-x_{4,q}^0}-\frac{1+\omega_{2q}}{x_{4,2q}^0-x_3^0}.$$

The polynomials in (7.88) give two solutions $\{0,\infty\}$ for $x_{3,k}^0$. Using these in (7.87), together with some algebra, yields two energy shifts

$$\begin{aligned}\delta P_-=\ \frac{2q^2\widetilde{\lambda}}{P_+\omega_q^2\omega_{2q}}\Big\{\ &\frac{3\omega_{2q}+(2\omega_q+\omega_{2q})(4\omega_q(1+\omega_q)+\omega_{2q})}{3+2\omega_q+\omega_{2q}}\,,\\ &-\frac{4\omega_q^2-(3-4\omega_q^2)\omega_{2q}-(1-2\omega_q)\omega_{2q}^2}{3+2\omega_q+\omega_{2q}}\ \Big\}.\end{aligned} \tag{8.9}$$

It is not immediately apparent that this equals the string Hamiltonian result (5.13) but after some work one can show that these two solutions are equal.

For the second state, $\alpha_1^+\theta_1^+\theta_1^+|0\rangle$, we have $K_3=2$ and the two roots $\{0,\infty\}$ for $x_{3,k}^0$ can only be distributed in one way. By doing analogously as above and using (7.77) in (7.87), we find

$$\delta P_-=\frac{2q^2\widetilde{\lambda}}{P_+}\frac{(\omega_q+\omega_{2q})}{\omega_q\omega_{2q}}, \tag{8.10}$$

which reproduces the string Hamiltonian result of (5.12).

8.3 The $\mathfrak{su}(1,1|2)$ sector

Now we turn to the larger $\mathfrak{su}(1,1|2)$ sector. The procedure is the same as above but now both sides of the Dynkin diagram gets excited and a general state has the three middle nodes K_3,K_4 and K_5 excited. We are allowed to pick the same solution, on the K_3 and K_5 node, but as before we must put distinct solutions on the fermionic nodes. In this sector a new feature appears: The states $\alpha_1^+\beta_1^+$ and $\theta_1^+\eta_1^+$ are allowed to mix. Also, in the case of confluent mode

numbers, it turns out that we have to make use of different gradings on some states to generate all the solutions from the string Hamiltonian.

Let us first investigate if the number of solutions from the string Hamiltonian and the Bethe equations match. A general $\mathfrak{su}(1,1|2)$ state with K_4 excitations and *distinct* mode numbers will yield a $2^{2K_4} \times 2^{2K_4}$ matrix and thus 2^{2K_4} energy shifts. The total number of solutions from the Bethe equations are just the square of (8.1), with $\nu = 0$, which equals the number of eigenvalues from the perturbative string Hamiltonian (5.14).

8.3.1 Two impurities

The Hamiltonian is a 16×16 matrix but it is only a 13×13 part which lies outside the already calculated $\mathfrak{su}(1|2)$ sector. There are seven different independent submatrices where the largest is a 4×4 matrix and is generated by the base kets $\alpha_1^+\beta_1^+|0\rangle$ and $\theta_1^+\eta_1^+|0\rangle$. There are three 2×2 submatrices, $\alpha_1^+\eta_1^+|0\rangle$, $\beta_1^+\theta_1^+|0\rangle$, $\beta_1^+\eta_1^+|0\rangle$ and three are scalar contributions $\beta_1^+\beta_1^+|0\rangle, \eta_1^+\eta_1^+|0\rangle$, $\theta_1^+\theta_1^+|0\rangle$. The latter will give the same results as presented in 8.1 so these we will ignore. The only part with mixing is the subpart generated by $\alpha_1^+\beta_1^+|0\rangle$ and $\theta_1^+\eta_1^+|0\rangle$. To calculate the energy shifts we start by solving (7.88) and, as before, the two solutions are $\{0, \infty\}$. With $\eta_1 = -1$ and $\eta_2 = 1$, so $K_4 = 3$ and $K_5 = K_3 = 1$, we have

$$p_k^1 = (1 + \omega_k)\Big(\frac{1}{x_{4,k}^0 - x_{3,k}^0} - \frac{1}{x_{4,k}^0 - x_{5,k}^0}\Big). \tag{8.11}$$

Whenever we pick the same solution for $x_{3,k}^0$ and $x_{5,k}^0$ we get zero and since we can do this in two ways we get two zero solutions.
The other two solutions are obtained by setting $\{x_{3,k}^0, x_{5,k}^0\} = \{0, \infty\}$ and $\{\infty, 0\}$ which gives $p_k^1 = \pm 2p_k^0$. Using this in (7.79) gives

$$\delta P_- = (0, 0, \pm\frac{2\widetilde{\lambda}}{P_+}\sum_{j=1}^{2}\frac{q_j^2}{\omega_{q_j}}), \tag{8.12}$$

which is in agreement with the string Hamiltonian result in (5.15).

For the three parts $\alpha^+\eta^+|0\rangle, \beta^+\theta^+|0\rangle$ and $\beta^+\eta^+|0\rangle$, we see that solving for the first state is analogous to the discussion after (8.2) but with $\eta_1 = 1$ and $\eta_2 = -1$. For the two other, the procedure will again be identical if we choose the opposite gradings. That is, for $\beta^+\theta^+|0\rangle$ we pick $\eta_1 = 1$ and $\eta_2 = -1$, while for $\beta^+\eta^+|0\rangle$ we choose $\eta_1 = -1$ and $\eta_2 = 1$ which give the same set of solution for all three states

$$\delta P_- = \pm\frac{2\widetilde{\lambda}}{P_+}\frac{q^2}{\omega_q}, \tag{8.13}$$

which is in agreement with (5.16).

8.3.2 Three impurities, distinct mode numbers

The full perturbative string Hamiltonian will now be a 64×64 matrix with non trivial 3×3 and 9×9 subsectors. Since the logic of solving the Bethe equation should be clear by now, we only present the obtained results in tabular form. Also, to make the comparison with the string Hamiltonian more transparent, we now also label the states by their charges $\{S_+, S_-, J_+, J_-\}$. The energy shifts for the 3×3 parts are given in table 12 and for the larger 9×9 subparts in

$\{\eta_1,\eta_2\}$	$\{K_1+K_3,K_4,K_5+K_7\}$	$\{S_+,S_-,J_+,J_-\}$	δP_-
$\{-,+\}$	$\{2,3,0\}$	$\{0,1,3,2\}_{\alpha_1^+\alpha_1^+\theta_1^+}$	Ω_3
$\{+,-\}$	$\{0,3,2\}$	$\{1,0,2,3\}_{\alpha_1^+\alpha_1^+\eta_1^+}$	$-\Omega_3$
$\{-,+\}$	$\{0,3,2\}$	$\{2,3,1,0\}_{\beta_1^+\beta_1^+\theta_1^+}$	Ω_3
$\{+,-\}$	$\{2,3,0\}$	$\{3,2,0,1\}_{\beta_1^+\beta_1^+\eta_1^+}$	$-\Omega_3$
$\{-,+\}$	$\{1,3,0\}$	$\{0,2,3,1\}_{\theta_1^+\theta_1^+\alpha_1^+}$	Λ_3
$\{-,+\}$	$\{0,3,1\}$	$\{1,3,2,0\}_{\theta_1^+\theta_1^+\beta_1^+}$	$-\Lambda_3$
$\{+,-\}$	$\{0,3,1\}$	$\{2,0,1,3\}_{\eta_1^+\eta_1^+\alpha_1^+}$	Λ_3
$\{+,-\}$	$\{1,3,0\}$	$\{3,1,0,2\}_{\eta_1^+\eta_1^+\beta_1^+}$	$-\Lambda_3$

Table 12: The states reproducing the 3×3 submatrices of the string Hamiltonian. Ω_3 and Λ_3, where the subscript indicate the number of solutions as given in (8.8) for Ω_3 and (8.6) for Λ_3.

$\{\eta_1,\eta_2\}$	$\{K_1+K_3,K_4,K_5+K_7\}$	$\{S_+,S_-,J_+,J_-\}$	δP_-
$\{+,+\}$	$\{1,3,1\}$	$\{1,1,2,2\}_{(\alpha_1^+\alpha_1^+\beta_1^+),(\alpha_1^+\theta_1^+\eta_1^+)}$	Ω_9
$\{-,-\}$	$\{1,3,1\}$	$\{2,2,1,1\}_{(\alpha_1^+\beta_1^+\beta_1^+),(\beta_1^+\theta_1^+\eta_1^+)}$	$-\Omega_9$
$\{-,+\}$	$\{1,3,1\}$	$\{1,2,2,1\}_{(\alpha_1^+\beta_1^+\theta_1^+),(\theta_1^+\theta_1^+\eta_1^+)}$	Λ_9
$\{+,-\}$	$\{1,3,1\}$	$\{2,1,1,2\}_{(\alpha_1^+\beta_1^+\eta_1^+,(\theta_1^+\eta_1^+\eta_1^+)}$	$-\Lambda_9$

Table 13: The states reproducing the 9×9 submatrices of the string Hamiltonian. Ω_9 and Λ_9, where the subscript indicate the number of solutions, is given by (8.14) and (8.15).

table 13. For the larger sectors we have a mixing between states of different boson and fermion number.
The functions Ω_9 and Λ_9 in table 13 depend on the mode numbers $\{q_1,q_2,q_3\}$ and are given by

$$\Omega_9=\frac{\widetilde{\lambda}}{P_+}\frac{1}{16\pi^2}\sum_{k=1}^{3}\frac{p^0_{q_k}}{\omega_{q_k}}\Big(\sum_{\substack{j=1\\ j\neq k}}^{3}\frac{2+\omega_{q_k}+\omega_{q_j}}{x^0_{4,q_k}-x^0_{4,q_j}}-\frac{1+\omega_{q_k}}{x^0_{4,q_k}-x^0_3}-\frac{1+\omega_{q_k}}{x^0_{4,q_k}-x^0_5})-p^0_{q_k}\Big) \tag{8.14}$$

$$\Lambda_9=-\frac{\widetilde{\lambda}}{P_+}\frac{1}{16\pi^2}\sum_{k=1}^{3}\frac{p^0_{q_k}}{\omega_{q_k}}\Big(\frac{1+\omega_{q_k}}{x^0_{4,q_k}-x^0_3}-\frac{1+\omega_{q_k}}{x^0_{4,q_k}-x^0_5}\Big). \tag{8.15}$$

To obtain the nine solutions for Ω_9 and Λ_9 one has to insert one of the three roots $\{0,\infty,y\}$ for each x^0_3 and x^0_5. We have not managed to match these results with the perturbative string Hamiltonian (5.14) analytically, but tested the agreement extensively numerically and found a precise agreement with the numerical string results of section 5.7.

8.3.3 Three impurities, confluent mode numbers

We will now look at three impurities with confluent mode numbers, $\{q,q,-2q\}$. With two distinct mode numbers we see from (7.88) that we have the two standard solutions $\{0,\infty\}$ for $x^0_{3,k}$ and $x^0_{5,k}$. The sectors exhibiting mixing, i.e. the states that span the 9×9 subparts of the previous section, now exhibit a new feature. The gradings are no longer equivalent and we will be forced to use both to generate all the desired solutions. The simpler states, that do not exhibit this feature, are presented in table 14 and the states where different gradings had to be used are presented in table 15.

$\{\eta_1,\eta_2\}$	$\{K_1+K_3,K_4,K_5+K_7\}$	$\{S_+,S_-,J_+,J_-\}$	δP_-
$\{+,+\}$	$\{1,3,0\}$	$\{0,1,3,2\}_{\alpha_1^+\alpha_1^+\theta_1^+}$	$\widetilde{\Omega}_2$
$\{+,+\}$	$\{0,3,1\}$	$\{1,0,2,3\}_{\alpha_1^+\alpha_1^+\eta_1^+}$	$\widetilde{\Omega}_2$
$\{-,-\}$	$\{0,3,1\}$	$\{2,3,1,0\}_{\beta_1^+\beta_1^+\theta_1^+}$	$-\widetilde{\Omega}_2$
$\{-,-\}$	$\{1,3,0\}$	$\{3,2,0,1\}_{\beta_1^+\beta_1^+\eta_1^+}$	$-\widetilde{\Omega}_2$
$\{+,+\}$	$\{2,3,0\}$	$\{0,2,3,1\}_{\theta_1^+\theta_1^+\alpha_1^+}$	$\widetilde{\lambda}_1$
$\{-,-\}$	$\{0,3,2\}$	$\{1,3,2,0\}_{\theta_1^+\theta_1^+\beta_1^+}$	$-\widetilde{\lambda}_1$
$\{+,+\}$	$\{0,3,2\}$	$\{2,0,1,3\}_{\eta_1^+\eta_1^+\alpha_1^+}$	$\widetilde{\lambda}_1$
$\{-,-\}$	$\{2,3,0\}$	$\{3,1,0,2\}_{\eta_1^+\eta_1^+\beta_1^+}$	$-\widetilde{\lambda}_1$

Table 14: The states reproducing the 2×2 submatrices for *confluent* mode numbers of the string Hamiltonian. $\widetilde{\Omega}_2$ and $\widetilde{\lambda}_2$, where the subscript indicate the number of solutions, is given by (8.9) and (8.10).

$\{\eta_1,\eta_2\}$	$\{K_1+K_3,K_4,K_5+K_7\}$	$\{S_+,S_-,J_+,J_-\}$	δP_-
$\{+,+\}$	$\{1,3,1\}$	$\{1,1,2,2\}_{(\alpha_1^+\alpha_1^+\beta_1^+),(\alpha_1^+\theta_1^+\eta_1^+)}$	Γ_4
$\{-,-\}$	$\{2,3,2\}$	$\{1,1,2,2\}_{(\alpha_1^+\alpha_1^+\beta_1^+),(\alpha_1^+\theta_1^+\eta_1^+)}$	$\widetilde{\Gamma}_1$
$\{-,-\}$	$\{1,3,1\}$	$\{2,2,1,1\}_{(\alpha_1^+\beta_1^+\beta_1^+),(\beta_1^+\theta_1^+\eta_1^+)}$	$-\Gamma_4$
$\{+,+\}$	$\{2,3,2\}$	$\{2,2,1,1\}_{(\alpha_1^+\beta_1^+\beta_1^+),(\beta_1^+\theta_1^+\eta_1^+)}$	$-\widetilde{\Gamma}_1$
$\{+,+\}$	$\{2,3,1\}$	$\{1,2,2,1\}_{(\alpha_1^+\beta_1^+\theta_1^+),(\theta_1^+\theta_1^+\eta_1^+)}$	$\widetilde{\Omega}_2$
$\{-,-\}$	$\{1,3,2\}$	$\{1,2,2,1\}_{(\alpha_1^+\beta_1^+\theta_1^+),(\theta_1^+\theta_1^+\eta_1^+)}$	$-\widetilde{\Omega}_2$
$\{-,-\}$	$\{2,3,1\}$	$\{2,1,1,2\}_{(\alpha_1^+\beta_1^+\eta_1^+,(\theta_1^+\eta_1^+\eta_1^+)}$	$-\widetilde{\Omega}_2$
$\{+,+\}$	$\{1,3,2\}$	$\{2,1,1,2\}_{(\alpha_1^+\beta_1^+\eta_1^+,(\theta_1^+\eta_1^+\eta_1^+)}$	$\widetilde{\Omega}_2$

Table 15: The states reproducing the larger submatrices, with *confluent* mode numbers, of the string Hamiltonian. The functions Γ_4 and $\widetilde{\Gamma}_1$ are given in (8.16) and $\widetilde{\Omega}_2$ is given in (8.9).

The energy shifts Γ_4 and $\widetilde{\Gamma}_1$ appearing in table 15 are given by

$$\begin{aligned}
\widetilde{\Gamma}_1 &= \frac{2q^2\widetilde{\lambda}}{P_+\omega_q^2\omega_{2q}}\Big(\frac{1}{\omega_q}+\frac{1}{\omega_{2q}}\Big)\,,\\
\Gamma_4 &= -\frac{2q^2\widetilde{\lambda}}{P_+\omega_q^2\omega_{2q}}\Big\{(\frac{1}{\omega_q}+\frac{1}{\omega_{2q}}),(\frac{1}{\omega_q}+\frac{1}{\omega_{2q}}),\frac{3\omega_{2q}+(2\omega_q+\omega_{2q})(\omega_{2q}+\omega_q(7+6\omega_q+\omega_{2q}))}{3+2\omega_q+\omega_{2q}},\\
&\quad\frac{3\omega_{2q}-(2\omega_q+\omega_{2q})(\omega_q(5+2\omega_q+3\omega_{2q})-\omega_{2q})}{3+2\omega_q+\omega_{2q}}\Big\}.
\end{aligned}\tag{8.16}$$

For the comparison to the eigenvalues of the string Hamiltonian we had to resort to numerical verifications, which have shown a precise agreement with the numerical string results of section 5.7.

8.3.4 Higher impurities

In going beyond three impurities numerical calculations on both sides, the Bethe equations and the string Hamiltonian, have been performed for a number of four and five impurity states. All numerical energy shifts match precisely, the tested configurations are listed in table 16.

$\{S_+, S_-, J_+, J_-\}$	State pattern			Number of solutions
$\{2,2,2,2\}$	$\theta_1^+\theta_1^+\eta_1^+\eta_1^+\|0\rangle$,	$\theta_1^+\eta_1^+\beta_1^+\alpha_1^+\|0\rangle$,	$\beta_1^+\beta_1^+\alpha_1^+\alpha_1^+\|0\rangle$	36 energy shifts
$\{2,2,3,3\}$	$\theta_1^+\theta_1^+\eta_1^+\eta_1^+\alpha_1^+\|0\rangle$,	$\theta_1^+\eta_1^+\beta_1^+\alpha_1^+\alpha_1^+\|0\rangle$,	$\beta_1^+\beta_1^+\alpha_1^+\alpha_1^+\alpha_1^+\|0\rangle$	100 energy shifts

Table 16: Checked 4 and 5 impurity states of $\mathfrak{su}(1,1|2)$.

8.4 The $\mathfrak{su}(2|3)$ sector

Now things become more complex. The polynomials (7.88) for a general state are highly non-linear, coupled and involve several variables. For this reason we will not be as thorough in our testing for the higher impurity cases as in the previous sections. The oscillators in this sector are $\alpha_1^+, \alpha_2^+, \theta_1^+$ and θ_2^+ where there is a mixing between $\alpha_1^+\alpha_2^+|0\rangle$ and $\theta_1^+\theta_2^+|0\rangle$. The string Hamiltonian is given in (5.17).

8.4.1 Two impurities

The $\mathfrak{su}(2|3)$ two impurity sector of the perturbative string Hamiltonian (5.17) will be a 12×12 matrix. Let us begin with the largest subpart, the one with mixing between $\alpha_1^+\alpha_2^+|0\rangle$ and $\theta_1^+\theta_2^+|0\rangle$. The excitation numbers, with grading $\eta_1=\eta_2=1$, for $\alpha_1^+\alpha_2^+|0\rangle$ are $K_1=K_2=K_3=1$ and $K_4=2$ while for $\theta_1^+\theta_2^+|0\rangle$ we have $K_2=1$ and $K_3=K_4=2$. Here the dynamically transformed version of the Bethe equations is advantageous, as it makes explicit that the relevant combination $K_1+K_3=2$ is the same for these two states. This is how the Bethe equations take care of the mixing. Solving for u_2^0 in (7.88), and using $u_{3,k}^0 = x_{3,k}^0 + \frac{\widetilde{\lambda}}{64\pi^2}\frac{1}{x_{3,k}^0}$, gives

$$u_2^0 = \frac{1}{2}(x_{3,1}^0 + x_{3,2}^0 + \frac{\widetilde{\lambda}}{64\pi^2}(\frac{1}{x_{3,1}^0}+\frac{1}{x_{3,2}^0})).$$

Plugging this into the second line of (7.88) gives

$$\begin{aligned}
&\frac{1}{x_{3,1}^0 - x_{3,2}^0 + \frac{\widetilde{\lambda}}{64\pi^2}(\frac{1}{x_{3,1}^0}-\frac{1}{x_{3,2}^0})} + \sum_{j=1}^{2}\frac{1+\omega_j}{x_{4,j}^0 - x_{3,1}^0} = 0, \qquad (8.17)\\
&\frac{1}{x_{3,2}^0 - x_{3,1}^0 + \frac{\widetilde{\lambda}}{64\pi^2}(\frac{1}{x_{3,2}^0}-\frac{1}{x_{3,1}^0})} + \sum_{j=1}^{2}\frac{1+\omega_j}{x_{4,j}^0 - x_{3,2}^0} = 0.
\end{aligned}$$

We can add these two equations above and see that four solutions are

$$(x_{3,1}^0, x_{3,2}^0) = (0,0),(0,\infty),(\infty,0),(\infty,\infty)\ .$$

This may at first glance seem strange since the seemingly equivalent state $\theta_1^+\theta_2^+|0\rangle$ only has the K_2 and K_3 node excited, implying that we can not pick the same solution twice for $x_{3,k}^0$ since K_3 is fermionic. However, the correct state to use is the $\alpha_1^+\alpha_2^+|0\rangle$ state. Here two different fermionic nodes K_1 and K_3 are excited and because of this we can use the same solutions on both nodes simultaneously.

Let us now turn to the calculation of the energy shifts for the these four states. We use the solutions from (8.17) in (7.77) and plug this into (7.79) which gives

$$\delta P_- = \{0,0,\pm\frac{\widetilde{\lambda}}{P_+}\frac{4q^2}{\omega_q}\} =: \chi_4, \qquad (8.18)$$

$\{\eta_1,\eta_2\}$	$\{K_1+K_3,K_2,K_4\}$	$\{S_+,S_-,J_+,J_-\}$	δP_-
$\{+,+\}$	$\{2,1,2\}$	$\{0,0,2,0\}_{(\alpha_1^+\alpha_2^+),(\theta_1^+\theta_2^+)}$	χ_4
$\{-,+\}$	$\{1,0,2\}$	$\{0,1,2,1\}_{\alpha_1^+\theta_1^+}$	κ_2
$\{-,+\}$	$\{1,0,2\}$	$\{0,-1,2,-1\}_{\alpha_2^+\theta_2^+}$	κ_2
$\{+,+\}$	$\{1,1,2\}$	$\{0,-1,2,1\}_{\alpha_1^+\theta_2^+}$	κ_2
$\{+,+\}$	$\{1,1,2\}$	$\{0,1,2,-1\}_{\alpha_2^+\theta_1^+}$	κ_2

Table 17: The two impurity states that fall into to the rank ≥ 1 sectors for $\mathfrak{su}(2|3)$. Here χ_4 is given by (8.18) and κ_2 is given by (8.2). For two of the states we have permutated the space-time indices.

which is in perfect agreement with (5.18). The energy shifts for the other states follows immediately and we present the results in table 17. From this table we see that all the energy shifts from (5.17), presented in (5.19) and (5.18), are reproduced.

8.4.2 Higher impurities

Due to the non linearity of the polynomials relating the Bethe roots we will only present results for excitations with $K_2 = K_3 = 1$, corresponding to states of the form $\alpha_1^+ \ldots \alpha_1^+\theta_2^+|0\rangle$ with space-time charge vector $\{S_+,S_-,J_+,J_-\} = \{0,-1,K_4,K_4-1\}$. From the first line in (7.88) we see that

$$\frac{1}{u_2^0 - (x_3^0 + \frac{\tilde{\lambda}}{64\pi^2}\frac{1}{x_3^0})} = 0,$$

and using this in the second line implies that the equation for x_3^0 reduces to the familiar form

$$\sum_{j=1}^{K_4} \frac{1+\omega_j}{x_{4,j}^0 - x_3^0} = 0. \tag{8.19}$$

Thus, the energy shift for this state is the same as for the $\alpha_1^+ \ldots \alpha_1^+\theta_1^+|0\rangle$ states. For $K_4 = 3$, the energy shift is presented in (8.6). For $K_4 - 1$ number of α_1^+ excitations and one θ_2^+ excitation, the energy shift, with gradings $\{+,+\}$, is given by

$$\Lambda_{K_4} = \frac{1}{16\pi^2}\sum_{k=1}^{K_4}\frac{p_k^0}{\omega_k}\Big(\sum_{\substack{j=1\\ j\neq k}}^{K_4}\frac{2+\omega_j+\omega_k}{x_{4,k}^0 - x_{4,j}^0} - \frac{1+\omega_k}{x_{4,k}^0 - x_3^0} - p_k^0(K_4-1)\Big). \tag{8.20}$$

This prediction we have verified numerically for $K_4 \leq 6$ with the energy shifts obtained by diagonalization of the string Hamiltonian (5.17).

$\{\eta_1,\eta_2\}$	$\{K_1+K_3,K_2,K_4\}$	$\{S_+,S_-,J_+,J_-\}$	δP_-
$\{+,+\}$	$\{1,1,K_4\}$	$\{0,-1,K_4,K_4-1\}_{(\alpha_1^+\ldots\alpha_1^+\theta_2^+)}$	Λ_{K_4}

Table 18: Higher impurity states from the $\mathfrak{su}(2|3)$ sector for states of the form $\alpha_1^+ \ldots \alpha_1^+\theta_2^+|0\rangle$. The function Λ_{K_4}, where K_4 indicates the number of solutions, is given in (8.20).

9 Conclusions and outlook

9.1 Comparison of Hamiltonian method and Bethe Ansatz

With the ABAKUS software at hand the calculation of arbitrary $AdS_5 \times S^5$ superstring spectra in the near plane-wave limit has become an easy task. Nevertheless the computation can only yield numerical eigenvalues of the Hamiltonian apart from very simple cases. The general solvability of the Bethe equations is highly restricted, due to its non linearity and coupled structure. On the one hand numerical solutions of the Bethe equations are hardly computable, on the other hand one can obtain analytic solutions in more cases than for the string Hamiltonian.

For example it is much easier to calculate the energy shifts analytically using the Bethe equations for generic $\mathfrak{su}(1,1|2)$ states. The characteristic polynomial from the perturbative string Hamiltonian is of degree 2^{2K_4} whereas the polynomials needed to be solved in the Bethe equations (7.88) are of degree $K_4 - 2$. Still, one generically deals with polynomials of a high degree, making it hard to explicitly find analytical results for states with large total excitation number K_4.

When it comes to the dynamical sector $\mathfrak{su}(2|2)$, a direct comparison is much more difficult due to the non-linearity and coupled structure of the Bethe equations. Here analytical results were established only for the case of two impurities. For three or more impurity states it turned out to be impossible to solve the Bethe equations, even numerically, except for one particular excitation pattern.

Table 19 provides an overview of the computed spectra by perturbative diagonalization of the string Hamiltonian and it shows where solutions of the Bethe equations are available.

impurities	$\mathfrak{su}(2)$ α_1^+	$\mathfrak{sl}(2)$ β_1^+	$\mathfrak{su}(1\|1)$ θ_1^+	$\mathfrak{su}(1\|2)$ θ_1^+, α_1^+	$\mathfrak{su}(1,1\|2)$ $\theta_2^+, \theta_1^+, \alpha_2^+, \alpha_1^+$	$\mathfrak{su}(2\|3)$ $\theta_1^+, \eta_1^+, \beta_1^+, \alpha_1^+$	$\mathfrak{psu}(2,2\|4)$ $\theta_4, ..., \eta_4, ...\beta_4, ..., \alpha_1$
2	√√	√√	√√	√√	√√	√√	
3	√√	√√	√√	√√	√√	√√	
4	√√	√√	√√		$\sqrt{}_{100,136}$	✓	
5	√√	√√	√√			✓	
6	√√	√√	√√			✓	
>6	√√	√√	√√				

√√ – solutions matched analytically √√ – solutions matched numerically

computed energy eigenvalues using ABAKUS software

Table 19: Schematic overview of the computed string eigenvalues and the available solutions of the Bethe equations. √√ indicates that we have calculated all possible charge configurations belonging to the sector and matched the Hamiltonian eigenvalues with the Bethe roots. $\sqrt{}_x$ indicates we have computed a matrix with x eigenvalues on the string side and matched them to solutions of the Bethe equations. If a ✓ is given, only few Bethe roots are available, which can be matched with string eigenvalues.

9.2 Summary

In this work we have explored the quantum integrability of the $AdS_5 \times S^5$ superstring by confronting the explicit diagonalization of the light-cone gauged string Hamiltonian with the solutions of a conjectured set of Bethe equations. All computations have been performed in the near plane-wave limit.

In the first part of the present diploma thesis the derivation of the quantized $AdS_5 \times S^5$ superstring Hamiltonian has been reviewed. It was argued that computing generic string spectra requires an especially designed computer algebra system. In particular it was shown, that in most cases the computational costs, for calculating the energy corrections, scale exponentially with respect to the number of different string modes. For this purpose the ABAKUS software, which computes the next to leading order correction of the energy, was designed as part of the present diploma thesis.

In the second part of this work, an introduction to the Bethe Ansatz has been given in order to motivate the presented Bethe equations. These equations were perturbatively converted into a set of light-cone Bethe equations which are expected to yield the spectrum of the $AdS_5 \times S^5$ superstring in the near plane-wave limit. Moreover, it was demonstrated how excited string states may be translated to distributions of spectral parameters in the Bethe equations. We have explicitly compared the predictions from the light-cone Bethe equations with direct diagonalization of the string Hamiltonian using the ABAKUS software. Perfect agreement has been found in all analyzed cases giving us a strong confidence in the validity of the light-cone Bethe equations for these classes of operators.

The presented results yield strong evidence that the energy spectrum of the $AdS_5 \times S^5$ superstring is described by a set of Bethe equations, which implies that the $AdS_5 \times S^5$ superstring is indeed an integrable quantum system, at least up to order $1/P_+$. The scaling dimension of composite gauge invariant operators of $\mathcal{N} = 4$ super Yang-Mills theory is encoded in the set of Bethe equations used. Thus this work represents a further highly non-trivial test of the conjectured AdS/CFT correspondence.

9.3 Outlook

In the light of this analysis it would be interesting to extend the perturbative studies of the string Hamiltonian to next order $\mathcal{O}(P_+^{-2})$. This is a very complicated problem due to normal ordering ambiguities. However, it might be tackled by making use of the symmetry algebra as discussed in [7] and [14].

Further on there are no Bethe equations at hand for finite size spin chains. In order to construct all loop equations for a finite size system, one has to incorporate interactions with a range that exceed the length of the chain, i.e. one is confronted with interactions wrapping around the spin chain. Bethe equations for such systems are currently not known.

Moreover it is still an open question how to quantize the $AdS_5 \times S^5$ superstring when all quantum numbers are finite. In the light of the presented results, it would be interesting to research if the energy spectrum of the superstring still matches the scaling dimension of the $\mathcal{N} = 4$ super Yang-Mills theory, which would yield even stronger evidence for the validity of the AdS/CFT correspondence.

A Further details on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ string Hamiltonian

This section provides further details on the derivation of the light-cone Hamiltonian as well as properties of the involved quantities such as the matrices Γ_c and the auxiliary functions f_n, g_n.

A.1 Solving the Virasoro constraint for the Hamiltonian

To resolve the second Virasoro constraint in (3.15) for p_- we rewrite it in the form

$$0 = (p_+^2 + p_-^2)G^{++} + 2p_+p_-G^{+-} + p_M p_N G^{MN} + \lambda G_{--}x'^{-}x'^{-} + \lambda G_{MN}x'^{M}x'^{N} \ . \tag{A.1}$$

The quantity x'^{-} is given by (3.13) as $x'^{-} = (p_M x'^{M})/P_+$. In order to expand in the number of fields we rescale

$$x_M \to \sqrt{\tfrac{2}{P_+}}\, x_M \quad , \qquad p_M \to \sqrt{\tfrac{P_+}{2}}\, p_M \ . \tag{A.2}$$

In the limit $P_+ \to \infty$ and in terms of the rescaled fields $x_M = \{z_a, y_s\}$ the metric components take the form

$$\begin{aligned}
G_{++} = G_{--} &= -\frac{y^2+z^2}{2P_+} + \frac{y^4-z^4}{2P_+^2} \ , \quad G_{+-} = \frac{1}{2} + \frac{-y^2+z^2}{2P_+} + \frac{y^4+z^4}{2P_+^2} \ , \\
G^{++} = G^{--} &= 2\,\frac{y^2+z^2}{P_+} + 2\,\frac{y^4-z^4}{P_+^2} \ , \quad G^{+-} = 2 + 2\,\frac{y^2-z^2}{P_+} + 2\,\frac{y^4+z^4}{P_+^2} \ , \\
G_{ab} &= \delta_{ab}\left(1 + \frac{z^2}{P_+} + \frac{3}{4}\frac{z^4}{P_+^2}\right) \ , \quad G_{su} = \delta_{su}\left(1 - \frac{y^2}{P_+} + \frac{3}{4}\frac{y^4}{P_+^2}\right) \ .
\end{aligned}$$

Since also the metric is expanded with respect to P_+, here after indices are contracted using Kronecker delta and for the sake of simplicity all indices are written as lower indices. In terms of the rescaled fields one finds the bosonic Hamitonian as solution of (A.1):

$$\begin{aligned}
\mathcal{H} = -p_- = \quad & \frac{1}{2}\left(p_a^{(z)}p_a^{(z)} + p_s^{(y)}p_s^{(y)} + z_a\, z_a + y_s\, y_s + \widetilde{\lambda}(z'_a\, z'_a + y'_s\, y'_s)\right) \\
& + \frac{1}{P_+}\Big(\quad p_s^{(y)}p_s^{(y)} z_a\, z_a + p_a^{(z)}p_a^{(z)} y_s\, y_s \\
& \qquad + \widetilde{\lambda}(y'_s\, y'_s\ z_a\, z_a - z'_a\, z'_a\ y_s\, y_s) + 2\widetilde{\lambda}(z'_a\, z'_a\ z_b\, z_b - y'_s\, y'_s\ y_u\, y_u)\Big) \quad .
\end{aligned} \tag{A.3}$$

The Hamiltonian (A.3) can be further simplified using a canonical transformation [7] generated by

$$V(x,p) = \frac{1}{2P_+}\big(p_s^{(y)} y_s\, z_a\, z_a - p_a^{(z)} z_a\, y_s\, y_s\big) \qquad\qquad (x = \{z_a, y_s\}).$$

The transformed coordinates $\widetilde{x}$ and momenta $\widetilde{p}$ are given by the standard expressions

$$\widetilde{x}_M = e^{\{V,\,\cdot\,\}} x_M \quad , \qquad \widetilde{p}_M = e^{\{V,\,\cdot\,\}} p_M \quad ,$$

where the poisson bracked is defined as

$$\{V(p,x), B(p,x)\} = \frac{\partial V}{\partial p_M}\frac{\partial B}{\partial x_M} - \frac{\partial V}{\partial x_M}\frac{\partial B}{\partial p_M} \quad . \tag{A.4}$$

In our case one finds

$$\widetilde{z}_a = z_a\big(1+\tfrac{1}{2P_+}y_s y_s\big) \quad , \qquad \widetilde{y}_s = y_s\big(1-\tfrac{1}{2P_+}z_a z_a\big)$$
$$\widetilde{p}_a^{(z)} = p_a^{(z)} + \frac{1}{2P_+}\big(p_a^{(z)}\ y_s\, y_s - 2\, z_a\ p_s^{(y)} y_s\big) \quad , \qquad \widetilde{p}_s^{(y)} = p_s^{(y)} - \frac{1}{2P_+}\big(p_s^{(y)}\ z_a\, z_a - 2\, y_s\ p_a^{(z)} z_a\big)\ .$$

Using the general property $\mathcal{H}(p^{(z)}, p^{(y)}, z, y) = \widetilde{\mathcal{H}}(\widetilde{p}^{(z)}, \widetilde{p}^{(y)}, \widetilde{z}, \widetilde{y})$ the transformed Hamiltonian $\widetilde{\mathcal{H}}$ acquires the form (3.19), where the tilde has been omitted.

A.2 Γ−matrices

Starting with the bosonic coordinates t, z_a and ϕ, y_s with $a, s = 1, \ldots, 4$ which parametrize AdS_5 and S^5 respectively, one has to do the field redefinition (3.20) in order to obtain fields Z_a, Y_a carrying definite charges. In the matrix valued $\mathfrak{psu}(2,2|4)$ charge $Q = Q(z_i\gamma_i,\, y_i\gamma_i,\, \theta, \eta)$ the bosonic coordinates z_i, y_i occur only multiplied with the 4×4 Dirac matrices γ_i, satisfying the $SO(5)$ Clifford algebra

$$\{\gamma_a, \gamma_b\} = \gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{a,b}\ . \tag{A.5}$$

To preserve the general structure of Q in terms of the new fields, the authors of [7] defined the Γ-matrices by the identity

$$z_i\gamma_i \stackrel{!}{=} Z_a\Gamma_a = Z_1\Gamma_1 + Z_2\Gamma_2 + Z_3\Gamma_3 + Z_4\Gamma_4\ , \tag{A.6}$$

which leads to

$$\begin{aligned} \Gamma_1 = \tfrac{1}{2}(\gamma_2 - i\gamma_1) \quad , \qquad \Gamma_2 = \tfrac{1}{2}(\gamma_4 - i\gamma_3) \quad , \\ \Gamma_3 = \tfrac{1}{2}(\gamma_4 + i\gamma_3) = \Gamma_2^\dagger \quad , \qquad \Gamma_4 = \tfrac{1}{2}(\gamma_2 + i\gamma_1) = \Gamma_1^\dagger \quad . \end{aligned} \tag{A.7}$$

In addition one defines

$$\Gamma_5 \equiv \Sigma = [\Gamma_1, \Gamma_4][\Gamma_2, \Gamma_3] = -\gamma_1\gamma_2\gamma_3\gamma_4\,, \quad \mathcal{P}_\pm = \frac{1}{2}(\mathbb{1} \pm \Gamma_5)\,. \tag{A.8}$$

One findes the anti-commutator relations

$$\begin{aligned} \Gamma_n\Gamma_m + \Gamma_m\Gamma_n \equiv \{\Gamma_n, \Gamma_m\} = \delta_{n,5-m} \qquad \text{for } n, m = 1, .., 4 \\ \{\Gamma_5, \Gamma_n\} = 0\ . \end{aligned} \tag{A.9}$$

Of course the form of the Γ_a is not unique, in fact one may choose every set of 4×4 matrices satisfying (A.9), because the trace of a product of Γ-matrices is fully determined by the anti-commutator relations and thus the Hamiltonian (3.40) is independent from the specific choise of Γ-matrices.
However, we will work with the explicit representation introduced in [7]:

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \Sigma \equiv \Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.10}$$

A.3 properties of f_n and g_n

In the mode decomposition (3.35) of the fermionic fields the two auxiliary functions

$$f_m = \sqrt{\frac{1}{2}\Big(1 + \frac{1}{\omega_m}\Big)}\ , \qquad g_m = \frac{\kappa\,\sqrt{\tilde{\lambda}}m}{1+\omega_m} f_m\ , \qquad \kappa = \pm 1\ .$$

appear. Here some usefull identeties are collected.

$$f_a^2 + g_a^2 = 1 \quad , \qquad f_a^2 - g_a^2 = \frac{1}{\omega_a} \quad , \qquad f_a\, g_a = \frac{\kappa\sqrt{\widetilde{\lambda}}}{2}\frac{a}{\omega_a}$$

$$f_a\, f_b - g_a\, g_b = \frac{\sqrt{(1+\omega_a)(1+\omega_b)}}{2\sqrt{\omega_a\omega_b}}\left(1 - \widetilde{\lambda}\frac{ab}{(1+\omega_a)(1+\omega_b)}\right)$$

$$f_a\, g_b - f_b\, g_a = \frac{\kappa\sqrt{\widetilde{\lambda}}}{2\,\sqrt{\omega_a\omega_b}}\frac{b(1+\omega_a) - a(1+\omega_b)}{2\sqrt{\omega_a\omega_b}}$$

B Systematic listing of numerical eigenvalues

In this section we systematically list numerical eigenvalues computed by the ABAKUS software. The tables contain values for all possible charge configurations $\{S_+, S_-, J_+, J_-\}$ of all closed subsectors of $\mathfrak{psu}(2,2|4)$ up to three impurities. In the case of four impurities only a certain selection is given. In order to convert the listed eigenvalues $-\delta P_-$ into the global energy consult chapter 6.

$$\text{For all computations } \widetilde{\lambda} = 0.1 \text{ and } P_+ = 100 \text{ was used.} \tag{B.1}$$

B.1 Two impurities

Using the analytic expressions of chapter 5 explicit values for the Hamiltonian eigenvalues $-\delta P_-$ are given. We also consider the charge configuration $\{0,0,0,0\}$, where the full Hamiltonian contributes. The mode numbers $n_1 = -n_2 = 3$ were used.

sector	charge	eigenvalues $-\delta P_-$
$\mathfrak{su}(2)$	{0,0,2,2}	0
$\mathfrak{sl}(2)$	{2,2,0,0}	0
$\mathfrak{su}(1\|1)$	{0,2,2,0}	0
$\mathfrak{su}(1\|2)$	{0,1,2,1}	± 0.0130586
$\mathfrak{su}(2\|3)$	{0,0,2,0}	± 0.0261171 $\quad 2\times 0$
	$\{0,\pm 1,2,\pm 1\}$	± 0.0130586
$\mathfrak{su}(1,1\|2)$	{1,1,1,1}	± 0.0261171 $\quad 2\times 0$
	$\{1,2,1,0\}$, $\{0,1,2,1\}$, $\{2,1,0,1\}$, $\{1,0,1,2\}$	± 0.0130586
full	{0,0,0,0}	$4\times \pm 0.0261171$ $\quad \pm 0.0522343$ $\quad 6\times 0$

Table 20: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for two impurity states. The charges are given in the convention $\{S_+, S_-, J_+, J_-\}$. The number in front of some eigenvalues denotes their multiplicity if unequal one.

The $\pm$ signs in front of some charges are just a shortform of writing several charge combinations all with the same eigenvalues. They are not related to the signatures of the eigenvalues in any sense.

B.2 Three impurities

For computing 3-impurity states a set of non-confluent mode numbers was used as well as a second set with confluent modes. For the non-confluent $n_1 = 2, n_2 = 1, n_3 = -3$ case the corresponding eigenvalues are listed in table 21, while the eigenvalues for the confluent modes $n_1 = n_2 = 3, n_3 = -6$ are given in table 22.

sector	charge	eigenvalues $-\delta P_-$
$\mathfrak{su}(2)$	{0,0,3,3}	-0.01063240
$\mathfrak{sl}(2)$	{3,3,0,0}	0.01063240
$\mathfrak{su}(1\|1)$	{0,3,3,0}	0

$\mathfrak{su}(1\|2)$ charge	eigenvalues $-\delta P_-$		
{0,1,3,2}	-0.0214958	0.000230962	0
{0,2,3,1}	-0.0108634	0.0108634	-0.0106324

$\mathfrak{su}(2\|3)$ charge	eigenvalues $-\delta P_-$			
{0,0 ,3,±3}	-0.0106324			
{0,±2,3,±1}	± 0.0108634	-0.0106324		
{0,±1,3,±2}	-0.0214958	0.000230962	0	
{0,±1,3,0}	0.0217267	3×-0.0214958	2×0.000230962	3×0
{0,0,3,±1}	-0.0323591	0.0110943	$2\times \pm 0.0108634$	3×-0.0106324

$\mathfrak{su}(1,1\|2)$ charge	eigenvalues $-\delta P_-$				
{1,0,2,3}	-0.0214958	0.000230962	0		
{1,1,2,2}	-0.0323591	0.0110943	$2\times \pm 0.0108634$	2×-0.0106324	0.0106324
{1,2,2,1}, {2,1,1,2}	± 0.0217267	± 0.0214958	± 0.000230962	3×0	
{1,3,2,0}, {3,1,0,2}	± 0.0108634	0.0106324			
{2,0,1,3}, {0,2,3,1}	± 0.0108634	-0.0106324			
{2,2,1,1}	0.0323591	-0.0110943	$2\times \pm 0.0108634$	2×0.0106324	-0.0106324
{2,3,1,0}, {3,2,0,1}	0.0214958	-0.000230962	0		

Table 21: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for three impurity states with non confluent mode numbers. The charges are given in the convention $\{S_+, S_-, J_+, J_-\}$. The number in front of some eigenvalues denotes their multiplicity if unequal one.

sector	charge	eigenvalues $-\delta P_-$
$\mathfrak{su}(2)$	{0,0,3,3}	-0.0155623
$\mathfrak{sl}(2)$	{3,3,0,0}	0.0155623
$\mathfrak{su}(1\|1)$	{0,3,3,0}	0

$\mathfrak{su}(1\|2)$ charge	eigenvalues $-\delta P_-$	
{0,1,3,2}	-0.0454059	0.0142814
{0,2,3,1}	-0.0155623	

$\mathfrak{su}(2\|3)$ charge	eigenvalues $-\delta P_-$		
{0,±1,3,0}	2×-0.0454059	2×0.0142814	
{0,0,3,±1}	-0.0752496	0.044125	3×-0.0155623
{0,±2,3,±1}, {0,0,3,±3}	-0.0155623		
{0,±1,3,±2}	-0.0454059	0.0142814	

$\mathfrak{su}(1,1\|2)$ charge	eigenvalues $-\delta P_-$			
{1,2,2,1},{2,1,1,2}	± 0.0454059	± 0.0142814		
{1,3,2,0},{3,2,0,1}	0.0155623			
{2,0,1,3}	-0.0155623			
{2,2,1,1}	0.0752496	-0.044125	2×0.0155623	-0.0155623
{1,1,2,2}	-0.0752496	0.044125	0.0155623	2×-0.0155623
{2,3,1,0},{3,2,0,1}	0.0454059	-0.0142814		
{1,0,2,3}	-0.0454059	0.0142814		

Table 22: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for three impurity states with confluent mode numbers. The charges are given in the convention $\{S_+, S_-, J_+, J_-\}$. The number in front of some eigenvalues denotes their multiplicity if unequal one.

B.3 Four impurities

Since the number of eigenvalues grows exponentially with the number of impurities, we have itemized only a certain selection of charges. Also the simple $su(2)$ and $sl(2)$ sectors are not listed anymore because closed analytical formulas are derived in section 5.1 and 5.2. Only non-confluent mode numbers $n_1 = 5, n_2 = 2, n_3 = -3, n_4 = -4$ were used. The eigenvalues are listed in table 23.

$\mathfrak{su}(2\|3)$ charge	eigenvalues $-\delta P_-$					
{0,±1,4,±3}	−0.0819702	−0.0254012	−0.0233732	−0.0155787		
{0,±2,4,±2}	−0.0585969	−0.056569	−0.0487744	0.00982251	0.00779455	0

$\mathfrak{su}(1,1\|2)$ charge	eigenvalues $-\delta P_-$				
{1,0,3,4}	−0.0819702	−0.0254012	−0.0233732	−0.0155787	
{3,1,1,3}	±0.0663915	±0.0585969	±0.056569		
	±0.00982251	±0.00779455	±0.00202796	4×0	
{2,2,2,2}	±0.0684194	$2 \times \pm 0.0663915$	±0.0643635	$2 \times \pm 0.0585969$	$2 \times \pm 0.056569$
	±0.0487744	$2 \times \pm 0.00982251$	$2 \times \pm 0.00779455$	$2 \times \pm 0.00202796$	6×0

Table 23: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for 4 impurity states with non confluent mode numbers. The charges are given in the convention $\{S_+, S_-, J_+, J_-\}$. The number in front of some eigenvalues denotes their multiplicity if unequal one.

B.4 Comments on higher impurities

To explore the possibility of computing even higher impurities with ABAKUS, we have picked a $\mathfrak{su}(1|2)$ charge which results in quite few eigenvalues compared to other 6 impurity charges as for example the $\mathfrak{su}(1,1|2)$ charge $\{4,3,2,3\}$ with 107 different eigenvalues. The results are shown in table 24, where the mode numbers $n_1 = 5, n_2 = 4, n_3 = 3, n_4 = 2, n_5 = 1, n_5 = -15$ have been used. Computations in the $\mathfrak{su}(1|2)$ sector have been performed up to 12 impurities generating about 1000 eigenvalues.

$\mathfrak{su}(1\|2)$ charge	eigenvalues $-\delta P_-$				
{0,3,6,3}	−0.242402	−0.235202	−0.233235	−0.230555	−0.228141
	−0.226174	−0.223493	−0.218973	−0.216293	−0.214326
	−0.0832433	−0.081276	−0.0785959	−0.0740759	−0.0713958
	−0.0694285	−0.0670145	−0.0643343	−0.0623671	−0.0551669

Table 24: Numerical results for the first order correction in $1/P_+$ of the string energy spectrum for 6 impurity states with non confluent mode numbers. The charges are given in the convention $\{S_+, S_-, J_+, J_-\}$. The number in front of some eigenvalues denotes their multiplicity if unequal one.

C CD containing the ABAKUS Software

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